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Gibbs measures for hardcore-SOS models on Cayley trees

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Abstract

We investigate the finite-state p-solid-on-solid model, for $p = \infty$, on Cayley trees of order $k \geq 2$ and establish a system of functional equations where each solution corresponds to a (splitting) Gibbs measure of the model. Our main result is that, for three states, $k = 2, 3$ and increasing coupling strength, the number of translation-invariant Gibbs measures behaves as $1 \to 3 \to 5 \to 6 \to 7$. This phase diagram is qualitatively similar to the one observed for three-state p-SOS models with $p > 0$ and, in the case of $k = 2$, we demonstrate that, on the level of the functional equations, the transition $p \to \infty$ is continuous.

1 Setting, models and functional equations

Statistical-mechanics models on trees are known to possess rich structural properties, see for example [5, 18] for general references. In the present note we contribute to this field by analyzing a hardcore model that emerges as a limit of the p-SOS model that was recently introduced in [3]. The setting for this ∞-*SOS model* is as follows.

Let $\Gamma^k = (V, L)$ be the uniform *Cayley tree* where each vertex has $k + 1$ neighbors with V being the set of vertices and L the set of edges. Endpoints x, y of an edge $\ell = \langle x, y\rangle$ are called *nearest neighbors*. On the Cayley tree there is a natural distance, to be denoted $d(x, y)$, being the smallest number of nearest-neighbors pairs in a path between the vertices x and y, where a *path* is a sequence of nearest-neighbor pairs of vertices where two consecutive pairs share at least one vertex. For a fixed $x^0 \in V$, the *root*, we let

$$V_n = \{x \in V \colon d(x, x^0) \leq n\} \quad \text{and} \quad W_n = \{x \in V \colon d(x^0, x) = n\}$$

denote the *ball* of radius n, respectively the *sphere* of radius n, both with center at x^0. Further, let $S(x)$ be the *direct successors* of x, i.e., for $x \in W_n$

$$S(x) = \{y \in W_{n+1} \colon d(x, y) = 1\}.$$

Next, we denote by $\Phi = \{0, 1, \dots, m\}$ the *local state space*, i.e., the space of values of the spins associated to each vertex of the tree. Then, a *configuration* on the Cayley tree is a collection $\sigma = \{\sigma(x) \colon x \in V\} \in \Phi^V = \Omega$.

Let us now describe hardcore interactions between spins of neighboring vertices. For this, let $G = (\Phi, K)$ be a graph with vertex set Φ, the set of spin values, and edge set K. A configuration σ is called G-*admissible* on a Cayley tree if $\{\sigma(x), \sigma(y)\} \in K$ is an edge of G for any pair of nearest neighbors $\langle x, y\rangle \in L$. We let Ω^G denote the sets of G-admissible configurations. The restriction of a configuration on a subset A of V is denoted by σ_A and Ω^G_A denotes the set of all G-admissible

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configurations on A. On a general level, we further define the *matrix of activity* on edges of G as a function

$$\lambda\colon \{i,j\} \in K \to \lambda_{i,j} \in \mathbb{R}_+,$$

where $\mathbb{R}_+$ denotes the positive real numbers and $\lambda_{i,j}$ is called the *activity* of the edge $\{i,j\} \in K$. In this note, we consider the graph G as shown in Figure 1, which is called a *hinge-graph*, see for example [2]. In words, in the hinge-graph G, configuration are admissible only if, for any pair of nearest-neighbor vertices $\{x,y\}$, we have that

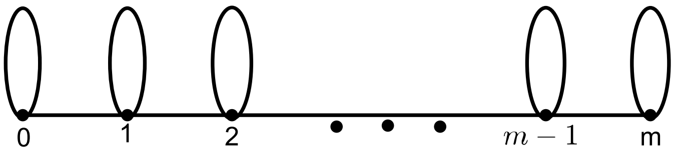

Figure 1: The hinge-graph G with $m+1$ vertices.

$$|\sigma(x) - \sigma(y)| \in \{0,1\}. \tag{1}$$

Let us also note that our choice of admissibilities generalizes certain finite-state random homomorphism models, see [6, 11], where only configurations with $|\sigma(x) - \sigma(y)| = 1$ are allowed.

Our main interest lies in the analysis of the set of *splitting Gibbs measures* (SGMs) defined on hinge-graph addmissible configurations. Let us start by defining SGMs for general admissibility graphs G. Let

$$z\colon x \mapsto z_x = (z_{0,x}, z_{1,x}, \ldots, z_{m,x}) \in \mathbb{R}_+^{m+1}$$

be a vector-valued function on V. Then, given $n = 1, 2, \ldots$ and an activity $\lambda = (\lambda_{i,j})_{\{i,j\}\in K}$, consider the probability distribution $\mu^{(n)}$ on $\Omega^G_{V_n}$, defined as

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{\langle x,y\rangle \in V_n} \lambda_{\sigma_n(x),\sigma_n(y)} \prod_{x\in W_n} z_{\sigma(x),x}, \tag{2}$$

where $\sigma_n = \sigma_{V_n}$. Here Z_n is the *partition function*

$$Z_n = \sum_{\widetilde{\sigma}_n \in \Omega^G_{V_n}} \prod_{\langle x,y\rangle \in V_n} \lambda_{\widetilde{\sigma}_n(x),\widetilde{\sigma}_n(y)} \prod_{x\in W_n} z_{\widetilde{\sigma}(x),x}.$$

The sequence of probability distributions $(\mu^{(n)})_{n\geq 1}$ is called *compatible* if, for all $n \geq 1$ and σ_{n-1}, we have that

$$\sum_{\omega_n \in \Omega^G_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)\mathbf{1}\{\sigma_{n-1} \vee \omega_n \in \Omega^G_{V_n}\} = \mu^{(n-1)}(\sigma_{n-1}), \tag{3}$$

where $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations σ_{n-1} and ω_n. Note that, by Kolmogorov's extension theorem, for a compatible sequence of distributions, there exists a unique measure μ on Ω^G such that, for all n and $\sigma_n \in \Omega^G_{V_n}$,

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n).$$

This motivates the following definition.

Definition 1. *We call the measure μ, defined by* (2) *and* (3)*, the* splitting Gibbs measure *corresponding to the activity λ and the function $z\colon\ x\in V\setminus\{x^0\}\mapsto z_x$.*

Let $A=A^G=\big(a_{ij}\big)_{\{i,j\}\in K}$ denote the adjacency matrix of G, i.e.,

$$a_{ij}=a_{ij}^G=\begin{cases}1, & \text{if } \{i,j\}\in K,\\ 0, & \text{if } \{i,j\}\notin K,\end{cases}$$

then, the following statement describes conditions on z_x guaranteeing compatibility of the distributions $(\mu^{(n)})_{n\geq 1}$.

Theorem 1. *The sequence of probability distributions $(\mu^{(n)})_{n\geq 1}$ in* (2) *are compatible if and only if, for any $x\in V$, the following system of equations holds*

$$z_{i,x}=\prod_{y\in S(x)}\frac{\sum_{j=0}^{m-1}a_{ij}\lambda_{i,j}z_{j,y}+a_{im}\lambda_{i,m}}{\sum_{j=0}^{m-1}a_{mj}\lambda_{m,j}z_{j,y}+a_{mm}\lambda_{m,m}},\quad i=0,1,\ldots,m-1. \tag{4}$$

Proof. The proof is similar to the proof of [17, Theorem 1] and [16, Proposition 2.1]. □

Note that, in (4), the normalization is at the spin state m, i.e., we assume that, for all $x\in V$, we have $z_{m,x}=1$.

In the remainder of the manuscript, we restrict our choice of activities in order to make contact to p-SOS models defined via the formal Hamiltonian

$$H(\sigma)=-J\sum_{\langle x,y\rangle}|\sigma(x)-\sigma(y)|^p, \tag{5}$$

for $p>0$ and coupling constant $J\in\mathbb{R}$, see [3, 5, 18, 19] and references therein. The present note then presents a continuation of previous investigations related to p-SOS models on trees with $p>0$, but now in the case where $p=\infty$. More precisely, we denote $\theta=\exp(J)$ and consider the activity $\lambda=(\lambda_{i,j})_{\{i,j\}\in K}$ defined as

$$\lambda_{i,j}=\begin{cases}1, & \text{if } i=j\\ \theta, & \text{if } |i-j|=1,\\ \infty, & \text{otherwise.}\end{cases} \tag{6}$$

We call the resulting hinge-graph model (with hinge-graph as in Figure 1) the ∞-*SOS model*. Consequently, for the ∞-SOS model with activity (6), the equation (4) reduces to

$$\begin{aligned} z_{0,x}&=\textstyle\prod_{y\in S(x)}\frac{z_{0,y}+\theta z_{1,y}}{\theta z_{m-1,y}+1}\\ z_{i,x}&=\textstyle\prod_{y\in S(x)}\frac{\theta z_{i-1,y}+z_{i,y}+\theta z_{i+1,y}}{\theta z_{m-1,y}+1},\quad i=1,\ldots,m-1\\ z_{m,x}&=1,\end{aligned} \tag{7}$$

and, by Theorem 1, for any $z=\{z_x\colon\ x\in V\}$ satisfying (7), there exists a unique SGM μ for the ∞-SOS. However, the analysis of solutions to (7) for an arbitrary m is challenging. We therefore restrict our attention to a smaller class of measures, namely the translation-invariant SGMs.

2 Translation-invariant SGMs for the ∞-SOS model with $m = 2$

Searching only for translation-invariant measures, the functional equation (7) reduces to

$$\begin{aligned} z_0 &= \left(\tfrac{z_0+\theta z_1}{\theta z_{m-1}+1}\right)^k, \\ z_i &= \left(\tfrac{\theta z_{i-1}+z_i+\theta z_{i+1}}{\theta z_{m-1}+1}\right)^k, \ \ i = 1, \dots, m-1, \\ z_m &= 1. \end{aligned} \tag{8}$$

In the following we restrict our attention to the case where $m = 2$. In this case, denoting $x = \sqrt[k]{z_0}$ and $y = \sqrt[k]{z_1}$, from (8) we get

$$x = \frac{x^k + \theta y^k}{\theta y^k + 1} \qquad \text{and} \qquad y = \frac{\theta x^k + y^k + \theta}{\theta y^k + 1}. \tag{9}$$

In particular, considering only the first equation of this system, we find the solutions $x = 1$ and

$$\theta y^k = x^{k-1} + x^{k-2} + \cdots + x. \tag{10}$$

We start by investigating the case $x = 1$.

2.1 Case $x = 1$

In this case, from the second equation in (9), we get that

$$\theta y^{k+1} - y^k + y - 2\theta = 0 \tag{11}$$

and hence, as a direct application of Descartes' rule of signs, the following statement follows.

Lemma 1. *For all $k \geq 2$, there exist at most three positive roots for* (11).

For small values of k, i.e., $k = 2, 3$, we can solve (11) explicitly and exhibit regions of θ where there are exactly three solutions.

2.1.1 Case $k = 2$

In this case, the equation (11) coincides with the corresponding equation for $p = \infty$ found in [7]. The solutions are presented in Section 3 below.

2.1.2 Case $k = 3$

In this case, we can rearrange (11) as

$$\theta = \frac{y^3 - y}{y^4 - 2} =: \alpha(y),$$

where we assumed $y \neq \sqrt[4]{2}$ since $y = \sqrt[4]{2}$ is not a solution. It follows from Figure 2 that up to three solutions y for (11) appear as follows. There exists $\theta_c \approx 0.206$ such that we have:

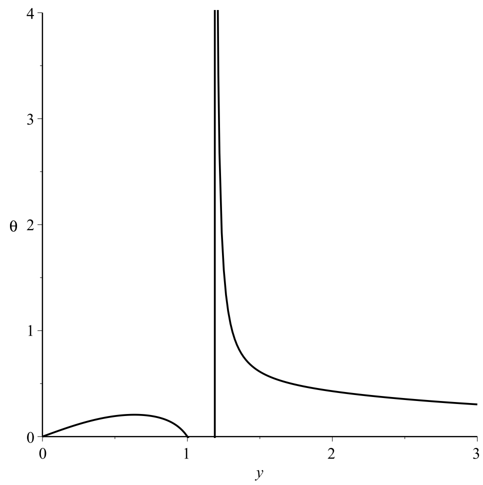

Figure 2: Graph of the function $\alpha(y)$.

1) If $\theta \in (0, \theta_c)$, then there are three solutions $y_1 > \sqrt[4]{2}$ and $y_2 < y_3 < 1$.
2) If $\theta = \theta_c$, then there are two solutions $y_1 > \sqrt[4]{2}$ and $y_2 < 1$.
3) If $\theta > \theta_c$, then there is a unique solution $y_1 > \sqrt[4]{2}$.

We can derive the value of θ_c explicitly as follows. The equation $\alpha'(y) = 0$ can be solved explicitly as

$$y_0 = \sqrt{1 - c + 1/c} \approx 0.635, \text{ with } c = \sqrt[3]{1 + \sqrt{2}}.$$

Since $y_0 \in (0, 1)$, the critical value θ_c is thus given by

$$\theta_c = \alpha(y_0) = \frac{(1-c)\sqrt{c(1+c-c^2)}}{c^4 - 3c^2 + 2c - 2\sqrt{2} - 1}. \tag{12}$$

2.2 Case $x \neq 1$

Let us consider the second situation as presented in (9).

2.2.1 Case $k = 2$

In this case, using (10) and the second equation in (9), we get

$$\theta^4 x^4 + (2\theta^2 - \theta)x^3 + (2\theta^4 - 2\theta + 1)x^2 + (2\theta^2 - \theta)x + \theta^4 = 0, \tag{13}$$

which is a polynomial with symmetric coefficients and hence, denoting $\xi = x + 1/x$, (13) can be rewritten as

$$\theta^4(\xi^2 - 2) + (2\theta^2 - \theta)\xi + (2\theta^4 - 2\theta + 1) = 0,$$

which is equivalent to

$$\theta^4\xi^2 + (2\theta^2 - \theta)\xi + 1 - 2\theta = 0. \tag{14}$$

But, this equation has two solutions $\xi_{1,2}$ given in (28) below and thus we have up to four additional solutions. Again, this case coincides with the corresponding equation for $p = \infty$ found in [7] and the overview of solutions is presented in Section 3 below.

2.2.2 Case $k = 3$

In this case, by (10) and the second equation in (9) we obtain

$$\theta^6x^8 + (-\theta^6 + 3\theta^4 - \theta^2)x^7 + \theta^6x^6 + (2\theta^6 - 3\theta^2 + 1)x^5 + (-2\theta^6 + 6\theta^4 - 7\theta^2 + 2)x^4 \\ + (2\theta^6 - 3\theta^2 + 1)x^3 + \theta^6x^2 + (-\theta^6 + 3\theta^4 - \theta^2)x + \theta^6 = 0 \quad (15)$$

and this equation may have up to eight positive solutions since, if $\theta < \theta_c'' \approx 0.605$, the number of sign changes in the coefficients is eight. However, by computer analysis we can show that there exists $\widehat{\theta}_c \approx 0.4812$ such that, if $\theta < \widehat{\theta}_c$, then (15) has precisely four positive solutions, if $\theta = \widehat{\theta}_c$, then there are precisely two solutions and, if $\theta > \widehat{\theta}_c$, then there exists no positive solution (see Figure 3 for an implicit plot). For example, if $\theta = 0.481$, then the four positive solutions are given approximately by

$$0.2072006567, 0.2260627940, 4.423549680, \text{ and } 4.826239530.$$

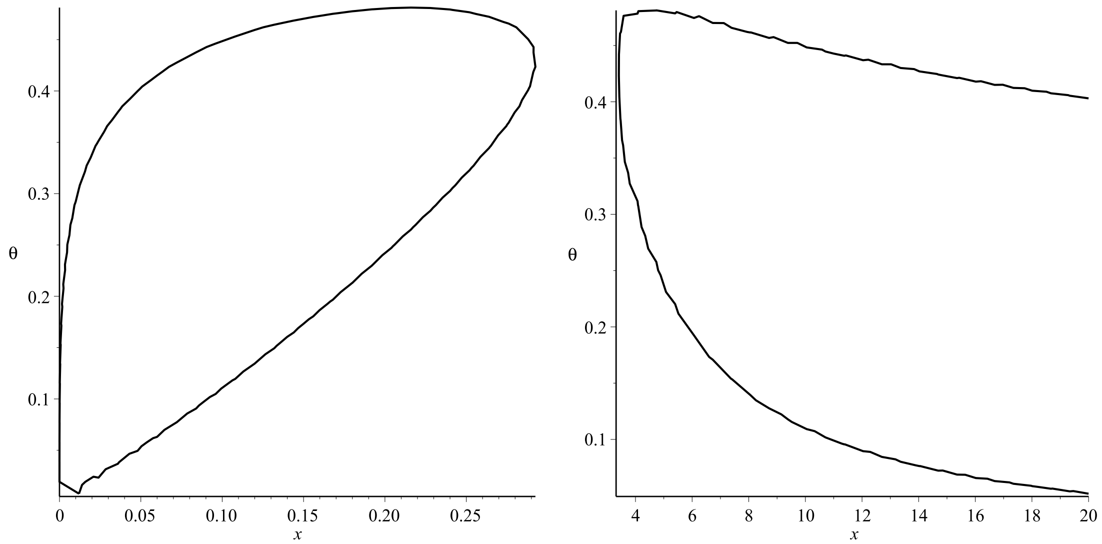

Figure 3: The implicit plot of (15). Left: For $\theta \in (0, \widehat{\theta}_c)$, there are two solutions in $(0, 0.2927)$. Right: For the same region of θ there are two fixed points in $(3.41, \infty)$ since both curves towards the right side converge to zero.

Let us provide an analytical proof of this as well. Since (15) is symmetric, we divide both sides by x^4 and denote $\eta = x + 1/x$. Then, we arrive at

$$\theta^6((\eta^2-2)^2-2)+(-\theta^6+3\theta^4-\theta^2)\eta(\eta^2-3)+\theta^6(\eta^2-2)+(2\theta^6-3\theta^2+1)\eta-2\theta^6+6\theta^4-7\theta^2+2 = 0$$

and consequently

$$\theta^6\eta^4 + (-\theta^6 + 3\theta^4 - \theta^2)\eta^3 - 3\theta^6\eta^2 + (5\theta^6 - 9\theta^4 + 1)\eta - 2\theta^6 + 6\theta^4 - 7\theta^2 + 2 = 0. \quad (16)$$

Thus, each solution $\eta > 2$ of (16) defines two positive solutions to (15). Denoting $t = \theta^2$, we write (16) as the following cubic equation with respect to t,

$$(\eta^4 - \eta^3 - 3\eta^2 + 5\eta - 2)t^3 + (3\eta^3 - 9\eta + 6)t^2 + (-\eta^3 - 7)t + \eta + 2 = 0,$$

which is equivalent to

$$(\eta-1)^3(\eta+2)t^3+3(\eta-1)^2(\eta+2)t^2-(\eta^3+7)t+\eta+2=0. \tag{17}$$

Now, again by the rule of sign changes, for each $\eta > 2$, this equation may have up to two positive solution $t=\theta^2=t(\eta)$. For $\eta > 2$ let us introduce the new variables

$$w=(\eta-1)t>0 \qquad \text{and} \qquad E=\frac{\eta^3+7}{(\eta-1)(\eta+2)}=: b(\eta). \tag{18}$$

With this, (17) can be expressed as

$$w^3+3w^2-Ew+1=0 \tag{19}$$

and we can solve the last equation with respect to E, which leads to

$$E=w^2+3w+1/w=: a(w).$$

Note that the function $a(w)$ is monotone decreasing between 0 and $1/2$ (because $a'(w)=0$ has a unique positive solution $w=1/2$ and $a(0)=a(+\infty)=\infty$) and increasing when $w>1/2$. Thus, the minimal value of E is given by $a(1/2)=15/4$ and hence, for each $E>15/4$, there are exactly two positive solutions $w_1=w_1(E)$ and $w_2=w_2(E)$, with $w_1<w_2$. If $E=15/4$, then there exists a unique $w_c=1/2$. If $E<15/4$, then there is no solution, see Figure 4. From Figure 4 it is also

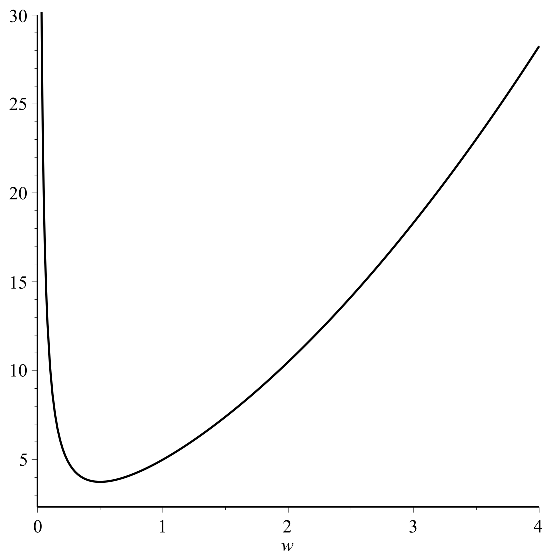

Figure 4: The graph of the function $E=a(w)$.

easy to see that $w_1(E)$ is a decreasing function with values in $(0,1/2]$ and $w_2(E)$ is an increasing function with values in $[1/2,\infty)$ with respect to the variable $E>15/4$. Moreover,

$$\lim_{E\to\infty} w_1(E)=0 \qquad \text{and} \qquad \lim_{E\to\infty} w_2(E)=\infty,$$

and from the function $b(\eta)$ given in (18), for $E=15/4$, we get that

$$\eta_c=(7+3\sqrt{57})/8\approx 3.707.$$

Note that, for the derivative of $\eta \mapsto b(\eta)$, we have

$$b'(\eta) = \frac{(\eta+1)^2(\eta^2-7)}{(\eta-1)^2(\eta+2)^2}.$$

Consequently, $b(\eta)$ is increasing for $\eta > \sqrt{7} \approx 2.65$ with a minimum value $b(\sqrt{7}) = 21/(1+2\sqrt{7}) \approx 3.34$. For each $E \geq 15/4$, since $b(\eta)$ is an increasing function, from $E = b(\eta)$ one obtains a unique $\eta \geq \eta_c$. As a consequence, $w_1(b(\eta))$ is a decreasing and $w_2(b(\eta))$ is an increasing functions of $\eta > \eta_c$. Thus, η is a solution to the following two independent equations, obtained from the first formula of (18),

$$w_1(b(\eta)) = (\eta - 1)\theta^2. \tag{20}$$

$$w_2(b(\eta)) = (\eta - 1)\theta^2. \tag{21}$$

Denoting by $\tilde{\theta}$ the positive solution of the equation

$$1/2 = (\eta_c - 1)\tilde{\theta}^2,$$

see (18), we have

$$\tilde{\theta} = \frac{2}{\sqrt{3\sqrt{57}-1}} \approx 0.4298.$$

Then, as can be seen in Figure 5, the following assertions hold.

i. If $\theta \in (0, \tilde{\theta})$, then (20) and (21) have a unique solution $\eta > 2$.

ii. If $\theta \in [\tilde{\theta}, \widehat{\theta}_c)$, then (20) has no solution, but (21) has two solutions greater than 2.

iii. If $\theta = \widehat{\theta}_c$, then (20) has no solution, but (21) has a unique solution.

iv. If $\theta > \widehat{\theta}_c$, then both equations have no solution.

Consequently, under the above mentioned Conditions i.-ii., (16) has two solutions greater then 2, which define four positive solutions for (15).

Remark 1. *To find the exact critical value $\widehat{\theta}_c$ mentioned above, one has to solve the following system of equation with respect to unknowns $\widehat{\theta}_c$ and η_1*

$$w_2'(b(\eta_1))b'(\eta_1) = \widehat{\theta}_c^2 \qquad \text{and} \qquad w_2(b(\eta_1)) = (\eta_1 - 1)\widehat{\theta}_c^2.$$

With respect to Theorem 1, we can summarize our results for $k = 3$ in the following statement.

Theorem 2. *For the ∞-SOS model with $m = 2$ and $k = 3$, there exist critical values $\theta_c \approx 0.206$ (given explicitly by* (12)*) and $\widehat{\theta}_c \approx 0.4812$ such that*

1. *If $\theta > \widehat{\theta}_c$, then there is unique translation-invariant SGM.*
2. *If $\theta = \widehat{\theta}_c$, then there are three translation-invariant SGMs.*
3. *If $\theta \in (\theta_c, \widehat{\theta}_c)$, then there are five translation-invariant SGMs.*
4. *If $\theta = \theta_c$, then there are six translation-invariant SGMs.*
5. *If $\theta \in (0, \theta_c)$, then there are seven translation-invariant SGMs.*

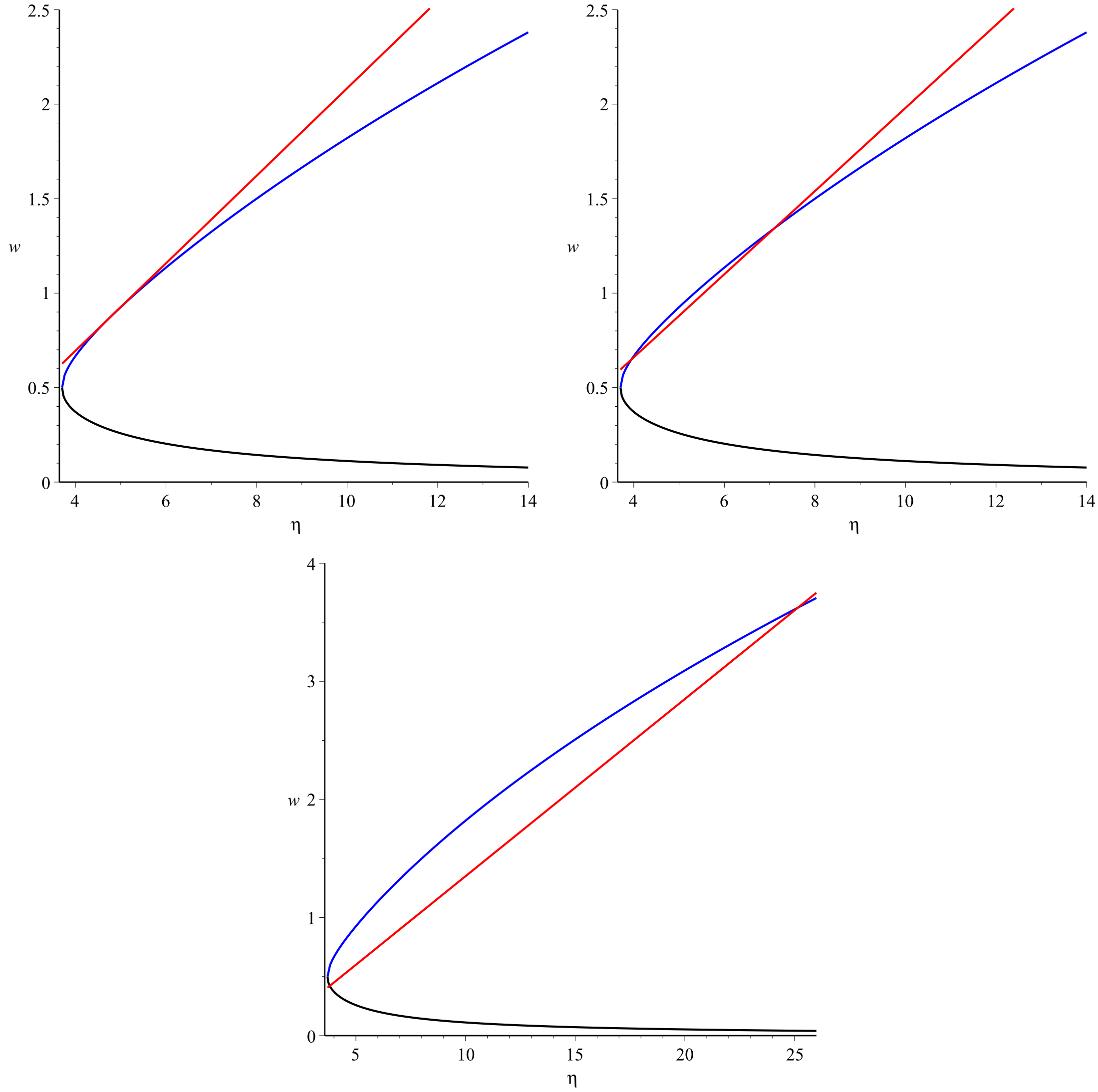

Figure 5: Up-left: The graphs of $\eta \mapsto (\eta-1)\widehat{\theta}_c^2$ (red),$\eta \mapsto w_2(b(\eta))$ (blue) and $\eta \mapsto w_1(b(\eta))$ (black). Up-right: The graphs of $\eta \mapsto (\eta-1)0.22$ (red), $\eta \mapsto w_2(b(\eta))$ (blue) and $\eta \mapsto w_1(b(\eta))$ (black). Down: The graphs of $\eta \mapsto (\eta-1)0.15$ (red), $\eta \mapsto w_2(b(\eta))$ (blue) and $w_1(b(\eta))$ (black).

3 The p-SOS model with $p \to \infty$

In this section we exhibit the functional equations corresponding to the p-SOS model defined by the Hamiltonian (5) and give results related to the case when $p \to \infty$. The limiting equations turn out to be the same equations as the ones for the ∞-SOS model. Assuming $k = m = 2$, [7] establishes and analyzes the translation-invariant SGMs of the p-SOS model corresponding to the positive solutions of the following system

$$x = \frac{x^2 + \theta y^2 + \theta^{2^p}}{\theta^{2^p} x^2 + \theta y^2 + 1}, \tag{22}$$

$$y = \frac{\theta x^2 + y^2 + \theta}{\theta^{2^p} x^2 + \theta y^2 + 1}. \tag{23}$$

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In the following, we establish limits for the obtained solutions when $p \to \infty$. First, from (22) we get $x = 1$ or

$$\theta y^2 = (1-\theta^{2^p})x - \theta^{2^p}(x^2+1). \tag{24}$$

Remark 2. *Since $x > 0$ we have that (24) can hold iff $\theta < 1$.*

3.1 Case $0 < \theta < 1$.

Let us distinguish two subcases.

3.1.1 Case $x = 1$

Solving by computer the cubic equation

$$\theta y^3 - y^2 + (\theta^{2^p}+1)y - 2\theta = 0 \tag{25}$$

and taking limits of each solution as $p \to \infty$, we see that the solutions have the limits $y_i(\theta)$, $i = 1, 2, 3$. The limiting functions have lengthy formulas, but their graphs can be simply plotted as shown in Figure 6. Moreover, the critical value of θ for existence of more than one solution is obtained from the discriminant of the cubic equation as $p \to \infty$, i.e.,

$$\Delta_0(\theta) = \frac{1}{27\theta^2}\left(4(1-3\theta)^3 - (2-9\theta+54\theta^3)^2\right) = 0.$$

Hence, by Figure 6 it is clear that there exists a unique $\theta_0 \approx 0.135$ such that $\Delta_0(\theta_0) = 0$.

3.1.2 Case $x \neq 1$

In this case, there are up to four solutions when $p > 0$ is fixed. These solutions are defined by the quantities $\xi_1(\theta, p) < \xi_2(\theta, p)$ given by

$$\xi_{1,2}(\theta, p) := \frac{q}{2}\frac{-3\theta q^2+2(\theta+1)q+2(\theta^2-1)\mp\theta\sqrt{q(q+2\theta-2)[(q-\theta-1)^2+(\theta+1)(3\theta-1)]}}{(q-\theta-1)[\theta q^2+(\theta^2-1)(q+\theta-1)]}, \tag{26}$$

where $q = 1-\theta^{2^p}$, see [7]. Moreover, if

$$2 < \xi_1(\theta, p) \le \xi_2(\theta, p), \tag{27}$$

one can find all four positive solutions $x_i = x_i(\theta, p)$, $i = 4, 5, 6, 7$ explicitly. In this case, as $p \to \infty$, we check the Condition (27). From (26) we get

$$\lim_{p\to\infty} \xi_{1,2}(\theta, p) = \xi_{1,2}(\theta) := \frac{1}{2\theta^3} \cdot \left(1 - 2\theta \mp \sqrt{(2\theta-1)(4\theta^2+2\theta-1)}\right) \tag{28}$$

and these numbers exist iff

$$(2\theta-1)(4\theta^2+2\theta-1) \ge 0 \quad \Leftrightarrow \quad \theta \in (0, (\sqrt{5}-1)/4] \cup [1/2, 1).$$

Thus, Condition (27) is satisfied iff $\theta \in (0, (\sqrt{5}-1)/4]$ (see Figure 7). Now using (28), for $\theta \in (0, (\sqrt{5}-1)/4))$, we obtain $x_i(\theta)$, $i = 4, 5, 6, 7$. Since the last x_i's exist, we get

$$y_i(\theta) = \sqrt{x_i(\theta)/\theta}, \quad i = 4, 5, 6, 7.$$

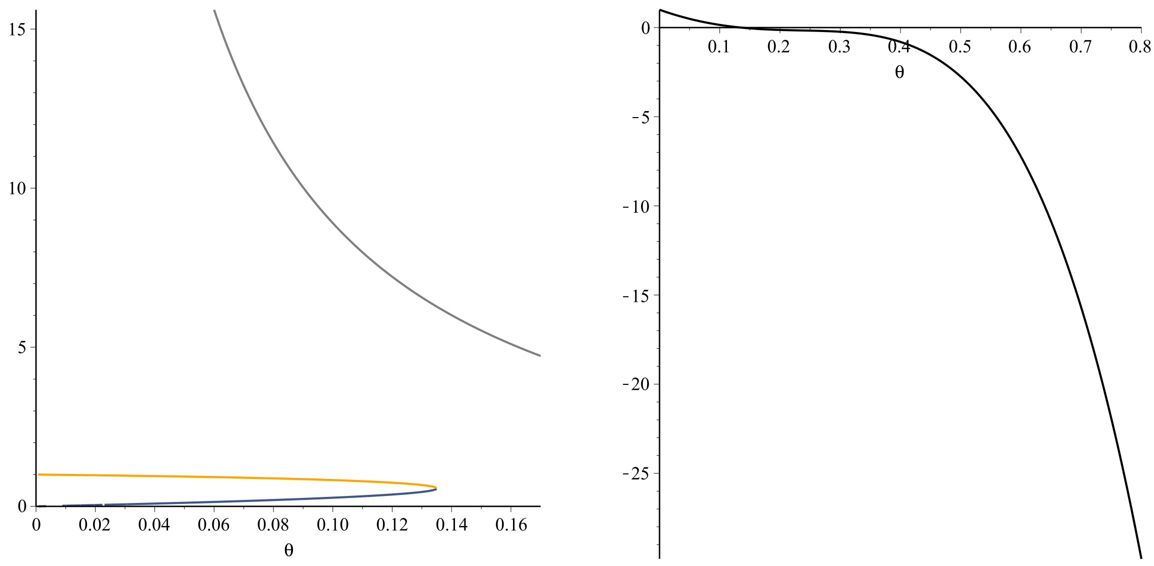

Figure 6: Left: The graphs of the functions $\theta \mapsto y_1(\theta)$ (gray), $\theta \mapsto y_2(\theta)$ (blue) and $\theta \mapsto y_3(\theta)$ (orange). Right: The graph of the function $\theta \mapsto \Delta_0(\theta)$ for $\theta \in (0, 1)$.

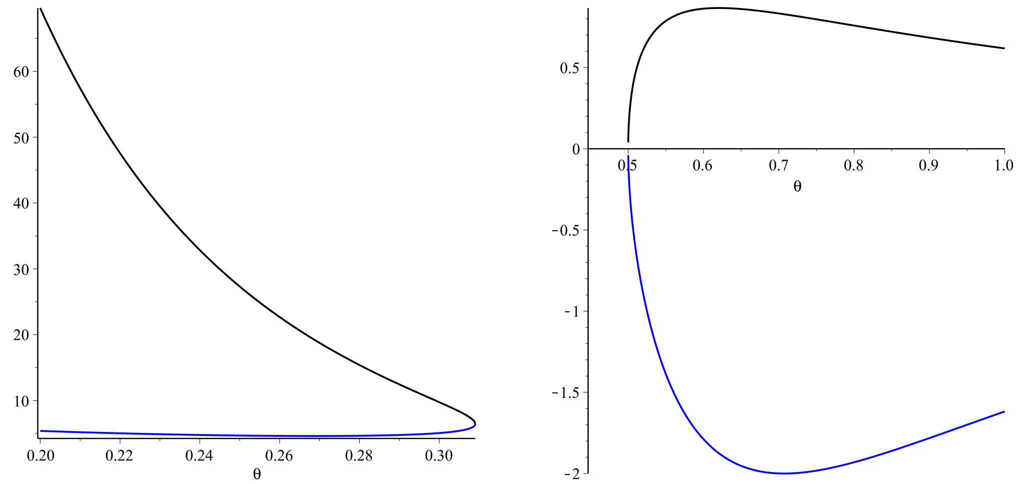

Figure 7: Left: The graphs of the functions $\theta \mapsto \xi_1(\theta)$ (blue) and $\theta \mapsto \xi_2(\theta)$ (black) for $\theta \in (0, (\sqrt{5}-1)/4]$. Right: The same graphs for $\theta \in [\frac{1}{2}, 1)$.

3.2 Case $\theta > 1$

In this case, assuming $x = 1$, from (25) we get a unique solution for large p, which has a limit as $p \to \infty$. If $\theta > 1$ and $x \neq 1$ then the statement of Remark 2 is satisfied for any $p > 0$ and therefore there is no solution. We summarize the results of this section in the following statement which essentially says that the number of translation-invariant SGMs remains unchanged in the limiting model as $p \to \infty$.

Proposition 1. *For the p-SOS model, as $p \to \infty$, there exist critical values $\theta_0 \approx 0.135$ and $\theta_0' = (\sqrt{5}-1)/4 \approx 0.309$ such that*

1. *If $\theta > \theta_0'$, then there is a unique translation-invariant SGM.*

2. *If $\theta = \theta_0'$, then there are three translation-invariant SGMs.*

3. *If $\theta \in (\theta_0, \theta_0')$, then there are five translation-invariant SGMs.*

4. *If $\theta = \theta_0$, then there are six translation-invariant SGMs.*

5. *If $\theta \in (0, \theta_0)$, then there are seven translation-invariant SGMs.*

4 Conditions for non-extremality of translation-invariant SGMs

It is known that a translation-invariant SGM corresponding to a vector $v = (x, y) \in \mathbb{R}^2$ (which is a solution to (9)) is a tree-indexed Markov chain with states $\{0, 1, 2\}$, see [5, Definition 12.2], and for the transition matrix

$$\mathbb{P} = \begin{pmatrix} \frac{x^k}{x^k+\theta y^k} & \frac{\theta y^k}{x^k+\theta y^k} & 0 \\ \frac{\theta x^k}{\theta x^k+y^k+\theta} & \frac{y^k}{\theta x^k+y^k+\theta} & \frac{\theta}{\theta x^k+y^k+\theta} \\ 0 & \frac{\theta y^k}{\theta y^k+1} & \frac{1}{\theta y^k+1} \end{pmatrix}. \tag{29}$$

Hence, for each given solution (x_i, y_i), $i = 1, \ldots, 7$ of (9), we need to calculate the eigenvalues of $\mathbb{P}$. The first eigenvalue is one since we deal with a stochastic matrix, the other two eigenvalues

$$\lambda_j(x_i, y_i, \theta, k), \qquad j = 1, 2, \tag{30}$$

can be found via symbolic computer analysis, but they have bulky formulas. For example, in the case $x = 1$, for each y the matrix (29) has three eigenvalues, 1 and

$$\lambda_1(1, y, \theta, k) = \frac{(1 - 2\theta^2)y^k}{\theta y^{2k} + (2\theta^2 + 1)y^k + 2\theta} \qquad \text{and} \qquad \lambda_2(1, y, \theta, k) = \frac{1}{\theta y^k + 1}.$$

However, we can still deduce the following relation.

Lemma 2. *If $\theta \in (0, 1)$, then, for any solution y of* (11)*, we have that*

$$|\lambda_1(1, y, \theta, k)| \leq \lambda_2(1, y, \theta, k).$$

Proof. Since $\lambda_2 > 0$, we have to show that

$$-\lambda_2(1, y, \theta, k) \leq \lambda_1(1, y, \theta, k) \leq \lambda_2(1, y, \theta, k). \tag{31}$$

It is easy to see that the inequality on the left is true for θ satisfying $1 - 2\theta^2 \geq 0$. If $1 - 2\theta^2 < 0$ then the inequality on the left is equivalent to

$$\theta(1 - \theta^2)y^{2k} + y^k + \theta \geq 0,$$

which is true for all $\theta < 1$. Next, the inequality on the right of (31) is equivalent to the inequality

$$(\theta y^k + 1)^2 \geq 0,$$

which is universally true, concluding the proof. □

Now, a sufficient condition for non-extremality of a Gibbs measure μ corresponding to $\mathbb{P}$ on a Cayley tree of order $k \geq 1$ is given by the Kesten–Stigum Condition $k\lambda^2 > 1$, where λ is the second-largest (in absolute value) eigenvalue of $\mathbb{P}$, see [8]. Hence, denoting for $i = 1, \dots, 7$,

$$\eta_i(\theta, k) = k\lambda_2^2(x_i, y_i, \theta, k) - 1 \qquad \text{and} \qquad \mathbb{K}_i = \{(\theta, k) \in (0,1) \times \mathbb{N} \colon \eta_i(\theta, k) > 0\},$$

using Lemma 2, we have the following criterion.

Proposition 2. *Let μ_i denote the translation-invariant SGM associated to the tuple (x_i, y_i, θ, k). If $(\theta, k) \in \mathbb{K}_i$ then μ_i is non-extremal.*

In order to employ the proposition, for $k = 2$ and $k = 3$, we find representations for $\mathbb{K}_i$. In case $k = 2$ and $x_i = 1$, we have for $i = 1, 2, 3$ that $\lambda_2(1, y_i, \theta, 2) = 1/(\theta y_i^2 + 1)$ and thus

$$\eta_i(\theta, 2) = \frac{2}{(\theta y_i^2 + 1)^2} - 1.$$

Hence, from Figure 8 it follows that, for μ_1, the Kesten–Stigum condition is never satisfied, but for μ_2

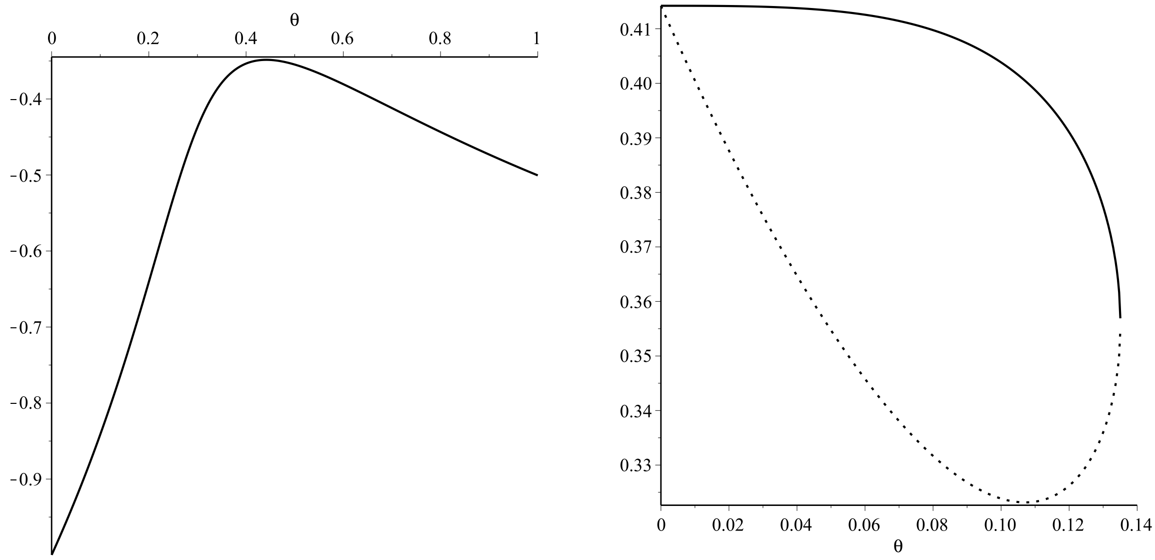

Figure 8: Left: The graph of the function $\eta_1(\theta, 2)$, $\theta \in (0,1)$. Right: The graphs of the functions $\eta_2(\theta, 2)$ (solid line) and $\eta_3(\theta, 2)$ (doted line) when $\theta \in (0, 0.14)$.

and μ_3 the condition is always satisfied, i.e., μ_2 and μ_3 are not-extreme.

In case $k = 3$ and $x_i = 1$, we have for $i = 1, 2, 3$, using $\lambda_2(1, y_i, \theta, 3) = 1/(\theta y_i^3 + 1)$, that

$$\eta_i(\theta, 3) = \frac{3}{(\theta y_i^3 + 1)^2} - 1.$$

But, as shown in Section 2.1.2, if $\theta < \theta_c \approx 0.206$, there exist two solutions $y_2, y_3 < 1$. For these solutions we have

$$\eta_i(\theta, 3) = \frac{3}{(\theta y_i^3 + 1)^2} - 1 > \frac{3}{(\theta + 1)^2} - 1.$$

But, since $\theta < \theta_c$, we have that $3/(\theta + 1)^2 - 1 > 0$ and we can formulate the following summarizing result.

Proposition 3. *For $\theta < 1$, $k = 2$ and $k = 3$ the translation-invariant SGMs corresponding to solutions of the form $(1, y)$ with $y < 1$ are not-extreme.*

Let us note that for $k = 3$ and $x_i = 1$ the translation-invariant SGM corresponding to the solution $y_1 > \sqrt[4]{2}$ does not satisfy the Kesten–Stigum condition if $\theta > (\sqrt{3} - 1)/\sqrt[4]{8} \approx 0.435$. Indeed, using $y_1 > \sqrt[4]{2}$ we get

$$\eta_i(\theta, 3) = \frac{3}{(\theta y_1^3 + 1)^2} - 1 < \frac{3}{(\theta \sqrt[4]{8} + 1)^2} - 1 < 0 \;\Leftrightarrow\; \theta > \frac{\sqrt{3} - 1}{\sqrt[4]{8}}.$$

Remark 3. *Let us finally discuss further extremality conditions for translation-invariant SGMs. Various approaches in the literature aim to establish sufficient conditions for extremality, which can be simplified to a finite-dimensional optimization problem based solely on the transition matrix. For instance, the percolation method proposed in [13] and [14], the symmetric-entropy method by [4], or the bound provided in [12] for the Ising model in the presence of an external field. Different techniques are employed also in [1] in order to demonstrate the sharpness of the Kesten-Stigum bound for an Ising channel with minimal asymmetry.*

However, since, in our case, the transition matrix corresponding to a translation-invariant SGM depends on the solutions (x_i, y_i), which have a very complex form, it appears challenging to apply the aforementioned methods to verify extremality. Furthermore, the difficulty increases when we only have knowledge of the existence of a solution but lack its explicit form. Nonetheless, our results could serve as a basis for numerical investigations of extremality in the future.

References

[1] C. Borgs, J. Chayes, E. Mossel, S. Roch, *The Kesten-Stigum reconstruction bound is tight for roughly symmetric binary channels*, FOCS, 2006, 47th Annual IEEE Conference on Foundations of Computer Science, 518–530. Preprint arXiv:math/0604366v1.

[2] G.R. Brightwell, P. Winkler, *Graph homomorphisms and phase transitions*, J. Combin. Theory Ser. B **77**(2) (1999), 221–262.

[3] L. Coquille, C. Külske, A. Le Ny, *Extremal inhomogeneous Gibbs states for SOS-models and finite-spin models on trees*, J. Stat. Phys. **190** (2023), 71–97.

[4] M. Formentin, C. Külske, *A symmetric entropy bound on the non-reconstruction regime of Markov chains on Galton-Watson trees*. Electron. Commun. Probab. 14 (2009), 587–596.

[5] H.-O. Georgii, Gibbs Measures and Phase Transitions, Second edition. de Gruyter Studies in Mathematics, 9. Walter de Gruyter, Berlin, 2011.

[6] B. Jahnel, C. Külske, U.A. Rozikov, Gradient Gibbs measures for the random homomorphism model on Cayley trees. In preparation.

[7] B. Jahnel, U.A. Rozikov, Three-state p-SOS models on binary Cayley trees. arXiv:2402.09839

[8] H. Kesten, B.P. Stigum, *Additional limit theorem for indecomposable multi-dimensional Galton–Watson processes*, Ann. Math. Statist. **37** (1966), 1463–1481.

[9] C. Külske, U.A. Rozikov, *Extremality of translation-invariant phases for a three-state SOS-model on the binary tree*, J. Stat. Phys. **160**(3) (2015), 659–680.

[10] C. Külske, U.A. Rozikov, *Fuzzy transformations and extremality of Gibbs measures for the Potts model on a Cayley tree*, Random Struct. Algorithms. **50**(4) (2017), 636–678.

[11] P. Lammers, F. Toninelli: *Height function localisation on trees*, Comb. Probab, **33**(1) (2024), 50–64.

[12] J.B. Martin, *Reconstruction thresholds on regular trees.* Discrete random walks (Paris, 2003), 191–204 (electronic), Discrete Math. Theor. Comput. Sci. Proc., AC, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2003.

[13] F. Martinelli, A. Sinclair, D. Weitz, *Fast mixing for independent sets, coloring and other models on trees*, Random Struct. Algorithms, **31** (2007), 134–172.

[14] E. Mossel, Y. Peres, *Information flow on trees*, Ann. Appl. Probab. **13**(3) (2003), 817–844.

[15] E. Mossel, *Survey: Information flow on trees*, Graphs, Morphisms and Statistical Physics, 155–170, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 63, Amer. Math. Soc., Providence, RI, 2004.

[16] U.A. Rozikov, Y.M. Suhov, *Gibbs measures for SOS model on a Cayley tree*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9**(3) (2006), 471–488.

[17] U.A. Rozikov, Sh.A. Shoyusupov, *Fertile three state HC models on Cayley tree*, Theor. Math. Phys. **156**(3) (2008), 1319–1330.

[18] U.A. Rozikov, *Gibbs Measures on Cayley Trees*, World Sci. Publ. Singapore. 2013.

[19] U.A. Rozikov, *Gibbs Measures in Biology and Physics: The Potts model*, World Sci. Publ. Singapore. 2022.