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Toyomu Matsuda, Willem van Zuijlen

Abstract

We construct random Schrödinger operators, called Anderson Hamiltonians, with Dirichlet and Neumann boundary conditions for a fairly general class of singular random potentials on bounded domains. Furthermore, we construct the integrated density of states of these Anderson Hamiltonians, and we relate the Lifschitz tails (the asymptotics of the left tails of the integrated density of states) to the left tails of the principal eigenvalues.

Contents

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1 Introduction

In this paper, we consider random Schrödinger operators of the form

$$-\Delta - \xi, \tag{1}$$

where $\Delta = \sum_{i=1}^{d} \partial_i^2$ is the Laplacian on $\mathbb{R}^d$ and ξ is a random potential. Such operators are also called *Anderson Hamiltonians*. We consider the construction of such operators for very irregular potentials ξ that do not need to be functions, hence there is –a priori– no obvious interpretation of (1). In fact, we consider a fairly general class of irregular potentials under the minimal assumption on the regularity of the potential ξ, which means that we assume that the regularity of ξ is $-2+\delta$ for some $\delta > 0$. Typical examples of potentials that are within this regularity regime include the *white noise*, namely the centered Gaussian field with delta correlation, in d-dimensions with $d \in \{1,2,3\}$. Another example is a Gaussian noise ξ whose covariance is formally given by

$$\mathbb{E}[\xi(x)\xi(y)] = c|x-y|^{-\alpha}, \quad c \in (0,\infty),\ \alpha \in (0, \min\{d,4\}).$$

We consider a bounded domain U and construct the Anderson Hamiltonian on U with both Dirichlet as well as Neumann boundary conditions. For the latter, besides that the domain needs also to be Lipschitz, we have to impose more restrictive assumptions on the potential. For example, these assumptions do not allow us to construct the Anderson Hamiltonian with Neumann boundary conditions for a white noise potential on a three dimensional domain. In order to construct this operator one expects – due to the work of Hairer and Gerencsér [29] for the parabolic Anderson model – the need to perform an additional renormalisation, but then only on the boundary.

After constructing the Anderson Hamiltonian it is natural to investigate its spectral properties. One of the most studied objects in the theory of random Schrödinger operators is the *integrated density of states* (IDS), see for example [18, Chapter VI] and [41] for overviews. The IDS is a nonrandom, increasing and right-continuous function on $\mathbb{R}$ and is often characterized as the vague limit of the normalized eigenvalue counting functions. The left tail asymptotics of the IDS are called *Lifschitz tails*. Relating the Lifschitz tails to the tail asymptotics of the principal eigenvalues is a classical result, see for example Kirsch and Martinelli [40] and Simon [59].

We construct the IDS of the Anderson Hamiltonian with a singular potential and we relate its left tail to those of the principal eigenvalues. In particular, by applying the work [36] by Hsu and Labbé, we derive the precise tail behaviour of the IDS for the white noise in d dimensions, for $d \in \{2,3\}$.

In Section 1.1 we discuss the history and references towards the Anderson Hamiltonians with irregular potentials. In Section 1.2 we state our assumptions for our main results, which are presented in Section 1.3. In Section 1.4 we discuss the strategies and techniques that we use to derive our results. In Section 1.5 we describe the outline of the rest of the paper and in Section 1.6 we give an overview of some notation that is used throughout the paper.

1.1 Literature on the Anderson Hamiltonian and its spectral properties

The mathematical study of Anderson Hamiltonians with singular potentials dates back to the work [28] by Fukushima and Nakao. They constructed the Anderson Hamiltonian with a white noise potential and with Dirichlet boundary conditions on the one dimensional domain $(-L, L)$, as the self-adjoint operator associated to the closed symmetric form on $H_0^1((-L, L))$, (formally) given by

$$(u, v) \mapsto \int_{(-L,L)} \nabla u \cdot \nabla v - \int_{(-L,L)} \xi uv.$$

For ξ being the white noise one has to make sense of the term $\int_{(-L,L)} \xi uv$. To do so, Fukushima and Nakao replaced it by

$$\int_{(-L,L)} (uv' + vu')B,$$

where B is the Brownian motion on $(-L, L)$ (as ξ is the derivative of B, this is an integration by parts identity). In general, as shown in Theorem 4.6 (a), for a bounded open set U in $\mathbb{R}^d$ and a potential V of regularity greater than -1, it is possible to make sense of

$$\int_U Vuv$$

for $u, v \in H_0^1(U)$. Therefore, in that case, one can construct the Anderson Hamiltonian by considering the associated symmetric form.

However, this approach fails to work if the regularity of ξ is below -1. The treatment of such singular ξ became possible only after the advent of the theory on *singular stochastic partial differential equations* (singular SPDEs), most notably the theory of *regularity structures* by Hairer [32] and the theory of *paracontrolled distributions* by Gubinelli, Imkeller and Perkowski [31].

Motivated by the theory of paracontrolled distributions, Allez and Chouk [2] constructed the Anderson Hamiltonian with white noise on the 2D torus as the limit of

$$-\Delta - \xi_\varepsilon + c_\varepsilon,$$

where ξ_ε is a regularized potential and c_ε is a suitably chosen constant. They obtained an explicit domain of the operator and its action. Subsequently, Gubinelli, Ugurcan and Zachhuber [30] constructed the Anderson Hamiltonian with white noise on the 2D and 3D torus and studied SPDEs whose linear part is given by (1). Chouk and van Zuijlen [20] constructed the Anderson Hamiltonian with white noise and with either Dirichlet or Neumann boundary conditions on 2D boxes. Mouzard [3] constructed the Anderson Hamiltonian with white noise on 2D compact manifolds, which can also be viewed as a generalisation of [2], based on the theory of higher order paracontrolled distributions [9]. Additionally, he proves a Weyl law for that Anderson Hamiltonian; we also prove this in Proposition 5.29. Ugurcan [66] constructed the Anderson Hamiltonian on $\mathbb{R}^2$ using the methods of paracontrolled distributions.

The works [20], [30], [20], [3] and [66] mentioned above use the techniques of the theory of paracontrolled distributions [31]. Labbé [43] used the theory of regularity structures to construct the Anderson

Hamiltonian with d-dimensional white noise ($d \leq 3$) on the box Q with Dirichlet or periodic boundary conditions. Instead of directly constructing the operator itself, he solved the resolvent equation

$$(a - \Delta - \xi)g = f, \quad f \in L^2(Q) \tag{2}$$

with Dirichlet boundary conditions for a large $a > 0$, and he defined the operator

$$G_a f := g, \quad \text{where } g \text{ solves (2).}$$

Then, the Anderson Hamiltonian with Dirichlet boundary conditions is defined as $G_a^{-1} - a$. Although this approach is robust, the construction is abstract and the domain of the operator is implicit.

Fukushima and Nakao [28] studied the integrated density of states (IDS) for the Anderson Hamiltonian with white noise potential on one dimensional intervals and derived the explicit formula that was predicted by physicists. The IDS for the Anderson Hamiltonian with white noise potential on two dimensional boxes was constructed by Matsuda in [47].

Besides the study of the IDS, quite related are the studies of the asymptotics of the eigenvalues. Chouk and van Zuijlen [20] showed the asymptotics of the eigenvalues in two dimensions for a white noise potential and Labbé and Hsu [36] extended this to three dimensions. Most recently, Bailleul, Dang and Mouzard [5] studied different properties of the Anderson Hamiltonian and its spectrum, for example the corresponding heat kernel and heat kernel estimates are studied, estimates of the norms of the eigenfunctions in terms of the size of their corresponding eigenvalues are given and a lower estimate on the spectral gap is given.

We remark that in one dimension with white noise, beyond the asymptotics of the eigenvalues and the study of the IDS, more is known about the spectrum properties.

Namely, McKean [48] showed that appropriately shifted and rescaled principal eigenvalues converge, as the segment size grows to infinity, to the Gumbel distribution in law. Cambronero and McKean [15] and Cambronero, Ramríez and Rider [16] derived precise tail asymptotics of the principal eigenvalue on the fixed torus. Dumaz and Labbé investigated the detailed statistics of the eigenvalues and the eigenfunctions in a series of works [25], [26] and [24]. No analogous results are known for singular potentials other than the white noise in one dimension (see the conjectures in the introduction of [36]).

1.2 Assumptions

The following will be assumed throughout the paper.

Assumption. We fix the dimension $d \in \mathbb{N} \setminus \{1\}$. We let $\Omega := \mathcal{S}'(\mathbb{R}^d)$, $\mathbb{P}$ a probability measure on the Borel-σ-algebra on Ω that is translation invariant. The random variable ξ is defined by $\xi(\omega) := \omega$. There exists a $\delta \in (0,1)$ such that for all $\sigma \in (0,\infty)$ one has $\mathbb{P}(\xi \in \mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)) = 1$. where $\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)$ is a weighted Besov-Hölder space, see Definition 2.2. A smooth, symmetric function $\rho \in \mathcal{S}(\mathbb{R}^d)$ with $\int \rho = 1$ is given and we set

$$\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x) \quad \text{and} \quad \xi_\varepsilon := \rho_\varepsilon * \xi.$$

Besides the above assumptions, the following three assumptions will appear in our main results.

- I Assumption I (see 3.10) guarantees the convergence of the BPHZ models associated to the regularity structure of the generalized parabolic Anderson model

$$\partial_t u = \Delta u + \sum_{i,j=1}^d g_{i,j}(u)\partial_i u \partial_j u + \sum_{i=1}^d h_i(u)\partial_i u + k(u) + f(u)\xi.$$

In principle, we assume this convergence to hold throughout the paper. However, since its precise formulation requires a labour, it will be stated at the end of Section 3.1.

II Assumption II (see 5.7), stated in Section 5.3, is necessary to construct the Neumann Anderson Hamiltonians by our approach (but not necessary for the Dirichlet Anderson Hamiltonian). This assumption allows the 2D white noise but excludes the 3D white noise.

III Assumption III (see 5.22), stated in Section 5.4, supposes that the probability measure $\mathbb{P}$ is ergodic under the action of translations. This assumption is very natural to construct the integrated density of states.

Remark 1.1. We do not allow d to be equal to one for the reason that in that case we are not guaranteed to have an admissible kernel for our green kernel G as mentioned in the beginning of Section 3.1.2. However, the use of the theory of regularity structures in this case is an overkill. Moreover, the case $d = 1$ has been well-studied (for a white noise potential).

1.3 Main results

Now we state our three main results of the paper. The first result concerns the construction of Anderson Hamiltonians on bounded domains, with Dirichlet boundary conditions. With a "domain" we mean an nonempty open subset of $\mathbb{R}^d$ (remember we assume $d \in \mathbb{N} \setminus \{1\}$).

Definition 1.2. Assume I (see Assumption 3.10). Let $\varepsilon > 0$ and c_ε be the constant defined as in (89).

(a) For a bounded domain U we define $\mathcal{H}_\varepsilon^{\mathrm{D},U}$ to be the self-adjoint operator on $L^2(U)$,

$$-\Delta - \xi_\varepsilon + c_\varepsilon \tag{3}$$

with Dirichlet boundary conditions.

(b) For a bounded Lipschitz domain U we define $\mathcal{H}_\varepsilon^{\mathrm{N},U}$ to be the self-adjoint operator (3) on $L^2(U)$ with Neumann boundary conditions.

Remark 1.3. Actually, in Section 5 we first define the operators $\mathcal{H}_\varepsilon^{\mathrm{D},U}$ and $\mathcal{H}_\varepsilon^{\mathrm{N},U}$ as those that correspond to symmetric forms given in terms of the stochastic terms that we introduce in Section 3. Then we show that these equal (3).

Definition 1.4. [54, Definition p. 284] Let $A, A_1, A_2, \ldots$ be self-adjoint operators on a Banach space $\mathfrak{X}$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ converges in *norm resolvent sense* and write

$$A_n \xrightarrow{\text{NR}}_{n\to\infty} A$$

if

$$\lim_{n\to\infty} \|(\mathrm{i} + A_n)^{-1} - (\mathrm{i} + A)^{-1}\|_{\mathfrak{X}\to\mathfrak{X}} = 0.$$

A sequence converges in norm resolvent sense if and only if the above convergence holds with "i" replaced by "λ" for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ [54, Theorem VIII.19].

Theorem 1.5 (Theorem 5.4)**.** *Assume I (see Assumption 3.10). Let U be a bounded domain. There exists a self-adjoint operator $\mathcal{H}^{\mathrm{D},U}$ on $L^2(U)$ such that*

$$\mathcal{H}_\varepsilon^{\mathrm{D},U} \xrightarrow{\text{NR}}_{\varepsilon\downarrow 0} \mathcal{H}^{\mathrm{D},U} \text{ in probability.}$$

Furthermore, each of the operators has a countable spectrum of eigenvalues and the eigenvalues of $\mathcal{H}_\varepsilon^{\mathrm{D},U}$ converge in probability to those of $\mathcal{H}^{\mathrm{D},U}$.

The limit $\mathcal{H}^{\mathrm{D},U}$ is independent of the mollifier ρ.

The second main result concerns Anderson Hamiltonians on bounded Lipschitz domains with Neumann boundary conditions.

Theorem 1.6 (Theorem 5.17)**.** *Assume I and II (see Assumptions 3.10 and 5.7). Let U be a bounded Lipschitz domain. There exists a self-adjoint operator $\mathcal{H}^{\mathrm{N},U}$ on $L^2(U)$ such that*

$$\mathcal{H}^{\mathrm{N},U}_\varepsilon \xrightarrow{\text{NR}}_{\varepsilon\downarrow 0} \mathcal{H}^{\mathrm{N},U} \text{ in probability.}$$

Furthermore, each of the operators has a countable spectrum of eigenvalues and the eigenvalues of $\mathcal{H}^{\mathrm{N},U}_\varepsilon$ converge in probability to those of $\mathcal{H}^{\mathrm{N},U}$.

The limit $\mathcal{H}^{\mathrm{N},U}$ is independent of the mollifier ρ.

Remark 1.7. The statement of Theorem 5.4 is actually slightly more general. Convergence in probability implies that there exists a subsequence and a set $\Omega_1 \subset \Omega$ of probability one such that the subsequence converges everywhere on Ω_1. For the convergence of the Dirichlet operators, this set Ω_1 can be chosen independently from the choice of bounded domain U.

The last main result concerns the integrated density of states (IDS) of Anderson Hamiltonians. For example, we show that the notion of the IDS for Anderson Hamiltonians with smooth potentials can be extended to irregular potentials.

For a bounded domain U and $L \in [1,\infty)$ we write $|U|$ for the Lebesgue measure of U and

$$U_L := LU = \{x \in \mathbb{R}^d \mid L^{-1}x \in U\}.$$

Recall that for the Anderson Hamiltonian with a smooth ergodic potential V the integrated density of states $\boldsymbol{N}_V$ is given by the right-continuous and increasing function $\mathbb{R} \to \mathbb{R}$ with $\lim_{\lambda\to-\infty} \boldsymbol{N}_V(\lambda) = 0$ for which, with $(\lambda_k^V(U))_{k\in\mathbb{N}}$ being the eigenvalues of $-\Delta - V$ with Dirichlet boundary conditions on U (counting multiplicities), for any bounded domain U and continuity point λ of $\boldsymbol{N}_V$,

$$\lim_{L\to\infty} \frac{1}{|U_L|} \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda_k^V(U_L)\le\lambda\}} = \boldsymbol{N}_V(\lambda).$$

Theorem 1.8. *Assume I and III (see Assumptions 3.10 and 5.22). There exists a right-continuous and increasing function $\boldsymbol{N} : \mathbb{R} \to \mathbb{R}$ with*

$$\lim_{\lambda\to-\infty} \boldsymbol{N}(\lambda) = 0,$$

such that the following holds:

(a) *For $(\lambda_k^{\mathrm{D}}(U))_{k\in\mathbb{N}}$ being the eigenvalues of $\mathcal{H}^{\mathrm{D},U}$ as in Theorem 1.5 (counting multiplicities), almost surely, one has for every bounded domain U and every continuity point λ of $\boldsymbol{N}$*

$$\lim_{L\to\infty} \frac{1}{|U_L|} \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda_k^{\mathrm{D}}(U_L)\le\lambda\}} = \boldsymbol{N}(\lambda).$$

($\boldsymbol{N}$ is called the integrated density of states of the Anderson Hamiltonian with potential ξ.)

(b) *Let $\boldsymbol{N}_\varepsilon$ be the integrated density of states of the Anderson Hamiltonian with potential $\xi_\varepsilon - c_\varepsilon$, where c_ε is the constant defined by (89).*
Then, $\boldsymbol{N}_\varepsilon$ converges vaguely to $\boldsymbol{N}$ (see Definition 5.33).

(c) *One has $\lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}(\lambda) = \frac{|B(0,1)|}{(2\pi)^d}$.*

(d) *For any bounded domain U and $\alpha \in (0,\infty)$, the following identities hold in $[-\infty,0]$:*

$$\limsup_{\lambda\to-\infty}(-\lambda)^{-\alpha}\log \boldsymbol{N}(\lambda) = \limsup_{\lambda\to-\infty}(-\lambda)^{-\alpha}\log \mathbb{P}(\lambda_1^{\mathrm{D}}(U)\le\lambda),$$
$$\liminf_{\lambda\to-\infty}(-\lambda)^{-\alpha}\log \boldsymbol{N}(\lambda) = \liminf_{\lambda\to-\infty}(-\lambda)^{-\alpha}\log \mathbb{P}(\lambda_1^{\mathrm{D}}(U)\le\lambda).$$

(e) *Assume furthermore II (see Assumption 5.7). For $(\lambda_k^{\mathrm{N}}(U))_{k\in\mathbb{N}}$ being the eigenvalues of $\mathcal{H}^{\mathrm{N},U}$ as in Theorem 1.6, for every bounded Lipschitz domain U and every continuity point λ of $\boldsymbol{N}$,*

$$\lim_{\substack{L\to\infty\\ L\in\mathbb{N}}}\frac{1}{|U_L|}\sum_{k\in\mathbb{N}}\mathbb{1}_{\{\lambda_k^{\mathrm{N}}(U_L)\le\lambda\}} = \boldsymbol{N}(\lambda), \quad \textit{almost surely.}$$

Proof. See Theorem 5.38, Theorem 5.41 and Theorem 5.42. □

With the above theorem in combination with [36] we obtain the precise tail behaviour of the IDS for the Anderson Hamiltonian with white noise potential in d dimensions.

Corollary 1.9. *Let $d\in\{2,3\}$ and ξ be the d-dimensional white noise. Then,*

$$\lim_{\lambda\to-\infty}(-\lambda)^{-\frac{4-d}{2}}\log \boldsymbol{N}(\lambda) = -\frac{8}{d^{d/2}(4-d)^{2-d/2}}\kappa_d^{-4},$$

where κ_d is the best constant of the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4(\mathbb{R}^d)} \le C\|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4}\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}.$$

Proof. This follows from Theorem 5.42 and [36, Theorem 2]. □

Remark 1.10. The case $d=1$ is of course known, see [28]. The case $d=2$ was proved in [47]. In physics literature, these tail behaviours have been already expected (e.g. [17], [12]).

1.4 About the strategies and techniques

Let us discuss our strategy of proving Theorem 1.5. The core idea is the following: instead of directly working on the operators themselves, we work on the symmetric forms associated to them. In fact, we are inspired by Gubinelli, Ugurcan and Zachhuber [30], where they figured out that the form domain of the Anderson Hamiltonian is quite simple on the torus with 2D or 3D white noise. To see how we can benefit from this idea, we observe the following elementary lemma (the proof follows by integration by parts, see also Lemma 4.5).

Lemma 1.11. *Let U be a bounded domain, $\zeta, w\in C^\infty(\bar U)$ and $u\in C_c^\infty(U)$. Set $u^\flat := e^{-w}u$. Then, one has*

$$\int_U (|\nabla u|^2-\zeta u^2) = \int_U e^{2w}|\nabla u^\flat|^2 - \int_U e^{2w}(\zeta+|\nabla w|^2+\Delta w)(u^\flat)^2. \quad (4)$$

As Lemma 1.11 suggests, to make sense of the symmetric form

$$(u,v)\mapsto \int_U (\nabla u\cdot\nabla v - \xi uv),$$

for the singular potential ξ, one hopes to find smooth functions w_ε such that the sequence of functions

$$e^{2w_\varepsilon}(\xi_\varepsilon + |\nabla w_\varepsilon|^2 + \Delta w_\varepsilon) \tag{5}$$

converges to some limit y of sufficient regularity such that $\langle yu^\flat, v^\flat \rangle$ makes sense for $u^\flat, v^\flat \in H_0^1(U)$. It turns out that this is possible, when we replace ξ_ε in (5) by $\xi_\varepsilon - c_\varepsilon$ for some constants c_ε that diverge as $\varepsilon \downarrow 0$. For a description on how to choose w_ε, see the discussion after Theorem 3.3.

As mentioned, our proof of Theorem 1.5 relies on the theory of symmetric forms. In particular, the work [42] by Kuwae and Shioya is important for us as it provides a correct notion of convergence of symmetric forms bounded below. Another key ingredient to the proof of Theorem 1.5 is the theory of regularity structures initiated by Hairer [32] and developed by Bruned, Hairer and Zambotti [13] and Chandra and Hairer [19]. Regarding this, see Section 3.

An important idea used in Lemma 1.11 is the exponential transformation. This is now a well-known technique in singular SPDEs. The most notable one is the Cole-Hopf transform of the KPZ equation as used by Bertini and Giacomin [10]. Hairer and Labbé [33] used the exponential transformation to simplify the 2D parabolic Anderson model. As already mentioned, Gubinelli, Ugrucan and Zachhuber [30] used it to construct the Anderson Hamiltonian with 2D or 3D white noise. Recently, Jagannath and Perkowski [38] applied it to simplify the construstruction of the dynamical Φ^4_3 model and Zachhuber [67] applied it to prove global well-posedness of multiplicative stochastic wave equations. A major drawback of the exponential transformation is the lack of robustness. For instance, it does not work if we replace the Laplacian with a fractional Laplacian.

The strategy of proving Theorem 1.6 in which the Neumann operator is considered is similar to that of Theorem 1.5 in which the Dirichlet operator is considered. However, as one has to deal with a boundary term in the Neumann setting (basically due to the integration by parts formula, see Lemma 4.5), we will have to impose an additional assumption.

Finally, let us discuss Theorem 1.8. There are two standard approaches to construct the IDS: the path integral approach [18, Section VI.1.2] and the functional analytic approach [18, Section VI.1.3]. In our framework, we cannot use the path integral approach. Indeed, it was shown in [47] that the 2D white noise is critical for this approach in that the Laplace transform of the IDS is finite only for small parameters. Therefore, if the regularity of the potential ξ is lower than that of the 2D white noise, we expect the blow-up of the Laplace transform of the IDS for any parameter. Hence, in this paper we adopt the functional analytic approach. This approach, introduced by Kirsch and Martinelli [40], is based on the super-(sub-)additivity of the Dirichlet (Neumann) eigenvalue counting functions and the ergodic theorem by Akcoglu and Krengel [1]. There is one significant problem in our situation. That is, without Assumption 5.7, we do not have Neumann Anderson Hamiltonians. To solve this problem, we introduce artificial Neumann Anderson Hamiltonians (see Definition 5.19). For this to be possible, it is crucial that we have a rather explicit representation of the symmetric form associated to the Anderson Hamiltonian. Many technical estimates here are inspired by Doi, Iwatsuka and Mine [23].

1.5 Outline

In Section 2, we introduce some notation related to the function spaces that we use. Technical estimates related to the objects introduced in Section 2 are postponed to Appendix A. In Section 3, we describe a theorem (Theorem 3.3) to construct some continuous functions W_N and some distributions Y_N that are required to define the symmetric forms associated to the Anderson Hamiltonians. We postpone the proof of this theorem to Appendix C: This is done because it requires the full-fledged theory of regularity structures [13], which will be reviewed in Appendix B. In Section 4, we cover some theory on (deterministic) symmetric forms that will be relevant to our problems. In Section 5, we give the definition of the Anderson Hamiltonians and prove the main theorems.

1.6 Notation

We set $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. We call a subset of $\mathbb{R}^d$ a *domain* if it is an open subset of $\mathbb{R}^d$. We denote by $\overline{U}$ the closure of a subset U of $\mathbb{R}^d$. Given a subset U of $\mathbb{R}^d$, $L \in (0, \infty)$ and $x \in \mathbb{R}^d$, we set

$$U_L := LU = \{y \in \mathbb{R}^d \mid L^{-1}y \in U\},$$

$d(x, U) := \inf\{|x - y| \quad | \; y \in U\}$, $B(U, R) := \{y \in \mathbb{R}^d \mid d(y, U) \leq R\}$ and $B(x, R) := B(\{x\}, R)$. We denote by $|U|$ the Lebesgue measure of a measurable set U.

We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions equipped with the locally convex topology generated by the Schwartz seminorms, and, by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, that is, the dual space of $\mathcal{S}(\mathbb{R}^d)$. We denote by $\mathrm{supp}(f)$ the support of a distribution or a continuous function f in $\mathbb{R}^d$. Let $k \in \mathbb{N} \cup \{\infty\}$. For a domain U, we write $C^k(U)$ for the k times continuously differentiable functions on U and $C_c^k(U)$ for those functions in $C^k(U)$ with compact support. For a closed set $V \subset \mathbb{R}^d$ (we will consider $\overline{U}$ and ∂U for domains U), we define

$$C^k(V) := \{f|_V : f \in C^k(\mathbb{R}^d)\}.$$

For a subset U of $\mathbb{R}^d$, either open or closed, we define

$$\|f\|_{C^k(U)} := \sup_{x \in U} \sum_{l \in \mathbb{N}_0^d : |l| \leq k} |\partial^k f(x)| \quad \text{if } k < \infty$$

We denote by $L^p(U)$, $p \in [1, \infty]$, the usual Lebesgue L^p-space on U. We denote by $\langle F, f \rangle$ the dual pairing of $F \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$ and the dual pairing of Besov spaces [57, Theorem 2.17]. We denote by $f * g$ the convolution of f and g. By duality, the convolution $f * g$ for $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{S}'(\mathbb{R}^d)$ is defined and represents a smooth function.

Let A, X be sets and $f, g : A \times X \to [0, \infty]$. We write $f(a, x) \lesssim_a g(a, x)$ if there exists a constant $C \in (0, \infty]$ (possibly) depending on a –for which we also write either $C = C(a)$ or $C = C_a$– such that $f(a, x) \leq Cg(a, x)$ for all x. We will not explicitly write the dependence on the dimension d, i.e., we write "$\lesssim_a$" instead of "$\lesssim_{d,a}$".

2 Function spaces

2.1 Besov spaces on $\mathbb{R}^d$

Here we describe definitions and important properties of Besov spaces on $\mathbb{R}^d$. Technical estimates related to Besov spaces will be given in Section A.1.

Definition 2.1. The *Fourier transform* of f is defined by

$$\mathcal{F}f(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot y} \, \mathrm{d}x$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. We define $\mathcal{F}f$ for $f \in \mathcal{S}'(\mathbb{R}^d)$ by duality: $\langle \mathcal{F}f, g \rangle := \langle f, \mathcal{F}g \rangle$ for $g \in \mathcal{S}(\mathbb{R}^d)$.

Definition 2.2. Let $\check{\chi}$, χ be smooth radial functions with values in $[0,1]$ on $\mathbb{R}^d$ with the following properties:

- $\operatorname{supp}(\check{\chi}) \subseteq B(0,\frac{4}{3})$, $\operatorname{supp}(\chi) \subseteq \{x \in \mathbb{R}^d \mid \frac{3}{4} \le |x| \le \frac{8}{3}\}$.
- $\check{\chi}(x) + \sum_{j=0}^{\infty} \chi(2^{-j}x) = 1$ for $x \in \mathbb{R}^d$ and $\sum_{j\in\mathbb{Z}} \chi(2^{-j}x) = 1$ for $x \in \mathbb{R}^d \setminus \{0\}$.

The existence of such $\check{\chi}$ and χ is guaranteed by [4, Proposition 2.10]. For $f \in \mathcal{S}'(\mathbb{R}^d)$ we set

$$\Delta_{-1} f = \mathcal{F}^{-1}(\check{\chi}\mathcal{F}f), \qquad \Delta_j f = \mathcal{F}^{-1}(\chi(2^{-j}\cdot)\mathcal{F}f), \qquad j \in \mathbb{N}_0.$$

Let $p,q \in [1,\infty]$ and $r \in \mathbb{R}$. For $\sigma \in \mathbb{R}$, we set

$$w_\sigma(x) := (1+|x|^2)^{-\frac{\sigma}{2}}.$$

The weighted nonhomogeneous Besov space $B^{r,\sigma}_{p,q}(\mathbb{R}^d)$ consists of those distributions f in $\mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)} < \infty$, where

$$\|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)} := \left\| \left(2^{-rj} \|w_\sigma \Delta_j f\|_{L^p(\mathbb{R}^d)} \right)_{j=-1}^{\infty} \right\|_{\ell^q}$$

Let us mention that the norm actually depends on the choice of $\check{\chi}$ and χ, though the space does not. See for example [4, Corollary 2.70]. [4, Lemma 2.69] implies that different choices of $\check{\chi}$ and χ as above give equivalent norms.

We set $\mathcal{C}^{r,\sigma}(\mathbb{R}^d) := B^{r,\sigma}_{\infty,\infty}(\mathbb{R}^d)$ and write $\mathcal{C}^r(\mathbb{R}^d) := \mathcal{C}^{r,0}(\mathbb{R}^d)$, $B^r_{p,q}(\mathbb{R}^d) = B^{r,0}_{p,q}(\mathbb{R}^d)$.

2.2 Sobolev–Slobodeckij spaces on bounded domains

Recall that a bounded domain U of $\mathbb{R}^d$ is called a *bounded Lipschitz domain* if its boundary can be locally approximated by Lipschitz functions (for the precise definition see [65, Definition 4.3]).

Definition 2.3. Let U be a domain in $\mathbb{R}^d$. Let $p \in [1,\infty]$ and $r \ge 0$.

(a) The space $W^r_p(U)$ is the completion of $\{f|_U \mid f \in C^\infty(\overline{U}), \|f|_U\|_{W^r_p(U)} < \infty\}$ with respect to the norm

$$\|f\|_{W^r_p(U)} := \sum_{\alpha\in\mathbb{N}_0^d, |\alpha|\le r} \|\partial^\alpha f\|_{L^p(U)} + \sum_{\alpha\in\mathbb{N}_0^d, |\alpha|=\lfloor r\rfloor} [\partial^\alpha f]_{W_p^{r-\lfloor r\rfloor}(U)},$$

where $[g]_{W^0_p(U)} := 0$ and for $s \in (0,1)$,

$$[g]_{W^s_p(U)} := \begin{cases} \left(\int_{U\times U} \frac{|g(x)-g(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} & p<\infty, \\ \sup_{x,y\in U, |x-y|\le 1} \frac{|g(x)-g(y)|}{|x-y|^s}, & p=\infty. \end{cases}$$

We set $H^r(U) := W^r_2(U)$. We denote by $W^r_{p,0}(U)$ the completion of $C^\infty_c(U)$ with respect to the norm $\|\cdot\|_{W^r_p(\mathbb{R}^d)}$ (not $\|\cdot\|_{W^r_p(U)}$) and we set $H^r_0(U) := W^r_{2,0}(U)$.

(b) Let U be a bounded Lipschitz domain and $r \in (0,1)$. The space $W^r_p(\partial U)$ is the completion of $C^\infty(\partial U)$ with respect to the norm

$$\|g|_{\partial U}\|_{W^r_p(\partial U)} := \|g\|_{L^p(\partial U)} + [g]_{W^r_p(\partial U)}$$

where

$$[g]_{W^r_p(\partial U)} := \begin{cases} \left(\int_{\partial U\times \partial U} \frac{|g(x)-g(y)|^p}{|x-y|^{d-1+pr}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} & p<\infty, \\ \sup_{x,y\in \partial U, |x-y|\le 1} \frac{|g(x)-g(y)|}{|x-y|^r} & p=\infty. \end{cases}$$

Remark 2.4 (Equivalent definitions)**.** For a bounded domain U, let $\tilde{W}_p^r(U)$ be the space of $f \in L^p(U)$ such that the distributional derivatives $\partial^\alpha f$ for $|\alpha| \le r$ are in $L^p(U)$ and $\|f\|_{W_p^r(U)} < \infty$.

Then $W_{p,0}^r(U)$ is the closure of $C_c^\infty(U)$ in $\tilde{W}_p^r(U)$ and if U is a bounded Lipschitz domain, then $W_p^r(U) = \tilde{W}_p^r(U)$, see for example [52, Theorem 1.2] or [50].

Definition 2.5. For U a domain in $\mathbb{R}^d$ and $r \ge 0$ we also write $C^r(U) = W_\infty^r(U)$ and $\|f\|_{C^r(U)} = \|f\|_{W_\infty^r(U)}$.

The following lemma relates the Sobolev–Slobodeckij spaces W_p^r (and C^r) for $U = \mathbb{R}^d$ to the Besov spaces.

Lemma 2.6. *Let $s \in (0,\infty)\setminus\mathbb{N}$ and $p \in [1,\infty]$. Then $W_p^s(\mathbb{R}^d) = B_{p,p}^s(\mathbb{R}^d)$, $C^s(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$ and the norms $\|\cdot\|_{W_p^s(\mathbb{R}^d)}$ and $\|\cdot\|_{B_{p,p}^s(\mathbb{R}^d)}$ are equivalent (hence $\|\cdot\|_{C^s(\mathbb{R}^d)}$ and $\|\cdot\|_{\mathcal{C}^s(\mathbb{R}^d)}$ are equivalent).*

Proof. This follows by [64, p.90]: For $p \in [1,\infty)$ one has $W_p^s(\mathbb{R}^d) = B_{p,p}^s(\mathbb{R}^d)$ with equivalent norms, see [64, p. 90 and p.113] ($W^{s,p}(\mathbb{R}^d)$ is written instead of $W_p^s(\mathbb{R}^d)$ and it is shown that $W_p^s(\mathbb{R}^d) = \Lambda_{p,p}^s(\mathbb{R}^d) = B_{p,p}^s(\mathbb{R}^d)$), for $C^s(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d) = B_{\infty,\infty}^s(\mathbb{R}^d)$ with equivalent norms, see [64, p.90 (9), (6) and p.113] (actually, in [64] $\mathcal{C}^s(\mathbb{R}^d)$ is defined differently but shown to be the same as $B_{\infty,\infty}^s(\mathbb{R}^d)$). □

Lemma 2.7. *Let U be a bounded Lipschitz domain.*

(a) *Set $\mathcal{D} := \cup_{p\in[1,\infty], r\in[0,\infty)} W_p^r(U)$. There exists an extension operator $\iota : \mathcal{D} \to \mathcal{S}'(\mathbb{R}^d)$ such that*

- $\iota(f) = f$ *as distributions on U for $f \in \mathcal{D}$,*
- $\|\iota(f)\|_{W_p^r(\mathbb{R}^d)} \lesssim_{U,p,r} \|f\|_{W_p^r(U)}$ *for every $p \in [1,\infty]$, $r \in [0,\infty)$ and $f \in \mathcal{D}$,*
- $\iota(f) \in C^\infty(\mathbb{R}^d)$ *for all $f \in C^\infty(\overline{U})$.*

(b) *Let $p \in (1,\infty)$ and $r \in (\frac{1}{p}, 1 + \frac{1}{p})$. Then, the map $C^\infty(\overline{U}) \to C^\infty(\partial U)$, $f \mapsto f|_{\partial U}$ extends uniquely to a bounded linear operator $\mathcal{T} = \mathcal{T}_{W_p^r(U)} : W_p^r(U) \to W_p^{r-\frac{1}{p}}(\partial U)$. Furthermore, there exists a bounded linear operator that is the right inverse of $\mathcal{T}$.*

Proof. For (a) see [61, Chapter 6] in combination with [63, Section 4], or, for $r \in [0,1)$, [22, Theorem 5.4]. For (b), see [45, Theorem 3]. □

Definition 2.8. An extension operator ι as in Lemma 2.7 (a) is called a *universal extension operator* from U to $\mathbb{R}^d$. The operator $\mathcal{T}$ as in Lemma 2.7 (b) is called the *trace operator*.

3 Stochastic terms for the Anderson Hamiltonian

As we motivated below Lemma 1.11, for a singular random potential ξ, in this section, see Theorem 3.3, we derive random functions W_ε and scalars c_ε such that

$$e^{2W_\varepsilon}(\xi_\varepsilon - c_\varepsilon + |\nabla W_\varepsilon|^2 + \Delta W_\varepsilon)$$

converges to some random Y of sufficient regularity.

In the rest of the section, i.e., in Section 3.1, we discuss the necessary definitions of the theory of regularity structures such that we can describe our main assumption: Assumption 3.10. The proof of Theorem 3.3 needs the full-fledged theory of regularity structures and is therefore postponed to Appendix C.

The W_ε are given in terms of a convolution with the Fourier cutoff of size N of the Green's function. We tune this N (randomly) in such a way that we can get desired bounds on W_ε. Therefore, we first introduce some notation on the Green's function and its Fourier cutoff of size N.

Let G be the Green's function of $-\Delta$ on $\mathbb{R}^d$ ($d \geq 2$), which means that $-\Delta G * f = f$ for $f \in \mathcal{S}(\mathbb{R}^d)$. That is, G is the distribution which is represented by the function defined for $x \neq 0$ by

$$G(x) = \begin{cases} \frac{1}{2\pi} \log|x|^{-1} & d = 2, \\ \frac{1}{d(d-2)\omega_d} |x|^{-(d-2)} & d \geq 3, \end{cases}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ (for $d \geq 3, G = \mathcal{F}^{-1}(|2\pi \cdot|^{-2})$).

Definition 3.1. Let $\check{\chi}$ be the function introduced in Definition 2.2. For $N \in \mathbb{N}_0$, we set

$$G_N := \mathcal{F}^{-1}((1 - \check{\chi}(2^{-N}\cdot))|2\pi\cdot|^{-2}).$$

Remark 3.2. Let $\phi \in C^\infty(\mathbb{R}^d)$ be such that $\phi\psi$ is a Schwartz function for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ (equivalently, ϕ and all its derivatives are of at most polynomial growth). If g is a tempered distribution, i.e., $g \in \mathcal{S}'(\mathbb{R}^d)$, then the product ϕg is defined by

$$\langle \phi g, \psi \rangle = \langle g, \phi\psi \rangle, \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

The function $\phi = (1 - \check{\chi}(2^{-N}\cdot))|\cdot|^{-2}$ is such a smooth function. Therefore we can define its Fourier multiplier, for which we use same notation as the convolution (as it generalises the convolution), i.e., we write $G_N * f = \mathcal{F}^{-1}((1 - \check{\chi}(2^{-N}\cdot))|\cdot|^{-2}\mathcal{F}f)$.

Theorem 3.3. *Under Assumption 3.10 stated below, for $\varepsilon \in (0,1)$ and $N \in \mathbb{N}_0$ there exist random variables X^ε and Y_N^ε in $C^\infty(\mathbb{R}^d)$, X in $\mathcal{S}'(\mathbb{R}^d)$ and Y_N in $\mathcal{S}'(\mathbb{R}^d)$ with the following properties:*

- *For every $\sigma \in (0,\infty)$, X is almost surely in $\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)$ and Y_N is almost surely in $\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)$, and for all $p \in [1,\infty)$*

$$\lim_{\varepsilon \downarrow 0} \|X^\varepsilon - X\|_{L^p(\mathbb{P}, \mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d))} = 0,$$
$$\lim_{\varepsilon \downarrow 0} \|Y_N^\varepsilon - Y_N\|_{L^p(\mathbb{P}, \mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d))} = 0, \qquad N \in \mathbb{N}.$$

Furthermore, for all $p \in [1,\infty)$, $X - \xi$ is an element of $L^p(\mathbb{P}, \mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d))$,

$$\lim_{\varepsilon \downarrow 0} \|X^\varepsilon - \xi_\varepsilon - (X - \xi)\|_{L^p(\mathbb{P}, \mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d))} = 0,$$

and there exists an integer $\mathfrak{b} = \mathfrak{b}(\delta) \in \mathbb{N}$, independent of σ and p such that

$$\mathfrak{a} := \mathfrak{a}(\delta,\sigma) := \sup_{N \in \mathbb{N}} 2^{-\mathfrak{b}N} \|Y_N\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)} \in L^p(\mathbb{P}). \tag{6}$$

The limits X and Y_N are independent of the mollifier ρ.

- *For $N \in \mathbb{N}_0$ set $W^\varepsilon_N := G_N * X^\varepsilon$ and $W_N := G_N * X$, where G_N is as in Definition 3.1. Let U be a bounded domain.*

  If M and ε are random variables (depending on U) with values in $\mathbb{N}_0$ and $(0,\infty)$, respectively, such that $\|W_M\|_{L^\infty(U)} \le 1$ and $\|W^\varepsilon_M - W_M\|_{L^\infty(U)} \le 1$ almost surely, then one has

  $$|\nabla W^\varepsilon_M|^2 + \Delta W^\varepsilon_M + e^{-2W^\varepsilon_M} Y^\varepsilon_M = -\xi_\varepsilon + c_\varepsilon \quad \text{on } U \text{ almost surely,}$$

  for some scalars c_ε that are defined in (89).

Proof. It follows from Proposition C.27 and Corollary C.36. □

Let us present the heuristic idea behind the derivation of W. For this purpose, we introduce a formal notion of degree $\deg$, which coincides with $|\cdot|_+$ given below. We set $\deg(\xi) = -2+\delta$, $\deg(\partial_i \tau) = \deg(\tau) - 1$, $(-\Delta)^{-1}\tau = \deg(\tau) + 2$ and $\deg(\tau_1 \cdot \tau_2) = \deg(\tau_1) + \deg(\tau_2)$. The degree $\deg(\tau)$ more or less reflects the regularity of τ.

Remember that we are going to construct a W such that $\xi + |\nabla W|^2 + \Delta W$ is sufficiently regular. Our strategy is to first neglect the $|\nabla W|^2$ term (as its degree $\deg(|\nabla W|^2)$ is greater than $\deg(\Delta W)$) and try to find a W such that ΔW compensates the irregularity of ξ. The most natural choice for this is $W = (-\Delta)^{-1}\xi$. Then,

$$\xi + |\nabla W|^2 + \Delta W = |\nabla(-\Delta)^{-1}\xi|^2 =: \tau_1.$$

Observe $\deg(\tau_1) = -2 + 2\delta$, which is greater than $\deg(\xi) = -2+\delta$. If the degree $-2+2\delta$ is too small to our taste, then we instead set

$$W := (-\Delta)^{-1}(\xi + \tau_1).$$

For this W we obtain

$$\xi + |\nabla W|^2 + \Delta W = \underbrace{2\nabla(-\Delta)^{-1}\xi \cdot \nabla(-\Delta)^{-1}\tau_1}_{\tau_2} + \underbrace{|\nabla(-\Delta)^{-1}\tau_1|^2}_{\tau_3},$$

where $\deg(\tau_2) = -2 + 3\delta$ and $\deg(\tau_3) = -2 + 4\delta$ are both greater than $-2+2\delta$. One can repeat this argument until one obtains a sum of terms for $\xi + |\nabla W|^2 + \Delta W$ such that each term has sufficiently large degree. (As Theorem 4.6 (a) shows, “sufficiently large” means that the degree is greater than -1.)

The above arguments are not yet mathematically rigorous, as for instance, the term $|\nabla(-\Delta)^{-1}\xi|^2$, that is the inner product of $\nabla(-\Delta)^{-1}\xi$ with itself, a priori does not make sense since $\nabla(-\Delta)^{-1}\xi$ is not a function in general. Moreover, it turns out that $|\nabla(-\Delta)^{-1}\xi_\varepsilon|^2$ itself does not converge as $\varepsilon \downarrow 0$, but if we take a “renormalization” of it, namely

$$|\nabla(-\Delta)^{-1}\xi_\varepsilon|^2 - \mathbb{E}[|\nabla(-\Delta)^{-1}\xi_\varepsilon|^2(0)],$$

then it does converge in probability. Then, we take the limit of it as our definition of τ_1 (instead of the nonrigorous definition $|\nabla(-\Delta)^{-1}\xi|^2$ above). The theory of regularity structures, which aims to solve singular stochastic partial differential equations, provides a correct framework for this operation of renormalization.

3.1 A brief discussion on regularity structures

The theory of regularity structures was first introduced by the seminal work [32] and developed by [13], [19] and [14]. It provides a general framework to solve singular stochastic partial differential equations. There are other theories to solve singular stochastic partial differential equations, most notably the theory of paracontrolled calculus [31], [9], [7], [8]. In this paper, we use a regularity structure $(A, \mathscr{T}, G)$ for the *generalized Parabolic Anderson model* (gPAM)

$$\partial_t u = \Delta u + \sum_{i,j=1}^{d} g_{i,j}(u)\partial_i u \partial_j u + \sum_{i=1}^{d} h_i(u)\partial_i u + k(u) + f(u)\xi,$$

as constructed in [13]. In what follows, we simply say that $\mathscr{T}$ is the regularity structure for the gPAM instead of the triplet. The vector space $\mathscr{T}$ is equipped with Hopf algebras

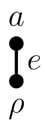
$$(\mathscr{T}_+, \mathscr{M}_{\mathscr{T}_+}, \mathbf{1}_+, \Delta_+, \mathbf{1}'_+, \mathscr{A}_+) \quad \text{and} \quad (\mathscr{T}_-, \mathscr{M}_{\mathscr{T}_-}, \mathbf{1}_-, \Delta_-, \mathbf{1}'_-, \mathscr{A}_-)$$

(written in the following order: vector space, product, unit, coproduct, counit and antipode) and coactions

$$\Delta_+^\circ : \mathscr{T} \to \mathscr{T} \otimes \mathscr{T}_+ \quad \text{and} \quad \Delta_-^\circ : \mathscr{T} \to \mathscr{T}_- \otimes \mathscr{T}.$$

For the precise definitions of them, see Definition B.26. In the terminology of [6], the pair $(\mathscr{T}, \mathscr{T}_+)$ is a concrete regularity structure and the pair $(\mathscr{T}, \mathscr{T}_-)$ is a renormalization structure.

A decorated forest is a 5-tuple $(F, \hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e})$ equipped with a type map $\mathfrak{t}$, see Definition B.3 for the precise definition. If F is a tree, we call it a decorated tree. We write $(F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}}$ for brevity. The vector space $\mathscr{T}$ has a canonical basis $\mathfrak{B}(\mathscr{T}) = \mathfrak{B}(H_\circ^R)$ (see Definition B.22), whose elements are decorated trees. The symbol Ξ, which represents the noise ξ, is identified with the decorated tree

$\circ :=$ 

(7)

with $\hat{F}(\rho) = \hat{F}(a) = \hat{F}(e) = \mathrm{N}(\rho) = \mathrm{N}(a) = \mathfrak{o}(\rho) = \mathfrak{o}(a) = \mathfrak{e}(e) = 0$ and $\mathfrak{t}(e) = \Xi$. This decorated tree belongs to $\mathfrak{B}(\mathscr{T})$. The polynomial X^k is identified with the decorated tree $\bullet$ with $\hat{F}(\bullet) = 0$, $\mathrm{N}(\bullet) = k$ and $\mathfrak{o}(\bullet) = 0$, which belongs to $\mathfrak{B}(\mathscr{T})$. We write $\mathbf{1} := X^0$.

A grading $|\cdot|_+$ (see Definition B.6) is assigned to each element τ of $\mathfrak{B}(\mathscr{T})$ and one has a grade decomposition $\mathscr{T} = \oplus_\beta \mathscr{T}_\beta$, where $\mathscr{T}_\beta$ is the subspace generated by $\{\tau \in \mathfrak{B}(\mathscr{T}) \mid |\tau|_+ = \beta\}$. In fact, the index set A of the regularity structure is identified with $\{|\tau|_+ \mid \tau \in \mathfrak{B}(\mathscr{T})\}$. We have $|\Xi|_+ = -2 + \delta$. One has integration operators $\mathscr{I} := \mathscr{I}_0, \mathscr{I}_k : \mathscr{T} \to \mathscr{T}$ for $k \in \mathbb{N}_0^d$ with $|k| = 1$ (see Definition B.5). We often write $\mathscr{I}_i := \mathscr{I}_{\boldsymbol{e}_i}$, where $\boldsymbol{e}_i$ is the ith unit vector of $\mathbb{R}^d$.

3.1.1 Models

Recall the notion of *models* $\mathscr{Z} = (\Pi, \Gamma)$ from [32, Definition 2.17]. In our situation, the scaling $\mathfrak{s}$ is uniform: $\mathfrak{s} = (1, 1, \dots, 1)$. We also need the functional $|||\cdot|||_{\gamma;\mathfrak{K}}$ and the pseudometric $|||\cdot;\cdot|||_{\gamma;\mathfrak{K}}$ from [32, (2.16) and (2.17)].

Definition 3.4. A smooth map $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ with $\operatorname{supp} K \subset B(0,1)$ is called an *admissible kernel* if it satisfies [32, Assumption 5.1] with $K(x,y) := K(x-y)$ and with $\beta = 2$ and if $\int_{\mathbb{R}^d} x^k K(x) dx = 0$ for $|k| \le 1$.

Definition 3.5 ([32, Definition 5.9])**.** Given an admissible kernel K, a model (Π,Γ) for $\mathscr{T}$ is said to *realize* K if one has

$$\Pi_x \mathscr{I}_k \tau = \partial^k K * \Pi_x \tau - \sum_{j\in\mathbb{N}_0^d : |\tau|_+ + 2 - |j| - |k| > 0} \frac{(\cdot - x)^j}{j!} [\partial^{k+j} K * \Pi_x \tau](x)$$

for every $\tau \in \mathfrak{B}(\mathscr{T})$, $k \in \mathbb{N}_0^d$ with $|k| \leq 1$ and $x \in \mathbb{R}^d$. The space $\bar{\mathscr{M}}(\mathscr{T}, K)$ of all K-admissible models is endowed with the topology induced by the collection of pseudometrics $(\|\!|\cdot\,;\cdot\|\!|_{\gamma,K})_{\gamma,K}$. In fact, the space $\bar{\mathscr{M}}(\mathscr{T}, K)$ is a complete metric space.

Definition 3.6 ([13, Definition 6.9])**.** We call a linear map $\mathbf{\Pi} : \mathscr{T} \to \mathcal{S}'(\mathbb{R}^d)$ a (ζ-)*realization* if

$$\mathbf{\Pi}\mathbf{1} = 1, \quad \mathbf{\Pi}\Xi = \zeta, \quad \mathbf{\Pi}(X^k\tau) = x^k \mathbf{\Pi}\tau \quad \text{for every } \tau \in \mathfrak{B}(\mathscr{T}).$$

A realization is called *smooth* if its image is a subset of $C^\infty(\mathbb{R}^d)$. Given an admissible kernel K, a realization $\mathbf{\Pi}$ is called K*-admissible* if it additionally satisfies

$$\mathbf{\Pi}\mathscr{I}_k(\tau) = \partial^k K * \tau \quad \text{for every } \tau \in \mathfrak{B}(\mathscr{T}) \text{ and } k \in \mathbb{N}_0^d \text{ with } |k| \leq 1.$$

Definition 3.7 ([13, Definition 6.8])**.** Let K be an admissible kernel. To a smooth K-admissible realization $\mathbf{\Pi}$, one can associate a model

$$\mathscr{Z}(\mathbf{\Pi}) := (\Pi, \Gamma)$$

realizing K as in [13, Definition 6.8]. We denote by $\mathscr{M}(\mathscr{T}, K)$ the closure in $\bar{\mathscr{M}}(\mathscr{T}, K)$ of

$$\{\mathscr{Z}(\mathbf{\Pi}) \mid \mathbf{\Pi} \text{ is a smooth } K\text{-admissible realization}\}.$$

Definition 3.8 ([13, Proposition 6.12])**.** A *(K-)canonical realization* $\mathbf{\Pi}^{\mathrm{can},\varepsilon}$ for ξ_ε is the smooth K-admissible ξ_ε-realization characterized by the identities

$$\mathbf{\Pi}^{\mathrm{can},\varepsilon}(\tau\sigma) = \mathbf{\Pi}^{\mathrm{can},\varepsilon}(\tau)\mathbf{\Pi}^{\mathrm{can},\varepsilon}(\sigma), \quad \mathbf{\Pi}^{\mathrm{can},\varepsilon}(\mathscr{R}_\alpha\tau) = \mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau,$$

where $\mathscr{R}_\alpha\tau$ is obtained from $\tau = (F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}}$ by resetting $\hat{F}(\rho_{\mathscr{R}\tau}) = 1$ and $\mathfrak{o}(\tau) = \alpha$. We set

$$\mathscr{Z}^{\mathrm{can},\varepsilon} := (\Pi^{\mathrm{can},\varepsilon}, \Gamma^{\mathrm{can}}) := \mathscr{Z}(\mathbf{\Pi}^{\mathrm{can},\varepsilon}).$$

3.1.2 BPHZ renormalization

In this section, we fix an admissible kernel $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ such that the function $K - G$ on $\mathbb{R}^d \setminus \{0\}$ extends to a smooth function on $\mathbb{R}^d$ and $K = K(-\cdot)$. The existence of such K is guaranteed by [32, Lemma 5.5]. All models below are supposed to realize this K.

In the situation of our interest, the model $\mathscr{Z}^{\mathrm{can},\varepsilon}$ does not converge as $\varepsilon \downarrow 0$. To obtain a limit, one has to "twist" the realization $\mathbf{\Pi}^{\mathrm{can},\varepsilon}$. This operation of twisting is called *renormalization*. The most natural renormalization is called the *BPHZ renormalization*, as introduced in [13].

Definition 3.9 ([13, Theorem 6.16])**.** The *BPHZ realization* $\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon}$ is a unique ξ_ε-realization characterized by the following properties:

- $\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon} = (g \otimes \mathbf{\Pi}^{\mathrm{can},\varepsilon})\Delta_-^\circ$ for some algebraic map $g : \mathscr{T}_- \to \mathbb{R}$;

- For every $\tau \in \mathscr{T}$ with $|\tau|_+ < 0$, one has $\mathbb{E}[\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon}\tau(0)] = 0$.

We set

$$\mathscr{Z}^{\mathrm{BPHZ},\varepsilon} := (\Pi^{\mathrm{BPHZ},\varepsilon}, \Gamma^{\mathrm{BPHZ},\varepsilon}) := \mathscr{Z}(\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon}).$$

Now we can state our important assumption of the noise ξ. Heuristically, it claims the convergence of $\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}$ in $L^p(\mathbb{P})$.

Assumption 3.10 (Assumption I)**.** As $\varepsilon \downarrow 0$, the family of models $(\mathscr{Z}^{\mathrm{BPHZ},\varepsilon})_{\varepsilon\in(0,1)}$ converges to some model $\mathscr{Z}^{\mathrm{BPHZ}} = (\Pi^{\mathrm{BPHZ}}, \Gamma^{\mathrm{BPHZ}})$, independent of the mollifier ρ, in $\mathscr{M}(\mathscr{T}, K)$ in probability. Furthermore, there exists a $\delta' \in (0,1)$ with the following property. For every $p \in 2\mathbb{N}$, there exist constants $C_p^{\mathrm{BPHZ}} \in (0,\infty)$ and a map $\boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}} : (0,1) \to (0,\infty)$ such that $\lim_{\varepsilon\downarrow 0} \boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}}(\varepsilon) = 0$ and the estimates

$$\mathbb{E}[|\langle \Pi_x^{\mathrm{BPHZ}}\tau, \phi_x^\lambda\rangle|^p] \le C_p^{\mathrm{BPHZ}} \lambda^{p(|\tau|_+ + \delta')},$$
$$\mathbb{E}[|\langle \Pi_x^{\mathrm{BPHZ}}\tau - \Pi_x^{\mathrm{BPHZ},\varepsilon}\tau, \phi_x^\lambda\rangle|^p] \le \boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}}(\varepsilon) \lambda^{p(|\tau|_+ + \delta')}$$

hold for all $x \in \mathbb{R}^d$, $\lambda \in (0,1)$, $\phi \in C^2(\mathbb{R}^d)$ with $\|\phi\|_{C^2(\mathbb{R}^d)} \le 1$ and with $\mathrm{supp}(\phi) \subseteq B(0,1)$ and $\tau = (T,0)_{\mathfrak{e}}^{\mathrm{N},0} \in \mathcal{T}$ with $|\tau|_+ < 0$. Here we write $\phi_x^\lambda := \lambda^{-d}\phi(\lambda^{-1}(\cdot - x))$.

Remark 3.11. The work [19], see especially Theorem 2.31 and Theorem 2.34 therein, gives conditions of the noise ξ under which Assumption 3.10 holds. It is worth observing that Assumption 3.10 holds for the 2D and the 3D white noise, and the Gaussian noise ξ whose covariance is formally given by

$$\mathbb{E}[\xi(x)\xi(y)] = \gamma(x-y),$$

where $\gamma : \mathbb{R}^d \setminus \{0\} \to [0,\infty)$ is smooth and bounded away from 0 and for some $\delta \in (0,1)$ we have

$$\sup_{\substack{k\in\mathbb{N}_0^d,\\ |k|\le 6d}} \sup_{x\in B(0,1)\setminus\{0\}} |\partial^k\gamma(x)||x|^{\min\{4,d\}-\delta+|k|} < \infty,$$

see [19, Theorem 2.15]. For example one could take γ to be given by

$$\gamma(x) = c|x|^{-\alpha}$$

for some $c \in (0,\infty)$ and $\alpha \in (0, \min\{d,4\})$.

4 Analysis of symmetric forms

It is common practice in the theory of rough paths [44] to first show the existence of sufficiently many stochastic objects and then apply deterministic analysis to derive results. In this section we consider the (deterministic) analysis of symmetric forms, which we use in Section 5 in combination with Theorem 3.3 to construct the Anderson Hamiltonian and derive its spectral properties.

First we recall the definition of a symmetric form and some related definitions in Definition 4.1, then we describe the symmetric forms $\mathcal{E}_{W,\mathcal{Z}}^U$ (in Definition 4.2) that we will study in Sections 4.2 and 4.3. We motivate the study of $\mathcal{E}_{W,\mathcal{Z}}^U$ from the viewpoint of the Anderson Hamiltonian, before we turn to the examples of bounded symmetric forms in Section 4.1, basic spectral properties of the symmetric forms and their associated self adjoint operators in Section 4.2 and finally consider estimates of eigenvalues in Section 4.3.

We recall some definitions of symmetric forms.

Definition 4.1. Let H be a Hilbert space over $\mathbb{R}$. A bilinear map $\mathcal{Q} : \mathcal{D}(\mathcal{Q})\times\mathcal{D}(\mathcal{Q}) \to \mathbb{R}$, with $\mathcal{D}(\mathcal{Q})$ a dense subspace of H, is called a *symmetric form* on H if $\mathcal{Q}(u, v) = \mathcal{Q}(v, u)$ for all $u, v \in \mathcal{D}(\mathcal{Q})$. Let $\mathcal{Q}$ be a symmetric form on H. We write

$$[\![\mathcal{Q}]\!]_H := \sup_{u\in\mathcal{D}(\mathcal{Q}),\|u\|_H=1} |\mathcal{Q}(u, u)|. \tag{8}$$

If $[\![\mathcal{Q}]\!]_H < \infty$, then we call $\mathcal{Q}$ a *bounded symmetric form*. In that case, without loss of generality we assume $\mathcal{D}(\mathcal{Q}) = H$. The set of bounded symmetric forms is a Banach space under the norm $[\![\cdot]\!]_H$. Then, a sequence $(\mathcal{Z}_n)_{n\in\mathbb{N}}$ of bounded symmetric forms converges to a bounded symmetric form $\mathcal{Z}$ if

$$[\![\mathcal{Z}_n - \mathcal{Z}]\!]_H \to 0.$$

Let $M > 0$. A symmetric form $\mathcal{Q}$ is called *M-bounded from below* if $\mathcal{Q}(u, u) + M\|u\|_H^2 \geq 0$ for all $u \in \mathcal{D}(\mathcal{Q})$. It is called *bounded from below* if it is M-bounded from below for some $M > 0$. If $\mathcal{Q}$ is M-bounded from below and $(\mathcal{D}(\mathcal{Q}), \mathcal{Q} + M\langle\cdot,\cdot\rangle_H)$ is a Hilbert space for some $M > 0$, then $\mathcal{Q}$ is said to be *closed*. If $\mathcal{Q}$ is a closed symmetric form and M is as above, then a subset of $\mathcal{D}(\mathcal{Q})$ is called a *core* for $\mathcal{Q}$ if it is dense in the Hilbert space $(\mathcal{D}(\mathcal{Q}), \mathcal{Q} + M\langle\cdot,\cdot\rangle_H)$.

Observe that a symmetric form is determined by its values on the diagonal of $H \times H$, i.e., $\mathcal{Q}(u, v) = \frac{1}{2}[\mathcal{Q}(u + v, u + v) - \mathcal{Q}(u, u) - \mathcal{Q}(v, v)]$. For this reason we often only define symmetric forms on the diagonal.

Definition 4.2. Let U be a bounded domain, $W \in L^\infty(U)$ and $\mathcal{Z}$ be a bounded symmetric form on $H^s(U)$ for some $s \in [0, 1)$. We define the symmetric form $\mathcal{E} = \mathcal{E}^U_{W,\mathcal{Z}}$ on $e^W H^1(U)$ as follows: for $u = e^W u^\flat$ with $u^\flat \in H^1(U)$, we set

$$\mathcal{E}(u, u) := \mathcal{E}^U_{W,\mathcal{Z}}(u, u) := \int_U e^{2W(x)}|\nabla u^\flat(x)|^2 \,\mathrm{d}x + \mathcal{Z}(u^\flat, u^\flat).$$

Let us turn to the motivation of studying the symmetric forms of the form $\mathcal{E}^U_{W,\mathcal{Z}}$ as above, by extending the motivation given below Lemma 1.11. As we also want to motivate the construction of the Anderson Hamiltonian with Neumann boundary conditions, in Lemma 4.5 we prove a generalisation of Lemma 1.11 that allows to consider smooth functions up to the boundary. For that we first recall an integration by parts formula on bounded Lipschitz domains.

Lemma 4.3 ([27, Theorem 5.6])**.** *Let U be a bounded Lipschitz domain. Then, the outer unit normal ν exists a.e. on ∂U and we have*

$$\int_U \nabla f(x) \cdot \nabla g(x) \,\mathrm{d}x = -\int_U f(x)\Delta g(x) \,\mathrm{d}x + \int_{\partial U} f(x)\nabla_\nu g(x) \,\mathrm{d}S(x) \tag{9}$$

for every $f \in H^1(U)$ and $g \in C^2(\overline{U})$, where S is the $(d-1)$-dimensional Lebesgue measure on ∂U and for $x \in \partial U$

$$\nabla_\nu g(x) := \lim_{h\to 0} \frac{g(x + h\nu(x)) - g(x)}{h}.$$

If U is not necessarily Lipschitz, then

$$\int_U \nabla f(x) \cdot \nabla g(x) \,\mathrm{d}x = -\int_U f(x)\Delta g(x) \,\mathrm{d}x$$

for every $f \in H^1(\mathbb{R}^d)$ and $\phi \in C^2_c(U)$.

Definition 4.4. Let U be a bounded Lipschitz domain and ν be the outer unit normal on ∂U. For measurable functions f and g such that g is continuously differentiable, and $f \nabla_\nu g$ is integrable on ∂U (with respect to S, see Lemma 4.3) we write

$$\int_{\partial U} f \nabla g \cdot \mathrm{d}\boldsymbol{S} := \int_{\partial U} f \nabla_\nu g \, \mathrm{d}S.$$

Lemma 4.5. *Let U be a bounded Lipschitz domain, $\zeta, w \in C^\infty(\bar{U})$ and $u \in C^\infty(\bar{U})$. Set $u^\flat := e^{-w}u$. Then, one has*

$$\begin{aligned}\int_U (|\nabla u|^2 - \zeta u^2) \\ &= \int_U e^{2w} |\nabla u^\flat|^2 - \int_U e^{2w}(\zeta + |\nabla w|^2 + \Delta w)(u^\flat)^2 + \int_{\partial U} e^{2w}(u^\flat)^2 \nabla w \cdot \mathrm{d}\boldsymbol{S}.\end{aligned}$$

Proof. One has

$$\int_U e^{2w}|\nabla u^\flat|^2 = \int_U e^{2w} |\nabla(e^{-w}u)|^2 = \int_U |\nabla w|^2 u^2 + \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla(u^2)$$

and by integration by parts (Lemma 4.3) one has

$$\int_U \nabla w \cdot \nabla(u^2) = \int_{\partial U} u^2 \nabla w \cdot \mathrm{d}\boldsymbol{S} - \int_U (\Delta w) u^2.$$ □

By this lemma (or Lemma 1.11) and by Theorem 3.3 (with the notation as therein) we see that the Anderson Hamiltonian with potential $\xi_\varepsilon - c_\varepsilon$,

$$-\Delta - \xi_\varepsilon + c_\varepsilon, \tag{10}$$

on U with Dirichlet boundary conditions, corresponds[1] to the symmetric form on $H_0^1(U)$ given by

$$\begin{aligned}\int_U |\nabla u|^2 - (\xi_\varepsilon - c_\varepsilon)u^2 &= \int_U e^{2W_M^\varepsilon} |\nabla u^\flat|^2 + \mathcal{Z}_M^\varepsilon(u^\flat, u^\flat), \quad u^\flat = e^{-W}u \\ &= \mathcal{E}^U_{W_M^\varepsilon, \mathcal{Z}_M^\varepsilon}(u,u),\end{aligned}$$

where, due to the identity

$$\xi_\varepsilon - c_\varepsilon + |\nabla W_M^\varepsilon|^2 + \Delta W_M^\varepsilon = -e^{-2W_M^\varepsilon} Y_M^\varepsilon,$$

the symmetric form $\mathcal{Z}_M^\varepsilon$ is given by

$$\mathcal{Z}_M^\varepsilon(v,v) = \int_U Y_M^\varepsilon v^2.$$

When we instead consider (10) on U with Neumann boundary conditions, we additionally have a boundary term, in the sense that the corresponding symmetric form on $H^1(U)$ is given by $\mathcal{E}^U_{W_M^\varepsilon, \mathcal{Z}_M^{\mathrm{N},\varepsilon}}$ where

$$\mathcal{Z}_M^{\mathrm{N},\varepsilon}(v,v) = \mathcal{Z}_M^\varepsilon(v,v) + \int_{\partial U} e^{2W_M^\varepsilon} v^2 \nabla W_M^\varepsilon \cdot \mathrm{d}\boldsymbol{S}.$$

[1] For a discussion on the correspondence between operators with Dirichlet (Neumann) boundary conditions and symmetric forms on H_0^s (on H^s), we refer to [21, Section 6 and 7].

The latter integral over the boundary ∂U will be decomposed into a few different terms. Let us recall from Theorem 3.3 that $W_M^\varepsilon = G_M * X^\varepsilon$. Then

$$\int_{\partial U} e^{2W_M^\varepsilon} v^2 \nabla W_M^\varepsilon \cdot \mathrm{d}\boldsymbol{S} = \tilde{\mathcal{Z}}_M^\varepsilon(v,v) + \widehat{\mathcal{Z}}_M^\varepsilon(v,v),$$

where

$$\begin{aligned}
\tilde{\mathcal{Z}}_M^\varepsilon(v,v) &= \int_{\partial U} e^{2W_M^\varepsilon} v^2 \nabla(G_0 * \xi_\varepsilon) \cdot \mathrm{d}\boldsymbol{S}, \\
\widehat{\mathcal{Z}}_M^\varepsilon(v,v) &= \int_{\partial U} e^{2W_M^\varepsilon} v^2 \nabla(G_M * (X^\varepsilon - \xi_\varepsilon)) \cdot \mathrm{d}\boldsymbol{S} \\
&\quad + \int_{\partial U} e^{2W_M^\varepsilon} v^2 \nabla((G_M - G_0) * \xi_\varepsilon) \cdot \mathrm{d}\boldsymbol{S}.
\end{aligned}$$

In order to study the convergence of the Anderson Hamiltonian with potential $\xi_\varepsilon - c_\varepsilon$ with Dirichlet and with Neumann boundary conditions in the resolvent sense, as we will show in the present section (see Lemma 4.15 and Theorem 4.17), it suffices to consider a certain continuity of $\mathcal{E}_{W,\mathcal{Z}}^U$ as a function of W and $\mathcal{Z}$.

Before we turn to that more general setting, let us elaborate on this a bit more for the limits we want to consider. By Theorem A.3, for all $\sigma \in (0,\infty)$ we have $\xi_\varepsilon \to \xi$ in $\mathcal{C}^{-2+\delta,\sigma}$ almost surely (as Assumption 3.10 actually guarantees that $\xi \in \mathcal{C}^{-2+\delta+\kappa,\sigma}$ for some $\kappa > 0$), and by Theorem 3.3 we have almost surely

$$\begin{aligned}
&Y_M^\varepsilon \to Y_M \text{ in } \mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d), \\
&X^\varepsilon \to X \text{ in } \mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d), \quad X^\varepsilon - \xi_\varepsilon \to X - \xi \text{ in } \mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d).
\end{aligned} \tag{11}$$

Therefore, by Corollary A.10 and Lemma A.6 we have the following convergences almost surely

$$W_M^\varepsilon \to W_M \quad \text{in } \mathcal{C}^{\delta,\sigma}(\mathbb{R}^d), \tag{12}$$

$$\nabla(G_0 * \xi_\varepsilon) \to \nabla(G_0 * \xi) \quad \text{in } \mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d), \tag{13}$$

$$\nabla(G_M * (X^\varepsilon - \xi_\varepsilon)) \to \nabla(G_M * (X - \xi)) \quad \text{in } \mathcal{C}^{-1+2\delta,\sigma}(\mathbb{R}^d), \tag{14}$$

$$\nabla((G_M - G_0) * \xi_\varepsilon) \to \nabla((G_M - G_0) * \xi) \quad \text{in } \mathcal{C}^{\delta,\sigma}(\mathbb{R}^d). \tag{15}$$

For the convergence of the Anderson Hamiltonian with potential $\xi_\varepsilon - c_\varepsilon$ on U with Dirichlet boundary conditions, (12) and (11) suffice (the latter convergence implies the convergence of the symmetric form $\mathcal{Z}_M^\varepsilon$).

If we instead consider Neumann boundary conditions, we additionally need $\tilde{\mathcal{Z}}_M^\varepsilon$ and $\widehat{\mathcal{Z}}_M^\varepsilon$ to converge. The convergence of $\widehat{\mathcal{Z}}_M^\varepsilon$ is guaranteed by (15) and by (14) if we assume $\delta > \frac{1}{2}$, as then both convergences imply convergence of the restricted function to ∂U in $L^\infty(\partial U)$. In order to deal with the convergence of $\tilde{\mathcal{Z}}_M^\varepsilon$, the convergence (13) does not seem to suffice, due to the integration over the $d-1$ dimensional boundary ∂U. Let us elaborate on this. We can write $\tilde{\mathcal{Z}}_M^\varepsilon = \langle \tilde{Y}_\varepsilon^U, e^{2W_M^\varepsilon} v^2 \rangle$, where $\tilde{Y}_\varepsilon^U$ is the distribution $\varphi \mapsto \int_{\partial U} \nabla(G_0 * \xi_\varepsilon) \varphi \, \mathrm{d}S$. The distribution $\tilde{Y}_\varepsilon^U$ could be formally interpreted as the product of $\nabla(G_0 * \xi_\varepsilon)$ with the distribution $\delta_{\partial U}$, given by $\varphi \mapsto \int_{\partial U} \varphi \, \mathrm{d}S$. The distribution $\delta_{\partial U}$ is of regularity -1 (e.g., for $F = [0,1]^{d-1} \times \{0\}$, δ_F is the tensor product of $\mathbb{1}_{[0,1]^{d-1}}$ and δ_0, which are in $\mathcal{C}^0(\mathbb{R}^{d-1})$ and $\mathcal{C}^{-1}(\mathbb{R})$, respectively; hence the tensor product is in $\mathcal{C}^{-1}(\mathbb{R}^d)$, see [58]). Hence the product of these two, and thus $\tilde{Y}_\varepsilon^U$, converges only in the space of regularity $-2+\delta$. Under additional assumptions (Assumptions 5.7) on ξ_ε we will basically show the latter for $\delta > \frac{1}{2}$. This turns out to be sufficient; and we do not need to have regularity > -1 but only $> -\frac{3}{2}$. This is due to the following identity

$$\tilde{\mathcal{Z}}_M^\varepsilon(v,v) = \langle \tilde{Y}_\varepsilon^U, \mathcal{R}(e^{2W_M^\varepsilon}\mathcal{T}(v)^2)\rangle.$$

The allowance of $\frac{1}{2}$ less/ regularity is due to the use of the trace operator $\mathcal{T}$ and the operator $\mathcal{R}$ (which is the composition of the right inverse trace operator and the extension operator); observe that $v \mapsto v^2$ forms a map $W_{2q}^\beta(U) \to W_q^{\beta-\varepsilon}(\mathbb{R}^d)$ whereas $v \mapsto \mathcal{R}[(\mathcal{T}v)^2]$ forms a map $W_{2q}^\beta(U) \to W_q^{\beta+\frac{1}{2q}-\varepsilon}(\mathbb{R}^d)$

4.1 Main examples of bounded symmetric forms

Recall the notation $[\![\cdot]\!]$ from (8) and the constants in Definition A.11.

Theorem 4.6. *Let $\delta \in (0,1)$, $\sigma \in (0,\infty)$ and $s \in (1-\delta, 1)$.*

(a) *Let $Y \in \mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)$. For any bounded domain U and $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi = 1$ on a neighborhood of U, the formula*

$$\mathcal{Z}_Y^U(v,v) = \langle \phi Y, \mathbb{1}_U v^2\rangle$$

defines a bounded symmetric form on $H_0^s(U)$ and if U is moreover Lipschitz, it also defines a bounded symmetric form on $H^s(U)$. The symmetric form $\mathcal{Z}_Y^U$ is independent of the choice of ϕ. Moreover, for $L \geq 1$

$$[\![\mathcal{Z}_Y^{U_L}]\!]_{H_0^s(U_L)} \lesssim_{\delta,\varepsilon,U} L^{2\sigma}\|Y\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}, \tag{16}$$

and if U is a bounded Lipschitz domain, then

$$[\![\mathcal{Z}_Y^{U_L}]\!]_{H^s(U_L)} \lesssim_{\delta,\varepsilon,U} L^{2\sigma}\|Y\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}. \tag{17}$$

(b) *Let U be a bounded Lipschitz domain. Suppose that $\delta \in (\frac{1}{2},1)$ and $s \in (\frac{3}{2}-\delta,1)$. Let $\varepsilon \in (0,\delta-\frac{1}{2})$, $p \in (2,\infty)$ and $q \in (1,2)$ be such that $\frac{1}{p}+\frac{1}{q} = 1$ and*

$$\beta := 2-\delta+\varepsilon-\frac{1}{q} \leq \frac{1}{2}, \qquad 2-\delta+\varepsilon-\frac{1}{2q}+\frac{d}{2p} \leq s. \tag{18}$$

Let $\tilde{Y} \in B_{p,p}^{-2+\delta}(\mathbb{R}^d)$ with $\operatorname{supp}(\tilde{Y}) \subseteq \partial U$, $\|\tilde{Y}_\varepsilon - \tilde{Y}\|_{B_{p,p}^{-2+\delta}(\mathbb{R}^d)} \xrightarrow{\varepsilon\downarrow 0} 0$ for some $\tilde{Y}_\varepsilon$ given by $\varphi \mapsto \int_{\partial U}\varphi f_\varepsilon \,\mathrm{d}S$ for $f_\varepsilon \in L^1(\partial U)$, and $V \in \mathcal{C}^\beta(U)$. Then, with $\mathcal{T} = \mathcal{T}_{W_{2q}^{\beta+\frac{1}{2q}}(U)}$, $\mathcal{R}$ a right inverse of $\mathcal{T}$ and ι a universal extension operator from U to $\mathbb{R}^d$ as in Lemma 2.7,

$$\tilde{\mathcal{Z}}(v,v) := \tilde{\mathcal{Z}}_{\tilde{Y},V}^U(v,v) := \langle \tilde{Y}, \iota\circ\mathcal{R}[V(\mathcal{T}v)^2]\rangle$$

defines a bounded symmetric form on $H^s(U)$ that is independent of the choice of $\mathcal{R}$ and ι.

If $\tilde{Y}_L \in B_{p,p}^{-2+\delta}(\mathbb{R}^d)$ with $\operatorname{supp}(\tilde{Y}_L) \subseteq \partial U_L$ and $V_L \in \mathcal{C}^\delta(U_L)$ for $L \geq 1$, then

$$[\![\tilde{\mathcal{Z}}_{\tilde{Y}_L,V_L}^{U_L}]\!]_{H^s(U_L)} \lesssim_{\delta,\varepsilon,p,U} L^{2\varepsilon}\|V_L\|_{C^\delta(U_L)}\|\tilde{Y}_L\|_{B_{p,p}^{-2+\delta}(\mathbb{R}^d)}. \tag{19}$$

(c) *Let U be a bounded Lipschitz domain. Suppose $\delta \in (0,1)$ and $s \in (\frac{1}{2},1)$. Let $\widehat{Y} \in \mathcal{C}^{1+\delta,\sigma}(\mathbb{R}^d)$ and $V \in \mathcal{C}^{\delta}(\mathbb{R}^d)$. Then, with $\mathcal{T} = \mathcal{T}_{H^s(U)}$ as in Lemma 2.7 (for the notation see Definition 4.4),*

$$\widehat{\mathcal{Z}}^{U}_{\widehat{Y},V}(v,v) := \int_{\partial U} V(\mathcal{T}v)^2 \nabla \widehat{Y} \cdot \mathrm{d}\boldsymbol{S}$$

defines a bounded symmetric form on $H^s(U)$ for every $s \in (\frac{1}{2},1)$. Moreover, for $L \geq 1$

$$[\![\widehat{\mathcal{Z}}^{U_L}_{\widehat{Y},V}]\!]_{H^s(U_L)} \lesssim_{\delta,\sigma,U} L^{\sigma} \|V\|_{C^{\delta}(U_L)} \|\widehat{Y}\|_{\mathcal{C}^{1+\delta,\sigma}(\mathbb{R}^d)}. \tag{20}$$

Proof. (a) Let us first consider U to be a bounded Lipschitz domain and $v \in H^s(U_L)$. We comment on how to obtain (16) afterwards. Let ϕ be as in the statement. It is rather straightforward to check that the definition of $\mathcal{Z}^U_Y$ does not depend on ϕ. Observe that therefore $\mathcal{Z}^{U_L}_Y(v,v) = \langle \phi(L^{-1}\cdot)Y, \mathbb{1}_U v^2\rangle$ for all $L > 0$. Choose a $p \in [1,\infty]$ and an $\varepsilon \in (0,\delta)$ such that $1-\delta+\varepsilon+\frac{d}{2p} \leq s$ and $p\sigma > d$. By the duality of Besov spaces [57, Theorem 2.17], we have

$$|\langle \phi(L^{-1}\cdot)Y, \mathbb{1}_{U_L} v^2\rangle| \lesssim_{\delta,p} \|\phi(L^{-1}\cdot)Y\|_{B^{-1+\delta}_{p,p}(\mathbb{R}^d)} \|\mathbb{1}_{U_L} v^2\|_{B^{1-\delta}_{q,q}(\mathbb{R}^d)}.$$

By Lemma A.4 and Lemma A.1 (more specifically, (62) using that $p\sigma > d$) we have

$$\|\phi(L^{-1}\cdot)Y\|_{B^{-1+\delta}_{p,p}(\mathbb{R}^d)} \lesssim_{p,\delta,\sigma,\phi} L^{2\sigma}\|Y\|_{B^{-1+\delta,2\sigma}_{p,p}(\mathbb{R}^d)} \lesssim_{p,\delta,\sigma} L^{2\sigma}\|Y\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}.$$

Then by Lemma 2.6 (see Definition A.11 for C_{Mult} and C_{Prod})

$$\begin{aligned}\|\mathbb{1}_{U_L} v^2\|_{B^{1-\delta}_{q,q}(\mathbb{R}^d)} &\lesssim_{\delta,p} \|\mathbb{1}_{U_L} v^2\|_{W^{1-\delta}_q(\mathbb{R}^d)} \leq C^{U_L}_{\mathrm{Mult}}[W^{1-\delta}_q]\|v^2\|_{W^{1-\delta}_q(U_L)} \\ &\leq C^{U_L}_{\mathrm{Mult}}[W^{1-\delta}_q] C^{U_L}_{\mathrm{Prod}}[W^{1-\delta+\varepsilon}_{2q} \to W^{1-\delta}_q]\|v\|^2_{W^{1-\delta+\varepsilon}_{2q}(U_L)}.\end{aligned}$$

Now we apply the embedding estimate (see Definition A.11 for C_{Embed}) and the estimate $\|u\|_{H^{1-\delta+\varepsilon+\frac{d}{2p}}(U_L)} \lesssim_{\delta,\varepsilon,p} \|u\|_{H^s(U_L)}$ (as $1-\delta+\varepsilon+\frac{d}{2p} \leq s$), we have

$$\|v\|_{W^{1-\delta+\varepsilon}_{2q}(U_L)} \lesssim_{\delta,\varepsilon,p} C^{U_L}_{\mathrm{Embed}}[H^{1-\delta+\varepsilon+\frac{d}{2p}} \to W^{1-\delta+\varepsilon}_{2q}]\|v\|_{H^s(U_L)}.$$

Hence

$$\begin{aligned}[\![\mathcal{Z}]\!]_{H^s(U_L)} \lesssim_{\alpha,\varepsilon,p,\sigma} L^{2\sigma} C^{U_L}_{\mathrm{Mult}}[W^{1-\alpha}_q] C^{U_L}_{\mathrm{Prod}}[H^{1-\alpha+\varepsilon} \to W^{1-\alpha}_1] \\ \times C^{U_L}_{\mathrm{Embed}}[H^s \to W^{1-\alpha+\varepsilon}_{2q}]\|Y\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}.\end{aligned}$$

Therefore (17) follows by Lemma A.15. If U is not necessarily Lipschitz but $v \in H^s_0(U)$, in the above estimates, we can replace the constant $C^U_{\mathrm{Mult}}[W^{1-\delta}_q]$ by 1 and the estimate (16) follows similarly.

(b) First, observe that the requirements for the existence of $\mathcal{T}$ and $\mathcal{R}$ as in Lemma 2.7 are satisfied. Indeed, $\beta + \frac{1}{2q} \in (\frac{1}{2q}, 1+\frac{1}{2q})$, or equivalently, $\beta \in (0,1)$, and $2-\delta \in (\frac{1}{q}, 1+\frac{1}{q})$: On the one hand we have $2-\delta \in (1,\frac{3}{2}) \subset (\frac{1}{q}, 1+\frac{1}{q})$ and because $\varepsilon \in (0, \delta-\frac{1}{2})$ and $\frac{1}{q} \in (\frac{1}{2},1)$, on the other hand we have $\beta = 2-\delta+\varepsilon-\frac{1}{q} < 2-\frac{1}{2}-\frac{1}{2} = 1$ and $\beta > 2-\delta-1 = 1-\delta > 0$.

Again by the duality, we have

$$|\langle \tilde{Y}, \iota \circ \mathcal{R}[V(\mathcal{T}v)^2]\rangle| \lesssim_{\delta,p} \|\tilde{Y}\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)} \|\iota \circ \mathcal{R}[V(\mathcal{T}v)^2]\|_{B^{2-\delta}_{q,q}(\mathbb{R}^d)}. \tag{21}$$

Again, by using Lemma 2.6,

$$\|\iota \circ \mathcal{R}[V(\mathcal{T}v)^2]\|_{B^{2-\delta}_{q,q}(\mathbb{R}^d)} \lesssim_{\delta,p} \|\iota\|_{W^{2-\delta}_q(U)\to W^{2-\delta}_q(\mathbb{R}^d)} \|\mathcal{R}\| \|V(\mathcal{T}v)^2\|_{W^{\beta-\varepsilon}_q(\partial U)}. \tag{22}$$

Now we estimate $\|V(\mathcal{T}v)^2\|_{W^{\beta-\varepsilon}_q(\partial U)}$. Recall the notation $[\cdot]$ from Definition 2.3 (b). If we set $\psi := (\mathcal{T}v)^2$, then

$$\begin{aligned}
[V\psi]_{W^{\beta-\varepsilon}_q(\partial U)} &\le \Big(\int_{\partial U}\int_{\partial U} \frac{|V(x)-V(y)|^q|\psi(x)|^q}{|x-y|^{d-1+q(\beta-\varepsilon)}} \,\mathrm{d}S(x)\,\mathrm{d}S(y)\Big)^{\frac{1}{q}} \\
&\quad + \Big(\int_{\partial U}\int_{\partial U} \frac{|\psi(x)-\psi(y)|^q|V(x)|^q}{|x-y|^{d-1+q(\beta-\varepsilon)}} \,\mathrm{d}S(x)\,\mathrm{d}S(y)\Big)^{\frac{1}{q}} \\
&\lesssim_\beta \|V\|_{W^{\beta}_\infty(U)} \Big(\int_{\partial U}\int_{\partial U} \frac{|\psi(x)|^q}{|x-y|^{d-1-q\varepsilon}} \,\mathrm{d}S(x)\,\mathrm{d}S(y)\Big)^{\frac{1}{q}} + \|V\|_{L^\infty(U)}[\psi]_{W^{\beta-\varepsilon}_q(\partial U)} \\
&\lesssim_{\beta,\delta} \|V\|_{\mathcal{C}^\delta(\mathbb{R}^d)} \Big(1 + \sup_{x\in\partial U}\int_{\partial U} \frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}}\Big)^{\frac{1}{q}} \|\psi\|_{W^{\beta-\varepsilon}_q(\partial U)},
\end{aligned}$$

where we used $\|V\|_{W^\beta_\infty(U)} \vee \|V\|_{L^\infty(U)} \lesssim_{\beta,\delta} \|V\|_{W^\delta_\infty(U)} = \|V\|_{\mathcal{C}^\delta(U)}$ which holds because $\beta \le \frac{1}{2} < \delta$.

Therefore, by observing

$$\begin{aligned}
\|(\mathcal{T}v)^2\|_{W^{\beta-\varepsilon}_q(\partial U)} &\le C^{\partial U}_{\mathrm{Prod}}[W^\beta_{2q} \to W^{\beta-\varepsilon}_q] \|\mathcal{T}v\|^2_{W^\beta_{2q}(\partial U)} \\
&\le C^{\partial U}_{\mathrm{Prod}}[W^\beta_{2q} \to W^{\beta-\varepsilon}_q] \|\mathcal{T}\|^2 \|v\|^2_{W^{\beta+\frac{1}{2q}}_{2q}(U)}
\end{aligned}$$

and $\|v\|_{W^{\beta+\frac{1}{2q}}_{2q}(U)} \le C^U_{\mathrm{Embed}}[H^s \to W^{\beta+\frac{1}{2q}}_{2q}] \|v\|_{H^s(U)}$, we obtain

$$\begin{aligned}
[\![\tilde{\mathcal{Z}}]\!]_{H^s(U)} &\lesssim_{\delta,\varepsilon,p} C^{\partial U}_{\mathrm{Prod}}[W^\beta_{2q} \to W^{\beta-\varepsilon}_q] C^U_{\mathrm{Embed}}[H^s \to W^{\beta+\frac{1}{2q}}_{2q}]^2 C^{\partial U}_{\mathrm{R}}[W^{2-\delta}_q] \\
&\quad \times \Big(1 + \sup_{x\in\partial U}\int_{\partial U} \frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}}\Big)^{\frac{1}{q}} C^U_{\mathrm{Ext}}[W^q_{2-\delta}] \|\mathcal{R}_{W^{\beta-\varepsilon}_q(\partial U)}\| \|\mathcal{T}_{W^{\beta+\frac{1}{2q}}_{2q}(U)}\|^2 \\
&\quad \times \|V\|_{\mathcal{C}^\delta(U)} \|\tilde{Y}\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)}.
\end{aligned}$$

Let us now check that $\tilde{\mathcal{Z}}$ is independent of the choice of $\mathcal{R}$ and ι. For $\varphi \in C^\infty(\overline{U}) \cap W^r_p(U)$ and $\mathcal{V} \in C^\infty(\overline{U})$ the function $\iota \circ \mathcal{R}[\mathcal{V}(\mathcal{T}\varphi)^2]$ is equal to $\mathcal{V}\varphi^2$ on ∂U and thus for $\varepsilon > 0$

$$\int_{\partial U} f_\varepsilon(\iota \circ \mathcal{R}[\mathcal{V}(\mathcal{T}\varphi)^2]) \,\mathrm{d}S = \int_{\partial U} f_\varepsilon \mathcal{V}\varphi^2 \,\mathrm{d}S.$$

By the above estimates we have already seen that $\tilde{\mathcal{Z}}^U_{\tilde{Y},V}(v)$ is continuous as a function of $\tilde{Y}$, V and v. As $C^\infty(\overline{U}) \cap W^r_p(U)$ is dense in $W^r_p(U)$, V is the limit of smooth functions in $C^\infty(\overline{U})$ and $\tilde{Y}$ is the limit of $\tilde{Y}_\varepsilon$, it therefore follows that $\tilde{\mathcal{Z}}$ is independent of the choice of $\mathcal{R}$ and ι.

Therefore, by (21) and (22) (as we may take the infimum over ι and $\mathcal{R}$),

$$|\langle \tilde{Y}, \iota \circ \mathcal{R}[V(\mathcal{T}v)^2]\rangle| \lesssim_{\delta,p} \|\tilde{Y}\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)} C^U_{\mathrm{Ext}}[W^q_{2-\delta}] \|V(\mathcal{T}v)^2\|_{W^{\beta-\varepsilon}_q(\partial U)}.$$

With this (19) follows from Lemma A.15 (for the estimate on the C_{Embed} we use the second inequality in (18)) and because

$$\int_{\partial U_L} \frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}} = \int_{\partial U} L^{d-1} \frac{\mathrm{d}S(y)}{|Lx-Ly|^{d-1-q\varepsilon}} = L^{q\varepsilon} \int_{\partial U} \frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}},$$

so that

$$\sup_{x\in\partial U_L}\left(1+\int_{\partial U_L}\frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}}\right)^{\frac{1}{q}}\leq L^\varepsilon\sup_{x\in\partial U}\left(1+\int_{\partial U}\frac{\mathrm{d}S(y)}{|x-y|^{d-1-q\varepsilon}}\right)^{\frac{1}{q}}.$$

The latter supremum is finite: $\int_{\partial U}\frac{dy}{|x-y|^{d-1-q\varepsilon}}$ is finite for all $x\in\partial U$ due to the fact that U is a Lipschitz domain, so that by the compactness of ∂U it follows that the supremum is finite as well. The other L^ε factor comes from $C^{\partial U_L}_{\mathrm{Prod}}[W^r_{2q}\to W^{r-\varepsilon}_q]\lesssim_{p,\varepsilon,U}L^\varepsilon$, see Lemma A.15.

(c) First, observe that (for ν being the outer normal on ∂U)

$$\left|\widehat{\mathcal{Z}}^U_{\widehat{Y},V}(v,v)\right|=\left|\int_{\partial U}V(\mathcal{T}v)^2\nabla_\nu\widehat{Y}\,\mathrm{d}S\right|\leq\|V\|_{L^\infty(\partial U)}\|\nabla\widehat{Y}\|_{L^\infty(\partial U)}\|\mathcal{T}v\|^2_{L^2(\partial U)}.$$

Secondly, use $\|\mathcal{T}v\|_{L^2(\partial U)}\leq\|\mathcal{T}v\|_{W_2^{s-\frac12}(\partial U)}\leq\|\mathcal{T}_{H^s(U)}\|\|v\|_{H^s(U)}$, $\|V\|_{L^\infty(\partial U)}\leq\|V\|_{C^\delta(U)}$ and $\|\nabla\widehat{Y}\|_{L^\infty(\partial U_L)}\leq\|\phi(L^{-1}\cdot)\nabla\widehat{Y}\|_{L^\infty(\mathbb{R}^d)}\lesssim\|\phi(L^{-1}\cdot)\nabla\widehat{Y}\|_{\mathcal{C}^\delta(\mathbb{R}^d)}\lesssim_{\delta,\sigma,\phi}L^\sigma\|\widehat{Y}\|_{\mathcal{C}^{1+\delta,\sigma}(\mathbb{R}^d)}$, where the last inequality is due to Lemma A.4. (20) then follows by Lemma A.15 (observe that we use that $s>\frac12$ in order to have $\sup_{L\geq1}\|\mathcal{T}_{H^s(U_L)}\|<\infty$). □

4.2 Basic spectral properties

In this section we study the spectral properties of the Dirichlet and Neumann operator corresponding to $\mathcal{E}$, which are introduced in Definition 4.9.

Assumptions for this section 4.7. In this section, U is a bounded domain.

Definition 4.8. Let $\mathcal{Q}$ be a symmetric form on a Hilbert space H. Let D be the set of $u\in\mathcal{D}(\mathcal{Q})$ such that there exists a $\tilde{u}\in H$ such that $\mathcal{Q}(u,v)=\langle\tilde{u},v\rangle_H$ for all $v\in\mathcal{D}(\mathcal{Q})$. For such u the element $\tilde{u}$ is unique, and we will write $Au=\tilde{u}$. Then A on D forms a linear operator on H, called the *operator associated with* $\mathcal{Q}$.

Definition 4.9. If $\mathcal{Z}$ is a bounded symmetric form on $H^1_0(U)$, then we write $\mathcal{E}^{\mathrm{D},U}_{W,\mathcal{Z}}$ for $\mathcal{E}^U_{W,\mathcal{Z}}$ with $\mathcal{D}(\mathcal{E}^{\mathrm{D},U}_{W,\mathcal{Z}})=e^WH^1_0(U)$ and let $\mathcal{H}^{\mathrm{D}}=\mathcal{H}^{\mathrm{D},U}=\mathcal{H}^{\mathrm{D},U}_{W,Z}$ be the operator associated with $\mathcal{E}^{\mathrm{D},U}_{W,\mathcal{Z}}$ on $L^2(U)$.

If $\mathcal{Z}$ is a bounded symmetric form on $H^1(U)$, then we write $\mathcal{E}^{\mathrm{N},U}_{W,\mathcal{Z}}$ for $\mathcal{E}^U_{W,\mathcal{Z}}$ with $\mathcal{D}(\mathcal{E}^{\mathrm{N},U}_{W,\mathcal{Z}})=e^WH^1(U)$ and let $\mathcal{H}^{\mathrm{N}}=\mathcal{H}^{\mathrm{N},U}=\mathcal{H}^{\mathrm{N},U}_{W,Z}$ be the operator associated with $\mathcal{E}^{\mathrm{N},U}_{W,\mathcal{Z}}$ on $L^2(U)$.

Definition 4.10. Let U be a bounded Lipschitz domain and $s\in(0,1)$. We define

$$C^U_{\mathrm{IP}}[H^s]:=\sup_{f\in H^1(U)\setminus\{0\}}\frac{\|f\|_{H^s(U)}}{\|f\|^{1-s}_{L^2(U)}\|f\|^s_{H^1(U)}}.$$

If U is a bounded domain that is not necessarily Lipschitz, we define $C^U_{\mathrm{IP}}[H^s]$ similarly as above by replacing "$H^a(U)$" by "$H^a_0(U)$" for a being either s or 1.

Lemma 4.11. *Let $s\in(0,1)$. Let $\mathcal{Z}$ be a bounded symmetric form on $H^s_0(U)$. Then, for any $\delta\in(0,1)$ and $v\in H^1_0(U)$, we have*

$$|\mathcal{Z}(v,v)|\leq\delta\int_U|\nabla v|^2+\left(\delta+\delta^{-\frac{s}{1-s}}C^U_{\mathrm{IP}}[H^s_0]^{\frac{2}{1-s}}[\![\mathcal{Z}]\!]^{\frac{1}{1-s}}_{H^s_0(U)}\right)\|v\|^2_{L^2(U)}.$$

If $\mathcal{Z}$ a bounded symmetric form on $H^s(U)$, then the above statement holds with H^s_0 replaced by H^s.

Proof. We only prove the claim for a bounded Lipschitz domain. One has $|\mathcal{Z}(v,v)| \le [\![\mathcal{Z}]\!]_{H^s(U)} \|v\|_{H^s(U)}^2$. By interpolation and Young's inequality (using that $a^s b^{1-s} \le a+b$),

$$\|v\|_{H^s(U)}^2 \le C_{\mathrm{IP}}^U[H^s]^2 \|v\|_{H^1(U)}^{2s} \|v\|_{L^2(U)}^{2(1-s)} \le C_{\mathrm{IP}}^U[H^s]^2 (\eta \|v\|_{H^1(U)}^2 + \eta^{-\frac{s}{1-s}} \|v\|_{L^2(U)}^2)$$

for any $\eta \in (0,\infty)$. Therefore,

$$|\mathcal{Z}(v,v)| \le \eta [\![\mathcal{Z}]\!]_{H^s(U)} C_{\mathrm{IP}}^U[H^s]^2 \|v\|_{H^1(U)}^2 + \eta^{-\frac{s}{1-s}} [\![\mathcal{Z}]\!]_{H^s(U)} C_{\mathrm{IP}}^U[H^s]^2 \|v\|_{L^2(U)}^2.$$

We can choose η so that $\delta = \eta [\![\mathcal{Z}]\!]_{H^s(U)} C_{\mathrm{IP}}^U[H^s]^2$ and use that $\|v\|_{H^1(U)}^2 = \|v\|_{L^2(U)}^2 + \int_U |\nabla v|^2$. □

Proposition 4.12. *Let $W \in L^\infty(U)$ and $\mathcal{Z}$ be a bounded symmetric form on $H_0^s(U)$ for some $s \in (0,1)$. Then $\mathcal{E}_{W,\mathcal{Z}}^{\mathrm{D},U}$ is closed and $e^W C_c^\infty(U)$ is a core. Consequently, H^{D} is self-adjoint.*

If $\mathcal{Z}$ instead is a bounded symmetric form on $H^s(U)$, then $\mathcal{E}_{W,\mathcal{Z}}^{\mathrm{N},U}$ is closed and $e^W C^\infty(\overline{U})$ is a core. Consequently, $\mathcal{H}^{\mathrm{N}}$ is self-adjoint.

Proof. In view of Lemma 4.11 and the symmetric form version of the Kato-Rellich theorem [39, Theorem 1.33 in Chapter VI], we can assume $\mathcal{Z} = 0$. Observe that for $u = e^W u^\flat$, $u^\flat \in H^1(U)$

$$e^{-2\|W\|_{L^\infty(U)}} \mathcal{E}_{0,0}(u^\flat, u^\flat) \le \mathcal{E}_{W,0}(u,u) \le e^{2\|W\|_{L^\infty(U)}} \mathcal{E}_{0,0}(u^\flat, u^\flat).$$

Therefore the claim follows as $\mathcal{E}_{0,0}$ is closed and $C_c^\infty(U)$ is a core for $\mathcal{E}_{0,0}$.

The self-adjointness of the corresponding operators follows as they are closed densely defined and symmetric, cf. [21, Section 4.4]). □

By applying a standard result from the spectral theory, we can easily show that the spectrum of $\mathcal{H}^{\mathrm{D}}$ on a bounded domain and that of $\mathcal{H}^{\mathrm{N}}$ on a bounded Lipschitz domain are discrete and that the min-max formula (also known as the Courant-Fischer formula) holds for the eigenvalues.

Proposition 4.13. *The spectrum of $\mathcal{H}^{\mathrm{D}}$ is given by a sequence of eigenvalues $(\lambda_k^{\mathrm{D}})_{k=1}^\infty$ (counting multiplicities), such that (with the notation $\sqsubset$ for "is a linear subspace of") $\lambda_1^{\mathrm{D}} \ge \lambda_2^{\mathrm{D}} \ge \cdots$,*

$$\begin{aligned}
\lambda_k^{\mathrm{D}} := \lambda_k^{\mathrm{D}}(U; W, \mathcal{Z}) &:= \inf_{\substack{L \sqsubset \mathcal{D}(\mathcal{H}^{\mathrm{D}}) \\ \dim L = k}} \sup_{\substack{u \in L \\ \|u\|_{L^2(U)}=1}} \langle \mathcal{H}^{\mathrm{D}} u, u \rangle_{L^2(U)}, \\
&= \inf_{\substack{L \sqsubset e^W H_0^1(U) \\ \dim L = k}} \sup_{\substack{u \in L \\ \|u\|_{L^2(U)}=1}} \mathcal{E}(u,u), \\
&= \inf_{\substack{L \sqsubset e^W C_c^\infty(U) \\ \dim L = k}} \sup_{\substack{u \in L \\ \|u\|_{L^2(U)}=1}} \mathcal{E}(u,u)
\end{aligned}$$

and $\lim_{k\to\infty} \lambda_k^{\mathrm{D}} = \infty$. In particular, $(\lambda - \mathcal{H}^{\mathrm{D}})^{-1}$ is a compact operator for all λ that are not in the spectrum of $\mathcal{H}^{\mathrm{D}}$. If U is a bounded Lipschitz domain, an analogous statement for $\mathcal{H}^{\mathrm{N}}$ holds if $H_0^1(U)$ and $C_c^\infty(\overline{U})$ are replaced by $H^1(U)$ and $C^\infty(\overline{U})$.

Proof. By well-known results of spectral theory (see e.g., [21, Corollary 4.2.3, Theorem 4.5.2, Theorem 4.5.3] in combination with Proposition 4.12), it suffices to show that the form domain is compactly embedded in $L^2(U)$, which follows from the compact embeddings of Sobolev spaces (see [11, Theorem 8.11.2] for the fact that the embedding $H_0^1(U) \hookrightarrow L^2(U)$ is compact for any bounded domain U [11, Theorem 8.11.4] for the fact that the embedding $H^1(U) \hookrightarrow L^2(U)$ is compact for any bounded Lipschitz domain U). □

We show continuous dependence of the spectral structure with respect to W and $\mathcal{Z}$. This follows from the result of [42].

Definition 4.14. Let H be a Hilbert space and $M > 0$. Let $(\mathcal{Q}_n)_{n=1}^{\infty}$ and $\mathcal{Q}$ be closed symmetric forms that are M-bounded from below. We use the following convention: if $u \notin \mathcal{D}(\mathcal{Q})$, then we set $\mathcal{Q}(u,u) := \infty$.

(a) [42, Definition 2.8] We say the sequence $(\mathcal{Q}_n)_{n=1}^{\infty}$ Γ-*converges* to $\mathcal{Q}$, if the following hold:

  (i) If the sequence $(u_n)_{n=1}^{\infty}$ converges to u in H, then

$$\mathcal{Q}(u,u) \leq \liminf_{n\to\infty} \mathcal{Q}_n(u_n,u_n). \tag{23}$$

  (ii) For any $u \in \mathcal{D}(\mathcal{Q})$, there exists a sequence $(u_n)_{n=1}^{\infty}$ in H such that

$$u_n \to u \text{ in } H \quad \text{and} \quad \lim_{n\to\infty} \mathcal{Q}_n(u_n,u_n) = \mathcal{Q}(u,u).$$

(b) [42, Definition 2.12] The sequence $(\mathcal{Q}_n)_{n=1}^{\infty}$ is said to be *compact* if the condition

$$\sup_{n\in\mathbb{N}} \mathcal{Q}_n(u_n,u_n) + (M+1)\|u_n\|_H^2 < \infty$$

implies $(u_n)_{n=1}^{\infty}$ is precompact in H, that is the sequence has a converges subsequence in H.

(c) [42, Definition 2.13] We say the sequence $(\mathcal{Q}_n)_{n=1}^{\infty}$ *converges compactly* to $\mathcal{Q}$ if $(\mathcal{Q}_n)_{n=1}^{\infty}$ Γ-converges to $\mathcal{Q}$ and if $(\mathcal{Q}_n)_{n=1}^{\infty}$ is compact and write

$$\mathcal{Q}_n \xrightarrow{compact}_{n\to\infty} \mathcal{Q}.$$

Lemma 4.15. *Let H be a Hilbert space and $M > 0$. Suppose that $(\mathcal{Q}_n)_{n=1}^{\infty}$ is a sequence of closed quadratic forms on H that are M-bounded from below and converges compactly to $\mathcal{Q}$. Let A_n (resp. A) be the self-adjoint operator associated with $\mathcal{Q}_n$ (resp. $\mathcal{Q}$).*

(a) [42, Theorem 2.4 and Theorem 2.5] *For any bounded continuous function f on $\mathbb{R}$, we have $\|f(A_n) - f(A)\|_{H\to H} \to 0$. In particular, $A_n \xrightarrow{\text{NR}}_{n\to\infty} A$ (see Definition 1.4).*

(b) [42, Corollary 2.5] *Let $\lambda_{k,n}$ (resp. λ_k) be the k-th eigenvalue of A_n (resp. A), counting multiplicities. Then, we have $\lim_{n\to\infty} \lambda_{k,n} = \lambda_k$ for any k. Moreover, for any k there exist (a choice of the) k-th eigenfunctions $\phi_{k,n}$ (resp. ϕ_k) of A_n (resp. A) such that $\phi_{k,n}$ converges to ϕ_k in H.*

Remark 4.16. In the proof of the following theorem we use the following elementary fact. If $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ and $\liminf_{n\to\infty} a_n < \infty$, then there exists a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\liminf_{n\to\infty} a_{\varphi(n)} = \liminf_{n\to\infty} a_n$ and $\sup_{n\in\mathbb{N}} a_{\varphi(n)} < \infty$.

Theorem 4.17. *Let $s \in [0,1)$. Suppose that $W_n \to W$ in $C(\overline{U})$.*

- *If $[\![\mathcal{Z}_n - \mathcal{Z}]\!]_{H_0^s(U)} \xrightarrow{n\to\infty} 0$, then*

$$\mathcal{E}^{\mathrm{D},U}_{W_n,\mathcal{Z}_n} \xrightarrow{compact}_{n\to\infty} \mathcal{E}^{\mathrm{D},U}_{W,\mathcal{Z}}.$$

- *If U is a bounded Lipschitz domain and $[\![\mathcal{Z}_n - \mathcal{Z}]\!]_{H^s(U)} \xrightarrow{n\to\infty} 0$, then*

$$\mathcal{E}^{\mathrm{N},U}_{W_n,\mathcal{Z}_n} \xrightarrow{compact}_{n\to\infty} \mathcal{E}^{\mathrm{N},U}_{W,\mathcal{Z}}.$$

Proof. We only prove the second statement. We first show that $(\mathcal{E}^{\mathrm{N}}_{W_n,\mathcal{Z}_n})_{n\in\mathbb{N}}$ is compact. Suppose $\sup_{n=1}^{\infty} \mathcal{E}^{\mathrm{N}}_{W_n,\mathcal{Z}_n}(u_n,u_n) + (M+1)\|u_n\|^2_{L^2(U)} < \infty$. We set $u_n^\flat := e^{-W_n}u_n$. By Lemma 4.11, we have $\sup_{n=1}^\infty \|u_n^\flat\|_{H^1(U)} < \infty$ because for any $\delta \in (0,1)$,

$$\begin{aligned}\sup_{n=1}^{\infty} \mathcal{E}_{W_n,\mathcal{Z}_n}(u_n,u_n) &= \sup_{n=1}^{\infty} \int_U e^{2W_n}|\nabla u_n^\flat|^2 + \mathcal{Z}_n(u_n,u_n) \\ &\geq \sup_{n=1}^{\infty} e^{-2\|W_n\|_{L^\infty}}(1-\delta)\int_U |\nabla u_n^\flat|^2 - \Big(\delta + C\delta^{-\frac{s}{1-s}} [\![\mathcal{Z}_n]\!]_{H^s(U)}^{\frac{1}{1-s}}\Big)\|u_n\|^2_{L^2(U)},\end{aligned}$$

and thus $\sup_{n=1}^\infty \|u_n\|_{H^1(U)} < \infty$ as $W_n \to W$ in $C(\overline{U})$. Since the embedding $H^1(U) \hookrightarrow L^2(U)$ is compact, the sequence $(u_n)_{n=1}^\infty$ is also precompact in $L^2(U)$.

Next we show that $(\mathcal{E}^{\mathrm{N}}_{W_n,\mathcal{Z}_n})_{n=1}^\infty$ Γ-converges to $\mathcal{E}^{\mathrm{N}}_{W,\mathcal{Z}}$. Since (ii) of Definition 4.14 (a) is trivial, we focus on showing (i) Definition 4.14 (a).

Suppose that $(u_n)_{n=1}^\infty$ converges to u in $L^2(U)$. As we want to show (23), we may assume $\liminf_{n\to\infty} \mathcal{E}^{\mathrm{N},U}_{W_n,\mathcal{Z}_n}(u_n,u_n) < \infty$ and by Remark 4.16 we may as well assume $\sup_{n\in\mathbb{N}} \mathcal{E}^{\mathrm{N},U}_{W_n,\mathcal{Z}_n}(u_n,u_n) < \infty$ (by possibly considering a subsequence), so that by the above $\sup_{n\in\mathbb{N}}\|u_n^\flat\|_{H^1(U)} < \infty$. It suffices to show

$$\int_U e^{2W}|\nabla u^\flat|^2 \leq \liminf_{n\to\infty} \int_U e^{2W_n}|\nabla u_n^\flat|^2, \tag{24}$$

$$\lim_{n\to\infty} \mathcal{Z}_n(u_n^\flat,u_n^\flat) = \mathcal{Z}(u^\flat,u^\flat). \tag{25}$$

Since the sequence $(u_n^\flat)_{n=1}^\infty$ is uniformly bounded in $H^1(U)$ and converges to $u^\flat$ in $L^2(U)$, by interpolation it converges to $u^\flat$ in $H^s(U)$ from which (25) follows. For $v \in C_c^\infty(U)$,

$$\langle \nabla u_n^\flat, v\rangle_{L^2(U)} = -\langle u_n^\flat, \nabla v\rangle_{L^2(U)} \to -\langle u^\flat, \nabla v\rangle_{L^2(U)} = \langle \nabla u^\flat, v\rangle_{L^2(U)},$$

which implies that $(\nabla u_n^\flat)_{n=1}^\infty$ converges weakly to $\nabla u^\flat$ in $L^2(U)$. Therefore, $(e^{W_n}\nabla u_n^\flat)_{n=1}^\infty$ converges weakly to $e^W \nabla u^\flat$ in $L^2(U)$ and this implies (24) (this follows for example by the dual representation of the norm on $L^2(U)$). □

4.3 Estimates of eigenvalues

In this section, we give deterministic estimates of the eigenvalues and the eigenvalue counting functions of the operators constructed in Section 4.2. The motivation comes from the study of the integrated density of states in Section 5.4.

Assumptions for this section 4.18. We assume the following throughout this section:

- W is a continuous function defined on $\mathbb{R}^d$.
- $s \in [0,1)$, $\mathcal{U}$ is the collection of bounded Lipschitz domains.
- $\mathcal{Z}^U$ is a bounded symmetric form on $H^s(U)$ for all $U \in \mathcal{U}$.

For each $U \in \mathcal{U}$, we let $\mathcal{E}^U = \mathcal{E}^U_{W,\mathcal{Z}^U}$ be the symmetric form defined in Definition 4.2. Recall the notations $\mathcal{H}^{\mathrm{N},U}$ and $\mathcal{H}^{\mathrm{D},U}$ from Definition 4.9 and $\lambda_k^{\mathrm{N}}(U)$, $\lambda_k^{\mathrm{D}}(U)$ from Proposition 4.13.

Remark 4.19. For Dirichlet boundary conditions we do not necessarily need to consider Lipschitz domains. Indeed, if $\mathcal{U}$ would instead be the collection of all bounded domains and $\mathcal{Z}^U$ a bounded symmetric form on $H_0^s(U)$ for all $U \in \mathcal{U}$, then the statements of Lemma 4.21, Lemma 4.27 (a) and Lemma 4.31 (a) remain valid.

Definition 4.20. For $\# \in \{\mathrm{N}, \mathrm{D}\}$ and $U \in \mathcal{U}$, we define the *eigenvalue counting functions* $\boldsymbol{N}^\#(U, \lambda)$ for $\lambda \in \mathbb{R}$ by

$$\boldsymbol{N}^\#(U,\lambda) := \boldsymbol{N}^\#_{W,\mathcal{Z}}(U,\lambda) := \sum_{k=1}^{\infty} \mathbb{1}_{\{\lambda_k^\#(U;W,\mathcal{Z}) \le \lambda\}}.$$

We set $\boldsymbol{N}_0^\#(U,\lambda) := \boldsymbol{N}^\#_{0,0}(U,\lambda)$, which is the eigenvalue counting function of the Neumann or the Dirichlet Laplacian on U.

For $L > R > 0$ we set

$$U_L^R := U_L \cap B(\partial U_L, R), \quad C(\partial U_L, R) := \{x \in U_L \mid d(x, \partial U_L) = R\}. \tag{26}$$

(Observe $C(\partial U_L, R) = \partial U_L^R \setminus \partial U_L$.) We denote by $H^1_{\mathfrak{m},R}(U_L^R)$ the closure in $H^1(U_L^R)$ with respect to H^1-norm of the set

$$\{\phi \in C^\infty(\overline{U_L^R}) \mid \phi = 0 \text{ on a neighborhood of } C(\partial U_L, R)\}.$$

Let $\boldsymbol{N}_0^{\mathfrak{m}}(U_L^R, \lambda)$ be the eigenvalue counting function of the operator associated with the symmetric form $(u,u) \mapsto \int_{U_L^R} |\nabla u|^2$ with the domain $H^1_{\mathfrak{m},R}(U_L^R)$.

Lemma 4.21. *Let $U, U_1, U_2 \in \mathcal{U}$, $U_1 \subseteq U_2$ and $\lambda \in \mathbb{R}$. Then*

$$\boldsymbol{N}^{\mathrm{D}}(U,\lambda) \le \boldsymbol{N}^{\mathrm{N}}(U,\lambda),$$
$$\boldsymbol{N}^{\mathrm{D}}(U_1,\lambda) \le \boldsymbol{N}^{\mathrm{D}}(U_2,\lambda).$$

Proof. Since $H_0^1(U) \subseteq H^1(U)$, the min-max formula (Lemma 4.13) implies $\lambda_k^{\mathrm{D}}(U) \ge \lambda_k^{\mathrm{N}}(U)$ for all k, and thus the first inequality. The second also follows by the min-max formula, as $H_0^1(U_1) \subseteq H_0^1(U_2)$. □

Lemma 4.22. *Let $U \in \mathcal{U}$, $s \in (0,1)$, $\theta \in (0,\infty)$ and $\lambda \in \mathbb{R}$. We set*

$$\Lambda^{\pm}_{\lambda,\theta}(W,\mathcal{Z}) := (1 \pm \theta) e^{\pm 4\|W\|_{L^\infty(U)}} (\lambda \pm A_\pm),$$

where

$$A_\pm := A_{\pm,\theta}^{W,\mathcal{Z}} := \theta + \left(\frac{\theta}{1 \pm \theta}\right)^{-\frac{s}{1-s}} C_{\mathrm{IP}}^U [H^s]^{\frac{2}{1-s}} e^{(2 \pm \frac{2s}{1-s})\|W\|_{L^\infty(U)}} [\![\mathcal{Z}]\!]_{H^s(U)}^{\frac{1}{1-s}}.$$

Then, one has

$$\boldsymbol{N}_0^{\mathrm{D}}(U, \Lambda^-_{\lambda,\theta}(W,\mathcal{Z})) \le \boldsymbol{N}^{\mathrm{D}}(U,\lambda) \le \boldsymbol{N}^{\mathrm{N}}(U,\lambda) \le \boldsymbol{N}_0^{\mathrm{N}}(U, \Lambda^+_{\lambda,\theta}(W,\mathcal{Z})).$$

Proof. We only prove $\boldsymbol{N}^{\mathrm{N}}(U,\lambda) \le \boldsymbol{N}_0^{\mathrm{N}}(U, \Lambda^+_{\lambda,\theta}(U, W, \mathcal{Z}))$; the other inequality follows similarly. By setting $\delta := \frac{\theta}{1+\theta} e^{-2\|W\|_{L^\infty(U)}}$, Lemma 4.11 yields

$$|\mathcal{Z}(u^\flat, u^\flat)| \le \delta \int_U |\nabla u^\flat|^2 + A_+ e^{-2\|W\|_{L^\infty(U)}} \int_U (u^\flat)^2.$$

One has $\int_U e^{2W}|\nabla u^\flat|^2 \geq e^{-2\|W\|_{L^\infty(U)}} \int_U |\nabla u^\flat|^2$. Therefore, by Proposition 4.13,

$$\begin{aligned}
\lambda_k^{\mathrm{N}}(U) &= \inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int e^{2W}(u^\flat)^2 = 1}} \int_U e^{2W}|\nabla u^\flat|^2 + \mathcal{Z}(u^\flat, u^\flat) \\
&\geq \inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int e^{2W}(u^\flat)^2 = 1}} \frac{e^{-2\|W\|_{L^\infty(U)}}}{1+\theta} \int_U |\nabla u^\flat|^2 - A_+ e^{-2\|W\|_{L^\infty(U)}} \int_U (u^\flat)^2 \\
&\geq \frac{e^{-2\|W\|_{L^\infty(U)}}}{1+\theta} \left\{ \inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int e^{2W}(u^\flat)^2 = 1}} \int_U |\nabla u^\flat|^2 \right\} - A_+.
\end{aligned}$$

We compute

$$\begin{aligned}
\inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int e^{2W}(u^\flat)^2 = 1}} \int_U |\nabla u^\flat|^2 &= \inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int e^{2W}(u^\flat)^2 \leq 1}} \int_U |\nabla u^\flat|^2 \\
&\geq \inf_{\substack{L \sqsubset H^1(U),\\ \dim L = k}} \sup_{\substack{u^\flat \in L,\\ \int (u^\flat)^2 \leq e^{-2\|W\|_{L^\infty(U)}}}} \int_U |\nabla u^\flat|^2 \\
&= e^{-2\|W\|_{L^\infty(U)}} \lambda_k^{\mathrm{N}}(U; 0, 0). \qquad (27)
\end{aligned}$$

Therefore,

$$\lambda_k^{\mathrm{N}}(U) \geq \frac{e^{-4\|W\|_{L^\infty(U)}}}{1+\theta} \lambda_k^{\mathrm{N}}(U; 0, 0) - A_+$$

and the claimed inequality follows. □

As Lemma 4.22 suggests, we need estimates of $\boldsymbol{N}_0^{\#}(U, \lambda)$. The following lemma is sufficient for our purpose.

Lemma 4.23. *Let U be a bounded Lipschitz domain.*

(a) *Then, there exist $C_U, R_U > 0$ such that*

$$\boldsymbol{N}_0^{\mathrm{m}}(U_L^R, \lambda) \leq C_U R_U^d (1+\lambda)^{\frac{d}{2}} L^{d-1}$$

for every $L \geq 1$, $\lambda \geq 0$ and $R \geq R_U$.

(b) [56, Theorem 3.1 and Theorem 3.2] *There exists a $C_U' > 0$ such that*

$$\begin{aligned}
\frac{|B(0,1)|}{(2\pi)^d} |U| \lambda^{\frac{d}{2}} - C_U' \lambda^{\frac{d-1}{2}} \log \lambda \leq \boldsymbol{N}_0^{\mathrm{D}}(U, \lambda) \\
\leq \boldsymbol{N}_0^{\mathrm{N}}(U, \lambda) \leq \frac{|B(0,1)|}{(2\pi)^d} |U| \lambda^{\frac{d}{2}} + C_U' \lambda^{\frac{d-1}{2}} \log \lambda
\end{aligned}$$

for every $\lambda \geq 2$.

Proof. The claim (a) follows from the proof of [23, Theorem 6.2]. Indeed, we can combine the estimates (6.20), (6.23), (6.24) and (6.25) therein. □

Definition 4.24. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be closed symmetric forms on Hilbert spaces H_1 and H_2 that are bounded from below. We write $\mathcal{Q}_1 \prec \mathcal{Q}_2$ if there exists an isometry $\Phi : H_2 \to H_1$ such that $\Phi(\mathcal{D}(\mathcal{Q}_2)) \subseteq \mathcal{D}(\mathcal{Q}_1)$ and $\mathcal{Q}_1(\Phi(f), \Phi(f)) \leq \mathcal{Q}_2(f, f)$ for every $f \in \mathcal{D}(\mathcal{Q}_2)$.

Lemma 4.25. *Let $\mathcal{Q}_1, \mathcal{Q}_2$ be as in Definition 4.24 and let A_1 and A_2 be the associated self-adjoint operators. Suppose that the spectrum of A_1 and that of B_2 are discrete and we denote them by $(\mu_k(A_1))_{k=1}^\infty$ and $(\mu_k(A_2))_{k=1}^\infty$ respectively. Then, $\mathcal{Q}_1 \prec \mathcal{Q}_2$ implies $\mu_k(A_1) \leq \mu_k(A_2)$ for every k.*

Proof. This follows from the min-max formula. □

In order to compare the eigenvalue counting functions on different domains, it will be convenient to introduce the following symmetric forms.

Definition 4.26. Let $J \in \mathbb{N}$. Let $\mathcal{E}_j$ be a symmetric form on a Hilbert space H_j for $j \in \{1, \ldots, J\}$. We define the symmetric form $\bigoplus_{j=1}^J \mathcal{E}_j$ on the Hilbert space $\bigoplus H_j$ by $\mathcal{D}(\bigoplus_{j=1}^J \mathcal{E}_j) = \bigoplus_{j=1}^J \mathcal{D}(\mathcal{E}_j)$ and for $v = \bigoplus_{j=1}^J v_j$ with $v_j \in \mathcal{D}(\mathcal{E}_j)$, $(\bigoplus_{j=1}^J \mathcal{E}_j)(v, v) := \sum_{j=1}^J \mathcal{E}_j(v_j, v_j)$.

Observe that if A_j is the operator associated with $\mathcal{E}_j$ for all j, then the operator $\bigoplus_{j=1}^J A_j$ defined by $\bigoplus_{j=1}^J A_j v = \bigoplus_{j=1}^J A_j v_j$ for $v = \bigoplus_{j=1}^J v_j \in \bigoplus_{j=1}^J \mathcal{D}(A_j) =: \mathcal{D}(\bigoplus_{j=1}^J A_j)$ is the operator associated with $\bigoplus_{j=1}^J \mathcal{E}_j$. In particular, the principal eigenvalue of $\bigoplus_{j=1}^J A_j$ is given by $\min_{j=1}^J \lambda_1(A_j)$, where $\lambda_1(A_j)$ is the principal eigenvalue of A_j for all j.

Moreover, if A_j has a countable spectrum for all j, then one has $\boldsymbol{N}_{\bigoplus_{j=1}^J \mathcal{E}_j} = \sum_{j=1}^J \boldsymbol{N}_{\mathcal{E}_j}$, where $\boldsymbol{N}_{\mathcal{Q}}$ is the eigenvalue counting function corresponding to the operator associated with $\mathcal{Q}$.

Observe that using this notation, one also has $\boldsymbol{N}_{a\mathcal{Q}+b\mathfrak{I}} =_{\mathcal{Q}} (\frac{\lambda-b}{a})$, where $\mathfrak{I}$ is the symmetric form $\mathfrak{I}(v, v) = \|v\|^2$. Moreover, $\mathcal{Q}_1 \prec \mathcal{Q}_2$ implies $\boldsymbol{N}_{\mathcal{Q}_1} \geq \boldsymbol{N}_{\mathcal{Q}_2}$.

Lemma 4.27. *Let $U, U_1, \ldots, U_J \in \mathcal{U}$, $U = \cup_{j=1}^J U_j$ with $\overline{U}_j \cap \overline{U}_k = \partial U_j \cap \partial U_k$ for $j \neq k$.*

(a) *If*

$$\mathscr{Z}^{U_j}(v, v) = \mathscr{Z}^U(v, v), \qquad v \in H_0^1(U_j), j \in \{1, \ldots, J\}, \tag{28}$$

then

$$\lambda_1^{\mathrm{D}}(U) \leq \min_{j=1}^J \lambda_1^{\mathrm{D}}(U_j) \quad \text{and} \quad \boldsymbol{N}^{\mathrm{D}}(U, \lambda) \geq \sum_{j=1}^J \boldsymbol{N}^{\mathrm{D}}(U_j, \lambda). \tag{29}$$

(b) *If*

$$\mathscr{Z}^U(v, v) = \sum_{j=1}^J \mathscr{Z}^{U_j}(v|_{U_j}, v|_{U_j}), \quad v \in H^1(U), \tag{30}$$

then

$$\lambda_1^{\mathrm{N}}(U) \geq \min_{j=1}^J \lambda_1^{\mathrm{N}}(U_j) \quad \text{and} \quad \boldsymbol{N}^{\mathrm{N}}(U, \lambda) \leq \sum_{j=1}^J \boldsymbol{N}^{\mathrm{N}}(U_j, \lambda).$$

Proof. (a) follows from the fact that $\oplus_{j=1}^J H_0^1(U_j) \subseteq H_0^1(U)$.

(b) As $L^2(U)$ and $\bigoplus_{j=1}^J L^2(U_j)$ are isometric, $H^1(U) \subseteq \oplus_{j=1}^J H^1(U_j)$ and $\mathcal{E}^U(u, u) = \sum_{j=1}^J \mathcal{E}^{U_j}(u|_{U_j}, u|_{U_j})$ for $u \in e^W H^1(U)$, we have $\oplus_{j=1}^J \mathcal{E}^{\mathrm{N},U_j} \prec \mathcal{E}^{\mathrm{N},U}$. Now both inequalities follow from Lemma 4.25 (see also the comments in Definition 4.26). □

Remark 4.28. Observe that (28) and (30) hold for $U, U_1, \dots, U_J$ as in Lemma 4.31 (a) if $\delta \in (0,1)$, $\sigma \in [0,\infty)$, $Y \in \mathcal{C}^{-1+\delta}(\mathbb{R}^d)$ and $\mathcal{Z}^U$ is given by $\mathcal{Z}^U_Y$ for $U \in \mathcal{U}$ as in Theorem 4.6 (a); or if $Y \in C^1(\mathbb{R}^d)$ and $\mathcal{Z}^U$ is given for $U \in \mathcal{U}$ by

$$\mathcal{Z}^U(v,v) := \int_{\partial U} v^2 \nabla Y \cdot \mathrm{d}\boldsymbol{S},$$

or if it is a linear combination of the above examples.

We can give a "reversed" inequality of (29). First we present an auxiliary lemma which is based on the IMS formula, see [60].

Lemma 4.29. *Let $J \in \mathbb{N}$ and $U, U_1, \dots, U_J \in \mathcal{U}$. Let $\eta_1, \dots, \eta_J$ be smooth functions $\mathbb{R}^d \to [0,1]$ such that there exists an $A > 0$ such that*

$$\|\nabla \eta_j\|^2_{L^\infty(\mathbb{R}^d)} \le A, \quad j \in \{1, \dots, J\}, \qquad \sum_{j=1}^J \eta_j^2 = 1 \text{ on } U.$$

Then

$$\mathcal{E}^U_{W,0}(u,u) \ge \sum_{j=1}^J \mathcal{E}^U_{W,0}(\eta_j u, \eta_j u) - A\|\eta_j u\|^2_{L^2}, \quad u \in e^W H^1(U).$$

Proof. Observe that $\sum_{j=1}^J \nabla(\eta_j)^2 = 0$. Let $u = e^W u^\flat$ with $u^\flat \in H^1(U)$. Then

$$\eta_j^2 |\nabla u^\flat|^2 = |\nabla(\eta_j u^\flat)|^2 - |\nabla \eta_j|^2 (u^\flat)^2 - \nabla(\eta_j^2) \cdot u^\flat \nabla u^\flat,$$

and therefore

$$\begin{aligned}\int_U e^{2W} |\nabla u^\flat|^2 &= \sum_{j=1}^J \int_U e^{2W} \eta_j^2 |\nabla u^\flat|^2 \\ &\ge \sum_{j=1}^J \Big\{ \int_U e^{2W} |\nabla(\eta_j u^\flat)|^2 - A\|\eta_j u^\flat\|^2_{L^2} \Big\},\end{aligned}$$

□

Remark 4.30. So far we have only considered the Anderson Hamiltonians on bounded domains, which means bounded open subsets of $\mathbb{R}^d$. However, whether one considers U or $\overline{U}$, does not intrinsically make a difference. In the following lemma and further on we will consider the Anderson Hamiltonian on closed boxes of the form $[0,L]^d$ for example. One may read $(0,L)^d$ instead in order to align with the rest of the text, though we write $[0,L]^d$ as this is more common in the literature.

Lemma 4.31. *Let $Z \in \mathcal{C}^{-1+\delta}(\mathbb{R}^d)$ with $\delta \in (0,1)$ and suppose $\mathcal{Z}^U = \mathcal{Z}^U_Z$ as in Theorem 4.6 (a) for every Lipschitz domain U.*

(a) *There exists a $K > 0$ (which depends only on d) such that for all $U \in \mathcal{U}$, all $l, L > 0$ with $L > 2l$ and $n \in \mathbb{N}$,*

$$\lambda_1^{\mathrm{D}}([0,nL]^d) \ge \min_{k \in \mathbb{Z}^d \cap [-1,n+1]^d} \lambda_1^{\mathrm{D}}(kL + [-l, L+l]^d) - \frac{K}{l^2},$$

$$\boldsymbol{N}^{\mathrm{D}}([0,nL]^d, \lambda) \le \sum_{k \in \mathbb{Z}^d \cap [-1,n+1]^d} \boldsymbol{N}^{\mathrm{D}}\Big(kL + [-l, L+l]^d, \lambda + \frac{K}{l^2}\Big).$$

(b) *There exists a $K > 0$ (depending only on d) such that for all $U \in \mathcal{U}$ and $s \in (1-\delta, 1)$ there exist $C_{s,U}, R_U > 0$ such that for all $L \geq 1$, $\lambda \in \mathbb{R}$ and $R \geq R_U$,*

$$\boldsymbol{N}^{\mathrm{N}}(U_L, \lambda) \leq \boldsymbol{N}^{\mathrm{D}}(U_L, \lambda + KR^{-2}) \\ + C_{s,U} R^d L^{d-1} e^{d\frac{3-2s}{1-s}\|W\|_{L^\infty(U_L)}} (1 + \max\{\lambda, 0\} + [\![\mathcal{Z}]\!]_{H^s(U)}^{\frac{1}{1-s}})^{\frac{d}{2}}.$$

Proof. (a) According to [20, Lemma 8.2], there exists a smooth function $\eta : \mathbb{R}^d \to [0,1]$ with $\eta = 1$ on $[0, L-2l]^d$ and $\operatorname{supp}(\eta) \subseteq [-2l, L]^d$ such that $\|\nabla \eta\|_{L^\infty(\mathbb{R}^d)}^2 \leq \frac{K}{l^2}$ and

$$\sum_{k \in \mathbb{Z}^d} \eta(x + kL)^2 = 1 \quad \text{for } x \in \mathbb{R}^d.$$

We set $\eta_k := \eta(\cdot + (l, l, \dots, l) + Lk)$ for $k \in \mathbb{Z}^d$. Observe that $\operatorname{supp}(\eta_k) \subseteq kL + [-l, L+l]^d$, and,

$$\sum_{k \in [-1, n+1]^d \cap \mathbb{Z}^d} \eta_k^2 = 1 \text{ on } [0, nL]^d.$$

Therefore, the map

$$\Phi : L^2([0, nL]^d) \to \bigoplus_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} L^2(kL + [-l, L+l]^d), \quad u \mapsto (\eta_k u)_{k \in [-1,n+1]^d \cap \mathbb{Z}^d}.$$

is an isometry and $\Phi(e^W H_0^1([0, nL]^d)) \subseteq \oplus_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} e^W H_0^1(kL + [-l, L+l]^d)$.

Observe that for $v \in H_0^1([0, nL]^d)$, for $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi = 1$ on a neighborhood of $[0, nL]^d$,

$$\begin{aligned} \mathcal{Z}_Z^{[0,nL]^d}(v, v) = \langle \phi Z, \mathbb{1}_{[0,nL]^d} v^2 \rangle &= \sum_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} \langle \phi Z, \mathbb{1}_{[0,nL]^d} \eta_k^2 v^2 \rangle \\ &= \sum_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} \mathcal{Z}_Z^{kL + [-l, L+l]^d}(\eta_k v, \eta_k v). \end{aligned}$$

Therefore, by Lemma 4.29,

$$\mathcal{E}^{[0,nL]^d}(u, u) \geq \sum_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} \Big\{ \mathcal{E}^{kL + [-l, L+l]^d}(\eta_k u, \eta_k u) - \frac{K}{l^2} \|\eta_k u\|_{L^2(kL + [-l, L+l]^d)}^2 \Big\}.$$

and thus $\mathcal{E}_{W,\mathcal{Z}}^{[0,nL]^d} \succ \bigoplus_{k \in [-1,n+1]^d \cap \mathbb{Z}^d} [\mathcal{E}_{W,\mathcal{Z}}^{kL + [-l, L+l]^d} - \frac{K}{l^2} \mathfrak{I}]$ (where $\mathfrak{I}$ is as in Definition 4.26), from which we conclude the estimates (use the discussion in Definition 4.26).

(b) As given in [23, Proposition 4.3], there exist smooth functions α_1 and α_2 on $\mathbb{R}^d$ and a $K > 0$ (only depending on d) such that

$$\operatorname{supp}(\alpha_1) \subseteq U_L \setminus B(\partial U_L, \tfrac{R}{2}), \quad \operatorname{supp}(\alpha_2) \subseteq B(\partial U_L, R),$$

$$\alpha_1^2 + \alpha_2^2 = 1 \text{ on a neighborhood of } U_L, \qquad \sum_{j=1}^{2} |\nabla \alpha_j|^2 \leq K R^{-2}.$$

Recall the definitions of U_L^R and $H_{\mathfrak{m},R}^1(U_L^R)$ from Definition 4.20. The map

$$\Phi : L^2(U_L) \to L^2(U_L) \oplus L^2(U_L^R), \quad u \mapsto \alpha_1 u \oplus \alpha_2 u$$

is an isometry and $\Phi(e^W H^1(U_L)) \subseteq e^W H^1_0(U_L) \oplus e^W H^1_{\mathfrak{m},R}(U_L^R)$.

Observe that $\mathcal{Z}^{U_L}(\alpha_2 v, \alpha_2 v) = \mathcal{Z}^{U_L^R}(\alpha_2 v, \alpha_2 v)$ as $\operatorname{supp} \alpha_2 \cap U_L \subset U_L^R$. Therefore, by Lemma 4.29

$$\mathcal{E}^{U_L}(u,u) \geq \mathcal{E}^{U_L}(\alpha_1 u, \alpha_1 u) + \mathcal{E}^{U_L^R}(\alpha_2 u, \alpha_2 u) - \sum_{j=1}^{2} KR^{-2} \|\alpha_j u\|^2_{L^2(U_L)}.$$

By applying Lemma 4.11 with $\delta = \frac{e^{-2\|W\|_{L^\infty(U_L)}}}{2}$, we obtain

$$\mathcal{E}^{U_L^R}(\alpha_2 u, \alpha_2 u) \geq \frac{e^{-2\|W\|_{L^\infty(U_L)}}}{2} \int_{U_L^R} |\nabla(\alpha_2 u^\flat)|^2 - A\|\alpha_2 u\|^2_{L^2(U_L^R)},$$

where

$$A := \frac{e^{-2\|W\|_{L^\infty(U_L)}}}{2} + 2^{\frac{s}{1-s}} e^{\frac{2s}{1-s}\|W\|_{L^\infty(U_L)}} C_{\mathrm{IP}}^{U_L}[H^s]^{\frac{2}{1-s}} [\![\mathcal{Z}]\!]^{\frac{1}{1-s}}_{H^s(U)}.$$

Therefore, $\mathcal{E}^{\mathrm{N},U_L} \succ (\mathcal{E}^{\mathrm{D},U_L} - KR^{-2}\mathfrak{I}) + (\mathcal{E}^{\mathfrak{m},R,U_L^R} - (KR^{-2}+A)\mathfrak{I})$, where $\mathcal{E}^{\mathfrak{m},R,U_L^R}$ is the restriction of $\mathcal{E}^{\mathrm{N},U_L^R}$ to $H_{\mathfrak{m},R}(U_L^R)$, and thus, by Lemma 4.25 (the additional factor $e^{2\|W\|_{L^\infty(U_L)}}$ is explained similarly as in (27)),

$$\boldsymbol{N}^{\mathrm{N}}(U_L,\lambda) \leq \boldsymbol{N}^{\mathrm{D}}(U_L, \lambda + KR^{-2}) + \boldsymbol{N}^{\mathfrak{m}}_0(U_L^R, 2e^{4\|W\|_{L^\infty(U_L)}}(\lambda + KR^{-2} + A)).$$

By Lemma 4.23 (a), for $R \geq R_U$,

$$\begin{aligned}
&\boldsymbol{N}^{\mathfrak{m}}_0(U_L^R, 2e^{4\|W\|_{L^\infty(U)}}(\lambda + KR^{-2} + A)) \\
&\qquad \lesssim_U R^d L^{d-1} \big\{ e^{4\|W\|_{L^\infty(U)}} (1 + \max\{\lambda, 0\} + KR^{-2} + A) \big\}^{\frac{d}{2}}.
\end{aligned}$$

It remains to apply Lemma A.15, more specifically (68): $C_{\mathrm{IP}}^{U_L}[H^s] \lesssim_{s,U} 1$. □

5 The Anderson Hamiltonian with Dirichlet and Neumann boundary conditions

Based on the results obtained in Section 3 and in Section 4, we can give the definition of the Anderson Hamiltonian $-\Delta - \xi$ with Dirichlet- and with Neumann boundary conditions, and show that it is the limit of the operators $-\Delta - \xi_\varepsilon + c_\varepsilon$, where the constants c_ε are defined in (89).

The construction of the Dirichlet Anderson Hamiltonian is given in Section 5.1.

For Neumann boundary conditions we have to deal with the additional boundary term (see the beginning of Section 4). We impose another assumption (Assumption 5.7) in order to deal with these boundary terms. More precisely, we are able to handle this by means of Theorem 4.6 (b) by showing that the terms $\tilde{Y}^U_\varepsilon$ (as mentioned in the beginning of Section 4) converge. Let us indicate that the 3D white noise does not satisfy the conditions of Assumption 5.7, though the 2D white noise does. The additional assumption and the convergence of the terms $\tilde{Y}^U_\varepsilon$ are considered in Section 5.2.

The construction of the Neumann Anderson Hamiltonian is given in Section 5.3. Finally, in Section 5.4 we consider the integrated density of states associated to the Anderson Hamiltonian.

Let us now introduce the random variable M with values in $\mathbb{N}_0$ such that the conditions of the second part of Theorem 3.3 are satisfied. For that we first observe the following:

Lemma 5.1. *Assume 3.10. Let U be a bounded domain, $\sigma \in (0,\infty)$ and $\delta_- \in (0,\delta)$. Then for $L \geq 1$ and $n \in \mathbb{N}_0$*

$$\|W_n\|_{C^{\delta_-}(U_L)} \lesssim_{U,\delta_-,\delta,\sigma} L^\sigma 2^{-(\delta-\delta_-)n} \|X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}.$$

Consequently, almost surely $\lim_{n\to\infty} \|W_n\|_{\mathcal{C}^{\delta_-}(U)} = 0$.

Proof. This follows by Lemma A.5 and Corollary A.10, because $W_n = G_n * X$. The consequence follows because $X \in \mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)$ almost surely. □

5.1 The Dirichlet Anderson Hamiltonian

Assumptions for this section 5.2. In this section we assume 3.10.

Definition 5.3. Let U be a bounded domain and let $r \in (1-\delta, 1)$. Using the notation of Theorem 4.6, for $N \in \mathbb{N}_0$ we define the following symmetric forms on $H_0^r(U)$:

$$\mathcal{Z}_N[U] := \mathcal{Z}^U_{Y_N}, \quad \mathcal{Z}^\varepsilon_N[U] := \mathcal{Z}^U_{Y^\varepsilon_N}. \tag{31}$$

For $\delta_- \in (0,\delta]$ and $\gamma \in (0,\infty)$, we set

$$M(U,\delta_-;\gamma) := \inf\{N \in \mathbb{N} \mid \|W_n\|_{\mathcal{C}^{\delta_-}(U)} \leq \gamma \text{ for all } n \geq N\}. \tag{32}$$

Recalling the notation from Definition 4.9, for $M = M(U,\delta;1)$ (which attains its values in $\mathbb{N}_0$ by Lemma 5.1), we set

$$\mathcal{H}^{\mathrm{D}} := \mathcal{H}^{\mathrm{D},U} := \mathcal{H}^{\mathrm{D},U}_{W_M,\mathcal{Z}_M[U]}, \quad \mathcal{H}^{\mathrm{D}}_\varepsilon := \mathcal{H}^{\mathrm{D},U}_\varepsilon := \mathcal{H}^{\mathrm{D},U}_{W^\varepsilon_M,\mathcal{Z}^\varepsilon_M[U]}.$$

Recalling Proposition 4.13, we set

$$\lambda^{\mathrm{D}}_k(U) := \lambda^{\mathrm{D}}_k(U; W_M, \mathcal{Z}_M[U]), \quad \lambda^{\mathrm{D}}_{k;\varepsilon}(U) := \lambda^{\mathrm{D}}_k(U; W^\varepsilon_M, \mathcal{Z}^\varepsilon_M[U]).$$

Theorem 5.4. *For $\varepsilon \in (0,1)$,*

$$\mathcal{H}^{\mathrm{D}}_\varepsilon = -\Delta - \xi_\varepsilon + c_\varepsilon. \tag{33}$$

Let $(\varepsilon_n)_{n=1}^\infty$ be a sequence in $(0,1)$ such that $\varepsilon_n \to 0$. Then, there exist a subsequence $(\varepsilon_{n_m})_{m=1}^\infty$ and a subset $\Omega_1 \subseteq \Omega$ of $\mathbb{P}$-probability 1 such that on Ω_1 the following holds: for any bounded domain U, one has

$$\mathcal{H}^{\mathrm{D},U}_{\varepsilon_{n_m}} \xrightarrow{\text{NR}}_{m\to\infty} \mathcal{H}^{\mathrm{D},U}. \tag{34}$$

and for all $k \in \mathbb{N}$

$$\lim_{m\to\infty} \lambda^{\mathrm{D}}_{k;\varepsilon_{n_m}}(U) = \lambda^{\mathrm{D}}_k(U).$$

Proof. (33) follows by our choice of M (see (32)) and by Lemma 1.11 (see also Lemma 4.5) and Theorem 3.3.

Let $\sigma \in (0,1)$. By Theorem 3.3, there exist a subsequence $(\varepsilon_{n_m})_{m=1}^\infty$ and a subset $\Omega_1 \subseteq \Omega$ of $\mathbb{P}$-probability 1 such that on Ω_1, for every $N \in \mathbb{N}_0$,

$$\lim_{m\to\infty} \|X^{\varepsilon_{n_m}} - X\|_{\mathcal{C}^{\delta,\sigma}(\mathbb{R}^d)} = 0, \qquad \lim_{m\to\infty} \|Y^{\varepsilon_{n_m}}_N - Y_N\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}.$$

Observe that by Lemma A.5 and Corollary A.10, like in the proof of Lemma 5.1, for all $\varepsilon \in (0,1)$,

$$\|W_M^\varepsilon - W_M\|_{\mathcal{C}^\delta(U)} \lesssim_{U,\delta,\sigma} \|X^\varepsilon - X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)},$$

and that by Theorem 4.6 (a), for $r \in (1-\delta, 1)$,

$$[\![\mathcal{Z}_M^\varepsilon - \mathcal{Z}_M]\!]_{H_0^r(U)} \lesssim_{\delta,p,U} \|Y_M^\varepsilon - Y_M\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}.$$

Therefore, the claim follows from Theorem 4.17 and Lemma 4.15. □

Remark 5.5. Let $\mathfrak{M}$ be a random variable with values in $\mathbb{N}_0$ such that $\mathfrak{M} \geq M(U,\delta;1)$. Then, almost surely, $\mathcal{H}^{\mathrm{D}}(U) = \mathcal{H}^{\mathrm{D}}_{W_{\mathfrak{M}},\mathcal{Z}_{\mathfrak{M}}[U]}(U)$, because $\mathcal{H}^{\mathrm{D}}_{W^\varepsilon_{\mathfrak{M}},\mathcal{Z}^\varepsilon_{\mathfrak{M}}[U]}(U) = -\Delta - \xi_\varepsilon + c_\varepsilon = \mathcal{H}^{\mathrm{D}}_\varepsilon(U)$ and similarly as in Theorem 5.4, $\mathcal{H}^{\mathrm{D}}_{W_{\mathfrak{M}},\mathcal{Z}_{\mathfrak{M}}[U]}(U)$ is the limit (in the sense of (34)) of $\mathcal{H}^{\mathrm{D}}_{W^\varepsilon_{\mathfrak{M}},\mathcal{Z}^\varepsilon_{\mathfrak{M}}[U]}(U)$. We will apply this in Section 5.4 with $\mathfrak{M} = M(U,\delta;\gamma)$ for $\gamma \in (0,1]$.

As will be needed in Section 5.4, we give estimates on $M(U,\delta_-;\gamma)$ and $\mathcal{Z}_M$.

Lemma 5.6. *Let U be a bounded domain, $0 < \delta_- < \delta$ and $\sigma \in (0,\infty)$. Then, there exists a $C = C(U,\delta_-,\sigma) \in (0,\infty)$ such that for all $L \geq 1$ and $\gamma \in (0,\infty)$ one has*

$$M(U_L,\delta_-;\gamma) \leq 1 + (\delta-\delta_-)^{-1}\log_2\left(C\gamma^{-1}L^\sigma\|X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}\right). \tag{35}$$

Moreover, for $r \in (1-\delta_-,1)$, $\gamma \in (0,\infty)$ and $L \geq 1$, one has for $M_{L,\delta_-,\gamma} := M(U_L,\delta_-;\gamma)$

$$[\![\mathcal{Z}_{M_{L,\delta_-,\gamma}}[U_L]]\!]_{H_0^r(U_L)} \lesssim_{U,\delta_-,\delta,\sigma} \gamma^{-(\delta-\delta_-)^{-1}\mathfrak{b}} L^{(2+\mathfrak{b}(\delta-\delta_-)^{-1})\sigma}\|X\|^{(\delta-\delta_-)^{-1}\mathfrak{b}}_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}\mathfrak{a}. \tag{36}$$

Proof. (35) is a direct consequence of Lemma 5.1.

(36) follows by Theorem 4.6 (a) (see (16)) since by definition of $\mathfrak{b}$ and $\mathfrak{a}$ (see (6)),

$$\|Y_N\|_{\mathcal{C}^{-1+\delta_-,\sigma}(\mathbb{R}^d)} \leq 2^{\mathfrak{b}N}\mathfrak{a}, \tag{37}$$

and by using (35). □

5.2 Stochastic terms for the Neumann Anderson Hamiltonian

Assumption 5.7 (Assumption II)**.** We assume $\delta \in (\frac{1}{2},1)$ and that there exists a $\delta' \in (0,1)$ such that, for each $p \in (1,\infty)$, there exist a constant $C_p^\partial \in (0,\infty)$ and a map $\boldsymbol{\varepsilon}_p^\partial : (0,1)^3 \times \mathbb{R}^d \to (0,\infty)$ with the following properties.

(i) One has

$$\sup_{\varepsilon_1,\varepsilon_2,\lambda\in(0,1),x\in\mathbb{R}^d} \boldsymbol{\varepsilon}_p^\partial(\varepsilon_1,\varepsilon_2;\lambda,x) < \infty \tag{38}$$

and for each fixed $\lambda \in (0,1)$ and $x \in \mathbb{R}^d$, one has $\lim_{\varepsilon_1,\varepsilon_2\downarrow 0}\boldsymbol{\varepsilon}_p^\partial(\varepsilon_1,\varepsilon_2;\lambda,x) = 0$.

(ii) For every $\varepsilon_1,\varepsilon_2,\lambda \in (0,1)$, $p \in (1,\infty)$, bounded Lipschitz domain U, $x \in B(U,1)$ and $\phi \in C^2(\mathbb{R}^d)$ with $\|\phi\|_{C^2(\mathbb{R}^d)} \leq 1$ and $\mathrm{supp}(\phi) \subseteq B(0,1)$, one has (for ϕ_x^λ see Assumption 3.10)

$$\mathbb{E}[|\langle\xi_{\varepsilon_1},\mathbb{1}_U\phi_x^\lambda\rangle|^p] \leq C_p^\partial\lambda^{(-2+\delta+\delta')p} \tag{39}$$

$$\mathbb{E}[|\langle\xi_{\varepsilon_1}-\xi_{\varepsilon_2},\mathbb{1}_U\phi_x^\lambda\rangle|^p] \leq \boldsymbol{\varepsilon}_p^\partial(\varepsilon_1,\varepsilon_2;\lambda,x)\lambda^{(-2+\delta+\delta')p}. \tag{40}$$

Furthermore, for every bounded domain U, we assume that, as $\varepsilon \downarrow 0$, the distributions $\mathbb{1}_U \xi_\varepsilon$ converge to some limit that is independent of the mollifier ρ, in $\mathcal{S}'(\mathbb{R}^d)$ in probability.

Remark 5.8. In the proof of the Lemma 5.9 we will use the following facts, which are straightforward to check. For functions $f, g, \phi : \mathbb{R}^d \to \mathbb{R}$ (for which the following expressions make sense)

$$\begin{aligned} \langle f * \phi, g \rangle_{L^2(\mathbb{R}^d)} &= \langle f, \phi(-\cdot) * g \rangle_{L^2(\mathbb{R}^d)}, \\ \langle f * \phi(\lambda \cdot), g \rangle_{L^2(\mathbb{R}^d)} &= \lambda^{-2d} \langle f(\tfrac{1}{\lambda} \cdot) * \phi, g(\tfrac{1}{\lambda} \cdot) \rangle_{L^2(\mathbb{R}^d)}, \quad \lambda > 0, \\ (f * g)(\cdot - w) &= [f(\cdot - w)] * g = f * [g(\cdot - w)], \quad w \in \mathbb{R}^d, \\ \langle f * \phi(\cdot - w), g \rangle_{L^2(\mathbb{R}^d)} &= \langle f(\cdot - w) * \phi, g \rangle_{L^2(\mathbb{R}^d)} = \langle f * \phi, g(\cdot + w) \rangle_{L^2(\mathbb{R}^d)}, \quad w \in \mathbb{R}^d. \end{aligned}$$

Lemma 5.9. *Let ξ be a centered Gaussian noise whose covariance is given by*

$$\mathbb{E}[\langle \xi, \varphi \rangle \langle \xi, \psi \rangle] = \langle \gamma * \varphi, \psi \rangle_{L^2(\mathbb{R}^d)}, \quad \varphi, \psi \in C_c^\infty(\mathbb{R}^d).$$

Suppose one has a bound

$$|\gamma| \le f + g \tag{41}$$

where $g \in L^\infty(\mathbb{R}^d)$ and f satisfies $f(\lambda x) = \lambda^{-\alpha} f(x)$ with $\alpha < 3$ for every $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^d$. Furthermore, suppose f is locally integrable. Then, ξ satisfies Assumption 5.7.

Proof. As ξ is Gaussian, so is for example $\langle \xi_{\varepsilon_1}, \mathbb{1}_U \phi_x^\lambda \rangle_{\mathbb{R}^d}$. As for Gaussian random variables Z one has $\mathbb{E}[|Z|^p] = \mathbb{E}[|Z|^2]^{\frac{p}{2}} \mathbb{E}[|X|^p]$, for X a standard normal random variable, it is sufficient to consider $p = 2$.

Let U be a bounded Lipschitz domain and ϕ be as in Assumption 5.7. Let $x \in \mathbb{R}^d$. Observe that (similarly to the first equality in Remark 5.8, using that ρ is symmetric)

$$\langle \xi_{\varepsilon_1} - \xi_{\varepsilon_2}, \psi \rangle = \langle \xi * \rho_{\varepsilon_1} - \xi * \rho_{\varepsilon_2}, \psi \rangle = \langle \xi * (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}), \psi \rangle = \langle \xi, (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) * \psi \rangle$$

Therefore,

$$\mathbb{E}[\langle \xi_{\varepsilon_1} - \xi_{\varepsilon_2}, \mathbb{1}_U \phi_x^\lambda \rangle^2] = \langle \gamma * (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) * (\mathbb{1}_U \phi_x^\lambda), (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) * (\mathbb{1}_U \phi_x^\lambda) \rangle_{L^2(\mathbb{R}^d)}.$$

We set

$$\boldsymbol{\varepsilon}_2^\partial(\varepsilon_1, \varepsilon_2; \lambda, x) := \lambda^\alpha \big| \langle \gamma * (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) * (\mathbb{1}_U \phi_x^\lambda), (\rho_{\varepsilon_1} - \rho_{\varepsilon_2}) * (\mathbb{1}_U \phi_x^\lambda) \rangle_{L^2(\mathbb{R}^d)} \big|$$

For fixed λ, one has

$$\lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} \langle \gamma * \rho_{\varepsilon_1} * (\mathbb{1}_U \phi_x^\lambda), \rho_{\varepsilon_2} * (\mathbb{1}_U \phi_x^\lambda) \rangle_{L^2(\mathbb{R}^d)} = \langle \gamma * (\mathbb{1}_U \phi_x^\lambda), (\mathbb{1}_U \phi_x^\lambda) \rangle_{L^2(\mathbb{R}^d)},$$

and hence $\lim_{\varepsilon_1, \varepsilon_2 \downarrow 0} \boldsymbol{\varepsilon}_2^\partial(\varepsilon_1, \varepsilon_2; \lambda, x) = 0$. To prove the bound (38) and (39), it suffices to show

$$\sup_{\varepsilon \in (0,1), \lambda \in (0,1), x \in \mathbb{R}^d} \lambda^\alpha \big| \langle \gamma * \rho_\varepsilon * (\mathbb{1}_U \phi_x^\lambda), \rho_\varepsilon * (\mathbb{1}_U \phi_x^\lambda) \rangle_{L^2(\mathbb{R}^d)} \big| < \infty.$$

Let us write

$$U_x^\lambda = \lambda^{-1}(U - x). \tag{42}$$

By Remark 5.8 and (41),

$$\sup_{x\in\mathbb{R}^d}\left|\langle \gamma * \rho_\varepsilon * (\mathbb{1}_U \phi_x^\lambda), \rho_\varepsilon * (\mathbb{1}_U \phi_x^\lambda)\rangle_{L^2(\mathbb{R}^d)}\right|$$
$$= \sup_{x\in\mathbb{R}^d}|\langle \gamma(\lambda\cdot) * (\mathbb{1}_{U_x^\lambda}\phi) * \rho_{\varepsilon/\lambda}, (\mathbb{1}_{U_x^\lambda}\phi) * \rho_{\varepsilon/\lambda}\rangle|$$
$$\le \lambda^{-\alpha}\langle f * |\phi| * |\rho_{\varepsilon/\lambda}|, |\phi| * |\rho_{\varepsilon/\lambda}|\rangle + \langle g(\lambda\cdot) * |\phi| * |\rho_{\varepsilon/\lambda}|, |\phi| * |\rho_{\varepsilon/\lambda}|\rangle.$$

Since, using Young's inequality one can bound the second term by

$$\|g\|_{L^\infty(\mathbb{R}^d)}\|\phi\|^2_{L^1(\mathbb{R}^d)}\|\rho\|^2_{L^1(\mathbb{R}^d)},$$

it comes down to showing

$$\sup_{\mu\in(0,\infty)}\langle f * |\phi| * |\rho_\mu|, |\phi| * |\rho_\mu|\rangle < \infty.$$

- Suppose $\mu \le 1$. Let $\sigma > d$. By the weighted Young's inequality, Theorem A.2, one has

$$\langle f * |\phi_U| * |\rho_\mu|, |\phi_U| * |\rho_\mu|\rangle \lesssim_\sigma \|w_\sigma f\|_{L^1(\mathbb{R}^d)}\|w_{-\sigma}\phi\|^2_{L^2(\mathbb{R}^d)}\|w_{-\sigma}\rho_\mu\|^2_{L^1(\mathbb{R}^d)}.$$

As ϕ is a continuous function with compact support, we have $\|w_{-\sigma}\phi\|_{L^2(\mathbb{R}^d)} < \infty$. Since f is locally integrable and satisfies the scaling property,

$$\|w_\sigma f\|_{L^1(\mathbb{R}^d)} = \int_{\partial B(0,1)}|f(x)|\,\mathrm{d}S(x)\int_0^\infty r^{d-1-\alpha}(1+r^2)^{-\frac{\sigma}{2}}\,\mathrm{d}r < \infty.$$

Then, we observe, as $\mu \le 1$,

$$\|w_{-\sigma}\rho_\mu\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(1+\mu^2|x|^2)^{\frac{\sigma}{2}}|\rho|(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d}(1+|x|^2)^{\frac{\sigma}{2}}|\rho|(x)\,\mathrm{d}x < \infty.$$

- Now suppose $\mu \ge 1$. By change of variables, see Remark 5.8,

$$\langle f * |\phi| * |\rho_\mu|, |\phi| * |\rho_\mu|\rangle = \mu^{-\alpha}\langle f * |\phi_{\mu^{-1}}| * |\rho|, |\phi_{\mu^{-1}}| * |\rho|\rangle.$$

Therefore, it reduces to the case $\mu \le 1$.

As Lemma 5.12 below shows, the estimate (40) implies the convergence of $\mathbb{1}_U\xi_\varepsilon$. To see the independence of the mollifier ρ, we note that the limit ξ^U is a centered Gaussian and

$$\mathbb{E}[\langle \xi^U, \phi\rangle^2] = \int_{U\times U}\gamma(x-y)\phi(x)\phi(y)\,\mathrm{d}x\,\mathrm{d}y. \qquad \square$$

Remark 5.10. An example of a γ that satisfies the conditions of Lemma 5.9 is the following. Let $n \in \{1,\dots,d\}$ and $d_1,\dots,d_n \in \mathbb{N}$ be such that $d = d_1+\dots+d_n$. Let $\alpha_1,\dots,\alpha_n \in (0,\infty)$ are such that $\alpha_j < d_j$ for all j and $\alpha_1+\ldots+\alpha_n < 3$. Then, for $x = (x_1,\dots,x_n) \in \mathbb{R}^{d_1}\times\dots\times\mathbb{R}^{d_n}$, we set

$$\gamma(x) := |x_1|^{-\alpha_1}\cdots|x_n|^{-\alpha_n}.$$

For this example, $f = \gamma$ and $g = 0$.

Remark 5.11. The 2D white noise ξ^{2D} does not satisfy the conditions of Lemma 5.9. However, one has

$$\mathbb{E}[\langle \xi^{\text{2D}}_{\varepsilon_1} - \xi^{\text{2D}}_{\varepsilon_2}, \mathbb{1}_U\phi_x^\lambda\rangle^2] = \lambda^{-2}\|(\mathbb{1}_{U_x^\lambda}\phi) * (\rho_{\varepsilon_1/\lambda} - \rho_{\varepsilon_2/\lambda})\|_{L^2(\mathbb{R}^d)},$$

where U_x^λ is as in (42). Therefore, the 2D white noise satisfies Assumption 5.7 as well.

Lemma 5.12. *Assume 5.7(i). If the estimate (40) holds, then there exists a ξ^U with values in $\mathcal{C}^{-2+\delta}(\mathbb{R}^d)$ such that the distributions $\mathbb{1}_U \xi_\varepsilon$ converge as follows, for all $p \in (\frac{d}{\delta'}+1,\infty)$,*

$$\lim_{\varepsilon\downarrow 0} \|\mathbb{1}_U\xi_\varepsilon - \xi^U\|_{L^p(\mathbb{P},\mathcal{C}^{-2+\delta}(\mathbb{R}^d))}.$$

If, furthermore, also the estimate (39) holds, then there exists $r \in (0,\infty)$ such that for all $p \in (\frac{d}{\delta'}+1,\infty)$,

$$\mathbb{E}[\|\xi^U\|^p_{\mathcal{C}^{-2+\delta}(\mathbb{R}^d)}] \lesssim_{p,\delta} |B(U,r)| C^\partial_p. \tag{43}$$

Proof. Let $\delta_+ \in (\delta,\delta+\delta')$. We need the wavelet characterization of Besov spaces given in Proposition C.30. Using the notation therein, we have

$$\mathbb{E}[\|\mathbb{1}_U\xi_{\varepsilon_1} - \mathbb{1}_U\xi_\varepsilon\|^p_{B^{-2+\delta_+}_{p,p}}] \lesssim_{p,\delta_+} \sum_{n\in\mathbb{N}_0} 2^{np(-2+\delta_+)}2^{-nd} \sum_{G\in\mathfrak{G}^n, m\in\mathbb{Z}^d} \mathbb{E}[|\langle \xi_{\varepsilon_1}-\xi_{\varepsilon_2}, \mathbb{1}_U 2^{nd/2}\Psi^{n,G}_m\rangle|^p]$$

Since $\psi_\mathfrak{f}$ and $\psi_\mathfrak{m}$ are compactly supported, there exists an $r\in(0,\infty)$ such that the sum with respect to m is over $\mathbb{Z}^d \cap 2^n B(U,r)$. Therefore, by (40), as $\sum_{m\in 2^n B(U,r)} \lesssim 2^{nd}|B(U,r)|$,

$$\mathbb{E}[\|\mathbb{1}_U\xi_{\varepsilon_1} - \mathbb{1}_U\xi_\varepsilon\|^p_{B^{-2+\delta_+}_{p,p}}] \lesssim_{p,\delta_+} \sum_{n\in\mathbb{N}_0} 2^{-n(\delta+\delta'-\delta_+)}2^{-nd} \sum_{m\in 2^n B(U,r)} \boldsymbol{\varepsilon}^\partial_p(\varepsilon_1,\varepsilon_2;2^{-n},2^{-\max\{n-1,0\}m}). \tag{44}$$

Because $\sum_{n\in\mathbb{N}_0} 2^{-n(\delta+\delta'-\delta_+)}2^{-nd}\sum_{m\in 2^nB(U,r)} 1 \lesssim |B(U,r)|\sum_{n\in\mathbb{N}_0} 2^{-n(\delta+\delta'-\delta_+)}$, in view of 5.7(i), the dominated convergence theorem yields that the right-hand side (and thus left-hand side) of (44) converges to 0. Now the convergence of $\mathbb{1}_U\xi_\varepsilon$ follows by the Besov embedding (61) of Lemma A.1 (we may choose and $\kappa > 0$ small enough such that $\frac{d}{p}+\kappa < \delta' < \delta_+ - \delta$). Due to 5.7 Using (39), one can similarly prove the estimate (43). □

Recall the notation G_N ($N\in\mathbb{N}_0$) from Definition 3.1.

Proposition 5.13. *Assume 5.7. Let U be a bounded Lipschitz domain, $\sigma\in(0,\infty)$ and*

$$p\in(d/\sigma+2,\infty).$$

Let $\tilde{Y}^U_\varepsilon$ be the distribution (see Definition 4.4 for the notation)

$$\varphi \mapsto \int_{\partial U} \varphi\nabla(G_0 * \xi_\varepsilon)\cdot \mathrm{d}\boldsymbol{S}.$$

Then there exists a $\tilde{Y}^U$ in $B^{-2+\delta}_{p,p}(\mathbb{R}^d)$ such that $\|\tilde{Y}^U_\varepsilon - \tilde{Y}^U\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)} \xrightarrow{\varepsilon\downarrow 0} 0$ in probability. Furthermore, for every $m\in(0,\infty)$, uniformly over $L\geq 1$ one has

$$\mathbb{E}[\|\tilde{Y}^{U_L}\|^m_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)}] \lesssim_{p,\delta,\sigma,U,m} L^{2m\sigma}\mathbb{E}[\|\xi\|^m_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}] + L^d C^\partial_m.$$

Proof. By the integration by parts formula (Lemma 4.3),

$$\langle \tilde{Y}^{U_L}_\varepsilon,\varphi\rangle = \int_{\partial U_L}\varphi\nabla(G_0*\xi_\varepsilon)\cdot\mathrm{d}\boldsymbol{S} = \int_{U_L}\nabla\varphi\cdot\nabla G_0 * \xi_\varepsilon - \int_{U_L}\varphi\Delta G_0*\xi_\varepsilon. \tag{45}$$

We first consider the first term. Let ϕ be a smooth function on $\mathbb{R}^d$ such that $\phi = 1$ on a neighborhood U. The map

$$\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}, \quad \varphi \mapsto \langle \mathbb{1}_{U_L} \nabla\varphi, \phi(L^{-1}\cdot)\nabla(G_0 * \xi)\rangle \tag{46}$$

is well-defined, is independent of ϕ and is an element of $B^{-2+\delta}_{p,p}(\mathbb{R}^d)$. Indeed, if $q \in (1,\infty)$ is such that $p^{-1} + q^{-1} = 1$, by the duality [57, Theorem 2.17],

$$|\langle \mathbb{1}_{U_L} \nabla\varphi, \phi(L^{-1}\cdot)\nabla(G_0 * \xi)\rangle| \le \|\mathbb{1}_{U_L}\nabla\varphi\|_{B^{1-\delta}_{q,q}(\mathbb{R}^d)} \|\phi(L^{-1}\cdot)\nabla(G_0*\xi)\|_{B^{-1+\delta}_{p,p}(\mathbb{R}^d)}.$$

By Lemma A.6 and Lemma A.15 (see Definition A.11 for C_{Mult}), as $1-\delta < \frac{1}{2} < 1 - \frac{1}{p} = \frac{1}{q}$,

$$\|\mathbb{1}_{U_L}\nabla\varphi\|_{B^{1-\delta}_{q,q}(\mathbb{R}^d)} \lesssim_{q,\delta} C^{U_L}_{\mathrm{Mult}}[W^{1-\delta}_q] \|\nabla\varphi\|_{B^{1-\delta}_{q,q}(\mathbb{R}^d)} \lesssim_{q,\delta,U} \|\varphi\|_{B^{2-\delta}_{q,q}(\mathbb{R}^d)}.$$

By Lemma A.1 (remember that we have $p\sigma > d$) and Lemma A.4,

$$\|\phi(L^{-1}\cdot)\nabla(G_0*\xi)\|_{B^{-1+\delta}_{p,p}(\mathbb{R}^d)} \lesssim_{p,\delta,\sigma} L^{2\sigma}\|\nabla(G_0*\xi)\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}.$$

By Lemma A.6 and Corollary A.10,

$$\|\nabla(G_0*\xi)\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)} \lesssim_{\delta,\sigma} \|\xi\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}.$$

Therefore, the distribution defined by (46) belongs to the dual space of $B^{2-\delta}_{q,q}(\mathbb{R}^d)$, which is identified with $B^{-2+\delta}_{p,p}(\mathbb{R}^d)$, and its norm in $B^{-2+\delta}_{p,p}(\mathbb{R}^d)$ is bounded by

$$C_{p,\delta,\sigma,U} L^{2\sigma}\|\xi\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}.$$

Now it is easy to see that this distribution is the limit of the first term of the right-hand side of (45) (as $\|\xi_\varepsilon - \xi\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)} \to 0$).

Now we consider the second term of the right-hand side of (45). Note

$$\int_U \varphi\Delta G_0 * \xi_\varepsilon = -\int_U \varphi\xi_\varepsilon + \int_U \varphi[\Delta(G_0 - G)] * \xi_\varepsilon.$$

The second term converges to

$$\int_U \varphi[\Delta(G_0 - G)] * \xi,$$

as $[\Delta(G_0 - G)] * \xi$ is a smooth function. The convergence and an estimate of the first term is provided by Lemma 5.12. □

5.3 The Neumann Anderson Hamiltonian

As described in the beginning of Section 4 (below Lemma 4.5), the boundary term will be dealt with by the decomposition into symmetric forms $\tilde{\mathcal{Z}}$ and $\widehat{\mathcal{Z}}$. Let us first consider the ingredients for the latter symmetric form.

Definition 5.14. Let U be a bounded Lipschitz domain. For $N \in \mathbb{N}_0$ we define

$$\begin{aligned}\widehat{Y}_N &:= G_N * (X - \xi) + (G_N - G_0) * \xi,\\ \widehat{Y}^\varepsilon_N &:= G_N * (X^\varepsilon - \xi_\varepsilon) + (G_N - G_0) * \xi_\varepsilon, \qquad \varepsilon \in (0,1).\end{aligned}$$

Lemma 5.15. *Let U be a bounded Lipschitz domain. Then for $\delta \in (\frac{1}{2},1)$ and $N \in \mathbb{N}_0$*

$$\|\widehat{Y}_N\|_{\mathcal{C}^{1+\delta,\sigma}} \lesssim_{U,\delta} \|X-\xi\|_{\mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d)} + 2^N \|\xi\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}.$$

In particular, $\|\widehat{Y}_N - \widehat{Y}_N^\varepsilon\|_{\mathcal{C}^{1+\delta,\sigma}}$ converges in probability to 0.

Proof. The estimate and a similar one for $\|\widehat{Y}_N - \widehat{Y}_N^\varepsilon\|_{\mathcal{C}^{1+\delta,\sigma}}$ from which the convergence follows by Theorem 3.3, follow from Corollary A.10. □

Definition 5.16. Assume 3.10 and 5.7, let U be a bounded Lipschitz domain. Let $r \in (1-\delta,1)$. Using the notation of Theorem 4.6 and of Proposition 5.13, we define the following symmetric forms on $H^r(U)$, for $N \in \mathbb{N}_0$

$$\tilde{\mathcal{Z}}_N[U] := \tilde{\mathcal{Z}}^U_{\tilde{Y}^U, e^{2W_N}}, \qquad \tilde{\mathcal{Z}}^\varepsilon_N[U] := \tilde{\mathcal{Z}}^U_{\tilde{Y}^U_\varepsilon, e^{2W^\varepsilon_N}},$$
$$\widehat{\mathcal{Z}}_N[U] := \widehat{\mathcal{Z}}^U_{\widehat{Y}_N, e^{2W_N}}, \qquad \widehat{\mathcal{Z}}^\varepsilon_N[U] := \widehat{\mathcal{Z}}^U_{\widehat{Y}^\varepsilon_N, e^{2W^\varepsilon_N}}.$$

We furthermore make abuse of notation (compared to the symmetric forms on $H^r_0(U)$ as in (31)) and define the following symmetric forms on $H^r(U)$,

$$\mathcal{Z}_N[U] := \mathcal{Z}^U_{Y_N}, \quad \mathcal{Z}^\varepsilon_N[U] := \mathcal{Z}^U_{Y^\varepsilon_N}. \tag{47}$$

Then we define

$$\mathcal{Z}^{\mathrm{N}}_N[U] := \mathcal{Z}_N[U] + \tilde{\mathcal{Z}}_N[U] + \widehat{\mathcal{Z}}_N[U],$$
$$\mathcal{Z}^{\mathrm{N},\varepsilon}_N[U] := \mathcal{Z}^\varepsilon_N[U] + \tilde{\mathcal{Z}}^\varepsilon_N[U] + \widehat{\mathcal{Z}}^\varepsilon_N[U], \qquad \varepsilon \in (0,1).$$

Recalling the notations from Definition 4.9 and Proposition 4.13, for $M = M(U,\delta;1)$ (see (32)) we set

$$\mathcal{H}^{\mathrm{N}} := \mathcal{H}^{\mathrm{N},U} := \mathcal{H}^{\mathrm{N},U}_{W_M, \mathcal{Z}^{\mathrm{N}}_M[U]}, \qquad \lambda^{\mathrm{N}}_k(U) := \lambda^{\mathrm{N}}_k(U; W_M, \mathcal{Z}^{\mathrm{N}}_M[U]),$$
$$\mathcal{H}^{\mathrm{N}}_\varepsilon := \mathcal{H}^{\mathrm{N},U}_\varepsilon := \mathcal{H}^{\mathrm{N}}_{W^\varepsilon_M, \mathcal{Z}^{\mathrm{N},\varepsilon}_M[U]}, \qquad \lambda^{\mathrm{N}}_{k;\varepsilon}(U) := \lambda^{\mathrm{N}}_k(U; W^\varepsilon_M, \mathcal{Z}^{\mathrm{N},\varepsilon}_M[U]).$$

Theorem 5.17. *Assume 3.10 and 5.7. Let U be a bounded Lipschitz domain. For $\varepsilon \in (0,1)$,*

$$\mathcal{H}^{\mathrm{N}}_\varepsilon = -\Delta - \xi_\varepsilon + c_\varepsilon \tag{48}$$

Then, one has (see Definition 1.4 for "$\xrightarrow{\text{NR}}$")

$$\mathcal{H}^{\mathrm{N},U}_\varepsilon \xrightarrow{\text{NR}}_{\varepsilon\downarrow 0} \mathcal{H}^{\mathrm{N},U} \text{ in probability,}$$
$$\lambda^{\mathrm{N}}_{k;\varepsilon}(U) \xrightarrow{\varepsilon\downarrow 0} \lambda^{\mathrm{N}}_k(U) \text{ in probability,} \qquad k \in \mathbb{N}.$$

Proof. (48) follows by Theorem 3.3 and Lemma 4.5.

The rest of the proof is similar to that of Theorem 5.4 and uses Proposition 5.13 and Lemma 5.15. □

Lemma 5.18. *Assume 3.10. Let U be a bounded Lipschitz domain and*

$$\frac{1}{2} < \delta_- < \delta.$$

Let $r \in (1-\delta_-,1)$. For $\gamma \in (0,\infty)$, we define $M_{L,\delta_-,\gamma} := M(U_L,\delta_-;\gamma)$.

(a) *Then, for every* $\sigma \in (0,\infty)$, $L \geq 1$ *and* $\gamma \in (0,\infty)$,

$$[\![\mathcal{Z}_{M_{L,\delta_-,\gamma}}[U_L]]\!]_{H^r(U_L)} \lesssim_{U,\delta_-,\delta,\sigma} \gamma^{-(\delta-\delta_-)^{-1}\mathfrak{b}} L^{(2+\mathfrak{b}(\delta-\delta_-)^{-1})\sigma} \|X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}^{(\delta-\delta_-)^{-1}\mathfrak{b}} \mathfrak{a}.$$

(b) *Additionally, assume* 5.7*. Suppose that* $r \in (\frac{3}{2}-\delta_-,1)$, $\varepsilon \in (0,\delta_- - \frac{1}{2})$, $p \in (2,\infty)$ *and* $q \in (1,2)$ *are such that* $\frac{1}{p}+\frac{1}{q} = 1$ *and* (18) *holds for* $s = r$*. Then, for every* $\sigma \in (0,\infty)$, $L \geq 1$ *and* $\gamma \in (0,1]$,

$$\begin{aligned}
&[\![\mathcal{Z}^{\mathrm{N}}_{M_{L,\delta_-,\gamma}}[U_L]]\!]_{H^r(U_L)} \\
&\quad\lesssim_{U,\delta_-,\delta,\sigma,\varepsilon} \gamma^{-(\delta-\delta_-)^{-1}\mathfrak{b}} L^{(2+(\delta-\delta_-)^{-1}\mathfrak{b})\sigma} \|X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}^{(\delta-\delta_-)^{-1}\mathfrak{b}} \big(\mathfrak{a} + \|\xi\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)}\big) \\
&\qquad + L^{2\varepsilon} \|\tilde{Y}^{U_L}\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)} + L^{\sigma}\|X-\xi\|_{\mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d)}.
\end{aligned}$$

Proof. (a) follows as in the proof of Lemma 5.6 by (37) and (35).

For (b) we use (a) and estimate $\tilde{\mathcal{Z}}_{M_{L,\gamma}}[U_L]$ and $\widehat{\mathcal{Z}}_{M_{L,\gamma}}[U_L]$. By Theorem 4.6 (b) and (c) we have

$$\begin{aligned}
\tilde{\mathcal{Z}}_{M_{L,\gamma}}[U_L] &\lesssim_{\delta,\varepsilon,p,U} L^{2\varepsilon} \|e^{2W_{M_{L,\gamma}}}\|_{C^\delta(U_L)} \|\tilde{Y}^{U_L}\|_{B^{-2+\delta}_{p,p}(\mathbb{R}^d)}, \\
\widehat{\mathcal{Z}}_{M_{L,\gamma}}[U_L] &\lesssim_{\delta,\sigma,U} L^{\sigma} \|e^{2W_{M_{L,\gamma}}}\|_{C^\delta(U_L)} \|\widehat{Y}_{M_{L,\gamma}}\|_{\mathcal{C}^{1+\delta,\sigma}(\mathbb{R}^d)}.
\end{aligned}$$

As for any $x,y \in \mathbb{R}$,

$$|e^x - e^y| = e^x|1-e^{y-x}| \leq Ce^x|y-x| \leq Ce^{x\vee y}|y-x|,$$

by definition of $M_{L,\gamma}$ we have $\|e^{2W_{M_{L,\gamma}}}\|_{\mathcal{C}^\delta(U)} \lesssim 2\gamma e^{2\gamma} \leq 2e^2$.

Therefore, we obtain the desired inequality by the estimate of $\|\widehat{Y}_{M_{L,\gamma}}\|_{\mathcal{C}^{1+\delta,\sigma}(\mathbb{R}^d)}$ from Lemma 5.15 and (35). □

Without Assumption 5.7, we can still construct an artificial Neumann Anderson Hamiltonian, which will be used in Section 5.4 as a technical tool.

Definition 5.19. Assume 3.10 and let U be a bounded Lipschitz domain and $r \in (1-\delta,1)$. For $M = M(U,\delta;1)$, we set

$$\overline{\mathcal{H}}^{\mathrm{N},U} := \mathcal{H}^{\mathrm{N},U}_{W_M,\mathcal{Z}_M[U]}, \quad \overline{\lambda}^{\mathrm{N}}_k(U) := \lambda^{\mathrm{N}}_k(U;W_M,\mathcal{Z}_M[U]).$$

Remark 5.20. Similar to Remark 5.5, for $\mathfrak{M}$ being a random variable with values in $\mathbb{N}_0$ such that $\mathfrak{M} \geq M(U,\delta,1)$, one has, almost surely, $\overline{\mathcal{H}}^{\mathrm{N},U} = \overline{\mathcal{H}}^{\mathrm{N},U}_{W_{\mathfrak{M}},\mathcal{Z}_{\mathfrak{M}}[U]}$ and, under Assumption 5.7, $\mathcal{H}^{\mathrm{N},U} = \mathcal{H}^{\mathrm{N},U}_{W_{\mathfrak{M}},\mathcal{Z}_{\mathfrak{M}}[U]}$.

Remark 5.21. To construct the natural Neumann Anderson Hamiltonian without Assumption 5.7, we conjecture that "boundary renormalisation" is necessary. For instance, if ξ is the 3D white noise, the recent work [29] suggests that the operators associated to the symmetric forms

$$(u,v) \mapsto \int_U \nabla u\cdot\nabla v - (\xi_\varepsilon - c_\varepsilon)uv + c'_\varepsilon \int_{\partial U} uv \,\mathrm{d}S$$

converge, where the constants c'_ε diverge logarithmically.

5.4 Integrated density of states

The aim of this section is to construct the integrated density of states associated to the Anderson Hamiltonian with potential ξ. For this sake, we need the following assumption.

Assumption 5.22 (Assumption III)**.** Recall that our probability space Ω is the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. Then, we have maps $T_x : \Omega \to \Omega$ ($x \in \mathbb{R}^d$) of translations $\omega \to \omega(\cdot - x)$. We assume the probability measure $\mathbb{P}$ to be ergodic with respect to $(T_x)_{x\in\mathbb{R}^d}$.

Lemma 5.23 ([55, Proposition 6.1])**.** *Let ξ be a centered Gaussian field such that*

$$\lim_{x\to\infty} \mathbb{E}[\xi(0)\xi(x)] = 0.$$

Then, Assumption 5.22 holds.

Assumption 5.24. (Assumptions for this section) Throughout this section, we assume 3.10 and 5.22. We fix $\delta_-, \delta, r \in (0,1)$ and $p \in (2,\infty)$ such that $\delta_- < \delta$ and (18) is satisfied for $q \in (1,2)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and some $\varepsilon \in (0, \delta_- - \frac{1}{2})$. We also fix $\sigma \in (0, \frac{1}{4})$ satisfying

$$(2 + (\delta - \delta_-)^{-1}\mathfrak{b}(\delta_-))d\sigma < 1 - r. \tag{49}$$

We set

$$\begin{aligned}
|||\boldsymbol{\xi}||| &:= 1 + \|X\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)} + \mathfrak{a} + \|X - \xi\|_{\mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d)},\\
\mathfrak{a}^\varepsilon &:= \mathfrak{a}^\varepsilon(\delta,\sigma) := \sup_{N\in\mathbb{N}} 2^{-\mathfrak{b}N}\|Y_N^\varepsilon\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)} \in L^p(\mathbb{P}),\\
|||\boldsymbol{\xi}_{\boldsymbol{\varepsilon}}||| &:= 1 + \|X^\varepsilon\|_{\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)} + \mathfrak{a}^\varepsilon + \|X^\varepsilon - \xi_\varepsilon\|_{\mathcal{C}^{-2+2\delta,\sigma}(\mathbb{R}^d)}, \qquad \varepsilon > 0.
\end{aligned}$$

Whenever we assume 5.7, we implicitly also assume $\delta_- > \frac{1}{2}$; and for any bounded Lipshitz domain U, we set (with $\tilde{Y}^U$ as in Proposition 5.13)

$$|||\xi|||_{\partial U} := \sup_{L\in\mathbb{N}} L^{-\frac{1}{4}}\|\tilde{Y}^{U_L}\|_{B_{p,p}^{-1+\delta}(\mathbb{R}^d)}, \quad |||\xi_\varepsilon|||_{\partial U} := \sup_{L\in\mathbb{N}} L^{-\frac{1}{4}}\|\tilde{Y}_\varepsilon^{U_L}\|_{B_{p,p}^{-1+\delta}(\mathbb{R}^d)}, \quad \varepsilon > 0.$$

Remark 5.25. By Theorem 3.3, $|||\boldsymbol{\xi}||| \in L^q(\mathbb{P})$ for every $q \in [1,\infty)$ and under Assumption 5.7, by Proposition 5.13, $|||\xi|||_{\partial U} \in L^q(\mathbb{P})$ for every $q \in [1,\infty)$. By Lemma 5.6, Lemma 5.18 and the condition on σ, (49), there exists an $m \in \mathbb{N}$ such that for all bounded Lipschitz domains U, for all $L \geq 1$ and $\gamma \in (0,1]$, for $M_{L,\gamma} = M(U_L, \delta_-, \gamma)$,

$$[\![\mathscr{Z}_{M_{L,\gamma}}[U_L]]\!]_{H^r(U_L)} \lesssim_U \gamma^{-(\delta-\delta_-)^{-1}\mathfrak{b}} L^{\frac{1-r}{d}} |||\boldsymbol{\xi}|||^m, \tag{50}$$

$$[\![\mathscr{Z}^{\mathrm{N}}_{M_{L,\gamma}}[U_L]]\!]_{H^r(U_L)} \lesssim_U \gamma^{-(\delta-\delta_-)^{-1}\mathfrak{b}} L^{\frac{1-r}{d}} (|||\boldsymbol{\xi}|||^m + |||\xi|||_{\partial U}) \quad \text{under Assumption 5.7.} \tag{51}$$

(For (51) observe that $\sigma \leq \frac{1-r}{d}$ and that we may choose $\varepsilon > 0$ as in Lemma 5.18 (b) such that $2\varepsilon < \frac{1-r}{d}$.) In (50) one may replace "$\mathscr{Z}$" and "$\boldsymbol{\xi}$" by "$\mathscr{Z}^\varepsilon$" and "$\boldsymbol{\xi}_{\boldsymbol{\varepsilon}}$" and in (51) one may replace "$\mathscr{Z}^{\mathrm{N}}$", "$\boldsymbol{\xi}$" and "ξ" by "$\mathscr{Z}^{\mathrm{N},\varepsilon}$", "$\boldsymbol{\xi}_{\boldsymbol{\varepsilon}}$" and "$\xi_\varepsilon$".

Definition 5.26. Recall the notation from Definition 4.20. For a bounded domain U, $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, we set $\boldsymbol{N}^{\mathrm{D}} := \boldsymbol{N}^{\mathrm{D}}_{W_M,\mathscr{Z}_M}$ and $\boldsymbol{N}^{\mathrm{D}}_\varepsilon := \boldsymbol{N}^{\mathrm{D}}_{W^\varepsilon_M,\mathscr{Z}^\varepsilon_M}$, i.e.,

$$\boldsymbol{N}^{\mathrm{D}}(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda_k^{\mathrm{D}}(U)\leq\lambda\}}, \qquad \boldsymbol{N}^{\mathrm{D}}_\varepsilon(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda^{\mathrm{D}}_{k;\varepsilon}(U)\leq\lambda\}}.$$

If U is a bounded Lipshitz domain, we set $\overline{\boldsymbol{N}}^{\rm N} := \boldsymbol{N}^{\rm N}_{W_M,\mathcal{Z}_M}$ and $\overline{\boldsymbol{N}}^{\rm N}_\varepsilon := \boldsymbol{N}^{\rm N}_{W^\varepsilon_M,\mathcal{Z}^\varepsilon_M}$, i.e.,

$$\overline{\boldsymbol{N}}^{\rm N}(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\overline{\lambda}^{\rm N}_k(U)\le\lambda\}}, \qquad \overline{\boldsymbol{N}}^{\rm N}(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\overline{\lambda}^{\rm N}_{k;\varepsilon}(U)\le\lambda\}},$$

and under Assumption 5.7 we set $\boldsymbol{N}^{\rm N} := \boldsymbol{N}^{\rm N}_{W_M,\mathcal{Z}^{\rm N}_M}$ and $\boldsymbol{N}^{\rm N}_\varepsilon := \boldsymbol{N}^{\rm N}_{W^\varepsilon_M,\mathcal{Z}^{{\rm N},\varepsilon}_M}$, i.e.,

$$\boldsymbol{N}^{\rm N}(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda^{\rm N}_k(U)\le\lambda\}}, \qquad \boldsymbol{N}^{\rm N}_\varepsilon(U,\lambda) := \sum_{k\in\mathbb{N}} \mathbb{1}_{\{\lambda^{\rm N}_{k;\varepsilon}(U)\le\lambda\}}.$$

Remark 5.27. In most of the following we restrict our statements to $\boldsymbol{N}^{\rm D}$, $\overline{\boldsymbol{N}}^{\rm N}$ and $\boldsymbol{N}^{\rm N}$. However, by 'adding some ε's' the statements are also valid by replacing the occurrences of "$\boldsymbol{N}^{\rm D}$", "$\overline{\boldsymbol{N}}^{\rm N}$" "$\boldsymbol{N}^{\rm N}$", "$\boldsymbol{\xi}$" and "$\xi$", and by "$\boldsymbol{N}^{\rm D}$", "$\overline{\boldsymbol{N}}^{\rm N}_\varepsilon$" "$\boldsymbol{N}^{\rm N}_\varepsilon$", "$\boldsymbol{\xi}_\varepsilon$" and "$\xi_\varepsilon$".

Lemma 5.28. *Let U be a bounded Lipschitz domain. Then, for any $\theta \in (0,1)$, there exist $\lambda_{U,\theta}$, $C_{U,\theta,r} \in (0,\infty)$ and an integer $l \in \mathbb{N}$ such that for every $\lambda \ge \lambda_{\theta,U}$*

$$\boldsymbol{N}^{\rm D}(U,\lambda) \ge (1-\theta)\frac{|B(0,1)|}{(2\pi)^d}|U|\{\lambda+\theta+C_{U,\theta,r}|\!|\!|\boldsymbol{\xi}|\!|\!|^l\}^{\frac{d}{2}}, \tag{52}$$

$$\overline{\boldsymbol{N}}^{\rm N}(U,\lambda) \le (1+\theta)\frac{|B(0,1)|}{(2\pi)^d}|U|\{\lambda+\theta+C_{U,\theta,r}|\!|\!|\boldsymbol{\xi}|\!|\!|^l\}^{\frac{d}{2}}. \tag{53}$$

In particular, $\mathbb{E}[\overline{\boldsymbol{N}}^{\rm N}(U,\lambda)^m] < \infty$ for every $m \in (0,\infty)$ and $\lambda \in \mathbb{R}$.

If we furthermore assume 5.7, then

$$\boldsymbol{N}^{\rm N}(U,\lambda) \le (1+\theta)\frac{|B(0,1)|}{(2\pi)^d}|U|\{\lambda+\theta+C_{U,\theta,r}(|\!|\!|\boldsymbol{\xi}|\!|\!| + |\!|\!|\xi|\!|\!|_{\partial U})^l\}^{\frac{d}{2}}. \tag{54}$$

Proof. The proof of (52) and (54) are similar to (53), hence we only give the proof of the latter. Remember that $\boldsymbol{N}^{\rm D}_0$ and $\boldsymbol{N}^{\rm N}_0$ are the eigenvalue counting functions of $-\Delta$ with Dirichlet and Neumann boundary conditions, respectively. By Lemma 4.23 (b), there exists a $\lambda_{U,\theta} > 0$ such that for $\lambda \ge \lambda_{U,\theta}$ we have

$$\boldsymbol{N}^{\rm N}_0(U,\lambda) \le (1+\theta)^{\frac{1}{2}}\frac{|B(0,1)|}{(2\pi)^d}|U|\lambda^{\frac{d}{2}}.$$

Let $\theta' \in (0,\infty), \gamma \in (0,1]$. By Lemma 4.22, and Remark 5.20, with $\Lambda^+_{\lambda,\theta'} := \Lambda^+_{\lambda,\theta'}(W_{M_\gamma},\mathcal{Z}_{M_\gamma})$ where M_γ is the random variable $M(U,\delta;\gamma)$ (see (32)), one has

$$\overline{\boldsymbol{N}}^{\rm N}(U,\lambda) \le (1+\theta)^{\frac{1}{2}}\frac{|B(0,1)|}{(2\pi)^d}|U|(\Lambda^+_{\lambda,\theta'})^{\frac{d}{2}}.$$

Recalling the definition of $\Lambda^+_{\lambda,\theta'}$, one observes that there exists a constant $C'_{U,\theta',r}$ such that

$$\Lambda^+_{\lambda,\theta'} \le (1+\theta')e^{(2+\frac{2r}{1-r})\gamma}(\lambda+\theta'+C'_{U,\theta',r}[\![\mathcal{Z}_{M_\gamma}]\!]^{\frac{1}{1-r}}_{H^r(U)}).$$

Therefore, if $\gamma := (2+\frac{2r}{1-r})^{-1}\log(1+\theta')$ and $\theta' := (1+\theta)^{\frac{1}{2d}} - 1 \in (0,\theta)$, and $C_{U,\theta,r} = C'_{U,\theta',r}\gamma^{\frac{1-r}{d\sigma}-2}$ (see (49)), using (50), one has

$$\Lambda^+_{\lambda,\theta'} \le (1+\theta)^{\frac{1}{d}}(\lambda+\theta+C_{U,\theta,r}|\!|\!|\boldsymbol{\xi}|\!|\!|^l),$$

which yields (53). □

As a direct consequence of Lemma 4.31, we obtain the following asymptotics. These asymptotics agree with the asymptotics of the eigenvalue counting function for the Laplacian operator, as proven by Weyl (also called Weyl's law) and later generalised for a class of Schrödinger operators by Kirsch and Martinelli [40, Proposition 2.3] (observe that our results agree with the work of Mouzard on two dimensional manifolds, see [3]).

Proposition 5.29. *Let U be a bounded Lipschitz domain, then*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}^{\mathrm{D}}(U,\lambda) = \lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \overline{\boldsymbol{N}}^{\mathrm{N}}(U,\lambda) = \frac{|B(0,1)|}{(2\pi)^d}|U|,$$

and under Assumption 5.7,

$$\lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}^{\mathrm{N}}(U,\lambda) = \frac{|B(0,1)|}{(2\pi)^d}|U|.$$

Proposition 5.30. *There exist functions $\mathbb{R} \to [0,\infty)$, $\boldsymbol{\mathcal{N}}^{\mathrm{D}}$ and $\overline{\boldsymbol{\mathcal{N}}}^{\mathrm{N}}$, such that for all $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^d$, $\mathbb{P}$-almost surely and in $L^1(\mathbb{P})$, with $Q = y + [-\frac{1}{2},\frac{1}{2}]^d$,*

$$\begin{aligned}
\boldsymbol{\mathcal{N}}^{\mathrm{D}}(\lambda) &= \lim_{L\to\infty} \frac{1}{L^d} \boldsymbol{N}^{\mathrm{D}}(Q_L,\lambda),\\
\overline{\boldsymbol{\mathcal{N}}}^{\mathrm{N}}(\lambda) &= \lim_{L\in\mathbb{Q}, L\to\infty} \frac{1}{L^d} \overline{\boldsymbol{N}}^{\mathrm{N}}(Q_L,\lambda)
\end{aligned}$$

exist. Moreover,

$$\begin{aligned}
\sup_{L>0} \frac{1}{L^d}\mathbb{E}[\boldsymbol{N}^{\mathrm{D}}(Q_L,\lambda)] &= \boldsymbol{\mathcal{N}}^{\mathrm{D}}(\lambda)\\
&\le \overline{\boldsymbol{\mathcal{N}}}^{\mathrm{N}}(\lambda) = \inf_{L>0} \frac{1}{L^d}\mathbb{E}[\overline{\boldsymbol{N}}^{\mathrm{N}}(Q_L,\lambda)]. \qquad (55)
\end{aligned}$$

Under Assumption 5.7, one may simultaneously replace $\overline{\boldsymbol{N}}^{\mathrm{N}}$ by $\boldsymbol{N}^{\mathrm{N}}$ and $\overline{\mathscr{N}}^{\mathrm{N}}$ by $\mathscr{N}^{\mathrm{N}}$ in the above definition and inequality.

Proof. This follows by the ergodic theorem by Akcoglu and Krengel [1], see [40, Section 3] for more details, for which applicability we use Assumption 5.22 and check the following. By Lemma 4.27 and Remark 4.28 $Q \mapsto \boldsymbol{N}^{\mathrm{D}}(Q,\lambda)$ is superadditive and $Q \mapsto \overline{\boldsymbol{N}}^{\mathrm{N}}(Q,\lambda)$ is subadditive. Furthermore, if $Q \subseteq [-1,1]^d$, Lemma 4.21 implies

$$\boldsymbol{N}^{\mathrm{D}}(Q,\lambda) \le \boldsymbol{N}^{\mathrm{D}}([-1,1]^d,\lambda) \le \overline{\boldsymbol{N}}^{\mathrm{N}}([-1,1]^d,\lambda),$$

and $\overline{\boldsymbol{N}}^{\mathrm{N}}([-1,1]^d,\lambda)$ is in $L^1(\mathbb{P})$ by Lemma 5.28. □

Remark 5.31. The cube Q in Proposition 5.30 does not need to be centered at the origin. This is important in the proof of Theorem 5.38.

Definition 5.32. We define

$$\boldsymbol{N}^{\mathrm{D}}(\lambda) := \inf_{\lambda'>\lambda} \boldsymbol{\mathcal{N}}^{\mathrm{D}}(\lambda') \text{ and } \quad \overline{\boldsymbol{N}}^{\mathrm{N}}(\lambda) := \inf_{\lambda'>\lambda} \overline{\boldsymbol{\mathcal{N}}}^{\mathrm{N}}(\lambda'),$$

respectively. Note that they are right-continuous functions that satisfy $\lim_{\lambda\to-\infty} \boldsymbol{N}^{\#}(\lambda) = 0$ for $\#$ denoting either D or N.

Definition 5.33. A sequence $(f_n)_{n\in\mathbb{N}}$ of increasing functions $\mathbb{R} \to [0,\infty)$, is said to *converge vaguely* to some function $f : \mathbb{R} \to [0,\infty)$ if $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \mathbb{R}$ that are continuity points of f.

Remark 5.34. If λ is a continuity point of $\boldsymbol{N}^{\mathrm{D}}(\lambda)$, then $\boldsymbol{N}^{\mathrm{D}}(\lambda) = \boldsymbol{\mathcal{N}}^{\mathrm{D}}(\lambda)$.

Remark 5.35. Observe that by Lemma 4.31 (b) and (50) for $\gamma = 1$, for all $\mu > 0$ and bounded Lipschitz domains U, there exists a $C_{U,\mu} > 0$ such that

$$\begin{aligned}\boldsymbol{N}^{\mathrm{D}}(U_L,\lambda) &\le \overline{\boldsymbol{N}}^{\mathrm{N}}(U_L,\lambda)\\ &\le \boldsymbol{N}^{\mathrm{D}}(U_L,\lambda+\mu) + C_{U,\mu} L^{d-\frac{1}{2}}[1+\max\{\lambda,0\} + \|\!|\boldsymbol{\xi}|\!\|^{\frac{m}{1-r}}]^{\frac{d}{2}},\end{aligned}$$

and under Assumption 5.7,

$$\begin{aligned}\boldsymbol{N}^{\mathrm{D}}(U_L,\lambda) &\le \boldsymbol{N}(U_L,\lambda)\\ &\le \boldsymbol{N}^{\mathrm{D}}(U_L,\lambda+\mu) + C_{U,\mu} L^{d-\frac{1}{2}}[1+\max\{\lambda,0\} + (\|\!|\boldsymbol{\xi}|\!\|^{m} + \|\!|\xi|\!\|_{\partial U})^{\frac{1}{1-r}}]^{\frac{d}{2}}.\end{aligned}$$

Proposition 5.36. *$\boldsymbol{N}^{\mathrm{D}} = \overline{\boldsymbol{N}}^{\mathrm{N}}$ and, under Assumption 5.7, $\boldsymbol{N}^{\mathrm{D}} = \boldsymbol{N}^{\mathrm{N}}$.*

Proof. Let λ be a continuity point of both $\boldsymbol{N}^{\mathrm{D}}$ and $\overline{\boldsymbol{N}}^{\mathrm{N}}$. By Remark 5.35 applied to $U = [-\frac{1}{2},\frac{1}{2}]^d$, by Proposition 5.30, because $\boldsymbol{\mathcal{N}}^{\mathrm{D}}(\lambda) = \boldsymbol{N}^{\mathrm{D}}(\lambda)$ and $\overline{\boldsymbol{\mathcal{N}}}^{\mathrm{N}}(\lambda) = \overline{\boldsymbol{N}}^{\mathrm{N}}(\lambda)$, see Remark 5.34, we have for all $\mu > 0$,

$$\boldsymbol{N}^{\mathrm{D}}(\lambda) \le \overline{\boldsymbol{N}}^{\mathrm{N}}(\lambda) \le \boldsymbol{N}^{\mathrm{D}}(\lambda+\mu),$$

so that the equality follows as both $\boldsymbol{N}^{\mathrm{D}}$ and $\overline{\boldsymbol{N}}^{\mathrm{N}}$ are right-continuous. Under Assumption 5.7 we can argue similarly with "$\boldsymbol{N}^{\mathrm{N}}$" instead of "$\overline{\boldsymbol{N}}^{\mathrm{N}}$". □

Definition 5.37. Thanks to Proposition 5.36, we may simply write

$$\boldsymbol{N} := \boldsymbol{N}^{\mathrm{D}} = \overline{\boldsymbol{N}}^{\mathrm{N}} (= \boldsymbol{N}^{\mathrm{N}} \quad \text{under Assumption 5.7}).$$

We call $\boldsymbol{N}$ the *integrated density of states* for the Anderson Hamiltonian with potential ξ.

Theorem 5.38. *Let U be a bounded domain. Then, almost surely,*

$$\lim_{L\to\infty} \frac{1}{|U_L|} \boldsymbol{N}^{\mathrm{D}}(U_L,\cdot) = \boldsymbol{N} \quad \textit{vaguely}. \tag{56}$$

If U is a bounded Lipschitz domain, then

$$\lim_{L\in\mathbb{N}, L\to\infty} \frac{1}{|U_L|} \overline{\boldsymbol{N}}^{\mathrm{N}}(U_L,\cdot) = \boldsymbol{N} \quad \textit{vaguely}. \tag{57}$$

Under Assumption 5.7, one can replace $\overline{\boldsymbol{N}}^{\mathrm{N}}$ by $\boldsymbol{N}^{\mathrm{N}}$ in (57).

Proof. Firstly, observe that we may assume U to be a bounded Lipschitz domain due to the monotonicity of $\boldsymbol{N}^{\mathrm{D}}(U,\lambda)$ as a function of U. By Remark 5.35 it suffices to prove (56) (also for $\boldsymbol{N}^{\mathrm{N}}$ under Assumption 5.7).

Let $\lambda \in \mathbb{R}$ be a continuity point of $\boldsymbol{N}$. We set

$$I_n := \{k \in \mathbb{Z}^d \mid k + [0,1]^d \subseteq 2^n U\}, \quad J_n := \{k \in \mathbb{Z}^d \mid (k+[0,1]^d) \cap (2^n U) \neq \varnothing\}.$$

By Lemma 4.21 and Lemma 4.27,

$$\sum_{k\in I_n} \boldsymbol{N}^{\mathrm{D}}(2^{-n}L(k+[0,1]^d),\lambda) \le \boldsymbol{N}^{\mathrm{D}}(U_L,\lambda) \le \sum_{k\in J_n} \overline{\boldsymbol{N}}^{\mathrm{N}}(2^{-n}L(k+[0,1]^d),\lambda).$$

Therefore, by Proposition 5.30 and Remark 5.34,

$$\frac{\#I_n}{2^{dn}|U|}\boldsymbol{N}^{\mathrm{D}}(\lambda) \le \liminf_{L\to\infty} \frac{1}{|U_L|}\boldsymbol{N}^{\mathrm{D}}(U_L,\lambda) \le \limsup_{L\to\infty} \frac{1}{|U_L|}\boldsymbol{N}^{\mathrm{D}}(U_L,\lambda) \le \frac{\#J_n}{2^{dn}|U|}\overline{\boldsymbol{N}}^{\mathrm{N}}(\lambda).$$

By Proposition 5.36, one has $\boldsymbol{N} = \boldsymbol{N}^{\mathrm{D}} = \overline{\boldsymbol{N}}^{\mathrm{N}}$. Thus, the proof is complete by letting $n\to\infty$. □

Remark 5.39. Recall Remark 5.27. Let $\varepsilon \in (0,1)$. There exist functions $\boldsymbol{\mathcal{N}}_\varepsilon^{\mathrm{D}}, \overline{\boldsymbol{\mathcal{N}}}_\varepsilon^{\mathrm{N}}$ and $\boldsymbol{\mathcal{N}}_\varepsilon^{\mathrm{N}}$ such that analogues statements as in Proposition 5.30 hold. Then we define $\boldsymbol{N}_\varepsilon^{\mathrm{D}}(\lambda) := \inf_{\lambda'>\lambda} \boldsymbol{\mathcal{N}}_\varepsilon^{\mathrm{D}}(\lambda')$ and similarly $\overline{\boldsymbol{N}}_\varepsilon^{\mathrm{N}}$ and $\boldsymbol{N}_\varepsilon^{\mathrm{N}}$. By analogous arguments as in Theorem 5.38 we also have $\boldsymbol{N}_\varepsilon^{\mathrm{D}} = \overline{\boldsymbol{N}}_\varepsilon^{\mathrm{N}}$ ($= \boldsymbol{N}_\varepsilon^{\mathrm{N}}$ under Assumption 5.7). In this case $\boldsymbol{N}_\varepsilon := \boldsymbol{N}_\varepsilon^{\mathrm{D}}$ is called the integrated density of states for the Anderson Hamiltonian with potential $\xi_\varepsilon - c_\varepsilon$.

For the convergence of $\boldsymbol{N}_\varepsilon$ to $\boldsymbol{N}$ as in Theorem 5.41, we introduce the following auxiliary lemma.

Lemma 5.40. *Let $\#$ denote either D or N. For all $L>0$, $\lambda\in\mathbb{R}$ and $\mu>0$,*

$$\liminf_{\varepsilon\downarrow 0} \mathbb{E}[\boldsymbol{N}_\varepsilon^{\#}(Q_L,\lambda)] \ge \mathbb{E}[\boldsymbol{N}^{\#}(Q_L,\lambda-\mu)].$$

Proof. First we observe that as $\lambda_{k;\varepsilon}^{\#}(Q_L) \to \lambda_k^{\#}(Q_L)$ in probability for all k (by Theorem 5.4 and Theorem 5.17), the following holds: For all $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n \downarrow 0$ as $n\to\infty$, there exists a subsequence $(\varepsilon_{n_m})_{m\in\mathbb{N}}$ and a $\Omega_1 \subset \Omega$ of $\mathbb{P}$-probability 1, such that on Ω_1, $\lambda_{k;\varepsilon_{n_m}}^{\#}(Q_L) \to \lambda_k^{\#}(Q_L)$ for all k, and therefore for all $\mu>0$

$$\liminf_{\varepsilon\downarrow 0} \boldsymbol{N}_\varepsilon^{\#}(Q_L,\lambda) \ge \boldsymbol{N}^{\#}(Q_L,\lambda-\mu)$$

(indeed, if $\mu_{1,m} \le \mu_{2,m} \le \cdots$ and $\mu_{k,m} \to \mu_k$ in $\mathbb{R}$ as $m\to\infty$, for all $k\in\mathbb{N}$, then $\liminf_{m\to\infty} \sum_{k=1}^\infty \mathbb{1}_{\{\mu_{k,m}\le\mu\}} \ge \sum_{k=1}^\infty \mathbb{1}_{\{\mu_k \le \mu-\varepsilon\}}$ for all $\mu\in\mathbb{R}$ and $\varepsilon>0$). Therefore, for all $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_n\downarrow 0$ as $n\to\infty$, there exists a subsequence $(\varepsilon_{n_m})_{m\in\mathbb{N}}$ such that by Fatou's lemma

$$\liminf_{m\to\infty} \mathbb{E}[\boldsymbol{N}_{\varepsilon_{n_m}}^{\#}(Q_L,\lambda)] \ge \mathbb{E}[\liminf_{m\to\infty} \boldsymbol{N}_{\varepsilon_{n_m}}^{\#}(Q_L,\lambda)] \ge \mathbb{E}[\boldsymbol{N}^{\#}(Q_L,\lambda-\mu)].$$

From this the inequality follows. □

Theorem 5.41. *$\boldsymbol{N}_\varepsilon \to \boldsymbol{N}$ vaguely.*

Proof. Let $Q := [-1/2,1/2]^d$, $L\in\mathbb{N}$ and $\mu\in(0,1)$. Let $\lambda\in\mathbb{R}$ be a continuity point of $\boldsymbol{N}$. By (55) (see also Remark 5.34)

$$\boldsymbol{N}(\lambda) - \boldsymbol{N}_\varepsilon(\lambda) \le \frac{1}{L^d}\mathbb{E}[\overline{\boldsymbol{N}}^{\mathrm{N}}(Q_L,\lambda) - \boldsymbol{N}_\varepsilon^{\mathrm{D}}(Q_L,\lambda)].$$

By Lemma 5.40, for all $L\ge 1$ and $\mu>0$,

$$A := \limsup_{\varepsilon\downarrow 0}\{\boldsymbol{N}(\lambda) - \boldsymbol{N}_\varepsilon(\lambda)\} \le \frac{1}{L^d}\mathbb{E}[\overline{\boldsymbol{N}}^{\mathrm{N}}(Q_L,\lambda) - \boldsymbol{N}^{\mathrm{D}}(Q_L,\lambda-\mu)].$$

Therefore, by (55), taking the infimum over $L\ge 1$ in the above inequality, we obtain $A \le \boldsymbol{N}(\lambda) - \boldsymbol{N}(\lambda-\mu)$ for all $\mu>0$. As λ is a continuity point of $\boldsymbol{N}$, it follows that $A\le 0$. Similarly, one can show

$$\liminf_{\varepsilon\downarrow 0}\{\boldsymbol{N}(\lambda) - \boldsymbol{N}_\varepsilon(\lambda)\} \ge 0.$$

□

Theorem 5.42. *One has the following tail estimates of the IDS.*

(a) *One has* $\lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}(\lambda) = \frac{|B(0,1)|}{(2\pi)^d}$.

(b) *For every bounded domain* U *and every* $\alpha \in (0,\infty)$*, one has*

$$\limsup_{\lambda\to-\infty} (-\lambda)^{-\alpha} \log \boldsymbol{N}(\lambda) = \limsup_{\lambda\to-\infty} (-\lambda)^{-\alpha} \log \mathbb{P}(\lambda_1^{\mathrm{D}}(U) \le \lambda), \tag{58}$$

$$\liminf_{\lambda\to-\infty} (-\lambda)^{-\alpha} \log \boldsymbol{N}(\lambda) = \liminf_{\lambda\to-\infty} (-\lambda)^{-\alpha} \log \mathbb{P}(\lambda_1^{\mathrm{D}}(U) \le \lambda). \tag{59}$$

Proof. (a) Let $Q := [0,1]^d$. By applying Fatou's lemma, then (55), using the definition of $\boldsymbol{N}^{\mathrm{D}}$ and using Lemma 5.28

$$\mathbb{E}[\liminf_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}^{\mathrm{D}}(Q,\lambda)] \le \liminf_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}(\lambda) \le \limsup_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}(\lambda) \le \frac{|B(0,1)|}{(2\pi)^d}.$$

Since $\lim_{\lambda\to\infty} \lambda^{-\frac{d}{2}} \boldsymbol{N}^{\mathrm{D}}(Q,\lambda) = \frac{|B(0,1)|}{(2\pi)^d}$ by Proposition 5.29, (a) follows.

(b) Let $\lambda < 0$. Thanks to the monotonicity of λ_1^{D} (see Proposition 4.13), we may and do assume $U = [0,L]^d$ for some $L \in (0,\infty)$. By (55) (see also Remark 5.34),

$$\mathbb{P}(\lambda_1^{\mathrm{D}}(U) \le \lambda) \le \mathbb{E}[\boldsymbol{N}^{\mathrm{D}}(U,\lambda)] \le L^d \boldsymbol{N}(\lambda).$$

Therefore, we establish that the left-hand sides are greater or equal to the right-hand sides of (58) and (59). By Lemma 4.31 (a), for $l \in (0, L/2)$ and $n \in \mathbb{N}$, one has

$$\boldsymbol{N}^{\mathrm{D}}(U_n,\lambda) \le \sum_{k\in\mathbb{Z}^d\cap[-1,n+1]^d} \boldsymbol{N}^{\mathrm{D}}(k+[-l,L+l]^d, \lambda + Kl^{-2})$$

and hence

$$\frac{1}{n^d}\mathbb{E}[\boldsymbol{N}^{\mathrm{D}}(U_n,\lambda)] \le \frac{(n+2)^d}{n^d}\mathbb{E}[\boldsymbol{N}^{\mathrm{D}}([0,L+2l]^d, \lambda + Kl^{-2})].$$

Letting $n \to \infty$, for $p, q \in (1,\infty)$ with $p^{-1} + q^{-1} = 1$, one obtains

$$\begin{aligned}
\boldsymbol{N}(\lambda) &\le \mathbb{E}[\boldsymbol{N}^{\mathrm{D}}([0,L+2l]^d, \lambda + Kl^{-2})] \\
&= \mathbb{E}[\boldsymbol{N}^{\mathrm{D}}([0,L+2l]^d, \lambda + Kl^{-2})\mathbb{1}_{\{\lambda_1^{\mathrm{D}}([0,L+2l]^d)\le\lambda+Kl^{-2}\}}] \\
&\le \mathbb{E}[\boldsymbol{N}^{\mathrm{D}}([0,L+2l]^d, Kl^{-2})^q]^{\frac{1}{q}} \mathbb{P}(\lambda_1^{\mathrm{D}}([0,L+2l]^d) \le \lambda + Kl^{-2})^{\frac{1}{p}}.
\end{aligned}$$

where we applied Hölder's inequality in the second inequality. Note that

$$\mathbb{E}[\boldsymbol{N}^{\mathrm{D}}([0,L+2l]^d, Kl^{-2})^q] < \infty$$

by Lemma 4.22 and Lemma 4.23 (b). Therefore, for $U = [0, L+2l]^d$, the left-hand sides are less or equal to the right-hand sides of (58) and (59). As L and $l \in (0, L/2)$ are can be chosen arbitrarily, the equalities follow. □

A Estimates related to function spaces

A.1 Estimates in Besov spaces

This subsection gives estimates in weighted Besov spaces (see Definition 2.2).

Lemma A.1. *Let $p, q \in [1, \infty]$, $r \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in [0, \infty)$. Then*

$$\|f\|_{\mathcal{C}^{r,\sigma_1}(\mathbb{R}^d)} \lesssim_{p,r,\sigma_1} \|f\|_{B^{r+\frac{d}{p},\sigma_1}_{p,\infty}(\mathbb{R}^d)}, \tag{60}$$

$$\|f\|_{\mathcal{C}^{r,\sigma_1}(\mathbb{R}^d)} \lesssim_{p,q,r,\kappa,\sigma_1} \|f\|_{B^{r+\frac{d}{p}+\kappa,\sigma_1}_{p,q}(\mathbb{R}^d)}, \qquad \kappa > 0. \tag{61}$$

If $p\sigma_2 > d$, then

$$\|f\|_{B^{r,\sigma_1+\sigma_2}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma_1,\sigma_2} \|f\|_{\mathcal{C}^{r,\sigma_1}(\mathbb{R}^d)}. \tag{62}$$

Proof. By [65, Theorem 6.5], one has $\|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)} \sim_{p,q,r,\sigma} \|w_\sigma f\|_{B^r_{p,q}(\mathbb{R}^d)}$. Therefore (60) and (61) follow by the (unweighted) Besov embedding, see [64, Section 2.7.1]. For (62), the product estimate in the Besov space (see [46, Corollary 2.1.35], which follows also from [53, Lemma 2.1]) yields

$$\|f\|_{B^{r,\sigma_1+\sigma_2}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma_1,\sigma_2} \|w_{\sigma_2}\|_{B^{|r|+\frac{1}{2}}_{p,q}(\mathbb{R}^d)} \|f\|_{\mathcal{C}^{r,\sigma_1}(\mathbb{R}^d)}.$$

Now $\|w_{\sigma_2}\|_{B^{|r|+\frac{1}{2}}_{p,q}(\mathbb{R}^d)} \lesssim \|w_{\sigma_2}\|_{B^{|r|+\frac{3}{4}}_{p,p}(\mathbb{R}^d)} \lesssim \|w_{\sigma_2}\|_{W^{|r|+\frac{3}{4}}_{p}(\mathbb{R}^d)} \lesssim \|w_{\sigma_2}\|_{W^{|r|+1}_{p}(\mathbb{R}^d)}$ (by Lemma 2.6). Since $|\partial^m w_{\sigma_2}| \lesssim_{m,\sigma_2} w_{\sigma_2 - |m|}$, we have $\|w_{\sigma_2}\|_{W^{|r|+1}_p(\mathbb{R}^d)} < \infty$ if $p\sigma_2 > d$. ☐

Theorem A.2 (Weighted Young's inequality)**.** *Let $p, q, r \in [1, \infty]$, $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and $\sigma \in [0, \infty)$. Then*

$$\|w_\sigma (f * g)\|_{L^r} \lesssim_\sigma \|w_{-\sigma} f\|_{L^p} \|w_\sigma g\|_{L^q}.$$

Proof. Using that $w_\sigma(x) \lesssim_\sigma w_\sigma(x-y) w_{-\sigma}(y)$, one can estimate $w_\sigma(x)|f * g|(x) \le [(|f|w_{-\sigma}) * (|g|w_\sigma)](x)$, see also [51, Theorem 2.4]. The rest follows by Young's inequality, [4, Theorem 1.4]. ☐

Theorem A.3. *Let $r \in \mathbb{R}$ and $\sigma \in [0, \infty]$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\int \varphi = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-d} \varphi(\varepsilon^{-1} x)$. Then, for all $\eta \in \mathcal{C}^{r,\sigma}(\mathbb{R}^d)$, $\delta > 0$,*

$$\|\varphi_\varepsilon * \eta - \eta\|_{\mathcal{C}^{r-\delta,\sigma+\delta}(\mathbb{R}^d)} \xrightarrow{\varepsilon \downarrow 0} 0.$$

Proof. In this proof we refrain from writing "$(\mathbb{R}^d)$". Let $\delta > 0$ and $p \in (\frac{d}{\delta}, \infty)$. By Lemma A.1, (62), η is an element of $B^{r,\sigma+\delta}_{p,1}$. As by Lemma A.1, (60),

$$\|\varphi_\varepsilon * \eta - \eta\|_{\mathcal{C}^{r-\delta,\sigma+\delta}(\mathbb{R}^d)} \lesssim \|\varphi_\varepsilon * \eta - \eta\|_{B^{r,\sigma+\delta}_{p,1}} = \sum_{j=-1}^{\infty} 2^{rj} \|w_{\sigma+\delta}(\varphi_\varepsilon * \Delta_j \eta - \Delta_j \eta)\|_{L^p}.$$

It suffices to show for all i that $\|w_{\sigma+\delta}(\varphi_\varepsilon * \Delta_j \eta - \Delta_j \eta)\|_{L^p} \xrightarrow{\varepsilon \downarrow 0} 0$ and

$$\|w_{\sigma+\delta}(\varphi_\varepsilon * \Delta_j \eta - \Delta_j \eta)\|_{L^p} \lesssim \|w_{\sigma+\delta} \Delta_j \eta\|_{L^p} \tag{63}$$

As $\varphi_\varepsilon * \Delta_j \eta$ converges to $\Delta_j \eta$ almost everywhere (as it does at every Lebesgue point, see [37, Proposition 2.3.8]), the converges follows from (63) by Lebesgue's dominated convergence theorem. By the weighted Young inequality, we have $\|w_{\sigma+\delta}(\varphi_\varepsilon * \Delta_j \eta)\|_{L^p} \lesssim_\sigma \|w_{-(\sigma+\delta)} \varphi_\varepsilon\|_{L^1} \|w_{\sigma+\delta} \Delta_j \eta\|_{L^p}$. As $w_{-(\sigma+\delta)}(\varepsilon x) \le w_{-(\sigma+\delta)}(x)$ for $\varepsilon \in (0,1)$, we have $\|w_{-(\sigma+\delta)} \varphi_\varepsilon\|_{L^1} \le \|w_{-(\sigma+\delta)} \varphi\|_{L^1}$ which is finite because $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This proves (63). ☐

Lemma A.4. *Let $p,q \in [1,\infty]$, $r \in \mathbb{R}$ and $\sigma \in [0,\infty)$. Let $Z \in B^{r,\sigma}_{p,q}(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then, one has*

$$\|\phi(L^{-1}\cdot)Z\|_{B^r_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma,\phi} L^\sigma \|Z\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}, \qquad L \geq 1.$$

Proof. By the product estimate in the Besov space [57, Theorem 4.37], we have

$$\|\phi(L^{-1}\cdot)Z\|_{B^r_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r} \|\phi(L^{-1}\cdot)w_{-\sigma}\|_{\mathcal{C}^{|r|+\frac{1}{2}}(\mathbb{R}^d)} \|w_\sigma Z\|_{B^r_{p,q}(\mathbb{R}^d)}.$$

Since $\|w_\sigma Z\|_{B^r_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma} \|Z\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}$ by [65, Theorem 6.5] and $\|\cdot\|_{\mathcal{C}^{|r|+\frac{1}{2}}} \lesssim \|\cdot\|_{W^{|r|+\frac{1}{2}}_\infty} \lesssim \|\cdot\|_{W^{|r|+1}_\infty}$ by Lemma 2.6, it suffices to show

$$\|\phi(L^{-1}\cdot)w_{-\sigma}\|_{W^{|r|+1}_\infty(\mathbb{R}^d)} \lesssim_{r,\sigma,\phi} L^\sigma.$$

For this, it suffices to show

$$\|\partial^m[w_{-\sigma}\phi(L^{-1}\cdot)]\|_{L^\infty(\mathbb{R}^d)} \lesssim_{\sigma,m,\phi} L^\sigma$$

for every $m \in \mathbb{N}_0^d$. By the Leibniz rule,

$$\partial^m[w_{-\sigma}\phi(L^{-1}\cdot)] = \sum_{l=0}^m \binom{m}{l} L^{-|m-l|}\partial^l w_{-\sigma}\partial^{m-l}\phi(L^{-1}\cdot).$$

Since $|\partial^l w_{-\sigma}(x)| \lesssim_{\sigma,l} (1+|x|^2)^{\frac{\sigma-|l|}{2}}$, we obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}|\partial^m[w_{-\sigma}\phi(L^{-1}\cdot)](x)| &\lesssim_{\sigma,m} \sum_{l=0}^m \binom{m}{l} L^{-|m-l|} \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{\frac{\sigma-|l|}{2}}|\partial^{m-l}\phi(L^{-1}x)| \\ &= \sum_{l=0}^m \binom{m}{l} L^{-|m-l|} \sup_{x\in\mathbb{R}^d}(1+|Lx|^2)^{\frac{\sigma-|l|}{2}}|\partial^{m-l}\phi(x)| \\ &\lesssim_{m,\sigma,\phi} L^\sigma.\end{aligned}$$

□

Lemma A.5. *Let U be a bounded domain, $r \in \mathbb{R}$ and $\sigma \in (0,\infty)$. Then for for $Z \in \mathcal{C}^{r,\sigma}(\mathbb{R}^d)$*

$$\|Z\|_{\mathcal{C}^r(U_L)} \lesssim_{U,\sigma} L^\sigma \|Z\|_{\mathcal{C}^{r,\sigma}(\mathbb{R}^d)}, \qquad L \geq 1.$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be 1 on a neighborhood of U. Then $\|Z\|_{\mathcal{C}^r(U_L)} = \|\phi(L\cdot)Z\|_{\mathcal{C}^r(U)} \leq \|\phi(L\cdot)Z\|_{\mathcal{C}^r(\mathbb{R}^d)}$. By Lemma 2.6,

$$\|\phi(L\cdot)Z\|_{\mathcal{C}^r(\mathbb{R}^d)} \lesssim_r \|\phi(L\cdot)Z\|_{\mathcal{C}^r(\mathbb{R}^d)}.$$

Therefore, we obtain the desired estimate by an application of Lemma A.4. □

Lemma A.6. *Let $p,q \in [1,\infty]$, $r \in \mathbb{R}$, $\sigma \in [0,\infty)$, $m \in \mathbb{N}_0^d$ and $a \in \mathbb{R}$.*

(a) *One has $\|\partial^m f\|_{B^{r-|m|,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma,m} \|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}$.*

(b) *Let $\tilde{\chi}$ be a smooth function on $\mathbb{R}^d$ such that $\tilde{\chi} = 0$ in a neighborhood of 0 and all the derivatives are bounded. Then, one has $\|\mathcal{F}^{-1}[|2\pi\cdot|^{2a}\tilde{\chi}\mathcal{F}f]\|_{B^{r-2a,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,\sigma,a,\tilde{\chi}} \|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}$.*

Proof. We only prove (a), as the proof of (b) is similar. We use the notations from Definition 2.2. Let ψ be a compactly supported smooth function on $\mathbb{R}^d$ such that $\psi = 1$ on a neighborhood of $\operatorname{supp}(\chi)$ and set $\psi_j := \psi(2^{-j}\cdot)$. Because $\Delta_j \partial^m f = \mathcal{F}^{-1}[\chi(2^{-j}\cdot)\psi_j] * \Delta_j f$ for $j \in \mathbb{N}_0$, by Theorem A.2 we have

$$\|w_\sigma(\Delta_j \partial^m f)\|_{L^p(\mathbb{R}^d)} \le \|w_{-\sigma}\mathcal{F}^{-1}[(-2\pi i\cdot)^m \psi_j]\|_{L^1(\mathbb{R}^d)} \|w_\sigma \Delta_j f\|_{L^p(\mathbb{R}^d)}. \tag{64}$$

It remains to observe that for all $j \in \mathbb{N}_0$

$$\begin{aligned} 2^{-j|m|}\|w_{-\sigma}\mathcal{F}^{-1}[(-2\pi i\cdot)^m\psi_j]\|_{L^1(\mathbb{R}^d)} &= \|w_{-\sigma}\mathcal{F}^{-1}[(-2^{-j+1}\pi i\cdot)^m\psi(2^{-j}\cdot)]\|_{L^1(\mathbb{R}^d)} \\ &= \|2^{jd}\Big[w_{-\sigma}(2^{-j}\cdot)[\mathcal{F}^{-1}[(-2\pi i\cdot)^m\psi\Big](2^j\cdot)\|_{L^1(\mathbb{R}^d)} \\ &\le \|w_{-\sigma}[\mathcal{F}^{-1}[(-2\pi i\cdot)^m\psi]\|_{L^1(\mathbb{R}^d)}, \end{aligned}$$

as $w_{-\sigma}(2^{-j}\cdot) \le w_{-\sigma}$. For $j = -1$ a similar estimate as (64) holds for a $\tilde{\psi} \in C_c^\infty(\mathbb{R}^d)$ with $\tilde{\psi} = 1$ on $\operatorname{supp}(\tilde{\chi})$. □

Definition A.7. For $J \in \mathbb{N}_0$ or $J = -1$ we write

$$\Delta_{\le J} f = \sum_{j=-1}^{J} \Delta_j f, \qquad \Delta_{\ge J} f = \sum_{j=J}^{\infty} \Delta_j f.$$

Remark A.8. Observe that by definition of $\check{\chi}$ and χ (Definition 2.2), for $N \in \mathbb{N}$

$$(1-\check{\chi})(2^{-N}x) = \sum_{j=N}^{\infty} \chi(2^{-j}x), \qquad x \in \mathbb{R}^d,$$

and therefore

$$\Delta_{\ge N} f = \mathcal{F}^{-1}\Big((1-\check{\chi})(2^{-N}\cdot)\mathcal{F}f\Big), \quad \Delta_{\le N} f = \mathcal{F}^{-1}\Big(\check{\chi}(2^{-N}\cdot)\mathcal{F}f\Big).$$

Lemma A.9. *Let $p, q \in [1,\infty]$, $r, s \in \mathbb{R}$ with $r \le s$, $\sigma \in [0,\infty)$ and $N \in \mathbb{N}_0$. Then, one has (observe the difference of the positions of r and s)*

$$\begin{aligned} \|\Delta_{\ge N} f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)} &\lesssim_{s-r,\sigma} 2^{-(s-r)N}\|f\|_{B^{s,\sigma}_{p,q}(\mathbb{R}^d)}, \\ \|\Delta_{\le N} f\|_{B^{s,\sigma}_{p,q}(\mathbb{R}^d)} &\lesssim_{s-r,\sigma} 2^{(s-r)N}\|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}. \end{aligned}$$

Proof. We first observe by Remark A.8 that $\Delta_{\ge N}\Delta_j f = [2^{Nd}\mathcal{F}^{-1}(1-\check{\chi})(2^N\cdot) * f$. Thus, by [51, Theorem 2.4 and Lemma 2.6], one has

$$\|w_\sigma(\Delta_j \Delta_{\ge N} f)\|_{L^p(\mathbb{R}^d)} \lesssim \|w_\sigma(\Delta_j f)\|_{L^p(\mathbb{R}^d)}$$

Therefore,

$$\begin{aligned} \|\Delta_{\ge N} f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)} &= \Big(\sum_{j=N-1}^{\infty} 2^{jqr}\|w_\sigma(\Delta_j\Delta_{\ge N} f)\|^q_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}} \\ &\le 2^{-(N-1)(s-r)}\Big(\sum_{j=N-1}^{\infty} 2^{jqs}\|w_\sigma(\Delta_j\Delta_{\ge N} f)\|^q_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}} \\ &\lesssim_{s-r,\sigma} 2^{-N(s-r)}\Big(\sum_{j=N-1}^{\infty} 2^{jqs}\|w_\sigma(\Delta_j f)\|^q_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}} \\ &\le 2^{-N(s-r)}\|f\|_{B^{s,\sigma}_{p,q}(\mathbb{R}^d)}. \end{aligned}$$

The second inequality can be proven similarly. □

Recall the definition of an admissible kernel and of G_N, see Definition 3.4 and Definition 3.1.

Corollary A.10. *Let $p, q \in [1, \infty]$, $r, s \in \mathbb{R}$ with $r \le s$, $\sigma \in [0, \infty)$ and $N \in \mathbb{N}_0$. Let K be an admissible kernel. Set $H_N := G_N - K$. Then, one has*

$$\|G_N * f\|_{B^{r+2,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,s,\sigma} 2^{-(s-r)N} \|f\|_{B^{s,\sigma}_{p,q}(\mathbb{R}^d)},$$
$$\|H_N * f\|_{B^{s+2,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,s,\sigma} 2^{(s-r)N} \|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)},$$
$$\|(G_N - G_0) * f\|_{B^{s+2,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,s,\sigma} 2^{(s-r)N} \|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}.$$

Proof. Suppose $\psi \in C_c^\infty(\mathbb{R}^d)$ is 0 in a neighborhood of 0 and is equal to 1 on $\operatorname{supp}(1 - \check{\chi})$. If we set $g := \mathcal{F}^{-1}[|2\pi\cdot|^{-2}\psi\mathcal{F}f]$, then $G_N * f = \Delta_{\ge N} g$. Therefore, the first claimed inequality follows from Lemma A.6 (b) and Lemma A.9.

To prove the second claimed inequality, recall that one has $H_N = (G_N - G_0) + (G_0 - K)$. By Lemma C.4 below, $G_0 - K$ belongs to $\mathcal{S}(\mathbb{R}^d)$. Therefore,

$$\|(G_0 - K) * f\|_{B^{s+2,\sigma}_{p,q}(\mathbb{R}^d)} \lesssim_{p,q,r,s,\sigma} \|f\|_{B^{r,\sigma}_{p,q}(\mathbb{R}^d)}.$$

On the other hand, one has $(G_N - G_0) * f = (\Delta_{\ge N} - \Delta_{\ge 0})g = (\Delta_{\le N-1} - \Delta_{-1})g$. Hence, by Lemma A.6 (b) and by Lemma A.9, the third inequality and thus the second follow. □

A.2 Estimates of constants of functional inequalities on bounded domains

In Definition 4.10 we have introduced the smallest constant that appears in interpolation inequalities. In this section we introduce also other constants that appear in functional inequalities and study their behaviour (also under scaling of the underlying domain).

Definition A.11. Let U be a bounded domain and $p, p_1, p_2 \in [1, \infty]$, $r_1, r_2, s \in [0, \infty)$, $r \in (0, \infty)$ and $\delta \in (0, r)$. We set

$$C^U_{\mathrm{Embed}}[W^{r_1}_{p_1} \to W^{r_2}_{p_2}] := \sup_{f \in W^{r_1}_{p_1}(U)\setminus\{0\}} \frac{\|f\|_{W^{r_2}_{p_2}(U)}}{\|f\|_{W^{r_1}_{p_1}(U)}},$$
$$C^U_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\delta}_p] := \sup_{f \in W^r_{2p}(U)\setminus\{0\}} \frac{\|f^2\|_{W^{r-\delta}_p(U)}}{\|f\|^2_{W^r_{2p}(U)}}.$$

Similarly, we set $C^U_{\mathrm{Embed}}[W^{r_1}_{p_1,0} \to W^{r_2}_{p_2,0}], \ldots$ by replacing the function spaces "W^r_p" to those with zero boundary conditions "$W^r_{p,0}$". If U is a bounded Lipschitz domain, for a universal extension operator ι from U to $\mathbb{R}^d$ as in Lemma 2.7, we set

$$C^U_{\mathrm{Ext}}[W^{r_1}_{p_1}, W^{r_2}_{p_2}] := \inf\{\|\iota\|_{W^{r_1}_{p_1}(U)\to W^{r_1}_{p_1}(\mathbb{R}^d)} + \|\iota\|_{W^{r_2}_{p_2}(U)\to W^{r_2}_{p_2}(\mathbb{R}^d)} \,|\, \iota \text{ is an universal extension operator}\},$$
$$C^{\partial U}_{\mathrm{R}}(W^r_p) := \inf\{\|\mathcal{R}\|_{W^r_p(\partial U)\to W^{r+\frac{1}{p}}_p(U)} \mid \mathcal{R} \text{ is a right inverse of } \mathcal{T}_{W^{r+\frac{1}{p}}_p(U)}\},$$
$$C^{\partial U}_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\delta}_p] := \sup_{f \in W^r_{2p}(\partial U)\setminus\{0\}} \frac{\|f^2\|_{W^{r-\delta}_p(\partial U)}}{\|f\|^2_{W^r_{2p}(\partial U)}},$$
$$C^U_{\mathrm{Mult}}[W^r_p] := \sup_{f \in W^r_p(U)\setminus\{0\}} \frac{\|\mathbb{1}_U f\|_{W^r_p(\mathbb{R}^d)}}{\|f\|_{W^r_p(U)}},$$

and $C^U_{\mathrm{Ext}}[W^r_p] := C^U_{\mathrm{Ext}}[W^r_p, W^r_p]$.

Lemma A.12. *Let U be a domain.*

(a) *Let $p_1, p_2 \in (1,\infty)$ with $p_1 \le p_2$ and $r_1, r_2 \in [0,\infty)$ with $r_2 = r_1 - d(\frac{1}{p_1} - \frac{1}{p_2})$. Then, one has $C^U_{\mathrm{Embed}}[W^{r_1}_{p_1,0} \to W^{r_2}_{p_2,0}] \lesssim_{p_1,p_2,r_1} 1$. If U is a bounded Lipschitz domain, one has $C^U_{\mathrm{Embed}}[W^{r_1}_{p_1} \to W^{r_2}_{p_2}] \lesssim_{p_1,p_2,r_1} C^U_{\mathrm{Ext}}[W^{r_1}_{p_1}]$.*

(b) *Let $s \in (0,1)$. Then, one has $C^U_{\mathrm{IP}}[H^s_0] \lesssim_s 1$. If U is a bounded Lipschitz domain, one has $C^U_{\mathrm{IP}}[H^s] \lesssim_s C^U_{\mathrm{Ext}}[L^2, H^1]$.*

Proof. We only prove the claim for a bounded Lipschitz domain U.

(a) Let ι be a universal extension operator from U to $\mathbb{R}^d$. Then, by using the Sobolev embedding in $\mathbb{R}^d$ [11, Theorem 8.12.6] for the second inequality,

$$\|f\|_{W^{r_2}_{p_2}(U)} \le \|\iota(f)\|_{W^{r_2}_{p_2}(\mathbb{R}^d)} \lesssim_{p_1,p_2,r_1} \|\iota(f)\|_{W^{r_1}_{p_1}(\mathbb{R}^d)} \le \|\iota\|_{W^{r_1}_{p_1}(U)\to W^{r_1}_{p_1}(\mathbb{R}^d)}\|f\|_{W^{r_1}_{p_1}(U)},$$

and thus $\|f\|_{W^{r_2}_{p_2}(U)} \lesssim_{p_1,p_2,r_1} C^U_{\mathrm{Extend}}[W^{r_1}_{p_1}]\|f\|_{W^{r_1}_{p_1}(U)}$.

(b) We can prove the claim similarly by using the inequality [4, Proposition 2.22]

$$\|f\|_{H^s(\mathbb{R}^d)} \lesssim_s \|f\|^{1-s}_{L^2(\mathbb{R}^d)}\|f\|^s_{H^1(\mathbb{R}^d)}.$$

□

Lemma A.13. *Let U be a bounded domain, $p \in [1,\infty)$, $r \in (0,1)$ and $\varepsilon \in (0,r)$. Then we have*

$$C^U_{\mathrm{Prod}}[W^r_{2p,0} \to W^{r-\varepsilon}_{p,0}] \lesssim_{p,\varepsilon} 1,$$

and if U is a bounded Lipschitz domain

$$C^U_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\varepsilon}_{p}] \lesssim_{p,\varepsilon} 1,$$

$$C^{\partial U}_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\varepsilon}_{p}] \lesssim_{p,\varepsilon} 1 + \sup_{x\in\partial U}\Big(\int_{\partial U}\frac{\mathrm{d}y}{|x-y|^{d-1-2p\varepsilon}}\Big)^{\frac{1}{2p}}.$$

Proof. We only prove the first inequality. Since

$$\int_{|x|\ge 1}\frac{1}{|x|^{d+p(r-\varepsilon)}}\,\mathrm{d}x < \infty,$$

one has

$$\|f^2\|_{W^{r-\varepsilon}_p(\mathbb{R}^d)} \lesssim_{p,r} \|f^2\|_{L^p(\mathbb{R}^d)} + \int_{|x-y|\le 1}\frac{|f(x)^2-f(y)^2|^p}{|x-y|^{d+p(r-\varepsilon)}}\,\mathrm{d}x\,\mathrm{d}y.$$

Observe that $\|f^2\|_{L^p(\mathbb{R}^d)} \le \|f\|^2_{L^{2p}(\mathbb{R}^d)}$. Furthermore, observe that

$$\frac{|f(x)^2-f(y)^2|^p}{|x-y|^{d+p(r-\varepsilon)}} = \Big(\frac{|f(x)+f(y)|}{|x-y|^{\frac{d}{2p}-\varepsilon}}\Big)^p\Big(\frac{|f(x)-f(y)|}{|x-y|^{\frac{d}{2p}+r}}\Big)^p,$$

so that, by Hölder's inequality

$$\int_{|x-y|\le 1}\frac{|f(x)^2-f(y)^2|^p}{|x-y|^{d+p(r-\varepsilon)}}\,\mathrm{d}x\,\mathrm{d}y \le \|f\|_{W^r_{2p}(\mathbb{R}^d)}\Big(\int_{|x-y|\le 1}\frac{|f(x)+f(y)|^{2p}}{|x-y|^{d-2p\varepsilon}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2p}}.$$

Now the latter integral can be estimated by $\|f\|_{L^{2p}}$ times the following integral over the unit ball that can be estimated as follows

$$\int_{|x|\le 1}\frac{\mathrm{d}x}{|x|^{d-2p\varepsilon}} \lesssim \int_0^1\frac{dr}{r^{1-2p\varepsilon}} = \frac{1}{2p\varepsilon}.$$

□

Lemma A.14 ([62, Proposition 5.3]). *Let $p \in (1,\infty)$, $r \in (0,\frac{1}{p})$ and let U be a bounded Lipschitz domain. Then, the map*

$$B^r_{p,p}(\mathbb{R}^d) \to B^r_{p,p}(\mathbb{R}^d), \quad f \mapsto f\mathbb{1}_U$$

is a bounded linear operator.

Lemma A.15. *Let U be a bounded Lipschitz domain. Then, we have*

$$\sup_{L\ge 1} C^{U_L}_{\mathrm{Ext}}[W^{r_1}_{p_1}, W^{r_2}_{p_2}] < \infty, \quad p_1,p_2 \in [1,\infty], r_1,r_2 \in [0,\infty), \tag{65}$$

$$\sup_{L\ge 1} \|\mathcal{T}_{W^r_p(U_L)}\| < \infty, \quad p \in (1,\infty), r \in (\tfrac{1}{p}, 1+\tfrac{1}{p}), \tag{66}$$

$$\sup_{L\ge 1} C^{\partial U_L}_{\mathrm{R}}[W^r_p] < \infty, \quad p \in (1,\infty), r \in (0,1), \tag{67}$$

$$\sup_{L\ge 1} C^{U_L}_{\mathrm{Embed}}[W^{r_1}_{p_1} \to W^{r_2}_{p_2}] < \infty, \quad \begin{cases} p_1,p_2 \in (1,\infty), r_1,r_2 \in [0,\infty), \\ p_1 \le p_2, r_2 = r_1 - d(\frac{1}{p_1} - \frac{1}{p_2}), \end{cases} \tag{68}$$

$$\sup_{L\ge 1} C^{U_L}_{\mathrm{IP}}[H^s] < \infty \quad s \in (0,1), \tag{69}$$

$$\sup_{L\ge 1} C^{U_L}_{\mathrm{Mult}}[W^r_p] < \infty, \quad p \in (1,\infty), r \in (0,\tfrac{1}{p}), \tag{70}$$

$$\sup_{L\ge 1} C^{U_L}_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\varepsilon}_p] < \infty \quad p \in [1,\infty), r \in (0,1), \varepsilon \in (0,r), \tag{71}$$

$$\sup_{L\ge 1} L^{-\varepsilon} C^{\partial U_L}_{\mathrm{Prod}}[W^r_{2p} \to W^{r-\varepsilon}_p] < \infty, \quad p \in [1,\infty), r \in (0,1), \varepsilon \in (0,r). \tag{72}$$

If U is a bounded domain (that is not necessarily Lipschitz), then (68) and (71) hold by replacing the occurrences of the form "W^a_b" by "$W^a_{b,0}$".

Proof. Let ι be a universal extension operator from U to $\mathbb{R}^d$ (Definition 2.8). For $L \ge 1$, we define a universal extension operator ι_L from U_L to $\mathbb{R}^d$ by $\iota_L(f) := \iota(f(L\cdot))(L^{-1}\cdot)$. By change of variables, and using that $\partial^\alpha \iota(f) = \iota(\partial^\alpha f)$ it is straightforward to check that $\|\iota_L\|_{W^r_p(U_L)\to W^r_p(\mathbb{R}^d)} \le \|\iota\|_{W^r_p(U)\to W^r_p(\mathbb{R}^d)}$. This implies (65). The two (66) and (67) estimates can be proven similarly.

(68) and (69) follow by Lemma A.12 and by (65).

(70) First observe $r \in (0,1)$. Set $F := \iota_L(f)$ for $f \in W^r_p(U_L)$. Then, $\mathbb{1}_{U_L} f = \mathbb{1}_{U_L} F$ and thus $\|\mathbb{1}_{U_L} f\|_{W^r_p(\mathbb{R}^d)} \le \|F\|_{L^p} + [\mathbb{1}_{U_L} F]_{W^r_p(\mathbb{R}^d)}$. By change of variables, one has $[g(L^{-1}\cdot)]_{W^s_p(\mathbb{R}^d)} = L^{\frac{d}{p}-s}[g]_{W^s_p(\mathbb{R}^d)}$ and thus

$$[\mathbb{1}_{U_L} F]_{W^r_p(\mathbb{R}^d)} = L^{\frac{d}{p}-r}[\mathbb{1}_U F(L\cdot)]_{W^r_p(\mathbb{R}^d)}.$$

By Lemma A.14 and Lemma 2.6, one has

$$\begin{aligned}[\mathbb{1}_U F(L\cdot)]_{W^r_p(\mathbb{R}^d)} \le \|\mathbb{1}_U F(L\cdot)\|_{W^r_p(\mathbb{R}^d)} &\lesssim_{U,p,r} \|F(L\cdot)\|_{L^p(\mathbb{R}^d)} + [F(L\cdot)]_{W^r_p(\mathbb{R}^d)} \\ &= L^{-\frac{d}{p}}\|F\|_{L^p(\mathbb{R}^d)} + L^{-\frac{d}{p}-r}[F]_{W^r_p(\mathbb{R}^d)}.\end{aligned}$$

The claim follows because $\|F\|_{W^p_r(\mathbb{R}^d)} \le \|\iota_L\|_{W^r_p(U_L)}\|f\|_{W^p_r(U_L)} \le \|\iota\|_{W^r_p(U)}\|f\|_{W^p_r(U_L)}$.

(71) and (72) follow from Lemma A.13. □

B A regularity structure for the gPAM

In this appendix, we consider a regularity structure for the generalized Parabolic Anderson model

$$\partial_t u = \Delta u + \sum_{i,j=1}^{d} g_{i,j}(u)\partial_i u \partial_j u + \sum_{i=1}^{d} h_i(u)\partial_i u + k(u) + f(u)\xi,$$

based on the abstract theory by Bruned, Hairer and Zambotti [13]. See [6] as well.

B.1 Terminologies

Here we review some terminologies from [13].

Definition B.1. We fix a *type set* $\mathfrak{L} := \{\Xi, \mathscr{I}\}$. The symbol Ξ represents the noise ξ and the symbol $\mathscr{I}$ represents an abstract integration operator.

Definition B.2. We define the following notions regarding graphs.

(a) A *rooted tree* is a finite connected simple graph without cycles, with a distinguished vertex called the *root*. We do not allow for an empty tree but we allow for a *trivial tree* $\bullet$ which consists of only one vertex. Vertices will be called *nodes*. Given a rooted tree T, the set of nodes and that of edges are denoted by $N = N_T$ and by $E = E_T$ respectively. We denote by ρ_T the root of T. Nodes of a rooted tree are endowed with a partial order $\le$ by their distances from the root. We orient edges $(x, y) \in E$ so that $x \le y$.

(b) A *forest* is a finite simple graph without cycles. We say a forest is *rooted* if every component of the forest is a rooted tree. We allow for an empty rooted forest. Given a rooted forest F, the set of nodes and that of edges are denoted by $N = N_F$ and by $E = E_F$ respectively.

(c) A tree or a forest is called *typed* if it is endowed with a map $\mathfrak{t} : E \to \mathfrak{L}$ where E is the set of edges.

(d) We say A is a *subforest* of a forest F, and write $A \subseteq F$, if $N_A \subseteq N_F$ and $E_A \subseteq E_F$ and if $(x, y) \in E_A$ implies $\{x, y\} \subseteq N_A$. We note that a *subtree* of a rooted tree is again a rooted tree whose root is the unique vertex which is closest to the root of the original rooted tree. Therefore, a subforest of a rooted forest is again a rooted forest. If a forest is typed, a subforest inherits types by restriction. If A and B are (rooted, typed) forests, we denote by $A \sqcup B$ the disjoint union of A and B with types naturally inherited.

Definition B.3. In this paper, a typed forest F is often equipped with a *colouring* $\hat{F}$ and *decorations* $\mathrm{N}, \mathfrak{o}, \mathfrak{e}$ as follows.

(a) A pair $(F, \hat{F})$ is called a *colourful forest* if the following hold:

- $F = (E_F, N_F, \mathfrak{t})$ is a typed rooted forest.
- One has $\hat{F} : E_F \sqcup N_F \to \{0, 1, 2\}$ such that if $\hat{F}((x, y)) = i > 0$ for $(x, y) \in E_F$ then $\hat{F}(x) = \hat{F}(y) = i$.

(b) If $(F, \hat{F})$ is a colourful forest and

- $\mathrm{N} : N_F \to \mathbb{N}_0^d$,

- $\mathfrak{o} : N_F \to \mathbb{Z} \oplus \mathbb{Z}[\mathfrak{L}]$ with $\operatorname{supp}(\mathfrak{o}) \subseteq \cup_{i>0} \hat{F}^{-1}(i)$,
- $\mathfrak{e} : E_F \to \mathbb{N}_0^d$ and $\operatorname{supp}(\mathfrak{e}) \subseteq E_F \setminus \operatorname{supp}(\hat{F})$,

then the 5-tuple $(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$, also written $(F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}}$, is called a *decorated forest*. We denote by $\mathfrak{F}$ the set of decorated forests.

(c) For $x, y \in N_F$, we write $x \sim y$ if they are connected in $\cup_{i>0} \hat{F}^{-1}(i)$.

(d) Given a decoreted forest $(F, \hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e})$, we view a subforest $A \subseteq F$ as a decorated forest by restricting the associated maps $(\hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e})$.

(e) We write $\bullet^m$ for the decorated tree $(\bullet, 2, m, 0, 0)$.

Many examples of colourful forests can be found in [13].

Definition B.4. Two notions of product for forests are defined as follows.

(a) For decorated forests $\tau_i = (F_i, \hat{F}_i, \mathrm{N}_i, \mathfrak{o}_i, \mathfrak{e}_i)$ $(i = 1, 2)$, we define the *forest product* by

$$\tau_1 \cdot \tau_2 := (F_1 \sqcup F_2, \hat{F}_1 + \hat{F}_2, \mathrm{N}_1 + \mathrm{N}_2, \mathfrak{o}_1 + \mathfrak{o}_2, \mathfrak{e}_1 + \mathfrak{e}_2)$$

where, for $i \neq j$, $(\hat{F}_i, \mathrm{N}_i, \mathfrak{o}_i, \mathfrak{e}_i)$ are set to 0 on F_j.

(b) For a a decorated forest $\tau = (F, \hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e})$, we denote by $\mathscr{J}(\tau)$ the decorated tree

$$(\mathscr{J}(F), [\hat{F}], [\mathrm{N}], [\mathfrak{o}], \mathfrak{e}),$$

where $\mathscr{J}(F)$ is the tree obtained by gluing all the roots of F,

$$[\hat{F}](\rho_{\mathscr{J}(F)}) := \max_{y \text{ is a root of } F} \hat{F}(y), \quad [\hat{F}](x) = \hat{F}(x) \text{ for } x \neq \rho_{\mathscr{J}(F)}$$

and $[\mathrm{N}]$ and $[\mathfrak{o}]$ are defined at the new root by summing the values at the roots of F, and are equal to N and $\mathfrak{o}$ elsewhere, respectively. The *tree product* is defined by

$$\tau_1 \tau_2 := \mathscr{J}(\tau_1 \cdot \tau_2).$$

Definition B.5 ([13, Section 4.3])**.** For a decorated tree $\tau = (T, \hat{T})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}}$ and $k \in \mathbb{N}_0^d$, we write $\mathscr{I}_k(\tau)$ for a decorated tree $\sigma = (S, \hat{S})_{\tilde{\mathfrak{e}}}^{\tilde{\mathrm{N}},\tilde{\mathfrak{o}}}$ obtained by connecting the old root ρ_τ to a new root ρ_σ with a new edge $e = (\rho_0, \rho_\tau)$ and by defining $\hat{S}$, $\tilde{\mathrm{N}}$, $\tilde{\mathfrak{o}}$ and $\tilde{\mathfrak{e}}$ as an extension of $\hat{T}$, N, $\mathfrak{o}$ and $\mathfrak{e}$ such that

$$\hat{S}(\rho_\sigma) = \tilde{\mathrm{N}}(\rho_\sigma) = \tilde{\mathfrak{o}}(\rho_\sigma) = 0, \quad \mathfrak{t}(e) = \mathscr{I}, \quad \hat{S}(e) = 0, \quad \tilde{\mathfrak{e}}(e) = k,$$

For $i \in \{1, \dots, d\}$, we write $\mathscr{I}_i(\tau) := \mathscr{I}_{\boldsymbol{e}_i}(\tau)$, where $\boldsymbol{e}_i$ is the ith standard basis of $\mathbb{R}^d$.

Definition B.6 ([13, Definition 5.3])**.** Let δ' be the constant appearing in Assumption 3.10 and we fix a small $\kappa \in (0, \delta')$ such that $\delta + \kappa$ is irrational. We assign the *degree* $|\cdot|$ to the types by[2]

$$|\Xi| := -2 + \delta + \kappa, \quad |\mathscr{I}| := 2. \tag{73}$$

We extend the degree to $(k, v) \in \mathbb{Z}^d \oplus \mathbb{Z}[\mathfrak{L}]$ by

$$|(k, v)| := \sum_{i=1}^d k_i + a|\Xi| + b|\mathscr{I}|$$

[2] In Section 3, we set $|\Xi| := -2 + \delta$, but the new definition (73) is more convenient in Section C.6.

where $v = a\Xi + b\mathscr{I}$. For a decorated tree $\tau = (F, \hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e})$, we set

$$\hat{E}_i := \hat{F}^{-1}(i) \cap E_F, \quad \hat{E} := \hat{E}_1 \cup \hat{E}_2, \quad \hat{N}_i := \hat{F}^{-1}(i) \cap N_F$$

and we define two notions of degrees $|\cdot|_-$ and $|\cdot|_+$ by

$$|\tau|_- := \sum_{e \in E_F \setminus \hat{E}} (|\mathfrak{t}(e)| - \mathfrak{e}(e)) + \sum_{x \in N_F} \mathrm{N}(x)$$
$$|\tau|_+ := \sum_{e \in E_F \setminus \hat{E}_2} (|\mathfrak{t}(e)| - \mathfrak{e}(e)) + \sum_{x \in N_F} \mathrm{N}(x) + \sum_{x \in N_F \setminus \hat{N}_2} |\mathfrak{o}(x)|.$$

Remark B.7. Since Schauder's estimate does not hold for integer exponents, we assume that $\delta + \kappa$ is irrational.

B.2 Hopf algebras on forests and trees

In this section, we introduce Hopf algebra structures on some spaces of forests and those of trees. For this purpose, we begin with introducing contraction operators.

Definition B.8 ([13, Definition 3.18])**.** We set

$$\mathscr{K}(F, \hat{F})^{\mathrm{N},\mathfrak{o}}_{\mathfrak{e}} := (\mathscr{K}_{\hat{F}} F, \hat{F})^{[\mathrm{N}],[\mathfrak{o}]}_{[\mathfrak{e}]},$$

where

- $\mathscr{K}_{\hat{F}} F$ is the quotient forest $F/\sim$, where the equivalent relation $\sim$ is in the sense of Definition B.3-(c);
- $\hat{F}$ and $[\mathfrak{e}]$ are natural "restrictions";
- one has $[\mathrm{N}](x) := \sum_{y \sim x} \mathrm{N}(y)$;
- one has

$$[\mathfrak{o}](x) := \sum_{y \sim x} \mathfrak{o}(y) + \sum_{e \in E(x)} \mathfrak{t}(e), \quad E(x) := \{(y, z) \in E \mid y \sim z \sim x\}.$$

For a decorated forest τ and $i \in \mathbb{N}$, one has a unique decomposition $\tau = \mu \cdot \nu$ such that on ν the map $\hat{F}$ is equal to i and on each component of μ the map $\hat{F}$ is not equal to i everywhere.

Definition B.9. Then, we set

$$k_i(\nu) := \begin{cases} (\bullet, i, \sum_{x \in N_\nu} \mathrm{N}(x), 0, 0) & \text{if } \sum_{x \in N_\nu} \mathrm{N}(x) > 0 \\ \varnothing & \text{otherwise} \end{cases}$$

and

$$\mathscr{K}_i(\tau) := \mathscr{K}(\mu) \cdot k_i(\nu).$$

In addition, we denote by $\hat{\mathscr{K}}_i(\tau)$ the decorated forest that is obtained from $\mathscr{K}_i(\tau)$ by setting $\mathfrak{o}$ to 0 on $\hat{F}^{-1}(i)$.

Remark B.10. ν is allowed to be an empty forest $\varnothing$ and $k_i(\varnothing) = \varnothing$.

With these operators, one can write Hopf algebras associated to regularity structures and renormalization structures.

Definition B.11. We define vector spaces $H_1, H_\circ$ as follows.

(a) We denote by H_1 the free vector space generated by

$$\mathfrak{B}(H_1) := \{(F, \hat{F})^{\mathrm{N},\mathfrak{o}}_{\mathfrak{e}} \mid \hat{F} \le 1,\ \mathscr{K}_1(F) = F\}.$$

(b) We denote by $H_\circ$ the free vector space generated by $\mathfrak{B}(H_\circ)$, where $\tau \in \mathfrak{B}(H_\circ)$ if and only if

- τ is a tree and $\hat{F} \le 1$;
- $\mathscr{K}(\tau) = \tau$.

Definition B.12 ([13, Definition 3.3])**.** Given a decorated forest $\tau = (F, \hat{F})^{\mathrm{N},\mathfrak{o}}_{\mathfrak{e}}$, we denote by $\mathfrak{U}_1(\tau)$ the set of all subforests of F which contains $\hat{F}^{-1}(1)$ and subforests of F that are disjoint from $\hat{F}^{-1}(2)$. We set

$$\begin{aligned}\Delta_1 \tau := \sum_{A \in \mathfrak{U}_1(\tau)} \sum_{\mathrm{N}_A : \mathrm{N}_A \le \mathrm{N}} \sum_{\varepsilon_A^F} \frac{1}{\varepsilon_A^F!} \binom{\mathrm{N}}{\mathrm{N}_A} (A, F|_A, \mathrm{N}_A + \pi \varepsilon_A^F, \mathfrak{o}|_{N_A}, \mathfrak{e}|_{E_A}) \\ \otimes (F, \hat{F} \cup_1 A, \mathrm{N} - \mathrm{N}_A, \mathfrak{o} + \mathrm{N}_A + \pi(\varepsilon_A^F - \mathfrak{e}\mathbb{1}_A), \mathfrak{e}\mathbb{1}_{E_F \setminus E_A} + \varepsilon_A^F),\end{aligned} \tag{74}$$

where

- ε_A^F runs over all maps $E_F \to \mathbb{N}_0^d$ supported on the (outgoing) boundary
$$\partial(A, F) := \{(e_+, e_-) \in E_F \setminus E_A \mid e_+ \in N_A\};$$
- for $\varepsilon : E_F \to \mathbb{N}_0^d$ one defines $\pi\varepsilon : N_F \to \mathbb{Z}^d$ by
$$\pi\varepsilon(x) := \sum_{\substack{e \in E_F \\ e = (x,y) \text{ for some } y}} \varepsilon(x);$$
- $\hat{F} \cup_1 A$ is the map defined by
$$\hat{F} \cup_1 A(x) := \begin{cases} 1 & \text{if } x \in A \\ \hat{F}(x) & \text{otherwise.} \end{cases}$$

Some of the main results from [13] are the following.

Proposition B.13 ([13, Proposition 4.11])**.** *The vector space H_1 is a Hopf algebra with multiplication*

$$\mathcal{M}(\tau_1 \otimes \tau_2) := \mathscr{K}_1(\tau_1 \cdot \tau_2),$$

with unit $\varnothing$, with coproduct $(\mathscr{K}_1 \otimes \mathscr{K}_1)\Delta_1$ and with counit

$$\mathbb{1}'_{H_1}((F, \hat{F})^{\mathrm{N},\mathfrak{o}}_{\mathfrak{e}}) := \mathbb{1}_{\{\varnothing\}}((F, \hat{F})^{\mathrm{N},\mathfrak{o}}_{\mathfrak{e}}),$$

The Hopf algebra H_1 is graded with respect to $|\cdot|_-$.

Proposition B.14 ([13, Proposition 4.14])**.** *The vector space $H_\circ$ is a left comodule over the Hopf algebra H_1 with coaction*

$$(\mathscr{K}_1 \otimes \mathscr{K})\Delta_1 : H_\circ \to H_1 \otimes H_\circ.$$

B.3 Rule

We set

$$\mathfrak{T} := \{(F, \hat{F}, \mathrm{N}, \mathfrak{o}, \mathfrak{e}) \in \mathfrak{F} \mid F \text{ is a tree, } \hat{F} \equiv 0, \ \mathfrak{o} \equiv 0\}.$$

The set $\mathfrak{T}$ is a monoid with the tree product and with the trivial tree as unit. We simply write $T^{\mathrm{N}}_{\mathfrak{e}}$ for $(T, 0, \mathrm{N}, 0, \mathfrak{e}) \in \mathfrak{T}$.

Definition B.15. Given a decorated tree $T^{\mathrm{N}}_{\mathfrak{e}} \in \mathfrak{T}$, we associate to each $x \in N_T$ a node type

$$\mathscr{N}_T(x) := \mathscr{N}(x) := \big((\mathfrak{t}(e_1), \mathfrak{e}(e_1)), \dots, (\mathfrak{t}(e_n), \mathfrak{e}(e_n))\big),$$

where $(e_1, \dots, e_n)$ are the edges leaving the node x, namely, for each j one can find a $y_j \in N_T$ such that $e_j = (x, y_j)$.

Definition B.16. Let $(\mathfrak{L} \times \mathbb{N}_0)^n / \sim_n$ be the set of unordered n-tuples valued in $\mathfrak{L} \times \mathbb{N}_0$ and let $\mathcal{PN}$ be the power set of $\cup_{n \in \mathbb{N}_0} (\mathfrak{L} \times \mathbb{N}_0)^n / \sim_n$. We define the *rule* $R : \mathfrak{L} \to \mathcal{PN}$ by

$$R(\Xi) := \{()\},$$
$$R(\mathscr{I}) := \{([\mathscr{I}]_n), ([\mathscr{I}]_n, \mathscr{I}_i), ([\mathscr{I}]_n, \mathscr{I}_i, \mathscr{I}_j), ([\mathscr{I}]_n, \Xi); n \in \mathbb{N}_0, i, j \in \{1, \dots, d\}\},$$

where we write $[\mathscr{I}]_n$ for the n-tuple of $(\mathscr{I}, 0)$ and write $\mathscr{I}_i$ for $(\mathscr{I}, \boldsymbol{e}_i)$, where $\boldsymbol{e}_i$ is the ith unit vector in $\mathbb{R}^d$.

It is not difficult to show that the rule R is subcritical in the sense of [13, Definition 5.14] and complete in the sense of [13, Definition 5.20].

Definition B.17 ([13, Definition 5.8])**.** Let $\tau = T^{\mathrm{N}}_{\mathfrak{e}} \in \mathfrak{T}$.

(a) We say τ *conforms to the rule* R *at the node* x if the following hold:

- if x is the root, then $\mathscr{N}(x) \in R(\Xi)$ or $\mathscr{N}(x) \in R(\mathscr{I})$;
- otherwise, one has $\mathscr{N}(x) \in R(\mathfrak{t}(e))$, where e is the edge such that $e = (y, x)$ for some node y.

(b) We say τ *conforms to the rule* R if τ comforms to R at every node, except possibly the root.

(c) We say τ *strongly conforms to the rule* R if τ comforms to R at every node.

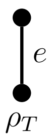
Definition B.18 ([13, Definition 5.13])**.** We define sets $\mathfrak{T}_\diamond$ ($\diamond \in \{\circ, 1, -\}$) as follows.

(a) We denote by $\mathfrak{T}_\circ \subseteq \mathfrak{T}$ the set of trees which strongly conform to R.

(b) We denote by $\mathfrak{T}_1 \subseteq \mathfrak{F}$ the smallest submonoid under the forest product which contains $\mathfrak{T}_\circ$.

(c) We denote by $\mathfrak{T}_- \subseteq \mathfrak{T}_\circ$ the set of trees $T^{\mathrm{N}}_{\mathfrak{e}}$ with the following properties:

- one has $|\tau|_- < 0$ and $\mathrm{N}(\rho_T) = 0$;
- if there exists only one edge containing ρ_T, then

$$T^{\mathrm{N}}_{\mathfrak{e}} =$$


$$, \quad \mathrm{N}(\rho_T) = \mathfrak{e}(e) = 0, \ \mathfrak{t}(e) = \Xi. \tag{75}$$

B.4 Definition of the regularity structure

The content of this section is parallel to [13, Section 5.5]. Our goal here is to construct subspaces of $H_\diamond$ ($\diamond \in \{1, \circ\}$) which provide a correct framework for the theory of regularity structures. Since we desire that elements of those spaces conform to the rule R, one might want to consider a subspace spanned by $\mathfrak{T}_\diamond$. However, $\mathfrak{T}_\diamond$ is not closed under the coproduct of $H_\diamond$. Therefore, we introduce the following definition.

Definition B.19. Recall the notation introduced in Definition B.12. For $\diamond \in \{1, \circ\}$, we denote by $\mathfrak{B}(H_\diamond^C) \subseteq \mathfrak{B}(H_\diamond)$ the set consisting of

$$\mathscr{K}_1(F, \hat{F} \cup_1 A, \mathrm{N} - \mathrm{N}_A, \mathrm{N}_A + \pi(\varepsilon_A^F - \mathfrak{e}\mathbb{1}_A), \mathfrak{e}\mathbb{1}_{F\setminus A} + \varepsilon_A^F) \tag{76}$$

for $\tau = (F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},0} \in \mathfrak{T}_\diamond$, $A \in \mathfrak{U}_1(\tau)$, $\mathrm{N}_A \leq \mathrm{N}$ with $\mathrm{supp}(\mathrm{N}_A) \subseteq N_A$ and $\varepsilon_A^F : E_F \to \mathbb{N}_0^d$ with $\mathrm{supp}(\varepsilon_A^F) \subseteq \partial(A, F)$. We denote by $H_\diamond^C$ the free vector space generated by $\mathfrak{B}(H_\diamond^C)$.

Remark B.20. By choosing $A = \varnothing$ one observes $\mathfrak{T}_\diamond \subseteq \mathfrak{B}(H_\diamond^C)$ for $\diamond \in \{1, \circ\}$. In fact, as Lemma B.21 below shows, H_1^C is the smallest subbialgebra of $H_\diamond$ both containing $\mathfrak{T}_\diamond$ and closed under the coactions.

Lemma B.21. *The subspace H_1^C is a subbialgebra of H_1. Furthermore, the statements of Proposition B.14 remain valid if one replaces $(H_1, H_\circ)$ by $(H_1^C, H_\circ^C)$.*

Proof. This is essentially proven in [13, Lemma 5.25 and Lemma 5.28]. □

Definition B.22. For $\diamond \in \{1, \circ\}$ we denote by $H_\diamond^R$ the free vector space generated by

$$\mathfrak{B}(H_\diamond^R) := \{(F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}} \in \mathfrak{B}(H_\diamond^C) \mid (F, \hat{F}\mathbb{1}_{\hat{F}\neq 1})_{\mathfrak{e}}^{\mathrm{N},0} \in \mathfrak{T}_\diamond\}.$$

Definition B.23. We denote by $\mathscr{T}_1$ the free vector space generated by

$$\mathfrak{B}(\mathscr{T}_1) := \{\tau \in \mathfrak{B}(H_1^C) \mid (F, 0)_{\mathfrak{e}}^{\mathrm{N},0} \in \mathfrak{T}_- \text{ for every connected component } (F, \hat{F})_{\mathfrak{e}}^{\mathrm{N},\mathfrak{o}} \text{ of } \tau\}$$

and by $\mathscr{T}_2$ the free vector space generated by $\mathfrak{B}(\mathscr{T}_2)$, where $\tau \in \mathfrak{B}(\mathscr{T}_2)$ if and only if

$$\hat{\mathscr{K}_2}\big[\mathscr{I}_{k_1}(\tau_1) \cdots \mathscr{I}_{k_n}(\tau_n) \bullet^m\big]$$

for $n \in \mathbb{N}_0$, $k_1, \ldots, k_n, m \in \mathbb{N}_0^d$ and $\tau_1, \ldots, \tau_n \in H_\circ^R$ such that $|\mathscr{I}_{k_j}(\tau_j)|_+ > 0$ for every $j = 1, \ldots, n$. We denote by $\mathfrak{p}_1^C : H_1^C \to \mathscr{T}_1$ the natural projection. We note that $\mathscr{T}_1$ is an algebra under the forest product and $\mathscr{T}_2$ is an algebra under the tree product.

Proposition B.24 ([13, Proposition 5.35])**.** *The linear map*

$$\Delta_1 := (\mathfrak{p}_1^C \otimes \mathfrak{p}_1^C)(\mathscr{K}_1 \otimes \mathscr{K}_1)\Delta_1 : \mathscr{T}_1 \to \mathscr{T}_1 \otimes \mathscr{T}_1$$

defines a coproduct over the algebra $\mathscr{T}_1$ (with the forest product as multiplication). With this coproduct and the counit as in Proposition B.13, $\mathscr{T}_1$ is a Hopf algebra. Furthermore, the vector space $H_\circ^R$ is a right comodule over $\mathscr{T}_1$ with coaction

$$\Delta_-^\circ := (\mathfrak{p}_1^C \otimes \mathrm{Id})(\mathscr{K}_1 \otimes \mathscr{K})\Delta_1 : H_\circ^R \to \mathscr{T}_1 \otimes H_\circ^R.$$

Remark B.25. As shown in [13, Proposition 5.34], one can view $\mathscr{T}_2$ as a Hopf algebra with grade $|\cdot|_+$ and one can define a coaction $\Delta_+^\circ : H_\circ^R \to H_\circ^R \otimes \mathscr{T}_2$, of which we do not need the precise definition here but will only use the recursive formula [13, Proposition 4.17].

Definition B.26. We set

$$\mathscr{T} := H_\circ^R, \quad \mathscr{T}_+ := \mathscr{T}_2, \quad \mathscr{T}_- := \mathscr{T}_1,$$

Then, in the language of [6], the pair $(\mathscr{T}, \mathscr{T}_+)$ is a concrete regularity structure and the pair $(\mathscr{T}_-, \mathscr{T})$ is a renormalization structure for the generalized PAM. For $\diamond \in \{-, +\}$, we denote

- by $\Delta_\diamond$ the coproduct of $\mathscr{T}_\diamond$,
- by $\mathbf{1}_\diamond$ the unit of $\mathscr{T}_\diamond$,
- by $\mathbf{1}'_\diamond$ the counit of $\mathscr{T}_\diamond$ and
- by $\mathscr{A}_\diamond$ the antipode of $\mathscr{T}_\diamond$.

Recall that the product $\mathscr{M}_-$ of $\mathscr{T}_-$ is the forest product while the product $\mathscr{M}_+$ of $\mathscr{T}_+$ is the tree product. We write $\mathbf{1} \in \mathscr{T}$ for $\bullet^0$. For $\diamond \in \{-,+\}$, the Hopf algebra $\mathscr{T}_\diamond$ is graded with $|\cdot|_\diamond$. The vector space $\mathscr{T}$ is graded both with $|\cdot|_-$ and with $|\cdot|_+$.

Definition B.27. As shown in [13, Proposition 5.39], if

$$A := \{|\tau|_+ \mid \tau \in \mathfrak{B}(\mathscr{T})\},$$

and we denote by G the character group of $\mathscr{T}_+$, the triplet $(A, \mathscr{T}, G)$ is a regularity structure in the sense of [32, Definition 2.1]. We have the graded decomposition

$$\mathscr{T} = \oplus_{\gamma\in A}\mathscr{T}_\gamma, \quad \mathscr{T}_\gamma := \operatorname{span}\{\tau \in \mathfrak{B}(\mathscr{T}) \mid |\tau|_+ = \gamma\}.$$

We write $\mathfrak{p}_{<\beta}$ for the natural projection from $\mathscr{T}$ to $\mathscr{T}_{<\beta} := \oplus_{\gamma<\beta}\mathscr{T}_\gamma$.

Definition B.28. If $\tau, \sigma \in \mathfrak{B}(\mathscr{T})$ are such that $\tau\sigma \in \mathfrak{B}(\mathscr{T})$, we write $\tau \star \sigma := \tau\sigma$. We extend the product $\star$ bilineary. Note that the product $\star$ is not defined for all pairs (τ, σ).

The following lemma essentially states that the product $\star$ is regular in the sense of [32, Definition 4.6].

Lemma B.29 ([13, Proposition 3.11])**.** *If $\tau, \sigma \in \mathfrak{B}(\mathscr{T})$ are such that $\tau\sigma \in \mathfrak{B}(\mathscr{T})$, then $\Delta^\circ_+(\tau\star\sigma) = \Delta^\circ_+(\tau)\Delta^\circ_+(\sigma)$.*

Definition B.30. Let V be the subspace of $\mathscr{T}$ generated by

$$\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n), \qquad \tau_1,\dots,\tau_n \in \mathscr{T}.$$

For $i \in \{1,\dots,d\}$, we define the linear map $\mathscr{D}_i : V \to \mathscr{T}$, called a *derivative*, by

$$\mathscr{D}_i[\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)] = \sum_{j=1}^{d}\mathscr{I}_i(\tau_j)\prod_{k\neq j}\mathscr{I}(\tau_k).$$

Proposition B.31 ([13, Section 6.1])**.** *Let $\mathfrak{i}_- : \mathscr{T}_- \to H_1^R$ be the natural projection. Then, there exists a unique algebra morphism $\hat{\mathscr{A}}_- : \mathscr{T}_- \to H_1^R$ such that*

$$\mathscr{M}_{H_1}(\hat{\mathscr{A}}_- \otimes \mathrm{Id}_{\hat{\mathscr{T}}_-})(\mathscr{K}_1\otimes\mathscr{K}_1)\Delta_1\mathfrak{i}_- = \mathbf{1}'_-(\cdot)\mathbf{1}_{H_1} \quad \text{on } \mathscr{T}_-,$$

where $\mathscr{M}_{H_1}$ is the product in H_1.

Definition B.32. We call $\hat{\mathscr{A}}_-$ a *negative twisted antipode*.

As for $\hat{\mathscr{A}}_-$, we only use the following property. As shown in [13, Proposition 6.6], one has the following recursive formula:

$$\hat{\mathscr{A}}_-\tau = -\mathscr{M}_{H_1}(\hat{\mathscr{A}}_- \otimes \mathrm{Id}_{H_1^R})(\hat{\Delta}_-\tau - \tau\otimes\mathbf{1}_-), \tag{77}$$

where $\hat{\Delta}_- := (\mathfrak{p}_1^C \otimes \mathrm{Id}_{H_1^R})(\mathscr{K}_1\otimes\mathscr{K}_1)\Delta_1$.

List of symbols from Appendix B

$\mathscr{A}_\diamond$ antipode of the Hopf algebra $\mathscr{T}_\diamond$ 53
$\hat{\mathscr{A}}_-$ negative twisted antipode 53
$\mathfrak{B}$ basis 49
$\mathscr{K}$ contraction operator for decorated trees 49
$|\cdot|$ degree or simply absolute value 48
$\mathscr{I}$ abstract symbol for integration 47
$\mathscr{J}$ joining trees at their roots 48
$\mathfrak{p}_{<\beta}$ the natural projection from $\mathscr{T}$ to $\oplus_{\gamma<\beta}\mathscr{T}_\gamma$ 53
$(\mathscr{T}, \mathscr{T}_+)$ concrete regularity structure for the gPAM 52
$(\mathscr{T}_-, \mathscr{T})$ renormalization structure for the gPAM 52
$\mathfrak{T}$ set of trees $(T,0)_{\mathfrak{e}}^{\mathfrak{n},0}$ 50
$\mathfrak{T}_-$ set of trees which conform to R and have negative homogeneity 51
$\mathfrak{T}_\circ$ set of trees which strongly conform to R 51
$\mathfrak{t}$ type map:$E \to \mathfrak{L}$ 47
$\mathfrak{L}$ type set $\{\Xi, \mathscr{I}\}$ 47
$\mathbf{1}'_\diamond$ counit of $\mathscr{T}_\diamond$ 53
$\mathfrak{T}_1$ free monoid generated by $\mathfrak{T}_\circ$ 51
$\mathbf{1}_\diamond$ unit of $\mathscr{T}_\diamond$ 53
$\bullet^m$ the decorated tree $(\bullet, 2, m, 0, 0)$ 48
E_F edge set of F 47
$(F, \hat{F})$ colourful forest 47
$(F, \hat{F}, \mathfrak{n}, \mathfrak{o}, \mathfrak{e})$ decorated forest 48
$H_\circ$ vector space of trees invariant under $\mathscr{K}$ 49
H_1 Hopf algebra of forests invariant under $\mathscr{K}_1$ 49
$H_\diamond^C$ subspace of $H_\diamond$ which contains $\mathfrak{T}_\diamond$ and is closed under coproducts 52
$H_\diamond^R$ subspace of $H_\diamond^C$ whoses basis belongs to $\mathfrak{T}_\diamond$ 52
$\mathscr{D}_i$ derivative in $\mathscr{T}$ 53
N_F node set of F 47
R rule 51
$\mathscr{N}_T(x)$ node type of T at x 51
Δ_-° coaction $\mathscr{T} \to \mathscr{T}_- \otimes \mathscr{T}$ 52
$\Delta_\diamond$ coproduct $\mathscr{T}_\diamond \to \mathscr{T}_\diamond \otimes \mathscr{T}_\diamond$ 53
Ξ abstract symbol for a noise 47
ρ_T root of a tree T 47
$\tau_1 \cdot \tau_2$ forest product of τ_1 and τ_2 48
$\tau_1\tau_2$ tree product of τ_1 and τ_2 48

C Proof of Theorem 3.3

Based on the framework discussed in Appendix B, here we provide the details to prove Theorem 3.3.

C.1 Modelled distributions

An important concept in the theory of regularity structures is the *modelled distribution* ([32, Definition 3.1]). We denote by $\mathcal{D}^\gamma(\mathscr{T}, \mathscr{Z}) = \mathcal{D}^\gamma(\mathscr{Z})$ the space of modelled distributions with respect to the model $\mathscr{Z}$ whose images are in $\mathscr{T}_{<\gamma} = \oplus_{\beta<\gamma}\mathscr{T}_\beta$. We set

$$\mathcal{D}^\gamma_\alpha(\mathscr{T},\mathscr{Z}) := \{f \in \mathcal{D}^\gamma(\mathscr{T},\mathscr{Z}) \mid f \text{ is } \oplus_{\alpha\le\beta<\gamma}\mathscr{T}_\beta\text{-valued}\}.$$

We will use the norm $|||\cdot|||_{\gamma;\mathfrak{K}}$ given by [32, (3.1)].

Definition C.1. Let $\mathscr{Z}$ be a model over $\mathscr{T}$ and let $\gamma, l > 0$. By [32, Theorem 3.10], there exists a unique continuous linear operator $\mathcal{R} = \mathcal{R}^{\mathscr{Z}} : \mathcal{D}^\gamma(\mathscr{T},\mathscr{Z}) \to \mathcal{C}^{\min A}_{\mathrm{loc}}(\mathbb{R}^d)$ with the following property: there exists a $C = C(\gamma, l, \mathscr{T}) > 0$ such that for every compact set $\mathfrak{K} \subseteq \mathbb{R}^d$

$$|(\mathcal{R}f - \Pi_x f(x))(\phi^\lambda_x)| \le C\lambda^\gamma \|\mathscr{Z}\|_{\gamma;B(\mathfrak{K},l)} |||f|||_{\gamma;B(\mathfrak{K},l)}, \quad \text{where } \phi^\lambda_x := \lambda^{-d}\phi(\lambda^{-1}(\cdot - x)), \tag{78}$$

uniformly over $\phi \in \mathcal{C}^2(B(0,l))$ with $\|\phi\|_{\mathcal{C}^r(\mathbb{R}^d)} \le 1$, $\lambda \in (0,1)$, $f \in \mathcal{D}^\gamma(\mathscr{T},\mathscr{Z})$ and $x \in \mathfrak{K}$. The operator $\mathcal{R}$ is called the *reconstruction operator*.

Proposition C.2 ([32, Theorem 4.7]). *Let V_1 and V_2 be subspaces of $\mathscr{T}$ closed under the action of the structure group. Suppose the product $\tau_1 \star \tau_2$ is well-defined for every $\tau_1 \in V_1$ and $\tau_2 \in V_2$. Let $\mathscr{Z}$ be a model for $\mathscr{T}$ and let $f_i \in \mathcal{D}^{\gamma_i}_{\alpha_i}(V_i,\mathscr{Z})$ for $i = 1, 2$. Then, if we set $\gamma := \min\{\gamma_1 + \alpha_2, \gamma_2 + \alpha_1\}$, one has $\mathfrak{p}_{<\gamma}(f_1 \star f_2) \in \mathcal{D}^\gamma_{\alpha_1+\alpha_2}(\mathscr{T},\mathscr{Z})$. Moreover, there exists a constant $C \in (0,\infty)$ which depends only on $\mathscr{T}$ such that*

$$|||\mathfrak{p}_{<\gamma}(f_1 \star f_2)|||_{\gamma;\mathfrak{K}} \le C(1 + |||\mathscr{Z}|||_{\gamma_1+\gamma_2;\mathfrak{K}})^2 |||f_1|||_{\gamma_1;\mathfrak{K}} |||f_2|||_{\gamma_2;\mathfrak{K}} \quad \textit{for every compact set } \mathfrak{K} \subseteq \mathbb{R}^d.$$

Definition C.3. Let $F \in C^\infty_b(\mathbb{R}^d)$ and let V be a subspace of $\{\tau \in \mathscr{T} \mid \mathfrak{p}_{<0}\tau = 0\}$ that is closed under the product $\star$ and under the action of the structure group. We define the map $F^\star : V \to V$ by

$$F^\star(\tau) := \sum_{k\in\mathbb{N}_0} \frac{D^k F(\bar\tau)}{k!}(\tau - \bar\tau)^{\star k}, \quad \bar\tau := \mathfrak{p}_0\tau.$$

According to [32, Theorem 4.16], if $\gamma > 0$ and $f \in \mathcal{D}^\gamma(V,\mathscr{Z})$, then one has

$$F^\star_\gamma(f)(x) := \mathfrak{p}_{<\gamma}F^\star(f(x)) \in \mathcal{D}^\gamma(V,\mathscr{Z}).$$

Furthermore, there exist a constant $C \in (0,\infty)$ and an integer $k \in \mathbb{N}$, which depend only on $\mathscr{T}$, F and γ, such that

$$|||\mathfrak{p}_{<\gamma}F(f)|||_{\gamma;\mathfrak{K}} \le C(1 + |||\mathscr{Z}|||_{\gamma;\mathfrak{K}} + |||f|||_{\gamma:\mathfrak{K}})^k \quad \text{for every compact set } \mathfrak{K} \subseteq \mathbb{R}^d. \tag{79}$$

C.2 Operations with kernels

In the rest, we fix an adimissible kernel K as in Section 3.1.2. Recall the notation G_N from Definition 3.1 and we set $H_N := G_N - K$.

Lemma C.4. *For every $N \in \mathbb{N}_0$, the function H_N belongs to $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Since the Fourier transform of $G_N - G$ has a compact support, we observe that $H_N = (G_N - G) + (G - K)$ is smooth. Thus, it comes down to showing that H_N decays rapidly or equivalently, as K is supported on $B(0,1)$, to showing that G_N decays rapidly. For $m \in \mathbb{N}^d_0$, one has

$$\partial^m G_N = \mathcal{F}^{-1}[(1-\check\chi)(2^{-N}\cdot)|\cdot|^{-2}(2\pi\mathrm{i}\cdot)^m]$$

for some polynomial P_m. Then, for $n = (n_1,\dots,n_d) \in \mathbb{N}^d_0$,

$$x^n\partial^m G_N = (2\pi i)^n\mathcal{F}^{-1}[(1-\check\chi)(2^{-N}\cdot)\partial^n(|\cdot|^{-2}P_m)] + R,$$

where $R \in \mathcal{S}(\mathbb{R}^d)$. Therefore, if $n_1,\dots,n_d$ are sufficiently large, $\partial^n(|\cdot|^{-2}P_m)$ is integrable. This means $x^n\partial^m G_N$ is bounded and hence G_N decays rapidly. □

Remark C.5. Thanks to Lemma C.4, the convolution $H_N * f$ is well-defined for $f \in \mathcal{S}'$ and the distribution $H_N * f$ represents a smooth function.

Definition C.6. For $f \in \mathcal{S}'(\mathbb{R}^d)$, $N \in \mathbb{N}_0$, we set

$$[\Delta(G_N - G)] * f := \mathcal{F}^{-1}[\check{\chi}(2^{-N}\cdot)] * f.$$

By considering their Fourier transforms, one observes

$$(\Delta G_N) * f = -f + [\Delta(G_N - G)] * f \tag{80}$$

We recall operations of kernels on modelled distributions from [32, Section 5].

Definition C.7. Let $\mathscr{Z} = (\Pi, \Gamma)$ be a model realizing K in the sense of [32, Definition 5.9].

(a) We set

$$\mathcal{J}(x)\tau := \mathcal{J}^{\mathscr{Z}}(x)\tau := \sum_{|k|<|\tau|_+ +2} \frac{X^k}{k!}\big[D^k K * \Pi_x \tau(x)\big], \quad x \in \mathbb{R}^d,$$

for $\tau \in \mathfrak{B}(\mathscr{T})$ and extend it linearly for $\tau \in \mathscr{T}$.

(b) Let $\gamma \in (0,\infty) \setminus \mathbb{N}$ and $f \in \mathcal{D}^\gamma(\mathscr{T}, \mathscr{Z})$. We set

$$\mathcal{N} f(x) := \mathcal{N}_\gamma^{\mathscr{Z}} f(x) := \sum_{|k|<\gamma+2} \frac{X^k}{k!} D^k K * (\mathcal{R}^{\mathscr{Z}} f - \Pi_x f(x))(x) \tag{81}$$

and

$$\mathcal{K} f(x) := \mathcal{K}_\gamma^{\mathscr{Z}} f(x) := (\mathscr{I} + \mathcal{J}^{\mathscr{Z}}(x)) f(x) + \mathcal{N}_\gamma^{\mathscr{Z}} f(x). \tag{82}$$

By [32, Theorem 5.12], $\mathcal{K}$ maps $\mathcal{D}^\gamma(\mathscr{T}, \mathscr{Z})$ to $\mathcal{D}^{\gamma+2}(\mathscr{T}, \mathscr{Z})$ and one has $\mathcal{R}\mathcal{K} f = K * \mathcal{R} f$. More precisely, one has

$$|||\mathcal{K} f|||_{\gamma+2;\mathfrak{K}} \lesssim_{\mathscr{T},\gamma} (1 + |||\mathscr{Z}|||_{\gamma+2;B(\mathfrak{K},1)})^2 |||f|||_{\gamma;B(\mathfrak{K},1)}. \tag{83}$$

uniformly over $\mathscr{Z} \in \mathscr{M}(\mathscr{T}, K)$, $f \in \mathcal{D}(\mathscr{T}, \mathscr{Z})$ and compact sets $\mathfrak{K} \subseteq \mathbb{R}^d$. See [34, Theorem 5.1].

(c) For a smooth function F on $\mathbb{R}^d$ and $\beta \in (0,\infty)$, we set

$$R_\beta F(x) := \sum_{|k|<\beta} \frac{X^k}{k!} D^k F(x), \quad x \in \mathbb{R}^d.$$

Then [32, Lemma 2.12] implies $R_\beta F \in \mathcal{D}^\beta(\mathscr{T}, \mathscr{Z})$.

Definition C.8. Suppose that the model $\mathscr{Z}$ realizes K. For $\gamma \in (0,\infty) \setminus \mathbb{N}$, $f \in \mathcal{D}^\gamma(\mathscr{T}, \mathscr{Z})$ and $N \in \mathbb{N}_0$, we set

$$\mathcal{G}_N f(x) := \mathcal{G}_{N,\gamma}^{\mathscr{Z}} f(x) := \mathcal{K}_\gamma^{\mathscr{Z}} f(x) + R_{\gamma+2}[H_N * \mathcal{R} f](x).$$

Note that one has $\mathcal{R}^{\mathscr{Z}} \mathcal{G}_N^{\mathscr{Z}} f = G_N * \mathcal{R}^{\mathscr{Z}} f$. For the meaning of the parameter N, see Remark C.15 below.

C.3 Definition of modelled distributions

Definition C.9. We define $\mathcal{T}, \mathcal{T}_- \subseteq \mathscr{T}$ and $\mathfrak{B}(\mathcal{T}), \mathfrak{B}(\mathcal{T}_-) \subseteq \mathfrak{B}(\mathscr{T})$ as follows.

(a) For $\tau_1, \tau_2 \in \mathscr{T}$ we write "$\nabla \mathscr{I}(\tau_1) \cdot \nabla \mathscr{I}(\tau_2)$" instead of "$\sum_{j=1}^d \mathscr{I}_j(\tau_1)\mathscr{I}_j(\tau_2)$".

(b) We denote by $\mathcal{T}$ the smallest subset of $\mathscr{T}$ with the following properties:

- $\Xi \in \mathcal{T}$ and
- if $\tau_1, \tau_2 \in \mathcal{T}$, then $\nabla \mathscr{I}(\tau_1) \cdot \nabla \mathscr{I}(\tau_2) \in \mathcal{T}$.

Furthermore, we associate $c(\tau) \in \mathbb{N}$ to each $\tau \in \mathcal{T}$ by setting $c(\Xi) := 1$ and by inductively setting for $\tau_1, \tau_2 \in \mathcal{T}$

$$c(\nabla \mathscr{I}(\tau_1) \cdot \nabla \mathscr{I}(\tau_2)) := \begin{cases} 2c(\tau_1)c(\tau_2) & \text{if } \tau_1 \neq \tau_2, \\ c(\tau_1)c(\tau_2) & \text{if } \tau_1 = \tau_2. \end{cases}$$

(c) One defines $\mathfrak{B}(\mathcal{T}) \subseteq \mathfrak{B}(\mathscr{T})$ as the minimal subset with the following properties:

- $\Xi \in \mathfrak{B}(\mathcal{T})$ and
- if $\tau_1, \tau_2 \in \mathfrak{B}(\mathcal{T})$ and $i \in \{1, \dots, d\}$, then $\mathscr{I}_i(\tau_1)\mathscr{I}_i(\tau_2) \in \mathfrak{B}(\mathcal{T})$.

(d) We set $\mathcal{T}_- := \{\tau \in \mathcal{T} \mid |\tau|_+ < 0\}$ and $\mathfrak{B}(\mathcal{T}_-) := \{\tau \in \mathfrak{B}(\mathcal{T}) \mid |\tau|_+ < 0\}$.

Definition C.10. Given a model $\mathscr{Z}$ realizing K, we associate $\tau^{\mathcal{K}} = \tau^{\mathcal{K},\mathscr{Z}} \in \mathcal{D}^{\gamma_\tau}(\mathscr{T}, \mathscr{Z})$ to each $\tau \in \mathcal{T}_-$ by setting $\Xi^{\mathcal{K}} := \Xi$ and by inductively setting

$$\gamma_\tau := \min\{\gamma_{\tau_1} + 1 + |\tau_2|_+, \gamma_{\tau_2} + 1 + |\tau_1|_+\}, \quad \tau^{\mathcal{K}} := \sum_{i=1}^d \mathscr{D}_i[\mathcal{K}\tau_1^{\mathcal{K}}] \star \mathscr{D}_i[\mathcal{K}\tau_2^{\mathcal{K}}]$$

for $\tau = \nabla \mathscr{I}(\tau_1) \cdot \nabla \mathscr{I}(\tau_2)$. The exponent γ_Ξ is choosen so that $\gamma_\tau > 2$ for every $\tau \in \mathcal{T}_-$.

Remark C.11. Thanks to Proposition C.2 and [32, Theorem 5.12], indeed one has $\tau^{\mathcal{K},\mathscr{Z}} \in \mathcal{D}^{\gamma_\tau}$. Furthermore, for $\tau = \nabla \mathscr{I}(\tau_1) \cdot \nabla \mathscr{I}(\tau_2)$ and a compact set $\mathfrak{K}$, one has

$$|||\tau^{\mathcal{K},\mathscr{Z}}|||_{\gamma_\tau;\mathfrak{K}} \lesssim_{\mathscr{T}} (1 + |||\mathscr{Z}|||_{\gamma_{\tau_1}+\gamma_{\tau_2}+2;B(\mathfrak{K},1)})^6 |||\tau_1^{\mathcal{K},\mathscr{Z}}|||_{\gamma_{\tau_1};B(\mathfrak{K},1)} |||\tau_2^{\mathcal{K},\mathscr{Z}}|||_{\gamma_{\tau_2};B(\mathfrak{K},1)}.$$

Therefore, there exist a constant $\gamma, C \in (0, \infty)$ and integers $k, l \in \mathbb{N}$, which depend only on $\mathscr{T}$, such that

$$|||\tau^{\mathcal{K},\mathscr{Z}}|||_{\gamma;\mathfrak{K}} \leq C(1 + |||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)})^k \tag{84}$$

uniformly over $\tau \in \mathcal{T}_-$, $\mathscr{Z} \in \mathscr{M}(\mathscr{T}, \mathscr{Z})$ and compact sets $\mathfrak{K} \subseteq \mathbb{R}^d$.

Definition C.12. Let $F \in C_c^\infty(\mathbb{R})$ be such that $F(x) = -e^{2x}$ if $|x| \leq 2$. Given $N \in \mathbb{N}$ and a model $\mathscr{Z}$ realizing K, we set

$$\boldsymbol{X} := \boldsymbol{X}^{\mathscr{Z}} := \sum_{\tau \in \mathcal{T}_-} c(\tau)\tau^{\mathcal{K},\mathscr{Z}},$$

$$\boldsymbol{W}_N := \boldsymbol{W}_N^{\mathscr{Z}} := \mathfrak{p}_{<2}\mathcal{G}_{N,2}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}}$$

and

$$\boldsymbol{Y}_N := \boldsymbol{Y}_N^{\mathscr{Z}}$$

$$:=\mathfrak{p}_{<\delta}\Big[F^{\star}(\boldsymbol{W}_N^{\mathscr{Z}})\star\Big\{\sum_{\substack{\tau_1,\tau_2\in\mathcal{T}_-,\\ |\tau_1|_++|\tau_2|_+>-2}}\sum_{i=1}^{d}c(\tau_1)c(\tau_2)\mathscr{D}_i[\mathcal{K}^{\mathscr{Z}}\tau_1^{\mathcal{K},\mathscr{Z}}]\star\mathscr{D}_i[\mathcal{K}^{\mathscr{Z}}\tau_2^{\mathcal{K},\mathscr{Z}}]$$
$$+2\sum_{i=1}^{d}\mathscr{D}_i[\mathcal{K}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}}]\star R_2[\partial_i\{H_N*(\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}})\}]\Big\}\Big].$$

Proposition C.13. *Suppose that a model $\mathscr{Z}$ realizes K. Let $N\in\mathbb{N}$. Then, one has $\boldsymbol{W}_N^{\mathscr{Z}}\in\mathcal{D}_0^2(\mathscr{T},\mathscr{Z})$ and $\boldsymbol{Y}_N^{\mathscr{Z}}\in\mathcal{D}_{-1+\delta}^{\delta}(\mathscr{T},\mathscr{Z})$. More precisely, there exist constants $\gamma,C\in(0,\infty)$ and integers $k,l\in\mathbb{N}$ such that the following estimates hold uniformly over $N\in\mathbb{N}$, $\mathscr{Z},\overline{\mathscr{Z}}\in\mathscr{M}(\mathscr{T},K)$ and convex compact sets $\mathfrak{K}\subseteq\mathbb{R}^d$:*

$$\begin{aligned}
|||\boldsymbol{X}^{\mathscr{Z}}|||_{2;\mathfrak{K}}&\le C(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)})^k,\\
|||\boldsymbol{W}_N^{\mathscr{Z}}|||_{2;\mathfrak{K}}&\le C\{(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)})^k+\|H_N*(\mathcal{R}^{\mathscr{Z}}X^{\mathscr{Z}})\|_{C^2(\mathfrak{K})}\},\\
|||\boldsymbol{Y}_N^{\mathscr{Z}}|||_{\delta;\mathfrak{K}}&\le C(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)}+\|H_N*(\mathcal{R}^{\mathscr{Z}}X^{\mathscr{Z}})\|_{C^2(\mathfrak{K})})^k,
\end{aligned}$$

and furthermore

$$\begin{aligned}
|||\boldsymbol{X}^{\mathscr{Z}};\boldsymbol{X}^{\overline{\mathscr{Z}}}|||_{2;\mathfrak{K}}&\le C(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)}|||\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)})^k|||\mathscr{Z};\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)},\\
|||\boldsymbol{W}_N^{\mathscr{Z}};\boldsymbol{W}_N^{\overline{\mathscr{Z}}}|||_{2;\mathfrak{K}}&\le C(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)}+|||\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)})^k|||\mathscr{Z};\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)}\\
&\quad+\|H_N*(\mathcal{R}^{\mathscr{Z}}X^{\mathscr{Z}}-\mathcal{R}^{\overline{\mathscr{Z}}}X^{\overline{\mathscr{Z}}})\|_{C^2(\mathfrak{K})}\\
|||\boldsymbol{Y}_N^{\mathscr{Z}};\boldsymbol{Y}_N^{\overline{\mathscr{Z}}}|||_{\delta;\mathfrak{K}}&\le C\Big(1+|||\mathscr{Z}|||_{\gamma;B(\mathfrak{K},l)}+|||\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)}\\
&\quad+\|H_N*(\mathcal{R}^{\mathscr{Z}}X^{\mathscr{Z}})\|_{C^2(\mathfrak{K})}+\|H_N*(\mathcal{R}^{\overline{\mathscr{Z}}}X^{\overline{\mathscr{Z}}})\|_{C^2(\mathfrak{K})}\Big)^k\\
&\quad\times(|||\mathscr{Z};\overline{\mathscr{Z}}|||_{\gamma;B(\mathfrak{K},l)}+\|H_N*(\mathcal{R}^{\mathscr{Z}}X^{\mathscr{Z}}-\mathcal{R}^{\overline{\mathscr{Z}}}X^{\overline{\mathscr{Z}}})\|_{C^2(\mathfrak{K})}).
\end{aligned}$$

Proof. The estimate for $\boldsymbol{X}^{\mathscr{Z}}$ follows from (84). As for the estimate of $\boldsymbol{W}_N^{\mathscr{Z}}$, the Schauder estimate (83) gives the estimate for $\mathcal{K}\boldsymbol{X}^{\mathscr{Z}}$. The estimate for $R_2[H_N*(\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}})]$ follows from the estimate

$$|||R_2[H_N*(\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}})]|||_{2;\mathfrak{K}}\le\sum_{m:|m|\le 2}\|\partial^m[H_N*(\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}})]\|_{L^\infty(\mathfrak{K})},$$

where the convexity of $\mathfrak{K}$ is used. The estimate for $\boldsymbol{Y}_N^{\mathscr{Z}}$ follows from Proposition C.2, the estimate (79) and the Schauder estimate (83).

For the estimates of the differences, let us just mention that for differences there exist analogue estimates to those in Proposition C.2, the estimate (79) and (83), see [32, Proposition 4.10], [35, Proposition 3.11] and [32, Theorem 5.12] respectively. Using them, we can prove the last three inequalities of the differences similarly. □

Definition C.14. Given a model $\mathscr{Z}$ realizing K and $N\in\mathbb{N}$, we set

$$X:=X^{\mathscr{Z}}:=\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}},\quad W_N:=W_N^{\mathscr{Z}}:=\mathcal{R}^{\mathscr{Z}}\boldsymbol{W}_N^{\mathscr{Z}}$$

and

$$Y_N:=Y_N^{\mathscr{Z}}:=\mathcal{R}^{\mathscr{Z}}\boldsymbol{Y}_N^{\mathscr{Z}}+F(W_N^{\mathscr{Z}})\Big\{|\nabla[H_N*(X^{\mathscr{Z}})]|^2+[\Delta(G_N-G)]*X^{\mathscr{Z}}\Big\}.$$

Remark C.15. The parameter N will be used to ensure W_N is bounded on a given bounded domain. Therefore, N will be random and will depend on the domain. The idea of introducing such parameter is also used in [3]. As noted in Definition C.8, one has $W_N = G_N * X$.

Lemma C.16. *Let $\varepsilon \in (0,1)$. To simplify notation, we write $X^{\mathrm{can}} := X^{\mathscr{Z}^{\mathrm{can},\varepsilon}}$ here for instance. Then, one has the following idenitty:*

$$|\nabla W_N^{\mathrm{can}}|^2 + \Delta W_N^{\mathrm{can}} = -\xi_\varepsilon + \sum_{\substack{\tau_1,\tau_2\in\mathcal{T}_-,\\ |\tau_1|_+ + |\tau_2|_+ > -2}} c(\tau_1)c(\tau_2)\nabla(K * \mathcal{R}^{\mathrm{can}}\tau_1^{\mathcal{K},\mathrm{can}})\cdot\nabla(K*\mathcal{R}^{\mathrm{can}}\tau_2^{\mathcal{K},\mathrm{can}}) + 2\nabla[K * X^{\mathrm{can}}]\cdot\nabla[H_N * (X^{\mathrm{can}})] + |\nabla[H_N*(X^{\mathrm{can}})]|^2 + [\Delta(G_N - G)] * X^{\mathrm{can}}$$

Proof. One has $W_N^{\mathrm{can}} = K * X^{\mathrm{can}} + H_N * X^{\mathrm{can}}$ and

$$|\nabla W_N|^2 = \sum_{\tau_1,\tau_2\in\mathcal{T}_-} c(\tau_1)c(\tau_2)\nabla[K*\mathcal{R}^{\mathrm{can}}\tau_1^{\mathcal{K},\mathrm{can}}]\cdot\nabla[K*\mathcal{R}^{\mathrm{can}}\tau_2^{\mathcal{K},\mathrm{can}}] + 2\nabla[K*X^{\mathrm{can}}]\cdot\nabla[H_N * X^{\mathrm{can}}] + |\nabla H_N * X^{\mathrm{can}}|^2$$

Furthermore,

$$\Delta W_N = -\sum_{\tau\in\mathcal{T}_-} c(\tau)\mathcal{R}^{\mathrm{can}}\tau^{\mathcal{K},\mathrm{can}} + [\Delta(G_N - G)] * X^{\mathrm{can}}.$$

Now it remains to observe

$$\sum_{\tau_1,\tau_2\in\mathcal{T}_-} c(\tau_1)c(\tau_2)\nabla[K*\mathcal{R}^{\mathrm{can}}\tau_1^{\mathcal{K},\mathrm{can}}]\cdot\nabla[K*\mathcal{R}^{\mathrm{can}}\tau_2^{\mathcal{K},\mathrm{can}}] - \sum_{\tau\in\mathcal{T}_-} c(\tau)\mathcal{R}^{\mathrm{can}}\tau^{\mathcal{K},\mathrm{can}} = -\xi_\varepsilon + \sum_{\substack{\tau_1,\tau_2\in\mathcal{T}_-,\\ |\tau_1|_+ + |\tau_2|_+ > -2}} c(\tau_1)c(\tau_2)\nabla(K*\mathcal{R}^{\mathrm{can}}\tau_1^{\mathcal{K},\mathrm{can}})\cdot\nabla(K*\mathcal{R}^{\mathrm{can}}\tau_2^{\mathcal{K},\mathrm{can}}). \quad \square$$

C.4 BPHZ renormalization for X

The goal of this section is to show $X^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} = X^{\mathscr{Z}^{\mathrm{can},\varepsilon}} - c_\varepsilon$ (Proposition C.24). To this end, our first goal is to obtain the basis expansion for modelled distributions $\tau^{\mathcal{K},\mathscr{Z}} \in \mathcal{T}_-$, which will be given in Lemma C.19.

Lemma C.17. *For every $\tau_1,\tau_2\in\mathcal{T}_-$ with $|\tau_1|_+, |\tau_2|_+ < -1$ and $i,j\in\{1,\dots,d\}$, one has*

$$\Delta_+^\circ[\mathscr{I}_i(\tau_1)] = \mathscr{I}_i(\tau_1)\otimes\mathbf{1}_+, \quad \Delta_+^\circ[\mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)] = [\mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)]\otimes\mathbf{1}_+.$$

In particular, the constant map $x\mapsto\mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)$ belongs to $\mathcal{D}^\infty_{|\tau_1|+|\tau_2|+2}(\mathscr{T},\mathscr{Z})$ for any model $\mathscr{Z} = (\Pi,\Gamma)$ and

$$\mathcal{R}[\mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)] = \Pi_x[\mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)],$$

where the right-hand side is independent of x.

Proof. In view of the recursive formula [13, Proposition 4.17], one can prove the claim by induction on $|\cdot|_+$. Indeed, suppose one is going to prove $\Delta_+^\circ\tau = \tau\otimes\mathbf{1}_+$, where $\tau = \mathscr{I}_i(\tau_1)\mathscr{I}_j(\tau_2)$ and

$\Delta_+^\circ \tau_k = \tau_k \otimes \mathbf{1}_+$. By Lemma B.29, $\Delta_+^\circ \tau = \Delta_+^\circ[\mathscr{I}_i(\tau_1)]\Delta_+^\circ[\mathscr{I}_j(\tau_2)]$. Therefore, it suffices to show $\Delta_+^\circ[\mathscr{I}_i(\tau_1)] = [\mathscr{I}_i(\tau_1)] \otimes \mathbf{1}_+$. By [13, Proposition 4.17], one has

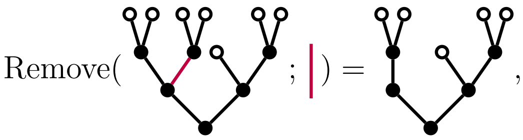
$$\Delta_+^\circ \mathscr{I}_i(\tau_1) = (\mathscr{I}_i \otimes \mathrm{Id})\Delta\tau_1 + \sum_{k:|\tau|_+ + 1 - |k| > 0} \frac{X^k}{k!} \otimes \hat{\mathscr{I}}_{e_i + k}(\tau_1).$$

It remains to observe that $(\mathscr{I}_i \otimes \mathrm{Id})\Delta\tau_1 = [\mathscr{I}_i(\tau_1)] \otimes \mathbf{1}_+$ by hypothesis of the induction and that the set over which k ranges is empty. □

Definition C.18. We use some notations from Section B.1. Let $\tau \in \mathfrak{B}(\mathcal{T})$ and let e be an edge of τ with $\mathfrak{t}(e) = \mathscr{I}$. By removing the edge e, we obtain a decorated forest with two connected components. We denote by

$$\mathrm{Remove}(\tau; e)$$

the component containing the root of τ, with decoration inherited from τ. For instance,



where ○ represents the noise (7). We set

$$\mathrm{Remove}(\mathfrak{B}(\mathcal{T})) := \{\mathrm{Remove}(\tau; e) \mid \tau \in \mathfrak{B}(\mathcal{T}), e \in E_\tau \text{ with } \mathfrak{t}(e) = \mathscr{I}\},$$
$$\mathrm{Remove}^{\mathfrak{n}}(\mathfrak{B}(\mathcal{T})) := \{(T, 0)_{\mathfrak{e}}^{\mathfrak{n},0} \mid (T, 0)_{\mathfrak{e}}^{0,0} \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))\}.$$

Lemma C.19. *Suppose $\mathscr{Z} = (\Pi, \Gamma)$ is a model realizing K. Then, one has a claim for $\tau \in \mathcal{T}_-$ as follows.*

(a) *If $\tau = \Xi$ or $\tau = \nabla\mathscr{I}(\tau_1) \cdot \nabla\mathscr{I}(\tau_2)$ with $|\tau_1|_+, |\tau_2|_+ < -1$, then $\tau^{\mathcal{K},\mathscr{Z}} = \tau$.*

(b) *If $\tau = \nabla\mathscr{I}(\tau_1) \cdot \nabla\mathscr{I}(\tau_2)$ with $|\tau_1|_+ > -1$ and $|\tau_2|_+ < -1$, then one has the expansion*

$$\tau^{\mathcal{K},\mathscr{Z}}(x) = \tau + \sum_{\sigma \in \mathfrak{V}(\tau)} a_{\tau,\sigma}^{\mathscr{Z}}(x)\sigma, \tag{85}$$

with the following properties:

- *$\mathfrak{V}(\tau)$ is a finite subset of $\mathrm{Remove}^{\mathfrak{n}}(\mathfrak{B}(\mathcal{T}))$ that is independent of $\mathscr{Z}$,*
- *one has*

$$\begin{aligned} &a_{\tau,\sigma}^{\mathscr{Z}}(x) \\ &= \sum_{\substack{j \in \{1,\dots,d\}, n \in \mathbb{N}_0, \rho \in \mathcal{T}_-, \\ l_1,\dots,l_n \in \mathbb{N}_0^d, \sigma_1,\dots,\sigma_n \in \mathrm{Remove}^{\mathfrak{n}}(\mathfrak{B}(\mathcal{T})), \\ |\sigma_k|_+ + 2 - l_k > 0, -1 < |\rho|_+ < |\tau|_+}} c_{\tau,\sigma,\rho}^{l_1,\dots,l_n,\sigma_1,\dots,\sigma_n}(\mathcal{P})[\partial_j K * \Pi_x \rho^{\mathcal{K},\mathscr{Z}}(x)] \prod_{k=1}^n [\partial^{l_k} K * \Pi_x \sigma_k](x) \\ &+ \sum_{\substack{n \in \mathbb{N}_0, \rho \in \mathcal{T}_-, \\ l, l_1,\dots,l_n \in \mathbb{N}_0^d, \sigma_1,\dots,\sigma_n \in \mathrm{Remove}^{\mathfrak{n}}(\mathfrak{B}(\mathcal{T})), \\ |\sigma_k|_+ + 2 - l_k > 0, -1 < |\rho|_+ < |\tau|_+}} c_{\tau,\sigma,\rho,l}^{l_1,\dots,l_n,\sigma_1,\dots,\sigma_n}(\mathcal{R})[\partial^l K * (\mathcal{R}^{\mathscr{Z}} \rho^{\mathcal{K},\mathscr{Z}} - \Pi_x \rho^{\mathcal{K},\mathscr{Z}}(x))](x) \\ &\qquad \times \prod_{k=1}^n [\partial^{l_k} K * \Pi_x \sigma_k](x), \end{aligned}$$

where the sum is actually finite and the constants

$$c_{\tau,\sigma,\rho}^{l_1,\dots,l_n,\sigma_1,\dots,\sigma_n}(\mathcal{P}) \quad \textit{and} \quad c_{\tau,\sigma,\rho,l}^{l_1,\dots,l_n,\sigma_1,\dots,\sigma_n}(\mathcal{R})$$

are independent of $\mathscr{Z}$.

Proof. To see the claim (a), if $|\tau|_+ < -1$, thanks to Lemma C.17, the identity (82) becomes

$$\mathcal{K}\tau = \mathscr{I}\tau + (K * \mathcal{R}\tau)(x)\mathbf{1}$$

and hence $\mathscr{D}_i\mathcal{K}\tau = \mathscr{I}_i\tau$. The claim (b) seems complicated but can be proven easily by induction. Suppose that one has $\tau = \nabla\mathscr{I}(\tau_1)\cdot\nabla\mathscr{I}(\tau_2)$ such that τ_1 has the expansions of the form (85) and $\tau_2^{\mathcal{K}} = \tau_2$. Furthermore, one has $-1 < |\tau_1|_+ < 0$ since $|\tau|_+ < 0$. Therefore, one has

$$\tau_1^{\mathcal{K}} = \tau_1 + \sum_{\sigma\in\mathrm{Remove}^{\mathrm{n}}(\mathfrak{B}(\mathcal{T}))} a_\sigma\sigma, \quad \tau_2^{\mathcal{K}} = \tau_2 \tag{86}$$

where a_σ has the desired property. By the definition (82) of $\mathcal{K}$, one has

$$\mathscr{D}_i\mathcal{K}\tau_1^{\mathcal{K}}(x) = \mathscr{I}_i\tau_1 + \sum_{\sigma\in\mathrm{Remove}^{\mathrm{n}}(\mathfrak{B}(\mathcal{T}))} a_\sigma(x)\mathscr{I}_i(\sigma) + [\partial_i K * \Pi_x\tau_1](x)\mathbf{1}$$

$$+ \sum_{\substack{\sigma\in\mathrm{Remove}^{\mathrm{n}}(\mathfrak{B}(\mathcal{T})), l\in\mathbb{N}_0^d \\ |\sigma|+1-|l|>0}} a_\sigma(x)[\partial^{e_i+l}K * \Pi_x\sigma](x)\frac{X^l}{l!} + \sum_{|l|<\gamma_{\tau_1}+1} [\partial^{e_i+l}K * (\mathcal{R}\tau_1^{\mathcal{K}} - \Pi_x\tau_1^{\mathcal{K}})](x)\frac{X^l}{l!},$$

where γ_{τ_1} is chosen so that $\tau_1^{\mathcal{K}} \in \mathcal{D}^{\gamma_{\tau_1}}(\mathscr{T},\mathscr{Z})$, see Remark C.11. Since $\mathscr{D}_i\mathcal{K}\tau_2^{\mathcal{K}} = \mathscr{I}_i\tau_2$ as shown in the part (a), one has

$$\mathscr{I}_i(\sigma)\mathscr{I}_i(\tau_2), X^l\mathscr{I}_i(\tau_2) \in \mathrm{Remove}^{\mathrm{n}}(\mathfrak{B}(\mathcal{T})).$$

Since $|\tau_1|_+ < |\tau|_+$, we complete the induction. □

We recall an explicit formula of the BPHZ realization.

Definition C.20 ([13, Theorem 6.18])**.** Let $\hat{\mathscr{T}}_-$ be the free algebra generated by $\mathscr{T}$ under the forest product. (In fact, recalling H_1^R from Definition B.22, we have $\hat{\mathscr{T}}_- = H_1^R$.) We define the algebra homomorphism $g_\varepsilon^- : \hat{\mathscr{T}}_- \to \mathbb{R}$ characterized by

$$g_\varepsilon^-(\mathfrak{i}_\circ\tau) := \mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau(0)],$$

where $\mathfrak{i}_\circ : \mathscr{T} \to \hat{\mathscr{T}}_-$ is the natural injection. Then, we have

$$\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon} = (g_\varepsilon^-\hat{\mathscr{A}}_- \otimes \mathbf{\Pi}^{\mathrm{can},\varepsilon}\Delta_-^\circ). \tag{87}$$

In view of the identity (87) and Lemma C.19, we need to understand $(g_\varepsilon^-\hat{\mathscr{A}}_- \otimes \mathbf{\Pi}^{\mathrm{can},\varepsilon})\Delta_-^\circ\tau$ for $\tau\in\mathcal{T}_-$ and $\tau \in \mathrm{Remove}^{\mathrm{n}}(\mathfrak{B}(\mathcal{T}))$. As one can easily guess from the definition of g_ε^-, it is necessary to estimate $\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau(0)]$ for such τ. The following simple lemma is a consequence of the symmetry of the noise ξ.

Lemma C.21. *For* $\tau \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$, *one has* $\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau(0)] = 0$.

Proof. Let $\tau = (T,0)^{0,0}_{\mathfrak{e}} \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$. Let $\mathbf{\Pi}^{\text{minus}}$ be the canonical realization for $\xi_\varepsilon(-\cdot)$. Since $\xi \overset{\mathrm{d}}{=} \xi(-\cdot)$, one has $\mathbf{\Pi}^{\text{minus}}\sigma \overset{\mathrm{d}}{=} \mathbf{\Pi}^{\mathrm{can},\varepsilon}\sigma$ for every $\sigma \in \mathscr{T}$. If we set

$$n(T) := \#\{e \in E_T \mid \mathfrak{t}(e) = \mathscr{I}\},$$

by using the identity

$$\partial_i K * [f(-\cdot)] = -[\partial_i K * f](-\cdot),$$

where the fact $K = K(-\cdot)$ is used, one has $\mathbf{\Pi}^{\text{minus}}\tau = (-1)^{n(T)}\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau$. However, since $\tau \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$, $n(T)$ is odd. Therefore, one has

$$\mathbf{\Pi}^{\text{minus}}\tau \overset{\mathrm{d}}{=} \mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau \quad \text{and} \quad \mathbf{\Pi}^{\text{minus}}\tau = -\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau,$$

and concludes $\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau(0)] = 0$. ☐

Lemma C.22. *For $\tau = (F,\hat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \in \mathfrak{B}(\mathcal{T}) \cup \mathrm{Remove}^{\mathfrak{n}}(\mathfrak{B}(\mathcal{T}))$ and $x \in \mathbb{R}^d$, one has*

$$\Delta^\circ_-\tau = \tau \otimes \mathbf{1} + \mathbf{1}_- \otimes \tau + \ker(g^-_\varepsilon \hat{\mathscr{A}}_- \otimes \Pi^{\mathrm{can},\varepsilon}_x) \cap \ker(g^-_\varepsilon \hat{\mathscr{A}}_- \otimes \mathbf{\Pi}^{\mathrm{can},\varepsilon}).$$

Proof. Recall from Definition B.2-(a) that edges are oriented. We call an edge $e = (a,b)$ a leaf if b is not followed by any edge. We call a node a of F true if there exists an edge $e = (a,b)$ such that $\mathfrak{t}(e) = \mathscr{I}$. We denote by N^{true} the set of all true nodes of F. For a subforest G of F, we set

$$N^j_G := \{a \in N_G \cap N^{\text{true}} \mid \text{ there exist exactly } j \text{ outgoing edges in } G \text{ at } a\}.$$

Recalling the coproduct formula (74), one has

$$\begin{aligned}\Delta^\circ_-\tau = \tau \otimes \mathscr{R}_{|\tau|_+}\mathbf{1} + \mathbf{1}_- \otimes \tau \\ + \sum_{G\subseteq F, G\neq\varnothing} \sum_{\mathfrak{n}_G \neq \mathfrak{n}, \varepsilon^F_G} \frac{1}{\varepsilon^F_G!}\binom{\mathfrak{n}}{\mathfrak{n}_G}(G,0)^{\mathfrak{n}_G+\pi\varepsilon^F_G,0}_{\mathfrak{e}} \otimes \mathscr{K}(F,\mathbb{1}_G)^{\mathfrak{n}-\mathfrak{n}_G,\pi(\varepsilon^F_A-\mathfrak{e}\mathbb{1}_G)}_{\mathfrak{e}\mathbb{1}_{E_F\setminus E_G}+\varepsilon^F_G},\end{aligned}$$

where $\mathscr{R}_\alpha$ is defined in Definition 3.8. However, note that $\mathbf{\Pi}^{\mathrm{can},\varepsilon}\mathscr{R}_\alpha\mathbf{1} = \mathbf{\Pi}^{\mathrm{can},\varepsilon}\mathbf{1}$. We fix $G \neq \varnothing$, $\mathfrak{n}_G \neq \mathfrak{n}$ and ε^F_G and set

$$\tau_1 := (G,0)^{\mathfrak{n}_G+\pi\varepsilon^F_G,0}_{\mathfrak{e}}, \quad \tau_2 := \mathscr{K}(F,\mathbb{1}_G)^{\mathfrak{n}-\mathfrak{n}_G,\pi(\varepsilon^F_A-\mathfrak{e}\mathbb{1}_G)}_{\mathfrak{e}\mathbb{1}_{E_F\setminus E_G}+\varepsilon^F_G}$$

We will prove $(g^-_\varepsilon \hat{\mathscr{A}}_- \otimes \Pi^{\mathrm{can},\varepsilon}_x)(\tau_1 \otimes \tau_2) = 0$ by considering various cases, which will complete the proof. When a case is studied, we exclude all cases considered before.

1. Suppose that $G \neq F$ and that a connected component T of G satisfies $N^0_T = \varnothing$ and $N^1_T = N^1_F \cap N_G$. Then, the forest τ_2 contains a leaf (a,ρ_T) of edge type $\mathscr{I}$ and hence $\Pi^{\mathrm{can},\varepsilon}_x\tau_2 = \mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau_2 = 0$.
2. Suppose G contains a leaf of edge type $\mathscr{I}$. Then, in view of the recursive formula (77), this is also the case for each forest appearing in $\hat{\mathscr{A}}_-\tau_1$ and hence $g^-_\varepsilon\hat{\mathscr{A}}_-\tau_1 = 0$.
3. Suppose $N^0_G \neq \varnothing$. If the case 2 is excluded, then a connected component of τ_1 is of the form $\bullet^{\mathfrak{n}_1,0}$ and hence $\tau_1 = 0$ (as an element of $\mathscr{T}_-$).
4. Suppose τ_1 contains a connected component $\tau_3 = (T,0)^{\mathfrak{n},0}_{\mathfrak{e}}$ such that $\#N^1_T \geq 2$. Let $a \in N^1_T$.
   - If a is the root of T, then $\tau_3 = \mathscr{I}_i(\tau_4)$ and hence $\tau_1 = 0$ (as an element of $\mathscr{T}_-$).

- If a is not the root of T, one can merge two consecutive edges (a_1, a) and (a, a_2) into a single edge (a_1, a_2) to obtain a new tree $\tau_5 \in \mathfrak{T}_\circ$ with $|\tau_3|_- = |\tau_5|_- + 1$. Since $|\sigma|_- \geq -2 + \delta$ for every $\sigma \in \mathfrak{T}_\circ$, if $\#(N_T^1 \setminus \{\rho_T\}) \geq 2$, then $|\tau_3|_- > 0$ and hence $\tau_1 = 0$ (as an element of $\mathscr{T}_-$).

5. Suppose that τ_1 contains a connected component $\tau_6 = (T_6, 0)_\mathfrak{e}^{\mathfrak{n}_6, 0}$ such that $N_{T_6}^0 = N_{T_6}^1 = \varnothing$. Then, $T_1 = T_6 = F$ and $\tau_1 \in \mathfrak{B}(\mathcal{T})$. However, this implies $\mathfrak{n} = \mathfrak{n}_G = 0$, which is excluded.
6. Therefore, it remains to consider the case where every connnected component $\tau_7 = (T_7, 0)_\mathfrak{e}^{\mathfrak{n}_7, 0}$ of τ_1 satisfies $\#N_{T_7}^1 = 1$ and $N_{T_7}^0 = \varnothing$ and all leaves of τ_7 are of type Ξ, namely $\tau_7 \in \mathrm{Remove}^\mathfrak{n}(\mathfrak{B}(\mathcal{T}))$. If $\mathfrak{n}_7 \neq 0$ on N_{T_7}, then $|\tau_7|_- > 0$. Thus, we suppose $\mathfrak{n}_7 = 0$. We will show $g_\varepsilon^- \hat{\mathscr{A}}_- \tau_7 = 0$, which implies $g_\varepsilon^- \hat{\mathscr{A}}_- \tau_1 = 0$ since the character $g_\varepsilon^- \hat{\mathscr{A}}_-$ is multiplicative. To apply the recursive formula (77), consider the expansion

$$\hat{\Delta}_- \tau_7 - \tau_7 \otimes \mathbf{1}_- = \mathbf{1} \otimes \tau_7 + \sum_{\tau_8} c_{\tau_8} \tau_8 \otimes \tau_9.$$

Then, one has

$$g_\varepsilon^- \hat{\mathscr{A}}_- \tau_7 = -\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau_7(0)] - \sum_{\tau_8} c_{\tau_8} \times \left(g_\varepsilon^- \hat{\mathscr{A}}_- \tau_8\right) \times \mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau_9(0)].$$

By the same reasoning as before, one can suppose that every component $\tau_{10} = (T_{10}, 0)_\mathfrak{e}^{0,0}$ of τ_8 belongs to $\mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$. However, since T_{10} has a strictly smaller number of edges than T_7 does, one can assume $g_\varepsilon^- \hat{\mathscr{A}}_- \tau_8 = 0$ by induction. Therefore, it remains to show $\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau_7(0)] = 0$. But this was shown in Lemma C.21. □

Corollary C.23. *If $\tau \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$, then $g_\varepsilon^- \hat{\mathscr{A}}_- \tau = 0$. If $\tau \in \mathcal{T}_-$, then*

$$g_\varepsilon^- \hat{\mathscr{A}}_- \tau = -\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau(0)].$$

Proof. The claim for $\tau \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$ is proved in the proof of Lemma C.22, see the case 6. If $\tau \in \mathcal{T}_-$, by Lemma C.22 one has

$$\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon} \tau = \mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau + g_\varepsilon^- \hat{\mathscr{A}} \tau.$$

However, since $|\tau|_- < 0$, one has $\mathbb{E}[\mathbf{\Pi}^{\mathrm{BPHZ},\varepsilon} \tau(0)] = 0$ by definition, which completes the proof. □

Proposition C.24. *For $\tau \in \mathcal{T}_-$, one has*

$$\begin{aligned} \Pi_x^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} \tau^{\mathcal{K},\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}(x) &= \Pi_x^{\mathscr{Z}^{\mathrm{can},\varepsilon}} \tau^{\mathcal{K},\mathscr{Z}^{\mathrm{can},\varepsilon}}(x) - \mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau(0)], \quad x \in \mathbb{R}^d, \\ \mathcal{R}^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} \tau^{\mathcal{K},\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} &= \mathcal{R}^{\mathscr{Z}^{\mathrm{can},\varepsilon}} \tau^{\mathcal{K},\mathscr{Z}^{\mathrm{can},\varepsilon}} - \mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau(0)]. \end{aligned} \tag{88}$$

In particular,

$$X^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} = X^{\mathscr{Z}^{\mathrm{can},\varepsilon}} - c_\varepsilon.$$

where

$$c_\varepsilon := \sum_{\tau \in \mathcal{T}_-} c(\tau) \mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon} \tau(0)]. \tag{89}$$

Proof. To simplify notation, we write $\mathcal{R}^{\mathrm{BPHZ}} := \mathcal{R}^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}$ here, for instance. Since

$$\mathcal{R}^{\#}\tau^{\mathcal{K},\#}(x) = [\Pi_x^{\#}\tau^{\mathcal{K},\#}(x)](x), \quad \# \in \{\mathrm{can}, \mathrm{BPHZ}\},$$

it suffices to prove (88). By Lemma C.19, one has the expansion

$$\tau^{\mathcal{K},\mathrm{BPHZ}}(x) = \tau + \sum_{\sigma} a_{\tau,\sigma}^{\mathrm{BPHZ}}(x)\sigma.$$

In the expression of $a_{\tau,\sigma}^{\mathrm{BPHZ}}$ given in Lemma C.19, every ρ in the sum satisfies $|\rho|_+ < |\tau|_+$. Therefore, one can assume $a_{\sigma}^{\mathrm{BPHZ}} = a_{\sigma}^{\mathrm{can}}$ by induction. By Lemma C.22 and Corollary C.23,

$$\Delta_-^{\circ}\tau^{\mathcal{K},\mathrm{BPHZ}}(x) = \tau \otimes \mathbf{1} + \mathbf{1}_- \otimes \tau + \sum_{\sigma} a_{\sigma}^{\mathrm{can}}(x)\mathbf{1}_- \otimes \sigma + \ker(g_\varepsilon^- \hat{\mathscr{A}}_- \otimes \Pi_x^{\mathrm{can}}).$$

Furthermore, by [13, Theorem 6.16], one has

$$\Pi_x^{\mathrm{BPHZ}} = (g_\varepsilon^- \hat{\mathscr{A}}_- \otimes \Pi_x^{\mathrm{can}})\Delta_-^{\circ}.$$

Therefore,

$$\begin{aligned}\Pi_x^{\mathrm{BPHZ}}\tau^{\mathcal{K},\mathrm{BPHZ}}(x) &= g_\varepsilon^- \hat{\mathscr{A}}_-\tau + \Pi_x^{\mathrm{can}}\tau + \sum_{\sigma} a_{\sigma}^{\mathrm{can}}(x)\Pi_x^{\mathrm{can}}\sigma \\ &= -\mathbb{E}[\mathbf{\Pi}^{\mathrm{can},\varepsilon}\tau(0)] + \Pi_x^{\mathrm{can}}\tau^{\mathcal{K},\mathrm{can}}(x),\end{aligned}$$

where we applied Corollary C.23 to get the last equality. □

C.5 BPHZ renormalization for $\boldsymbol{Y}_N$

The goal of this section is to compare $\boldsymbol{Y}_N^{\mathscr{Z}^{\mathrm{can},\varepsilon}}$ and $\boldsymbol{Y}_N^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}$, as we did for $\boldsymbol{X}$ in the previous section. Again, we need to obtain the basis expansion for $\boldsymbol{Y}_N$.

Lemma C.25. *Let $\tau_1, \tau_2 \in \mathcal{T}_-$, $i \in \{1, \dots, d\}$ and $N \in \mathbb{N}$. Let $\mathscr{Z}$ be a model realizing K. Assume $|\tau_1|_+ + |\tau_2|_+ > -2$. Then, for $x \in \mathbb{R}^d$, one has*

$$\begin{aligned}&\mathfrak{p}_{<\delta}\{F(\boldsymbol{W}_N^{\mathscr{Z}})(x) \star \mathscr{D}_i[\mathcal{K}\tau_1^{\mathcal{K},\mathscr{Z}}](x) \star \mathscr{D}_i[\mathcal{K}\tau_2^{\mathcal{K},\mathscr{Z}}](x)\} \\ &\qquad = \mathfrak{p}_{<\delta}\Big\{\sum_{k\in\mathbb{N}_0} \frac{D^kF(W_N^{\mathscr{Z}}(x))}{k!}\Big(\sum_{\tau\in\mathcal{T}_-}\mathscr{I}\tau\Big)^{\star k} \star \mathscr{D}_i[\mathcal{K}\tau_1^{\mathcal{K},\mathscr{Z}}](x) \star \mathscr{D}_i[\mathcal{K}\tau_2^{\mathcal{K},\mathscr{Z}}](x)\Big\}\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{p}_{<\delta}\{F(\boldsymbol{W}_N^{\mathscr{Z}})(x) \star \mathscr{D}_i[\mathcal{K}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}}](x) \star R_2[\partial_i\{H_N * (\mathcal{R}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}})\}](x)\} \\ &\qquad = \mathfrak{p}_{<\delta}\Big\{\sum_{k\in\mathbb{N}_0} \frac{D^kF(W_N^{\mathscr{Z}}(x))}{k!}\partial_i[H_N * (X^{\mathscr{Z}})](x)\Big(\sum_{\tau\in\mathcal{T}_-}\mathscr{I}\tau\Big)^{\star k} \star \mathscr{D}_i[\mathcal{K}^{\mathscr{Z}}\boldsymbol{X}^{\mathscr{Z}}](x)\Big\}.\end{aligned}$$

Proof. By Lemma C.19, one has

$$\boldsymbol{W}_N^{\mathscr{Z}}(x) = \sum_{\tau\in\mathcal{T}_-}\mathscr{I}\tau + W_N^{\mathscr{Z}}(x)\mathbf{1} + \boldsymbol{W}_N^{\mathscr{Z},+}(x),$$

where $\boldsymbol{W}_N^{\mathscr{Z},+}(x) \in \oplus_{\alpha\geq 1}\mathscr{T}_\alpha$. Recalling Definition C.3, one has

$$F(\boldsymbol{W}_N^{\mathscr{Z}})(x) = \sum_{k\in\mathbb{N}_0} \frac{D^k F(W_N^{\mathscr{Z}}(x))}{k!}\Big(\sum_{\tau\in\mathcal{T}_-} \mathscr{I}\tau + \boldsymbol{W}_N^{\mathscr{Z},+}(x)\Big)^{\star k}.$$

Since Lemma C.19 implies that

$$\mathscr{D}_i[\mathcal{K}\tau_1^{\mathcal{K},\mathscr{Z}}](x) \star \mathscr{D}_i[\mathcal{K}\tau_2^{\mathcal{K},\mathscr{Z}}](x)$$

is $\oplus_{\alpha\geq -1+\delta}\mathscr{T}_\alpha$-valued, one can ignore the contribution from $\boldsymbol{W}_N^{\mathscr{Z},+}(x)$ when the projection $\mathfrak{p}_{<\delta}$ is applied. This observation proves the claimed identities. □

Lemma C.26. *Let $N \in \mathbb{N}$. Then, one has*

$$\begin{aligned}\mathcal{R}^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}\boldsymbol{Y}_N^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}} &= F(W_N^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}) \\ \times \Big\{ &\sum_{\tau_1,\tau_2\in\mathcal{T}_-,|\tau_1|_+ + |\tau_2|_+ > -2} c(\tau_1)c(\tau_2)\nabla(K * \mathcal{R}^{\mathscr{Z}^{\mathrm{can},\varepsilon}}\tau_1^{\mathcal{K},\mathscr{Z}^{\mathrm{can},\varepsilon}}) \cdot \nabla(K * \mathcal{R}^{\mathscr{Z}^{\mathrm{can},\varepsilon}}\tau_2^{\mathcal{K},\mathscr{Z}^{\mathrm{can},\varepsilon}}) \\ &\qquad + 2\nabla[K * X^{\mathscr{Z}^{\mathrm{can},\varepsilon}}] \cdot \nabla[H_N * X^{\mathrm{can},\varepsilon}]\Big\}\end{aligned}$$

Proof. To simplify notation, we write $\Pi_x^{\mathrm{BPHZ},\varepsilon} := \Pi_x^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}$ here, for instance. One has

$$\mathcal{R}^{\mathrm{BPHZ},\varepsilon}\boldsymbol{Y}_N^{\mathrm{BPHZ},\varepsilon}(x) = [\Pi_x^{\mathrm{BPHZ},\varepsilon}\boldsymbol{Y}_N^{\mathrm{BPHZ},\varepsilon}(x)](x).$$

In view of Lemma C.19, Proposition C.24 and Lemma C.25, it suffices to show

$$\begin{aligned}\Pi_x^{\mathrm{BPHZ},\varepsilon}[\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)\mathscr{I}_i(\tau_{n+1})] &= \Pi_x^{\mathrm{can},\varepsilon}[\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)\mathscr{I}_i(\tau_{n+1})], \\ \Pi_x^{\mathrm{BPHZ},\varepsilon}[\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)\mathscr{I}_i(\tau_{n+1})\mathscr{I}_i(\tau_{n+2})] &= \Pi_x^{\mathrm{can},\varepsilon}[\mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)\mathscr{I}_i(\tau_{n+1})\mathscr{I}_i(\tau_{n+2})],\end{aligned} \tag{90}$$

for $\tau_1,\dots,\tau_n,\tau_{n+1} \in \mathcal{T}_-$ and $\tau_{n+2} \in \mathrm{Remove}(\mathfrak{B}(\mathcal{T}))$. We only prove the second identity of (90). We set

$$\boldsymbol{\tau} := (F,0)_{\mathfrak{e}}^{0,0} := \mathscr{I}(\tau_1)\cdots\mathscr{I}(\tau_n)\mathscr{I}_i(\tau_{n+1})\mathscr{I}_i(\tau_{n+2}), \quad (F_j,0)_{\mathfrak{e}}^{0,0} := \tau_j.$$

The proof of (90) follows the argument in the proof of Lemma C.22. We claim

$$\begin{aligned}\Delta_-^\circ\boldsymbol{\tau} = \mathbf{1}_-\otimes\boldsymbol{\tau} + &\sum_{J\subseteq\{1,\dots,n\}} \Big[\mathscr{I}_i(\tau_{n+1})\prod_{j\in J}\mathscr{I}(\tau_j)\Big] \otimes \Big[\mathscr{I}_i(\tau_{n+2})\prod_{j\notin J}\mathscr{I}(\tau_j)\Big] \\ &+ \sum_{J\subseteq\{1,\dots,n\}} \Big[\mathscr{I}_i(\tau_{n+1})\mathscr{I}_i(\tau_{n+2})\prod_{j\in J}\mathscr{I}(\tau_j)\Big] \otimes \prod_{j\notin J}\mathscr{I}(\tau_j).\end{aligned} \tag{91}$$

Indeed, let $\sigma\otimes\sigma'$ be a basis appearing in the coproduct formula (74) for $\Delta_-^\circ\boldsymbol{\tau}$. If we set $(G,0)_{\mathfrak{e}}^{\mathfrak{n},0} := \sigma$ and $\sigma_k := (G\cap F_j,0)_{\mathfrak{e}}^{\mathfrak{n},0}$, by repeating the argument in the proof of Lemma C.22, the forest σ_k is either $\varnothing$, τ_k or $\mathrm{Remove}(\rho_k;e_k)$ for some ρ_k and e_k.

- If $\sigma_k = \varnothing$, then $\sigma = 0$ in $\mathscr{T}_-$ unless $(\rho_{\boldsymbol{\tau}},\rho_{\tau_k}) \notin E_\sigma$.
- If $\sigma_k = \tau_k$, then σ' has a leaf of type $\mathscr{I}$ unless $(\rho_{\boldsymbol{\tau}},\rho_{\tau_k}) \in E_\sigma$.
- If $\sigma_k = \mathrm{Remove}(\rho_k;e_k)$, then $|\sigma|_+ > 0$ and hence $\sigma = 0$ in $\mathscr{T}_-$.

Therefore, the claimed identity (91) is established. It remains to show

$$g_\varepsilon^- \hat{\mathscr{A}}_- \Big[\mathscr{I}_i(\tau_{n+1}) \prod_{j\in J} \mathscr{I}(\tau_j)\Big] = 0, \quad g_\varepsilon^- \hat{\mathscr{A}}_- \Big[\mathscr{I}_i(\tau_{n+1})\mathscr{I}_i(\tau_{n+2}) \prod_{j\in J} \mathscr{I}(\tau_j)\Big] = 0. \tag{92}$$

Without loss of generality, we can suppose $J = \{1, \dots, n\}$. The proof is based on induction. We only consider the first identity of (92). As for the case $n = 0$, the first identity of (92) is shown in Lemma C.21. Similarly to (91), one can show

$$\hat{\Delta}_- \boldsymbol{\tau} = \mathbf{1}_- \otimes \boldsymbol{\tau} + \sum_{J\subseteq\{1,\dots,n\}} \Big[\mathscr{I}_i(\tau_{n+1}) \prod_{j\in J} \mathscr{I}(\tau_j)\Big] \otimes \prod_{j\notin J} \mathscr{I}(\tau_j)$$

In view of the recursive formula (77) and the hypothesis of the induction, it remains to show

$$\mathbb{E}[\boldsymbol{\Pi}^{\mathrm{can},\varepsilon} \boldsymbol{\tau}(0)] = 0.$$

However, this can be proved as in Lemma C.21, since $\boldsymbol{\tau}$ has an odd number of edges e such that $\mathfrak{t}(e) = \mathscr{I}$ and $|\mathfrak{e}(e)| = 1$. □

Proposition C.27. *Let U be a bounded domain.*

Suppose that M and ε are random variables (depending on U) with values in $\mathbb{N}_0$ and $(0,\infty)$, respectively, such that $|W_M^{\mathscr{Z}\mathrm{BPHZ},\varepsilon}| \le 2$ on U and $\|W_M^{\mathrm{AH},\varepsilon} - W_M^{\mathrm{AH}}\|_{L^\infty(U)} \le 1$ almost everywhere.

Then,

$$|\nabla W_N^{\mathscr{Z}\mathrm{BPHZ},\varepsilon}|^2 + \Delta W_N^{\mathscr{Z}\mathrm{BPHZ},\varepsilon} + e^{-2W_N^{\mathscr{Z}\mathrm{BPHZ},\varepsilon}} Y_N^{\mathscr{Z}\mathrm{BPHZ},\varepsilon} = -\xi_\varepsilon + c_\varepsilon \quad \text{on } U, \tag{93}$$

where the constant c_ε is defined in (89).

Proof. By Proposition C.24, one has $W_N^{\mathscr{Z}\mathrm{BPHZ},\varepsilon} = W_N^{\mathscr{Z}\mathrm{can},\varepsilon}$. Therefore, by Lemma C.16 and Lemma C.26, the left-hand side of (93) is equal to

$$-\xi_\varepsilon - [\Delta(G_N - G)] * c_\varepsilon = -\xi_\varepsilon + c_\varepsilon. \qquad \square$$

C.6 Stochatstic estimates and Besov regularity

Proposition C.13 gives pathwise estimates for the modelled distributions $\boldsymbol{X}$, $\boldsymbol{W}_N$ and $\boldsymbol{Y}_N$. Here we give stochastic estimates for X and Y_N in suitable Besov spaces. To this end, we will need a wavelet characterization of weighted Besov spaces.

Theorem C.28 ([49], [65, Theorem 1.61])**.** *For any $k \in \mathbb{N}$, there exist $\psi_{\mathfrak{f}}, \psi_{\mathfrak{m}} \in C_c^k(\mathbb{R})$ with the following properties.*

- *For $n \in \mathbb{N}_0$, if we denote by V_n the subspace of $L^2(\mathbb{R})$ spanned by*

  $$\{\psi_{\mathfrak{f}}(2^n \cdot - m) \mid m \in \mathbb{Z}\},$$

  then the inclusions $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots$ hold and $L^2(\mathbb{R})$ is the closure of $\cup_{n\in\mathbb{N}_0} V_n$.

- *The set*
$$\{\psi_{\mathfrak{f}}(\cdot - m) \mid m \in \mathbb{Z}\} \cup \{\psi_{\mathfrak{m}}(\cdot - m) \mid m \in \mathbb{Z}\}$$
forms an orthonormal basis of V_1. Therefore, the set
$$\{\psi_{\mathfrak{f}}(\cdot - m) \mid m \in \mathbb{Z}\} \cup \{2^{\frac{n}{2}}\psi_{\mathfrak{m}}(2^n \cdot - m) \mid n \in \mathbb{N}_0, m \in \mathbb{Z}\}$$
forms an orthonormal basis of $L^2(\mathbb{R})$.
- *One has $\int_{\mathbb{R}} x^l \psi_{\mathfrak{m}}(x)\,\mathrm{d}x = 0$ for every $l \in \{1, 2, \ldots, k\}$.*

One can build an orthonormal basis of $L^2(\mathbb{R}^d)$ as follows.

Proposition C.29 ([65, Proposition 1.53])**.** *Let $k \in \mathbb{N}$ and let $\psi_{\mathfrak{f}}, \psi_{\mathfrak{m}} \in C_c^k(\mathbb{R}^d)$ be as in Theorem C.28. For $n \in \mathbb{N}_0$, we define the sets of d-tuples by*

$$\mathfrak{G}^n := \begin{cases} \{(\mathfrak{f}, \ldots, \mathfrak{f})\} & \text{if } n = 0, \\ \{(G_1, \ldots, G_d) \in \{\mathfrak{f}, \mathfrak{m}\}^d \mid \exists j \text{ s.t. } G_j = \mathfrak{m}\} & \text{if } n \geq 1. \end{cases}$$

For $n \in \mathbb{N}_0$, $G \in \mathfrak{G}^n$, $m \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$, we set

$$\Psi_m^{n,G}(x) := 2^{\frac{d\max\{n-1,0\}}{2}} \prod_{j=1}^d \psi_{G_j}(2^{\max\{n-1,0\}}x_j - m_j). \tag{94}$$

The set $\{\Psi_m^{n,G} \mid n \in \mathbb{N}_0, G \in \mathfrak{G}^n, m \in \mathbb{Z}^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$.

With the expansion by the basis $\{\Psi_m^{n,G} \mid n \in \mathbb{N}_0, G \in \mathfrak{G}^n, m \in \mathbb{Z}^d\}$, one can give a wavelet characterization of weighted Besov spaces.

Proposition C.30 ([65, Theorem 6.15])**.** *Let $p, q \in [1, \infty]$, $r \in \mathbb{R}$ and $\sigma \in (0, \infty)$. Suppose*

$$k > \max\left\{r, \frac{2d}{p} + \frac{d}{2} - r\right\}$$

and let $\{\Psi_m^{n,G} \mid n \in \mathbb{N}_0, G \in \mathfrak{G}^n, m \in \mathbb{Z}^d\}$ be as in Proposition C.29. Then, there exists a constant $C \in (0, \infty)$ such that for every $f \in B_{p,q}^{r,\sigma}(\mathbb{R}^d)$ one has

$$\begin{aligned} C^{-1}&\|f\|_{B_{p,q}^{r,\sigma}(\mathbb{R}^d)} \\ &\leq \left\| \left(2^{n(r-d/p)}\Big(\sum_{G\in\mathfrak{G}^n, m\in\mathbb{Z}^d} w_\sigma(2^{-n}m)^p |2^{nd/2}\langle f, \Psi_m^{n,G}\rangle|^p\Big)^{1/p}\right)_{n\in\mathbb{N}_0}\right\|_{l^q(\mathbb{N}_0)} \\ &\leq C\|f\|_{B_{p,q}^{r,\sigma}(\mathbb{R}^d)}. \end{aligned}$$

We fix $k \in \mathbb{N}$ such that $k > \frac{5d}{2} + 2$, and we consider the orthonormal basis $\{\Psi_m^{n,G}\}$ given by (94). We set $\Psi := \Psi_0^{0,(\mathfrak{f},\ldots,\mathfrak{f})}$.

Definition C.31. Let $\mathscr{Z} = (\Pi, \Gamma), \overline{\mathscr{Z}} = (\overline{\Pi}, \overline{\Gamma}) \in \mathscr{M}(\mathscr{T}, K)$. Given a compact set $\mathfrak{K} \subseteq \mathbb{R}^d$, we set

$$\llbracket \mathscr{Z} \rrbracket_{\mathfrak{K}} := \sup_{\tau=(T,0)_{\mathfrak{e}}^{\mathfrak{n},0}\in\mathfrak{B}(\mathscr{T})\cap\mathscr{T}_{<0}} \sup_{n\in\mathbb{N}} \sup_{x\in\mathfrak{K}\cap 2^{-n}\mathbb{Z}^d} 2^{n|\tau|_+} |\langle \Pi_x\tau, 2^{nd}\Psi(2^n(\cdot - x))\rangle_{\mathbb{R}^d}|,$$

$$\llbracket \mathscr{Z}; \overline{\mathscr{Z}} \rrbracket_{\mathfrak{K}} := \sup_{\tau=(T,0)_{\mathfrak{e}}^{\mathfrak{n},0}\in\mathfrak{B}(\mathscr{T})\cap\mathscr{T}_{<0}} \sup_{n\in\mathbb{N}} \sup_{x\in\mathfrak{K}\cap 2^{-n}\mathbb{Z}^d} 2^{n|\tau|_+} |\langle \Pi_x\tau - \overline{\Pi}_x\tau, 2^{nd}\Psi(2^n(\cdot - x))\rangle_{\mathbb{R}^d}|.$$

Lemma C.32. *For each $\gamma \in \mathbb{R}$, there exist a constant $C \in (0,\infty)$ and an integer $k \in \mathbb{N}$ such that the following estimates hold uniformly over $\mathscr{Z}, \overline{\mathscr{Z}} \in \mathscr{M}(\mathscr{T}, K)$ and compact sets $\mathfrak{K} \subseteq \mathbb{R}^d$:*

$$|||\mathscr{Z}|||_{\gamma;\mathfrak{K}} \leq C(1+[\![\mathscr{Z}]\!]_{\mathfrak{K}})^k, \quad |||\mathscr{Z};\overline{\mathscr{Z}}|||_{\gamma;\mathfrak{K}} \leq C(1+[\![\mathscr{Z}]\!]_{\mathfrak{K}})^k([\![\mathscr{Z};\overline{\mathscr{Z}}]\!]_{\mathfrak{K}} + [\![\mathscr{Z};\overline{\mathscr{Z}}]\!]_{\mathfrak{K}}^k).$$

Proof. Using the recursive formula [13, Proposition 4.17], one can prove the claim as in [43, Lemma 2.3]. □

Lemma C.33. *Let $L \in [1,\infty)$ and set $Q_L := [-L,L]^d$. Let $p \in 2\mathbb{N}$. Under Assumption 3.10, if $p\delta' > d+1$, one has*

$$\mathbb{E}[[\![\mathscr{Z}^{\mathrm{BPHZ}}]\!]_{Q_L}^p] \leq C_p^{\mathrm{BPHZ}} L^d, \quad \mathbb{E}[[\![\mathscr{Z}^{\mathrm{BPHZ}};\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}]\!]_{Q_L}^p] \leq \boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}}(\varepsilon) L^d.$$

Proof. The proof is essentially the repetition of [43, Lemma 4.11]. Set

$$\mathfrak{B}_0(\mathscr{T}) := \{\tau = (T,0)_{\mathfrak{e}}^{\mathfrak{n},0} \in \mathfrak{B}(\mathscr{T}) \mid |\tau|_+ < 0\}.$$

If we write $\Psi_x^\lambda := \lambda^{-d}\Psi(\lambda^{-1}(\cdot - x))$, one has

$$\begin{aligned}\mathbb{E}[[\![\mathscr{Z}^{\mathrm{BPHZ}}]\!]_{Q_L}^p] &= \mathbb{E}[\sup_{\tau\in\mathfrak{B}_0(\mathscr{T})}\sup_{n\in\mathbb{N}}\sup_{x\in Q_L\cap 2^{-n}\mathbb{Z}^d} 2^{n|\tau|+p}|\langle \Pi_x\tau, \Psi_x^{2^{-n}}\rangle_{\mathbb{R}^d}|^p] \\ &\lesssim \sum_{\tau\in\mathfrak{B}_0(\mathscr{T})}\sum_{n\in\mathbb{N}} 2^{nd}L^d 2^{n|\tau|+p}\mathbb{E}[|\langle \Pi_0\tau, \Psi_0^{2^{-n}}\rangle_{\mathbb{R}^d}|^p],\end{aligned}$$

where the stationarity of the noise ξ and the estimate $\#(Q_L \cap 2^{-n}\mathbb{Z}^d) \lesssim 2^{nd}L^d$ are used. By Assumption 3.10,

$$\mathbb{E}[|\langle \Pi_0\tau, \Psi_0^{2^{-n}}\rangle_{\mathbb{R}^d}|^p] \lesssim_{\psi_{\mathfrak{f}}} C_p^{\mathrm{BPHZ}} 2^{-np(|\tau|_+ + \delta')}.$$

Therefore,

$$\mathbb{E}[[\![\mathscr{Z}^{\mathrm{BPHZ}}]\!]_{Q_L}^p] \lesssim C_p^{\mathrm{BPHZ}} L^d |\mathfrak{B}_0(\mathscr{T})|(2^{p\delta'-d}-1)^{-1}.$$

The estimate for the second claimed inequality is similar. □

Lemma C.34. *Let $\mathfrak{K} \subseteq \mathbb{R}^d$ be a compact set and $\sigma \in (0,\infty)$. Then, there exists a constant $C \in (0,\infty)$ such that for all $N \in \mathbb{N}$*

$$\|H_N * X\|_{C^2(\mathfrak{K})} \leq C 2^{3N}\|X\|_{\mathcal{C}^{-2,\sigma}(\mathbb{R}^d)}.$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be such that $\phi \equiv 1$ on $\mathfrak{K}$. By Lemma A.4, one has

$$\|H_N * X\|_{C^2(\mathfrak{K})} \lesssim \|\phi(H_N * X)\|_{\mathcal{C}^2(\mathbb{R}^d)} \lesssim_\sigma \|H_N * X\|_{\mathcal{C}^{2,\sigma}(\mathbb{R}^d)}.$$

It remains to apply Corollary A.10. □

Recall from Definition B.6 that we have, for instance, $|\Xi|_+ = -2+\delta+\kappa$ for some $\kappa \in (0,\delta')$.

Proposition C.35. *Under Assumption 3.10, there exist a deterministic integer $k = k(\delta_-) \in \mathbb{N}$ such that for all $\sigma \in (0,\infty)$, $p \in 2\mathbb{N}$ with $p > (d+1)/\min\{\delta'-\kappa,\sigma\}$ and $N \in \mathbb{N}$ we have the following:*

$$\begin{aligned}\mathbb{E}[\|X^{\mathscr{Z}^{\mathrm{BPHZ}}}\|^p_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] &\lesssim_{\delta,\delta',\kappa,\sigma,p} C_{kp}^{\mathrm{BPHZ}}, \\ \mathbb{E}[\|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}\|^p_{B_{p,p}^{-1+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] &\lesssim_{\delta,\delta',\kappa,\sigma,p} C_{kp}^{\mathrm{BPHZ}} 2^{kpN}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\|X^{\mathscr{Z}^{\mathrm{BPHZ}}} - X^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}\|^p_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] &\lesssim_{\delta,\delta',\kappa,\sigma,p} C_{kp}^{\mathrm{BPHZ}}[\boldsymbol{\varepsilon}_{kp}^{\mathrm{BPHZ}}(\varepsilon) + \boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}}(\varepsilon)], \\ \mathbb{E}[\|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}} - Y_N^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}}\|^p_{B_{p,p}^{-1+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] &\lesssim_{\delta,\delta',\kappa/2,\sigma,p} C_{kp}^{\mathrm{BPHZ}} 2^{kpN}[\boldsymbol{\varepsilon}_{kp}^{\mathrm{BPHZ}}(\varepsilon) + \boldsymbol{\varepsilon}_p^{\mathrm{BPHZ}}(\varepsilon)].\end{aligned}$$

Proof. Set $\mathscr{Z} := \mathscr{Z}^{\mathrm{BPHZ}}$. In the proof, we drop superscripts for BPHZ. Natural numbers k, l, γ depend only on $\mathscr{T}$ and they vary from line to line. We will not write down the dependence on $\mathscr{T}, \delta, \delta_-, p, \sigma$. Recall the notation $\Psi_m^{n,G}$ from (94).

Suppose we are given a modelled distribution $f \in \mathcal{D}_\alpha^\gamma(\mathscr{T}, \mathscr{Z})$ with $\alpha < 0 < \gamma$. We decompose

$$\langle \mathcal{R}f, 2^{nd/2}\Psi_m^{n,G}\rangle_{\mathbb{R}^d} = \langle \mathcal{R}f - \Pi_{2^{-n}m} f(2^{-n}m), 2^{nd/2}\Psi_m^{n,G}\rangle_{\mathbb{R}^d} + \langle \Pi_{2^{-n}m} f(2^{-n}m), 2^{nd/2}\Psi_m^{n,G}\rangle_{\mathbb{R}^d}.$$

Using (78), the first term is bounded by a constant times

$$2^{-n\gamma} |||f|||_{\gamma;B(2^{-n}m,l)} |||\mathscr{Z}|||_{\gamma;B(2^{-n}m,l)}.$$

To estimate the second term, consider the basis expansion

$$f(x) = \sum_\sigma a_\sigma(x)\sigma.$$

One has $|a_\sigma(2^{-n}m)| \le |||f|||_{\gamma;B(2^{-n}m,l)}$ and

$$|\langle \Pi_{2^{-n}m}\sigma, 2^{nd/2}\Psi_m^{n,G}\rangle_{\mathbb{R}^d}| \lesssim 2^{-n\alpha} |||\mathscr{Z}|||_{\gamma;B(2^{-n}m,l)}.$$

Therefore,

$$|\langle \mathcal{R}f, 2^{nd/2}\Psi_m^{n,G}\rangle_{\mathbb{R}^d}| \lesssim 2^{-n\alpha} |||f|||_{\gamma;B(2^{-n}m,l)} |||\mathscr{Z}|||_{\gamma;B(2^{-n}m,l)}. \quad (95)$$

Applying the estimate (95) to $\boldsymbol{X}$ and $\boldsymbol{Y}_N$, by Proposition C.30, we get

$$\|X\|^p_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)} \lesssim \sum_{n\in\mathbb{N}_0} 2^{-n(d+\kappa/2)} \sum_{G\in G^n, m\in\mathbb{Z}^d} w_\sigma(2^{-n}m)^p |||\boldsymbol{X}|||^p_{2;B(2^{-n}m,l)} |||\mathscr{Z}|||^p_{2;B(2^{-n}m,l)},$$

$$\|Y_N\|^p_{B_{p,p}^{-1+\delta+\kappa/2,\sigma}(\mathbb{R}^d)} \lesssim \sum_{n\in\mathbb{N}_0} 2^{-n(d+\kappa/2)} \sum_{G\in G^n, m\in\mathbb{Z}^d} w_\sigma(2^{-n}m)^p |||\boldsymbol{Y}_N|||^p_{\delta;B(2^{-n}m,l)} |||\mathscr{Z}|||^p_{\delta;B(2^{-n}m,l)}.$$

To estimate $\|X\|_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}$, we use Lemma C.13 and stationarity to obtain

$$\mathbb{E}[\|X\|^p_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] \lesssim \sum_{n\in\mathbb{N}_0} 2^{-n(d+(\delta-\delta_-))} \sum_{G\in G^n, m\in\mathbb{Z}^d} w_\sigma(2^{-n}m)^p \mathbb{E}[(1 + |||\mathscr{Z}|||_{\gamma;B(0,l)})^{kp}].$$

Since

$$\sum_{m\in\mathbb{Z}^d} w_\sigma(2^{-n}m)^p \lesssim \int_{\mathbb{R}^d} (1+|2^{-n}x|^2)^{-\frac{p\sigma}{2}} dx = 2^{nd}\|w_\sigma\|^p_{L^p(\mathbb{R}^d)},$$

and by Lemma C.32 and by Lemma C.33

$$\mathbb{E}[(1 + |||\mathscr{Z}|||_{\gamma;B(0,l)})^{kp}] \lesssim C^{\mathrm{BPHZ}}_{k'p}$$

for some $k' \in \mathbb{N}$, we conclude

$$\mathbb{E}[\|X\|^p_{B_{p,p}^{-2+\delta+\kappa/2,\sigma}(\mathbb{R}^d)}] \lesssim C^{\mathrm{BPHZ}}_{kp}.$$

The estimate of Y_N is similar by using Lemma C.34. The estimates of the differences can be proved similarly by using [32, (3.4)]. □

Corollary C.36. *Under Assumption 3.10, let $\sigma \in (0,\infty)$, $p \in [1,\infty)$ and $N \in \mathbb{N}$. Then, as $\varepsilon \downarrow 0$, $(X^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}})_{\varepsilon\in(0,1)}$ converges in $L^p(\mathbb{P})$ to $X^{\mathscr{Z}^{\mathrm{BPHZ}}}$ in $\mathcal{C}^{-2+\delta,\sigma}(\mathbb{R}^d)$, and $(Y_N^{\mathscr{Z}^{\mathrm{BPHZ},\varepsilon}})_{\varepsilon\in(0,1)}$ converges in $L^p(\mathbb{P})$ to $Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}$ in $\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)$. Furtheremore, there exists a deterministic $k = k(\delta) \in \mathbb{N}$, independent of σ and N, such that*

$$\sup_{N\in\mathbb{N}} 2^{-kN} \|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}\|_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)} \in L^p(\mathbb{P}). \tag{96}$$

Proof. The claim on the convergence follows from Proposition C.35 and by applying Besov embeddings. To show (96), let $p \in 2\mathbb{N}$ be such that $d/p < \kappa/2$. By Proposition C.35 and the Besov embedding, for some $k' \in \mathbb{N}$,

$$\mathbb{E}[\|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}\|^p_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}] \lesssim_{p,\delta,\sigma} \mathbb{E}[\|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}\|^p_{B^{-1+\delta+d/p,\sigma}_{p,p}(\mathbb{R}^d)}] \lesssim_{p,\delta,\kappa,\sigma} 2^{pk'N}.$$

Therefore, if $k > k'$,

$$\sum_{N\in\mathbb{N}} 2^{-kpN} \mathbb{E}[\|Y_N^{\mathscr{Z}^{\mathrm{BPHZ}}}\|^p_{\mathcal{C}^{-1+\delta,\sigma}(\mathbb{R}^d)}] < \infty. \qquad \square$$

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