Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Local well-posedness and global stability of one-dimensional shallow water equations with surface tension and constant contact angle

Jiaxu Li[1], Xin Liu[2], Dirk Peschka[3]

submitted: January 9, 2024

[1] The Institute of Mathematical Sciences
The Chinese University of Hong Kong
Shatin, N.T.
Hong Kong, CN
E-Mail: jiaxvlee@gmail.com

[2] Department of Mathematics
Texas A&M University
College Station, US
E-Mail: xliu23@tamu.edu

[3] Weierstrass Institute
Mohrenstr. 39
10117 Berlin, DE
E-Mail: dirk.peschka@wias-berlin.de

No. 3084
Berlin 2023

2020 *Mathematics Subject Classification.* 76N10, 35R35, 76B15, 74K35.

Key words and phrases. Shallow water equations, thin films, surface tension, contact lines, physical vacuum.

Dirk Peschka was supported by the German Research Foundation (DFG) through the project 422792530 within the Priority Program SPP 2171. Jiaxu Li was supported in part by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK-14301421, CUHK-14300819, CUHK-14302819, CUHK-14300917, the key project of NSFC (Grant No. 12131010) the Shun Hing Education and Charity Fund.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Local well-posedness and global stability of one-dimensional shallow water equations with surface tension and constant contact angle

Jiaxu Li, Xin Liu, Dirk Peschka

Abstract

We consider the one-dimensional shallow water problem with capillary surfaces and moving contact lines. An energy-based model is derived from the two-dimensional water wave equations, where we explicitly discuss the case of a stationary force balance at a moving contact line and highlight necessary changes to consider dynamic contact angles. The moving contact line becomes our free boundary at the level of shallow water equations, and the depth of the shallow water degenerates near the free boundary, which causes singularities for the derivatives and degeneracy for the viscosity. This is similar to the physical vacuum of compressible flows in the literature. The equilibrium, the global stability of the equilibrium, and the local well-posedness theory are established in this paper.

1 Introduction

The shallow water problem is a system of nonlinear partial differential equations that characterizes the motion of thin fluid layers, considering gravitational, viscous, and Coriolis forces. It is commonly employed to model the behavior of surface waves in oceans, lakes, and other geophysical flows and was first derived by Saint-Venant [2]. Here, we consider the one-dimensional shallow water problem for a film height $h := h(x,t)$ and a vertically averaged horizontal velocity $u := u(x,t)$ in a moving domain $x \in \omega = (a,b)$ with $a = a(t)$ and $b = b(t)$, which move together with the flow. About their three-dimensional variants, we call the points a, b *contact lines*, see Figure 1. Combining all the unknowns into a state vector $q = (a,b,h,u)$, this system has a Hamiltonian

$$\mathscr{H}(q) := \int_a^b \left(\tfrac{1}{2} h|u|^2 + U(h, \partial_x h) \right) \mathrm{d}x, \quad U(h,\xi) := \tfrac{g}{2} h^2 + \tfrac{\gamma}{2}(|\xi|^2 + \alpha^2). \tag{1}$$

The first integrand of $\mathscr{H}$ is the kinetic energy, and the internal energy U has contributions from gravity and surface energy. Here, g, γ, α denote the constant of gravity, surface tension, and the contact slope (the tangent of the contact angle). The height is non-negative and vanishes at the contact lines, i.e. $h(x,t) > 0$ for $a(t) < x < b(t)$ and $h\big(a(t),t\big) = h\big(b(t),t\big) = 0$. The film height is zero outside the domain $\omega(t)$. Alternatively, one can formulate this problem using the momentum $p := hu$, which vanishes outside the domain $\omega(t)$.

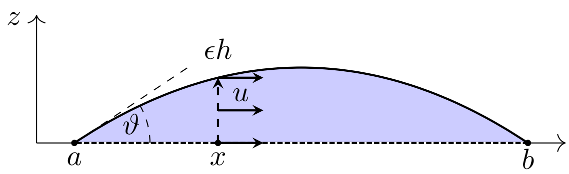

Figure 1: Sketch of fluid film height $h = h(x,t)$, horizontally uniform velocity field $u = u(x,t)$, contact lines $a = a(t)$, $b = b(t)$, contact angle $0 \leq \vartheta \ll 1$.

Then, for given initial data $(h,u)(t=0) = (h_0,u_0)$, $a(0) = a_0$, and $b(0) = b_0$, the free boundary problem describing the shallow water evolution $q(t)$ is

$$\partial_t h + \partial_x(hu) = 0, \tag{2a}$$

$$\partial_t(hu) + \partial_x(hu^2) + h\partial_x\pi - 4\mu\partial_x(h\partial_x u) = 0, \tag{2b}$$

$$\dot{a}(t) = u\big(a(t),t\big) \quad \text{and} \quad \dot{b}(t) = u\big(b(t),t\big), \tag{2c}$$

where $\pi = \partial_h U - \partial_x(\partial_{\partial_x h} U) = gh - \gamma\partial_{xx}h$ and $\dot{a} := \frac{\mathrm{d}}{\mathrm{d}t}a$, $\dot{b} := \frac{\mathrm{d}}{\mathrm{d}t}b$. Here $\mu \in (0,\infty)$ is a constant. These equations are satisfied by h, u at time t for all $x \in (a,b)$ and ensure conservation of mass (2a) and

DOI 10.20347/WIAS.PREPRINT.3084

conservation of momentum (2b). At the contact lines $a(t) < b(t)$, we have the following kinematic and constant contact angle boundary conditions

$$h\big(a(t),t\big) = 0 \quad \text{and} \quad h\big(b(t),t\big) = 0, \tag{2d}$$

$$\partial_x h\big(a(t),t\big) = \alpha \quad \text{and} \quad \partial_x h\big(b(t),t\big) = -\alpha. \tag{2e}$$

Note that (2d), upon differentiation with respect to time, is equivalent to kinematic conditions of the form

$$\begin{aligned} \partial_t h\big(a(t),t\big) + \dot{a}(t)\,\partial_x h\big(a(t),t\big) &= 0, \\ \partial_t h\big(b(t),t\big) + \dot{b}(t)\,\partial_x h\big(b(t),t\big) &= 0, \end{aligned} \tag{2d'}$$

which, together with (2a) and (2e), imply (2c). Later, we will show that solutions of (2) with Hamiltonian (1) satisfy an energy-dissipation balance

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}\big(q(t)\big) = -\int_a^b 4\mu h|\partial_x u|^2\,\mathrm{d}x \le 0, \tag{3}$$

and therefore obtain consistency with the second law of thermodynamics. With (2c) we ensure the conservation of mass at the contact lines. The constant contact angle in (2e) emerges from a stationary force balance when taking the derivative of the Hamiltonian $\mathscr{H}$.

A rigorous derivation of viscous shallow water equations without surface tension can be found in [10]. Formal derivations of shallow water equations including surface tension based on asymptotic expansions can be found in [16, 44, 46, 48, 58]. However, even without surface tension it was realized already by Lynch and Gray [43] that the shallow water problem is a free boundary problem where *wet* regions $\{x : h(x,t) > 0\}$ can advance into or recede from *dry* regions $\{x : h(x,t) = 0\}$. This has led to the development of more complex numerical methods to treat the corresponding free boundary problem, cf. [3] and references therein. The class of methods for the free boundary shallow water problem is mainly divided into Lagrangian and Eulerian methods: In the Lagrangian approach, the free boundary problem is mapped to a fixed domain and solved there [1]. Such approaches result in very precise but highly nonlinear partial differential equations, but can be difficult to solve in higher dimensions and for topological transitions. Eulerian methods solve the shallow water problem on a fixed domain and then try to maintain good properties using specialized techniques [45], e.g. non-negativity of the height or density. Length scales L of most geophysical problems are way above typical capillary length $\lambda = \sqrt{\gamma/g}$ and thus surface tension can be neglected. Alternatively, the contact angle can be treated by regularizing the surface energy with a wetting potential [36, 46, 54], which avoids topological transitions and maintains the global positivity of solutions.

However, for microfluidic wetting and dewetting problems surface tension plays a vital role [6, 13] but the considered fluids are often very viscous. Any viscous hydrodynamic model needs to address the so-called "no-slip paradox" discovered by Huh & Scriven and Dussan & Davis [15, 31] for example by modification of the no-slip boundary condition with an appropriate Navier-slip or free-slip boundary condition. Corresponding formal asymptotic techniques result in *thin-film models* [6, 48] of the form

$$\partial_t h - \partial_x\big[m(h)\,\partial_x(-\gamma\partial_{xx}h + \partial_h U_{\mathrm{int}})\big] = 0, \tag{4}$$

where $U_{\mathrm{int}} = U_{\mathrm{int}}(h)$ is an intermolecular potential with a similar role as U in (1) and $m(h) = \frac{1}{3}|h|^3 + bh^2$ is a degenerate mobility with Navier-slip length b. The thin-film free boundary problem with moving contact lines is well-understood mathematically [4, 5, 21, 23, 35], in particular regarding the (lack of) regularity near a moving contact line, e.g. cf. [18, 22, 24]. Numerical algorithms with stationary and dynamic force balance at a moving contact line have been investigated in dimension $d = 1$ [51, 52] and $d = 2$ [53] based on energy-variational arguments. In particular, the importance of dynamic contact angles [57] based on microscopic arguments and formulated in a thermodynamic framework should be emphasized [55, 56].

Without surface tension, the shallow water problem can be seen as an important special case of the compressible isentropic Navier-Stokes equations (viscous Saint-Venant system). Here, the height is replaced by the density $h \equiv \rho$ and one assumes a density-dependent viscosity coefficient, such that

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{5a}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P - 2\partial_x\big(\mu(\rho)\partial_x u\big) = 0, \tag{5b}$$

with $P = \rho^2$ and $\mu(\rho) = \mu\rho$. For the general isentropic Navier-Stokes equations one considers $P = \rho^\kappa$ and $\mu(\rho) = \rho^\alpha$ with the adiabatic index $\kappa > 1$ and $\alpha \geq 0$.

Dry regions $\{x : h(x,t) = 0\}$ in the shallow water equations correspond to *vacuum* $\{x : \rho(x,t) = 0\}$ in the compressible Navier-Stokes system. There is a large amount of literature about the long-time existence and asymptotic behavior of solutions to the system (5) in the case $\mu(\rho)$ is constant $(\alpha = 0)$. When the initial density is strictly away from the vacuum, see Kazhikhov [34] and Hoff [28] for strong solutions, Hoff and Smoller [30] for weak solutions. When the initial density contains a vacuum, this leads to some singular behaviors of solutions, such as the failure of continuous dependence of weak solutions on initial data [29] and the finite time blow-up of smooth solutions [33, 59], and even non-existence of classical solutions with finite energy [37].

Therefore, one may consider density-dependent viscosity case $(\alpha > 0)$. It is reasonable for compressible Navier-Stokes equations, see Liu-Xin-Yang [39], a viscous Saint-Venant system for the shallow waters, see Gerbeau-Perthame [19], and some geophysical flows, see [7–9]. In particular, Didier-Benoît-Lin studied a compressible fluid model of Korteweg type in [9]:

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{6a}$$

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) = \mathrm{div}\,(-P\mathbb{I} + 2\mu\rho\mathbb{D}u) + \gamma\rho\nabla\Delta\rho, \tag{6b}$$

see also Danchin-Desjardins [12], Hao-Hsiao-Li [26], Germain-LeFloch [20].

The vacuum-free boundary problem of (5) has attracted a vast of attractions in recent years. In the case that the viscosity is constant, Luo-Xin-Yang [40] studied the global regularity and behavior of the weak solutions near the interface when the initial density connects to vacuum states in a very smooth manner. Zeng [61] showed the global existence of smooth solutions for which the smoothness extends all the way to the boundary. In the case that the viscosity is density-dependent, the global existence of weak solutions was studied by many authors, see [60] without external force, and [14, 47, 62] with external force and the references therein. By taking the effect of external force into account, Ou-Zeng [49] obtained the global well-posedness of strong solutions and the global regularity uniformly up to the vacuum boundary. When the viscosity coefficient vanishes at vacuum, Li-Wang-Xin [38] first establishes the local well-posedness of classic solutions of system (5) without surface tension.

In this paper, we provide the ingredients to combine well-established models for moving contact lines that are valid on microscopic length scales with the shallow water problem on intermediate scales, where the capillary length is still relevant. We develop a theory to describe phenomena that combine capillarity, moving contact lines, and inertia. The major difficulty lies in the moving boundary and the degeneracy near the vacuum. We first investigate the stability of the stationary equilibrium. In particular, we analyze the linearized system (58) and find the key energy functional (76), in which the concavity of the equilibrium plays an important role. Then we move on to investigate the nonlinear stability theory, showing that the weighted, degenerate energy functional is strong enough to control the nonlinearities globally in time, thanks to Hardy's inequality. Finally, we sketch the local well-posedness theory for general initial data.

This paper is organized as follows: In Section 2, we give a brief derivation of system (2) from two-dimensional viscous water wave equations and summarize our main results in Section 2.5. We recall some weighted embedding inequalities in Section 3. In Section 4, we reformulate the free boundary problem (2) in the Lagrangian coordinates and state the main results of this paper. Then we present the linear stability and nonlinear stability in Sections 5 to 7 and therefore finish the proof of asymptotic stability of the stationary equilibrium. Section 8 is devoted to the local well-posedness theory for general initial data.

2 Shallow water equations with surface tension

2.1 Shallow water approximation

In the following, we will provide a systematic derivation of the one-dimensional shallow water equations with surface tension and moving contact lines from the two-dimensional water wave equations, which we state below. For the water wave model we follow the mathematical models presented and analyzed, for example, in [25, 55].

Let $0 < \epsilon \ll 1$ be the asymptotically small thickness of the liquid film, $\mathbf{u} := (u, w)(x, z, t)$ be the two-dimensional velocity field, $p := p(x, z, t)$ be the pressure potential, and $\epsilon h := \epsilon h(x, t)$ be the height of the liquid film. Using the height, similar to Figure 1, we define the time-dependent domain

$$\Omega_\epsilon(t) := \{(x, z) : x \in \omega(t), 0 < z < \epsilon h(x, t)\} \quad \text{for} \quad \omega(t) := \big(a(t), b(t)\big).$$

Additionally, we define the two free boundaries

$$\begin{aligned} &\Gamma_h(t) := \{(x, \epsilon h(x, t)) : a(t) < x < b(t)\} \subset \Gamma(t) \qquad \text{and} \\ &\Gamma_0(t) := \{(x, 0) : a(t) < x < b(t)\} \subset \Gamma(t), \end{aligned}$$

i.e. the top and bottom part of $\Gamma(t) = \partial\Omega_\epsilon(t)$. The outer normal on $\Gamma_h(t)$ is

$$\mathbf{n}_\epsilon(x, z, t) := \frac{1}{(1 + (\epsilon\partial_x h)^2)^{1/2}} \begin{pmatrix} -\epsilon\partial_x h(x, t) \\ 1 \end{pmatrix}.$$

With these definitions for a shallow domain, the viscous water wave equations can be written as

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \operatorname{div}\big(p\mathbb{I}_2 - \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)\big) = -g_\epsilon \mathbf{e}_z \qquad \text{in} \quad \Omega_\epsilon(t), \tag{7a}$$

$$\operatorname{div} \mathbf{u} = \partial_x u + \partial_z w = 0 \qquad \text{in} \quad \Omega_\epsilon(t), \tag{7b}$$

with kinematic conditions for the evolution of the boundaries $\Gamma_h(t), \Gamma_0(t)$

$$\epsilon(\partial_t h + u|_{z=\epsilon h}\partial_x h) - w|_{z=\epsilon h} = 0, \tag{7c}$$

$$\dot{a}(t) = u(a(t), 0, t), \quad \dot{b}(t) = u(b(t), 0, t). \tag{7d}$$

The stress boundary condition on the moving boundary $\Gamma_h(t)$ is

$$\Big(p\mathbb{I}_2 - \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)\Big)\, \mathbf{n}_\epsilon = -\gamma_\epsilon \partial_x \left(\frac{\epsilon\partial_x h}{\sqrt{1 + \epsilon^2|\partial_x h|^2}}\right) \mathbf{n}_\epsilon. \tag{7e}$$

On the bottom boundary $\Gamma_0(t)$ we have an impermeability boundary condition and a free slip boundary condition

$$w = 0 \qquad \text{on} \quad \Gamma_0(t), \tag{7f}$$

$$\partial_z u = 0 \qquad \text{on} \quad \Gamma_0(t). \tag{7g}$$

Instead of free slip (7g), also a Navier-slip condition $u - \mathscr{b}_\epsilon \partial_z u = 0$ with slip length $\mathscr{b}_\epsilon \geq 0$ in the sense of [10, 46] are possible boundary conditions on $\Gamma_0(t)$. However, a no-slip boundary condition $u = 0$ would be infeasible on $\Gamma_0(t)$ since this generates a logarithmic singularity in the energy dissipation. The final missing condition for the contact angle $0 \leq \vartheta < \pi/2$ is

$$\epsilon\, \partial_x h(a(t), t) = \tan\vartheta \quad \text{and} \quad \epsilon\, \partial_x h(b(t), t) = -\tan\vartheta. \tag{7h}$$

Note that this system has the Hamiltonian

$$\mathscr{H}_\epsilon := \int_{\Omega_\epsilon} \left(\tfrac{1}{2}|\mathbf{u}|^2 + g_\epsilon z\right) \mathrm{d}x\, \mathrm{d}z + \int_{\Gamma_h} \gamma_\epsilon \left(\sqrt{1 + \epsilon^2|\partial_x h|^2} - \cos\vartheta\right) \mathrm{d}x, \tag{8}$$

as a driving energy functional for the evolution with $\frac{\mathrm{d}}{\mathrm{dt}}\mathscr{H}_\epsilon \le 0$. Moreover, in the shallow water regime, we employ a smallness assumption of the contact slope α of the form

$$1 - \cos\vartheta = \tfrac{1}{2}\epsilon^2\alpha^2 \ll 1, \tag{9}$$

for some given $\alpha = \mathcal{O}(1)$. Then one can check that as $\epsilon \to 0$, the leading order of $\mathscr{H}_\epsilon$ is exactly the Hamiltonian $\mathscr{H}$ of the shallow water system, i.e., (1).

For convenience, note that (7a) can be rewritten component-wisely as,

$$\partial_t u + u\partial_x u + w\partial_z u + \partial_x(p - 2\mu\partial_x u) - \mu\partial_z(\partial_x w + \partial_z u) = 0, \tag{10a}$$

$$\partial_t w + u\partial_x w + w\partial_z w + \partial_z(p - 2\mu\partial_z w) - \mu\partial_x(\partial_z u + \partial_x w) = -g_\epsilon, \tag{10b}$$

and at $z = \epsilon h(x,t)$ (7e) can be rewritten using the two components

$$(p - 2\mu\partial_x u)(-\epsilon\partial_x h) - \mu(\partial_x w + \partial_z u) = -\gamma_\epsilon\partial_x\left(\tfrac{\epsilon\partial_x h}{\sqrt{1+\epsilon^2|\partial_x h|^2}}\right)(-\epsilon\partial_x h), \tag{10c}$$

$$-\mu(\partial_z u + \partial_x w)(-\epsilon\partial_x h) + (p - 2\mu\partial_z w) = -\gamma_\epsilon\partial_x\left(\tfrac{\epsilon\partial_x h}{\sqrt{1+\epsilon^2|\partial_x h|^2}}\right). \tag{10d}$$

We will use an asymptotic scaling for surface tension γ_ϵ and gravity g_ϵ in the shallow water approximation's final stages. Guided by [50], we derive shallow water equations with surface tension and moving contact lines from (7).

Step 1: Re-scaling the vertical variables. Owing to the vertical scale of the shallow domain, it is natural to introduce the changes of variables

$$z := \epsilon Z \quad \text{and} \quad w := \epsilon W. \tag{11}$$

The system (7) can be recast, using (u, W, p) in the (x, Z, t)-coordinates in the rescaled fluid domain $\Omega(t) := \{(x,Z) : x \in \omega(t), 0 < Z < h(x,t)\}$. Equations (7a)–(7b) transform into

$$\partial_t u + u\partial_x u + W\partial_Z u + \partial_x(p - 2\mu\partial_x u) - \tfrac{1}{\epsilon}\mu\partial_Z(\epsilon\partial_x W + \tfrac{1}{\epsilon}\partial_Z u) = 0, \tag{12a}$$

$$\epsilon^2(\partial_t W + u\partial_x W + W\partial_Z W) + \partial_Z(p - 2\mu\partial_Z W) - \mu\partial_x(\partial_Z u + \epsilon^2\partial_x W) = -\epsilon g_\epsilon, \tag{12b}$$

$$\partial_x u + \partial_Z W = 0, \tag{12c}$$

and the kinematic condition becomes

$$\partial_t h + u|_{Z=h}\partial_x h - W|_{Z=h} = 0. \tag{12d}$$

Meanwhile, boundary conditions (7e, 7f, 7g) can be rewritten as

$$\begin{aligned}&(p - 2\mu\partial_x u)(-\epsilon\partial_x h) - \mu(\epsilon\partial_x W + \frac{1}{\epsilon}\partial_Z u)\\ &= -\gamma_\epsilon\partial_x\left(\frac{\epsilon\partial_x h}{\sqrt{1+\epsilon^2|\partial_x h|^2}}\right)(-\epsilon\partial_x h) \quad \text{on} \quad \{(x, Z = h(x,t))\},\end{aligned} \tag{12e}$$

$$\begin{aligned}-\mu(\frac{1}{\epsilon}\partial_Z u + \epsilon\partial_x W)(-\epsilon\partial_x h) + (p - 2\mu\partial_Z W) = -\gamma_\epsilon\partial_x\left(\frac{\epsilon\partial_x h}{\sqrt{1+\epsilon^2|\partial_x h|^2}}\right)\\ \text{on} \quad \{(x, Z = h(x,t))\},\end{aligned} \tag{12f}$$

$$W|_{Z=0} = 0, \quad \partial_Z u|_{Z=0} = 0. \tag{12g}$$

In the following, the barotropic component (vertical average) of an arbitrary function $f(x,Z,t)$ is defined by

$$\overline{f}(x,t) := \frac{1}{h}\int_0^{h(x,t)} f(x,Z,t)\,\mathrm{d}Z. \tag{13}$$

This definition allows for trivial identities of the form

$$\partial_t(h\bar{f}) = (\partial_t h)f|_{Z=h} + h\overline{\partial_t f} \quad \text{and} \quad \partial_x(h\bar{f}) = (\partial_x h)f|_{Z=h} + h\overline{\partial_x f}. \tag{14}$$

Furthermore, for $f = u$ we have an exact continuity equation

$$\partial_t h + \partial_x(h\overline{u}) = 0, \tag{15}$$

for any solution of (7). This follows from the short computation

$$\begin{aligned}\partial_x(h\overline{u}) &\overset{(13)}{=} \partial_x \int_0^h u(x,Z,t)\mathrm{d}Z = (\partial_x h)u|_{Z=h} + \int_0^h \partial_x u(x,Z,t)\mathrm{d}Z \\ &\overset{(12c)}{=} (\partial_x h)u|_{Z=h} - \int_0^h \partial_Z W(x,Z,t)\mathrm{d}Z \overset{(7f,12d)}{=} -\partial_t h.\end{aligned}$$

Step 2: Multiscale analysis. The leading ϵ^{-2}-order of (12a) implies

$$\partial_{ZZ} u = \mathcal{O}(\epsilon^2), \tag{16}$$

and integrating that in Z using (12g) gives $\partial_Z u = \mathcal{O}(\epsilon^2)$. Integrating (12b) in Z and using (12f) yields that at the leading order

$$\begin{aligned}\big(p - 2\mu\partial_Z W - \mu\partial_x u\big)(x,Z,t) &= -\epsilon g_\epsilon Z + C(x,t) + \mathcal{O}(\epsilon^2), \\ \text{where} \quad C &= \epsilon g_\epsilon h - \mu\partial_x u|_{Z=h} - \epsilon\gamma_\epsilon \partial_{xx} h,\end{aligned} \tag{17}$$

at (x,Z,t). Therefore, integrating (17) in Z from 0 to $h(x,t)$ yields

$$\begin{aligned}h\overline{p} + \mu\partial_x(h\overline{u}) - \mu u|_{Z=h}\partial_x h = &-\epsilon\gamma_\epsilon h\partial_{xx} h - \mu h\partial_x u|_{Z=h} \\ &+ \tfrac{\epsilon}{2} g_\epsilon h^2 + \mathcal{O}(\epsilon^2),\end{aligned} \tag{18}$$

where we have used (12c) and (12d). Using (12c) we can rewrite (12a) as

$$\partial_t u + \partial_x(u^2) + \partial_Z(Wu) + \partial_x(p - 2\mu\partial_x u) - \epsilon^{-2}\mu\partial_{ZZ} u - \mu\partial_{Zx} W = 0,$$

where integration in Z from 0 to h using (12d), (12g), and (12e) yields

$$\begin{aligned}\partial_t(h\overline{u}) + \partial_x(h\overline{u^2}) + \partial_x(h\overline{p} - 2\mu h\overline{\partial_x u}) &= \big[(p - 2\mu\partial_x u)\partial_x h + \mu\partial_x W + \tfrac{1}{\epsilon^2}\mu\partial_Z u\big]\big|_{Z=h} \\ &\quad + \big[u|_{Z=h}\underbrace{(\partial_t h + u\partial_x h - W)|_{Z=h}}_{=0 \text{ via (12d)}}\big] \\ &\overset{(12e)}{=} -\epsilon\gamma_\epsilon(\partial_x h)\partial_{xx} h + \mathcal{O}(\epsilon^2).\end{aligned} \tag{19}$$

Substituting $h\overline{p}$ from (18) into (19) yields

$$\begin{aligned}&\partial_t(h\overline{u}) + \partial_x(h\overline{u^2}) + \partial_x(\tfrac{\epsilon}{2} g_\epsilon h^2) - \epsilon\gamma_\epsilon h\partial_{xxx} h \\ &\quad = \partial_x(2\mu h\overline{\partial_x u} + \mu\partial_x(h\overline{u}) - \mu u|_{Z=h}\partial_x h + \mu h\partial_x u|_{Z=h}) + \mathcal{O}(\epsilon^2) \\ &\quad = \partial_x(3\mu\partial_x(h\overline{u}) - 3\mu u|_{Z=h}\partial_x h + \mu h\partial_x u|_{Z=h}) + \mathcal{O}(\epsilon^2),\end{aligned} \tag{20}$$

where we have used the identity $h\overline{\partial_x u} = \partial_x(h\overline{u}) - u|_{Z=h}\partial_x h$.

Step 3: Barotropic approximation. This step will finish the formal derivation of shallow water equations with surface tension. Thanks to (16), one can derive that

$$u(x,Z,t) = \overline{u}(x,t) + \int_0^Z \partial_Z u(x,Z',t)\,\mathrm{d}Z' - \frac{1}{h}\int_0^h \left(\int_0^Z \partial_Z u(x,Z',t)\,\mathrm{d}Z'\right)\mathrm{d}Z$$

$$= \overline{u}(x,t) + \mathcal{O}(\epsilon^2).$$

From this approximation we deduce in particular that $u|_{Z=h} = \bar{u} + \mathcal{O}(\epsilon^2)$ and that $\overline{u^2} = \overline{u}^2 + \mathcal{O}(\epsilon^2)$ and thus (20) yields

$$\partial_t(h\overline{u}) + \partial_x(h\overline{u}^2) + \partial_x(\tfrac{\epsilon}{2}g_\epsilon h^2) - \epsilon\gamma_\epsilon h\partial_{xxx}h = 4\mu\partial_x(h\partial_x\overline{u}) + \mathcal{O}(\epsilon^2). \tag{21}$$

Finally, consider a scaling where $\epsilon g_\epsilon = g = \mathcal{O}(1)$ and $\epsilon\gamma_\epsilon = \gamma = \mathcal{O}(1)$. Formally passing the limit $\epsilon \to 0^+$ in (15) and (21) and renaming $u(x,t) := \bar{u}(x,t)$ leads to the shallow water equations with surface tension

$$\partial_t h + \partial_x(hu) = 0, \tag{22a}$$

$$\partial_t(hu) + \partial_x(hu^2) + \partial_x(\tfrac{g}{2}h^2) - \gamma h\partial_{xxx}h = 4\mu\partial_x(h\partial_x u). \tag{22b}$$

to be satisfied for $x \in \omega(t)$, and therefore we have recovered (2a) and (2b). By expanding (7h) for small slopes we obtain

$$\partial_x h(a(t),t) = \alpha \qquad \text{and} \qquad \partial_x h(b(t),t) = -\alpha, \tag{22c}$$

and the kinematic conditions remain as they were. A rigorous derivation of viscous shallow water equations without surface tension can be found in [10]. A rigorous derivation of system (22) in the case when $h > 0$, i.e., the non-degenerate case, follows similarly.

In this work, our goal is to investigate the case when system (22) degenerates. In particular, we focus on the case when $\omega = \{h > 0\} = \omega(t) \subset \mathbb{R}$ is a domain evolving together with the flow, and h has compact support. At the boundary of the support, these equations are supplemented by the boundary conditions (2c,2d,2e), which were untouched by the shallow water approximation.

2.2 Conservation laws and contact angle

We start by deriving a few conservation laws and energy balances for solutions to the shallow water problem.

Conservation of mass: Taking the time derivative of the volume of the incompressible fluid, i.e. the integral of h over $\omega(t)$, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} h\,\mathrm{d}x \overset{\text{(22a)}}{=} -hu|_a^b + h(b)\dot{b} - h(a)\dot{a} \overset{\text{(2c)}}{=} 0. \tag{23}$$

Balance of momentum: Assuming constant contact angle, integrating the momentum $p = hu$ and using the divergence form of (22b) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} hu\,\mathrm{d}x = -\int_a^b \gamma\partial_x h\partial_{xx}h\,\mathrm{d}x = -\frac{\gamma}{2}|\partial_x h|^2\big|_a^b = 0. \tag{24}$$

Balance of energy: Taking the L^2-inner product of (22b) with u and integrating by part in the resultant lead to, thanks to (22a),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}\int_{a(t)}^{b(t)} h|u|^2 + g|h|^2\,\mathrm{d}x\right\} - \gamma\int_a^b \partial_t h\partial_{xx}h\,\mathrm{d}x + \int_a^b 4\mu h|\partial_x u|^2\,\mathrm{d}x = 0. \tag{25}$$

Moreover, by applying integration by parts further, one can calculate that

$$\begin{aligned} -\gamma\int_a^b \partial_t h\partial_{xx}h\,\mathrm{d}x &= \gamma\int_a^b \partial_{tx}h\partial_x h\,\mathrm{d}x - \gamma(\partial_t h\partial_x h)\big|_a^b \\ &= \frac{\gamma}{2}\int_a^b \partial_t|\partial_x h|^2\,\mathrm{d}x + \gamma(u|\partial_x h|^2)\big|_a^b \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \tfrac{\gamma}{2}|\partial_x h|^2\,\mathrm{d}x + \tfrac{\gamma}{2}(u|\partial_x h|^2)\big|_a^b, \end{aligned} \tag{26}$$

where we have used (2d') and (2c). At this point, we want to explore the impact of using a constant contact angle (2e) on the energy balance. For constant contact angle (2e), using (2c) we find

$$(u|\partial_x h|^2)\big|_a^b = (\dot{b}-\dot{a})\alpha^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \alpha^2\,\mathrm{d}x. \tag{27}$$

Next, substituting (26,27) into (25) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_a^b h|u|^2 + gh^2 + \gamma(|\partial_x h|^2+\alpha^2)\,\mathrm{d}x\right] = -\int_a^b 4\mu h|\partial_x u|^2\,\mathrm{d}x. \tag{28}$$

This is the energy balance that we stated before in (3). The static contact angle produces a thermodynamic consistent shallow water model in the sense $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} \le 0$ with the Hamiltonian (1).

Remark 1. Formally, different equivalent variants of boundary and kinematic conditions at a, b are possible for the water wave equation, i.e. (2e,2d) or (2d') or combinations thereof. Similar arguments as in [38, remark 2] show that, classical solution to (22) with moving boundary satisfy

$$\partial_x u|_{x=a,b} = 0. \tag{29}$$

For $s \in \{a, b\}$, taking the time-derivative of the slope at $x = s$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\partial_x h(s(t),t)\big|_{s=a,b} &= (\partial_t\partial_x h + \dot{s}\partial_x(\partial_x h))\big|_s \overset{\text{(22a)}}{=} (-\partial_x^2(uh) + u\partial_{xx}h)\big|_s \\ &= \big(-2\partial_x h\partial_x u\big)_s \overset{\text{(29)}}{=} 0.\end{aligned} \tag{30}$$

and therefore imposing (29) implies that the contact angle does not change from its initial value. While the rest of the manuscript centers on a constant contact angle and free slip, we now briefly discuss a more general contact line model and the impact of a finite Navier slip length $\mathscr{b}_\epsilon$.

2.3 Dynamic contact angle and Navier slip

Models with dynamic contact angle use a dynamic stress balance at the contact line, a well-established concept in hydrodynamic models using variational arguments and dissipative processes. A general thermodynamic consistent model for a dynamic contact angle in the spirit of [52, 55] but applied to the shallow water problem replaces (2e) by

$$\nu\dot{a}(t) = \big[\alpha^2 - \big(\partial_x h\big(a(t),t\big)\big)^2\big] \quad \text{and} \quad \nu\dot{b}(t) = -\big[\alpha^2 - \big(\partial_x h\big(b(t),t\big)\big)^2\big]. \tag{31}$$

For example, in [46] it is shown that a scale $\mathscr{b}_\epsilon\epsilon^2 = \mathscr{b} = \mathcal{O}(1)$ leads to a modified momentum balance, where instead of (2b) one has

$$\partial_t(hu) + \partial_x(hu^2) + h\partial_x\pi - 4\mu\partial_x(h\partial_x u) + \mathscr{b}^{-1}u = 0, \tag{32}$$

while all other equations remain as they were. Redoing the previous computation for the energy-dissipation balance (25) for (32) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}\int_{a(t)}^{b(t)} h|u|^2 + g|h|^2\,\mathrm{d}x\right\} = \int_a^b \gamma\partial_t h\partial_{xx}h - 4\mu h|\partial_x u|^2 + \mathscr{b}^{-1}u^2\,\mathrm{d}x. \tag{33}$$

Similar as in (27) but now with dynamic contact angle (31) we get

$$\begin{aligned}(u|\partial_x h|^2)\big|_a^b &= \dot{b}\Big\{(\partial_x h)\big|_{x=b}^2 - \alpha^2 + \alpha^2\Big\} - \dot{a}\Big\{(\partial_x h)\big|_{x=a}^2 - \alpha^2 + \alpha^2\Big\} \\ &\overset{\text{(31)}}{=} \nu(\dot{a}^2 + \dot{b}^2) + (\dot{b}-\dot{a})\alpha^2 = \nu(\dot{a}^2+\dot{b}^2) + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \alpha^2\,\mathrm{d}x.\end{aligned} \tag{34}$$

Substituting (26,34) into (33) leads to the energy-dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = -\mathscr{D} \quad \text{where} \quad \mathscr{D} = \int_a^b \left[4\mu h|\partial_x u|^2 + \ell^{-1}u^2\right] \mathrm{d}x + \nu(\dot a^2 + \dot b^2). \tag{35}$$

By this computation, we have identified two new terms in the dissipation $\mathscr{D}$. Reconsidering the previous conservation of momentum (24) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} hu\,\mathrm{d}x = -\int_a^b \ell^{-1}u\,\mathrm{d}x - \frac{\gamma}{2}|\partial_x h|^2\Big|_a^b = -\int_a^b \ell^{-1}u\,\mathrm{d}x - \frac{\gamma\nu}{2}(\dot a + \dot b).$$

At least formally, in the limit $\ell^{-1}, \nu \to 0$ we recover the energy and momentum balance for the constant contact angle, whereas with finite ν, ℓ^{-1} the momentum is not conserved due to friction with the supporting interface at $z = 0$. The shallow water system with dynamic angle (31) is thermodynamic consistent in the sense that the Hamiltonian (free energy) decreases (35). While it is an interesting mathematical question to consider the regularity of solutions as $x \to a, b$ in the (singular) limits of vanishing or infinite μ, ν, ℓ^{-1}, in the following we set $\nu, \ell^{-1} = 0$.

2.4 Equilibrium

In this subsection, we look for a stationary equilibrium $q(x,t) = q_\mathrm{s}(x)$ with $q_\mathrm{s} = (a_\mathrm{s}, b_\mathrm{s}, h_\mathrm{s}, u_\mathrm{s})$ of (22) with constant/static contact angle (2e), i.e. $u_\mathrm{s} = 0$ and $\alpha > 0$. That is

$$\partial_x(\tfrac{g}{2}h_\mathrm{s}^2) - \gamma h_\mathrm{s}\partial_{xxx}h_\mathrm{s} = 0 \qquad \text{and} \qquad \partial_x h_\mathrm{s}(b_\mathrm{s}) = -\partial_x h_\mathrm{s}(a_\mathrm{s}) = -\alpha, \tag{36}$$

in $\omega_\mathrm{s} := (a_\mathrm{s}, b_\mathrm{s}) = \{h_\mathrm{s} > 0\}$. One can explicitly solve for h_s from (36)

$$h_\mathrm{s}(x) = \alpha\lambda\left(\frac{e^{2R/\lambda} + 1 - (e^{(x+R)/\lambda} + e^{(R-x)/\lambda})}{e^{2R/\lambda} - 1}\right). \tag{37}$$

where $\lambda = \sqrt{\gamma/g}$ is the capillary length and $b_\mathrm{s} = -a_\mathrm{s} = R$ sets the center of mass to the origin. With dynamic contact angle, this solution is the unique long-time limit up to translation and the droplet radius R is determined by the mass of the droplet. For static contact angle, the long-time behavior can be characterized by translation and by allowing arbitrary constant droplet speeds $u(x,t) = \bar u_\mathrm{s} \in \mathbb{R}$, $h(x,t) = h_\mathrm{s}(x - \bar u_\mathrm{s} t)$, $a(t) = -R + \bar u_\mathrm{s} t$, $b(t) = R + \bar u_\mathrm{s} t$. Generic equilibrium solutions for various R are shown in Figure 2. For $R \ll \lambda$ the droplet shape is parabolic $h_\mathrm{s}(x) \approx \frac{\alpha R}{2\lambda}\big(1 - (\frac{x}{R})^2\big)$, whereas for $L \gg \lambda$ it has a pancake shape $h_\mathrm{s}(x) \approx \alpha\lambda$ away from the contact line.

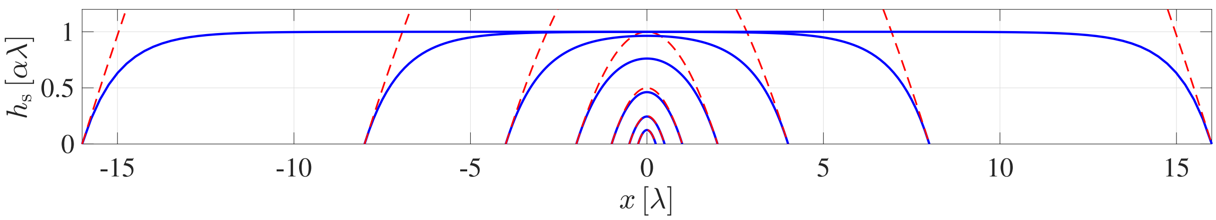

Figure 2: Equilibrium solutions h_s for increasing $R/\lambda \in \{\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16\}$ (blue lines) and parabolic shapes with same R/λ (red dashed lines).

2.5 Summary of shallow water equations and main results

We summarize the shallow water equations with surface tension in this subsection. To simplify the presentation, we assume that

$$g = 2,\ \gamma = 2,\ \alpha = 1,\ \mu = \frac{1}{2}. \tag{38}$$

Then the shallow water equations (22) with surface tension and constant contact angle, together with (2c,2d, 2e), read

$$\partial_t h + \partial_x(hu) = 0, \qquad \text{in} \quad \omega(t), \tag{39a}$$

$$\partial_t(hu) + \partial_x(hu^2) + \partial_x(h^2) - 2h\partial_{xxx}h = 2\partial_x(h\partial_x u), \qquad \text{in} \quad \omega(t), \tag{39b}$$

$$\partial_x h\big(b(t),t\big) = -\partial_x h\big(a(t),t\big) = -1, \qquad \text{for} \quad t \in (0,\infty), \tag{39c}$$

$$\partial_x u\big(b(t),t\big) = \partial_x u\big(a(t),t\big) = 0, \qquad \text{for} \quad t \in (0,\infty), \tag{39d}$$

$$\dot a = u\big(a(t),t\big), \quad \dot b = u\big(b(t),t\big), \qquad \text{for} \quad t \in (0,\infty), \tag{39e}$$

where $\omega(t) = (a(t), b(t))$. Alternatively, (39b) can be written in the conservative form:

$$\partial_t(hu) + \partial_x(hu^2) + \partial_x\big[h^2 - 2h\partial_{xx}h + |\partial_x h|^2\big] = 2\partial_x(h\partial_x u). \tag{39b'}$$

The equilibrium is given by

$$h_{\mathrm{s}}(x) = \frac{e^2+1}{e^2-1} - \frac{e^{x+1}+e^{1-x}}{e^2-1} \qquad \forall x \in (-1,1), \tag{40}$$

satisfying

$$h_{\mathrm{s}}''' = h_{\mathrm{s}}'. \tag{41}$$

Here we have chosen

$$a_{\mathrm{s}} = -1, b_{\mathrm{s}} = 1. \tag{42}$$

The main results of this paper can be summarized, in an informal way, as follows:

Theorem 1 (Informal statement of main theorems)**.** *(i) Given general initial data* $(a,b,h,u)|_{t=0} = (a_0,b_0,h_0,u_0)$*, as long as* h_0 *satisfies some convexity condition, there exists a unique local-in-time strong solution to system* (39)*. See Theorem 3, below, in Section 8 for full description*

(ii) With small enough perturbation, the equilibrium state (40) *is asymptotically stable. In particular, there exists a global unique strong solution to system* (39) *near* (40)*, and the solution converges to the equilibrium state as time goes to infinity. See Theorem 2, below, in Section 4.2 for a full description.*

3 Preliminaries

In this section, we recall some weighted embedding inequalities. The following general version of the Hardy inequality whose proof can be found in [27] will be used often in this paper:

Lemma 1 (Hardy's inequality, L^p version)**.** *Let* k *and* $p > 1$ *be given real numbers, and* g *be a function with bounded right-hand side in the following inequalities. Then*

- *if* $k + \frac{1}{p} > 1$*, one has*

$$\int_0^1 \big(s^{k-1}g(s)\big)^p \,\mathrm{d}s \le C_{k,p} \int \Big(\big(s^k|g'(s)|\big)^p + \big(s^k|g(s)|\big)^p\Big)\,\mathrm{d}s; \tag{43}$$

- *if* $k + \frac{1}{p} < 1$*, one has*

$$\int_0^1 \big(s^{k-1}(g(s)-g(0))\big)^p \,\mathrm{d}s \le C_{k,p} \int \big(s^k|g'(s)|\big)^p \,\mathrm{d}s. \tag{44}$$

Next, the following weighted Poincare inequality can be found in [17]:

Lemma 2 (Weighted Poincare's inequality). *Let h_{s} be given as in* (40). *Suppose that g satisfies*

$$\int_{-1}^{1} h_{\mathrm{s}} g \,\mathrm{d}\xi = 0.$$

Then one has

$$\int_{-1}^{1} h_{\mathrm{s}} |g|^2 \,\mathrm{d}\xi \leq C \int_{-1}^{1} h_{\mathrm{s}} |\partial_\xi g|^2 \,\mathrm{d}\xi \tag{45}$$

and

$$\int_{-1}^{1} |g|^2 \,\mathrm{d}\xi \leq C \int_{-1}^{1} h_{\mathrm{s}} |\partial_\xi g|^2 \,\mathrm{d}\xi. \tag{46}$$

4 Lagrangian formulation and main results

4.1 Lagrangian coordinates with reference to the equilibrium profile

To investigate the stability profile (40) of the shallow water equations with surface tension (39), we introduce the Lagrangian coordinates (ξ, t) in a way that captures the equilibrium profile as coefficients. Define $x = \eta(\xi, t)$ by

$$\int_{a(t)}^{\eta(\xi,t)} h(x,t) \,\mathrm{d}x = \int_{-1}^{\xi} h_{\mathrm{s}}(x) \,\mathrm{d}x. \tag{47}$$

In particular, we assume

$$\int_{a(t)}^{b(t)} h(x,t) \,\mathrm{d}x = \int_{-1}^{1} h_{\mathrm{s}}(x) \,\mathrm{d}x, \tag{48}$$

which, thanks to the conservation of mass (23), is a restriction on the initial data. Then taking the space and time derivatives (∂_ξ and ∂_t) in the (ξ, t)-coordinates leads to, for $\xi \in I = (-1, 1)$,

$$h(\eta(\xi,t),t)\partial_\xi \eta(\xi,t) = h_{\mathrm{s}}(\xi), \tag{49}$$

$$\begin{aligned} h(\eta(\xi,t),t)\partial_t \eta(\xi,t) &= -\int_{a(t)}^{\eta(\xi,t)} \partial_t h(x,t) \,\mathrm{d}x = \int_{a(t)}^{\eta(\xi,t)} \partial_x (hu)(x,t) \,\mathrm{d}x \\ &= h(\eta(\xi,t),t) u(\eta(\xi,t),t), \end{aligned} \tag{50}$$

where (39a) is used. Notably, (50) is equivalent to the Lagrangian flow map [11, 32], and the benefit of (47) is that it captures the equilibrium [41, 42]. Therefore, system (39) can be recast in the (ξ, t)-coordinates as

$$h(\xi,t) = \frac{h_{\mathrm{s}}(\xi)}{\partial_\xi \eta(\xi,t)}, \quad \partial_t \eta(\xi,t) = u(\xi,t), \tag{51a}$$

$$\begin{aligned} h_{\mathrm{s}} \partial_t u + \partial_\xi \Big(\big(\frac{h_{\mathrm{s}}}{\partial_\xi \eta}\big)^2 \Big) - 2 h_{\mathrm{s}} \frac{\partial_\xi}{\partial_\xi \eta} \Big(\frac{\partial_\xi}{\partial_\xi \eta} \Big(\frac{\partial_\xi}{\partial_\xi \eta} \Big(\frac{h_{\mathrm{s}}}{\partial_\xi \eta} \Big) \Big) \Big) \\ = 2 \partial_\xi \Big(\frac{h_{\mathrm{s}}}{\partial_\xi \eta} \frac{\partial_\xi u}{\partial_\xi \eta} \Big), \end{aligned} \tag{51b}$$

$$\partial_\xi \eta|_{\xi=-1,1} = 1, \quad \partial_\xi u|_{\xi=-1,1} = 0, \tag{51c}$$

where we have used the same notations for the variables h, u in the Lagrangian coordinates as in the original Euclidean coordinates. Notice that (51c) is equivalent to (39c), which can be seen from Remark 1.

We consider the solution η to system (51) near the equilibrium $\eta_{\rm s}(\xi) = \xi$. Thanks to (51a) and (51c), we consider $\eta(\xi, t)$ as follows:

$$\eta(\xi,t) = \xi + \theta \qquad \text{with} \quad \partial_\xi \theta|_{\xi=-1,1} = 0. \tag{52}$$

Notice that $u = \partial_t \eta = \partial_t \theta$. Then in terms of the perturbation variable θ, system (51) can be written as

$$\begin{aligned} h_{\rm s}\partial_{tt}\theta + \left((\frac{h_{\rm s}}{1+\theta_\xi})^2 \right)_\xi - 2\frac{h_{\rm s}}{1+\theta_\xi}\left(\frac{1}{1+\theta_\xi}\left(\frac{1}{1+\theta_\xi}\left(\frac{h_{\rm s}}{1+\theta_\xi} \right)_\xi \right)_\xi \right)_\xi \\ = 2\left(\frac{h_{\rm s}}{(1+\theta_\xi)^2}\theta_{\xi t} \right)_\xi, \qquad \text{with} \quad \theta_\xi|_{\xi=-1,1} = 0. \end{aligned} \tag{51'}$$

or equivalently, in the conservative form

$$\begin{aligned} h_{\rm s}\partial_{tt}\theta + \Bigg\{ \left(\frac{h_{\rm s}}{1+\theta_\xi} \right)^2 - 2\frac{h_{\rm s}}{1+\theta_\xi}\frac{1}{1+\theta_\xi}\left[\frac{1}{1+\theta_\xi}\left(\frac{h_{\rm s}}{1+\theta_\xi} \right)_\xi \right]_\xi \\ + \left[\frac{1}{1+\theta_\xi}\left(\frac{h_{\rm s}}{1+\theta_\xi} \right)_\xi \right]^2 \Bigg\}_\xi = 2\left(\frac{h_{\rm s}}{(1+\theta_\xi)^2}\theta_{\xi t} \right)_\xi, \qquad \text{with} \quad \theta_\xi|_{\xi=-1,1} = 0. \end{aligned} \tag{53}$$

4.2 Main result: Asymptotic stability

We now state the main stability result in this paper. For simplicity, we denote $I = (-1, 1)$ in the rest of the paper. The asymptotic stability of the stationary state of system (51) can be stated in the Lagrangian coordinates as follows:

Theorem 2 (Asymptotic Stability). *Let $h_{\rm s}$ be the equilibrium defined in* (40)*. There exists a constant $0 < \varepsilon \ll 1$ such that if the total initial energy $\mathfrak{E}_0$ defined in* (156)*, below, satisfies*

$$\mathfrak{E}_0 < \varepsilon, \tag{54}$$

then the system (51) *admits a unique global strong solution (η, u) in $I \times [0, \infty)$ with*

$$\begin{cases} \eta, \eta_\xi - 1 \in C^1([0,\infty); L^2(I)), \ \ \eta_{\xi\xi}, h_{\rm s}\partial_\xi^3\eta \in C([0,\infty); L^2(I)), \\ h_s^{3/2}\partial_\xi^4\eta \in L^\infty((0,\infty); L^2(I)), \ \ u, u_\xi, h_{\rm s}^{1/2}u_t \in C([0,\infty); L^2(I)), \\ u_{\xi t}, u_{\xi\xi}, h_{\rm s}^{1/2}u_{tt}, h_{\rm s}\partial_\xi^3 u \in L^\infty([0,\infty); L^2(I)), \end{cases} \tag{55}$$

Moreover, we have the following asymptotic stability:

$$\mathcal{E}_{\rm NL}(t) \triangleq \mathcal{E}_{\rm NL,1}(t) + \mathcal{E}_{\rm NL,2}(t) < Ce^{-C_1 t}, \tag{56}$$

for any $t \in [0, \infty)$ and some constants $C, C_1 \in (0, \infty)$, where $\mathcal{E}_{\rm NL,1}$ and $\mathcal{E}_{\rm NL,2}$ are defined in (100) *and* (101)*, below, respectively.*

Remark 2. For general initial height $h_0 > 0$ in $[a, b]$, we can define a diffeomorphism η_0 from $[-1, 1]$ to $[a, b]$ such that

$$\int_a^{\eta_0(\xi)} h_0(x)\, dx = \int_{-1}^{\xi} h_s(x)\, dx, \tag{57}$$

which is agree with (47).

5 Linear stability analysis

Assuming $\theta = \mathcal{O}(\varepsilon)$ with small ε representing the size of perturbation, from (51'), one can write down the following linear equation:

$$\begin{aligned} h_{\rm s}\partial_{tt}\theta_1 - 2\big(h_{\rm s}^2\theta_{1,\xi}\big)_\xi + 2h_{\rm s}\theta_{1,\xi}\partial_{\xi\xi\xi}h_{\rm s} + 2h_{\rm s}\partial_\xi\big(\theta_{1,\xi}\partial_{\xi\xi}h_{\rm s}\big) + 2h_{\rm s}\partial_{\xi\xi}\big(\theta_{1,\xi}\partial_\xi h_{\rm s}\big) \\ +2h_{\rm s}\partial_{\xi\xi\xi}\big(h_{\rm s}\theta_{1,\xi}\big) = 2\partial_\xi\big(h_{\rm s}\theta_{1,\xi t}\big), \qquad \text{with} \quad \theta_{1,\xi}|_{\xi=-1,1} = 0, \end{aligned} \tag{58}$$

where we have substituted (36) with (38) and (42), i.e.,

$$\begin{aligned} \partial_\xi h_{\rm s} = \partial_{\xi\xi\xi}h_{\rm s} \quad \text{with} \\ \partial_\xi h_{\rm s}(1) = -\partial_\xi h_{\rm s}(-1) = -1 \quad \text{and} \quad h_{\rm s}(1) = h_{\rm s}(-1) = 0, \end{aligned} \tag{59}$$

which can be solved as (40), and satisfies

$$\partial_{\xi\xi}h_{\rm s} = h_{\rm s} - \frac{e^2+1}{e^2-1} = -\frac{e^{\xi+1}+e^{1-\xi}}{e^2-1} < 0. \tag{60}$$

Moreover, one can rewrite the 'surface tension' terms as

$$\begin{aligned} 2h_{\rm s}\theta_{1,\xi}\partial_{\xi\xi\xi}h_{\rm s} + 2h_{\rm s}\partial_\xi\big(\theta_{1,\xi}\partial_{\xi\xi}h_{\rm s}\big) + 2h_{\rm s}\partial_{\xi\xi}\big(\theta_{1,\xi}\partial_\xi h_{\rm s}\big) + 2h_{\rm s}\partial_{\xi\xi\xi}\big(h_{\rm s}\theta_{1,\xi}\big) \\ = 2\partial_\xi\big(4h_{\rm s}\partial_{\xi\xi}h_{\rm s}\theta_{1,\xi} - 2|\partial_\xi h_{\rm s}|^2\theta_{1,\xi} + \partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})\big). \end{aligned} \tag{61}$$

Therefore, (58) can be written as

$$\begin{aligned} h_{\rm s}\partial_{tt}\theta_1 - 2\partial_\xi[(h_{\rm s}^2 + 2|\partial_\xi h_{\rm s}|^2 - 4h_{\rm s}\partial_{\xi\xi}h_{\rm s})\theta_{1,\xi}] + 2\partial_{\xi\xi}(h_{\rm s}^2\theta_{1,\xi\xi}) \\ = 2\partial_\xi\big(h_{\rm s}\theta_{1,\xi t}\big), \qquad \text{with} \quad \theta_{1,\xi}|_{\xi=-1,1} = 0. \end{aligned} \tag{58'}$$

5.1 L^2 stability

Taking the L^2-inner product of (58') with $\theta_{1,t}$ yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg\{\frac{1}{2}\int h_{\rm s}|\theta_{1,t}|^2 \,\mathrm{d}\xi + \int (2|\partial_\xi h_{\rm s}|^2 - 4h_{\rm s}\partial_{\xi\xi}h_{\rm s} + h_{\rm s}^2)|\theta_{1,\xi}|^2 \,\mathrm{d}\xi \\ + \int |h_{\rm s}|^2|\theta_{1,\xi\xi}|^2 \,\mathrm{d}\xi\bigg\} + 2\int h_{\rm s}|\theta_{1,\xi t}|^2 \,\mathrm{d}\xi = 0. \end{aligned} \tag{62}$$

Moreover, by taking the time derivatives in (58') and performing the same arguments as above, one can derive the same estimates as in (62) with θ_1 replaced by $\partial_t^k\theta_1$, for any $k \in \mathbb{N}$; that is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\bigg\{\frac{1}{2}\int h_{\rm s}|\partial_t^{k+1}\theta_1|^2 \,\mathrm{d}\xi + \int (2|\partial_\xi h_{\rm s}|^2 - 4h_{\rm s}\partial_{\xi\xi}h_{\rm s} + h_{\rm s}^2)|\partial_t^k\theta_{1,\xi}|^2 \,\mathrm{d}\xi \\ + \int |h_{\rm s}|^2|\partial_t^k\theta_{1,\xi\xi}|^2 \,\mathrm{d}\xi\bigg\} + 2\int h_{\rm s}|\partial_t^{k+1}\theta_{1,\xi}|^2 \,\mathrm{d}\xi = 0. \end{aligned} \tag{63}$$

On the other hand, from (58'), one can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int h_{\rm s}\theta_{1,t} \,\mathrm{d}\xi = \int h_{\rm s}\theta_{1,tt} \,\mathrm{d}\xi = 0. \tag{64}$$

Without loss of generality, we assume that

$$\int h_{\rm s}\theta_{1,t} \,\mathrm{d}\xi = 0 \qquad \text{and} \qquad \int h_{\rm s}\theta_1 \,\mathrm{d}\xi = 0. \tag{65}$$

In addition, integrating (58') from $\xi = -1$ to ξ yields

$$\begin{aligned} 2h_{\mathrm{s}}\partial_t\theta_{1,\xi} - 2\partial_\xi(h_{\mathrm{s}}^2\theta_{1,\xi\xi}) + 2(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2)\theta_{1,\xi} \\ = \int_{-1}^{\xi}(h_{\mathrm{s}}\partial_{tt}\theta_1)(\sigma)\,d\sigma. \end{aligned} \tag{66}$$

Taking the L^2-inner product of (66) with $\theta_{1,\xi}$ yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int h_{\mathrm{s}}|\theta_{1,\xi}|^2\ \mathrm{d}\xi + 2\int h_{\mathrm{s}}^2|\theta_{1,\xi\xi}|^2\ \mathrm{d}\xi + 2\int\big(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2\big)|\theta_{1,\xi}|^2\ \mathrm{d}\xi \\ &= -\int h_{\mathrm{s}}\partial_{tt}\theta_1\cdot\theta_1\ \mathrm{d}\xi = -\frac{\mathrm{d}}{\mathrm{d}t}\int h_{\mathrm{s}}\partial_t\theta_1\cdot\theta_1\ \mathrm{d}\xi + \int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi. \end{aligned} \tag{67}$$

Notice that, thanks to (65), applying (45) imples that

$$\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi \lesssim \int h_{\mathrm{s}}|\partial_\xi\theta_{1,t}|^2\ \mathrm{d}\xi \quad \text{and} \quad \int h_{\mathrm{s}}|\theta_1|^2\ \mathrm{d}\xi \lesssim \int h_{\mathrm{s}}|\partial_\xi\theta_1|^2\ \mathrm{d}\xi. \tag{68}$$

Therefore, there exists a constant $\mathfrak{c}_1 \in (0,\infty)$ such that, after adding (62) with $\mathfrak{c}_1 \times$ (67),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_0 + \mathscr{D}_0 = 0, \tag{69}$$

where

$$\begin{aligned} \mathscr{E}_0 :=& \frac{1}{2}\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi + \int(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2)|\theta_{1,\xi}|^2\ \mathrm{d}\xi \\ &+ \int |h_{\mathrm{s}}|^2|\theta_{1,\xi\xi}|^2\ \mathrm{d}\xi + \mathfrak{c}_1\int h_{\mathrm{s}}|\theta_{1,\xi}|^2\ \mathrm{d}\xi + \mathfrak{c}_1\int h_{\mathrm{s}}\partial_t\theta_1\cdot\theta_1\ \mathrm{d}\xi, \end{aligned} \tag{70}$$

$$\begin{aligned} \mathscr{D}_0 :=& 2\int h_{\mathrm{s}}|\theta_{1,\xi t}|^2\ \mathrm{d}\xi - \mathfrak{c}_1\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi + 2\mathfrak{c}_1\int h_{\mathrm{s}}^2|\theta_{1,\xi\xi}|^2\ \mathrm{d}\xi \\ &+ 2\mathfrak{c}_1\int\big(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2\big)|\theta_{1,\xi}|^2\ \mathrm{d}\xi, \end{aligned} \tag{71}$$

satisfying, thanks to (68),

$$\begin{aligned} \mathscr{E}_0 \geq& \frac{1}{2}\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi + \int(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2)|\theta_{1,\xi}|^2\ \mathrm{d}\xi \\ &+ \int |h_{\mathrm{s}}|^2|\theta_{1,\xi\xi}|^2\ \mathrm{d}\xi + \mathfrak{c}_1\int h_{\mathrm{s}}|\theta_{1,\xi}|^2\ \mathrm{d}\xi \\ &- \mathfrak{c}_1\Big(\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi\Big)^{1/2}\cdot\Big(\int h_{\mathrm{s}}|\theta_1|^2\ \mathrm{d}\xi\Big)^{1/2} \\ \geq& \Big(\frac{1}{2} - \frac{\mathfrak{c}_1}{4\sigma}\Big)\int h_{\mathrm{s}}|\theta_{1,t}|^2\ \mathrm{d}\xi + \int(2|\partial_\xi h_{\mathrm{s}}|^2 - 4h_{\mathrm{s}}\partial_{\xi\xi}h_{\mathrm{s}} + h_{\mathrm{s}}^2)|\theta_{1,\xi}|^2\ \mathrm{d}\xi \\ &+ \int |h_{\mathrm{s}}|^2|\theta_{1,\xi\xi}|^2\ \mathrm{d}\xi + \mathfrak{c}_1\Big(\int h_{\mathrm{s}}|\theta_{1,\xi}|^2\ \mathrm{d}\xi - \sigma\int h_{\mathrm{s}}|\theta_1|^2\ \mathrm{d}\xi\Big) \\ &\gtrsim \|h_{\mathrm{s}}^{1/2}\theta_{1,t}\|_{L^2}^2 + \|\theta_{1,\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}\theta_{1,\xi\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}^{1/2}\theta_{1,\xi}\|_{L^2}^2 \end{aligned} \tag{72}$$

and

$$\mathscr{D}_0 \gtrsim \|h_{\mathrm{s}}^{1/2}\theta_{1,\xi t}\|_{L^2}^2 + \|h_{\mathrm{s}}\theta_{1,\xi\xi}\|_{L^2}^2 + \|\theta_{1,\xi}\|_{L^2}^2 \gtrsim \mathscr{E}_0. \tag{73}$$

Therefore, (69) yields the exponential decay in time, i.e.,

$$\|\theta_1(t)\|_{L^\infty}^2 \lesssim \|h_{\mathrm{s}}^{1/2}\theta_{1,t}(t)\|_{L^2}^2 + \|\theta_{1,\xi}(t)\|_{L^2}^2 + \|h_{\mathrm{s}}\theta_{1,\xi\xi}(t)\|_{L^2}^2 \lesssim e^{-\lambda_0 t} \tag{74}$$

and

$$\int e^{\lambda_0 s}\big\{\|h_{\mathrm{s}}^{1/2}\theta_{1,\xi t}(s)\|_{L^2}^2 + \|h_{\mathrm{s}}\theta_{1,\xi\xi}(s)\|_{L^2}^2 + \|\theta_{1,\xi}(s)\|_{L^2}^2\big\}\,\mathrm{d}s \lesssim 1. \tag{75}$$

for some $\lambda_0 \in (0,\infty)$.

5.2 Linear Elliptic estimates

To estimate the nonlinearities in the original equation (51'), it is important to capture the higher-order estimates. Repeating the same arguments as in (63), one can easily derive that the same estimates of the types (74) and (75) for θ_1 replaced by $\partial_t^k\theta_1$, for any $k \in \mathbb{N}$, hold, i.e.,

$$\|h_{\rm s}^{1/2}\partial_t^{k+1}\theta_1(t)\|_{L^2}^2 + \|\partial_t^k\theta_{1,\xi}(t)\|_{L^2}^2 + \|h_{\rm s}\partial_t^k\theta_{1,\xi\xi}(t)\|_{L^2}^2 \leq C_k e^{-\lambda_k t} \tag{76}$$

and

$$\int e^{\lambda_k s}\big\{\|h_{\rm s}^{1/2}\partial_t^{k+1}\theta_{1,\xi}(s)\|_{L^2}^2 + \|h_{\rm s}\partial_t^k\theta_{1,\xi\xi}(s)\|_{L^2}^2 + \|\partial_t^k\theta_{1,\xi}(s)\|_{L^2}^2\big\}\,\mathrm{d}s \leq C_k \tag{77}$$

for some $\lambda_k, C_k \in (0,\infty)$. Without loss of generality, we assume that $0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$.

Next, we aim to show the estimates by shifting the time derivatives to space derivatives, i.e., the Elliptic estimates of (58').

Let

$$m_{\rm s} := 2|\partial_\xi h_{\rm s}|^2 - 4h_{\rm s}\partial_{\xi\xi}h_{\rm s} + h_{\rm s}^2 > 0. \tag{78}$$

Then (66) can be written as

$$h_{\rm s}\partial_t\theta_{1,\xi} - \partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi}) + m_{\rm s}\theta_{1,\xi} = \frac{1}{2}\int_{-1}^{\xi}(h_{\rm s}\partial_{tt}\theta_1)(\sigma)\,d\sigma. \tag{79}$$

5.2.1 Estimate of $\theta_{1,\xi\xi\xi}$

Let $\varepsilon \in (0,1)$ be an arbitrarily small constant. Unless stated explicitly, are independent of ε.

After rearranging the equation, (79) can be written as

$$\begin{aligned} -\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi}) + 2|h_{\rm s}'|^2\theta_{1,\xi} &= (4h_{\rm s}h_{\rm s}'' - h_{\rm s}^2)\theta_{1,\xi} - h_{\rm s}\partial_t\theta_{1,\xi} \\ &\quad + \frac{1}{2}\int_{-1}^{\xi}(h_{\rm s}\partial_{tt}\theta_1)(\sigma)\,d\sigma. \end{aligned} \tag{80}$$

Thanks to the fact that

$$\int_{-1}^{1} h_{\rm s}\partial_{tt}\theta_1\,\mathrm{d}\xi = 0,$$

one can write

$$\begin{aligned} \Big|\int_{-1}^{\xi}(h_{\rm s}\partial_{tt}\theta_1)(\sigma)\,d\sigma\Big| &= \begin{cases} |\int_{-1}^{\xi}(h_{\rm s}\partial_{tt}\theta_1)(\sigma)\,d\sigma| & \xi \leq 0, \\ |-\int_{\xi}^{1}(h_{\rm s}\partial_{tt}\theta_1)(\sigma)\,d\sigma| & \xi > 0 \end{cases} \\ &\lesssim h_{\rm s}\|h_{\rm s}^{1/2}\partial_{tt}\theta_1\|_{L^2}, \end{aligned} \tag{81}$$

where the last inequality follows by applying Hölder's inequality in the two intervals $(-1,0)$ and $(0,1)$, and using the fact that $h_{\rm s}(\xi) \simeq (1+\xi)(1-\xi)$.

After dividing (80) with $h_{\rm s}^{1/2}$, taking the square of both sides and integrating the resultant, i.e., $\|\frac{(80)}{h_{\rm s}^{1/2}}\|_{L^2}^2$, yield, thanks to (81),

$$\begin{aligned} \Big\|\frac{-\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi}) + 2|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\Big\|_{L^2}^2 &\lesssim \|\theta_{1,\xi}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 \\ &\quad + \|h_{\rm s}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2. \end{aligned} \tag{82}$$

The left hand side of (82) can be calculated as below:

$$\begin{aligned}
&\|\frac{-\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})+2|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 = 4\|\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 + \|\frac{\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})}{h_{\rm s}^{1/2}}\|_{L^2}^2 \\
&-4\int \frac{\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})\cdot|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}}\,{\rm d}\xi = 4\|\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 + \|h_{\rm s}^{3/2}\theta_{1,\xi\xi\xi}\|_{L^2}^2 \\
&+4\|h_{\rm s}^{1/2}h_{\rm s}'\theta_{1,\xi\xi}\|_{L^2}^2 + 4\int h_{\rm s}^2h_{\rm s}'\theta_{1,\xi\xi}\theta_{1,\xi\xi\xi}\,{\rm d}\xi + 4\int h_{\rm s}^2\theta_{1,\xi\xi}\cdot\partial_\xi\Big(\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}}\Big)\,{\rm d}\xi \\
&= 4\|\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 + \|h_{\rm s}^{3/2}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + 4\|h_{\rm s}^{1/2}h_{\rm s}'\theta_{1,\xi\xi}\|_{L^2}^2 \\
&-2\int h_{\rm s}^2h_{\rm s}''|\theta_{1,\xi\xi}|^2\,{\rm d}\xi + 4\int\big(2h_{\rm s}h_{\rm s}'h_{\rm s}''-(h_{\rm s}')^3\big)\theta_{1,\xi}\theta_{1,\xi\xi}\,{\rm d}\xi \\
&= 4\|\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 + \|h_{\rm s}^{3/2}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + 4\|h_{\rm s}^{1/2}h_{\rm s}'\theta_{1,\xi\xi}\|_{L^2}^2 \\
&-2\int h_{\rm s}^2h_{\rm s}''|\theta_{1,\xi\xi}|^2\,{\rm d}\xi - 2\int\big(2h_{\rm s}h_{\rm s}'h_{\rm s}''-(h_{\rm s}')^3\big)'|\theta_{1,\xi}|^2\,{\rm d}\xi.
\end{aligned}\tag{83}$$

Therefore, (82) and (83) yield

$$\begin{aligned}
\|h_{\rm s}^{3/2}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + \|h_{\rm s}^{1/2}\theta_{1,\xi\xi}\|_{L^2}^2 + \|\frac{\theta_{1,\xi}}{h_{\rm s}^{1/2}}\|_{L^2}^2 &\lesssim \|\theta_{1,\xi}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 \\
&+\|h_{\rm s}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2.
\end{aligned}\tag{84}$$

On the other hand, after dividing (80) with $h_{\rm s}+\varepsilon$, taking the square of both sides and integrating the resultant, i.e., $\|\frac{(80)}{h_{\rm s}+\varepsilon}\|_{L^2}^2$, yield, thanks to (81),

$$\begin{aligned}
\|\frac{-\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})+2|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}+\varepsilon}\|_{L^2}^2 &\lesssim \|\theta_{1,\xi}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 \\
&+\|h_{\rm s}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2.
\end{aligned}\tag{85}$$

Meanwhile, the left hand side of (85) can be calculated as below:

$$\begin{aligned}
\|\frac{-\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})+2|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}+\varepsilon}\|_{L^2}^2 &= 4\|\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{h_{\rm s}+\varepsilon}\|_{L^2}^2 + \|\frac{\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})}{h_{\rm s}+\varepsilon}\|_{L^2}^2 \\
&\underbrace{-4\int\frac{\partial_\xi(h_{\rm s}^2\theta_{1,\xi\xi})\cdot|h_{\rm s}'|^2\theta_{1,\xi}}{(h_{\rm s}+\varepsilon)^2}\,{\rm d}\xi}_{I_{01}}.
\end{aligned}\tag{86}$$

To calculate I_{01}, applying integration by parts yields

$$\begin{aligned}
I_{01} &= 4\int h_{\rm s}^2\theta_{1,\xi\xi}\cdot\partial_\xi\Big(\frac{|h_{\rm s}'|^2\theta_{1,\xi}}{(h_{\rm s}+\varepsilon)^2}\Big)\,{\rm d}\xi = 4\int\frac{h_{\rm s}^2|h_{\rm s}'|^2|\theta_{1,\xi\xi}|^2}{(h_{\rm s}+\varepsilon)^2}\,{\rm d}\xi \\
&+4\int\Big(\frac{2h_{\rm s}^2h_{\rm s}'h_{\rm s}''}{(h_{\rm s}+\varepsilon)^2} - \frac{2h_{\rm s}^2(h_{\rm s}')^3}{(h_{\rm s}+\varepsilon)^3}\Big)\theta_{1,\xi\xi}\theta_{1,\xi}\,{\rm d}\xi = 4\int\frac{h_{\rm s}^2|h_{\rm s}'|^2|\theta_{1,\xi\xi}|^2}{(h_{\rm s}+\varepsilon)^2}\,{\rm d}\xi \\
&-4\int\Big(\frac{h_{\rm s}^2h_{\rm s}'h_{\rm s}''}{(h_{\rm s}+\varepsilon)^2} - \frac{h_{\rm s}^2(h_{\rm s}')^3}{(h_{\rm s}+\varepsilon)^3}\Big)'|\theta_{1,\xi}|^2\,{\rm d}\xi \\
&= 4\int\frac{h_{\rm s}^2|h_{\rm s}'|^2|\theta_{1,\xi\xi}|^2}{(h_{\rm s}+\varepsilon)^2}\,{\rm d}\xi + 4\int\Big(\frac{-2h_{\rm s}|h_{\rm s}'|^2h_{\rm s}'' - h_{\rm s}^2|h_{\rm s}''|^2 - h_{\rm s}^2h_{\rm s}'h_{\rm s}'''}{(h_{\rm s}+\varepsilon)^2} \\
&+\frac{2h_{\rm s}|h_{\rm s}'|^4 + 5h_{\rm s}^2|h_{\rm s}'|^2h_{\rm s}''}{(h_{\rm s}+\varepsilon)^3} - \frac{3h_{\rm s}^2|h_{\rm s}'|^4}{(h_{\rm s}+\varepsilon)^4}\Big)|\theta_{1,\xi}|^2\,{\rm d}\xi.
\end{aligned}\tag{87}$$

Sending $\varepsilon \to 0$, thanks to (59), leads to

$$\lim_{\varepsilon\to 0^+} I_{01} = 4\int |h_{\mathrm{s}}'|^2|\theta_{1,\xi\xi}|^2 \,\mathrm{d}\xi \\ +4\int \big(-|h_{\mathrm{s}}''|^2 - h_{\mathrm{s}}' h_{\mathrm{s}} + \frac{3|h_{\mathrm{s}}'|^2 h_{\mathrm{s}}''}{h_{\mathrm{s}}} - \frac{|h_{\mathrm{s}}'|^4}{h_{\mathrm{s}}^2}\big)|\theta_{1,\xi}|^2 \,\mathrm{d}\xi. \tag{88}$$

Therefore sending $\varepsilon \to 0^+$ in (85) and (86) yields

$$4\int \big(-|h_{\mathrm{s}}''|^2 - h_{\mathrm{s}}' h_{\mathrm{s}} + \frac{3|h_{\mathrm{s}}'|^2 h_{\mathrm{s}}''}{h_{\mathrm{s}}}\big)|\theta_{1,\xi}|^2 \,\mathrm{d}\xi + 4\|h_{\mathrm{s}}'\theta_{1,\xi\xi}\|_{L^2}^2 + \underbrace{\|\frac{\partial_\xi(h_{\mathrm{s}}^2\theta_{1,\xi\xi})}{h_{\mathrm{s}}}\|_{L^2}^2}_{I_{02}} \\ \lesssim \|\theta_{1,\xi}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2. \tag{89}$$

Further more, I_{02} in (89) can be calculated as

$$I_{02} = \|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + 4\|h_{\mathrm{s}}'\theta_{1,\xi\xi}\|_{L^2}^2 + 4\int h_{\mathrm{s}}h_{\mathrm{s}}'\theta_{1,\xi\xi}\theta_{1,\xi\xi\xi}\,\mathrm{d}\xi \\ = \|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + 2\|h_{\mathrm{s}}'\theta_{1,\xi\xi}\|_{L^2}^2 - 2\int h_{\mathrm{s}}h_{\mathrm{s}}''|\theta_{1,\xi\xi}|^2\,\mathrm{d}\xi. \tag{90}$$

Therefore, (89) implies

$$\|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + \|\theta_{1,\xi\xi}\|_{L^2}^2 \lesssim \|\frac{\theta_{1,\xi}}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2, \tag{91}$$

which, together with (84), implies

$$\|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}\|_{L^2}^2 + \|\theta_{1,\xi\xi}\|_{L^2}^2 + \|\frac{\theta_{1,\xi}}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2 \lesssim \|\theta_{1,\xi}\|_{L^2}^2 + \|\partial_t\theta_{1,\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}^{1/2}\partial_{tt}\theta_1\|_{L^2}^2. \tag{92}$$

To sum up, thanks to (76) and (77) with $k=1$, (92) implies

$$e^{\lambda_1 t}\Big(\|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}(t)\|_{L^2}^2 + \|\theta_{1,\xi\xi}(t)\|_{L^2}^2 + \|\frac{\theta_{1,\xi}(t)}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big) \\ + \int_0^t e^{\lambda_1 s}\Big(\|h_{\mathrm{s}}\theta_{1,\xi\xi\xi}(s)\|_{L^2}^2 + \|\theta_{1,\xi\xi}(s)\|_{L^2}^2 + \|\frac{\theta_{1,\xi}(s)}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big)\,\mathrm{d}s \lesssim C_1. \tag{93}$$

Repeating the same argument with θ_1 replaced by $\partial_t\theta_1$, one can conclude that

$$e^{\lambda_2 t}\Big(\|h_{\mathrm{s}}\partial_t\theta_{1,\xi\xi\xi}(t)\|_{L^2}^2 + \|\partial_t\theta_{1,\xi\xi}(t)\|_{L^2}^2 + \|\frac{\partial_t\theta_{1,\xi}(t)}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big) \\ + \int_0^t e^{\lambda_2 s}\Big(\|h_{\mathrm{s}}\partial_t\theta_{1,\xi\xi\xi}(s)\|_{L^2}^2 + \|\partial_t\theta_{1,\xi\xi}(s)\|_{L^2}^2 + \|\frac{\partial_t\theta_{1,\xi}(s)}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big)\,\mathrm{d}s \lesssim C_2. \tag{94}$$

In particular, this implies that

$$\|\partial_t\theta_{1,\xi}(t)\|_{L^\infty}^2 \lesssim \|\partial_t\theta_{1,\xi\xi}(t)\|_{L^2}^2 \lesssim C_2 e^{-\lambda_2 t}, \tag{95}$$

and therefore the flow map is invertible. Moreover, the linearized system is asymptotically stable.

5.2.2 Estimate of $\theta_{1,\xi\xi\xi\xi}$

From (58'), one can write

$$\begin{aligned} h_s^2\theta_{1,\xi\xi\xi\xi} + 4h_s h_s'\theta_{1,\xi\xi\xi} &= -(h_s^2)''\theta_{1,\xi\xi} + \partial_\xi[(2|\partial_\xi h_s|^2 - 4h_s\partial_{\xi\xi}h_s)\theta_{1,\xi}] \\ &\quad + (h_s^2\theta_{1,\xi})_\xi + \partial_\xi(h_s\theta_{1,\xi t}) - \frac{1}{2}h_s\partial_{tt}\theta_1 \\ &= \big(-6h_s h_s'' + h_s^2\big)\theta_{1,\xi\xi} + \big(-4h_s h_s''' + 2h_s h_s'\big)\theta_{1,\xi} \\ &\quad + h_s\theta_{1,\xi\xi t} + h_s'\theta_{1,\xi t} - \frac{1}{2}h_s\partial_{tt}\theta_1. \end{aligned} \tag{96}$$

Therefore, after dividing (96) with $h_s^{1/2}$ and taking the L^2-norm of the resultant, one has that

$$\begin{aligned} \|\frac{h_s^2\theta_{1,\xi\xi\xi\xi} + 4h_s h_s'\theta_{1,\xi\xi\xi}}{h_s^{1/2}}\|_{L^2}^2 &\lesssim \|h_s^{1/2}\theta_{1,\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_{1,\xi}\|_{L^2}^2 \\ &\quad + \|h_s^{1/2}\theta_{1,\xi\xi t}\|_{L^2}^2 + \|\frac{\theta_{1,\xi t}}{h_s^{1/2}}\|_{L^2}^2 + \|h_s^{1/2}\partial_{tt}\theta_1\|_{L^2}^2. \end{aligned} \tag{97}$$

Meanwhile, the left hand side of (97) can be calculated as below:

$$\begin{aligned} \|\frac{h_s^2\theta_{1,\xi\xi\xi\xi} + 4h_s h_s'\theta_{1,\xi\xi\xi}}{h_s^{1/2}}\|_{L^2}^2 &= \|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}\|_{L^2}^2 + 16\|h_s^{1/2}h_s'\theta_{1,\xi\xi\xi}\|_{L^2}^2 \\ + 8\int h_s^2 h_s'\theta_{1,\xi\xi\xi}\theta_{1,\xi\xi\xi\xi}\,\mathrm{d}\xi &= \|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}\|_{L^2}^2 + 16\|h_s^{1/2}h_s'\theta_{1,\xi\xi\xi}\|_{L^2}^2 \\ -4\int (h_s^2 h_s')'|\theta_{1,\xi\xi\xi}|^2\,\mathrm{d}\xi &= \|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}\|_{L^2}^2 \\ + \int \big(8h_s(h_s')^2 - 4h_s^2 h_s''\big)|\theta_{1,\xi\xi\xi}|^2\,\mathrm{d}\xi &\gtrsim \|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_{1,\xi\xi\xi}\|_{L^2}^2. \end{aligned} \tag{98}$$

Therefore, thanks to (76), (77), (93), and (94), (97) and (98) imply

$$\begin{aligned} & e^{\lambda_2 t}\big(\|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}(t)\|_{L^2}^2 + \|h_s^{1/2}\theta_{1,\xi\xi\xi}(t)\|_{L^2}^2\big) \\ & + \int e^{\lambda_2 s}\big(\|h_s^{3/2}\theta_{1,\xi\xi\xi\xi}(s)\|_{L^2}^2 + \|h_s^{1/2}\theta_{1,\xi\xi\xi}(s)\|_{L^2}^2\big)\,\mathrm{d}s \lesssim C_2. \end{aligned} \tag{99}$$

6 Nonlinear elliptic estimates

In this section, we demonstrate the nonlinear elliptic estimates for the solution to (53), i.e., shifting the temporal derivative to the spatial derivation. Let

$$\mathcal{E}_{\mathrm{NL},1} := \sum_{k=0,1,2}\Big\{\|h_s^{1/2}\partial_t^{k+1}\theta\|_{L^2}^2 + \|h_s\partial_t^k\theta_{\xi\xi}\|_{L^2}^2 + \|\partial_t^k\theta_\xi\|_{L^2}^2\Big\}. \tag{100}$$

We will show that

$$\begin{aligned} \mathcal{E}_{\mathrm{NL},2} &:= \|h_s^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 \\ &+ \sum_{k=0,1}\Big\{\|h_s\partial_t^k\theta_{\xi\xi\xi}\|_{L^2}^2 + \|\partial_t^k\theta_{\xi\xi}\|_{L^2}^2 + \|\frac{\partial_t^k\theta_\xi}{h_s^{1/2}}\|_{L^2}^2\Big\}, \end{aligned} \tag{101}$$

i.e, the (weighted) estimates of $\theta_{\xi\xi\xi\xi}$, $\partial_t\theta_{\xi\xi\xi}$, and $\theta_{\xi\xi\xi}$, can be bounded in terms of $\mathcal{E}_{\mathrm{NL},1}$ provided that $\mathcal{E}_{\mathrm{NL},2}$ is small.

We first rewrite the equation (53), by separating the linear and nonlinear parts. Notice that

$$\frac{1}{1+\theta_\xi} = 1-\theta_\xi + \underbrace{\frac{\theta_\xi^2}{1+\theta_\xi}}_{=:g_1(\theta_\xi)} \quad \text{and} \quad \frac{1}{(1+\theta_\xi)^2} = 1-2\theta_\xi + \underbrace{\frac{2\theta_\xi+3}{(1+\theta_\xi)^2}\theta_\xi^2}_{=:g_2(\theta_\xi)}, \tag{102}$$

where, for small θ_ξ,

$$g_1(\theta_\xi), g_2(\theta_\xi) = \mathcal{O}(\theta_\xi^2). \tag{103}$$

With these notations, (53) can be written as

$$\begin{aligned} h_s\partial_{tt}\theta - 2\big[(h_s^2+2|h_s'|^2-4h_sh_s'')\theta_\xi\big]_\xi + 2\big[h_s^2\theta_{\xi\xi}\big]_{\xi\xi} - 2\big(h_s\theta_{\xi t}\big)_\xi \\ +\big[N_1\big]_\xi - 2\big[h_s(-2\theta_\xi+g_2(\theta_\xi))\theta_{\xi t}\big]_\xi = 0, \end{aligned} \tag{104}$$

where

$$\begin{aligned} N_1 :=\ & h_s^2 g_2(\theta_\xi) - 4h_s\theta_\xi\big[h_s'\theta_\xi + (h_s\theta_\xi)_\xi\big]_\xi \\ & -2h_s(1-2\theta_\xi+g_2(\theta_\xi))\big\{\theta_\xi(h_s\theta_\xi)_\xi + (1-\theta_\xi+g_1(\theta_\xi))[h_sg_1(\theta_\xi)]_\xi \\ & \quad +g_1(\theta_\xi)[h_s(1-\theta_\xi)]_\xi\big\}_\xi \\ & -2h_sg_2(\theta_\xi)\big[h_s' - h_s'\theta_\xi - (h_s\theta_\xi)_\xi\big]_\xi + \big[h_s'\theta_\xi + (h_s\theta_\xi)_\xi\big]^2 \\ & +\big\{2h_s' - 2h_s'\theta_\xi - 2(h_s\theta_\xi)_\xi + \theta_\xi(h_s\theta_\xi)_\xi + (1-\theta_\xi+g_1(\theta_\xi))(h_sg_1(\theta_\xi))_\xi \\ & +g_1(\theta_\xi)[h_s(1-\theta_\xi)]_\xi\big\} \times \big\{\theta_\xi(h_s\theta_\xi)_\xi + (1-\theta_\xi+g_1(\theta_\xi))[h_sg_1(\theta_\xi)]_\xi \\ & +g_1(\theta_\xi)[h_s(1-\theta_\xi)]_\xi\big\}, \end{aligned} \tag{105}$$

or, by denoting $f_i = f_i(y) \in C^\infty(-1/2,1/2)$, $i=1,2,\cdots,6$,

$$\begin{aligned} N_1 = h_s^2\big\{f_1(\theta_\xi)\theta_\xi^2 + f_2(\theta_\xi)\theta_\xi\theta_{\xi\xi\xi} + f_3(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi}\big\} \\ +h_sh_s'f_4(\theta_\xi)\theta_\xi\theta_{\xi\xi} + (h_s')^2f_5(\theta_\xi)\theta_\xi^2 + h_sh_s''f_6(\theta_\xi)\theta_\xi^2. \end{aligned} \tag{106}$$

To simplify the presentation, hereafter we use

$$f = f(y) \in C^\infty(-1/2,1/2) \quad \text{and} \quad F = F(y) \geq 0 \in C^\infty[0,\infty) \tag{107}$$

to denote the smooth functions of the argument, which is different from line to line.

6.1 Embedding inequalities

We summarize the weighted-L^p embedding inequalities used in this section. These inequalities are consequences of Hardy's inequalities (Lemma 1) and the Sobolev embedding inequalities.

Lemma 3. *The following inequalities hold:*

$$\begin{aligned} \|\theta_\xi\|_{L^\infty} \lesssim \|\theta_{\xi\xi}\|_{L^2} \lesssim \|h_s\theta_{\xi\xi\xi}\|_{L^2} + \|h_s\theta_{\xi\xi}\|_{L^2}, \\ \|\theta_{\xi t}\|_{L^\infty} \lesssim \|\theta_{\xi\xi t}\|_{L^2} \lesssim \|h_s\theta_{\xi\xi\xi t}\|_{L^2} + \|h_s\theta_{\xi\xi t}\|_{L^2}, \\ \|\frac{\theta_{\xi t}}{h_s^{1/2}}\|_{L^2} \lesssim \|\frac{\theta_{\xi t}}{h_s}\|_{L^2} \lesssim \|\theta_{\xi\xi t}\|_{L^2} \lesssim \|h_s\theta_{\xi\xi\xi t}\|_{L^2} + \|h_s\theta_{\xi\xi t}\|_{L^2}, \\ \|h_s\theta_{\xi\xi}\|_{L^\infty} \lesssim \|h_s\theta_{\xi\xi\xi}\|_{L^2} + \|\theta_{\xi\xi}\|_{L^2}, \\ \|h_s\theta_{\xi\xi t}\|_{L^\infty} \lesssim \|h_s\theta_{\xi\xi\xi t}\|_{L^2} + \|\theta_{\xi\xi t}\|_{L^2}, \\ \|h_s^{1/2}\theta_{\xi\xi}\|_{L^\infty} \lesssim \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^{3/2}} + \|h_s^{-1/2}\theta_{\xi\xi}\|_{L^{3/2}} \\ \lesssim \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2} + \|\theta_{\xi\xi}\|_{L^2}. \end{aligned} \tag{108}$$

Proof. This is a direct consequence of applying the Sobolev embedding inequalities and Hardy's inequalities. □

Lemma 4. *The following inequalities hold:*

$$\|\frac{\theta_\xi}{h_s}\|_{L^4} + \|\theta_{\xi\xi}\|_{L^4} \lesssim \|h_s^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2} + \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2} + \|\theta_{\xi\xi}\|_{L^2}, \tag{109}$$

$$\|h_s^{1/2}\theta_{\xi tt}\|_{L^4} \lesssim \|h_s\theta_{\xi\xi tt}\|_{L^2} + \|\theta_{\xi tt}\|_{L^2}. \tag{110}$$

Proof. Recall that $\theta_\xi|_{\xi=-1,1} = 0$. Thanks to Hardy's inequalities and the Sobolev inequalities one has that

$$\begin{aligned}\|\frac{\theta_\xi}{h_s}\|_{L^4}^4 &\underbrace{\lesssim}_{(44) \text{ with } k=0,p=4} \|\theta_{\xi\xi}\|_{L^4}^4 \lesssim \|h_s\theta_{\xi\xi\xi}\|_{L^4}^4 + \|h_s\theta_{\xi\xi}\|_{L^4}^4 \\ &\lesssim \|h_s^{3/2}\theta_{\xi\xi\xi}\|_{L^\infty}^2 \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 + \|h_s\theta_{\xi\xi}\|_{L^\infty}^2 \|h_s\theta_{\xi\xi}\|_{L^2}^2 \\ &\lesssim \left(\|h_s^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 + \|\theta_{\xi\xi}\|_{L^2}^2\right)^2,\end{aligned} \tag{111}$$

which proves (109).

To prove (110), applying Hardy's inequalities and the Sobolev inequalities yields

$$\|h_s^{1/2}\theta_{\xi tt}\|_{L^4}^4 \lesssim \|h_s\theta_{\xi tt}\|_{L^\infty}^2 \|\theta_{\xi tt}\|_{L^2}^2 \lesssim (\|h_s\theta_{\xi\xi tt}\|_{L^2}^2 + \|\theta_{\xi tt}\|_{L^2}^2)\|\theta_{\xi tt}\|_{L^2}^2. \tag{112}$$

This finishes the proof of (110). □

6.2 Estimates of $\theta_{\xi\xi\xi\xi}$

Similar to (96), one can write, from (104), that

$$\begin{aligned}h_s^2\theta_{\xi\xi\xi\xi} + 4h_sh_s'\theta_{\xi\xi\xi} &= (-6h_sh_s'' + h_s^2)\theta_{\xi\xi} + (-4h_sh_s''' + 2h_sh_s')\theta_\xi \\ &+h_s\theta_{\xi\xi t} + h_s'\theta_{\xi t} - \frac{1}{2}h_s\partial_{tt}\theta - \frac{1}{2}\left[N_1\right]_\xi + \left[h_s(-2\theta_\xi + g_2(\theta_\xi))\theta_{\xi t}\right]_\xi.\end{aligned} \tag{113}$$

Then applying the same arguments as in (97)–(98) yields

$$\begin{aligned}\|h_s^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 &\lesssim \|\frac{h_s^2\theta_{\xi\xi\xi\xi} + 4h_sh_s'\theta_{\xi\xi\xi}}{h_s^{1/2}}\|_{L^2}^2 \\ \lesssim \|h_s^{1/2}\theta_{\xi\xi}\|_{L^2}^2 + \|h_s^{1/2}\theta_\xi\|_{L^2}^2 &+ \|h_s^{1/2}\theta_{\xi\xi t}\|_{L^2}^2 + \|\frac{\theta_{\xi t}}{h_s^{1/2}}\|_{L^2}^2 + \|h_s^{1/2}\theta_{tt}\|_{L^2}^2 \\ +\|\frac{(N_1)_\xi}{h_s^{1/2}}\|_{L^2}^2 &+ F(\|\theta_\xi\|_{L^\infty})\|\theta_\xi\|_{L^\infty}^2\|\frac{\theta_{\xi t}}{h_s^{1/2}}\|_{L^2}^2 \\ +F(\|\theta_\xi\|_{L^\infty})\|\theta_\xi\|_{L^\infty}^2\|h_s^{1/2}\theta_{\xi\xi t}\|_{L^2}^2 &+ F(\|\theta_\xi\|_{L^\infty})\|\theta_{\xi t}\|_{L^\infty}^2\|h_s^{1/2}\theta_{\xi\xi}\|_{L^2}^2.\end{aligned} \tag{114}$$

To calculate $\|\frac{(N_1)_\xi}{h_s^{1/2}}\|_{L^2}^2$, from (106), one can calculate

$$\begin{aligned}(N_1)_\xi = h_s^2\{f(\theta_\xi)\theta_\xi\theta_{\xi\xi} &+ f(\theta_\xi)\theta_\xi\theta_{\xi\xi\xi\xi} + f(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi\xi} + f(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi}\theta_{\xi\xi}\} \\ &+h_sh_s'\{f(\theta_\xi)\theta_\xi^2 + f(\theta_\xi)\theta_\xi\theta_{\xi\xi\xi} + f(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi}\} \\ &+((h_s')^2 + h_sh_s'')\{f(\theta_\xi)\theta_\xi\theta_{\xi\xi}\} + ((h_s')^2 + h_sh_s'')'f(\theta_\xi)\theta_\xi^2.\end{aligned} \tag{115}$$

Therefore one has that

$$
\begin{aligned}
\|\frac{(N_1)_\xi}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2 \lesssim F(\|\theta_\xi\|_{L^\infty})\Big\{ & \|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}^{3/2}\theta_{\xi\xi}\|_{L^2}^2 + \|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2}^2 \\
+\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}^2\|h_{\mathrm{s}}^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 &+ \|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}^2\|h_{\mathrm{s}}^{1/4}\theta_{\xi\xi}\|_{L^4}^4 + \|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}^{1/2}\theta_\xi\|_{L^2}^2 \\
+\|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 &+ \|h_{\mathrm{s}}^{1/4}\theta_{\xi\xi}\|_{L^4}^4 + \|\frac{\theta_\xi}{h_{\mathrm{s}}}\|_{L^4}^2\|h_{\mathrm{s}}^{1/2}\theta_{\xi\xi}\|_{L^4}^2 \\
&+\|\theta_\xi\|_{L^\infty}^2\|\frac{\theta_\xi}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big\}.
\end{aligned}
\tag{116}
$$

Consequently, one can conclude from (114)–(116) that

$$
\begin{aligned}
\|h_{\mathrm{s}}^{3/2}\theta_{\xi\xi\xi\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}^{1/2}\theta_{\xi\xi\xi}\|_{L^2}^2 \lesssim \|\theta_{\xi\xi}\|_{L^2}^2 + \|\theta_{\xi\xi t}\|_{L^2}^2 + \|\frac{\theta_{\xi t}}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2 \\
+\mathcal{E}_{\mathrm{NL},1} + F(\mathcal{E}_{\mathrm{NL},2})\mathcal{E}_{\mathrm{NL},2}^2,
\end{aligned}
\tag{117}
$$

thanks to (108) and (109).

6.3 Estimates of $\theta_{\xi\xi\xi t}$ and $\theta_{\xi\xi\xi}$

Integrating (104) from $\xi = -1$ to ξ yields

$$
\begin{aligned}
&2h_{\mathrm{s}}\partial_t\theta_\xi - 2\partial_\xi(h_{\mathrm{s}}^2\theta_{\xi\xi}) + 2(2|h_{\mathrm{s}}'|^2 - 4h_{\mathrm{s}}h_{\mathrm{s}}'' + h_{\mathrm{s}}^2)\theta_\xi \\
&= \int_{-1}^{\xi}(h_{\mathrm{s}}\partial_{tt}\theta)(\sigma)\,d\sigma + N_1 - 2h_{\mathrm{s}}(-2\theta_\xi + g_2(\theta_\xi))\theta_{\xi t}.
\end{aligned}
\tag{118}
$$

Then, similar to (80), (118) can be written as

$$
\begin{aligned}
-\partial_\xi(h_{\mathrm{s}}^2\theta_{\xi\xi}) + 2|h_{\mathrm{s}}'|^2\theta_\xi = (4h_{\mathrm{s}}h_{\mathrm{s}}'' - h_{\mathrm{s}}^2)\theta_\xi - h_{\mathrm{s}}\partial_t\theta_\xi + \frac{1}{2}\int_{-1}^{\xi}(h_{\mathrm{s}}\partial_{tt}\theta)(\sigma)\,d\sigma \\
+\frac{1}{2}N_1 - h_{\mathrm{s}}(-2\theta_\xi + g_2(\theta_\xi))\theta_{\xi t}.
\end{aligned}
\tag{119}
$$

Repeating the same arguments as in (82)–(92) leads to

$$
\begin{aligned}
\sum_{k=0,1}\Big\{\|h_{\mathrm{s}}\partial_t^k\theta_{\xi\xi\xi}\|_{L^2}^2 + \|\partial_t^k\theta_{\xi\xi}\|_{L^2}^2 + \|\frac{\partial_t^k\theta_\xi}{h_{\mathrm{s}}^{1/2}}\|_{L^2}^2\Big\} \lesssim \mathcal{E}_{\mathrm{NL},1} \\
+F(\mathcal{E}_{\mathrm{NL},2})\mathcal{E}_{\mathrm{NL},2}^2 + \|\frac{N_1}{h_{\mathrm{s}}}\|_{L^2}^2 + \|\frac{\partial_t N_1}{h_{\mathrm{s}}}\|_{L^2}^2.
\end{aligned}
\tag{120}
$$

It suffices to calculate $\|\frac{\partial_t N_1}{h_{\mathrm{s}}}\|_{L^2}^2$. The estimate of $\|\frac{N_1}{h_{\mathrm{s}}}\|_{L^2}^2$ follows similarly.

From (106), direct calculation yields

$$
\begin{aligned}
\partial_t N_1 = h_{\mathrm{s}}^2\{&f(\theta_\xi)\theta_\xi\theta_{\xi t} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi\xi} + f(\theta_\xi)\theta_\xi\theta_{\xi\xi\xi t} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi}\theta_{\xi\xi} \\
&+f(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi t}\} \\
&+h_{\mathrm{s}}h_{\mathrm{s}}'f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi} + h_{\mathrm{s}}h_{\mathrm{s}}'f(\theta_\xi)\theta_\xi\theta_{\xi\xi t} + (|h_{\mathrm{s}}'|^2 + h_{\mathrm{s}}h_{\mathrm{s}}'')f(\theta_\xi)\theta_\xi\theta_{\xi t}.
\end{aligned}
\tag{121}
$$

Therefore one has that

$$
\begin{aligned}
\|\frac{\partial_t N_1}{h_{\mathrm{s}}}\|_{L^2}^2 \lesssim F(\|\theta_\xi\|_{L^\infty})\Big\{ & \|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}\theta_{\xi t}\|_{L^2}^2 + \|\theta_{\xi t}\|_{L^\infty}^2\|h_{\mathrm{s}}\theta_{\xi\xi\xi}\|_{L^2}^2 \\
+\|\theta_\xi\|_{L^\infty}^2\|h_{\mathrm{s}}\theta_{\xi\xi\xi t}\|_{L^2}^2 + \|\theta_{\xi t}\|_{L^\infty}^2 &\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}^2\|\theta_{\xi\xi}\|_{L^2}^2 + \|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}^2\|\theta_{\xi\xi t}\|_{L^2}^2 \\
+\|\theta_{\xi t}\|_{L^\infty}^2\|\theta_{\xi\xi}\|_{L^2}^2 &+ \|\theta_\xi\|_{L^\infty}^2\|\theta_{\xi\xi t}\|_{L^2}^2 + \|\frac{\theta_\xi}{h_{\mathrm{s}}}\|_{L^4}^2\|\theta_{\xi t}\|_{L^4}^2\Big\}.
\end{aligned}
\tag{122}
$$

Consequently, one can conclude from (120)–(122) that

$$\sum_{k=0,1}\Big\{\|h_{\rm s}\partial_t^k\theta_{\xi\xi\xi}\|_{L^2}^2+\|\partial_t^k\theta_{\xi\xi}\|_{L^2}^2+\|\frac{\partial_t^k\theta_\xi}{h_{\rm s}^{1/2}}\|_{L^2}^2\Big\}\lesssim \mathcal{E}_{\rm NL,1} \\ +F(\mathcal{E}_{\rm NL,2})\mathcal{E}_{\rm NL,2}^2. \tag{123}$$

In summary, from (117) and (123), we have shown that

$$\mathcal{E}_{\rm NL,2}\lesssim \mathcal{E}_{\rm NL,1}+F(\mathcal{E}_{\rm NL,2})\mathcal{E}_{\rm NL,2}^2. \tag{124}$$

7 Nonlinear *a priori* estimates and asymptotic stability

7.1 Energy estimates

We start with rewrite (53) as

$$h_{\rm s}\partial_{tt}\theta+\Big\{\frac{h_{\rm s}^2}{(1+\theta_\xi)^2}+\frac{(h_{\rm s}')^2}{(1+\theta_\xi)^4}-\frac{2h_{\rm s}h_{\rm s}''}{(1+\theta_\xi)^4}+\frac{5}{4}h_{\rm s}^2\Big[\Big(\frac{1}{(1+\theta_\xi)^2}\Big)_\xi\Big]^2 \\ -2h_{\rm s}\Big(\frac{1}{1+\theta_\xi}\Big)_t\Big\}_\xi-\Big\{\frac{h_{\rm s}^2}{2}\Big(\frac{1}{(1+\theta_\xi)^4}\Big)_\xi\Big\}_{\xi\xi}=0, \\ \text{with}\quad \theta_\xi|_{\xi=-1,1}=0. \tag{125}$$

Similar to (102), one has

$$\frac{1}{(1+\theta_\xi)^k}=1-k\theta_\xi+g_k(\theta_\xi),\qquad i=1,2,3,4, \tag{126}$$

where, for small θ_ξ,

$$g_k(\theta_\xi)=\mathcal{O}(\theta_\xi^2). \tag{127}$$

Then one can separate (125) into the linear and nonlinear parts, by writing

$$h_{\rm s}\partial_{tt}\theta-2\big\{(h_{\rm s}^2+2(h_{\rm s}')^2-4h_{\rm s}h_{\rm s}'')\theta_\xi+h_{\rm s}\theta_{\xi t}\big\}_\xi+2(h_{\rm s}^2\theta_{\xi\xi})_{\xi\xi} \\ +(M_1)_\xi+(M_2)_{\xi\xi}=0, \tag{128}$$

where

$$M_1:=h_{\rm s}^2g_2(\theta_\xi)+[(h_{\rm s}')^2-2h_{\rm s}h_{\rm s}'']g_4(\theta_\xi) \\ +\frac{5}{4}h_{\rm s}^2[(-2\theta_\xi+g_2(\theta_\xi))_\xi]^2+2h_{\rm s}(g_1(\theta_\xi))_t, \tag{129}$$

$$M_2:=-\frac{h_{\rm s}^2}{2}(g_4(\theta_\xi))_\xi, \tag{130}$$

or using (107)

$$M_1=h_{\rm s}^2\{f(\theta_\xi)\theta_\xi^2+f(\theta_\xi)\theta_\xi\theta_{\xi\xi}+f(\theta_\xi)\theta_{\xi\xi}^2\}+[(h_{\rm s}')^2-2h_{\rm s}h_{\rm s}'']f(\theta_\xi)\theta_\xi^2 \\ +h_{\rm s}f(\theta_\xi)\theta_\xi\theta_{\xi t}, \tag{131}$$

$$M_2=h_{\rm s}^2f(\theta_\xi)\theta_\xi\theta_{\xi\xi}. \tag{132}$$

Now we are ready to establish the estimates of $\mathcal{E}_{\rm NL,1}$. In particular, let

$$\mathcal{E}_k:=\frac{1}{2}\int h_{\rm s}|\partial_t^{k+1}\theta|^2\,{\rm d}\xi+\int(2|h_{\rm s}'|^2-4h_{\rm s}h_{\rm s}''+h_{\rm s}^2)|\partial_t^k\theta_\xi|^2\,{\rm d}\xi \\ +\int|h_{\rm s}|^2|\partial_t^k\theta_{\xi\xi}|^2\,{\rm d}\xi+\mathfrak{c}_1\int h_{\rm s}|\partial_t^k\theta_\xi|^2\,{\rm d}\xi+\mathfrak{c}_1\int h_{\rm s}\partial_t^{k+1}\theta\cdot\partial_t^k\theta\,{\rm d}\xi, \tag{133}$$

$$\mathscr{D}_k := 2\int h_\mathrm{s}|\partial_t^{k+1}\theta_\xi|^2 \, \mathrm{d}\xi - \mathfrak{c}_1 \int h_\mathrm{s}|\partial_t^{k+1}\theta|^2 \, \mathrm{d}\xi + 2\mathfrak{c}_1 \int h_\mathrm{s}^2 |\partial_t^k \theta_{\xi\xi}|^2 \, \mathrm{d}\xi \\ + 2\mathfrak{c}_1 \int \big(2|h_\mathrm{s}'|^2 - 4h_\mathrm{s}h_\mathrm{s}'' + h_\mathrm{s}^2\big)|\partial_t^k\theta_\xi|^2 \, \mathrm{d}\xi. \tag{134}$$

Then similar to (72) and (73)

$$\mathscr{D}_k \gtrsim \|h_\mathrm{s}^{1/2}\partial_t^{k+1}\theta_\xi\|_{L^2}^2 + \|h_\mathrm{s}\partial_t^k\theta_{\xi\xi}\|_{L^2}^2 + \|\partial_t^k\theta_\xi\|_{L^2}^2 \\ \gtrsim \mathscr{E}_k \gtrsim \|h_\mathrm{s}^{1/2}\partial_t^{k+1}\theta\|_{L^2}^2 + \|h_\mathrm{s}\partial_t^k\theta_{\xi\xi}\|_{L^2}^2 + \|\partial_t^k\theta_\xi\|_{L^2}^2, \tag{135}$$

and therefore

$$\sum_{k=0,1,2} \mathscr{E}_k \lesssim \mathscr{E}_{\mathrm{NL},1} \lesssim \sum_{k=0,1,2} \mathscr{E}_k. \tag{136}$$

We calculate the estimate of $\mathscr{E}_2$. The estimates of $\mathscr{E}_0$ and $\mathscr{E}_1$ follow with similar arguments.

After applying ∂_t^2 to (118) and repeating the same arguments as in (62)–(69), one can conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}_2 + \mathscr{D}_2 = \int (M_1)_{tt}\theta_{\xi ttt} \, \mathrm{d}\xi - \int (M_2)_{tt}\theta_{\xi\xi ttt} \, \mathrm{d}\xi \\ + \mathfrak{c}_1 \int (M_1)_{tt}\theta_{\xi tt} \, \mathrm{d}\xi - \mathfrak{c}_1 \int (M_2)_{tt}\theta_{\xi\xi tt} \, \mathrm{d}\xi. \tag{137}$$

Therefore, it suffices to estimate the right-hand side of (137).

Estimates of $\int (M_1)_{tt}\theta_{\xi ttt} \, \mathrm{d}\xi$ and $\int (M_1)_{tt}\theta_{\xi tt} \, \mathrm{d}\xi$

Applying Hölder's inequality yields

$$\Big|\int (M_1)_{tt}\theta_{\xi ttt} \, \mathrm{d}\xi\Big| + \Big|\int (M_1)_{tt}\theta_{\xi tt} \, \mathrm{d}\xi\Big| \\ \lesssim \Big\|\frac{(M_1)_{tt}}{h_\mathrm{s}^{1/2}}\Big\|_{L^2}\big(\|h_\mathrm{s}^{1/2}\theta_{\xi ttt}\|_{L^2} + \|h_\mathrm{s}^{1/2}\theta_{\xi tt}\|_{L^2}\big). \tag{138}$$

From (131), one can calculate

$$(M_1)_{tt} = h_\mathrm{s}^2\{f(\theta_\xi)\theta_\xi\theta_{\xi tt} + f(\theta_\xi)\theta_{\xi t}^2 + f(\theta_\xi)\theta_\xi\theta_{\xi\xi tt} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi t} \\ + f(\theta_\xi)\theta_{\xi tt}\theta_{\xi\xi} + f(\theta_\xi)\theta_{\xi t}^2\theta_{\xi\xi} + f(\theta_\xi)\theta_{\xi\xi}\theta_{\xi\xi tt} + f(\theta_\xi)\theta_{\xi\xi t}^2 \\ + f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi}\theta_{\xi\xi t} + f(\theta_\xi)\theta_{\xi tt}\theta_{\xi\xi}^2 + f(\theta_\xi)\theta_{\xi t}^2\theta_{\xi\xi}^2\} \\ + h_\mathrm{s}\{f(\theta_\xi)\theta_\xi\theta_{\xi ttt} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi tt} + f(\theta_\xi)\theta_{\xi t}^3\} \\ + [(h_\mathrm{s}')^2 - 2h_\mathrm{s}h_\mathrm{s}'']\{f(\theta_\xi)\theta_\xi\theta_{\xi tt} + f(\theta_\xi)\theta_{\xi t}^2\}. \tag{139}$$

Therefore, one can calculate

$$\Big\|\frac{(M_1)_{tt}}{h_\mathrm{s}^{1/2}}\Big\|_{L^2} \lesssim F(\|\theta_\xi\|_{L^\infty})\Big\{\|\theta_\xi\|_{L^\infty}\|h_\mathrm{s}^{3/2}\theta_{\xi tt}\|_{L^2} + \|\theta_{\xi t}\|_{L^\infty}\|h_\mathrm{s}^{3/2}\theta_{\xi t}\|_{L^2} \\ + \|\theta_\xi\|_{L^\infty}\|h_\mathrm{s}^{3/2}\theta_{\xi\xi tt}\|_{L^2} + \|\theta_{\xi t}\|_{L^\infty}\|h_\mathrm{s}^{3/2}\theta_{\xi\xi t}\|_{L^2} + \|h_\mathrm{s}^{1/2}\theta_{\xi\xi}\|_{L^\infty}\|h_\mathrm{s}\theta_{\xi tt}\|_{L^2} \\ + \|\theta_{\xi t}\|_{L^\infty}^2\|h_\mathrm{s}^{3/2}\theta_{\xi\xi}\|_{L^2} + \|h_\mathrm{s}^{1/2}\theta_{\xi\xi}\|_{L^\infty}\|h_\mathrm{s}\theta_{\xi\xi tt}\|_{L^2} + \|h_\mathrm{s}\theta_{\xi\xi t}\|_{L^\infty}\|h_\mathrm{s}^{1/2}\theta_{\xi\xi t}\|_{L^2} \\ + \|\theta_{\xi t}\|_{L^\infty}\|h_\mathrm{s}\theta_{\xi\xi}\|_{L^\infty}\|h_\mathrm{s}^{1/2}\theta_{\xi\xi t}\|_{L^2} + \|h_\mathrm{s}^{1/2}\theta_{\xi\xi}\|_{L^\infty}^2\|h_\mathrm{s}^{1/2}\theta_{\xi tt}\|_{L^2} \\ + \|\theta_{\xi t}\|_{L^\infty}^2\|h_\mathrm{s}\theta_{\xi\xi}\|_{L^\infty}\|h_\mathrm{s}^{1/2}\theta_{\xi\xi}\|_{L^2} \\ + \|\theta_\xi\|_{L^\infty}\|h_\mathrm{s}^{1/2}\theta_{\xi ttt}\|_{L^2} + \|\theta_{\xi t}\|_{L^\infty}\|h_\mathrm{s}^{1/2}\theta_{\xi tt}\|_{L^2} + \|\theta_{\xi t}\|_{L^\infty}^2\|h_\mathrm{s}^{1/2}\theta_{\xi t}\|_{L^2} \\ + \Big\|\frac{\theta_\xi}{h_\mathrm{s}}\Big\|_{L^4}\|h_\mathrm{s}^{1/2}\theta_{\xi tt}\|_{L^4} + \|\theta_{\xi t}\|_{L^\infty}\Big\|\frac{\theta_{\xi t}}{h_\mathrm{s}^{1/2}}\Big\|_{L^2}\Big\}. \tag{140}$$

Therefore, (138)–(140) implies

$$
\begin{gathered}
|\int (M_1)_{tt}\theta_{\xi ttt}\,\mathrm{d}\xi| + |\int (M_1)_{tt}\theta_{\xi tt}\,\mathrm{d}\xi| \\
\lesssim P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}) \sum_{k=0,1,2} \mathcal{D}_k,
\end{gathered}
\tag{141}
$$

thanks to (108)–(110). Hereafter P is a function of the argument such that

$$
P = P(y) \geq 0 \in C^\infty[0,\infty) \quad \text{and} \quad P(0) = 0. \tag{142}
$$

Estimates of $\int (M_2)_{tt}\theta_{\xi\xi ttt}\,\mathrm{d}\xi$ and $\int (M_2)_{tt}\theta_{\xi\xi tt}\,\mathrm{d}\xi$

From (130) and (132), one can calculate

$$
\begin{gathered}
(M_2)_{tt} = -\frac{h_{\mathrm{s}}^2}{2} g_4'(\theta_\xi)\theta_{\xi\xi tt} \\
\underbrace{-\frac{h_{\mathrm{s}}^2}{2}\{g_4''(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi t} + g_4''(\theta_\xi)\theta_{\xi tt}\theta_{\xi\xi} + g_4'''(\theta_\xi)\theta_{\xi t}^2\theta_{\xi\xi}\}}_{=:M_3}.
\end{gathered}
\tag{143}
$$

Therefore,

$$
\int (M_2)_{tt}\theta_{\xi\xi ttt}\,\mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\delta + \int h_{\mathrm{s}}^2\{\frac{1}{4} g_4''(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi tt}^2\}\,\mathrm{d}\xi + \int (M_3)_t\theta_{\xi\xi tt}\,\mathrm{d}\xi, \tag{144}
$$

where

$$
\mathcal{E}_\delta := -\frac{1}{4}\int h_{\mathrm{s}}^2 g_4'(\theta_\xi)|\theta_{\xi\xi tt}|^2\,\mathrm{d}\xi - \int M_3\theta_{\xi\xi tt}\,\mathrm{d}\xi. \tag{145}
$$

Direct calculation yields

$$
\begin{gathered}
\mathcal{E}_\delta \lesssim F(\|\theta_\xi\|_{L^\infty})\{\|\theta_\xi\|_{L^\infty}\|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2}^2 \\
+\|\theta_{\xi t}\|_{L^\infty}\|h_{\mathrm{s}}\theta_{\xi\xi t}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2} + \|\theta_{\xi tt}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}\|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2} \\
+\|\theta_{\xi t}\|_{L^\infty}^2\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2}\} \lesssim P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2})\mathcal{E}_{\mathrm{NL},1}.
\end{gathered}
\tag{146}
$$

On the other hand, one has that

$$
\begin{gathered}
(M_3)_t = h_{\mathrm{s}}^2\{f(\theta_\xi)\theta_{\xi t}\theta_{\xi t}\theta_{\xi\xi t} + f(\theta_\xi)\theta_{\xi tt}\theta_{\xi\xi t} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi tt} \\
+f(\theta_\xi)\theta_{\xi t}\theta_{\xi tt}\theta_{\xi\xi} + f(\theta_\xi)\theta_{\xi ttt}\theta_{\xi\xi} + f(\theta_\xi)\theta_{\xi t}\theta_{\xi t}\theta_{\xi t}\theta_{\xi\xi} \\
+f(\theta_\xi)\theta_{\xi t}\theta_{\xi tt}\theta_{\xi\xi}\}.
\end{gathered}
\tag{147}
$$

Therefore,

$$
\begin{gathered}
|\int h_{\mathrm{s}}^2\{\frac{1}{4} g_4''(\theta_\xi)\theta_{\xi t}\theta_{\xi\xi tt}^2\}\,\mathrm{d}\xi + \int (M_3)_t\theta_{\xi\xi tt}\,\mathrm{d}\xi| \\
\lesssim F(\|\theta_\xi\|_{L^\infty})\|\theta_{\xi t}\|_{L^\infty}\|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2}^2 \\
+F(\|\theta_\xi\|_{L^\infty})\{\|\theta_{\xi t}\|_{L^\infty}^2\|h_{\mathrm{s}}\theta_{\xi\xi t}\|_{L^2} + \|\theta_{\xi tt}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi t}\|_{L^\infty} \\
+\|\theta_{\xi t}\|_{L^\infty}\|\theta_{\xi tt}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty} + \|h_{\mathrm{s}}^{1/2}\theta_{\xi ttt}\|_{L^2}\|h_{\mathrm{s}}^{1/2}\theta_{\xi\xi}\|_{L^\infty} \\
+\|\theta_{\xi t}\|_{L^\infty}^3\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^2} + \|\theta_{\xi t}\|_{L^\infty}\|\theta_{\xi tt}\|_{L^2}\|h_{\mathrm{s}}\theta_{\xi\xi}\|_{L^\infty}\} \times \|h_{\mathrm{s}}\theta_{\xi\xi tt}\|_{L^2} \\
\lesssim P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}) \sum_{k=0,1,2} \mathcal{D}_k
\end{gathered}
\tag{148}
$$

Following similar arguments implies that

$$
|\int (M_2)_{tt}\theta_{\xi\xi tt}\,\mathrm{d}\xi| \lesssim P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}) \sum_{k=0,1,2} \mathcal{D}_k. \tag{149}
$$

Summary

Collecting (137), (141), (144)–(149) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}_2 + \mathcal{E}_\delta) + \mathcal{D}_2 \leq P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}) \sum_{k=0,1,2} \mathcal{D}_k. \tag{150}$$

Repeating the same arguments for $\mathcal{E}_0$ and $\mathcal{E}_1$, one can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{k=0,1,2} \mathcal{E}_k + \mathcal{E}_\delta) + (1 - P(\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2})) \sum_{k=0,1,2} \mathcal{D}_k \leq 0. \tag{151}$$

7.2 Continuity argument and proof of Theorem 2

Now we are at the right place to demonstrate the continuity arguments, which lead to the global stability and asymptotic stability theory. We first start with the *a priori* assumption; that is, for some $\epsilon \in (0,1)$, such that for $\forall T \in (0,\infty)$,

$$\sup_{0 \leq t \leq T} \{\mathcal{E}_{\mathrm{NL},1}(t) + \mathcal{E}_{\mathrm{NL},2}(t)\} \leq \varepsilon. \tag{152}$$

Then, for ε small enough, (124) and (151), together with (135), (136), and (146), imply that

$$\sum_{k=0,1,2} \mathcal{E}_k + \mathcal{E}_\delta \geq \mathfrak{d}_1 \sum_{k=0,1,2} \mathcal{E}_k, \qquad \mathcal{E}_{\mathrm{NL},2} \leq \mathfrak{d}_2 \mathcal{E}_{\mathrm{NL},1}, \tag{153}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{k=0,1,2} \mathcal{E}_k + \mathcal{E}_\delta) + \mathfrak{d}_3(\sum_{k=0,1,2} \mathcal{E}_k + \mathcal{E}_\delta) \leq 0 \tag{154}$$

for some constants $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3 \in (0,\infty)$, and therefore

$$\sup_{0 \leq t \leq T} e^{\mathfrak{d}_3 t}(\sum_{k=0,1,2} \mathcal{E}_k(t) + \mathcal{E}_\delta(t)) \leq (\sum_{k=0,1,2} \mathcal{E}_k(0) + \mathcal{E}_\delta(0)) =: \mathfrak{E}_0. \tag{155}$$

Here $\mathfrak{E}_0$ is the total initial energy, i.e.,

$$\begin{aligned} \mathfrak{E}_0 := \sum_{k=0,1,2} \Big\{ \frac{1}{2} \int h_{\mathrm{s}} |\partial_t^{k+1} \theta_{\mathrm{in}}|^2 \,\mathrm{d}\xi + \int (2|h_{\mathrm{s}}'|^2 - 4h_{\mathrm{s}} h_{\mathrm{s}}'' + h_{\mathrm{s}}^2) |\partial_t^k \theta_{\mathrm{in},\xi}|^2 \,\mathrm{d}\xi \\ + \int |h_{\mathrm{s}}|^2 |\partial_t^k \theta_{\mathrm{in},\xi\xi}|^2 \,\mathrm{d}\xi + \mathfrak{c}_1 \int h_{\mathrm{s}} |\partial_t^k \theta_{\mathrm{in},\xi}|^2 \,\mathrm{d}\xi \\ + \mathfrak{c}_1 \int h_{\mathrm{s}} \partial_t^{k+1} \theta_{\mathrm{in}} \cdot \partial_t^k \theta_{\mathrm{in}} \,\mathrm{d}\xi \Big\} + \mathcal{E}_\delta(0). \end{aligned} \tag{156}$$

where $\mathcal{E}_\delta(0) = \mathcal{E}_\delta|_{t=0}$ is the error term due to the nonlinearity, defined in (145). Here θ_{in} and $\partial_t \theta_{\mathrm{in}}$ are the initial data for equation (51'). The higher-order derivatives in time are defined inductively using the equation.

On the other hand, estimates (136), (153), and (155) imply

$$\sup_{0 \leq t \leq T} e^{\mathfrak{d}_3 t} \{\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}\} \leq \mathfrak{d}_4 \mathfrak{E}_0, \tag{157}$$

for some constant $\mathfrak{d}_4 \in (0,\infty)$. Therefore, for small enough initial data

$$\mathfrak{E}_0 \leq \frac{\varepsilon}{2\mathfrak{d}_4}, \tag{158}$$

estimate (157) implies

$$\sup_{0\le t\le T} e^{\mathfrak{d}_3 t}\{\mathcal{E}_{\mathrm{NL},1}+\mathcal{E}_{\mathrm{NL},2}\}\le \frac{\varepsilon}{2}, \tag{159}$$

and consequently, this closes the *a priori* assumption (152).

With the above *a priori* estimates with initial data satisfying (158), one can apply standard Galerkin's method to construct a local-in-time solution. Furthermore, one can apply the standard continuity argument and conclude the proof of theorem 2.

8 Local well-posedness for general initial data

8.1 Lagrangian formulation and main theory: Local well-posedness

The goal of this section is to investigate the local-in-time well-posedness theory of system (39) with general initial data; that is, with initial data not close to the equilibrium given by (40). Indeed, we only assume h_0 satisfies some regularity and convexity condition, see (164), below. To formulate the shallow water equations in the Lagrangian coordinates with reference to the initial height profile, define $x=\eta(\xi,t)$ by

$$\int_{a(t)}^{\eta(\xi,t)} h(x,t)\,\mathrm{d}x=\int_{-1}^{\xi} h_0(x)\,\mathrm{d}x. \tag{160}$$

Then repeating the derivation from (47) to (51), one can write down the shallow water equations in the (ξ,t)-coordinates as follows:

$$h(\xi,t)=\frac{h_0(\xi)}{\eta_\xi(\xi,t)},\quad \partial_t\eta(\xi,t)=u(\xi,t), \tag{161a}$$

$$h_0\partial_t u+\partial_\xi\left(\left(\frac{h_0}{\eta_\xi}\right)^2\right)-2h_0\frac{\partial_\xi}{\eta_\xi}\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{h_0}{\eta_\xi}\right)\right)\right)=2\partial_\xi\left(\frac{h_0}{\eta_\xi}\frac{u_\xi}{\eta_\xi}\right), \tag{161b}$$

$$\partial_\xi\eta|_{\xi=-1,1}=1,\quad \partial_\xi u|_{\xi=-1,1}=0. \tag{161c}$$

Here we have abused the notation and used $u=u(\xi,t)=u(\eta(\xi,t),t)$ to denote the velocity in both the Lagrangian and the Euclidean coordinates. Moreover, to be consistent with our estimates, inspired by [9], we rewrite the surface tension term as follows

$$\begin{aligned}-2h_0\frac{\partial_\xi}{\eta_\xi}\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{h_0}{\eta_\xi}\right)\right)\right)&=-\partial_\xi\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{h_0^2}{\eta_\xi^2}\right)\right)-3\left(\frac{\partial_\xi}{\eta_\xi}\left(\frac{h_0}{\eta_\xi}\right)\right)^2\right)\\&=\partial_{\xi\xi}\left(\frac{2h_0^2\eta_{\xi\xi}}{\eta_\xi^5}\right)+\partial_\xi\left(\frac{5h_0^2\eta_{\xi\xi}^2}{\eta_\xi^6}\right)+\partial_\xi\left(\frac{|\partial_\xi h_0|^2-2h_0\partial_\xi^2h_0}{\eta_\xi^4}\right).\end{aligned} \tag{161d}$$

Therefore, (161b) can be written as

$$\begin{aligned}h_0\partial_t u+\left\{\frac{h_0^2}{\eta_\xi^2}+\frac{|h_0'|^2-2h_0h_0''}{\eta_\xi^4}+\frac{5h_0^2\eta_{\xi\xi}^2}{\eta_\xi^6}\right\}_\xi+\left\{\frac{2h_0^2\eta_{\xi\xi}}{\eta_\xi^5}\right\}_{\xi\xi}&\\-2\partial_\xi\left(\frac{h_0}{\eta_\xi}\frac{u_\xi}{\eta_\xi}\right)=0.&\end{aligned} \tag{161b'}$$

Notice that (161b') is consistent with (125). Notice that, since h_0 is no longer the equilibrium profile, compared to (104), the linear part of (161b') has an extra term, i.e.,

$$(h_0^2+|h_0'|^2-2h_0h_0'')'=2h_0(h_0'-h_0''')\ne 0. \tag{162}$$

Fortunately, this term is $\mathcal{O}(h_0)$ near the boundary, and therefore the weighted estimates involving negative power of h_0 in section 5.2 (i.e., a weight of h_0^{-1} or $h_0^{-1/2}$) are bounded. Therefore one can expect the elliptic estimates of (161b') to be similar to those of (104) as in section 6. For the energy estimate, instead of using the smallness of perturbation, one should use the smallness of time to control the nonlinearities. In particular, we prove the following local well-posedness result:

Theorem 3 (Local Well-posedness)**.** *Suppose the initial data (u_0, h_0) satisfies*

$$u_0 \in H^3(I), \;\; \partial_\xi u_0 \in H_0^1(I), \tag{163}$$

$$h_0 \in H^4(I), \;\; \partial_{\xi\xi} h_0 \leq 0, \;\; C_1 d(\xi) \leq h_0 \leq C_2 d(\xi) \;\; \forall \xi \in \bar{I}, \tag{164}$$

for some positive constants C_1 and C_2, where $d(\xi) = d(\xi, \partial I)$ is the distant function from ξ to the initial boundary. $\Phi(t)$ defined in (167) *below is a energy functional, suppose $\Phi(0) \leq \infty$. Then there exist a small time $T^* > 0$ such that system* (51) *admits a unique strong solution (η, u) in $I \times [0, T]$, with*

$$\begin{cases} \eta, \eta_\xi \in C^1([0,T^*]; L^2(I)), \;\; \eta_{\xi\xi}, h_0 \partial_\xi^3 \eta \in C([0,T^*]; L^2(I)), \\ h_0^{3/2} \partial_\xi^4 \eta \in L^\infty([0,T^*]; L^2(I)), \;\; u, u_\xi, h_0^{1/2} u_t \in C([0,T^*]; L^2(I)), \\ u_{\xi t}, u_{\xi\xi}, h_0^{1/2} u_{tt}, h_0 \partial_\xi^3 u \in L^\infty([0,T^*]; L^2(I)), \end{cases} \tag{165}$$

and

$$\sup_{0 \leq t \leq T^*} \Phi(t) \leq C, \tag{166}$$

where C depends on $\|h_0\|_{H^4}$, $\|u_0\|_{H^3}$ and T^*.

For the completeness of this paper, we will sketch the local-in-time estimates, which lead to the local-in-time well-posedness theory, in the following.

The energy functional for the solution to (161b') is defined as

$$\begin{aligned} \Phi(t) = \Phi(u, \eta, t) := & \sum_{k=0}^{2} \|h_0^{1/2} \partial_t^k u(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^{2} \|h_0 \partial_t^k \partial_\xi^2 \eta(\cdot, t)\|_{L^2}^2 \\ & + \sum_{k=0}^{2} \|h_0 \partial_t^k \partial_\xi \eta(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^{2} \|(|\partial_\xi h_0|^2 - 2h_0 \partial_\xi^2 h_0)^{1/2} \partial_t^k \partial_\xi \eta(\cdot, t)\|_{L^2}^2 \\ & + \sum_{k=0}^{1} \|h_0 \partial_t^k \partial_\xi^3 \eta(\cdot, t)\|_{L^2}^2 + \|h_0^{3/2} \partial_\xi^4 \eta(\cdot, t)\|_{L^2}^2. \end{aligned} \tag{167}$$

Notice that Φ is the analogy of $\mathcal{E}_{\mathrm{NL},1} + \mathcal{E}_{\mathrm{NL},2}$ defined in (100) and (101). Then similar to Lemmas 3 and 4, one has the following embedding inequalities:

Lemma 5. *The following inequalities hold:*

$$\begin{aligned} \|u_\xi\|_{L^\infty} + \|\frac{u_\xi}{h_0^{1/2}}\|_{L^2} + \|\frac{u_\xi}{h_0}\|_{L^2} \lesssim \|u_{\xi\xi}\|_{L^2} \lesssim \|h_0 \partial_\xi^3 u\|_{L^2} + \|h_0 u_{\xi\xi}\|_{L^2} \\ \lesssim \Phi(t)^{1/2}, \\ \|u_{\xi t}\|_{L^2} \lesssim \|h_0 u_{\xi t}\|_{L^2} + \|h_0 \partial_\xi^2 u_t\|_{L^2} \lesssim \Phi(t)^{1/2}, \\ \|h_0^{1/2} u_{\xi t}\|_{L^4} \lesssim \|h_0 u_{\xi\xi t}\|_{L^2} + \|u_{\xi t}\|_{L^2} \lesssim \Phi(t)^{1/2}, \\ \|h_0 u_{\xi\xi}\|_{L^\infty} \lesssim \|h_0 \partial_\xi^3 u\|_{L^2} + \|u_{\xi\xi}\|_{L^2} \lesssim \Phi(t)^{1/2}, \end{aligned} \tag{168}$$

similarly,

$$\begin{aligned}\|\eta_{\xi\xi}\|_{L^2} &\lesssim \|h_0\eta_{\xi\xi}\|_{L^2} + \|h_0\partial_\xi^3\eta\|_{L^2}\Phi(t)^{1/2},\\ \|h_0^{1/2}\partial_\xi^3\eta\|_{L^2} &\lesssim \|h_0^{3/2}\partial_\xi^3\eta\|_{L^2} + \|h_0^{3/2}\partial_\xi^4\eta\|_{L^2} \lesssim \Phi(t)^{1/2},\\ \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty} &\lesssim \|h_0^{1/2}\partial_\xi^3\eta\|_{L^2} + \|\eta_{\xi\xi}\|_{L^2} \lesssim \Phi(t)^{1/2},\\ \|\eta_{\xi\xi}\|_{L^4} &\lesssim \|h_0^{3/2}\partial_\xi^4\eta\|_{L^2} + \|\eta_{\xi\xi}\|_{L^2} \lesssim \Phi(t)^{1/2},\\ \|h_0\partial_\xi^3\eta\|_{L^4} &\lesssim \|h_0^{3/2}\partial_\xi^4\eta\|_{L^2} + \|h_0^{1/2}\partial_\xi^3\eta\|_{L^2} \lesssim \Phi(t)^{1/2}.\end{aligned} \tag{169}$$

Moreover,

Lemma 6. *If $\eta(\xi,t)$ and ξ satisfy* (160) *, then it holds that*

$$\|\eta_{\xi\xi}\|_{L^2} + \|h_0\partial_\xi^3\eta\|_{L^2} \lesssim t \sup_{0\le s\le t}\Phi(t)^{1/2}. \tag{170}$$

Proof. It follows from (160) that $\eta(\xi,0)=\xi$, thus

$$\eta_{\xi\xi}(\xi,0) = \partial_\xi^3\eta(\xi,0) = 0. \tag{171}$$

Direct calculation together with Minkowski's inequality yields

$$\|\eta_{\xi\xi}(\cdot,t)\|_{L^2} \le \int_0^t \|u_{\xi\xi}(\cdot,s)\|_{L^2}ds \le t\sup_{0\le s\le t}\Phi(s)^{1/2}, \tag{172}$$

$$\|h_0\partial_\xi^3\eta(\cdot,t)\|_{L^2} \le \int_0^t \|h_0\partial_\xi^3 u(\cdot,s)\|_{L^2}ds \le t\sup_{0\le s\le t}\Phi(s)^{1/2}. \tag{173}$$

□

8.2 *A priori* estimate

The main aim of this section is to derive the key *a priori* bound, i.e., there exists $T\in(0,\infty)$ such that

$$\sup_{0\le t\le T}\Phi(t) \le 2M, \tag{174}$$

where $M := P(\Phi(0))$ for some polynomial P, to be determined later.

8.2.1 The *a priori* assumption

Assume that there exists a suitably small $T\in(0,1)$, to be determined, such that

$$\sup_{0\le t\le T}\Phi(t) \le M, \tag{175}$$

for some $M\in(0,\infty)$. It follows from (161a) that

$$\eta(\xi,t) = \xi + \int_0^t u(\xi,s)\,\mathrm{d}s, \quad (\xi,t)\in(I\times[0,T]), \tag{176}$$

which leads to, for $t\in(0,T)$, thanks to (168), that

$$|\eta_\xi(\xi,t)-1| \le \int_0^t \|u_\xi(\cdot,s)\|_{L^\infty}ds \le C_1M^{1/2}T \le \frac{1}{2}, \tag{177}$$

provided that T is small enough. Therefore, without loss of generality, we assume that

$$\frac{1}{2} \leq \eta_\xi(\xi,t) \leq \frac{3}{2}, \ \ (\xi,t) \in (I \times [0,T]), \tag{178}$$

for the remaining part of this section.

To simplify the notation, we use $P = P(\cdot)$ to represent a generic polynomial, which will be determined in the end.

8.2.2 Temporal derivative estimates

Basic energy estimate. Thanks to the boundary condition (161c), taking the L^2-inner product of (161b') with u yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int\Big\{\frac{1}{2}h_0u^2 + \frac{h_0^2}{\eta_\xi} + \frac{(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)}{3\eta_\xi^3} + \frac{h_0^2\eta_{\xi\xi}^2}{\eta_\xi^5}\Big\}d\xi \\ +2\int\frac{h_0u_\xi^2}{\eta_\xi^2}d\xi = -\int\partial_\xi\Big(\frac{u|\partial_\xi h_0|^2}{\eta_\xi^4}\Big)d\xi, \end{aligned} \tag{179}$$

where

$$\begin{aligned} -\int\partial_\xi\Big(\frac{u|\partial_\xi h_0|^2}{\eta_\xi^4}\Big)d\xi &\lesssim \|h_0\|_{H^3}^2(\|u\|_{H^1}^2 + \|\eta_{\xi\xi}\|_{L^2}^2 + 1) \\ &\lesssim \Phi(t) + 1. \end{aligned} \tag{180}$$

Therefore, integrating (179) from 0 to t, together with (178) and (180), yields

$$\begin{aligned} \|h_0^{1/2}u\|_{L^2}^2 + \|h_0\eta_\xi\|_{L^2}^2 + \|(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)^{1/2}\eta_\xi\|_{L^2}^2 + \|h_0\eta_{\xi\xi}\|_{L^2}^2 \\ \lesssim \Phi(0) + t(\sup_{0\leq s\leq t}\Phi(s) + 1), \end{aligned} \tag{181}$$

for any $t \in [0,T]$.

Estimate of $\partial_t^2 u$. After applying ∂_t^2 to (161b'), taking the L^2-inner product of the resultant with $\partial_t^2 u$ yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int\Big\{\frac{1}{2}h_0|\partial_t^2 u|^2 + \frac{h_0^2u_{\xi t}^2}{\eta_\xi^3} + \frac{2(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)u_{\xi t}^2}{\eta_\xi^5} + \frac{h_0^2|\partial_\xi^2 u_t|^2}{\eta_\xi^5}\Big\}\,\mathrm{d}\xi \\ &\qquad +2\int\frac{h_0|\partial_t^2 u_\xi|^2}{\eta_\xi^2}d\xi \\ &= \int\Big\{-\frac{3h_0^2u_{\xi t}^2u_\xi}{\eta_\xi^4} - \frac{10(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)u_{\xi t}^2u_\xi}{\eta_\xi^6} - \frac{5h_0^2|\partial_\xi^2 u_t|^2u_\xi}{\eta_\xi^6} \\ &\qquad + \frac{6h_0^2u_\xi^2\partial_t^2 u_\xi}{\eta_\xi^4} + \frac{20(|h_0'|^2 - 2h_0h_0'')u_\xi^2\partial_t^2 u_\xi}{\eta_\xi^6} \\ &\qquad +5\partial_{tt}\Big(\frac{h_0^2\eta_{\xi\xi}^2}{\eta_\xi^6}\Big)\partial_t^2 u_\xi d\xi - \frac{12h_0u_\xi u_{\xi t}\partial_t^2 u_\xi}{\eta_\xi^3} + \frac{12h_0u_\xi^3\partial_t^2 u_\xi}{\eta_\xi^4} \\ &\quad -[\frac{20h_0^2u_{\xi\xi}u_\xi}{\eta_\xi^6} + \frac{10h_0^2\eta_{\xi\xi}\partial_t u_\xi}{\eta_\xi^6} - \frac{60h_0^2\eta_{\xi\xi}u_\xi^2}{\eta_\xi^7}]_\xi\partial_t^2 u_\xi\Big\}\,\mathrm{d}\xi =: \sum_{i=1}^{9}\int I_i\,\mathrm{d}\xi. \end{aligned} \tag{182}$$

Thanks to (178), with the help of (168), (169) and Young's inequality, one can calculate that, for any $\varepsilon \in (0,1)$ and $C_\varepsilon \simeq \frac{1}{\varepsilon}$,

$$\sum_{i=1}^{3}\int I_i\,\mathrm{d}\xi \leq C\|u_\xi\|_{L^\infty}\Phi(t) \leq CP(\Phi(t)), \tag{183}$$

$$\begin{aligned}\int I_4 \ \mathrm{d}\xi \leq& \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon\|u_\xi\|_{L^\infty}^2\|u_\xi\|_{L^2}^2 \\ &\leq \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon P(\Phi(t)),\end{aligned} \tag{184}$$

$$\begin{aligned}\int I_5 \ \mathrm{d}\xi \leq& \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon\|u_\xi\|_{L^\infty}^2\|\frac{u_\xi}{h_0^{1/2}}\|_{L^2}^2 \\ &\leq \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon P(\Phi(t)),\end{aligned} \tag{185}$$

$$\begin{aligned}\int I_6 \ \mathrm{d}\xi \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 \\ &+ C_\varepsilon(1+\|u_\xi\|_{L^\infty}^8 + \|h_0 u_{\xi\xi}\|_{L^\infty}^8 + \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^8) \\ &\times(\|\eta_{\xi\xi}\|_{L^2}^2 + \|h_0\partial_\xi^2 u_t\|_{L^2}^2 + \|u_{\xi\xi}\|_{L^2}^2 + \|u_{\xi t}\|_{L^2}^2) \\ \leq& \ \varepsilon\|h_0^{1/2}u_{\xi t}\|_{L^2}^2 + C_\varepsilon P\big(\Phi(t)\big),\end{aligned} \tag{186}$$

$$\begin{aligned}\int I_7 \ \mathrm{d}\xi \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon\|u_\xi\|_{L^\infty}^2\|u_{\xi t}\|_{L^2}^2 \\ \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon P(\Phi(t)),\end{aligned} \tag{187}$$

$$\begin{aligned}\int I_8 \ \mathrm{d}\xi \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon\|u_\xi\|_{L^\infty}^4\|u_\xi\|_{L^2}^2 \\ \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 + C_\varepsilon P(\Phi(t)),\end{aligned} \tag{188}$$

$$\begin{aligned}\int I_9 \ \mathrm{d}\xi \leq& \ \varepsilon\|h_0^{1/2}\partial_t^2 u_\xi\|_{L^2}^2 \\ &+ C_\varepsilon(1+\|u_\xi\|_{L^\infty}^8 + \|h_0 u_{\xi\xi}\|_{L^\infty}^8 + \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^8) \\ &\times(\|h_0\partial_\xi^3 u\|_{L^2}^2 + \|\eta_{\xi\xi}\|_{L^2}^2 + \|h_0\partial_\xi^3\eta\|_{L^2}^2 + \|h_0\partial_\xi^3\eta\|_{L^4}^4 \\ &+ \|h_0^{1/2}u_{\xi t}\|_{L^4}^4 + \|u_{\xi\xi}\|_{L^2}^2 + \|h_0\partial_\xi^2 u_t\|_{L^2}^2 + \|u_{\xi t}\|_{L^2}^2) \\ \leq& \ \varepsilon\|h_0^{1/2}u_{\xi t}\|_{L^2}^2 + C_\varepsilon P\big(\Phi(t)\big).\end{aligned} \tag{189}$$

Thanks to (183)–(189), after choosing ε small enough, integrating (182) in t yields that, for any $t \in [0,T]$,

$$\begin{aligned}\|h_0^{1/2}\partial_t^2 u\|_{L^2}^2 + \|h_0 u_{\xi t}\|_{L^2}^2 + \|(|h_0'|^2 - 2h_0 h_0'')^{1/2}u_{\xi t}\|_{L^2}^2 + \|h_0\partial_\xi^2 u_t\|_{L^2}^2 \\ \lesssim \Phi(0) + tP\big(\sup_{0\leq s\leq t}\Phi(s)\big).\end{aligned} \tag{190}$$

Estimate of $\partial_t u$. Since

$$\partial_t u(\xi,t) = \partial_t u(\xi,0) + \int_0^t \partial_t^2 u(\xi,s)ds, \tag{191}$$

it then follows from Cauchy's inequality and Fubini's theorem that, for any $t \in [0,T)$,

$$\|h_0^{1/2}u_t\|_{L^2}^2 \lesssim \Phi(0) + t\int_0^t \|h_0^{1/2}u_{tt}(s)\|_{L^2}^2 ds \lesssim \Phi(0) + tP\big(\sup_{0\leq s\leq t}\Phi(s)\big). \tag{192}$$

Similar arguments also imply that, for any $t \in [0,T]$,

$$\begin{aligned}\|h_0 u_\xi\|_{L^2}^2 + \|(|h_0'|^2 - 2h_0 h_0'')^{1/2}u_\xi\|_{L^2}^2 + \|h_0\partial_\xi^2 u\|_{L^2}^2 \\ \lesssim \Phi(0) + tP\big(\sup_{0\leq s\leq t}\Phi(s)\big).\end{aligned} \tag{193}$$

Summary of temporal derivative estimates. Combing (181), (190), (192), and (193) leads to

$$\begin{aligned} & \sum_{k=0}^{2} \|h_0^{1/2}\partial_t^k u(\cdot,t)\|_{L^2}^2 + \sum_{k=0}^{2} \|h_0\partial_t^k\partial_\xi^2\eta(\cdot,t)\|_{L^2}^2 \\ & + \sum_{k=0}^{2} \|h_0\partial_t^k\partial_\xi\eta(\cdot,t)\|_{L^2}^2 + \sum_{k=0}^{2} \|(|h_0'|^2 - 2h_0h_0'')^{1/2}\partial_t^k\partial_\xi\eta(\cdot,t)\|_{L^2}^2 \\ & \lesssim \Phi(0) + tP\big(\sup_{0\le s\le t}\Phi(s)\big), \end{aligned} \tag{194}$$

for any $t \in [0,T]$. Moreover, Hardy's inequality (43) also gives

$$\|\eta_\xi\|_{L^2}^2 + \|u_\xi\|_{L^2}^2 + \|u_{\xi t}\|_{L^2}^2 \lesssim \Phi(0) + tP\big(\sup_{0\le s\le t}\Phi(s)\big). \tag{195}$$

8.2.3 Elliptic estimates

We now turn to the elliptic estimates. First, notice that, integrating (161b') from -1 to 1 yields

$$\int_{-1}^{1} h_0\partial_t u \,\mathrm{d}\xi + (|h_0'|^2)\big|_{\xi=-1}^{1} = 0, \tag{196}$$

and therefore

$$\int_{-1}^{1} h_0\partial_t^2 u d\xi = 0, \tag{197}$$

thanks to (164) and (161c).

Estimate of $\partial_\xi^3 u$. After applying ∂_t to (161b'), one has

$$\begin{aligned} & h_0\partial_t^2 u - 2\partial_\xi\big(h_0^2 u_\xi \eta_\xi^{-3}\big) - 4\partial_\xi\big((|h_0'|^2 - 2h_0h_0'')u_\xi\eta_\xi^{-5}\big) \\ & -2\partial_\xi\big(h_0 u_{\xi t}\eta_\xi^{-2}\big) + 4\partial_\xi\big(h_0 u_\xi^2\eta_\xi^{-3}\big) \\ & +2\partial_\xi^2\big(h_0^2 u_{\xi\xi}\eta_\xi^{-5}\big) - 10\partial_\xi^2\big(h_0^2\eta_{\xi\xi}u_\xi\eta_\xi^{-6}\big) + 10\partial_\xi\big(h_0^2 u_{\xi\xi}\eta_{\xi\xi}\eta_\xi^{-6}\big) \\ & -30\partial_\xi\big(h_0^2\eta_{\xi\xi}^2 u_\xi\eta_\xi^{-7}\big) = 0. \end{aligned} \tag{198}$$

Integrating (198) from $y=-1$ to ξ and multiplying the resulting equation by $\eta_\xi^5\partial_\xi^3 u$ yields, after a complex but straightforward calculation, that

$$\begin{aligned} & L := 2h_0^2|\partial_\xi^3 u|^2 + 4h_0\partial_\xi h_0 u_{\xi\xi}\partial_\xi^3 u - 4(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)u_\xi\partial_\xi^3 u \\ & = -\eta_\xi^5\partial_\xi^3 u\int_{-1}^{\xi} h_0\partial_t^2 u(y)dy + 2h_0^2 u_\xi\eta_\xi^2\partial_\xi^3 u + 2h_0 u_{\xi t}\eta_\xi^3\partial_\xi^3 u - 4h_0 u_\xi^2\eta_\xi^2\partial_\xi^3 u \\ & +10h_0^2\partial_\xi^3 u u_{\xi\xi}\eta_{\xi\xi}\eta_\xi^{-1} - 30h_0^2\partial_\xi^3 u\eta_{\xi\xi}^2 u_\xi\eta_\xi^{-2} + 10\partial_\xi^3 u\partial_\xi\big(h_0^2\eta_{\xi\xi}\big)u_\xi\eta_\xi^{-1} =: R. \end{aligned} \tag{199}$$

Since h_0 is concave after integration by parts one gets

$$\begin{aligned} \int L(\xi,t)\,\mathrm{d}\xi = & 2\|h_0\partial_\xi^3 u\|_{L^2}^2 + \int 2h_0\partial_\xi h_0\partial_\xi(u_{\xi\xi}^2)d\xi \\ & + \int 4(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)u_{\xi\xi}^2 d\xi - \int 4h_0\partial_\xi^3 h_0\partial_\xi(u_\xi^2)d\xi \\ = & 2\|h_0\partial_\xi^3 u\|_{L^2}^2 + 2\|\partial_\xi h_0 u_{\xi\xi}\|_{L^2}^2 \\ & - \int 10h_0\partial_\xi^2 h_0 u_{\xi\xi}^2 d\xi + \int \partial_\xi(h_0\partial_\xi^3 h_0)u_\xi^2 d\xi \\ \ge & 2\|h_0\partial_\xi^3 u\|_{L^2}^2 + 2\|\partial_\xi h_0 u_{\xi\xi}\|_{L^2}^2 \\ & - \int 10h_0\partial_\xi^2 h_0 u_{\xi\xi}^2 d\xi - C\|h_0\|_{H^4}^2\|u_\xi\|_{L^2}^2. \end{aligned} \tag{200}$$

and

$$\|h_0\|_{H^4}^2\|u_\xi\|_{L^2}^2 \lesssim \Phi(0) + tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big), \tag{201}$$

due to (195).

Next, to estimate $\int R(\xi,t)\ \mathrm{d}\xi$, thanks to (178), (168), (169), (170), (195) and (161b), (81) we have

$$\begin{aligned}\|h_0^{-1}\eta_x^5\int_{-1}^{\xi}h_0\partial_t^2u(y)dy\|_{L^2}^2 \lesssim&\|\eta_x\|_{L^\infty}^5\|h_0^{1/2}\partial_t^2u\|_{L^2}^2\\ \lesssim&P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big),\end{aligned} \tag{202}$$

$$\begin{aligned}\|h_0u_\xi\eta_\xi^2+u_{\xi t}\eta_\xi^3\|_{L^2}^2 \lesssim&\|h_0u_\xi\|_{L^2}^2\|\eta_\xi\|_{L^\infty}^4+\|u_{\xi t}\|_{L^2}^2\|\eta_\xi\|_{L^\infty}^6\\ \lesssim&P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big),\end{aligned} \tag{203}$$

$$\begin{aligned}\|u_\xi^2\eta_\xi^2\|_{L^2}^2 \lesssim&\|\eta_\xi\|_{L^\infty}^4\|u_\xi\|_{L^2}^2\|u_\xi\|_{L^\infty}^2 \lesssim \|u_\xi\|_{L^2}^3\|u_{\xi\xi}\|_{L^2}\\ \lesssim&\|u_\xi\|_{L^2}^3(\|h_0u_{\xi\xi}\|_{L^2}+\|h_0\partial_\xi^3u\|_{L^2})\\ \le&\varepsilon\|h_0\partial_\xi^3u\|_{L^2}^2+P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big),\end{aligned} \tag{204}$$

where we have used the simple fact that

$$u_\xi^2(\xi)=2\int_{-1}^{\xi}u_\xi(y)u_{\xi\xi}(y)dy \lesssim \|u_\xi\|_{L^2}\|u_{\xi\xi}\|_{L^2}, \tag{205}$$

due to boundary condition (161c).

$$\begin{aligned}\|h_0u_{\xi\xi}\eta_{\xi\xi}\eta_\xi^{-1}+h_0\eta_{\xi\xi}^2u_\xi\eta_\xi^{-2}\|_{L^2}^2 \lesssim&\|h_0(u_{\xi\xi}+\eta_{\xi\xi})\|_{L^\infty}^2\|\eta_{\xi\xi}\|_{L^2}^2\\ \lesssim&tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big),\end{aligned} \tag{206}$$

$$\begin{aligned}\|h_0^{-1}\partial_\xi\big(h_0^2\eta_{\xi\xi}\big)u_\xi\eta_\xi^{-1}\|_{L^2}^2 \lesssim& (1+\|h_0\|_{H^2}^2)\|u_\xi\|_{L^\infty}^2(\|\eta_{\xi\xi}\|_{L^2}^2+\|h_0\partial_\xi^3\eta\|_{L^2}^2)\\ \lesssim& tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big).\end{aligned} \tag{207}$$

Then choosing ε small enough, it follows from (201)-(204), (206), (207) and Young's inequality that

$$\int R\ \mathrm{d}\xi \lesssim \|h_0\partial_\xi^3u\|_{L^2}^2+P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big). \tag{208}$$

Therefore (208) together with (200) and (201) yields

$$\|h_0\partial_\xi^3u\|_{L^2}^2 \lesssim P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big), \tag{209}$$

which together with (193) and Hardy's inequality (43) gives

$$\|u_{\xi\xi}\|_{L^2}^2 \lesssim \|h_0u_{\xi\xi}\|_{L^2}^2+\|h_0\partial_\xi^3u\|_{L^2}^2 \lesssim P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big), \tag{210}$$

and due to (161c) and (44) one gets

$$\|h_0^{-1}u_\xi\|_{L^2}^2 \lesssim \|u_{\xi\xi}\|_{L^2}^2 \lesssim P\big(\Phi(0)\big)+tP\big(\sup_{0\le s\le t}\Phi(t)^{1/2}\big). \tag{211}$$

Estimate of $\partial_\xi^4\eta$. Following the estimates in section 6.2, after multiplying (161b') with $h_0^{-1/2}\eta_\xi^5$ and rearranging the resultant, one has that

$$
\begin{aligned}
2(h_0^{3/2}\eta_{\xi\xi\xi\xi} + 4h_0^{1/2}h_0'\eta_{\xi\xi\xi}) = {} & 2\underbrace{\left(-h_0^{1/2}h_0'\eta_\xi^3 + h_0^{1/2}h_0'''\eta_\xi\right)}_{:=J_1} - h_0^{1/2}u_t\eta_\xi^5 \\
& + 2h_0^{3/2}\eta_\xi^2\eta_{\xi\xi} - 12h_0^{1/2}h_0''\eta_{\xi\xi} + 2h_0^{1/2}\eta_\xi^3 u_{\xi\xi} + 2h_0^{-1/2}h_0'\eta_\xi^3 u_\xi \\
& + 20\frac{h_0^{3/2}\eta_{\xi\xi}\eta_{\xi\xi\xi}}{\eta_\xi} + 20\frac{h_0^{1/2}h_0'\eta_{\xi\xi}^2}{\eta_\xi} + \frac{10\eta_\xi^5}{h_0^{1/2}}\left(\frac{h_0^2\eta_{\xi\xi}}{\eta_\xi^6}\right)_\xi \eta_{\xi\xi} \\
& - 5\eta_\xi^5 h_0^{3/2}\left(\frac{\eta_{\xi\xi}^2}{\eta_\xi^6}\right)_\xi - 4h_0^{1/2}\eta_\xi^2\eta_{\xi\xi}u_\xi.
\end{aligned}
\tag{212}
$$

Repeating calculation similar to (98), since h_0 is concave, the L^2-norm of the left hand side of (212) satisfies

$$
\|h_0^{3/2}\partial_\xi^4\eta + 4h_0^{1/2}\partial_\xi h_0\partial_\xi^3\eta\|_{L^2}^2 \gtrsim \|h_0^{3/2}\partial_\xi^4\eta\|_{L^2}^2, \tag{213}
$$

and thanks to (178), (168), (169), (170), (193), (210) and (211) we have that the right hand side of (212) can be estimated as follows:

$$
\|J_1\|_{L^2}^2 \lesssim \|h_0\|_{H^4}^2\|\eta_\xi\|_{L^2}^2 \lesssim P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{214}
$$

$$
\|h_0^{1/2}u_t\eta_\xi^5\|_{L^2}^2 \lesssim \|h_0^{1/2}u_t\|_{L^2}^2 \lesssim P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{215}
$$

$$
\|h_0^{3/2}\eta_{\xi\xi}\eta_\xi^2\|_{L^2}^2 \lesssim \|h_0\|_{H^2}^3\|\eta_\xi\|_{H^1}^2 \lesssim P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{216}
$$

$$
\|h_0^{1/2}\partial_\xi^2 h_0\eta_{\xi\xi}\|_{L^2}^2 \lesssim \|h_0\|_{H^2}^3\|\eta_{\xi\xi}\|_{L^2}^2 \lesssim tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{217}
$$

$$
\|h_0^{1/2}u_{\xi\xi}\eta_\xi^3\|_{L^2}^2 \lesssim \|h_0\|_{H^1}\|u_{\xi\xi}\|_{L^2}^2 \lesssim P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{218}
$$

$$
\begin{aligned}
\|h_0^{-1/2}\partial_\xi h_0 u_\xi\eta_\xi^3\|_{L^2}^2 \lesssim {} & \|h_0\|_{H^2}^3\|h_0^{-1}u_\xi\|_{L^2}^2 \\
\lesssim {} & P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big),
\end{aligned}
\tag{219}
$$

$$
\|h_0^{3/2}\eta_{\xi\xi}\eta_{\xi\xi\xi}\eta_\xi^{-1}\|_{L^2}^2 \lesssim \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^2\|h_0\partial_\xi^3\eta\|_{L^2}^2 \lesssim tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{220}
$$

$$
\|h_0^{1/2}\partial_\xi h_0\eta_{\xi\xi}^2\eta_\xi^{-1}\|_{L^2}^2 \lesssim \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^2\|\eta_{\xi\xi}\|_{L^2}^2 \lesssim tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big), \tag{221}
$$

$$
\begin{aligned}
& \|h_0^{-1/2}\eta_\xi^5\eta_{\xi\xi}\partial_\xi\big(h_0^2\eta_{\xi\xi}\eta_\xi^{-6}\big)\|_{L^2}^2 + \|h_0^{3/2}\eta_\xi^5\partial_\xi\big(\eta_{\xi\xi}^2\eta_\xi^{-6}\big)\|_{L^2}^2 \\
\lesssim {} & (\|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^2 + \|h_0^{1/2}\eta_{\xi\xi}\|_{L^\infty}^4)(\|\eta_{\xi\xi}\|_{L^2}^2 + \|h_0\partial_\xi^3\eta\|_{L^2}^2) \\
\lesssim {} & tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big),
\end{aligned}
\tag{222}
$$

$$
\begin{aligned}
\|h_0^{1/2}u_\xi\eta_{\xi\xi}\eta_\xi^2\|_{L^2}^2 \lesssim {} & \|h_0\|_{H^1}\|u_\xi\|_{L^\infty}\|\eta_{\xi\xi}\|_{L^2}^2 \\
\lesssim {} & P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big),
\end{aligned}
\tag{223}
$$

Then it follows from (212) and (213)-(223) that

$$
\|h_0^{3/2}\partial_\xi^4\eta\|_{L^2}^2 \lesssim P\big(\Phi(0)\big) + tP\big(\sup_{0\le s\le t}\Phi(s)^{1/2}\big). \tag{224}
$$

8.2.4 Summary: The local-in-time *a priori* estimates

Now combing (170), (194), (209) and (224) together, we finally arrive at

$$\Phi(t) \le P\big(\Phi(0)\big) + CtP\big(\sup_{0\le s\le t} \Phi(s)^{1/2}\big), \tag{225}$$

for any $t \in [0,T]$, where C is a positive constant only depends on $\|h_0\|_{H^3}$, and P is a generic polynomial satisfying all the estimates above. Thus for sufficiently small $T \ge 0$,

$$\sup_{0\le t\le T} \Phi(t) \le 2P\big(\Phi(0)\big). \tag{226}$$

8.3 Proof of theorem 3

With the above *a priori* estimates and initial condition (163), (164), suppose the initial functional energy $\Phi(0) < \infty$, one can apply standard Galerkin's method to construct a strong solution (η, u) satisfying (165) and (166), see [38] for further details.

8.4 Uniqueness

Let (u,η) and (v,ζ) be two solutions to the system on $[0,T]$ with initial data (h_0, u_0) satisfying the same estimate. Their corresponding relationships are:

$$(\eta,\zeta)(\xi,t) = \xi + \int_0^t (u,v)(\xi,s)\,\mathrm{d}s. \tag{227}$$

Let

$$w = u - v, \ \ \chi = \eta - \zeta. \tag{228}$$

Then (w,χ) satisfies

$$\begin{aligned} h_0\partial_t w + \partial_\xi\bigg(\frac{h_0^2(\eta_\xi+\zeta_\xi)\chi_\xi}{\eta_\xi^2\zeta_\xi^2}\bigg) - 2\partial_\xi\bigg(\frac{h_0 w_\xi}{\eta_\xi^2} + \frac{h_0 v_\xi(\eta_\xi+\zeta_\xi)\chi_\xi}{\eta_\xi^2\zeta_\xi^2}\bigg) \\ = -\partial_\xi\bigg(\partial_\xi\bigg(\frac{2h_0^2\chi_{\xi\xi}}{\eta_\xi^5} + \frac{2h_0^2\zeta_{\xi\xi}P_1(\eta_\xi,\zeta_\xi)\chi_\xi}{\eta_\xi^5\zeta_\xi^5}\bigg)\bigg) \\ -\partial_\xi\bigg(\frac{5h_0^2(\eta_{\xi\xi}+\zeta_{\xi\xi})\chi_{\xi\xi}}{\eta_\xi^6} + \frac{5h_0^2\zeta_{\xi\xi}^2P_2(\eta_\xi,\zeta_\xi)\chi_\xi}{\eta_\xi^6\zeta_\xi^6}\bigg) \\ -\partial_\xi\bigg((|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)\frac{(\eta_\xi+\zeta_\xi)(\eta_\xi^2+\zeta_\xi^2)\chi_\xi}{\eta_\xi^4\zeta_\xi^4}\bigg), \end{aligned} \tag{229}$$

with initial data

$$(w,\chi)(\xi,0) = (0,0), \tag{230}$$

and boundary condition

$$(w_\xi,\chi_\xi)(-1,t) = (w_\xi,\chi_\xi)(1,t) = (0,0), \tag{231}$$

where polynomial

$$P_1(\eta_\xi,\zeta_\xi) = \eta_\xi^4 + \eta_\xi^3\zeta_\xi + \eta_\xi^2\zeta_\xi^2 + \eta_\xi\zeta_\xi^3 + \zeta_\xi^4,$$

and

$$P_2(\eta_\xi,\zeta_\xi) = (\eta_\xi^3+\zeta_\xi^3)(\eta_\xi^2+\eta_\xi\zeta_\xi+\zeta_\xi^2).$$

Define

$$\begin{aligned}\Phi_0(w,\chi,t) = \|\sqrt{h}_0 w(\cdot,t)\|_{L^2}^2 + \|h_0\chi_\xi(\cdot,t)\|_{L^2}^2 + \|h_0\chi_{\xi\xi}(\cdot,t)\|_{L^2}^2 \\ \|(|\partial_\xi h_0|^2 - 2h_0\partial_\xi^2 h_0)^{1/2}\chi_\xi(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{232}$$

Multiplying (229) both sides by w, integrating the resulting equation on $(-1,1)\times[0,T]$, using $\chi_t = w$ after integration by parts we finally find that

$$\begin{aligned}\sup_{0\le s\le t}\Phi_0(w,\chi,s) \le Ct\sup_{0\le s\le t}\Phi_0(w,\chi,s) + Ct\sup_{0\le s\le t}\int_\Omega h_0\chi_\xi^2(s)\,\mathrm{d}\xi \\ \le Ct\sup_{0\le s\le t}\Phi_0(w,\chi,s),\end{aligned} \tag{233}$$

for all $t\in[0,T]$, where we have used Hardy's inequality and C depends on $\Phi(u,\eta,t)$ and $\Phi(v,\zeta,t)$. Finally $w=0$ thus $\chi=0$ follows from (164) and the fact $(u,v)(\xi,t)\in C([-1,1]\times[0,T])$.

References

[1] A. A Akanbi and N. D Katopodes, *Model for flood propagation on initially dry land*, Journal of Hydraulic Engineering **114** (1988), no. 7, 689–706.

[2] A. J.-C. Barré de Saint-Venant, *Théorie du mouvement non-permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit*, Comptes rendus de l'Académie des Sciences de Paris **73** (1871), 147–154.

[3] P. D Bates and J.-M. Hervouet, *A new method for moving–boundary hydrodynamic problems in shallow water*, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences **455** (1999), no. 1988, 3107–3128.

[4] A. L Bertozzi, *The mathematics of moving contact lines in thin liquid films*, Notices of the AMS **45** (1998), no. 6, 689–697.

[5] M. Bertsch, L. Giacomelli, and G. Karali, *Thin-film equations with "partial wetting" energy: existence of weak solutions*, Physica D: Nonlinear Phenomena **209** (2005), no. 1-4, 17–27.

[6] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, and E. Rolley, *Wetting and spreading*, Reviews of Modern Physics **81** (2009), no. 2, 739.

[7] D. Bresch and B. Desjardins, *Existence of global weak solutions for a 2d viscous shallow water equations and convergence to the quasi-geostrophic model*, Communications in mathematical physics **238** (2003), 211–223.

[8] ———, *On the construction of approximate solutions for the 2d viscous shallow water model and for compressible Navier–Stokes models*, Journal de mathématiques pures et appliquées **86** (2006), no. 4, 362–368.

[9] D. Bresch, B. Desjardins, and C.-K. Lin, *On some compressible fluid models: Korteweg, lubrication, and shallow water systems* (2003).

[10] D. Bresch and P. Noble, *Mathematical derivation of viscous shallow-water equations with zero surface tension*, Indiana University Mathematics Journal **60** (2011), no. 4, 1137–1170.

[11] D. Coutand and S. Shkoller, *Well-posedness in Smooth Function Spaces for Moving-Boundary 1-D Compressible Euler Equations in Physical Vacuum*, Communications on Pure and Applied Mathematics **LXIV** (2011), 0328–0366.

[12] R. Danchin and B. Desjardins, *Existence of solutions for compressible fluid models of korteweg type*, Annales de l'Institut Henri Poincare C, Analyse non lineaire **18** (2001), no. 1, 97–133.

[13] P.-G. De Gennes, *Wetting: statics and dynamics*, Reviews of Modern Physics **57** (1985), no. 3, 827.

[14] Q. Duan, *On the dynamics of Navier–Stokes equations for a shallow water model*, Journal of Differential Equations **250** (2011), no. 6, 2687–2714.

[15] E. B. Dussan V. and S. H. Davis, *On the motion of a fluid-fluid interface along a solid surface*, Journal of Fluid Mechanics **65** (1974), no. 1, 71–95.

[16] T. Erneux and S. H Davis, *Nonlinear rupture of free films*, Physics of Fluids A: Fluid Dynamics **5** (1993), no. 5, 1117–1122.

[17] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, 2004. Series Title: Oxford Lecture Series in Mathematics and its Applications, 26 Publication Title: Oxford University Press.

[18] J. C. Flitton and J. R. King, *Surface-tension-driven dewetting of Newtonian and power-law fluids*, Journal of Engineering Mathematics **50** (2004), 241–266.

[19] J.-F. Gerbeau and B. Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation*, Ph.D. Thesis, 2000.

[20] P. Germain and P. LeFloch, *Finite Energy Method for Compressible Fluids: The Navier-Stokes-Korteweg Model*, Communications on Pure and Applied Mathematics **69** (2016), no. 1, 3–61.

[21] A. Ghosh, B. Niethammer, and J. J. Velázquez, *Revisiting shikhmurzaev's approach to the contact line problem*, Acta Applicandae Mathematicae **181** (2022), no. 1, 12.

[22] L. Giacomelli, M. V Gnann, and F. Otto, *Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3*, European Journal of Applied Mathematics **24** (2013), no. 5, 735–760.

[23] L. Giacomelli and F. Otto, *Rigorous lubrication approximation*, Interfaces and Free boundaries **5** (2003), no. 4, 483–529.

[24] M. V Gnann, *On the regularity for the Navier-slip thin-film equation in the perfect wetting regime*, Archive for Rational Mechanics and Analysis **222** (2016), 1285–1337.

[25] Y. Guo and I. Tice, *Stability of contact lines in fluids: 2d navier–stokes flow*, Journal of the European Mathematical Society (2023).

[26] C. Hao, L. Hsiao, and H.-L. Li, *Cauchy problem for viscous rotating shallow water equations*, Journal of Differential Equations **247** (2009), no. 12, 3234–3257.

[27] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 1. paperback ed., repr., transferred to digital printing, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2001.

[28] D. Hoff, *Global existence for 1d, compressible, isentropic Navier-Stokes equations with large initial data*, Transactions of the American Mathematical Society **303** (1987), no. 1, 169–181.

[29] D. Hoff and D. Serre, *The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow*, SIAM Journal on Applied Mathematics **51** (1991), no. 4, 887–898.

[30] D. Hoff and J. Smoller, *Non-formation of vacuum states for compressible Navier–Stokes equations*, Communications in Mathematical Physics **216** (2001), 255–276.

[31] C. Huh and L. E Scriven, *Hydrodynamic model of steady movement of a solid/liquid/fluid contact line*, Journal of Colloid and Interface Science **35** (1971), no. 1, 85–101.

[32] J. Jang and N. Masmoudi, *Well-posedness for compressible Euler equations with physical vacuum singularity*, Communications on Pure and Applied Mathematics **62** (2009), 1327–1385.

[33] Q. Jiu, Y. Wang, and Z. Xin, *Remarks on blow-up of smooth solutions to the compressible fluid with constant and degenerate viscosities*, Journal of Differential Equations **259** (2015), no. 7, 2981–3003.

[34] A. V. Kazhikhov and V. V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas: Pmm vol. 41, no.2, 1977, pp. 282–291*, Journal of Applied Mathematics and Mechanics **41** (1977), no. 2, 273–282.

[35] H. Knüpfer, *Well-posedness for the Navier slip thin-film equation in the case of partial wetting*, Communications on Pure and Applied Mathematics **64** (2011), no. 9, 1263–1296.

[36] J. Lallement, P. Villedieu, P. Trontin, and C. Laurent, *A shallow water type model to describe the dynamic of thin partially wetting films*, CFM 2017-23ème Congrès Français de Mécanique, 2017.

[37] H.-L. Li, Y. Wang, and Z. Xin, *Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier–Stokes equations*, Archive for Rational Mechanics and Analysis **232** (2019), 557–590.

[38] ______, *Well-posedness of classical solutions to the vacuum free boundary problem of the viscous Saint-Venant system for shallow waters*, arXiv, 2022. arXiv:2202.06340 [math].

[39] T.-P. Liu, Z. Xin, and T. Yang, *Vacuum states for compressible flow*, Discrete and Continuous Dynamical Systems **4** (1997), no. 1, 1–32.

[40] T. Luo, Z. Xin, and T. Yang, *Interface behavior of compressible Navier–Stokes equations with vacuum*, SIAM Journal on Mathematical Analysis **31** (2000), no. 6, 1175–1191.

[41] T. Luo, Z. Xin, and H. Zeng, *Nonlinear Asymptotic Stability of the Lane-Emden Solutions for the Viscous Gaseous Star Problem with Degenerate Density Dependent Viscosities*, Communications in Mathematical Physics **347** (Nov. 2016), no. 3, 657–702. arXiv: 1507.01069.

[42] ______, *On nonlinear asymptotic stability of the Lane–Emden solutions for the viscous gaseous star problem*, Advances in Mathematics **291** (Mar. 2016), 90–182. arXiv: 1506.03906.

[43] D. R. Lynch and W. G. Gray, *Finite element simulation of flow in deforming regions*, Journal of Computational Physics **36** (1980), no. 2, 135–153.

[44] F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects*, European Journal of Mechanics - B/Fluids **26** (2007), no. 1, 49–63.

[45] L. A. Monthe, F. Benkhaldoun, and I. Elmahi, *Positivity preserving finite volume Roe: schemes for transport-diffusion equations*, Computer Methods in Applied Mechanics and Engineering **178** (1999), no. 3, 215–232.

[46] A. Münch, B. Wagner, and T. P Witelski, *Lubrication models with small to large slip lengths*, Journal of Engineering Mathematics **53** (2005), 359–383.

[47] M. Okada, *Free boundary problem for one-dimensional motions of compressible gas and vacuum*, Japan Journal of Industrial and Applied Mathematics **21** (2004), no. 2, 109.

[48] A. Oron, S. H Davis, and S G. Bankoff, *Long-scale evolution of thin liquid films*, Reviews of Modern Physics **69** (1997), no. 3, 931.

[49] Y. Ou and H. Zeng, *Global strong solutions to the vacuum free boundary problem for compressible Navier–Stokes equations with degenerate viscosity and gravity force*, Journal of Differential Equations **259** (2015), no. 11, 6803–6829.

[50] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer New York, New York, NY, 1987.

[51] D. Peschka, *Thin-film free boundary problems for partial wetting*, Journal of Computational Physics **295** (2015), 770–778.

[52] ______, *Variational approach to dynamic contact angles for thin films*, Physics of Fluids **30** (Aug. 2018), no. 8, 082115.

[53] D. Peschka and L. Heltai, *Model hierarchies and higher-order discretisation of time-dependent thin-film free boundary problems with dynamic contact angle*, Journal of Computational Physics **464** (2022), 111325.

[54] D. Peschka, A. Münch, and B. Niethammer, *Self-similar rupture of viscous thin films in the strong-slip regime*, Nonlinearity **23** (2010), no. 2, 409.

[55] W. Ren and W. E, *Boundary conditions for the moving contact line problem*, Physics of Fluids **19** (Feb. 2007), no. 2, 022101.

[56] Y. D Shikhmurzaev, *Moving contact lines in liquid/liquid/solid systems*, Journal of Fluid Mechanics **334** (1997), 211–249.

[57] J. H Snoeijer and B. Andreotti, *Moving contact lines: scales, regimes, and dynamical transitions*, Annual Review of Fluid Mechanics **45** (2013), 269–292.

[58] D. Vaynblat, J. R Lister, and T. P Witelski, *Rupture of thin viscous films by van der Waals forces: Evolution and self-similarity*, Physics of Fluids **13** (2001), no. 5, 1130–1140.

[59] Z. Xin, *Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **51** (1998), no. 3, 229–240.

[60] T. Yang and C. Zhu, *Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum*, Communications in Mathematical Physics **230** (2002), 329–363.

[61] H. Zeng, *Global-in-time smoothness of solutions to the vacuum free boundary problem for compressible isentropic Navier–Stokes equations*, Nonlinearity **28** (2015), no. 2, 331.

[62] T. Zhang and D. Fang, *Global behavior of compressible Navier-Stokes equations with a degenerate viscosity coefficient*, Archive for Rational Mechanics and Analysis **182** (2006), 223–253.