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Existence of subcritical percolation phases for generalised weight-dependent random connection models

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Existence of subcritical percolation phases for generalised weight-dependent random connection models

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Abstract

We derive a sufficient condition for the existence of a subcritical percolation phase for a wide range of continuum percolation models where each vertex is embedded into Euclidean space and carries an independent weight. In contrast to many established models, the presence of an edge is not only allowed to depend on the distance and weights of its end vertices but can also depend on the surrounding vertex set. Our result can be applied in particular to models combining heavy-tailed degree distributions and long-range effects, which are typically well connected. Moreover, we establish bounds on the tail-distribution of the number of points and the diameter of the subcritical component of a typical point. The proofs rest on a multi-scale argument.

1 Introduction

The standard objects studied in continuum percolation theory are random graphs $\mathscr{G}^\lambda$ on the points of a homogeneous Poisson point process on $\mathbb{R}^d$ of intensity $\lambda > 0$. The spatial embedding of the vertices enters the connection probability in a way that vertices at a short distance are likelier connected by an edge than far apart vertices. Many well established models belong to that framework, i.e., Gilbert's disc model [6], the random connection model [18, 20], the Poisson–Boolean model [14, 7] and its soft version [9], continuum scale-free percolation [3, 4] or the age-dependent random connection model [12]. The standard question in percolation theory then is whether there exists a critical Poisson intensity $\lambda_c \in (0, \infty)$ such that the connected component of the origin (added to the graph if necessary) is infinite with a positive probability for all $\lambda > \lambda_c$ but is finite almost surely for $\lambda < \lambda_c$. We call the regime $(0, \lambda_c)$ the *subcritical percolation phase* and (λ_c, ∞) the *supercritical percolation phase*. Often, by ergodicity of the underlying Poisson point process and the way edges are drawn, if $\lambda > \lambda_c$ then $\mathscr{G}^\lambda$ contains an infinite connected component almost surely and if $\lambda < \lambda_c$ there cannot be an infinite connected component somewhere in the graph. Moreover, under very mild assumptions on the distribution of $\mathscr{G}^\lambda$, an existing infinite component is almost surely unique [2].

In dimension $d \geq 2$ percolation models typically contain a supercritical phase [19] which is essentially a consequence of the existence of a supercritical percolation phase in nearest-neighbour Bernoulli percolation on $\mathbb{Z}^2$ [13]. Therefore, for $d \geq 2$, the proof of existence of a non-trivial phase-transition $\lambda_c \in (0, \infty)$ reduces to prove the existence of a subcritical phase. In this article we present sufficient conditions for the existence of a subcritical percolation phase in a quite general setting. Under these conditions we are able to exhibit estimates for the tail behavior of the distribution of the Euclidean diameter and the number of points of the component of a typical point in the subcritical regime.

In a recent paper, Gracar et al. introduce a new coefficient δ_{eff} which the authors use to identify whether one-dimensional percolation models contain a supercritical phase [12]. We show how this coefficient can be used to derive the existence of a subcritical phase in all dimensions by generalising arguments

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of Gouéré for the Poisson–Boolean model [7]. In [12], the coefficient δ_{eff} is derived for the *weight-dependent random connection model* [10]. In this class of models, containing all the aforementioned models, each vertex carries an independent mark. The connection mechanism is such that edges are drawn independently given the vertex locations and their marks. Additionally connections to spatially close vertices or vertices with small marks are preferred, where the first preference leads to clustering and the latter can be used to get heavy-tailed degree distributions. This is done in a way that clustering and degree-distribution are modelled independently, i.e., the degree distribution depends only on the way the vertex marks enter the connection probability whereas the strength of clustering is determined by the geometric restrictions alone. We build on their work but extend the setting to models where both effects are allowed to depend on each other. Moreover, we additionally allow the edges to depend on local neighbourhoods of their end vertices.

1.1 Framework

We consider graphs where the vertex set is given by a standard Poisson point process on $\mathbb{R}^d$ of intensity $\lambda > 0$. Each vertex carries an independent mark distributed uniformly on $(0, 1)$. We denote a vertex by $\mathbf{x} = (x, u_x)$ and refer to $x \in \mathbb{R}^d$ as the vertex's *location* and to $u_x \in (0, 1)$ as the vertex's *mark*. We denote the set of all marked vertices by $\mathcal{X} = \mathcal{X}^\lambda$ and remark that $\mathcal{X}$ is a standard Poisson point process on $\mathbb{R}^d \times (0, 1)$ of intensity $\lambda > 0$ [17]. We denote by $\mathbf{N} = \mathbf{N}(\mathbb{R}^d \times (0, 1))$ the set of all at most countably subsets of $\mathbb{R}^d \times (0, 1)$ so that $\mathcal{X}$ is a random element of $\mathbf{N}$. We denote its law and expectation by $\mathbb{P}^\lambda$ and $\mathbb{E}^\lambda$. Given $\mathcal{X}$, a pair of vertices $\mathbf{x} = (x, u_x), \mathbf{y} = (y, u_y) \in \mathcal{X}$ is connected by an edge with probability $\mathbf{p}(\mathbf{x}, \mathbf{y}, \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\}) = \mathbf{p}(\mathbf{y}, \mathbf{x}, \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\})$. That is, whether an edge is drawn depends not only on the potential end vertices of the edge but may also depend on all other vertices. We assume that $\mathbf{p}$ fulfills the following homogeneity condition when integrating with respect to the underlying Poisson process: For two deterministically given vertices $\mathbf{x}$ and $\mathbf{y}$, we have

$$\mathbb{E}^\lambda[\mathbf{p}(\mathbf{x}, \mathbf{y}, \mathcal{X})] = \mathbb{E}^\lambda[\varphi(u_x, u_y, |x - y|^d, \mathcal{X})], \tag{1}$$

where $\varphi : (0, 1) \times (0, 1) \times (0, \infty) \times \mathbf{N} \to [0, 1]$ is measurable. Note that the deterministically given vertices are no elements of the Poisson point process with probability one. We make the following assumptions on the function φ:

(i) φ is symmetric in the first two arguments and non-increasing in the first three arguments. As a result, connections to spatially close vertices or vertices with a small mark are preferred. We also assume that φ is translation invariant and isotropic jointly in the third and fourth argument.

(ii) The integral
$$\int_0^1 \int_0^1 \int_0^\infty \mathbb{E}^\lambda[\varphi(s, t, r, \mathcal{X})] \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}t$$
is finite. This then ensures that expected degrees are finite.

Condition (1) essentially says that the annealed probability of two given vertices being connected only depends on the given vertices' distance and marks. Indeed, since $\mathcal{X}$ is a homogeneous process and φ is translation invariant and isotropic jointly in the third and fourth argument, the Poisson point process and its influence on the connection probability looks in expectation everywhere the same.

We denote the resulting undirected graph by $\mathscr{G} = \mathscr{G}^\lambda$ and also write now $\mathbb{P}^\lambda$ and $\mathbb{E}^\lambda$ for the underlying probability measure and its expectation. We denote the event that $\mathbf{x}$ and $\mathbf{y}$ are connected by an edge by $\mathbf{x} \sim \mathbf{y}$ and that they belong to the same connected component by $\mathbf{x} \leftrightarrow \mathbf{y}$. For a given vertex we denote by $\mathscr{C}(\mathbf{x})$ the connected component of $\mathbf{x}$.

1.2 Main result

To formulate our main result, we work on the Palm version [17] of the model. That is, a distinguishable typical vertex $\mathbf{o} = (o, U_o)$ is placed at the origin, marked with an independent uniform random variable U_0 and then added to the vertex set. The graph $\mathscr{G}_o = \mathscr{G}_o^\lambda$ is then constructed as above but now with the additional vertex $\mathbf{o}$. We denote its law by $\mathbb{P}_o = \mathbb{P}_o^\lambda$. We define the random variables

$$\begin{aligned}&\mathscr{C} = \mathscr{C}(\mathbf{o}) := \{\mathbf{x} \in \mathcal{X} : \mathbf{o} \leftrightarrow \mathbf{x}\},\\ &\mathscr{M} = \mathscr{M}(\mathbf{o}) := \sup\{|x|^d : \mathbf{x} \in \mathscr{C}(\mathbf{o})\} \text{ and}\\ &\mathscr{N} = \mathscr{N}(\mathbf{o}) := \sharp\mathscr{C}(\mathbf{o}),\end{aligned}$$

where $\sharp A$ denotes the number of elements in a countable set A. To ensure the existence of a subcritical percolation phase, we rely on two features our graph has to provide: First, the number of 'long edges' should be sufficiently small which then yields that percolation must happen locally in some sense. Second, the influence of the whole vertex set on the connection mechanism should be driven by spatially close vertices only to ensure that local percolation in two balls at a large distance can be seen as independent. To measure the intensity of 'long edges' in $\mathscr{G}$, the key quantity is given by the following limit

$$-\lim_{\mu\downarrow 0}\liminf_{n\to\infty}\frac{\log\int_{n^{-1-\mu}}^1\int_{n^{-1-\mu}}^1 \mathbb{E}^\lambda[\varphi(s,t,n,\mathcal{X})]\mathrm{d}s\mathrm{d}t}{\log n}.$$

Choosing subsequences if needed, it is no loss of generality to assume in the following that this limit exists. We then write

$$\begin{aligned}\psi(\mu) := \psi(\mu,\varphi) &= \lim_{n\to\infty}\frac{\log\int_{n^{-1-\mu}}^1\int_{n^{-1-\mu}}^1 \mathbb{E}^\lambda[\varphi(s,t,n,\mathcal{X})]\mathrm{d}s\mathrm{d}t}{\log n} \quad \text{and}\\ \delta_{\text{eff}} := \delta_{\text{eff}}(\varphi) &= -\lim_{\mu\downarrow 0}\psi(\mu,\varphi) = -\inf_{\mu>0}\psi(\mu,\varphi)\end{aligned} \tag{2}$$

and call δ_{eff} the *effective decay exponent* (associated with φ). The effective decay exponent δ_{eff} quantifies the occurrence of 'long edges' in a way comparable to classical long-range percolation where each pair of vertices x and y is connected by an edge independently with probability proportional to $|x-y|^{-d\delta}$ for some $\delta > 1$. Indeed, in that scenario we have $\delta_{\text{eff}} = \delta$. For more background on δ_{eff}, we refer to [12].

Let us write $\mathcal{B}_r(x)$ for the open ball of radius r centered in x and $\mathcal{B}_r := \mathcal{B}_r(o)$. To specify local percolation and quantify the influence of far apart vertices on the connection mechanism, we need to introduce some notation. For measureable domains $D \subset \mathbb{R}^d$ and $I \subset (0,1)$ we write

$$\mathcal{X}(D \times I) = \{\mathbf{x} = (x, u_x) \in \mathcal{X} \colon x \in D, u_x \in I\}.$$

If $I = (0,1)$, we simply write $\mathcal{X}(D) = \mathcal{X}(D \times (0,1))$. Further, we denote by $\mathscr{C}_D(\mathbf{x})$ the connected component of $\mathbf{x}$ restricted to the vertices located in D. For a given location $x \in \mathbb{R}^d$ and $\alpha > 1$, we define the event

$$G_\alpha(x) = \big\{\exists \mathbf{y} \in \mathcal{X}(\mathcal{B}_{\alpha^{1/d}}(x)) : \mathscr{C}_{\mathcal{B}_{10\alpha^{1/d}}(x)}(\mathbf{y}) \not\subset \mathcal{X}(\mathcal{B}_{8\alpha^{1/d}}(x))\big\}. \tag{3}$$

That is, the vertex $\mathbf{y}$ located close to x reaches with a path a vertex located in the annulus $\mathcal{B}_{10\alpha^{1/d}}(x) \setminus \mathcal{B}_{8\alpha^{1/d}}$ without using vertices located outside $\mathcal{B}_{10\alpha^{1/d}}(x)$. We abbreviate $G_\alpha = G_\alpha(o)$. We say that $\mathscr{G}^\lambda$ is *mixing* if there exist $\zeta > 0$ and $C_{\text{mix}} > 0$ such that for all $\lambda > 0$ and all $|x| > 30\alpha^{1/d}$, we have

$$\big|\operatorname{Cov}\big(\mathbb{1}_{G_\alpha}, \mathbb{1}_{G_\alpha(x)}\big)\big| \le C_{\text{mix}}\,\lambda\alpha^{-\zeta}. \tag{4}$$

Note that, in the examples mentioned in the introduction, $\mathbf{p}(\mathbf{y},\mathbf{x},\mathcal{X}\setminus\{\mathbf{x},\mathbf{y}\})=\mathbf{p}(\mathbf{y},\mathbf{x})$ and hence all of them are mixing in our sense.

Theorem 1.1 (Existence of a subcritical phase)**.** *If $\delta_{\text{eff}}>2$ and $\mathscr{G}$ is mixing, then there exists a critical intensity $\lambda_c>0$ such that for all $\lambda<\lambda_c$*

$$\mathbb{P}_o^\lambda(\mathscr{N}<\infty)=1 \qquad \textit{and} \qquad \mathbb{P}_o^\lambda(\mathscr{M}<\infty)=1.$$

Let us remark that for the proof of Theorem 1.1 it suffices if the right-hand side in (4) is replaced by $\lambda g(\alpha)$ where $g(\alpha)$ tends to zero at an arbitrary speed. However, to derive bounds on the decay of $\mathbb{P}_o^\lambda(\mathscr{N}\geq y)$ and $\mathbb{P}_o^\lambda(\mathscr{M}\geq y)$ we need that the graph mixes fast enough.

Theorem 1.2 (Decay properties in the subcritical phase)**.** *Let $s>0$ and assume that $\psi(s+1)<-(s+3)$ as well as $\zeta>s+1$, then there exists $\lambda_c'>0$ such that for all $t<s$ and all $\lambda<\lambda_c'$,*

$$\int_1^\infty y^t\mathbb{P}_o(\mathscr{M}\geq y)\mathrm{d}y<\infty \qquad \textit{and} \qquad \int_1^\infty y^t\mathbb{P}_o(\mathscr{N}\geq y)\mathrm{d}y<\infty.$$

We present the proofs of both theorems in Section 2.

1.3 Examples

The weight-dependent random connection model This model was introduced in [10] and further studied in [11, 12, 9]. Here, edges are drawn conditionally independent given $\mathcal{X}$ and we have

$$\mathbf{p}(\mathbf{x},\mathbf{y},\mathcal{X}\setminus\{\mathbf{x},\mathbf{y}\})=\rho\big(\beta^{-1}g(u_x,u_y)|x-y|^d\big), \tag{5}$$

where $\rho:(0,\infty)\to[0,1]$ is an integrable and non increasing *profile function* and $g:(0,1)^2\to(0,\infty)$ is a non increasing *kernel function* which is symmetric in both arguments. Here, $\beta>0$ controls the edge intensity of the graph by scaling the vertices' distance in the connection probability. By the Poisson point process mapping theorem [17] it is no loss of generality to fix $\beta=1$ and only vary the Poisson intensity λ or doing the opposite and fixing $\lambda=1$ whilst varying β. Two types of profile functions have been established in the literature. The *long-range* profile function $\rho(x):=p(1\wedge|x|^{-\delta})$ for $\delta>1$ or the *short-range* profile function $\rho(x):=p\mathbb{1}\{0\leq x\leq 1\}$ for some $p\in(0,1]$. These profile functions together with the *interpolation-kernel*

$$g(s,t):=(s\wedge t)^\gamma(s\vee t)^{\gamma'},\ \text{for}\ \gamma\in[0,1),\gamma'\in[0,2-\gamma),$$

introduced in [12], represent many of the literature's model such as the Poisson–Boolean model and its soft version, scale-free percolation and the age-dependent random connection model, cf. Table 1.

The model (5) has also been studied under the name *geometric inhomogeneous random graphs* in a similar yet slightly different parametrisation in [16, 1, 23]. Since $\delta_{\text{eff}}>2$ in case that $\delta>2,\gamma<1-1/\delta$ and $\gamma'<1-\gamma$, there always exists a subcritical percolation phase in these cases, cf. Figure 1. The model (5) also shows that the question of non existence of a subcritical phase cannot be answered with δ_{eff} alone. Indeed, in [11] it is shown that for $\gamma<\delta/(\delta+1)$ and $\gamma'\leq 1-\gamma$ there always exists a subcritical phase for all $\delta>1$ which also includes parameter regimes with $\delta_{\text{eff}}\leq 2$.

Table 1: Various choices for γ, γ' and δ for the weight-dependent random connection model and the models they represent in the literature. Here, to shorten notation, $\delta = \infty$ represents models constructed with ρ being the indicator function

Parameters	**Names and references**
$\gamma = 0, \gamma' = 0, \delta = \infty$	random geometric graph, Gilbert's disc model [6]
$\gamma = 0, \gamma' = 0, \delta < \infty$	random connection model [18, 20],
	long-range percolation [21]
$\gamma > 0, \gamma' = 0, \delta = \infty$	Boolean model [14, 7], scale-free Gilbert graph [15]
$\gamma > 0, \gamma' = 0, \delta < \infty$	soft Boolean model [9]
$\gamma = 0, \gamma' > 1, \delta = \infty$	ultra-small scale-free geometric network [24]
$\gamma > 0, \gamma' = \gamma, \delta \leq \infty$	scale-free percolation [3, 4],
	geometric inhomogeneous random graphs [1]
$\gamma > 0, \gamma' = 1 - \gamma, \delta \leq \infty$	age-dependent random connection model [8]

Let us mention that, in order to get bounds for the decay of $\mathscr{M}$ and $\mathscr{N}$, we identify for $\delta > 2$ and $\gamma' > 1/\delta$ that

$$\delta_{\text{eff}} > s + 3 \quad \Leftrightarrow \quad \gamma < \frac{\delta(1-\gamma') - s - 1}{\delta}.$$

Since we also know from above that $\gamma < 1 - 1/\delta$ this inequality has a valid solution whenever $\gamma' > s/\delta$. This is particularly satisfied for all $s \leq 1$. If however $\gamma' < 1/\delta$, we infer for $\gamma > 1/\delta$ that

$$\delta_{\text{eff}} > s + 3 \quad \Leftrightarrow \quad \gamma < \frac{\delta - s - 2}{\delta},$$

which has a valid solution whenever $s \leq 1$. This case includes in particular the soft Boolean model ($\gamma' = 0$). Finally, if also $\gamma < 1/\delta$, then simply $\delta_{\text{eff}} = \delta$.

Soft Boolean model with local interference We also present an example of a mixing graph where the edge probabilities indeed dependent on the surrounding point cloud. The idea is to combine the soft Boolean model [9] with local inference and noise in the spirit of SINR percolation [5, 22]. To formulate the model let us denote, for a given vertex $\mathbf{y} = (y, s)$ and $\xi \geq 0$, the random variable

$$N^\xi(\mathbf{y}, \mathcal{X}) := \sharp\big\{\mathbf{z} \in \mathcal{X} : |z - y|^d \leq s^{-\xi}\big\}.$$

The graph $\mathscr{G} = \mathscr{G}^\lambda$ is then generated by connecting $\mathbf{x} = (x, t)$ and $\mathbf{y} = (y, s)$ with probability

$$\mathbf{p}(\mathbf{x}, \mathbf{y}, \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\}) = \mathbb{1}_{\{s<t\}} \frac{1 \wedge s^{-\gamma\delta}|x - y|^{-d\delta}}{1 + N^\xi(\mathbf{y}, \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\})} + \mathbb{1}_{\{s \geq t\}} \frac{1 \wedge t^{-\gamma\delta}|x - y|^{-d\delta}}{1 + N^\xi(\mathbf{x}, \mathcal{X} \setminus \{\mathbf{x}, \mathbf{y}\})}, \tag{6}$$

where again $\gamma \in (0, 1)$ and $\delta > 2$. Since $N^\xi_\lambda((y, s), \mathcal{X}) \overset{d}{\sim} N^\xi_\lambda((o, s), \mathcal{X})$, Condition (1) is satisfied with

$$\varphi(s, t, r, \mathcal{X}) = \mathbb{1}_{\{s<t\}} \frac{1 \wedge s^{-\gamma\delta} r^{-\delta}}{1 + N^\xi((o, s), \mathcal{X})} + \mathbb{1}_{\{s \geq t\}} \frac{1 \wedge t^{-\gamma\delta} r^{-\delta}}{1 + N^\xi((o, t), \mathcal{X})}.$$

Let us note that the model (6) is a combination of the soft Boolean model, a special instance of the weight-dependent random connection model above, with random interference coming from the vertices surrounding the vertices that are to be connected. More precisely, each vertex $\mathbf{y} = (y, s)$ has a sphere of influence of radius $s^{-\gamma/d}$ and a sphere of interference of radius $s^{-\xi/d}$. The mark s can be understood as an inverse attraction parameter and the smaller s the more attractive the vertex

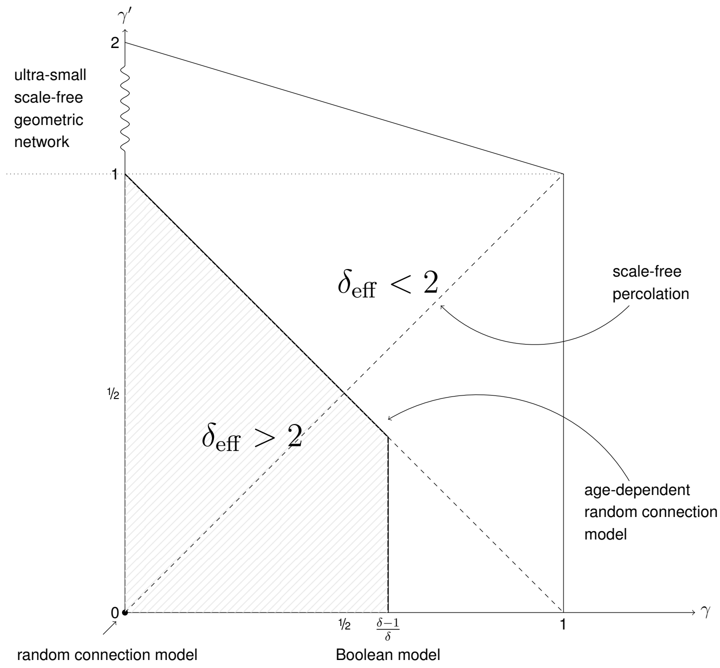

Figure 1: Phase diagram in γ and γ' for the weight-dependent random connection model constructed with the interpolation kernel and a profile function of polynomial decay at rate $\delta > 2$. The solid lines marks the phase transition $\delta_{\text{eff}} = 2$. Dashed lines represent no change of behaviour.

is. Note that either both spheres are big (for small s) or small (for large s). Now, the vertex $\mathbf{y}$ likes to connect to each vertex located within its sphere of influence, enlarged by an independent Pareto random variable for each candidate to include long-range effects (cf. the description of the soft Boolean model in [9]). However, $\mathbf{y}$ gets distracted by all vertices contained in its sphere of interference which makes it more difficult to form edges. For $\xi = 0$, each sphere of interference is of radius one and the model reduces to a version of the soft Boolean model with some additional yet on large scales insignificant fluctuations.

We start by showing that $\mathscr{G}$ is mixing for $\xi < 1$. To this end, observe that the left-hand side in (4) is bounded by some constant times the probability of the complement of the largest event on which the covariance is zero. Further observe that G_α and $G_\alpha(x)$ are independent whenever there is no pair of vertices in the involved balls such that their spheres of interference intersect. In other words, the covariance in (4) differs from zero when there are vertices $\mathbf{y} \in \mathcal{X}(\mathcal{B}_{10\alpha^{1/d}})$ and $\mathbf{z} \in \mathcal{X}(\mathcal{B}_{10\alpha^{1/d}}(x))$ such that

$$u_y^{-\xi} + u_z^{-\xi} \geq |z - y|^d.$$

We recall that $|x|^d \geq 30^d \alpha$ and hence $|z-y|^d \geq c\alpha$ for some constant c. Moreover, $u_y^{-\xi} + u_z^{-\xi} < 2(u_y^{-\xi} \vee u_z^{-\xi})$. Hence, the question reduces to the question whether there exists a vertex in one of the two balls with a mark smaller than $c\alpha^{-1/\xi}$. But, the expected number of such vertices and hence also the probability of existence of at least one such vertex is bounded by

$$C\lambda\alpha^{1-1/\xi}$$

for some constant C. Now, since $\xi < 1$, the graph is mixing with exponent $\zeta = 1/\xi - 1$. Let us further calculate some values of δ_{eff}, see Figure 2. We write $f \asymp g$ for positive functions if f/g is uniformly bounded from zero and infinity. Since $N^\xi((o,s),\mathcal{X})$ is Poisson distributed with parameter of order

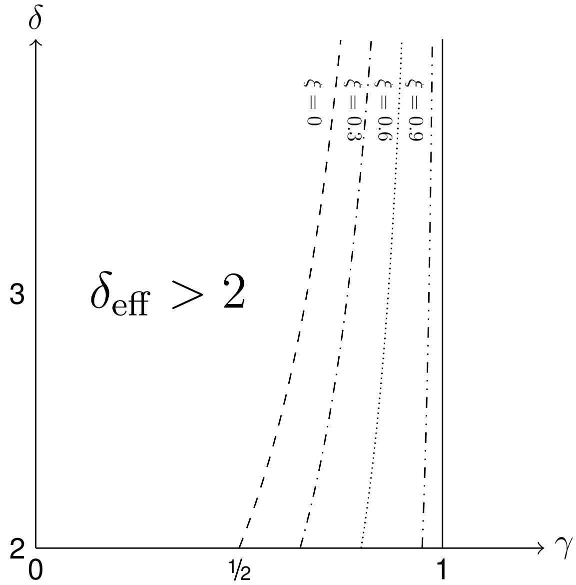

Figure 2: Phase diagram for γ and δ for the soft Boolean model with local interference. The dashed, dashed-dotted, dotted and dashed-double dotted lines represent the phase transition for $\delta_{\text{eff}} > 2$ for $\xi = 0, 0.3, 0.6$ and 0.9.

$s^{-\xi}$, we infer

$$n^{-\delta}\int_{1/n}^{1} s^{-\gamma\delta}\mathbb{E}^{\lambda}\Big[\frac{1}{1+N^{\xi}((o,s),\mathcal{X})}\Big]\mathrm{d}s \asymp n^{-\delta}\int_{1/n}^{1} s^{-\gamma\delta+\xi}\mathrm{d}s \asymp n^{-\delta} \vee n^{-\delta(1-\gamma)-1-\xi}.$$

Hence, if $\gamma < {}^{(1+\xi)}/_{\delta}$, then $\delta_{\text{eff}} = \delta$. This in particular always true if $\xi > \delta - 2$. In the case $\gamma > {}^{(1+\xi)}/_{\delta}$, we have

$$\delta_{\text{eff}} > 2 \quad \Leftrightarrow \quad \delta(1-\gamma)+\xi > 1 \quad \Leftrightarrow \quad \gamma < \frac{\delta+\xi-1}{\delta}.$$

For $\xi = 0$ we recover the bound for the soft Boolean model found in the previous paragraph. For $\xi > 0$ we observe that the local distraction indeed makes it harder to percolate. To get bounds on the tail distribution of $\mathscr{M}$ and $\mathscr{N}$ when $\delta_{\text{eff}} \neq \delta$, we observe for the mixing parameter

$$\frac{1}{\xi} - 1 > s+1 \quad \Leftrightarrow \quad \xi > \frac{1}{\delta+2}$$

and for $\gamma > {}^{(1+\xi)}/_{\delta}$, we infer

$$\delta_{\text{eff}} > s+3 \quad \Leftrightarrow \quad \gamma < \frac{\delta+\xi-s-2}{\delta},$$

which has a valid solution if the right-hand side is positive, which is equivalent to $s < \delta + \xi - 2$.

2 Proofs

We employ a multiscale argument similar to the one used for the Poisson–Boolean model in [7]. Recall the notation of

$$\mathcal{X}(D\times I) = \{\mathbf{x} = (x,u_x) \in \mathcal{X} \colon x \in D, u_x \in I\}$$

and $\mathcal{X}(D) = \mathcal{X}(D \times (0,1))$. Further recall the event $G_\alpha(x)$ and G_α introduced in (3). Let us write $A^c = \mathbb{R}^d \setminus A$, for the complement of any $A \subset \mathbb{R}^d$, and define two further events

$$H_\alpha = \{\exists \mathbf{x} \in \mathcal{X}(\mathcal{B}^c_{10\alpha^{1/d}}) \text{ and } \mathbf{y} \in \mathcal{X}(\mathcal{B}_{9\alpha^{1/d}})\colon \mathbf{x} \sim \mathbf{y}\} \quad \text{and}$$
$$F_\alpha = \{\exists \mathbf{y} \in \mathcal{X}(\mathcal{B}_{100\alpha^{1/d}}) \text{ and } \mathbf{x} \in \mathcal{X}\colon \mathbf{x} \sim \mathbf{y} \text{ and } |x-y|^d \geq \alpha\}.$$

We start by proving that both $\mathbb{P}(H_\alpha)$ and $\mathbb{P}(F_\alpha)$ tend to zero as $\alpha \to \infty$ whenever $\delta_{\text{eff}} > 2$.

Lemma 2.1.

(i) If $\delta_{\text{eff}} > 2$, then for all $\varepsilon > 0$ such that $\delta_{\text{eff}} - \varepsilon > 2$ there exists $\mu > 0$ and a constant C only depending on the dimension d and the choice of ε and μ such that

$$\mathbb{P}(H_\alpha) \leq C(\lambda \vee \lambda^2)\alpha^{-\eta}$$

with $\eta = (\delta_{\text{eff}} - 2 - \varepsilon) \wedge \mu$.

(ii) If $\delta_{\text{eff}} > 2$, then for all $\varepsilon > 0$ such that $\delta_{\text{eff}} - \varepsilon > 2$ there exists $\mu > 0$ and a constant C' only depending on the dimension d and the choice of ε and μ such that

$$\mathbb{P}(F_\alpha) \leq C'(\lambda \vee \lambda^2)\alpha^{-\eta}$$

with $\eta = (\delta_{\text{eff}} - 2 - \varepsilon) \wedge \mu$.

Proof. Define the event

$$\widetilde{H}_\alpha := \{\exists \mathbf{x} \in \mathcal{X}(\mathcal{B}^c_{10\alpha^{1/d}}) \text{ and } \mathbf{y} \in \mathcal{X}(\mathcal{B}_{9\alpha^{1/d}})\colon u_x \geq |x|^{-d(1+\mu)}, u_y \geq |x|^{-d(1+\mu)} \text{ and } \mathbf{x} \sim \mathbf{y}\}$$

and note that

$$H_\alpha \subset \widetilde{H}_\alpha \cup \{\exists \mathbf{x} \in \mathcal{X}(\mathcal{B}^c_{10\alpha^{1/d}})\colon |u_x| < |x|^{-d(1+\mu)}\} \cup \{\exists \mathbf{y} \in \mathcal{X}(\mathcal{B}_{9\alpha^{1/d}})\colon |u_y| < 10^d \alpha^{-(1+\mu)}\}.$$

By standard Poisson process properties, the last two events have probabilities of order

$$\lambda \int_{|x|^d > 10^d\alpha} \mathrm{d}x\, |x|^{-d(1+\mu)} = \tfrac{\pi(d)}{d\mu 10^\mu}\lambda\alpha^{-\mu} \qquad \text{and} \qquad \lambda \int_{|y|^d < 9^d\alpha} \mathrm{d}y\, (10^d\alpha)^{-(1+\mu)} = \tfrac{9\pi(d)}{10^{d(1+\mu)}}\lambda\alpha^{-\mu},$$

where $\pi(d)$ denotes the volume of the d-dimensional unit ball. For the event $\widetilde{H}_\alpha$, for any $\varepsilon > 0$ and all sufficiently large α and sufficiently small μ, we calculate using the Mecke-equation [17], (1) and the definition of δ_{eff},

$$\begin{aligned}\mathbb{P}(\widetilde{H}_\alpha) &\le \mathbb{E}^\lambda\Big[\sum_{\mathbf{y}\in\mathcal{X}(\mathcal{B}_{9\alpha^{1/d}})}\sum_{\mathbf{x}\in\mathcal{X}(\mathcal{B}^c_{10\alpha^{1/d}})}\mathbb{1}\{u_x,u_y>|x|^{-d(1+\mu)}\}\mathbf{p}(\mathbf{x},\mathbf{y},\mathcal{X}\setminus\{\mathbf{x},\mathbf{y}\})\Big]\\
&=\lambda^2\int\limits_{|y|^d<9^d\alpha}\mathrm{d}y\int\limits_{|x|^d>10^d\alpha}\mathrm{d}x\int\limits_{|x|^{-d(1+\mu)}}\mathrm{d}u_y\int\limits_{|x|^{-d(1+\mu)}}\mathrm{d}u_x\,\mathbb{E}^\lambda[\mathbf{p}(\mathbf{x},\mathbf{y},\mathcal{X})]\\
&=\lambda^2\int\limits_{|y|^d<9^d\alpha}\mathrm{d}y\int\limits_{|x|^d>10^d\alpha}\mathrm{d}x\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_y\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_x\mathbb{E}^\lambda\big[\varphi(u_x,u_y,|x-y|^d,\mathcal{X})\big]\\
&\le C\lambda^2\pi(d)\alpha\int\limits_{|x|^d>\alpha}\mathrm{d}x\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_y\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_x\,\mathbb{E}^\lambda[\varphi(u_x,u_y,|x|^d,\mathcal{X})]\\
&\le C\lambda^2\pi(d)\alpha\int\limits_{|x|^d>\alpha}|x|^{-d(\delta_{\mathrm{eff}}-\varepsilon)}\mathrm{d}x\\
&\le \tfrac{C\pi(d)^2}{\delta_{\mathrm{eff}}-\varepsilon-1}\lambda^2\alpha^{2-\delta_{\mathrm{eff}}+\varepsilon}.\end{aligned}$$

The proof of (ii) works analogous to the one of (i). We again use the same cut-off of the vertex marks and consider the coinciding event $\widetilde{F}_\alpha$ and have

$$\begin{aligned}\mathbb{P}(\widetilde{F}_\alpha) &\le \lambda^2\int\limits_{|y|^d<100^d\alpha}\mathrm{d}y\int\limits_{|x-y|^d>\alpha}\mathrm{d}x\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_y\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_x\,\mathbf{E}^\lambda[\varphi(s,t,|x-y|^d,\mathcal{X})]\\
&\le C\lambda^2\pi(d)\alpha\int\limits_{|x|^d>\alpha}\mathrm{d}x\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_y\int\limits_{|x|^{-d(1+\mu)}}^1\mathrm{d}u_x\,\mathbf{E}^\lambda[\varphi(s,t,|x|^d,\mathcal{X})]\\
&\le C'\lambda^2\alpha^{2-\delta_{\mathrm{eff}}+\varepsilon},\end{aligned}$$

as desired. □

Proof of Theorem 1.1. Note that, as in [7, Proposition 3.1], for the diameter we have that

$$\mathbb{P}_o(\mathscr{M}\ge 9^d\alpha)\le\mathbb{P}(G_\alpha)+\mathbb{P}(F_\alpha). \tag{7}$$

On the other hand, for the number of points, observe that

$$G_\alpha^c\cap H_\alpha^c\subset\{\mathscr{C}\subset\mathcal{X}(\mathcal{B}_{10\alpha^{1/d}})\}\subset\{\mathscr{N}\le\#\mathcal{X}(\mathcal{B}_{10\alpha^{1/d}})\}.$$

Moreover, by a standard Chernoff bound for the Poisson point process, there exists constants $c,c'>0$ such that

$$\mathbb{P}^\lambda(\sharp\mathcal{X}(B_{10\alpha^{1/d}})>c\lambda\alpha)\le e^{-c'\alpha/\lambda}$$

and hence

$$\begin{aligned}\mathbb{P}_o(\mathscr{N}>c\lambda\alpha)&\le\mathbb{P}_o(\mathscr{N}>\sharp\mathcal{X}(\mathcal{B}_{10\alpha^{1/d}}))+\mathbb{P}^\lambda(\sharp\mathcal{X}(B_{10\alpha^{1/d}})>c\lambda\alpha)\\
&\le\mathbb{P}(G_\alpha)+\mathbb{P}(H_\alpha)+e^{-c'\alpha/\lambda}.\end{aligned}\tag{8}$$

Hence, it suffices to prove that $\mathbb{P}(G_\alpha)$ tends to zero as α tends to infinity for all sufficiently small λ. From now on, we always assume that $\lambda < 1$ and therefore in particular $\lambda^2 < \lambda$. We write S_r for the sphere of radius r and define two finite sets $\mathcal{K}, \mathcal{L} \subset \mathbb{R}^d$ satisfying $\mathcal{K} \subset S_{10}$ and $\mathcal{L} \subset S_{80}$ as well as

$$S_{10} \subset \mathcal{K} + \mathcal{B}_1 \text{ and } S_{80} \subset \mathcal{L} + \mathcal{B}_1.$$

The key observation for the remaining proof is that

$$G_{10^d\alpha} \setminus F_\alpha \subset \Big(\bigcup_{k\in\mathcal{K}} G_\alpha(\alpha k)\Big) \cap \Big(\bigcup_{l\in\mathcal{L}} G_\alpha(\alpha l)\Big).$$

This is because on $G_{10^d\alpha} \setminus F_\alpha$ there exists a path from a vertex located in $\mathcal{B}_{10\alpha^{1/d}}$ to some vertex located in the annulus $\mathcal{B}_{100\alpha^{1/d}} \setminus \mathcal{B}_{80\alpha^{1/d}}$ using only vertices located in $\mathcal{B}_{100\alpha^{1/d}}$ and edges no longer than $\alpha^{1/d}$. Obviously, the sphere $S_{10\alpha^{1/d}}$ as well as the sphere $S_{80\alpha^{1/d}}$ are crossed by an edge. By the covering property of $\mathcal{K}$ and the fact that all edges are shorter than $\alpha^{1/d}$, one of the end vertices of the edge crossing $S_{10\alpha^{1/d}}$ is located in $\mathcal{B}_{\alpha^{1/d}}(\alpha k)$ for some $k \in \mathcal{K}$. Let's denote this vertex by $\mathbf{x}_k$. As the path needs to reach $\mathbf{x}_k$ and all edges are shorter than $\alpha^{1/d}$, the path also has to pass a vertex $\mathbf{x}_{\text{annul}}$ located in $\mathcal{B}_{10\alpha^{1/d}}(\alpha k) \setminus \mathcal{B}_{8\alpha^{1/d}}(\alpha k)$ that is connected by a path to $\mathbf{x}_k$ using only vertices located in $\mathcal{B}_{10\alpha^{1/d}}$. Put differently, $G_\alpha(\alpha k)$ occurs, see Figure 3. Using the same arguments, $G_\alpha(\alpha l)$ occurs for some $l \in \mathcal{L}$ and as a result,

$$\mathbb{P}(G_{10\alpha} \setminus F_\alpha) \le \sum_{k\in\mathcal{K},\, l\in\mathcal{L}} \mathbb{P}(G_\alpha(\alpha k) \cap G_\alpha(\alpha l)).$$

Therefore, there exists a constant C_1 depending on the choice of $\mathcal{K}$ and $\mathcal{L}$ such that

$$\mathbb{P}(G_{10\alpha}) \le C_1\, \mathbb{P}(G_\alpha)^2 + \mathbb{P}(F_\alpha) + C_1 \max_{k\in\mathcal{K},\, l\in\mathcal{L}} \Big(\mathbb{P}\big(G_\alpha(\alpha k) \cap G_\alpha(\alpha l)\big) - \mathbb{P}(G_\alpha)^2\Big).$$

Now, by the mixing assumption (4) and translation invariance, we have

$$\max_{k\in\mathcal{K},\, l\in\mathcal{L}} \Big(\mathbb{P}\big(G_\alpha(\alpha k) \cap G_\alpha(\alpha l)\big) - \mathbb{P}(G_\alpha)^2\Big) \le C_{\text{mix}}\lambda\alpha^{-\zeta}.$$

Combining this with Lemma 2.1, we find a constant C_2 such that

$$\mathbb{P}(G_{10\alpha}) \le C_2\mathbb{P}(G_\alpha)^2 + C_2\lambda\alpha^{-(\eta\wedge\zeta)}. \tag{9}$$

Let us define the functions $f(\alpha) = C_2\mathbb{P}(G_\alpha)$ and $g(\alpha) = C_2^2\lambda\alpha^{-(\eta\wedge\zeta)}$ and set

$$\lambda_0 = \frac{1}{2\cdot 10^d C_2 \pi(d)} \wedge \frac{1}{4C_2^2}.$$

Then, we have for all $\lambda < \lambda_0$ that

$$C_2\mathbb{P}(G_\alpha) \le C_2\mathbb{E}^\lambda[\sharp\mathcal{X}(\mathcal{B}_{10\alpha^{1/d}})] = 10^d C_2\pi(d)\lambda\alpha \le 1/2,$$

for all $\alpha \in [1, 10^d]$ as well as $g(\alpha) \le 1/4$ for all $\alpha \ge 1$. Moreover, by (9)

$$f(\alpha) \le f(\alpha/10^d)^2 + g(\alpha)$$

and therefore $\mathbb{P}(G_\alpha) \to 0$ as $\alpha \to \infty$ by [7, Lemma 3.7] as desired. □

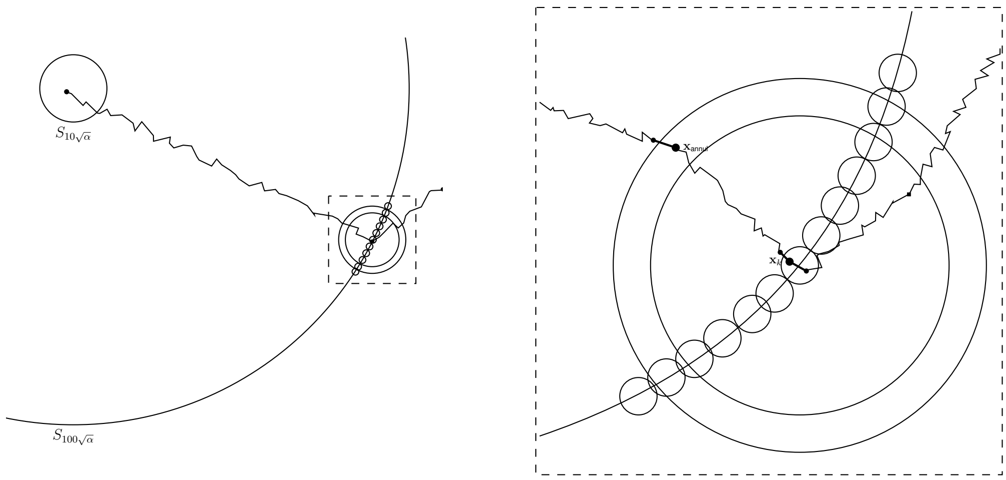

Figure 3: A path starting inside $\mathcal{B}_{10\alpha^{1/d}}$ leaving $\mathcal{B}_{100\alpha^{1/d}}$ using no edge longer than $\alpha^{1/d}$ as part of the event $G_{10\alpha} \setminus F_\alpha$. Enlarged the situation where the path crosses the sphere. In bold the edge entering the annulus $\mathcal{B}_{10\alpha^{1/d}}(\alpha k) \setminus \mathcal{B}_{8\alpha^{1/d}}(\alpha k)$, the edge entering $\mathcal{B}_{\alpha^{1/d}}(\alpha k)$ and the edge crossing the sphere $S_{100\alpha^{1/d}}$. Since all three edges are no longer than $\alpha^{1/d}$ they cannot 'jump over' the involved region. In the enlarged picture the event $G_\alpha(x_k)$ occurs.

Proof of Theorem 1.2. First note that, since $\mu \mapsto \psi(\mu)$ is increasing, we have

$$\delta_{\text{eff}} \geq -\psi(s+1) > s+3$$

and hence, for $\varepsilon = \delta_{\text{eff}} - 3 - s > 0$, it holds that $\delta_{\text{eff}} - 2 - \varepsilon = s+1$. Now, define

$$\mu(\varepsilon) := \sup\{\mu > 0\colon \psi(\mu) + \delta_{\text{eff}} \leq \varepsilon\}$$

and note that $\psi(s+1)+\delta_{\text{eff}} = \psi(s+1)+s+3+\varepsilon < \varepsilon$ and hence, by monotonicity, also $\mu(\epsilon) \geq s+1$. But then, $\eta = (\delta_{\text{eff}} - 2 - \varepsilon) \wedge \mu(\varepsilon) \wedge \zeta \geq s+1$. Now, recall the functions $f(\alpha) = C_2\mathbb{P}(G_\alpha)$ and $g(\alpha) = C_2^2\lambda\alpha^{-\eta}$ from the previous proof and fix $\lambda < \lambda_0$. By the choice of η we have for all $t < s$ that

$$\int_1^\infty \alpha^t g(\alpha)\mathrm{d}\alpha \leq C\int_1^\infty \alpha^{t-(s-1)}\mathrm{d}\alpha < \infty$$

and also

$$\int_1^\infty \alpha^t\big(\mathbb{P}(H_\alpha) + \mathbb{P}(F_\alpha)\big)\mathrm{d}\alpha < \infty.$$

From the integrability of g we derive by [7, Lemma 3.7]

$$\int_1^\infty \alpha^t f(\alpha)\mathrm{d}\alpha < \infty$$

and therefore $\int_1^\infty \alpha^t\mathbb{P}(G_\alpha) < \infty$. From (7) we conclude

$$\int_1^\infty \alpha^t\mathbb{P}_o(\mathscr{M} \geq \alpha)\mathrm{d}\alpha \leq C\int_1^\infty \alpha^t\big(\mathbb{P}(G_\alpha) + \mathbb{P}(F_\alpha)\big)\mathrm{d}\alpha < \infty.$$

For $\mathscr{N}$, we derive with (8)

$$\int_1^\infty \alpha^t\mathbb{P}_o(\mathscr{N} \geq \alpha)\mathrm{d}\alpha \leq C\lambda\int_1^\infty \alpha^t\big(\mathbb{P}(G_\alpha) + \mathbb{P}(H_\alpha) + e^{-c'\alpha/\lambda}\big)\mathrm{d}\alpha < \infty.$$

This finishes the proof. □

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