Eigen Mode Solver for Microwave Transmission Lines

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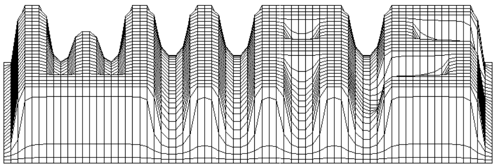

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Abstract

The electromagnetic properties of microwave transmission lines can be described using Maxwell's equations in the frequency domain. Applying a finite-volume scheme this results in an algebraic eigenmode problem. In this paper, an improved numerical computation of the eigenmodes is presented.

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1 Introduction

The design of microwave integrated circuits and packages requires efficient CAD tools for a three-dimensional electromagnetic simulation because the coupling effects become critical with growing packaging density and increasing frequency. Usually, a circuit is described in terms of its scattering matrix [1], [2], [3], [4], [5]. In this approach, the structure under investigation is embedded in a set of longitudinally homogeneous transmission lines (see Figure 1), which are assumed to be infinitely long. As a first step one has to calculate the field distribution

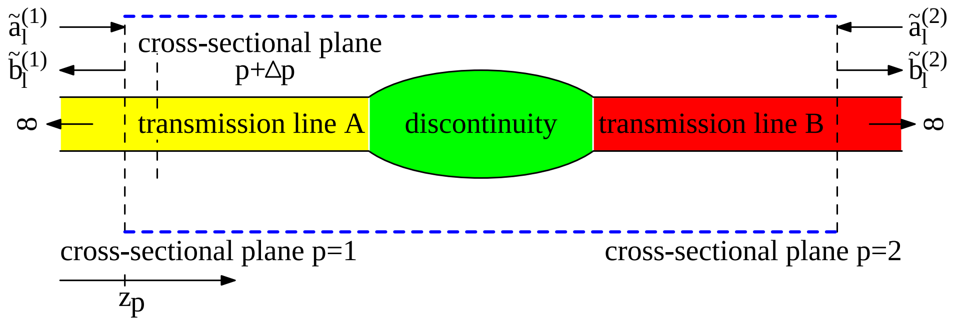

Figure 1: Structure under investigation

in these transmission lines. Because the transmission lines are longitudinally homogeneous, the electromagnetic field can be expanded into a sum of wave modes with each of them varying exponentially in the longitudinal direction. The corresponding ansatz results in an eigenvalue problem.
In a second step, the eigenfunctions determine the boundary values for the calculation of the fields in the three-dimensional structure.
In order to speed up the design process, the calculations have to be performed as effectively as possible. In this paper, we treat the numerical solution of the eigenmode problem. Applying the finite-volume method to discretize the Maxwellian equations results in a standard eigenvalue problem with a large sparse matrix. We describe a suitable method to calculate the set of interesting eigenvalues and eigenvectors.

2 Matrix Representation of Maxwell's Equations

In the frequency domain, we consider fields that vary with time according to the complex exponential function $e^{\jmath\omega t}$. Thus, the integral form of the Maxwellian equations reads

$$\oint_{\partial\Omega} \frac{1}{\tilde{\mu}\mu_0}\vec{B}\cdot d\vec{s} = \int_{\Omega} (\jmath\omega\tilde{\epsilon}\epsilon_0\vec{E})\cdot d\vec{\Omega}, \tag{1a}$$

$$\oint_{\partial\Omega} \vec{E}\cdot d\vec{s} = \int_{\Omega} (-\jmath\omega\vec{B})\cdot d\vec{\Omega}, \tag{1b}$$

$$\oint_{\cup\Omega} (\tilde{\epsilon}\epsilon_0\vec{E})\cdot d\vec{\Omega} = 0, \tag{1c}$$

$$\oint_{\cup\Omega} \vec{B}\cdot d\vec{\Omega} = 0, \tag{1d}$$

taking into account the constitutive relations

$$\vec{B} = \mu\vec{H}, \;\; \vec{D} = \epsilon\vec{E} \text{ with } \underline{\epsilon} = \epsilon + \frac{\kappa}{\jmath\omega}, \;\; \mu = \tilde{\mu}\mu_0, \;\; \underline{\epsilon} = \tilde{\epsilon}\epsilon_0. \tag{2}$$

The field vectors $\vec{E}$, $\vec{H}$, $\vec{D}$, and $\vec{B}$ (electric and magnetic field intensity, electric and magnetic flux density, respectively) are complex functions of the spatial coordinates only. ω is the circular frequency and $\jmath^2 = -1$. The permeability μ, the permittivity ϵ, and the conductivity κ are assumed to be scalar functions of the spatial coordinates. $\underline{\epsilon}$ is the complex permittivity.
Introducing the finite-volume scheme, the region is divided into rectangular parallelepipeds (see Figure 2) using a three-dimensional nonequidistant Cartesian grid. The edges of the cells are parallel to the coordinate axes. The grid nodes (i, j, k), the lower left front corners of the parallelepipeds, are numbered by

$$\ell = (k-1)n_x n_y + (j-1)n_x + i, \;\; i = 1(1)n_x, \;\; j = 1(1)n_y, \;\; k = 1(1)n_z. \tag{3}$$

n_x, n_y, and n_z denote the numbers of rectangular parallelepipeds in the x-, y- and z-direction, respectively. The field vectors are expressed as

$$\vec{M} = M_x\vec{i}_x + M_y\vec{i}_y + M_z\vec{i}_z, \quad \vec{M} \in \{\vec{E}, \vec{D}, \vec{H}, \vec{B}\}. \tag{4}$$

$\vec{i}_x$, $\vec{i}_y$, and $\vec{i}_z$ are the unit vectors in $x-$, $y-$ and $z-$direction of the Cartesian coordinate grid, respectively. The components E_x, E_y, and E_z of the electric field $\vec{E}$ are located in the centers of the edges of the elementary cells. The components B_x, B_y, and B_z, on the other hand, are normal to the face centers [6], [7]. Thus, the electric field components form a primary grid, and the magnetic flux density components a dual grid (see Figure 3). The Maxwellian equations are now applied to each cell. We use the lowest-order integration formulae

$$\oint_{\partial\Omega} \vec{f} \cdot d\vec{s} \approx \sum(\pm f_i s_i), \quad \int_{\Omega} \vec{f} \cdot d\vec{\Omega} \approx f\Omega \tag{5}$$

in order to approximate the left-hand and the right-hand sides of the first and the second Maxwellian equation (see (1a), (1b)).
The closed path $\partial\Omega$ of the integration in the primary grid consists of 4 straight lines of length s_i and is the path around the periphery of an unit cell face in the grid. f_i denotes the function value in the center of the side s_i. In the dual grid the closed path $\partial\Omega$ of the integration consists of 8 straight lines, and f_i denotes the function value in the center of the corresponding face in the primary grid (see Figure 3). Ω is the area of any cell face. f denotes the function value in the center of this face.
Let be

$$\vec{e} = \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}, \qquad \begin{array}{ll} \vec{e}_x = (e_{x_1}, e_{x_2}, \ldots, e_{x_{n_{xyz}}})^T, & e_{x_l} = E_{x_{i,j,k}}, \\ \vec{e}_y = (e_{y_1}, e_{y_2}, \ldots, e_{y_{n_{xyz}}})^T, & e_{y_l} = E_{y_{i,j,k}}, \\ \vec{e}_z = (e_{z_1}, e_{z_2}, \ldots, e_{z_{n_{xyz}}})^T, & e_{z_l} = E_{z_{i,j,k}}, \end{array} \tag{6}$$

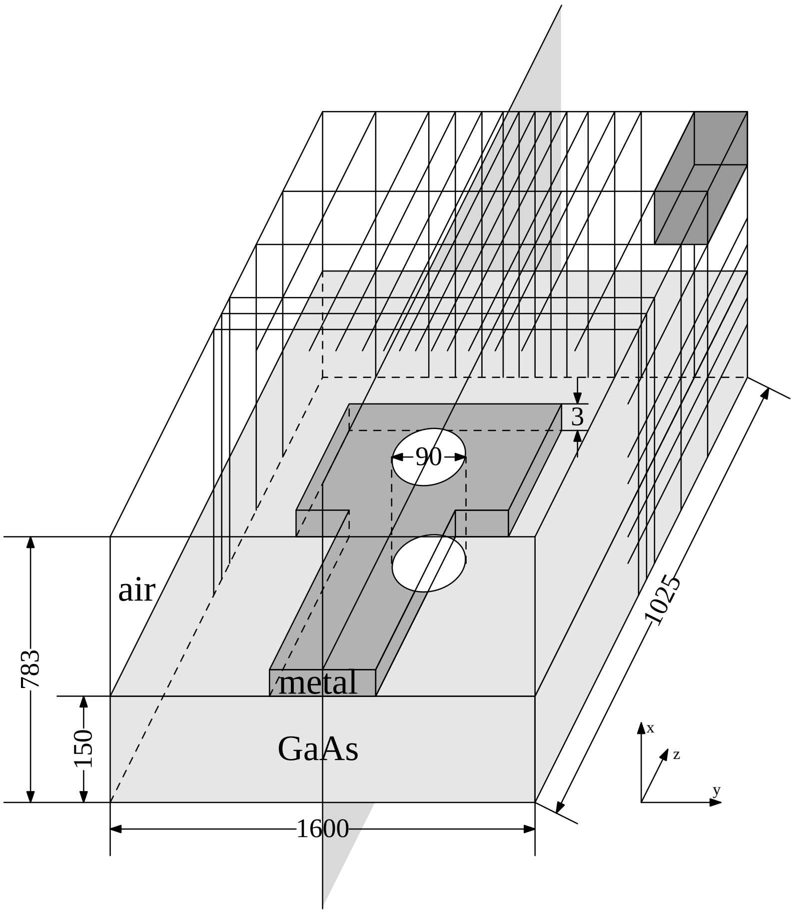

Figure 2: Via hole

$$\vec{b} = \begin{pmatrix} \vec{b}_x \\ \vec{b}_y \\ \vec{b}_z \end{pmatrix}, \qquad \begin{array}{l} \vec{b}_x = (b_{x_1}, b_{x_2}, \ldots, b_{x_{n_{xyz}}})^T, \\ \vec{b}_y = (b_{y_1}, b_{y_2}, \ldots, b_{y_{n_{xyz}}})^T, \\ \vec{b}_z = (b_{z_1}, b_{z_2}, \ldots, b_{z_{n_{xyz}}})^T, \end{array} \qquad \begin{array}{l} b_{x_\ell} = B_{x_{i,j,k}}, \\ b_{y_\ell} = B_{y_{i,j,k}}, \\ b_{z_\ell} = B_{z_{i,j,k}} \end{array} \tag{7}$$

with (see (3))

$$\ell = (k-1)n_{xy} + (j-1)n_x + i, \quad n_{xy} = n_x n_y, \quad n_{xyz} = n_x n_y n_z, \tag{8}$$

the vectors containing the electric and the magnetic field of the elementary cells, respectively.
Let be D_s and D_A diagonal matrices and A a matrix which represents the operator of the line integral in the second Maxwellian equation (see (1b)), using the primary grid [4] the following matrix representation of the second Maxwellian

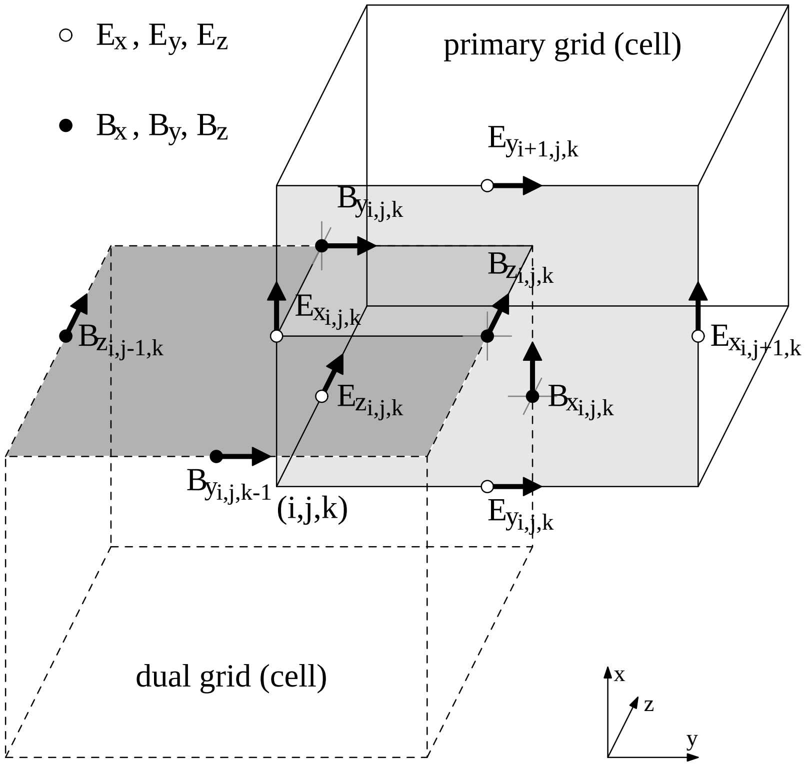

Figure 3: Primary and dual grid

Equation (1b) results:

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{s} = \int_{\Omega} (-\jmath\omega\vec{B}) \cdot d\vec{\Omega} \quad \Rightarrow \quad A D_s \vec{e} = -\jmath\omega D_A \vec{b}. \tag{9}$$

The diagonal matrices D_s and D_A contain the information on all dimensions for the specified structure and the corresponding mesh.
Let be $D_{A_{\tilde{\epsilon}}}$ and $D_{s/\tilde{\mu}}$ diagonal matrices and A^T the transposed matrix of A, using the dual grid the following matrix representation of the modified first Maxwellian equation (see (1a)) results:

$$\oint_{\partial\Omega} \frac{\vec{B}}{\tilde{\mu}} \cdot d\vec{s} = \int_{\Omega} (\jmath\omega\tilde{\epsilon}\epsilon_0\mu_0\vec{E}) \cdot d\vec{\Omega} \quad \Rightarrow \quad A^T D_{s/\tilde{\mu}} \vec{b} = \jmath\omega\epsilon_0\mu_0 D_{A\tilde{\epsilon}} \vec{e}. \tag{10}$$

The diagonal matrices $D_{s/\tilde{\mu}}$ and $D_{A_{\tilde{\epsilon}}}$ contain the information on dimension and material for the structure and the corresponding mesh.

Let be B the matrix which represents the operator of the surface integral in Equation (1c) we get the matrix representation of the electric-field divergence equation:

$$\oint_{\cup\Omega} \tilde{\epsilon}\epsilon_0 \vec{E} \cdot d\vec{\Omega} = 0 \quad \Rightarrow \quad B D_{A\tilde{\epsilon}} \vec{e} = 0. \tag{11}$$

The matrices A and B are sparse and consist only of the elements -1, 0 and 1. Combining (9) and (10) we get the matrix representation of the system of linear algebraic equations:

$$(A^T D_{s/\tilde{\mu}} D_A^{-1} A D_s - k_0^2 D_{A\tilde{\epsilon}})\vec{e} = 0, \quad k_0 = \omega\sqrt{\epsilon_0 \mu_0}. \tag{12}$$

k_0 is the wave number in vacuo.
The Maxwellian grid equations are a consistent discrete representation of the analytical equations in the sense that basic properties of the analytical fields are maintained [8].

3 The Eigenvalue Problem

In the following, we consider a longitudinally homogeneous transmission line, which means that $\tilde{\epsilon}$ and $\tilde{\mu}$ are functions of transverse position but are independent of the longitudinal direction z. With these assumptions, any field can be expanded into a sum of so-called modal fields

$$\vec{E}_{k_z}(x, y, z) = \vec{E}_{k_z}(x, y) e^{-jk_z z} \tag{13}$$

which vary exponentially in the longitudinal direction.
The $\vec{E}_{k_z}(x, y)$ are eigenfunctions of a partial differential equation of second order and the propagation constants k_z are related to the eigenvalues (see (14)).
To find the corresponding solutions of the discretized Maxwellian equations we consider the field components in three consecutive elementary cells (see Figure 4). Each cell is of lenght $2h$ in the z-direction where h must be chosen small enough so that $2h|k_z| << 1$. The electric field components $E_{x_{i,j,k+1}}$, $E_{x_{i,j,k-1}}$, $E_{y_{i,j,k+1}}$, $E_{y_{i,j,k-1}}$, $E_{z_{i,j,k-1}}$, $E_{z_{i+1,j,k-1}}$, and $E_{z_{i,j+1,k-1}}$ in Equation (12) are expressed by the values of cell k using ansatz (13). The longitudinal electric field components E_z can be eliminated by means of the Equation (11). So we get an eigenvalue problem for the transverse electric field on the transmission line region. $E_{x_{i,j,k}}$, $E_{y_{i,j,k}}$, $k = const$, are the $2n_{xy}$ components of the eigenfunctions with the eigenvalues

$$\gamma(h) = e^{-jk_z 2h} + e^{+jk_z 2h} - 2 = -4\sin^2(k_z h). \tag{14}$$

Let be

$$\underline{\vec{e}} = \begin{pmatrix} \underline{\vec{e}_x} \\ \underline{\vec{e}_y} \end{pmatrix}, \qquad \begin{aligned} \underline{\vec{e}_x} &= (\underline{e}_{x_1}, \underline{e}_{x_2}, \ldots, \underline{e}_{x_{n_{xy}}})^T, \\ \underline{\vec{e}_y} &= (\underline{e}_{y_1}, \underline{e}_{y_2}, \ldots, \underline{e}_{y_{n_{xy}}})^T, \end{aligned} \qquad \begin{aligned} \underline{e}_{x_\ell} &= E_{x_{i,j,k}}, \\ \underline{e}_{y_\ell} &= E_{y_{i,j,k}} \end{aligned} \tag{15}$$

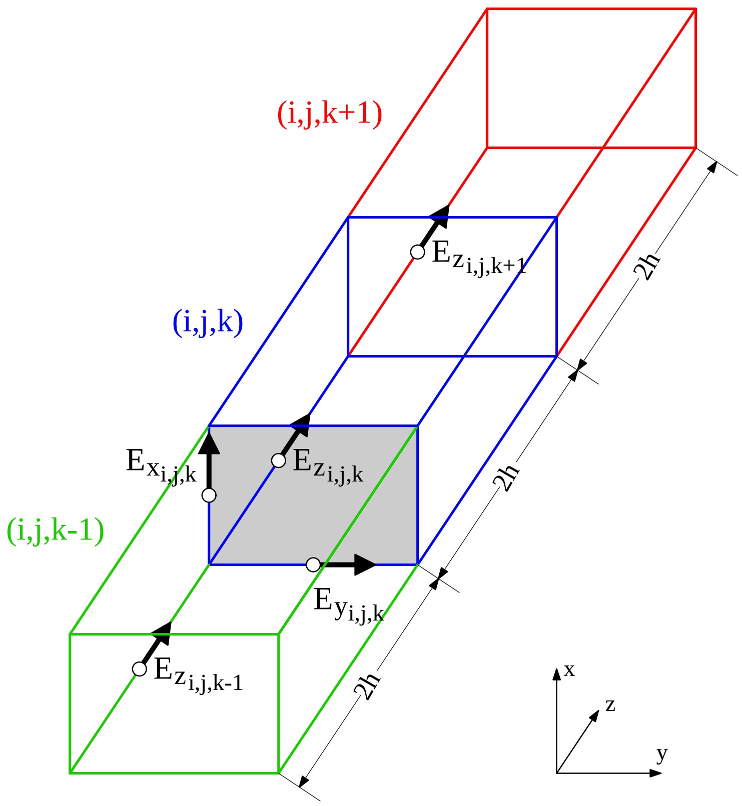

Figure 4: Reduction of the dimension

with

$$\begin{aligned} &\ell = (j-1)n_x + i, \quad i = 1(1)n_x, \quad j = 1(1)n_y, \\ &n_{xy} = n_x n_y, \quad \text{and} \quad k = 1 \quad \text{or} \quad k = n_z. \end{aligned} \tag{16}$$

The assumption $k = 1$ corresponds to the case, in which the cross-sectional plane (see Figure 1) is located on the left-handed (x, y)-plane of the enclosure. Let be $\bar{A}$, $\dim(\bar{A}) = 2n_{xy}$, the matrix of the eigenvalue problem. Because of boundary conditions on the left-hand side and the bottom of the port, the dimension of the eigenvalue problem is reduced to $n = 2n_{xy} - n_b$, $n_b = n_x + n_y$. In most cases we have to take into account also boundary conditions at interior boundaries, which further reduce the dimension. We denote the matrix of the reduced eigenvalue problem by A and the corresponding vector of the eigenfunction components by

$\vec{\tilde{e}}$. Thus, we have to solve the eigenvalue problem

$$A\vec{\tilde{e}} = \gamma\vec{\tilde{e}} \ , \quad \text{type}(A) = (n,n), \quad n = 2n_{xy} - n_b. \tag{17}$$

The size of n_b depends on the boundary conditions.
The sparse matrix A is nonsymmetric. There are n eigenvalues $\gamma = u + \jmath v$ and corresponding propagation constants $k_z = \beta - \jmath\alpha$:

$$\gamma_\iota = -4\sin^2(hk_{z_\iota}) = u_\iota + \jmath v_\iota, \quad k_{z_\iota} = \beta_\iota - \jmath\alpha_\iota, \quad \iota = 1(1)n \ . \tag{18}$$

4 The Propagation Constants

The propagation constants k_z can be computed from γ after the solution of the eigenvalue problem (17). We get from (14)

$$k_{z_\iota} = \frac{1}{h}\arcsin\left(\frac{\jmath}{2}\sqrt{\gamma_\iota}\right) = \frac{\jmath}{2h}\ln\left(\frac{\gamma_\iota}{2} + 1 + \sqrt{\frac{\gamma_\iota}{2}\left(\frac{\gamma_\iota}{2} + 2\right)}\right), \ \iota = 1(1)n. \tag{19}$$

Using the principal values of the functions, we have always $\beta > 0$ and $\alpha \geq 0$.
A propagation constant k_z and its corresponding eigenfunction are called a mode.
In technical applications only a small number m of modes are of interest. For real values of k_z, $\Im(k_z) = 0$, the modal field of Equation (13) describes a non-attenuated wave, which can be used to transmit signals. On the other hand, the amplitude of waves with $\alpha = -\Im(k_z) > 0$ decreases with increasing z. The larger the magnitude of $\Im(k_z)$ the stronger is the decay. Such waves are called evanescent waves, and can be neglected after a given length d of the transmission line. Therefore, we use to select the interesting modes the
Criterion 1: The propagation constants k_{z_ι}, $\iota = 1(1)n$, are sorted in ascending order of $|\alpha_\iota|$. In the case if some $|\alpha_\iota|$ have the same value the constants k_{z_ι} are sorted in descending order of $|\beta_\iota|$.
In the discussion to follow we distinguish between sets $\mathcal{A}$ of all eigenvalues and $\bar{\mathcal{A}}$ of all corresponding propagation constants, and sets $\mathcal{E}$ and $\bar{\mathcal{E}}$ of computed eigenvalues and propagation constants, respectively. The number of elements of a set $\mathcal{B}$ is denoted by $|\mathcal{B}|$.
Let be $\bar{\mathcal{A}}$, $|\bar{\mathcal{A}}| = n$, the set of propagation constants. $\bar{\mathcal{A}}$ is split into the subset $\bar{\mathcal{A}}^{(r)}$ of real, the subset $\bar{\mathcal{A}}^{(i)}$ of imaginary, and the subset $\bar{\mathcal{A}}^{(c)}$ of complex propagation constants:

$$\begin{aligned} &\bar{\mathcal{A}}^{(r)} \cup \bar{\mathcal{A}}^{(i)} \cup \bar{\mathcal{A}}^{(c)} = \bar{\mathcal{A}}, \\ &\bar{\mathcal{A}}^{(r)} \cap \bar{\mathcal{A}}^{(i)} = \emptyset, \quad \bar{\mathcal{A}}^{(r)} \cap \bar{\mathcal{A}}^{(c)} = \emptyset, \quad \bar{\mathcal{A}}^{(i)} \cap \bar{\mathcal{A}}^{(c)} = \emptyset. \end{aligned} \tag{20}$$

The set $\mathcal{A}$ of the corresponding eigenvalues γ_ι, $\iota = 1(1)n$, is split into the subset $\mathcal{A}^{(r_p)}$ of positive, the subset $\mathcal{A}^{(r_n)}$ of negative, the subset $\mathcal{A}^{(i)}$ of imaginary, and the subset $\mathcal{A}^{(c)}$ of complex eigenvalues:

$$\begin{aligned} &\mathcal{A}^{(r_p)} \cup \mathcal{A}^{(r_n)} \cup \mathcal{A}^{(i)} \cup \mathcal{A}^{(c)} = \mathcal{A}, \quad \mathcal{A}^{(r)} = \mathcal{A}^{(r_p)} \cup \mathcal{A}^{(r_n)}, \\ &\mathcal{A}^{(r_p)} \cap \mathcal{A}^{(r_n)} = \emptyset, \ \mathcal{A}^{(r)} \cap \mathcal{A}^{(i)} = \emptyset, \ \mathcal{A}^{(r)} \cap \mathcal{A}^{(c)} = \emptyset, \ \mathcal{A}^{(i)} \cap \mathcal{A}^{(c)} = \emptyset. \end{aligned} \tag{21}$$

The real propagation constants of the set $\bar{\mathcal{A}}^{(r)}$ are the propagation constants with the smallest magnitude of imaginary part. In most applications with lossless materials (that means with real $\tilde{\epsilon}$ and $\tilde{\mu}$) one has at least one real propagation constant. These propagation constants have to be taken into account anyway. It is important, however, to know also at least one propagation constant with the smallest nonvanishing $|\alpha|$, in order to decide whether the eigenfunctions decrease strongly enough in a given distance d.

5 The Numerical Problem

In an earlier version of the program [2] the complete set of propagation constants was computed and sorted in order to select the set $\bar{\mathcal{A}}^{(f)}$ of the m first propagation constants of $\bar{\mathcal{A}}$. This way is very time-consuming. The sparse matrix was stored as a dense matrix.
In the following we show that one can avoid the computation of all eigenvalues to find the few required propagation constants using an iterative method which has to be carried out twice.
We use the nonsymmetric version of the implicitly restarted Arnoldi iteration [9], [10], [11], [12]. The standard eigenvalue problem $Ax = \lambda x$ can be solved using the regular mode $Ax = \lambda x$ or the inverse mode $A^{-1}x = \frac{1}{\lambda}x$. The Arnoldi algorithm is applied iteratively to solve one of these problems generating Arnoldi vectors. Using the regular mode most of the cost in generating each Arnoldi vector is in the matrix-vector product. Using the inverse mode we have to solve a system of linear algebraic equations on each iteration step.
In general the method does not converge in the regular mode for our eigenvalue problem. Thus, the Arnoldi algorithm is called iteratively to solve the standard eigenvalue problem using the inverse mode $A^{-1}x = \frac{1}{\lambda}x$ with the more time-consuming solution of linear algebraic equations.
We use a direct method, a Gaussian elimination for sparse matrices, to solve the ill-conditioned system of linear equations. The process of solving the systems of linear algebraic equations consists of the factorization of the coefficient matrix and the subsequent forward and backward substitution. Because the matrix A does not change using the Arnoldi iteration, we have to factorize A only once at the beginning of the method. The forward and backward substitution is done on each iteration step. The coefficient matrix of the system of linear algebraic equations is sparse. The Gaussian elimination is carried out with a special pivoting which permits a compromise between a minimum fill-in (increase of the number of non-zero elements during elimination) and numerical stability (connection of topological and numerical pivoting). We use the maximum numerical stability in this criterion.

6 Selection of the Interesting Eigenvalues

By means of the Arnoldi iteration we can compute a set of eigenvalues of largest or smallest magnitude, real part or imaginary part, provided that the method converges, but we can not find in one step the set of eigenvalues according to criterion 1. Therefore, we must proceed in two steps. We estimate a number $m_1 \geq m$ of interesting modes. The we compute in a first run a subset $\mathcal{E}$ of m_1 eigenvalues γ of smallest magnitude using the Arnoldi method in inverse mode looking for eigenvalues of largest magnitude, and compute the corresponding set $\bar{\mathcal{E}}$, $|\bar{\mathcal{E}}| = m_1$, of propagation constants. However, we want to find all modes which are able to transmit signals. That means, we have to find a set $\bar{\mathcal{A}}^{(f)}$ of propagation constants with the smallest magnitude of the imaginary part, but possible with large real part. In general, we have $\bar{\mathcal{A}}^{(f)} \cap \bar{\mathcal{E}} \neq \emptyset$ but $\bar{\mathcal{A}}^{(f)} \not\subseteq \bar{\mathcal{E}}$. To search for these additional eigenvalues, we need a second run of the Arnoldi method with a modified matrix.

7 Estimation of the Maximum Propagation Constant

We can give an upper bound $k^{(max)}$ for the real part of the interesting propagation constants in the following way:
In a homogeneous lossless material (with real $\tilde{\epsilon}$ and $\tilde{\mu}$) the wave number k_f of an electromagnetic wave equals

$$k_f = \omega\sqrt{\epsilon\mu} = k_0\sqrt{\tilde{\epsilon}\tilde{\mu}}. \tag{22}$$

This value is also an upper bound for the (real) propagation constants of undamped modes in a waveguide that is completely filled with the same material. In case of an inhomogeneously filled waveguide the quantities $\tilde{\epsilon}$ and $\tilde{\mu}$ can be different from cell to cell. We select the maxima of these quantities

$$\tilde{\epsilon}^{(max)} = \max_{i,j,k}\{\tilde{\epsilon}_{i,j,k}\}, \quad \tilde{\mu}^{(max)} = \max_{i,j,k}\{\tilde{\mu}_{i,j,k}\} \tag{23}$$

and set

$$k^{(max)} = k_0\sqrt{\tilde{\epsilon}^{(max)}\tilde{\mu}^{(max)}} \tag{24}$$

as an upper bound for the propagation constants of propagating modes.
If we change to materials with small losses (that means with small imaginary parts of $\tilde{\epsilon}$ and $\tilde{\mu}$), we expect that the former real propagation constants become complex with nearly the same real part as in the lossless case and with small imaginary part. Hence in this case

$$k^{(max)} = k_0\sqrt{|\tilde{\epsilon}^{(max)}||\tilde{\mu}^{(max)}|} \tag{25}$$

approximates the maximum possible real part of the interesting propagation constants.

8 Distribution of Eigenvalues in the Complex Plane

The relation between γ and k_z is nonlinear. If we choose h small enough (which is necessary anyway to get small discretization errors), we can set $\sin(hk_z) \approx hk_z$ and have

$$\gamma = -4\sin^2(hk_z) \approx -4(hk_z)^2 = -4[(h\beta)^2 - (h\alpha)^2] + 8jh^2\alpha\beta. \tag{26}$$

$$|\gamma| = 4h^2(\beta^2 + \alpha^2) \quad \text{and} \quad |k_z| = \sqrt{\beta^2 + \alpha^2} \tag{27}$$

are monotonically increasing functions of the argument $\beta^2 + \alpha^2$ in the intervalls considered. Thus, the set $\bar{\mathcal{E}}$ found in the first run is a set of propagation constants of smallest magnitude.

We consider special values of $\gamma = u + jv$ and $k_z = \beta - j\alpha$:

$$\begin{aligned}
&\alpha = 0: && \gamma = -4(h\beta)^2, && u \le 0, \quad v = 0, && \text{(28a)}\\
&\alpha \ne 0, \quad \beta = 0: && \gamma = 4(h\alpha)^2 && u > 0, \quad v = 0, && \text{(28b)}\\
&\alpha \ne 0, \quad u = 0: && |\beta| = |\alpha|, && \gamma = \pm 8jh^2\beta^2. && \text{(28c)}
\end{aligned}$$

We split the set $\mathcal{E}$ into the subset $\mathcal{E}^{(r)}$ of real, the subset $\mathcal{E}^{(i)}$ of imaginary, and the subset $\mathcal{E}^{(c)}$ of complex eigenvalues. Similarly the set of the corresponding propagation constants $\bar{\mathcal{E}}$ is split into $\bar{\mathcal{E}}^{(r)}$, $\bar{\mathcal{E}}^{(i)}$, and $\bar{\mathcal{E}}^{(c)}$. The set $\mathcal{E}^{(r)}$ of real eigenvalues is divided into the subset $\mathcal{E}^{(r_p)}$ of positive and the subset $\mathcal{E}^{(r_n)}$ of negative eigenvalues:

$$\mathcal{E}^{(r_p)} \cup \mathcal{E}^{(r_n)} \cup \mathcal{E}^{(i)} \cup \mathcal{E}^{(c)} = \mathcal{E}, \quad \bar{\mathcal{E}}^{(r)} \cup \bar{\mathcal{E}}^{(i)} \cup \bar{\mathcal{E}}^{(c)} = \bar{\mathcal{E}}. \tag{29}$$

The subsets are pairwise disjoint in analogy to (20) and (21). Because of (26), (28a), (28b) and (28c) we have for $\mathcal{B} = \mathcal{E}$ and $\mathcal{B} = \mathcal{A}$ generally the mappings

$$\bar{\mathcal{B}}^{(r)} \to \mathcal{B}^{(r_n)}, \quad \bar{\mathcal{B}}^{(i)} \to \mathcal{B}^{(r_p)}, \quad \bar{\mathcal{B}}^{*(c)} \to \mathcal{B}^{(i)}, \quad \bar{\mathcal{B}}^{**(c)} \to \mathcal{B}^{(c)} \tag{30}$$

with

$$\bar{\mathcal{B}}^{(c)} = \bar{\mathcal{B}}^{*(c)} \cup \bar{\mathcal{B}}^{**(c)}, \quad \bar{\mathcal{B}}^{*(c)} \cap \bar{\mathcal{B}}^{**(c)} = \emptyset. \tag{31}$$

We consider the real, the imaginary, and the complex propagation constants in the discussion to follow.

Real propagation constants ($\alpha = 0$)

The subset of eigenvalues $\mathcal{E}^{(r_n)}$ (see (28a) and (30)) which corresponds with $\bar{\mathcal{E}}^{(r)}$ consists of $|\mathcal{E}^{(r_n)}| = |\bar{\mathcal{E}}^{(r)}| = m_r$ negative numbers. Let be $k_{z_\rho} = \beta_\rho$ the real propagation constant of smallest magnitude of $\bar{\mathcal{E}}^{(r)}$. The question is according to criterion 1: Are there other real propagation constants

$$k_{z_\sigma} = \beta_\sigma \in \bar{\mathcal{A}}^{*(r)} \subseteq \bar{\mathcal{A}}^{(r)} \text{ with } \bar{\mathcal{A}}^{*(r)} \cap \bar{\mathcal{E}}^{(r)} = \emptyset, \; |\beta_\sigma| > |\beta_\rho|? \tag{32}$$

To decide this question, we extend the matrix A by some, for example a set $\mathcal{E}^{(a)}$, $|\mathcal{E}^{(a)}| = m_a$, of negative elements $\gamma_\tau^{(a)}$, $\tau = 1(1)m_a$, adding m_a rows and m_a columns such that the elements $\gamma_\tau^{(a)}$ are diagonal elements of the extended matrix A^*. The other elements of the new rows and columns are chosen to be zero:

$$A^* = \begin{pmatrix} \gamma_1^{(a)} & & & \\ & \ddots & & \\ & & \gamma_{m_a}^{(a)} & \\ & & & \begin{pmatrix} & A & \end{pmatrix} \end{pmatrix} \tag{33}$$

The spectrum of A^* consists of the spectrum of A extended by the added m_a eigenvalues $\gamma_\tau^{(a)}$.
How should one choose the m_a additional eigenvalues?
We can give a value

$$\gamma^{(max)} = -4(hk^{(max)})^2 \tag{34}$$

(with $k^{(max)}$ from Equation (24) or Equation (25)) for the greatest element of the set $\mathcal{E}^{(a)}$ such that all interesting negative eigenvalues of A^* are greater than this bound. We use the following set $\mathcal{E}^{(a)}$ of additional eigenvalues

$$\gamma_\tau^{(a)} = \gamma^{(max)}(1 + \frac{\tau}{10}), \quad \tau = 1(1)m_a, \tag{35}$$

to build A^* and compute a set $\mathcal{E}^{(l)}$ of $|\mathcal{E}^{(l)}| = m_a + m_r$ eigenvalues of smallest real part of A^* using the Arnoldi method in inverse mode. Eigenvalues of A which fulfill the condition (32) belong to $\mathcal{E}^{(l)}$ rather than the eigenvalues of $\mathcal{E}^{(a)}$, and we can separate the m_n new eigenvalues from the set $\mathcal{E}^{(l)}$. If we find $m_n < m_a$ new eigenvalues, we need not change m_a. Otherwise we have to increase m_a for a new computation because we do not know whether more than m_a new negative eigenvalues of A exist.
We note it is essentially not to demand more eigenvalues of smallest real part of A^* than eigenvalues with negative real parts of A^* exist, because the Arnoldi method does not converge for our problem in this case.
The $m_r + m_n$ negative eigenvalues of A fulfill the criterion 1.
We note the factorization of the sparse matrix A^* (see (33)) is obviously a modification of the factorization of A, i.e., we can avoid a second matrix factorization.
<u>Imaginary propagation constants</u> ($\beta = 0$)
The subset of eigenvalues $\mathcal{E}^{(r_p)}$ which corresponds with $\bar{\mathcal{E}}^{(i)}$ consists of positive numbers (see (28b), (30)). Because of $|\gamma| = 4(h\alpha)^2$ and $|k_z| = |\alpha|$ there cannot exist another imaginary propagation constant in $\bar{A}^{(i)}$ with a smaller magnitude than we find in the set $\bar{\mathcal{E}}^{(i)}$.
If we do not find an imaginary propagation constant in the set $\bar{\mathcal{E}}^{(i)}$, we know the

magnitude of any imaginary propagation constant in $\bar{\mathcal{A}}^{(i)}$ can not be smaller than the magnitudes of the propagation constants of $\bar{\mathcal{E}}^{(r)}$.

Complex propagation constants

Because of (30), (31) we have to consider the two subsets of complex propagation constants $\bar{\mathcal{E}}^{*(c)}$ and $\bar{\mathcal{E}}^{**(c)}$.

$\bar{\mathcal{E}}^{*(c)}$: Let be $k_z^* = \alpha^*(\pm 1 - \jmath)$ (see (28c)) the propagation constant with the smallest magnitude in $\bar{\mathcal{E}}^{*(c)}$. We have $|\gamma^*| = 8(h\alpha^*)^2$ in this case. That means, complex propagation constants $k_{z_\sigma} \in \bar{\mathcal{A}}^{*(c)}$ with the property $|\alpha_\sigma| < |\alpha^*|$ belong to $\bar{\mathcal{E}}^{*(c)}$ rather than k_z^*.

$\bar{\mathcal{E}}^{**(c)}$: Let be $\gamma_1 \in \mathcal{E}^{**(c)}$ the complex eigenvalue with the largest magnitude and $k_{z_1} = \beta_1 + \jmath\alpha_1$ the corresponding propagation constant, and let be $\gamma_2 \in \mathcal{A}^{**(c)}$ an eigenvalue with the property

$$|\alpha_2| < |\alpha_1| \quad \text{and} \quad |\gamma_2| > |\gamma_1|. \tag{36}$$

If $\Re(\gamma_2) = -4[(h\beta_2)^2 - (h\alpha_2)^2] < 0$, we find the eigenvalue γ_2 because of $|\beta_2| > |\beta_1|$ (see (36)) in this case computing the set $\mathcal{E}^l$ of A^*.

If $\Re(\gamma_2) > 0$, we have

$$(\beta_1)^2 < (\beta_2)^2 < (\alpha_2)^2. \tag{37}$$

That means, the computed value $|\beta_1|$ is a lower bound of $|\alpha_2|$.

Numerical Results

Comparisons between the formerly used computation of all propagation constants and storage of the full matrix with our present method are given. The reduction of the computing times is demonstrated for the microstrip transmission line connecting the via hole shown in Figure 2. Because the structure has a symmetry plane, using appropriate boundary conditions it suffices to discretize its right-hand side only. This right-hand side is subdivided into $n_{xy} = n_x n_y$ elementary cells, $n_x = 33$, $n_y = 28$. The dimension of the eigenvalue problem is $2n_x n_y - n_b = 1668$. n_b is related by the boundary conditions. The total storage requirement is reduced by a factor of 20 in the new version for this example, since the sparse storage technique is applied. The reduction of the computing time essentially depends on the number of required eigenvalues and eigenvectors. The reduction of the computing time is represented in Figure 5 for the calculation of 4, 6, and 9 eigenvectors. The measured times involve matrix generation, solving the eigenvalue problem and computation of the mode fields. The examples were computed on a SUN SPARC Server 630.

9 Conclusions

The application of the finite-volume scheme to the boundary value problem of the Maxwellian equations which describes the electromagnetic properties of microwave transmission lines results in an eigenmode problem of high dimension. We avoid the time-consuming computation of all eigenvalues in order to calculate a selected set of propagation constants using an iterative method which is carried

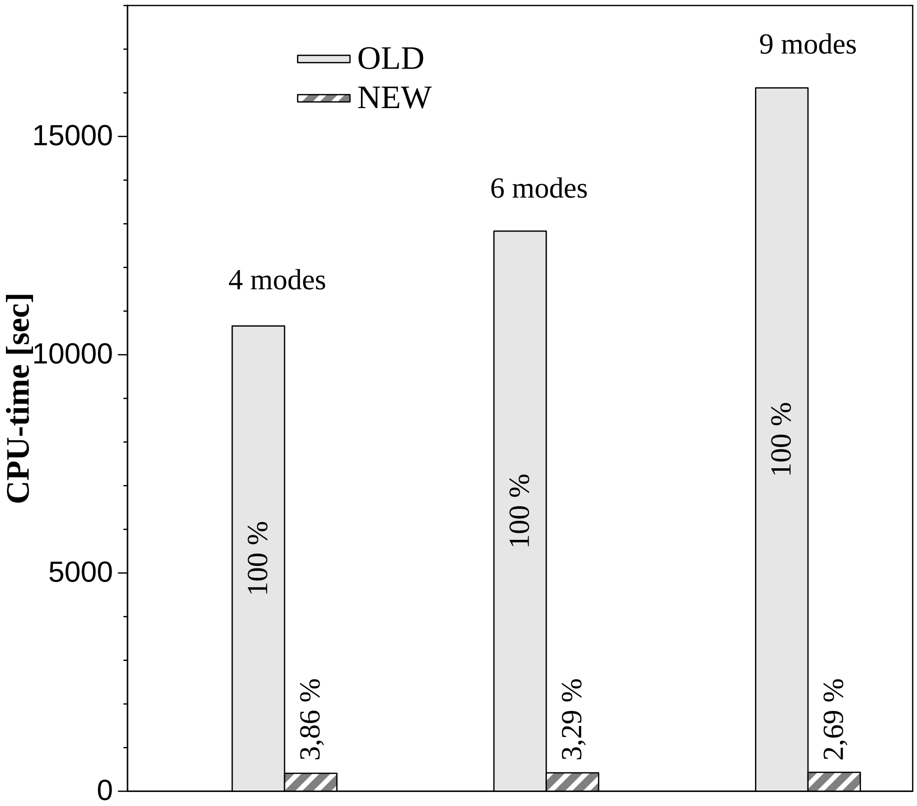

Figure 5: Computing times

out twice. The numerical effort and the storage requirements can be reduced considerably.

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