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Stochastic homogenization of Hamilton–Jacobi–Bellman equations on continuum percolation clusters

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Abstract

We prove homogenization properties of random Hamilton–Jacobi–Bellman (HJB) equations on continuum percolation clusters, almost surely w.r.t. the law of the environment when the origin belongs to the unbounded component in the continuum. Here, the viscosity term carries a degenerate matrix, the Hamiltonian is convex and coercive w.r.t. the degenerate matrix and the underlying environment is non-elliptic and its law is non-stationary w.r.t. the translation group. We do not assume uniform ellipticity inside the percolation cluster, nor any finite-range dependence (i.i.d.) assumption on the percolation models and the effective Hamiltonian admits a variational formula which reflects some key properties of percolation. The proof is inspired by a method of Kosygina–Rezakhanlou–Varadhan [KRV06] developed for the case of HJB equations with constant viscosity and uniformly coercive Hamiltonian in a stationary, ergodic and elliptic random environment. In the non-stationary and non-elliptic set up, we leverage the coercivity property of the underlying Hamiltonian as well as a relative entropy structure (both being intrinsic properties of HJB, in any framework) and make use of the random geometry of continuum percolation.

1 Introduction

Consider a continuum percolation model resulting from the realizations $\omega \in \Omega$ of a point process in $\mathbb{R}^d$ with $d \geq 2$. The translation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ acts on Ω and it is natural to assume that the underlying law $\mathbb{P}$ of the point process is invariant and ergodic under this action. If there is a unique *infinite unbounded component* $\mathcal{C}_\infty(\omega)$ containing the origin 0, for $\mathbb{P}$-almost every $\omega \in \Omega$, then the event $\Omega_0 = \{0 \in \mathcal{C}_\infty\}$ has strictly positive probability, allowing us to define the *conditional probability*

$$\mathbb{P}_0(\cdot) = \mathbb{P}(\cdot \,|\, \Omega_0). \tag{1.1}$$

Note that, because of conditioning on 0 being in the infinite cluster, the probability measure $\mathbb{P}_0$, in contrast to its unconditional counterpart $\mathbb{P}$, is *not* invariant under the action of $\{\tau_x\}_{x\in\mathbb{R}^d}$.

For $\omega \in \Omega_0$, we now consider the *Hamilton-Jacobi-Bellman* (HJB) equation on the infinite cluster $\mathcal{C}_\infty(\omega)$

$$\partial_t u_\varepsilon = \frac{\varepsilon}{2}\mathrm{div}\Big(a\big(\frac{x}{\varepsilon},\omega\big)\,\nabla u_\varepsilon\Big) + H\Big(\frac{x}{\varepsilon}, \nabla u_\varepsilon, \omega\Big), \qquad \text{in } (0,T)\times\varepsilon\mathcal{C}_\infty(\omega) \tag{1.2}$$

with an initial condition $f(\cdot)$ – we refer to (2.10) for a precise formulation. Here the diffusion coefficient $a(x,\omega) = a(\tau_x\omega)$ is *degenerate elliptic* in the sense that the support of $x \mapsto a(\tau_x\omega)$ is contained in the

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closure of the infinite cluster $\overline{\mathcal{C}}_\infty(\omega)$, the Hamiltonian $H(x,p,\omega) = H(p,\tau_x\omega)$ is convex and coercive in $p \in \mathbb{R}^d$ w.r.t. the semi-norm induced by the degenerate matrix a and the initial condition f is uniformly continuous. These conditions are natural and they guarantee that, for any fixed $\varepsilon > 0$ and $T > 0$, there is actually a unique viscosity solution u_ε of (1.2).

Given this background, the goal of the present article is to develop a method for studying homogenization of u_ε almost surely w.r.t. the conditional probability $\mathbb{P}_0$ – we will show that, $\mathbb{P}_0$-almost surely and as $\varepsilon \to 0$, $u_\varepsilon \to u_{\mathrm{hom}}$ with u_{hom} solving the homogenized equation

$$\begin{cases} \partial_t u_{\mathrm{hom}} = \overline{H}(\nabla u_{\mathrm{hom}}), & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u_{\mathrm{hom}}(0,x) = f(x), & \text{on } \mathbb{R}^d. \end{cases}$$

The *effective Hamiltonian* $\overline{H}$ admits a variational representation

$$\overline{H}(\theta) = \inf_G \left(\operatorname*{ess\,sup}_{\mathbb{P}_0} \left[\frac{1}{2}\mathrm{div}\big(a(G+\theta)\big) + H(G+\theta) \right] \right) \qquad \forall \theta \in \mathbb{R}^d, \tag{1.3}$$

and determines the effective equation u_{hom} as a viscosity solution

$$u_{\mathrm{hom}}(t,x) = \sup_{y\in\mathbb{R}^d} \Big[f(y) - t\mathcal{I}\big(\frac{y-x}{t}\big) \Big], \qquad \text{with} \quad \mathcal{I}(y) := \sup_{\theta\in\mathbb{R}^d} [\langle\theta,y\rangle - \overline{H}(\theta)],$$

we refer to Theorem 2.1 for a precise statement, and to Corollary 2.2 for an application to an $\mathbb{P}_0$-a.s. large deviation principle for a degenerate diffusion with a random drift and on a percolation cluster.

The study of homogenization of HJB equations was initiated in a fundamental work of Lions, Papanicolaou and Varadhan [LPV87] which treated first order Hamilton-Jacobi equations in the periodic setting – that is, when $a \equiv 0$ and $H(\cdot + z, \cdot) \equiv H(\cdot,\cdot)$ for all $z \in \mathbb{Z}^d$. Since then, there have been very important works in the field by Souganidis [So99], Ishii [I99], Evans [E92], Rezakhanlou-Tarver [RT00], Lions-Souganidis [LS05, LS10], Kosygina-Rezakhanlou-Varadhan [KRV06] (see also Kosygina-Varadhan [KV08] for time-dependent case), Armstrong-Souganidis [AS12, AS13] and Armstrong-Tran [AT14, AT15]. While particular conditions vary from paper to paper, the main assumptions in these works on the Hamiltonian H involve convexity and super-linearity in p, some regularity in p and x as well as uniform continuity on the initial condition f. In all these works, homogenization holds in an almost sure (i.e., *quenched*) sense w.r.t. the law of the random environment, which is assumed to be stationary and ergodic under the translation group. For aforementioned reasons, the latter framework does not cover the conditional measure $\mathbb{P}_0$, which is relevant for studying almost sure behavior of (1.2) on percolation clusters, where homogenization of elliptic equations of the form $-\mathrm{div}(a(\nabla u + e)) = 0$ in a reversible and discrete framework have been studied quite extensively in the recent years [SS04, MP07, BB07, PRS15, LNO15, AD18].

In this context and to the best of our knowledge, the present paper is the first instance where homogenization of HJB equations have been studied on percolation clusters – a fundamental class of models for studying statistical mechanics of random media. As we will see, their inherent properties like *non-translation-invariance* and *non-ellipticity* pervade through the sequel (including in the variational formula (1.3) of the homogenized limit) and manifest into fundamental difficulties – we refer to Section 2.3 for the main ideas of the proof. Also, the current method does not require any *finite-range dependence* (i.e. i.i.d.) assumption on the percolation models which are allowed to have long-range correlations, we refer to Appendix A for concrete examples of such models. Before turning to formal statements of the main results (cf. Section 2), it is instructive to first provide the precise mathematical layout of the point processes.

1.1 Point processes and Palm measures.

1.1.1 Point processes.

Fix an integer $d \geq 2$, and let Ω be the space of all locally finite subsets of $\mathbb{R}^d$. We denote by $\mathcal{B}(\mathbb{R}^d)$ to its Borel σ-algebra. The Lebesgue measure will be denoted by λ (or by λ_d when we need to emphasize on the dimension). We endow Ω with the smallest σ-algebra $\mathcal{G}$ that makes the maps $\omega \mapsto \#(\omega \cap A)$ measurable for all $A \in \mathcal{B}(\mathbb{R}^d)$, where $\#(\omega \cap A)$ denotes the cardinality of $\omega \cap A$. A *Point process* is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{G})$.

On Ω, the group of translations $(\tau_x)_{x \in \mathbb{R}^d}$ acts naturally as

$$\tau_x \omega := \omega - x = \big\{ y - x : y \in \omega \big\}.$$

We say a point process is stationary if

$$\mathbb{P} \circ \tau_x = \mathbb{P} \qquad \forall x \in \mathbb{R}^d. \tag{1.4}$$

A stationary point process is ergodic with respect to $(\tau_x)_{x \in \mathbb{R}^d}$ if

$$\forall\, A \in \mathcal{G}\, \forall\, x \in \mathbb{R}^d: \qquad \tau_x A = A \quad \Longrightarrow \quad \mathbb{P}(A) \in \{0, 1\}. \tag{1.5}$$

We also define the *intensity measure* of $\mathbb{P}$ as the measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ given by

$$\Theta(A) := \int \#(\omega \cap A) \mathbb{P}(\mathrm{d}\omega) = \mathbb{E}[\#(\omega \cap A)]. \tag{1.6}$$

Here and throughout the sequel, $\mathbb{E}$ will denote expectation w.r.t. $\mathbb{P}$. Notice that when $\mathbb{P}$ is stationary and Θ is locally finite, then there exists some $\zeta \in (0, \infty)$ such that $\Theta = \zeta\lambda$. We call ζ the *intensity* of the point process.

1.1.2 Palm measures.

We now turn to the definition of *Palm measures*, which, on an intuitive level, formalizes the idea of the distribution of a Point process conditioned on containing some fixed point $x \in \mathbb{R}^d$. First, we define the measure $\mathfrak{C}$ on $\mathbb{R}^d \otimes \Omega$ as

$$\mathfrak{C}(A) := \mathbb{E}\big[\sum_{x \in \omega} \mathbb{1}_A(x, \tau_x \omega) \big] \quad \text{ for } A \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}. \tag{1.7}$$

The measure $\mathfrak{C}$ can be decomposed when $\mathbb{P}$ is stationary: Indeed, by [SW08, Theorem 3.3.1], if $\mathbb{P}$ is a stationary point process (i.e., if (1.4) holds) with intensity $\zeta \in (0, \infty)$, then there exists a unique measure $\mathbb{P}^{(0)}$ on $(\Omega, \mathcal{G})$ such that

$$\mathfrak{C} = \zeta\lambda \otimes \mathbb{P}^{(0)}. \tag{1.8}$$

We call $\mathbb{P}^{(0)}$ the *Palm measure* corresponding to $\mathbb{P}$. It can be seen as the distribution of a point process conditioned on containing the origin (see [LP18, Proposition 9.5]). In particular, $\mathbb{P}^0(0 \notin \omega) = 0$ (see [LP18, Eq. (9.7)]. We can define more generally

$$\mathbb{P}^{(x)} := \mathbb{P}^{(0)} \circ \tau_x \qquad \text{for } x \in \mathbb{R}^d.$$

The aforementioned decomposition allows us to disintegrate $\mathbb{P}$ in terms of $(\mathbb{P}^{(x)})_{x\in\mathbb{R}^d}$: Indeed, by [SW08, Theorem 3.3.3], if $\mathbb{P}$ is a stationary point process with intensity $\zeta \in (0,\infty)$, then for all $f \in L^1(\mathbb{R}^d \times \Omega)$, $\omega \mapsto \sum_{x\in\omega} f(x,\omega)$ is measurable, and

$$\mathbb{E}\big[\sum_{x\in\omega} f(x,\omega)\big] = \zeta \int_{\mathbb{R}^d} \mathbb{E}^{(0)}[f(x,\tau_{-x}\omega)]\mathrm{d}x = \zeta \int_{\mathbb{R}^d} \mathbb{E}^{(x)}[f(x,\omega)]\mathrm{d}x. \tag{1.9}$$

Similarly, one can define the n-fold Palm distribution $\mathbb{P}^{(x_1,\cdots,x_n)}$ for $x_1,\cdots,x_n \in \mathbb{R}^d$. In this case, we have the equality

$$\mathbb{E}\Big[\sum_{x_1\cdots,x_n\in\omega}^{\neq} f(x_1,\cdots,x_n,\omega)\Big] = \zeta^n \int_{(\mathbb{R}^d)^n} \mathbb{E}^{(x_1,\cdots,x_n)}[f(x_1,\cdots,x_n,\omega)]\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{1.10}$$

for $f \in L^1((\mathbb{R}^d)^n \times \Omega)$, where the sign $\neq$ above the sum indicates that the sum is taken over pairwise distinct elements. Also, $\mathbb{E}^{(x_1,\cdots,x_n)}$ stands for expectation w.r.t. the n-fold Palm distribution $\mathbb{P}^{(x_1,\cdots,x_n)}$.

1.1.3 Assumptions on the point process $\mathbb{P}$.

For any $\omega \in \Omega$, which denotes a locally finite point set in $\mathbb{R}^d$, we define a random domain $\mathcal{C}(\omega)$, which is an open set

$$\mathcal{C}(\omega) := \bigcup_{x\in\omega} B_{\frac{1}{2}}(x) \subset \mathbb{R}^d, \tag{1.11}$$

where $B_r(x) = \{y \in \mathbb{R}^d : |y-x| < r\}$ denotes an open ball centered at x of radius $r > 0$. The set $\mathcal{C}(\omega)$ can be decomposed into a disjoint union of connected components. If there is a unique unbounded connected component, then this component is denoted by $\mathcal{C}_\infty(\omega) \subset \mathcal{C}(\omega) \subset \mathbb{R}^d$. Its boundary will be denoted by $\partial\mathcal{C}_\infty$, with $\mathrm{int}(\mathcal{C}_\infty)$ denoting the interior. Moreover, we define

$$\Omega_0 := \big\{\omega \in \Omega \colon \mathcal{C}_\infty(\omega) \text{ exists, } 0 \in \mathcal{C}_\infty(\omega)\big\} \subset \Omega. \tag{1.12}$$

If $\mathbb{P}(\Omega_0) > 0$ (which we will assume in condition **(P3)** stated below), then we can define the conditional probability measure $\mathbb{P}_0$ on Ω_0 via

$$\mathbb{P}_0(\cdot) := \mathbb{P}(\,\cdot\,|\Omega_0), \quad \text{viz. } \mathbb{P}_0(A) = \frac{\mathbb{P}(A)}{\mathbb{P}(\Omega_0)} \text{ for all } A \in \mathcal{G} \cap \Omega_0.$$

By the openness and the connectedness of $\mathcal{C}_\infty(\omega)$, every two points $x,y \in \mathcal{C}_\infty$ can be connected by a curve in $C^1([0,1];\mathbb{R}^d)$ such that the interior distance d_ω is defined on $\mathcal{C}_\infty(\omega)$ via

$$\mathrm{d}_\omega(x,y) = \inf\Big\{ \int_0^1 |\dot{r}(s)|\mathrm{d}s \,:\, r \in C^1([0,1];\mathbb{R}^d),\ r(0)=x,\ r(1)=y, \\ \text{and } r(s) \in \mathcal{C}_\infty(\omega) \text{ for all } s \in [0,1]\Big\}.$$

To state the condition **(P6)** below we define $n(\omega, e) \in \mathbb{N}$ for all $e \in \mathbb{Z}^d$ with $|e| = 1$ and $\omega \in \Omega_0$ to be the “successive arrivalsöf $\mathcal{C}_\infty$ along a certain direction e, i.e., we set

$$n(\omega, e) = \min\{k \in \mathbb{N} : ke \in \mathcal{C}_\infty(\omega)\}. \tag{1.13}$$

We are now ready to state the following assumptions on the point process $\mathbb{P}$:

(P1) $\mathbb{P}$ is stationary ergodic with respect to $(\tau_x)_{x\in\mathbb{R}^d}$. Moreover, $\mathbb{P}$ is also stationary ergodic with respect to τ_e for all $e \in \mathbb{Z}^d$ with $|e|_1 = 1$ (namely, any $A \in \mathcal{G}$ such that $\tau_e A = A$ satisfies $\mathbb{P}(A) \in \{0, 1\}$).

(P2) Recall the definition of Θ from (1.6). Then for any compact set $A \subset \mathcal{B}(\mathbb{R}^d)$ we have $\Theta(A) < \infty$. In particular, $\Theta = \zeta\lambda$ for some $\zeta \in (0, \infty)$.

(P3) With the above definition of $\mathcal{C}(\omega)$ and Ω_0 we assume $\mathbb{P}(\Omega_0) > 0$, i.e. with with positive $\mathbb{P}$-probability, the set $\mathcal{C}(\omega)$ has a unique open, unbounded and connected component $\mathcal{C}_\infty(\omega)$ containing the origin $0 \in \mathbb{R}^d$.

(P4) There are constants $c_0, c_1, c_2 > 0$ such that for each $x, y \in \mathbb{R}^d$,

$$\mathbb{P}^{(x,y)}(\mathrm{d}_\omega(x, y) \geq c_0|x - y|_\infty; 0, x, y \in \mathcal{C}_\infty) \leq c_1 \mathrm{e}^{-c_2|x-y|_\infty}. \tag{1.14}$$

where $\mathbb{P}^{(x,y)}$ refers to the two-fold Palm distribution defined above (1.10).

(P5) The FKG-inequality is satisfied, i.e., if $A_1, A_2 \subset \Omega$ are increasing events (meaning that if $\omega_1 \in A_i$ and $\omega_1 \subset \omega_2$, then $\omega_2 \in A_i$ for $i = 1, 2$), then $\mathbb{P}(A_1 \cap A_2) \geq \mathbb{P}(A_1)\mathbb{P}(A_2)$.

(P6) We let $\mathfrak{v}_e(\omega) := n(\omega, e)e$ and assume that there exist constants $c_3, c_4 > 0$ such that

$$\forall\, \varrho > 0\, \forall\, e \in \mathbb{Z}^d \text{ with } |e|_1 = 1 : \quad \mathbb{P}_0\big(|\mathfrak{v}_e(\omega)| > \varrho\big) \leq c_3 \mathrm{e}^{-c_4\varrho}. \tag{1.15}$$

Condition **(P1)**-**(P3)** are self-explanatory, **(P4)** guarantees that d_ω is comparable to the Euclidean distance with high probability, while **(P5)** is a standard monotonicity inequality satisfied by many models. Since $\mathbb{P}$ is ergodic with respect to τ_e (by **(P1)**) and since $\mathbb{P}(0 \in \mathcal{C}_\infty) > 0$, by the Poincaré recurrence theorem (cf. [P89, Sec, 2.3]) we have $n(\omega, e) < \infty$. Then **(P6)** implies that moving along the coordinate axes has good recurrence properties and that $\mathfrak{v}_e$ possesses all moments under $\mathbb{P}_0$. These assumptions are natural and are satisfied by many well-studied models – we refer to Appendix A.

2 Main results

In Section 2.1, we will introduce the equation (1.2) in a precise form and record the necessary assumptions. In Section 2.2, we will announce our main results, while in Section 2.3 we will outline the principal ingredients of the proof.

2.1 The HJB equation. We now state the assumptions on the diffusion coefficient a, the Hamiltonian H and the initial value f appearing in (1.2), which guarantee existence and uniqueness of a viscosity solution (cf. Proposition 3.3). Stating these assumptions require some further notation.

Denote by $\mathcal{S}_d$ the space of $d \times d$ symmetric matrices. There is a natural partial order in $\mathcal{S}_d$: we say that for $A, B \in \mathcal{S}_d, A \leq B$ if $B - A$ is positive semidefinite, i.e., all its eigenvalues are nonnegative. For any symmetric, positive semidefinite matrix a (which will be defined below in **(F1)**) denote by $\sigma \in \mathcal{S}_d$ the unique (symmetric and positive semidefinite) matrix such that

$$a = \frac{1}{2}\sigma\sigma \qquad \text{on } \Omega_0.$$

We also define the inner product $\langle \cdot, \cdot \rangle_a = \langle \cdot, \cdot \rangle_{a(\omega)}$ as

$$\begin{aligned} \langle x, y \rangle_a &:= \langle a(\omega)x, y \rangle = \langle x, a(\omega)y \rangle \quad \forall x, y \in \mathbb{R}^d, \qquad \text{which defines a semi-norm} \\ \|x\|_a &:= \sqrt{\langle x, x \rangle_a}. \end{aligned} \tag{2.1}$$

We impose the following assumptions on a, H and f:

(F1) The matrix $a : \Omega \to \mathcal{S}_d$ is positive semidefinite, and

$$a(x, \omega) := a(\tau_x \omega)$$

defines a stationary process with respect to $\{\tau_x\}_{x \in \mathbb{R}^d}$. The restriction of a to Ω_0 (defined in (1.12)) satisfies the following: there exists $c_5 \in (0, \infty)$ and a measurable function $\xi : \Omega_0 \to (0, \infty)$ such that $\mathbb{P}_0$-a.s.

$$\xi(\omega)|x|^2 \leq \langle a(\omega)x, x \rangle \leq c_5 |x|^2 \quad \forall x \in \mathbb{R}^d. \tag{2.2}$$

Furthermore, there exists $\delta > 0, \alpha > 1 + \delta, \gamma > d$ such that

$$\mathbb{E}_0\big[\xi(\omega)^{-\chi}\big] < \infty, \tag{2.3}$$

where

$$\chi = \chi(\alpha, \gamma, \delta) := \frac{\alpha}{2} \max\left\{ \frac{1+\delta}{\alpha - (1+\delta)}, \frac{\gamma}{\alpha - 1} \right\}. \tag{2.4}$$

For $\mathbb{P}_0$-a.s. ω, the maps $x \ni \mathbb{R}^d \mapsto \sigma(x, \omega) := \sigma(\tau_x \cdot) \in \mathcal{S}_d$ and $x \ni \mathbb{R}^d \mapsto \operatorname{div} a(x, \omega) \in \mathbb{R}^d$ are Lipschitz continuous, with Lipschitz constant independent on ω. Moreover,

$$\operatorname{supp}(a) \subset \overline{\mathcal{C}}_\infty$$

and $|\text{div } a|$ is uniformly bounded. Since a is Lipschitz, we can assume that $x \to \xi(\tau_x \omega)$ is also Lipschitz by taking the minimum eigenvalue of a.

(F2) The Hamiltonian $H : \mathbb{R}^d \times \Omega \to \mathbb{R}$ satisfies for each $\omega \in \Omega$ that $p \to H(\pm p, \omega)$ is convex. Moreover, there are constants $c_6, \ldots, c_9 > 0$ such that for all $(p, \omega) \in \mathbb{R}^d \times \Omega_0$,

$$c_6 \|p\|_a^\alpha - c_7 \leq H(p, \omega) \leq c_8 \|p\|_a^\alpha + c_9, \tag{2.5}$$

and $H(\cdot,\omega) \equiv 0$ outside Ω_0. Here, $\alpha > 1+\delta$ and $\delta > 0$ are arbitrary (but specified in (2.3)). Equivalently, for $\alpha' := \frac{\alpha}{\alpha-1}$ and constants $c_{10}, \dots, c_{13}$,

$$c_{10}\|q\|_a^{\alpha'} - c_{11} \le L(q,\omega) \le c_{12}\|q\|_a^{\alpha'} + c_{13}, \tag{2.6}$$

where $L(q,\omega) := \sup_{p\in\mathbb{R}^d}[\langle p,q\rangle_a - H(p,\omega)]$. In particular, $L(\cdot,\omega) \equiv 0$ outside Ω_0.

(F3) The map $x \mapsto H(x,p,\omega) := H(p,\tau_x\omega)$ defines a stationary process with respect to translations. Moreover, there are constants $c_{14}, c_{15}, c_{16} > 0$ such that for any $\omega \in \Omega_0$, $x, y \in \mathcal{C}_\infty(\omega)$ and $p \in \mathbb{R}^d$,

$$|H(x,p,\omega) - H(y,p,\omega)| \le (c_{14}|p|^\alpha + c_{15})|x-y|, \tag{2.7}$$
$$|H(x,p,\omega) - H(x,q,\omega)| \le c_{16}(|p|+|q|+1)^{\alpha-1}|p-q|. \tag{2.8}$$

(F4) The initial condition $f : \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous. In particular, for any $\delta > 0$, there exists some $K_\delta > 0$ such that for any $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \le K_\delta|x-y| + \delta. \tag{2.9}$$

2.2 Main results.

For any $\varepsilon > 0$, $T > 0$ and $\omega \in \Omega_0$, consider the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \partial_t u_\varepsilon = \frac{\varepsilon}{2}\mathrm{div}\big(a\big(\frac{x}{\varepsilon},\omega\big)\,\nabla u_\varepsilon\big) + H\big(\frac{x}{\varepsilon},\nabla u_\varepsilon,\omega\big), & \text{in } (0,T)\times\varepsilon\mathcal{C}_\infty(\omega), \\ u_\varepsilon(t,x) = f(x), & \text{on } \big(\{0\}\times\varepsilon\mathcal{C}_\infty(\omega)\big) \cup \big((0,T)\times\varepsilon\partial\mathcal{C}_\infty(\omega)\big). \end{cases} \tag{2.10}$$

We are now ready to state our first main result.

Theorem 2.1. *Assume* ***(P1)-(P6)*** *on the point process $\mathbb{P}$,* ***(F1)-(F4)*** *on a, H and f, and let u_ε be the unique viscosity solution of* (2.10). *Then $\mathbb{P}_0$-almost surely and as $\varepsilon \to 0$, we have that $u_\varepsilon \to u_{\mathrm{hom}}$ uniformly on compact sets, where u_{hom} is the unique viscosity solution of*

$$\begin{cases} \partial_t u_{\mathrm{hom}} = \overline{H}(\nabla u_{\mathrm{hom}}), & \text{in } (0,\infty)\times\mathbb{R}^d, \\ u_{\mathrm{hom}}(0,x) = f(x), & \text{on } \mathbb{R}^d. \end{cases} \tag{2.11}$$

Here, the effective Hamiltonian $\overline{H}$ is given by the variational formula

$$\overline{H}(\theta) = \inf_{G\in\mathcal{G}_\delta}\left(\operatorname*{ess\,sup}_{\mathbb{P}_0}\left[\frac{1}{2}\mathrm{div}\big(a(G+\theta)\big) + H(G+\theta)\right]\right), \tag{2.12}$$

where the class $\mathcal{G}_\delta$ contains $L^{1+\delta}(\mathbb{P}_0)$ functions (with $\delta > 0$ being arbitrary, but specified in (2.3)*) defined in Section 4. Moreover, the homogenized limit u_{hom} is given by the Hopf-Lax formula*

$$u_{\mathrm{hom}}(t,x) = \sup_{y\in\mathbb{R}^d}\left(f(y) - t\,\mathcal{I}\big(\frac{y-x}{t}\big)\right), \quad \text{with} \quad \mathcal{I}(y) := \sup_{\theta\in\mathbb{R}^d}\big[\langle\theta,y\rangle - \overline{H}(\theta)\big]. \tag{2.13}$$

Remark 1 The equation (1.2) can also be rewritten in a non-divergence form as

$$\partial_t u_\varepsilon = \frac{1}{2}\mathrm{Trace}\Big(a\big(\frac{x}{\varepsilon},\omega\big)\mathrm{Hess}_x u_\varepsilon\Big) + \hat{H}\Big(\frac{x}{\varepsilon},\nabla u_\varepsilon,\omega\Big), \qquad \text{with}$$
$$\hat{H}(x,p,\omega) = H(x,p,\omega) + \frac{1}{2}\mathrm{div}(a(x,\omega))\cdot p.$$

By **(F1)**, $|\mathrm{div}\ a| \le C$ is bounded and therefore our assumptions on H translate to that of $\hat{H}$. Consequently, homogenization of the above equation is covered also by Theorem 2.1. □

A particular case of H, which is appealing from a probabilistic viewpoint, is the quadratic Hamiltonian

$$H_b(p,\omega) := \frac{1}{2}\|p\|_a^2 + \langle b(\omega), p\rangle_a. \tag{2.14}$$

With this choice, Theorem 2.1 leads to the following result. For any configuration $\omega \in \Omega$, let P_0^ω denote the law of the diffusion

$$\mathrm{d}X_t = \sigma(X_t,\omega)\mathrm{d}W_t + \mathrm{div}\, a(X_t,\omega)\mathrm{d}t + a(X_t,\omega)b(X_t,\omega)\,\mathrm{d}t \tag{2.15}$$

starting at 0 in the environment ω, where $(W_t)_{t\ge 0}$ is a standard Brownian motion in $\mathbb{R}^d$ (whose law is independent of $\mathbb{P}$). Our next main result is a quenched large deviation principle for the a degenerate diffusion with a random drift on a continuum percolation cluster:

Corollary 2.2. *Assume **(P1)**-**(P6)** on the point process $\mathbb{P}$, and **(F1)** on $a(\cdot)$ such that* (2.3) *holds with $\chi = \frac{\alpha(1+\delta)}{2(\alpha-(1+\delta))}$. Let $b:\Omega\to\mathbb{R}^d$ be a map so that $x\mapsto b(\tau_x\omega)$ defines a stationary process w.r.t. translations and H_b defined in* (2.14) *satisfies* (2.7)-(2.8) *(for these to hold, it suffices to assume that $x \mapsto b(x,\omega)$ is bounded and Lipschitz). Then for $\mathbb{P}_0$-almost every realization $\omega\in\Omega_0$, the distribution $P_0^\omega[\frac{X_t}{t}\in\cdot]$ satisfies a large deviation principle with rate function*

$$I(x) = \sup_{\theta\in\mathbb{R}^d}\{\langle\theta,x\rangle - \overline{H}(\theta)\}, \quad \textit{with} \tag{2.16}$$
$$\overline{H}_b(\theta) = \inf_{G\in\mathcal{G}_\delta}\ \mathrm{ess\,sup}_{\mathbb{P}_0}\Big[\frac{1}{2}\mathrm{div}(a\,G) + \langle b, G+\theta\rangle_a + \frac{1}{2}\|G+\theta\|_a^2\Big], \tag{2.17}$$

being defined as in (2.12)*. In other words, for $\mathbb{P}_0$-a.e. $\omega\in\Omega_0$ and every open set $G\subset\mathbb{R}^d$ and closed set $F\subset\mathbb{R}^d$, $\liminf_{t\to\infty}\frac{1}{t}\log P_0^\omega[\frac{X_t}{t}\in G] \ge -\inf_G I(\cdot)$ and $\limsup_{t\to\infty}\frac{1}{t}\log P_0^\omega[\frac{X_t}{t}\in G] \le -\inf_F I(\cdot)$.*

Remark 2 We can also consider Hamiltonians of the type

$$H_{b,V}(p,\omega) := \frac{1}{2}\|p\|_a^2 + \langle b(\omega),p\rangle_a - V(\omega),$$

and show an $\mathbb{P}_0$-almost sure large deviation principle for the distribution of X_t/t under the measure $\mathrm{d}Q_0^\omega \propto \mathrm{e}^{-\int_0^t V(X_s,\omega)\mathrm{d}s}\mathrm{d}P_0^\omega$ if we assume some moment condition on the *potential* V w.r.t. $\mathbb{P}_0$, which provides an *absorbing* random environment. We refer to a Armstrong-Tran [AT14, Corollary 2] where such a result has been obtained (in a stationary ergodic setting) using the sub-additive ergodic theorem, which was developed in a detailed study for Brownian motion ($a\equiv\mathrm{Id}$, $b\equiv 0$) in a Poissonian potential by Sznitman [S94] (see also Kosygina [K08, Section 7]). □

Remark 3 Note that the moment condition (2.3) would hold, for instance, for $\delta \sim 0$, $\alpha = 2$ and any $\gamma > d$ (so that $\chi = \max\{\frac{1+\delta}{2-\delta}, \gamma\} = \gamma > d$). We also remark that (2.3) with the exponent $\chi = (\alpha\gamma)/2(\alpha - 1)$ for $\gamma > d$ is needed for Theorem 2.1 at one step in its lower bound (cf. the discussion on p.10 in Section 2.3), while the other exponent with $\chi = \frac{\alpha(1+\delta)}{2(\alpha-(1+\delta)}$ is used to obtain a weak limit $G \in \mathcal{G}_\delta$. Note that the former assumption (carrying the term $\gamma > d$) is not needed for Corollary 2.2 (see Section 5.4.2 for its proof). Also, if we required $\xi(\cdot) \geq c_0 > 0$ on Ω_0 (i.e., if a were uniformly elliptic just inside the cluster, which we do not assume), then the condition (2.5) would hold for any $\alpha > 1$. Finally, we remark that when the framework is discrete and $b \equiv 0$ (i.e., reversible), a more specific case of Corollary 2.2 corresponds to studying large deviations for simple random walk on percolation clusters in $\mathbb{Z}^d$ [K12, BMO16], see also [SS04, MP07, BB07] for CLT results. However, by definition, this set up is automatically uniformly elliptic inside the cluster (the transition probability $\pi_\omega(0 \to e) \geq 1/2d$ if the edge $0 \leftrightarrow e$ in the discrete lattice is present in the environment ω) and also reversible where [KV86] plays a crucial role (this is different from treating HJB equations). Also, for large deviations one uses a change of measure argument that is not applicable to a general Hamiltonian as in Theorem 2.1. □

2.3 Ingredients of the proof.

The goal of this section is to underline the main ingredients of the proof, for which, as a guiding philosophy we will follow a novel method developed by Kosygina-Rezakhanlou-Varadhan [KRV06] for treating viscous HJB equation (when $a(\omega) \equiv \mathrm{Id}$) in a stationary ergodic setting, see also Kosygina [K08, Sec. 6] for a review on this approach and Kosygina-Varadhan [KV08] for an extension of this method to a time-dependent set up. The root of this approach goes back to the seminal work of [LPV87] and the framework of *environment seen from the particle* developed in [PV81, K85, KV86]. In the present scenario, fundamental difficulties stem from a combination of non-translation invariance and degeneracy of HJB equations on percolation clusters. For the convenience of the reader, we will briefly outline the [KRV06] approach and subsequently underline the new input of the current method.

The previous approach of [KRV06]: Let us denote by $\mathbb{P}$ the law of a stationary and ergodic random environment, with $\tilde{u}_\varepsilon$ solving $\partial_t \tilde{u}_\varepsilon = \frac{\varepsilon}{2}\Delta\tilde{u}_\varepsilon + \tilde{H}(\varepsilon^{-1}x, \nabla\tilde{u}_\varepsilon, \omega)$ with $\tilde{u}_\varepsilon(0, x) = \tilde{f}(x)$ being uniformly continuous and $p \mapsto \tilde{H}(p, \omega)$ being convex and satisfying *uniformly* in ω, $\tilde{H}(p, \omega) \sim |p|^\alpha$ for $\alpha > 1$ suitably large. To avoid technicalities we drop recalling further conditions which were assumed earlier. The method consists of three main steps.

Lower bound. The starting point is the optimal control representation

$$\tilde{u}_\varepsilon(t, x, \omega) = \sup_{c \in \mathbf{C_b}} E^{Q^c_{x/\varepsilon}}\left[\tilde{f}(\varepsilon\tilde{X}(t/\varepsilon)) - \int_0^{t/\varepsilon} \tilde{L}(\tilde{X}(s), c(s, \tilde{X}(s)), \omega)\mathrm{d}s\right]. \tag{2.18}$$

Here $\mathbf{C_b}$ denotes the space of all *bounded controls* $c : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and Q^c_x denotes the law of the $\mathbb{R}^d$-valued diffusion $\tilde{X}(t) = x + \int_0^t c(s, \tilde{X}(s))\mathrm{d}s + B(t)$ starting at $x \in \mathbb{R}^d$. Then for fixed $x = 0$, a lower bound on $\tilde{u}_\varepsilon(t, 0, \omega)$ is obtained by restricting to controls of the form $c(s, x, \omega) = b(\tau_x\omega)$ for some $b \in L^\infty(\mathbb{P})$ and invoking the ergodic theorem:

$$\liminf_{\varepsilon \to 0} \tilde{u}_\varepsilon(t, 0, \omega) \geq \sup_{(b,\phi)} \left[f\left(t\int \mathbb{P}(\mathrm{d}\omega)\phi(\omega)b(\omega)\right) - t\int \mathbb{P}(\mathrm{d}\omega)\phi(\omega)\tilde{L}(b, \omega)\right]. \tag{2.19}$$

Here the supremum is taken over those pairs (b, ϕ) such that $\phi \mathrm{d}\mathbb{P}$ is an invariant measure for $\mathcal{A}_b := \frac{1}{2}\Delta + b \cdot \nabla$ with $b, \phi, \nabla\phi, \nabla^2\phi \in L^\infty(\mathbb{P})$.[11] The above lower bound (at $x = 0$) is extended to a locally uniform bound using the uniform ellipticity of the matrix $a(\omega) \equiv \mathrm{Id}$ in this set up and also translation-invariance of $\mathbb{P}$.

Convex variational analysis: Note that, for linear initial data $f(x) = \langle p, x\rangle$, the lower bound (2.19) is of the form $t\overline{H}(p)$, where $\overline{H}(p) := \sup_{(b,\phi)} \mathbb{E}^{\mathbb{P}}\big[\phi(\omega)\big(\langle p, b(\omega)\rangle - L(b(\omega), \omega)\big)\big]$. Because $\inf_u \mathbb{E}^{\mathbb{P}}[\phi(\omega)\mathcal{A}_b u(\omega)] = -\infty$ unless $\phi\mathrm{d}\mathbb{P}$ is an invariant measure for $\mathcal{A}_b$, the supremum over (b, ϕ) can be decoupled by adding a Lagrange multiplier, leading to

$$\overline{H}(p) = \sup_\phi \sup_b \inf_u \mathbb{E}^{\mathbb{P}}\big[\phi(\omega)\big(\langle p, b(\omega)\rangle - L(b(\omega), \omega) + \mathcal{A}_b u(\omega)\big)\big]. \tag{2.21}$$

Starting from the above observation, [KRV06] developed convex variational analysis by successively applying min-max theorems. The success of this min-max method relies on, among other requirements, "compactnessöf the underlying variational problems. In the stationary ergodic setting of [KRV06], this compactness becomes readily available if one restricts the relevant variational problem(s) (e.g. the infimum over u, the supremum over ϕ etc.) to bounded regions. Then one can successively pass to further lower bounds $\overline{H}_k(p)$ at finite truncation level k, which leads to approximate gradients $v_k := \nabla u_k$. The uniform super-linearity assumption $p \mapsto H(p, \omega) \gtrsim |p|^\alpha$ leads to a moment condition which implies existence of a weak limit point $v = \lim_{k\to\infty} v_k \in L^\alpha(\mathbb{P})$, which (in the stationary ergodic set up), is a stationary gradient and satisfies a mean-zero property $\mathbb{E}^{\mathbb{P}}[v] = 0$. It is worth noting that, in this set up, both properties are direct consequences of the *invariant* action of τ_x w.r.t. the environment law $\mathbb{P}$. Construction of such a v and successive application of min-max theorems then lead to a suitable variational lower bound on $\overline{H}(p)$.

Upper bound: Using the stationary gradients v constructed above, one then considers the path integral $V(\omega, x) = \int_{0\to x}\langle v, \mathrm{d}z(s)\rangle$ with the normalization $V(\omega, 0) = 0$ a.s. An important step for obtaining a matching upper bound then entails showing the sub-linear growth property $V(x, \omega) = o(|x|)$ as $|x| \to \infty$ almost surely w.r.t. $\mathbb{P}$. In the stationary ergodic set up, this result was shown using the aforementioned mean-zero property of v (w.r.t. $\mathbb{P}$) and the ergodic theorem, combined with uniform coercivity and further assumptions imposed on H (e.g. by using uniform gradient estimates for sufficiently regular H, or by the Sobolev embedding theorem when $\alpha > d$, or by a perturbation method when $\alpha > 2$ and H satisfies $D^2H(p, \omega) \geq cI$ on $\{p \in \mathbb{R}^d : |p| \geq k\}$ for $c, k > 0$). Then by comparison with a super-solution $\hat{u}_\varepsilon(t, x, \omega) := \langle p, x\rangle + t\overline{H}(p) + \varepsilon V(x/\varepsilon, \omega)$ and using that the perturbation caused by V is negligible, thanks to its sub-linear growth, provide a "matching" upper bound.

[11] Since the supremum in (2.18) is taken for every fixed ω, one can allow the control $c \in \mathbf{C_b}$ to be ω-dependent, and for a lower bound, restrict to those c which are independent of the time variable s and stationary in the space variable x, i.e., $c(s, x, \omega) = b(\tau_x\omega)$ for some $b \in L^\infty(\mathbb{P})$. Working with such controls allows one to a study the *environment seen from the particle*, which is a diffusion process $\overline{\omega}(t) = \tau_{X(t)}\omega \in \Omega$ taking values in the environment space Ω, starting at $\omega \in \Omega$ with generator $\mathcal{A}_b$. By restricting further to those $b \in L^\infty(\mathbb{P})$ with an invariant density ϕ (for the generator $\mathcal{A}_b$) with $\nabla\phi, \nabla^2\phi \in L^\infty(\mathbb{P})$, one uses ergodic properties of the environment process $\overline{\omega}(t)$. These ergodic properties then translate also to those of the original diffusion $X(t)$, leading to (2.19). Let us also note that, here gradient $\nabla = (\nabla_i)_{i=1}^d$ (and likewise Δ) is defined in a *weak* sense: ∇_i is the infinitesimal generator of the translation group $\{\tau_x\}_{x\in\mathbb{R}^d}$ acting on $L^2(\Omega, \mathcal{F}, \mathbb{P})$ via

$$(\nabla_i f)(\omega) = \lim_{h\to 0} \frac{f(\tau_{e_i h}\omega) - f(\omega)}{h}. \tag{2.20}$$

The current method: In the current set up, we also follow the earlier philosophy for the lower bound and consider a variational representation (3.8) of the solution u_ε of (1.2). For reasons to be explained below, instead of deterministic and bounded $c \in \mathbf{C_b}$, we choose progressively measurable controls sampled from an auxiliary probability space $(\mathscr{X}, \mathcal{F}, P)$ [22] and work with Lipschitz maps $b \in L^1_a(\phi \mathrm{d}\mathbb{P}_0)$ (instead of bounded b), see Section (3.2.2). Due to the non-invariance and non-ellipticity of $\mathbb{P}_0$, the ergodic theorem (shown in Theorem 3.4) needs extra care which leads to a lower bound on $\liminf_{\varepsilon\to 0} u_\varepsilon(t,0,\omega)$ at $x = 0$, see Lemma 3.9 and Lemma 3.10. The usual step then is to obtain (uniform in ε) Lipschitz estimates for the solutions of (1.2) in order to upgrade this inequality to locally uniform convergence. However, since the Hamiltonian is not uniformly coercive in our case (recall (2.5)), the Lipschitz estimates are only local. Nevertheless, one can control the oscillations of u_ε around $x = 0$ uniformly in t and ε for $|x|$ small by applying Morrey's inequality and a comparison principle from [AT15] and [D19], cf. Lemma 3.11. To apply this inequality, we need the moment assumption (2.3) with the exponent $\chi = (\alpha\gamma)/2(\alpha-1)$ for $\gamma > d$. The locally uniform lower bound, which is first obtained for smooth initial condition, is extended to any uniformly continuous f in Lemma 3.12-Lemma 3.15.

Now for the variational analysis, we are not allowed to apply the min-max route using restriction to bounded regions. Indeed, first note that, one does not expect a mean-zero property of a prospective weak limit w.r.t. $\mathbb{P}_0$ which is not shift-invariant. However, one can refine the mean-zero condition by studying shifts defined by the successive arrivals of the continuum cluster along coordinate directions (see (1.13)). But then, restriction to a bounded region is incompatible with non-translation invariance of $\mathbb{P}_0$ – for any prospective limit point, the latter would deny the gradient condition (recall from (2.20) that gradients are defined on Ω_0 with respect to the *usual* shifts τ_x that leaves $\mathbb{P}_0$ non-invariant), while the former would be discordant with the refined mean-zero condition which requires keeping track of arbitrarily long cluster excursions. Therefore, we use a different route. For the first step of the min-max theorem, we exploit the intrinsic coercivity properties of the Hamiltonian (w.r.t. a), which propagates to the accompanying variational formula. At this step, our choice of the class $(b, \phi) \in \mathcal{E}$ in (3.17) is important where we work with Lipschitz maps $b \in L^1_a(\phi \mathrm{d}\mathbb{P}_0)$. If we were to use uniformly bounded maps $b \in L^\infty(\mathbb{P}_0)$ as considered previously, this class would not be closed in $L^p(\mathbb{P}_0)$ for $p \geq 1$. This closeness is crucial for showing weak compactness in the first min-max step in Lemma 5.3. For the second step, we introduce a subtractive relative entropy term which is structurally well-suited to the optimal control variational formula accompanying from the preceding steps. This entropy term provides the requisite coercivity in order to apply the second min-max theorem, see Lemma 5.4. Combined with the moment assumption (2.3) for $\chi = \frac{\alpha(1+\delta)}{2(\alpha-(1+\delta)}$, subsequently we are able to deduce existence of a weak limit $G \in L^{1+\delta}(\mathbb{P}_0)$.

An advantage of this approach is that, the weak limit G is now *both* curl-free and refined mean-zero, being conformant to the properties of $\mathbb{P}_0$ and that of the cluster, see Lemma 5.6-Lemma 5.8 for details. We note that, while this technique seems to be a natural approach for treating degenerate HJB for non-stationary set up, it is also unifying with the earlier [KRV06] approach in the sense that the coercivity of H, used in the first min-max step, is an *intrinsic* assumption for HJB (regardless of the set up). Similarly, the relative entropy structure invoked in the second min-max step is well-suited to the preceding variational formulas from here that are applicable to both frameworks, and the desired limiting properties are established in a natural way,

[22] For conceptual reasons it might be useful to note that the progressively measurable control c is sampled from a fixed auxiliary probability space, and therefore, the SDE (3.7) admits a *strong* solution which is not Markovian. In contrast, the SDE $\tilde{X}$ underneath (2.18) admits a weak solution for $c \in \mathbf{C_b}$.

the properties being determined by the respective set up.

Turning to the upper bound, an important step here also involves showing sub-linearity of the path integral $V_G(\omega, x) = \int_{0\to x}\langle G, \mathrm{d}z(s)\rangle = o(|x|)$, $\mathbb{P}_0$-a.s. Note that our assumptions on a, H, the geometry of the continuum percolation as well as properties of the limit points G are different from [KRV06]. Therefore, the proof of this step is also quite different here for which we build on the assumptions **(P1)**-**(P6)**. Using these, quite some technical effort is needed to show also this step in the current scenario, which constitutes Section 4. Note that in the continuum framework we are not allowed to invoke arguments based on combinatorial counting, neither do not assume uniform ellipticity inside the infinite cluster, which are key properties used in the aforementioned works on limit theorems for simple random walks on discrete percolation clusters (in a reversible set up, which is different from studying HJB equations). Then using a mollification and continuity argument, and combined with the arguments from the lower bound part, the requisite upper bound is shown in Proposition 5.9 and Section 5.4.

An orthogonal approach to the [KRV06] method for treating HJB in a stationary ergodic set up involves sub-additivity [So99, RT00, LS05, LS10, AT14]. We believe that such a method could also be extended to the current percolation set up (with extra work). While sub-additivity does not immediately yield a variational formula for the effective Hamiltonian (in contrast to the present method), we refer to the very interesting work [AT14, Remark 3] in the stationary ergodic where such a variational formula has been obtained also using the sub-additive ergodic theorem.

Organization of the rest of the article: In Section 3 we will provide the lower bound for the solution of HJB equations. Section 4 is devoted to studying properties of the “correctorsänd in Section 5 we will carry out the variational analysis and complete the proof of the upper bound and that of Theorem 2.1 and Corollary 2.2. In Appendix A we will provide examples of percolation models covered by our set up and in Appendix B-C we will collect some auxiliary arguments that are used in the sequel.

3 Viscosity solutions, ergodic theorem and the lower bound

3.1 Viscosity solutions of H-J-B equations on percolation clusters.

The goal of the section is to provide existence and uniqueness of the PDE (2.10), by *explicitly* representing its solution as the value function of an optimal control problem. Notice that by **(F1)**, the HJB equation (2.10) with $\varepsilon = 1$ can be written as, for any $\omega \in \Omega_0$,

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{H}(x, \nabla u, \mathrm{Hess}_x u) & \text{in } V, \\ u(t,x) = f(x) & \text{on } \partial V \end{cases} \tag{3.1}$$

for the open set

$$\begin{aligned} &V := (0,T) \times \mathcal{C}_\infty(\omega), \qquad \text{and the Hamiltonian} \\ &\mathcal{H}(x,p,q) := \frac{1}{2}\mathrm{div}(a(x,\omega)) \cdot p + \frac{1}{2}\mathrm{Trace}(a(x,\omega)q) + H(x,p,\omega). \end{aligned} \tag{3.2}$$

Here and it what follows, we denote by ∂V the parabolic boundary. In the case when $V = (0,T) \times U$ for an open set $U \subset \mathbb{R}^d$, $\partial V = ((0,T) \times \partial U) \cup (\{0\} \times U)$.

To define the notion of viscosity solutions, we need some further notation. First, we recall the definition of the upper and lower semicontinuous envelopes. For a set $V \subset \mathbb{R}_+ \times \mathbb{R}^d$, denote by $\mathrm{USC}(\overline{V})$ the set of upper semicontinuous functions $w : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$. Similarly, $\mathrm{LSC}(\overline{V})$ denotes the space of lower semicontinuous functions $w : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$.

Definition 3.1. *Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $u : \overline{V} \to \mathbb{R}$ be a locally bounded function. We define the upper semicontinuous envelope $u_* : \overline{V} \mapsto \mathbb{R} \cup \{\infty\}$ as*

$$u^*(x) := \inf \big\{ w(x) : w \in \mathrm{USC}(\overline{V}) \text{ and } w \geq u \big\}. \tag{3.3}$$

The lower semicontinuous envelope is defined as $u_* := -(-u)^*$.

It follows directly from the definition that $u_* \leq u \leq u^*$. Moreover, $u_* \in \mathrm{LSC}(\overline{V})$ while $u^* \in \mathrm{USC}(\overline{V})$.

Definition 3.2 (Viscosity sub/super-solutions)**.** *Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$.*

- *We say that a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity subsolution of*

$$\frac{\partial u}{\partial t} = \mathcal{H}(x, \nabla u, \mathrm{Hess}_x u) \qquad \text{in } V \tag{3.4}$$

  if for all $(s, y) \in V$ and smooth function ϕ in a neighborhood of (s, y) such that the map $(t, x) \mapsto (u^* - \phi)$ has a local maximum at (s, y), one has

$$\frac{\partial \phi(t_0, x_0)}{\partial t} - \mathcal{H}(x_0, \nabla \phi(t_0, x_0), \mathrm{Hess}_x \phi(t_0, x_0)) \leq 0.$$

- *Similarly, we say a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity supersolution of* (3.4) *if for all $(s, y) \in V$ and smooth function ϕ in a neighborhood of (s, y) such that the map $(t, x) \mapsto (u_* - \phi)$ has a local minimum at (s, y), one has*

$$\frac{\partial \phi(t_0, x_0)}{\partial t} - \mathcal{H}(x_0, \nabla \phi(t_0, x_0), \mathrm{Hess}_x \phi(t_0, x_0)) \geq 0.$$

- *We say a locally bounded function $u : \overline{V} \to \mathbb{R}$ is a viscosity solution of* (3.4) *if it is both a subsolution and supersolution.*

The existence of viscosity solution itself follows from Perron's method (see for example [I87]). However, as mentioned above, our goal is to characterize the solution as the value function of an optimal control problem of diffusions on percolation clusters. This is carried out as follows: we fix an auxiliary probability space $(\mathscr{X}, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ on $\mathcal{F}$ and an auxiliary d-dimensional Brownian motion $(B_t)_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ (whose law P is independent of the law $\mathbb{P}$ of the point process). Let

$$\mathbf{C}_T = \Big\{ c : [0, T] \times \mathscr{X} \mapsto \mathbb{R}^d : c \text{ is progressively measurable and (3.6) holds.} \Big\}, \tag{3.5}$$

where

$$E^P\left[\int_0^T c(s)^2 \mathrm{d}s\right] < \infty, \tag{3.6}$$

By **(F1)**, for each $c \in \mathbf{C}_T$ and $x_0 \in \mathcal{C}_\infty$, there is a unique strong solution to the stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t \sigma(X_s)\mathrm{d}B_s + \int_0^t (\operatorname{div} a)(X_s)\mathrm{d}s + \int_0^t a(X_s)c(s)\mathrm{d}s \quad \text{a.s.} \quad \forall t \geq 0, \tag{3.7}$$

see [T13, Theorem 3.1]. Let us also mention that the above display is understood to hold in a pointwise sense for every fixed ω (and for each realization of the auxiliary probability space $\mathscr{X}$) which is suppressed from the notation. Now, note that by our assumptions, $X_t \in \overline{\mathcal{C}}_\infty$ for all $t \geq 0$ (cf. [BGHJ21, Lemma 3.4]). Denote by $P_x^{c,\omega}$ the law of the solution of (3.7) and set

$$u(t,x,\omega) := \sup_{c \in \mathbf{C}_T} J(t,x,c), \tag{3.8}$$

where

$$J(t,x,c) := E^{P_x^{c,\omega}}\left[f(X_t) - \int_0^t L(X_s, c(s))\mathrm{d}s\right]. \tag{3.9}$$

The characterization of the solution to (2.10) as an optimal control problem is given by the next proposition.

Proposition 3.3. *Assume **(F1)-(F4)**. Then for $\mathbb{P}_0$-almost every realization $\omega \in \Omega_0$, the function u in* (3.8) *is the unique viscosity solution (cf. 3.2) of*

$$\begin{cases} \partial_t u = \frac{1}{2}\operatorname{div}\big(a\big(x,\omega\big)\,\nabla u\big) + H\big(x, \nabla u, \omega\big), & \text{in } (0,T) \times \mathcal{C}_\infty(\omega), \\ u(t,x,\omega) = f(x), & \text{on } \big(\{0\} \times \mathcal{C}_\infty(\omega)\big) \cup \big((0,T) \times \partial\mathcal{C}_\infty(\omega)\big) \end{cases} \tag{3.10}$$

of at most linear growth.

Proof. The existence and uniqueness follow from dynamic programming and a comparison principle, respectively, see Appendix C for details. □

3.2 The ergodic theorem for the environment process on percolation clusters. The goal of this section is to prove an ergodic theorem (cf. Proposition 3.4 below) for the so-called *environment process* which, for homogenization of stationary ergodic random media (at least in the elliptic setting), goes back to the works of Kozlov [K85] and Papanicolau-Varadhan [PV82]. In our context, this environment process is a diffusion taking values in the space of *conditioned* environments Ω_0.

3.2.1 The environment process.

Recall that the group $\{\tau_x\}_{\mathbb{R}^d}$ acts on $(\Omega, \mathcal{G}, \mathbb{P})$ via translations. This action allows us to define, for any $u : \Omega \to \mathbb{R}$, its *weak gradient* via

$$(\nabla_i u)(\omega) := \lim_{\varepsilon \to 0} \frac{u(\tau_{\varepsilon e_i}(\omega) - u(\omega)}{\varepsilon}, \qquad i = 1, \dots, d.$$

Likewise, we also define the corresponding *divergence*. Now for $a : \Omega \to \mathbb{R}^{d\times d}$ satisfying **(F1)**, we set

$$(\mathscr{L}^{(b)} u)(\omega) := \frac{1}{2}\mathrm{div}\big(a(\omega)\nabla u(\omega)\big) + \langle b(\omega), \nabla u(\omega)\rangle_a \qquad \forall \omega \in \Omega_0. \tag{3.11}$$

For a reasonable class of maps $b : \Omega \to \mathbb{R}^d$ (which do not depend on the probability space $(\mathscr{X}, \mathcal{F}, P)$) and class of test functions u, $\mathscr{L}^{(b)}$ is the generator of a Markov process taking values on Ω_0 which can be defined as follows. Set $b(x,\omega) := b(\tau_x\omega)$ and let X_t denote the $\mathbb{R}^d$-valued diffusion solving the SDE

$$X_t = \int_0^t \sigma(X_s)\mathrm{d}B_s + \int_0^t (\mathrm{div}\, a)(X_s)\mathrm{d}s + \int_0^t a(X_s)b(X_s)\mathrm{d}s \quad \text{a.s.} \quad \forall t \geq 0,$$

with quenched law $P_0^{b,\omega}$ and generator

$$(\mathcal{L}^{(b,\omega)} u)(x) = \frac{1}{2}\mathrm{div}\big(a(x,\omega)\nabla u(x)\big) + \langle b(x,\omega), \nabla u(x)\rangle_a. \tag{3.12}$$

Then

$$\overline{\omega}_t := \tau_{X_t}\omega \tag{3.13}$$

is the Ω_0-valued diffusion process with generator $\mathscr{L}^{(b)}$ defined in (3.11). We call $(\overline{\omega}_t)_{t\geq 0}$ the *environment process* with generator $\mathscr{L}^{(b)}$, and its law with initial condition δ_ω is denoted by $Q^{b,\omega}$.

3.2.2 Invariant density for the environment process.

Recall that $\mathbb{P}_0 = \mathbb{P}(\cdot|\Omega_0)$. We write $L^1_+(\mathbb{P}_0)$ for the space of all non-negative and $\mathbb{P}_0$-integrable functions on Ω. Any probability density $\phi \in L^1_+(\mathbb{P}_0)$ with $\int \phi \,\mathrm{d}\mathbb{P}_0 = 1$ is an invariant density with respect to $Q^{b,\omega}$ if

$$\frac{1}{2}\mathrm{div}(a\nabla\phi) = \mathrm{div}(\phi(ab)), \quad \text{i.e., } (\mathscr{L}^{(b)})^\star\phi = 0, \quad \text{in } \Omega_0, \tag{3.14}$$

with the generator $\mathscr{L}^{(b)}$ defined in (3.11). For any probability density ϕ, we also set

$$L^1_a(\phi\mathrm{d}\mathbb{P}_0) := \Big\{ b : \Omega_0 \to \mathbb{R}^d \text{ measurable: } \int \mathrm{d}\mathbb{P}_0\phi \; \|b\|_a < \infty \Big\}. \tag{3.15}$$

where we remind the reader from (2.1) that

$$\|b\|_{L^1_a}(\phi\mathrm{d}\mathbb{P}_0) := \int \mathrm{d}\mathbb{P}_0\phi \; \|b\|_a = \int \mathbb{P}_0(\mathrm{d}\omega)\phi(\omega)\sqrt{\big|\langle b(\omega), a(\omega)b(\omega)\rangle\big|}. \tag{3.16}$$

As usual, $L^1_a(\phi \mathrm{d}\mathbb{P}_0)$ can be turned into a Banach space with the norm defined in (3.16) by taking the quotient w.r.t the subspace of functions with zero L^1_a-norm. Finally, for a suitable space X (which will be specified later on depending on the context), we will denote by $\mathrm{Lip} = \mathrm{Lip}(X)$ the set of 1-Lipschitz functions from $X \to \mathbb{R}^d$. With this background, we define the class

$$\mathcal{E} = \Big\{ (b,\phi) \in L^1_a(\phi \mathrm{d}\mathbb{P}_0) \times L^1_+(\mathbb{P}_0) \colon \mathbb{R}^d \ni x \mapsto b(x,\omega) \in \mathrm{Lip}\ \forall \omega \in \Omega_0, \\ \int \phi \mathrm{d}\mathbb{P}_0 = 1,\ (\mathscr{L}^{(b)})^* \phi = 0 \Big\}. \tag{3.17}$$

3.2.3 The ergodic theorem.

We are now ready to state the main result of this subsection:

Proposition 3.4. *Suppose that there exists ϕ such that $(b,\phi) \in \mathcal{E}$. Let $\mathbb{Q}(\mathrm{d}\omega) := \phi(\omega)\mathbb{P}_0(\mathrm{d}\omega)$. If $\mathbb{Q} \ll \mathbb{P}_0$, then the following three implications hold:*

- $\mathbb{Q} \sim \mathbb{P}_0$.
- *$\mathbb{Q}$ is ergodic with respect to the Markov process $Q^{b,\omega}$.*
- *There can be at most one such measure $\mathbb{Q}$.*

The proof of the above result will need a simple fact, for which we recall that $\Omega_0 = \{\omega \in \Omega \colon 0 \in \mathcal{C}_\infty(\omega)\}$, and also from (1.13) that $n(\omega,e) = \min\{k \in \mathbb{N} \colon ke \in \mathcal{C}_\infty(\omega)\}$. We then define the *induced shift* $\sigma_e : \Omega_0 \to \Omega_0$ by setting

$$\sigma_e(\omega) = \tau_{n(\omega,e)}\omega. \tag{3.18}$$

Then σ_e satisfies the following property:

Proposition 3.5. *For every $e \in \mathbb{Z}^d$ with $|e|_1 = 1$, the induced shift $\sigma_e : \Omega_0 \to \Omega_0$ is measure preserving and ergodic with respect to $\mathbb{P}_0$.*

Proof. See Appendix B. □

We now turn to the

Proof of Proposition 3.4: We first show that $\mathbb{Q}$ is equivalent to $\mathbb{P}_0$. Let $A := \{\phi > 0\}$. We need to show that $\mathbb{P}_0(A) = 1$. Since ϕ is a density, we know that $\mathbb{P}_0(A) > 0$. As $\phi \mathrm{d}\mathbb{P}_0$ is invariant with respect to the environmental process, we have

$$0 = \int_{A^c} \phi \mathrm{d}\mathbb{P}_0 = \int \mathbb{1}_{A^c} \phi \mathrm{d}\mathbb{P}_0 = \int E^{b,\omega}(\mathbb{1}_{A^c}(\tau_{X_1}\omega))\phi(\omega)\mathrm{d}\mathbb{P}_0 \\ = \int_A E^{b,\omega}(\mathbb{1}_{A^c}(\tau_{X_1}\omega))\phi(\omega)\mathrm{d}\mathbb{P}_0.$$

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Thus, for $\mathbb{P}_0(|A)$-a.s ω, $E^{b,\omega}(\mathbb{1}_{A^c}(\tau_{X_1}\omega)) = 0$. Equivalently, for $\mathbb{P}_0(|A)$-a.s ω, $E^{b,\omega}(\mathbb{1}_A(\tau_{X_1}\omega)) = 1$. In particular, for $\mathbb{P}_0(|A)$-a.s ω, $\mathbb{1}_A(\tau_{X_1}\omega) = 1$ $Q^{b,\omega}$-a.s. We claim that this implies that A is $\mathbb{P}_0$-a.s. invariant under the induced shift, so $\mathbb{P}_0(A) \in \{0,1\}$. Since $\mathbb{P}_0(A) > 0$, the equivalence between $\mathbb{Q}$ and $\mathbb{P}_0$ would be complete. To show the claim, notice that for ω as above, $\tau_x\omega \in A$ for almost all $x \in \mathcal{C}_\infty(\omega)$. Indeed, if there is a subset V of $\mathcal{C}_\infty(\omega)$ of positive Lebesgue measure satisfying $\tau_x\omega \notin A$ for $x \in V$, then since the diffusion visits every set of positive Lebesgue measure inside $\mathcal{C}_\infty(\omega)$, we would have $P^{b,\omega}(X_1 \in V) > 0$, so that $Q^{b,\omega}(\tau_{X_1}\omega \notin A) > 0$, which would be a contradiction. Thus, for $\mathbb{P}_0$-a.s. $\omega \in A$ and almost all $x \in \mathcal{C}_\infty(\omega)$, we have $\tau_x\omega \in A$. In other words,

$$\int_A \int_{\mathbb{R}^d} \mathbb{1}_{\{x:\tau_x\omega\notin A\}} \mathrm{d}x \mathrm{d}\mathbb{P}_0 = 0.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^d} \int_A \mathbb{1}_{A^c}(\tau_x\omega) \mathrm{d}\mathbb{P}_0 \mathrm{d}x = 0.$$

Hence, for almost all $x \in \mathbb{R}^d$,

$$\int_A \mathbb{1}_{A^c}(\tau_x\omega)\mathrm{d}\mathbb{P}_0 = \frac{1}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \int_\Omega \mathbb{1}_A(\omega)\mathbb{1}_{A^c}(\tau_x\omega)\mathrm{d}\mathbb{P} = 0.$$

By the continuity of the map $\mathbb{R}^d \ni y \mapsto \mathbb{1}_A(\omega)\mathbb{1}_{A^c}(\tau_x\omega) \in L^1(\mathbb{P})$, we deduce that for all $x \in \mathbb{R}^d$, $\mathbb{1}_A(\omega)\mathbb{1}_{A^c}(\tau_x\omega) = 0$ $\mathbb{P}_0$-a.s. In particular, $\mathbb{P}_0$-a.s., for all $x \in \mathbb{Q}^d$ we have $\mathbb{1}_A(\omega)\mathbb{1}_{A^c}(\tau_x\omega) = 0$. By definition of the induced shift (see 3.18), $n(\omega, e) \in \mathbb{Q}^d$ and we conclude that $\mathbb{P}_0$-a.s., $\mathbb{1}_A(\omega)\mathbb{1}_{A^c}(\sigma_e(\omega)) = 0$. In other words, A is invariant under the induced shift $\mathbb{P}_0$-a.s., which proves that $\mathbb{Q} \sim \mathbb{P}_0$. The other two assertions follow from standard arguments. □

The following consequence of the last theorem is a law of large numbers for the trajectory of the diffusion.

Corollary 3.6. *Fix $(b,\phi) \in \mathcal{E}$. Then $\mathbb{P}_0 \times P_0^{b,\omega}$-a.s.,*

$$\lim_{t\to\infty} \frac{X_t}{t} = \mathbb{E}_0\left[\phi(\omega)\left(\frac{1}{2}\mathrm{div}\, a(\omega) + a(\omega)b(\omega)\right)\right]. \tag{3.19}$$

Proof. By definition, X_t satisfies

$$X_t = \int_0^t \sigma(X_s)\,\mathrm{d}\mathcal{B}_s + \int_0^t (\frac{1}{2}\mathrm{div}\, a + ab)(X_s)\,\mathrm{d}s. \tag{3.20}$$

Since σ is bounded, the stochastic integral divided by t goes to 0 $\mathbb{P}_0\times P_0^{b,\omega}$-a.s. Moreover, Proposition 3.4 yields

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Big(\frac{1}{2}\mathrm{div}\, a + ab\Big)(X_s)\,\mathrm{d}s = \mathbb{E}_0\left[\Big(\frac{1}{2}\mathrm{div}\, a + ab\Big)\phi\right] \quad \mathbb{P}_0\times P_0^{b,\omega}\text{-}a.s. \tag{3.21}$$

This finishes the proof. □

The following immediate consequence of Proposition 3.4 and Corollary 3.6 will be used several times in the sequel:

Corollary 3.7. *Fix $(b,\phi)\in\mathcal{E}$. Then for $\mathbb{P}_0$ almost every $\omega\in\Omega_0$ and $P_0^{b,\omega}$-a.s. and in $L^1(P_0^{b,\omega})$, we have*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\int_0^{t/\varepsilon} b(X_s,\omega)\mathrm{d}s = t\int \mathbb{P}_0(\mathrm{d}\omega)\phi(\omega)b(\omega),\\
&\lim_{\varepsilon\to 0}\varepsilon\int_0^{t/\varepsilon} L(X_s,b(X_s,\omega),\omega)\mathrm{d}s =: th(b,\phi) = t\int \mathbb{P}_0(\mathrm{d}\omega)\phi(\omega)L(b(\omega),\omega)\phi(\omega)\\
&\lim_{\varepsilon\to 0}\varepsilon X_{t/\varepsilon} =: tm(b,\phi) = t\int \mathbb{P}_0(\mathrm{d}\omega)\phi(\omega)\Big(\frac{1}{2}\mathrm{div}(a(\omega))+b(\omega)\Big),
\end{aligned}\tag{3.22}$$

uniformly on $[0,T]$.

3.3 The lower bound. The main result of this section is the following the lower bound:

Theorem 3.8. *Under **(F1)-(F4)**, let $u_\varepsilon(t,x)$ be the solution of* (1.2) *and u_{hom} as in* (2.13)*. Then $\mathbb{P}_0$-a.s., for any $T,\ell>0$,*

$$\liminf_{\varepsilon\to 0}\inf_{0\le t\le T}\inf_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell}(u_\varepsilon(t,x,\omega)-u_{\mathrm{hom}}(t,x))\ge 0,\tag{3.23}$$

where

$$u_{\mathrm{hom}}(t,x)=\sup_{y\in\mathbb{R}^d}\big[f(y)-t\mathcal{I}\big(\frac{y-x}{t}\big)\big],\qquad \mathcal{I}(x)=\sup_{\theta\in\mathbb{R}^d}[\langle\theta,x\rangle-\overline{H}(\theta)],\tag{3.24}$$

and

$$\overline{H}(\theta):=\sup_{(b,\phi)\in\mathcal{E}}\left(\int\phi d\mathbb{P}_0\left[\frac{1}{2}\mathrm{div}(a\theta)+\langle\theta,b\rangle_a-L(b,\omega)\right)\right].\tag{3.25}$$

The rest of this section is devoted to the proof of the above theorem, for which we will need some preliminary results contained in Lemmas 3.9-3.11. First, we recall the definition of the space $\mathbf{C}_T$ of progressively measurable functions $c:[0,T]\times\mathscr{X}\mapsto\overline{\mathcal{C}}_\infty$ such that (3.6) holds. Then (cf. (3.7))

$$v(t,x,\omega):=\sup_{c\in\mathbf{C}_T}E^{P_x^{c,\omega}}\left[f(X_t)-\int_0^t L(X_s,c(s))\mathrm{d}s\right]$$

solves (3.10). Let v_ε be the solution of (3.10) with initial data $\varepsilon^{-1}f(\varepsilon x)$ and domain $(0,\frac{T}{\varepsilon})\times\mathcal{C}_\infty$. Then

$$u_\varepsilon(t,x,\omega):=\varepsilon v_\varepsilon\Big(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\omega\Big)\tag{3.26}$$

solves (2.10). By the uniqueness of the viscosity solution (recall Proposition 3.3) u_ε can then be written as

$$u_\varepsilon(t,x,\omega)=\sup_{c\in\mathbf{C}_T}E^{P_{x/\varepsilon}^{c,\omega}}\Big[f(\varepsilon X_{t/\varepsilon})-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\Big].\tag{3.27}$$

An alternative representation of the above expression is given by

$$u_\varepsilon(t,x,\omega)=\sup_{c\in\mathbf{C}_T}E^{P_x^{\varepsilon,c,\omega}}\Big[f(X_t)-\int_0^t L\Big(\frac{X_s}{\varepsilon},c\big(\frac{s}{\varepsilon}\big)\Big)\mathrm{d}x\Big],\tag{3.28}$$

where $P_x^{\varepsilon,c,\omega}$ is the law of the diffusion satisfying the SDE

$$X_t = x + \sqrt{\varepsilon}\int_0^t \sigma\left(\frac{X_s}{\varepsilon}\right)\mathrm{d}B_s + \int_0^t (\operatorname{div} a)\left(\frac{X_s}{\varepsilon}\right)\mathrm{d}s + \int_0^t a\left(\frac{X_s}{\varepsilon}\right)c\left(\frac{s}{\varepsilon}\right)\mathrm{d}s. \tag{3.29}$$

In this section, we use constants C, C'' independent on ω, t, ε that may change from line to line.

Lemma 3.9. *Assume* ***(F1)***, ***(F2)*** *and* ***(F4)****. Then we can replace the supremum of $c \in \mathbf{C}_T$ in* (3.28) *by a supremum over $c \in \mathbf{C}_T^* \subset \mathbf{C}_T$ of functions satisfying the following: for each $\delta > 0$, there exists a constant C_δ depending only on δ and the constants α, α' appearing in* ***(F2)*** *and* ***(F4)*** *such that for all $\omega \in \Omega_0$,*

$$\sup_{x\in\varepsilon\mathcal{C}_\infty} \varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \big|L(X_s, c(s))\big|\mathrm{d}s\right] \le C_\delta(t+\sqrt{\varepsilon t}) + 2\alpha\delta. \tag{3.30}$$

In particular, for all $c \in \mathbf{C}_T^*$,

$$\sup_{x\in\varepsilon\mathcal{C}_\infty} \varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a^{\alpha'}\mathrm{d}s\right] \le C_\delta(t+\sqrt{\varepsilon t}) + 2\alpha\delta. \tag{3.31}$$

Proof. First, recall that under $P^{c,\omega}_{x/\varepsilon}$, the diffusion satisfies

$$\varepsilon X_{t/\varepsilon} = x + \varepsilon\int_0^{t/\varepsilon} \sigma(X_s)\mathrm{d}B_s + \varepsilon\int_0^{t/\varepsilon} (\operatorname{div} a)(X_s)\mathrm{d}s + \varepsilon\int_0^{t/\varepsilon} a(X_s)c(s)\mathrm{d}s. \tag{3.32}$$

We will now use the upper bound (2.2) from **(F1)** to deduce that

$$|a(X_s)c(s)| \le C|\sigma(X_s)c(s)| = C\|c(s)\|_a.$$

Note that the norm above implicitly depends on X_s. Also using **(F1)** we have uniformly $|\mathrm{div}\ a| \le C'$ for some $C' < \infty$. Using these two bounds,

$$E^{P^{c,\omega}_{x/\varepsilon}}[|\varepsilon X_{t/\varepsilon} - x|] \le \varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\left(\int_0^{t/\varepsilon} \sigma(X_s)\mathrm{d}B_s\right)^2\right]^{1/2} + C't + \varepsilon C E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a\mathrm{d}s\right].$$

Using Itô isometry, followed by employing the upper bound from (2.2), we have

$$E^{P^{c,\omega}_{x/\varepsilon}}\left[\left(\int_0^{t/\varepsilon} \sigma(X_s)\mathrm{d}B_s\right)^2\right] \le C''t/\varepsilon.$$

Hence,

$$E^{P^{c,\omega}_{x/\varepsilon}}[|\varepsilon X_{t/\varepsilon} - x|] \le C(t+\varepsilon\sqrt{t}) + \varepsilon C E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a\mathrm{d}s\right]. \tag{3.33}$$

By Hölder's inequality and (2.6), we obtain the inequalities

$$\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a\mathrm{d}s\right] \le t^{1/\alpha}\left(\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a^{\alpha'}\mathrm{d}s\right]\right)^{1/\alpha'}, \tag{3.34}$$

$$\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \|c(s)\|_a^{\alpha'}\mathrm{d}s\right] \le c_{10}^{-1}\left(\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[\int_0^{t/\varepsilon} \left|L\Big(X_s, c(s)\Big)\right|\mathrm{d}s\right] + c_{11}t\right). \tag{3.35}$$

Notice that by (3.30) and (3.35) we obtain (3.31). Thus, we only need to prove (3.30).

Using the formula (3.28) with $c \equiv 0$, by (2.6), (2.9) and (3.33), for any $\delta > 0$ we obtain the lower bound

$$\begin{aligned} u_\varepsilon(t,x,\omega) - f(x) &\geq E^{P_x^{\varepsilon,0,\omega}}\left[f(X_t) - f(x) - \int_0^t L\left(\frac{X_s}{\varepsilon}, 0\right)\mathrm{d}s\right] \\ &\geq -K_\delta E^{P_x^{\varepsilon,0,\omega}}[|X_t - x|] - c_{13}t - \delta \\ &\geq -K_\delta(\sqrt{\varepsilon t} + t) - c_{13}t - \delta. \end{aligned} \tag{3.36}$$

So, we only need to consider $c \in \mathbf{C}_T$ such that

$$E^{P_x^{\varepsilon,c,\omega}}\left[f(X_t) - f(x) - \int_0^t L\left(\frac{X_s}{\varepsilon}, c\left(\frac{s}{\varepsilon}\right)\right)\mathrm{d}x\right] \geq -K_\delta(\sqrt{\varepsilon t} + t) - c_{13}t - \delta,$$

and because of (2.9), such c has to fulfill

$$E^{P_x^{\varepsilon,c,\omega}}\left[K_\delta|X_t - x| - \int_0^t L\left(\frac{X_s}{\varepsilon}, c\left(\frac{s}{\varepsilon}\right)\right)\mathrm{d}x\right] \geq -K_\delta(\sqrt{\varepsilon t} + t) - c_{13}t - 2\delta.$$

Set $\Theta(t) := E^{P_x^{\varepsilon,c,\omega}}\left[\int_0^t L\left(\frac{X_s}{\varepsilon}, c\left(\frac{s}{\varepsilon}\right)\right)\mathrm{d}s\right] + c_{11}t$. Applying the inequalities (3.33),(3.34) and (3.35) to the last display, we only need to consider $c \in \mathbf{C}_T$ satisfying

$$K_\delta(t + \sqrt{\varepsilon t} + 2c_9^{-1/\alpha'}t^{1/\alpha}\Theta(t)^{1/\alpha'}) - \Theta(t) + K_\delta(\sqrt{\varepsilon t} + t) + (c_{11} + c_{13})t + 2\delta \geq 0.$$

If $A := 2K_\delta c_9^{-1/\alpha'}t^{1/\alpha}$ and $B := 2K_\delta(\sqrt{\varepsilon t} + t) + (c_{11} + c_{13})t + 2\delta$, we can write the last inequality as

$$A\Theta(t)^{1/\alpha'} - \Theta(t) + B \geq 0.$$

By Young's inequality and using that $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$, we deduce that

$$\frac{A^\alpha}{\alpha} + \frac{\Theta(t)}{\alpha'} - \Theta(t) + B = \frac{A^\alpha}{\alpha} + B - \frac{\Theta(t)}{\alpha} \geq 0,$$

so that $\Theta(t) \leq \alpha B + A^\alpha$. Recalling the definitions of A and B, we deduce (3.30), finishing the proof of the lemma. □

Lemma 3.10. *Assume **(F1)**, **(F2)** and **(F4)**. For any $\eta > 0$, there exists a set N_η with $\mathbb{P}_0(N_\eta) > 1 - \eta$ such that for any $(b,\phi) \in \mathcal{E}$ (recall 3.17),*

$$\liminf_{\varepsilon\to 0} \inf_{\omega\in N_\eta} \inf_{0\leq t\leq T}[u_\varepsilon(t,0,\omega) - f(m(b,\phi)t) + th(b,\phi)] \geq 0,$$

where $h(b,\phi)$ and $m(b,\phi)$ are defined in (3.22).

Proof. For each $\varepsilon > 0$, $(b, \phi) \in \mathcal{E}$ and $(t, \frac{x}{\varepsilon}) \in (0, T) \times \mathcal{C}_\infty$,

$$u_\varepsilon(t, 0, \omega) \geq E^{P_0^{b,\omega}}\left[f(\varepsilon X_{t/\varepsilon}) - \varepsilon \int_0^{t/\varepsilon} L(X_s, b(X_s))\mathrm{d}s\right] \qquad \mathbb{P}_0 - a.s.$$

Since f is assumed to be uniformly continuous, recalling (3.22) we obtain

$$\liminf_{\varepsilon\to 0} u_\varepsilon(t, 0, \omega) \geq \liminf_{\varepsilon\to 0} E^{P_0^{c,\omega}}\left[f(\varepsilon X_{t/\varepsilon}) - \varepsilon \int_0^{t/\varepsilon} L(X_s, c(s))\mathrm{d}s\right] = f(tm(b, \phi)) - th(b, \phi)$$

for any $(b, \phi) \in \mathcal{E}$. The lemma now follows from Egorov's theorem. □

The next lemma requires Lipschitz estimates to control the oscillation of u_ε around zero, uniformly in ε, on balls of radius r, as $r \to 0$. We will need a stronger condition on f. For proving Theorem 3.8 for any f satisfying **(F4)**, this condition will be relaxed in Lemmas 3.12-3.14.

Lemma 3.11. *Assume* ***(F1)-(F3)****, and that the initial condition $f \in C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. Then for any $(b, \phi) \in \mathcal{E}$,*

$$\liminf_{r\to 0} \liminf_{\varepsilon\to 0} \inf_{0\leq t\leq T} \inf_{y\in\varepsilon\mathcal{C}_\infty:|y|\leq r} [u_\varepsilon(t, y, \omega) - f(tm(b, \phi)) + th(b, \phi)] \geq 0 \quad \mathbb{P}_0\text{-a.s.}$$

Proof. By Lemma 3.10, it is enough to prove that $\mathbb{P}_0$-a.s.,

$$\limsup_{r\to 0} \limsup_{\varepsilon\to 0} \sup_{0\leq t\leq T} \sup_{y\in\varepsilon\mathcal{C}_\infty:|y|\leq r} |u_\varepsilon(t, y, \omega) - u_\varepsilon(t, 0, \omega)| = 0. \tag{3.37}$$

Recall that $u_\varepsilon(t, x, \omega) = \varepsilon v_\varepsilon(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega)$, where $v_\varepsilon(t, x, \omega)$ solves (3.10) with initial condition $\varepsilon^{-1} f(\varepsilon x)$. Then

$$\sup_{y\in\varepsilon\mathcal{C}_\infty:|y|\leq r} |u_\varepsilon(t, y, \omega) - u_\varepsilon(t, 0, \omega)| = \varepsilon \sup_{y\in\mathcal{C}_\infty:|y|\leq\frac{r}{\varepsilon}} \left|v_\varepsilon\left(\frac{t}{\varepsilon}, y, \omega\right) - v_\varepsilon\left(\frac{t}{\varepsilon}, 0, \omega\right)\right|.$$

By Morrey's inequality (see [E10, Section 5.6.2]), for any $\gamma > d$, there is a constant $C = C(\gamma, d)$ such that

$$\sup_{y\in\mathcal{C}_\infty:|y|\leq\frac{r}{\varepsilon}} \left|v_\varepsilon\left(\frac{t}{\varepsilon}, y, \omega\right) - v_\varepsilon\left(\frac{t}{\varepsilon}, 0, \omega\right)\right| \leq \frac{Cr}{\varepsilon}\left(\frac{\int_{B_{r/\varepsilon}(0)} |\nabla v_\varepsilon(t/\varepsilon, x, \omega)dx|^\gamma}{\lambda_d(B_{r/\varepsilon}(0))}\right)^{1/\gamma}.$$

Since $|\nabla v_\varepsilon(t/\varepsilon, x, \omega)| = |\nabla f(\varepsilon x)| \leq Cr$ if $x \in B_{r/\varepsilon}(0) \setminus \mathcal{C}_\infty$, the main contributing part in the integral comes from $B_{r/\varepsilon}(0) \cap \mathcal{C}_\infty$. In view of the assumption that $f \in C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$, we can apply [D19, Theorem 2.5], and the comparison principle Theorem C.1 to conclude that uniformly on t, $|\nabla v_\varepsilon(t/\varepsilon, x, \omega)| \leq C\xi(x, \omega)^{-\alpha/2(\alpha-1)}$, where C is a constant depending on d, α, the (uniform) Lipschitz constant of σ, and the constants $c_6, \cdots, c_{16}$ defined in Eqs. (2.5)-(2.8). Therefore, by the ergodic theorem and using (2.3), we obtain

$$\limsup_{\varepsilon\to 0} \sup_{0\leq t\leq T} \varepsilon \sup_{y\in\mathcal{C}_\infty:|y|\leq\frac{r}{\varepsilon}} \left|v_\varepsilon\left(\frac{t}{\varepsilon}, y, \omega\right) - v_\varepsilon\left(\frac{t}{\varepsilon}, 0, \omega\right)\right| \leq Cr\mathbb{E}_0\left[\xi(\omega)^{-\alpha\gamma/2(\alpha-1)}\right]^{1/\gamma}.$$

We let $r \to 0$ to conclude. □

We are now ready to provide the

Proof of Theorem 3.8: Let us first prove the result assuming that the initial condition $f \in C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$.

For any $\ell \geq 1$, we can consider the family of functions $\{f^{(y)}, |y| \leq \ell\}$, where $f^{(y)}(x) := f(x+y)$. Notice that $f^{(y)}$ is uniformly continuous with a constant K_f as in **(F4)**. Then exactly as in the proof of Lemma 3.10, we find a family of functions u_ε^y such that for any $\eta > 0$, there exists some N_η with $\mathbb{P}_0(N_\eta) \geq 1-\eta$ and

$$\liminf_{\varepsilon\to 0} \inf_{\omega\in N_\eta} \inf_{0\leq t\leq T} \inf_{y\in\varepsilon\mathcal{C}_\infty:|y|\leq\ell} [u_\varepsilon^y(t,0,\omega) - f(y+m(b,\phi)t) + th(b,\phi)] \geq 0.$$

By the ergodic theorem, in a set N of $\mathbb{P}_0$-probability 1 (we can assume it is contained in $\bigcup_{\eta>0} N_\eta$), it holds that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1}, \tau_x\omega\in N_\eta\})}{\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1})\}} \\
&= \lim_{\varepsilon\to 0} \frac{\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1}, \tau_x\omega\in N_\eta\})\big/\lambda_d(\{x : |x|\leq \ell\varepsilon^{-1}\})}{\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1}\})\big/\lambda_d(\{x : |x|\leq \ell\varepsilon^{-1}\})} \\
&= \frac{\mathbb{P}[N_\eta\cap\Omega_0]}{\mathbb{P}[\Omega_0]} = \mathbb{P}_0(N_\eta) \geq 1-\eta.
\end{aligned}$$

For each $\omega \in N$, and $\varepsilon \leq \varepsilon_0(\eta)$,

$$\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1}, \tau_x\omega\in N_\eta\}) \geq (1-2\eta)\lambda_d(\{x\in\mathcal{C}_\infty : |x|\leq \ell\varepsilon^{-1})\}.$$

In particular, every $x \in \mathcal{C}_\infty$ satisfying $|x| \leq \varepsilon^{-1}\ell$ is within distance $\ell\varepsilon^{-1}(3\delta)^{1/d}$ from some $x' \in \mathcal{C}_\infty$ satisfying $\tau_{x'}\omega \in N_\eta$. Thus, by Lemma 3.11, and noting that $u_\varepsilon^x(t,0,\omega) = u_\varepsilon(t,x,\tau_{-x/\varepsilon}\omega)$, we deduce that for each $\omega \in N$,

$$\liminf_{\varepsilon\to 0} \inf_{0\leq t\leq T} \inf_{x\in\varepsilon\mathcal{C}_\infty:|x|\leq\ell} [u_\varepsilon(t,x,\omega) - f(x+m(b,\phi)t) + th(b,\phi))] \geq 0.$$

Let

$$u(t,x) := \sup_{(b,\phi)\in\mathcal{E}} [f(x+m(b,\phi)t) - th(b,\phi)]. \tag{3.38}$$

We claim that

$$u(t,x) = u_{\mathrm{hom}}(t,x), \qquad \text{with } u_{\mathrm{hom}} \text{ defined in (3.24)}. \tag{3.39}$$

Indeed, note first that by definition of $\overline{H}$ and (3.22),

$$\begin{aligned}
\overline{H}(\theta) &= \sup_{(b,\phi)\in\mathcal{E}} [\langle\theta, m(b,\phi)\rangle - h(b,\phi)] \\
&= \sup_{y\in\mathbb{R}^d} \sup_{\substack{(b,\phi)\in\mathcal{E}:\\ m(b,\phi)=y}} [\langle\theta, y\rangle - h(b,\phi)] \\
&= \sup_{y\in\mathbb{R}^d} \left[\langle\theta, y\rangle - \inf_{\substack{(b,\phi)\in\mathcal{E}:\\ m(b,\phi)=y}} h(b,\phi)\right].
\end{aligned}$$

On the other hand, since $\mathcal{I}$ is the convex conjugate of $\overline{H}$, we conclude that

$$\mathcal{I}(y) = \inf_{\substack{(b,\phi)\in\mathcal{E}:\\ m(b,\phi)=y}} h(b,\phi). \tag{3.40}$$

As a result, and using (3.40),

$$\begin{aligned} u(t,x) &= \sup_{(b,\phi)\in\mathcal{E}} [f(x+m(b,\phi)t) - th(b,\phi)] \\ &= \sup_{y\in\mathbb{R}^d} \sup_{\substack{(b,\phi)\in\mathcal{E}:\\ m(b,\phi)=y}} [f(x+yt) - th(b,\phi)] \\ &= \sup_{y\in\mathbb{R}^d} [f(x+yt) - \mathcal{I}(y)] \\ &= u_{\mathrm{hom}}(t,x), \end{aligned}$$

which proves the claim. As a consequence,

$$\liminf_{\varepsilon\to 0} \inf_{0\le t\le T} \inf_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell} [u_\varepsilon(t,x,\omega) - u_{hom}(t,x)] \ge 0 \; \mathbb{P}_0 - a.s.$$

This finishes the proof as long as the initial condition $f \in C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$. The next three lemmas will extend the result to a uniformly continuous initial condition f (i.e., f satisfying **(F4)**), concluding the proof of Theorem 3.8. □

Lemma 3.12. *Given any uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$, there exists a sequence $(f_k)_k \subset C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ such that $f_k \to f$ uniformly.*

Proof. First, we check that f can be approximated by Lipschitz functions. An explicit construction is given by

$$f_k(x) := \inf_{y\in\mathbb{R}^d} \{f(y) + k|x-y|\}.$$

Since f_k is the infimum of k-Lipschitz functions over a convex set, then f_k is also k-Lipschitz. Clearly $f_k \le f$. To verify that $f_k \to f$ uniformly, we note that

$$f(x) - f_k(x) = \sup_{y\in\mathbb{R}^d} [f(x) - f(y) - k|x-y|].$$

For each $\delta > 0$, let $K_\delta > 0$ such that $|f(x) - f(x)| \le K_\delta|x-y| + \delta$ for any $x, y \in \mathbb{R}^d$. Then

$$f(x) - f_k(x) \le \sup_{y\in\mathbb{R}^d} [(K_\delta - k)|x-y|] + \delta.$$

Therefore, for every $\delta > 0$, if $k > K_\delta$, we have $f(x) - f_k(x) \le \delta$ for all $x \in \mathbb{R}^d$. Since δ is arbitrary, this shows that f can be approximated by Lipschitz functions. On the other hand, any Lipschitz function can be approximated by functions in $C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ by mollification. □

Lemma 3.13. *Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ be an open set. Let f_1, f_2 be two uniformly continuous functions in V. Let u_1, u_2 be the corresponding viscosity solutions to* (3.1) *with initial conditions f_1 and f_2 respectively. Then*

$$\sup_{(t,x)\in V} |u_1(t,x) - u_2(t,x)| \le \sup_{(t,x)\in \partial V} |f_1(x) - f_2(x)|. \tag{3.41}$$

Proof. If the right-hand side is infinite, the claim is trivial. Otherwise, note that it is enough to show the inequality

$$\sup_{(t,x)\in V} [u_1(t,x) - u_2(t,x)] \le \sup_{(t,x)\in \partial V} [f_1(x) - f_2(x)].$$

By subtracting a constant, we can assume that $\sup_{(t,x)\in\partial V}[f_1(x) - f_2(x)] = 0$, that is, $f_1(x) \le f_2(x)$ for all $(t,x) \in \partial V$. Then the result follows from the comparison principle, Theorem C.1. □

Lemma 3.14. *Let $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ be an open set, $f : \mathbb{R}^d \to \mathbb{R}$, and $(f_n)_{n\in\mathbb{N}}, f_n : \mathbb{R}^d \to \mathbb{R}$ such that f is uniformly continuous and $(f_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ such that $f_n \to f$ uniformly. For each $n \in \mathbb{N}$, let u_n be the viscosity solution to* (3.1) *with initial condition f_n and u the viscosity solution to* (3.1) *with initial condition f. Then $u_n \to u$ uniformly in V.*

Proof. By Lemma 3.13, we have for each n

$$\sup_{(t,x)\in V} |u_n(t,x) - u(t,x)| \le \sup_{(t,x)\in \partial V} |f_n(x) - f(x)| \le \sup_{x\in\mathbb{R}^d} |f_n(x) - f(x)|.$$

Letting $n \to \infty$ and using that $f_n \to f$ uniformly finishes the proof. □

The following lemma will conclude the proof of Theorem 3.8.

Lemma 3.15. *The conclusion of Theorem 3.8 holds if the initial condition f is uniformly continuous.*

Proof. Using Lemma 3.12, let $(f_n)_{n\in\mathbb{N}} \subset C^\infty(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d)$ satisfy such that $f_n \to f$ uniformly. Given $\varepsilon > 0$, let u_ε^n, u be the solutions to (2.10) with initial conditions f_n and f respectively. Similarly, let $u_{\mathrm{hom}}^n, u_{\mathrm{hom}}$ be the solutions to (2.11) with initial conditions f_n and f respectively. By Lemma 3.14, $\mathbb{P}_0$-a.s., $u_\varepsilon^n \to u_\varepsilon$ uniformly (and uniformly on ε). Similarly, $u_{\mathrm{hom}}^n \to u_{\mathrm{hom}}$ uniformly. Moreover, by Theorem 2.1, $\mathbb{P}_0$-a.s. we know that u_ε^n converges as $\varepsilon \to 0$ to u_{hom}^n uniformly on compact sets. By the triangle inequality, we can deduce that $\mathbb{P}_0$-a.s., u_ε converges u_{hom} uniformly on compact sets. □

4 Correctors

Given any $\delta > 0$, we start this section by defining the class of gradients $G \in \mathcal{G}_\delta$ and the corresponding "correctors"$V_G : \mathbb{R}^d \times \Omega_0 \to \mathbb{R}^d$. Let $\mathcal{G}_\delta$ be the class of functions $G : \Omega_0 \to \mathbb{R}^d$ satisfying the following properties:

- $L^{1+\delta}(\mathbb{P}_0)$**-boundedness**: The following inequalities hold:

$$\|G\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty, \tag{4.1}$$

and

$$\operatorname*{ess\,sup}_{\mathbb{P}_0}\left[\frac{1}{2}\mathrm{div}(a(G+\theta))+H(G+\theta)\right]<\infty. \tag{4.2}$$

• **Curl-free property on the cluster**: Given any $G:\Omega_0\to\mathbb{R}^d$, with a slight abuse of notation we will continue to write

$$G:\Omega_0\times\mathbb{R}^d\to\mathbb{R}^d,\quad \text{with}\quad G(\omega,x)=G(\tau_x\omega).$$

Now, for $\mathbb{P}_0$-almost every $\omega\in\Omega_0$, we require that G is curl-free, meaning $\nabla\times G(\omega,\cdot)=0$ on $\mathcal{C}_\infty$, or simply, for $\mathbb{P}_0$-almost every $\omega\in\Omega_0$, we have

$$\int_{\mathrm{C}} G(\omega,\cdot)\ \cdot\mathrm{d}\mathbf{r}=0 \tag{4.3}$$

for every rectifiable simple closed path C on $\mathcal{C}_\infty$. For any G satisfying (4.3) we define $V_G:\Omega_0\times\mathbb{R}^d\to\mathbb{R}^d$ by

$$V_G(\omega,x):=\int_{0\rightsquigarrow x} G(\omega,\cdot)\ \cdot\mathrm{d}\mathbf{r}, \tag{4.4}$$

where $0\rightsquigarrow x$ is any piecewise smooth curve contained in $\mathcal{C}_\infty$ (and 0 when $x\notin\mathcal{C}_\infty$). Note that the choice of the smooth curve is irrelevant, thanks to (4.3).

• **Zero induced mean:** Recall the definition of $n(\omega,e)$ from (1.13) and set $\mathfrak{v}_e=\mathfrak{v}_e(\omega)=n(\omega,e)e$. Then we require that

$$\mathbb{E}_0[V_G(\cdot,\mathfrak{v}_e)]=0. \tag{4.5}$$

Definition 4.1. *For any $\delta>0$, we say that $G\in\mathcal{G}_\delta$ if* (4.1)-(4.3) *and* (4.5) *hold. Similarly, we declare that $G\in\mathcal{G}_\infty$ if the above conditions hold, but replace* (4.1)*, by*

$$\mathrm{ess\,sup}_{\mathbb{P}_0}|G(\omega)|<\infty. \tag{4.6}$$

In this section we will prove the following result.

Theorem 4.2. *Fix $d\geq 2$ and $G\in\mathcal{G}_\infty$. Then for $\mathbb{P}_0$-a.e. $\omega\in\Omega_0$ and every $\ell>0$, we have*

$$\lim_{\varepsilon\to 0}\sup_{\substack{x\in\varepsilon\mathcal{C}_\infty\\ |x|\leq\ell}}\varepsilon\left|V_G\left(\frac{x}{\varepsilon},\omega\right)\right|=0\qquad \mathbb{P}_0-\mathrm{a.s.}$$

The rest of this section is devoted to the proof of Theorem 4.2, which will be carried out in few steps.

4.1 Controlling directional growth.

The main result of this section, Theorem 4.3 stated below provides a control on the growth of V_G along coordinate directions. For this purpose, we fix a unit coordinate vector e and for $\omega\in\Omega_0$, define the *successive arrivals* $(n_k(\omega))_{k\in\mathbb{N}}$ of the cluster recursively as follows: Recall (1.13) and define

$$n_1(\omega)=n(\omega,e),\quad \text{and for } k\geq 1 \text{ we set}\quad n_{k+1}(\omega):=\min\{l\in\mathbb{N}:l>n_k(\omega),le\in\mathcal{C}_\infty(\omega)\}.$$

Theorem 4.3. *Let e be any unit coordinate vector. If $G \in \mathcal{G}_\delta$, then $\mathbb{P}_0$-a.s.,*

$$\lim_{k\to\infty} \frac{|V_G(n_k(\omega)e,\omega)|}{k} = 0.$$

The proof of Theorem 4.3 will need the following result.

Proposition 4.4. *For any unit coordinate vector e, recall that we denote the first successive arrival in direction e by $\mathfrak{v}_e = \mathfrak{v}_e(\omega) = n(\omega,e)e$. Then for any $G \in \mathcal{G}_\delta$, we have $\mathbb{E}_0|V_G(\mathfrak{v}_e,\cdot)| < \infty$. More precisely, there is a constant $C = C(d,\delta,\mathbb{P}_0)$ such that for any $G \in \mathcal{G}_\delta$, $\mathbb{E}_0|V_G(\mathfrak{v}_e,\cdot)| \le C\|G\|_{L^{1+\delta}(\mathbb{P}_0)}$.*

Proof of Theorem 4.3 (assuming Proposition 4.4). For each $k \in \mathbb{N}$, set $x_0 = 0$ and $x_j = n_j e$ for $1 \le j \le k$. We choose a path $0 \rightsquigarrow x_k$ from 0 to x_k contained in $\mathcal{C}_\infty(\omega)$ such that, for some $0 = t_0 < t_1 < \ldots < t_k = 1$ and $r : [0,1] \to (x_0 \rightsquigarrow x_k)$, it holds $r(t_j) = x_j$. Then by the definition of V_G in (4.4),

$$\begin{aligned} V_G(n_k e,\omega) &= \int_{x_0\rightsquigarrow x_k} G(r,\omega)\mathrm{d}r = \sum_{j=0}^{k-1}\int_{x_j\rightsquigarrow x_{j+1}} G(r,\omega)\mathrm{d}r = \sum_{j=0}^{k-1}\int_{0\rightsquigarrow(x_{j+1}-x_j)} G(r,\tau_{x_j}\omega)\mathrm{d}r \\ &= \sum_{j=0}^{k-1} V_G(x_{j+1}-x_j,\tau_{x_j}\omega) = \sum_{j=1}^{k-1} V_G((n_j(\omega)-n_{j-1}(\omega))e,\tau_{n_{j-1}e}\omega) \\ &= \sum_{j=0}^{k-1} V_G(n_1(\sigma_e^j(\omega))e,\sigma_e^j(\omega)). \end{aligned}$$

Recall (3.18) for the definition of the induced shift and Definition (4.4) for that of the corrector V_G. We define the function $F(\omega) = V_G(n(\omega,e)e,\omega)$, so that

$$V_G(n_k(\omega)e,\omega) = \sum_{j=0}^{k-1} F\circ\sigma_e^j(\omega). \tag{4.7}$$

From Proposition 3.5, the induced shift σ_e is $\mathbb{P}_0$-preserving and ergodic. Furthermore, from Proposition 4.4, the function $F \in L^1(\mathbb{P}_0)$. Then by Birkhoff's Ergodic Theorem,

$$\lim_{k\to\infty}\frac{\sum_{j=0}^{k-1} F\circ\sigma_e^j(\omega)}{k} = \mathbb{E}_0[V_G(n(\omega,e)e,\omega)] = 0, \tag{4.8}$$

where the last equality comes from the induced mean-zero property (4.5) of $G \in \mathcal{G}_\delta$. □

To show Proposition 4.4 we require first a lemma.

Lemma 4.5. *Let $\ell = \ell(\omega) = \mathrm{d}_\omega(0,\mathfrak{v}_e(\omega))$ be the graph distance between 0 and $\mathfrak{v}_e = n(\omega,e)e$. Then there exist constants $a, C > 0$ such that for any $t > 0$,*

$$\mathbb{P}_0\bigg(\sup_{0\le s\le n_1(\omega)} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se) > t\bigg) \le C\mathrm{e}^{-at}. \tag{4.9}$$

In particular,

$$\mathbb{P}_0(\ell > t) < C\mathrm{e}^{-at}.$$

Proof. Let $\varepsilon > 0$. For $t > 0$ we write $t_\varepsilon := \lfloor \varepsilon t \rfloor$. Then

$$\mathbb{P}_0\left(\sup_{0\le s\le n_1} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se) > t\right)$$
$$\le \mathbb{P}_0\left(n_1(\omega) \ge t\varepsilon\right) + \mathbb{P}_0\left(\sup_{0\le s\le t\varepsilon} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se) > t\right).$$

By **(P6)**, the claim follows once we prove that the second term goes to zero at an exponential rate. This probability is bounded above by $\sum_{i=1}^{\lfloor t\varepsilon\rfloor} \mathbb{P}_0\left(\sup_{i-1\le s\le i} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se) > t\right)$. Since the number of summands is growing only polynomially in t, it suffices to show that each summand there decays exponentially in t. We will proceed as follows:

We define

$$m := \min\{l \in \mathbb{N} : l > t_\varepsilon,\ -le \in \mathcal{C}_\infty\}, \quad A_{x,y} = \{\mathrm{d}_\omega(x,y) \ge t/2,\ x,y \in \mathcal{C}_\infty\}.$$

Now we observe that on the event $\{\sup_{i-1\le s\le i} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se) > t\}$, one of the following cases must hold:

- $m > 2t_\varepsilon$, or
- at least one of the points le with $l \in \mathbb{Z}$ and $|l| \le 2t_\varepsilon$ is in $\mathcal{C}_\infty$ and for some $i-1 \le s \le i$, $\max\{\mathrm{d}_\omega(0,-le), \mathrm{d}_\omega(-le,se)\} \ge t/2$.

In the first of the two cases above we have $|\mathfrak{v}_{-e} \circ \sigma_{-e}^m| > t_\varepsilon$ for at least one $m = 1, ..., t_\varepsilon$. Hence,

$$\mathbb{P}_0\big(\sup_{i-1\le s\le i} \mathbb{1}\big\{se \in \mathcal{C}_\infty(\omega)\big\}\mathrm{d}_\omega(0,se) > t\big)$$
$$\le \sum_{m=1}^{t_\varepsilon} \mathbb{P}_0\big(\sigma_{-e}^m(\{|\mathfrak{v}_{-e}| \ge t_\varepsilon\})\big) + \sum_{\ell=t_\varepsilon}^{2t_\varepsilon} \mathbb{P}_0\big(\exists i-1 \le s \le i : A_{0,-le} \cup A_{-le,se}\big).$$

By **(P6)**, the probabilities of the events in the first sum are equal and exponentially small. The second sum is bounded by

$$t_\varepsilon \mathbb{P}_0(A_{0,-le}) + \sum_{\ell=t_\varepsilon}^{2t_\varepsilon} \mathbb{P}_0\left(\exists i-1 \le s \le i : A_{-le,se}\right).$$

To bound the first term, we use (1.10) and **(P4)** to obtain the bound

$$\mathbb{P}_0(A_{0,-le}) \le \frac{1}{\mathbb{P}(0\in\mathcal{C}_\infty)}\mathbb{P}\left(\exists x \ne y \in \mathcal{C}_\infty(\omega) : |x| \le \frac{1}{2}, |y-le| \le \frac{1}{2}, \mathrm{d}_\omega(x,y) \ge \frac{t}{2}+1; 0,x,y \in \mathcal{C}_\infty\right)$$
$$\le \frac{1}{\mathbb{P}(0\in\mathcal{C}_\infty)}\mathbb{E}\Big[\sum_{x,y\in\omega}^{\ne} \mathbb{1}\Big\{|x| \le 1/2, |y-le| \le 1/2, \mathrm{d}_\omega(x,y) \ge t/2+1; 0,x,y \in \mathcal{C}_\infty\Big\}\Big]$$
$$= \frac{\zeta^2}{\mathbb{P}(0\in\mathcal{C}_\infty)}\int_{[-1/2,1/2]^d}\int_{[le-1/2,le+1/2]^d} \mathbb{P}^{x,y}\left(\mathrm{d}_\omega(x,y) \ge \frac{t}{2}+1; 0,x,y \in \mathcal{C}_\infty\right)\mathrm{d}x\mathrm{d}y$$
$$\le C\mathrm{e}^{-C' t_\varepsilon}$$

for some constants $C, C' > 0$ which are independent of l, and for $\varepsilon > 0$ small enough, with ζ defined in (**(P2)**). Following the same calculations as in the last display, we can also show that

$$\mathbb{P}_0\left(\exists i-1 \le s \le i : A_{-le,se}\right) \le C\mathrm{e}^{-C' t_\varepsilon}$$

for constants $C, C' > 0$ independent of i and l, for $\varepsilon > 0$ small enough. After estimating the probabilities of all events by an exponential upper bound, from the unions we get another factor that is linear in t, which can be absorbed by the exponential bound for t large enough. Thus the proof of Lemma 4.5 is complete.

□

Now we are ready to show Proposition 4.4.

Proof of Proposition 4.4. Let $\mathscr{B} = \mathscr{B}(\omega)$ be an enumeration of the balls that appear in the construction of $\mathscr{C}(\omega)$ (recall **(P3)**). Then define the random variable

$$\tilde{\mathrm{d}}_\omega(x,y) := \min\Big\{ n \in \mathbb{N} : \exists\, (B_i)_{i=1}^n \subset \mathscr{B} \text{ such that } x \in B_1, y \in B_n, \text{ and } B_{i-1} \cap B_i \neq \emptyset \; \forall\, 1 \le i \le n \Big\} \tag{4.10}$$

and set

$$\tilde{\ell} := \tilde{\mathrm{d}}_\omega(0, \mathfrak{v}_e). \tag{4.11}$$

Note that there is some constant $c > 0$ such that for all $n > 0$,

$$\mathbb{P}_0\left(\tilde{\ell} > n\right) \le \mathrm{e}^{-cn}. \tag{4.12}$$

For $j \in \mathbb{N}$, let $N_j := \mathbb{Z}/2 \cap [-j,j]^d$. We consider this set as a graph, where for $x, y \in N_j$, $x \sim y$ iff $|x-y|_1 = \frac{1}{2}$. Note that if $\tilde{\ell} = j$, then there is a nearest-neighbor path on N_j of length $k \le 3^d j$ such that for all $1 \le i \le k-1$, the line segment between x_i and x_{i+1} is contained in the cluster. Thus, we write

$$\{\tilde{\ell} = j\} = \bigcup_{k=1}^{3^d j} \bigcup_{x_1,\cdots,x_k} \left\{ \tilde{\ell} = j \cap A(x_1,\cdots,x_k) \right\},$$

where

$$A(x_1,\cdots,x_k) := \Big\{ x_1,\cdots,x_k \text{ is a nearest-neighbor path on } N_j, \; \forall 1 \le i \le k-1 \text{ the line segment between } x_{i-1}, x_i \text{ is inside } \mathcal{C}_\infty \Big\}. \tag{4.13}$$

If $\tilde{\ell} = j$, then one can write for some nearest neighbor path $0 = x_0, x_1, \cdots, x_k$ on N_j ($1 \le k \le 3^d j$) such that the line segment between x_i and x_{i+1} is inside $\mathcal{C}_\infty$ for all $0 \le i \le k-1$,

$$\begin{aligned}\int_{0 \rightsquigarrow \mathfrak{v}_e} |G(\omega, \cdot)| \cdot \mathrm{d}\mathbf{r} &\le \sum_{i=0}^{k-1} \int_0^1 |G(\tau_{x_i}\omega, t(x_{i+1} - x_i))| \mathrm{d}t \\ &\le 2 \sum_{i=0}^{k-1} \sum_{|e|=1} \int_0^{1/2} |G(\tau_{x_i}\omega, te)| \mathrm{d}t \\ &\le 2(3^d j) \sum_{x \in \mathcal{C}_\infty \cap N_j : |x| \le 3^d j} \sum_{|e|=1} \int_0^{1/2} |G(\tau_x \omega, te)| \mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}_0 |V_G(\mathfrak{v}_e, \cdot)| = \sum_{j=1}^{\infty} \mathbb{E}_0 \Big[\Big| \int_{0 \rightsquigarrow \mathfrak{v}_e} G(\omega, \cdot) \cdot \mathrm{d}\mathbf{r} \Big|, \tilde{\ell} = j \Big] \\ &\le 2 \sum_{j=1}^{\infty} \sum_{|e|=1} \sum_{x \in N_j : |x| \le 3^d j} (3^d j) \int_0^{1/2} \mathbb{E}_0 \Big[|G(\tau_x \omega, te)|, x \rightsquigarrow x + e \subset \mathcal{C}_\infty, \tilde{\ell} = j \Big] \mathrm{d}t \qquad (4.14) \\ &\le 2 \sum_{j=1}^{\infty} \sum_{|e|=1} \sum_{x \in N_j : |x| \le 3^d j} (3^d j) \int_0^{1/2} \mathbb{E}_0 \Big[|G(\tau_x \omega, te)|^{1+\delta}, x \rightsquigarrow x + e \subset \mathcal{C}_\infty \Big]^{1/(1+\delta)} \mathbb{P}_0(\tilde{\ell} = j)^{\frac{\delta}{1+\delta}} \mathrm{d}t.\end{aligned}$$

Since $G \in \mathcal{G}_\delta$, then for any $x \in \mathbb{R}^d$,

$$\mathbb{E}_0 \left[|G(\omega, x)|^{1+\delta}, x \in \mathcal{C}_\infty \right] \le \|G\|_{L^{1+\delta}(\mathbb{P}_0)}.$$

As a consequence, (4.14) can be bounded by

$$C(d) \|G\|_{L^{1+\delta}(\mathbb{P}_0)} \sum_{j=1}^{\infty} j^2 \mathbb{P}_0(\tilde{\ell} = j)^{\frac{\delta}{1+\delta}} \le C(d, \delta, \mathbb{P}_0) \|G\|_{L^{1+\delta}(\mathbb{P}_0)}$$

due to (4.12). This finishes the proof of the proposition.

□

Corollary 4.6. *Let $G \in \mathcal{G}_\infty$. Then for any unit coordinate vector e and $\mathbb{P}_0$-a.s.,*

$$\lim_{s \to \infty} \mathbb{1}\{se \in \mathcal{C}_\infty(\omega)\} \frac{|V_G(se, \omega)|}{s} = 0.$$

Proof. If $se \in \mathcal{C}_\infty(\omega)$, then there exists $k \ge 0$ such that $n_k(\omega) \le s < n_{k+1}(\omega)$. Note that $k \nearrow \infty$ as $s \nearrow \infty$. Then we have

$$\frac{|V_G(se, \omega)|}{s} \le \frac{|V_G(n_k(\omega)e, \omega)|}{n_k(\omega)} + \frac{|V_G((s - n_k(\omega))e, \tau_{n_k(\omega)e}\omega)|}{n_k(\omega)}.$$

By the ergodic theorem (as in the proof of Theorem 4.3) and **(P6)**, $\lim_{k\to\infty}\frac{n_k(\omega)}{k}=\mathbb{E}_0[n_1]<\infty$ $\mathbb{P}_0$-a.s. This fact, together with Theorem 4.3, allow us to deduce that the first term in the sum above goes to zero as $s\to\infty$. It remains to bound the second term. Note that it suffices to show that

$$\lim_{k\to\infty}\sup_{n_k(\omega)\le s\le n_{k+1}(\omega)}\mathbb{1}\{se\in\mathcal{C}_\infty(\omega)\}\frac{|V_G((s-n_k(\omega))e,\tau_{n_k(\omega)e},\omega)|}{k}=0\ \mathbb{P}_0\text{-}a.s.$$

Since $G\in\mathcal{G}_\infty$, it is enough to prove that

$$\lim_{k\to\infty}\sup_{n_k\le s\le n_{k+1}}\mathbb{1}\{se\in\mathcal{C}_\infty(\omega)\}\frac{\mathrm{d}_\omega(n_k(\omega)e,se)}{k}=0\ \mathbb{P}_0\text{-}a.s. \tag{4.15}$$

By the Borel-Cantelli lemma, it suffices to verify that for any $\varepsilon>0$,

$$\sum_{k=1}^\infty\mathbb{P}_0\left(\sup_{n_k(\omega)\le s\le n_{k+1}(\omega)}\mathbb{1}\{se\in\mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(n_k(\omega)e,se)>k\varepsilon\right)<\infty. \tag{4.16}$$

Since $\mathbb{P}_0$ is invariant under $\tau_{n_k e}$, the sum above is equal to

$$\sum_{k=1}^\infty\mathbb{P}_0\left(\sup_{0\le s\le n_1(\omega)}\mathbb{1}\{se\in\mathcal{C}_\infty(\omega)\}\mathrm{d}_\omega(0,se)>k\varepsilon\right).$$

By Lemma 4.5, this sum is finite for each $\varepsilon>0$, concluding the proof. □

4.2 Controlling density of growth.

The main result of this section is the following result:

Proposition 4.7. *Let $d\ge 2$ and $G\in\mathcal{G}_\infty$. Then for all $\varepsilon>0$ and $\mathbb{P}_0$-almost all ω,*

$$\limsup_{r\to\infty}\frac{1}{(2r)^d}\int_{x\in\mathcal{C}_\infty(\omega),|x|\le r}\mathbb{1}\{|V_G(x,\omega)|\ge\varepsilon r\}\mathrm{d}x=0. \tag{4.17}$$

The proof of Proposition 4.7 consists of three main steps.

Step 1: We start this section with a definition: Given $K>0$ and $\varepsilon>0$, we say that a point $x\in\mathbb{R}^d$ belongs to $\mathscr{G}_{K,\varepsilon}(\omega)$ for $\omega\in\Omega$ if $x\in\mathcal{C}_\infty(\omega)$ and

$$|V_G(x+te,\omega)-V_G(x,\omega)|\le K+\varepsilon|t| \tag{4.18}$$

for each $t\in\mathbb{R}$, and e is a unit coordinate vector such that $x+te\in\mathcal{C}_\infty(\omega)$. We will use the following consequence of Corollary 4.6 in the sequel: for every $\varepsilon>0$, $\mathbb{P}(0\in\mathcal{C}_\infty)=\lim_{K\to\infty}\mathbb{P}(0\in\mathscr{G}_{K,\varepsilon})$. For $k\in\{1,...,d\}$, let us also define

$$\Lambda_r^k=\{x\in\mathbb{R}^k:|x|_\infty\le r\}, \tag{4.19}$$

which is the k-dimensional section of the d-dimensional box $\{x\in\mathbb{R}^d:|x|_\infty\le r\}$, and set

$$\begin{aligned}\varrho_{k,\varepsilon}(\omega)&:=\limsup_{r\to\infty}\inf_{y\in\mathcal{C}_\infty(\omega)\cap\Lambda_r^1}\frac{1}{|\Lambda_r^k|}\int_{x\in\mathcal{C}_\infty(\omega)\cap\Lambda_r^k}\mathbb{1}\{|V_G(x,\omega)-V_G(y,\omega)|\ge\varepsilon r\}\mathrm{d}x,\\ \varrho_k(\omega)&:=\lim_{\varepsilon\searrow 0}\varrho_{k,\varepsilon}(\omega).\end{aligned} \tag{4.20}$$

Lemma 4.8. *Let $1 \le k < d$. If $\varrho_k = 0$ $\mathbb{P}$-almost surely, then also $\varrho_{k+1} = 0$ $\mathbb{P}$-almost surely.*

Step 2: Proof of Lemma 4.8. For $k \le d$, we consider the k-dimensional Lebesgue measure on $\mathbb{R}^k$ and we call it λ_k. We assume that $\mathbb{P}$-a.s. $\varrho_1 = 0$. In particular, for each $\varepsilon > 0$ and large enough r, there is some set $\Delta \subset \mathcal{C}_\infty \cap \Lambda_r^1$ satisfying

$$\lambda_1(\Lambda_r^1 \cap \mathcal{C}_\infty \setminus \Delta) \le \varepsilon \lambda_1(\Lambda_r^1),$$
$$|V_G(x,\omega) - V_G(y,\omega)| \le \varepsilon r \quad x, y \in \Delta.$$

Moreover, for $K > 0$ large enough (but deterministic), replacing Δ by $\Delta \cap \mathscr{G}_{K,\varepsilon}$ grant us the following properties for large r :

(i) $\lambda_1(\Lambda_r^1 \cap \mathcal{C}_\infty \setminus \Delta) \le \varepsilon \lambda_1(\Lambda_r^1)$,

(ii) $|V_G(x,\omega) - V_G(y,\omega)| \le \varepsilon r \quad x, y \in \Delta$,

(iii) $\Delta \subset \mathscr{G}_{K,\varepsilon}$, and

(iv) $\Delta \cap \Lambda_r^1 \ne \emptyset$.

This is a consequence of the fact that $\lim_{K\to\infty} \mathbb{P}(0 \in \mathcal{C}_\infty \setminus \mathscr{G}_{K,\varepsilon}) = 0$, and the ergodic theorem. We stress that even though these conditions are easily satisfied in dimension one, the construction will allow us to obtain the same properties in larger dimensions. In particular, we want that the "base" Δ is contained in each successive step, so that (iv) will be always valid.

Next, for $L \in \mathbb{N}$ and $r > 0$, define

$$\Xi_{L,r}(\omega) := \{x \in \Lambda_r^1 : \#\{0 \le i \le L-1 : x + ie_2 \in \mathcal{C}_\infty(\omega\} > 0\}. \tag{4.21}$$

we claim that for each $\delta > 0$, there exists some $L = L(\delta)$ (deterministic) that satisfies $\mathbb{P}$-a.s. $\lambda(\Xi_{L,r}) \ge (1-\delta)\lambda(\Lambda_r^1)$ for large r (which may depend on ω). Indeed, by the ergodic theorem, the following equality holds $\mathbb{P}$-a.s. for all $L \in \mathbb{N}$:

$$\lim_{r\to\infty} \frac{\lambda(\Xi_{L,r})(\omega)}{\lambda(\Lambda_r^1)} = \mathbb{P}(\#\{0 \le i \le L : ie_2 \in \mathcal{C}_\infty(\omega\} > 0). \tag{4.22}$$

Since

$$\lim_{L\to\infty} \frac{1}{L} \#\{i \in \{0, \cdots, L-1\} : ie_2 \in \mathcal{C}_\infty(\omega)\} = \mathbb{P}(0 \in \mathcal{C}_\infty) > 0 \ \mathbb{P}\text{-a.s.}$$

as $L \to \infty$, the probability on the right in (4.22) converges to 1, so the claim holds. For fixed L, choose $K > 0$ large enough so that $\mathbb{P}$-a.s., for all $i = 0, \cdots, L-1$ the conditions (i)-(iv) above will hold some $\Delta_i \subset \tau_{ie_2}(\Lambda_r^1)$ (replacing Λ_r^1 with $\tau_{ie_2}(\Lambda_r^1)$ in (i) and (iv), for r large enough. Next, we define for $r > 0$ (and setting $\Delta_0 := \Delta$)

$$\Lambda = \Lambda_r := \{x \in \Lambda_r^2 \cap \mathcal{C}_\infty \,\big|\, \exists\, 0 \le i \le L-1,\ (y,t) \in [-r,r]^2 :\ x = ye_1 + te_2 \text{ and } ye_1 + ie_2 \in \Delta_i\}. \tag{4.23}$$

In words, Λ represents the points in $x \in \Lambda_r^2$ which have some $\tilde{x} \in \Delta_i$ that shares the same projection over $\mathbb{R}e_1$. Note that $\Delta \subset \Lambda$, so in particular, $\Lambda \cap \Lambda_r^1 \neq \emptyset$ for large r. We show that the density Λ is close to 1. More precisely, if $x \in (\Lambda_r^2 \cap \mathcal{C}_\infty) \setminus \Lambda$, then $x = ye_1 + te_2$ for some $(y,t) \in [-r,r]^2$, and either $ye_1 \notin \Xi_{L,r}$ or $ye_1 \in \Xi_{L,r}$ and $ye_1 + ie_2 \in \mathcal{C}_\infty \setminus \Delta_i$ for all $i = 0, \cdots, L-1$. Therefore, for large enough r,

$$\begin{aligned}\frac{\lambda_2((\Lambda_r^2 \cap \mathcal{C}_\infty) \setminus \Lambda)}{\lambda_2(\Lambda_r^2)} &\leq \frac{1}{2r}\int_{-r}^{r} \mathbb{1}\{ye_1 \in \Lambda_r^1 \setminus \Xi_{L,r}\}\mathrm{d}y + \sum_{i=0}^{L-1} \frac{1}{2r}\int_{-r}^{r} \mathbb{1}\{ye_1 + se_2 \in \Lambda_r^1 \setminus \Delta_i\}\mathrm{d}y \\ &= \frac{1}{2r}\left[\lambda_1(\Lambda_r^1 \setminus \Xi_{L,r}) + \sum_{i=0}^{L-1} \lambda_1((\Lambda_1^r \cap \mathcal{C}_\infty) \setminus \Delta_i)\right] \leq L\varepsilon + \delta.\end{aligned} \tag{4.24}$$

At this point, we choose ε and δ. Let $\varepsilon, \delta > 0$ small enough so that $L\varepsilon + \delta < \frac{1}{2}\mathbb{P}(0 \in \mathcal{C}_\infty)^2$. By the FKG-inequality in **(P5)** (note that $\{x \in \mathcal{C}_\infty\}$ is an increasing event), for every $x, y \in \mathbb{R}^d$ we have

$$\mathbb{P}(x \in \mathcal{C}_\infty(\omega), y \in \mathcal{C}_\infty(\omega)) \geq \mathbb{P}(x \in \mathcal{C}_\infty(\omega))\mathbb{P}(y \in \mathcal{C}_\infty(\omega)) = \mathbb{P}(0 \in \mathcal{C}_\infty)^2.$$

Moreover, for K large enough, by the ergodic theorem we have for any $s, t \in \{0, \cdots, L-1\}$

$$\lim_{r\to\infty} \frac{1}{\lambda_1(\Lambda_r^1)}\lambda_1(x \in \Lambda_r^1 : x + se_2 \in \mathscr{G}_{K,\varepsilon}, x + te_2 \in \mathscr{G}_{K,\varepsilon}) = \mathbb{P}(se_2 \in \mathscr{G}_{K,\varepsilon}, te_2 \in \mathscr{G}_{K,\varepsilon}) > L\varepsilon + \delta. \tag{4.25}$$

Thus, for large enough r, for every $s, t \in \{0, \cdots, L-1\}$, the density of points $x \in \Lambda_r^1$ such that $x + se_2 \in \Delta_s$ and $x + te_2 \in \Delta_t$ is positive. To finish the proof, we verify that for each $u, v \in \Lambda$, $|V(u,\omega) - V(v,\omega)| \leq 7\varepsilon r$ for large r such that all the above holds (in particular, (4.24), (4.25)). Indeed, if $u = x_1e_1 + y_1e_2$ and $v = x_2e_1 + y_2e_2 \in \Lambda$, then there are $s, t \in \{0, \cdots, L-1\}$ such that if $u' := x_1e_1 + se_2$ and $v' := x_2e_1 + te_2$, then $u', v' \in \mathscr{G}_{K,\varepsilon}(\omega)$ (for $K = K(\omega)$ independent on r that satisfies the conditions listed above). Moreover, by (4.25), there exists some $x_3 \in \Lambda_r^1$ satisfying $u'', v'' \in \mathscr{G}_{K,\varepsilon}$, where $u'' := x_3e_1 + se_2$ and $v'' = x_3e_1 + te_2$. Putting all together, we have

$$\begin{aligned}|V_G(u,\omega) - V_G(v,\omega)| &\leq |V_G(u,\omega) - V_G(u',\omega)| + |V_G(u',\omega) - V_G(u'',\omega)| + |V_G(u'',\omega) - V_G(v'',\omega)| \\ &\quad + |V_G(v'',\omega) - V_G(v',\omega)| + |V_G(v',\omega) - V_G(v,\omega)| \\ &\leq K + \varepsilon|x_2 - s| + K + \varepsilon|x_1 - x_3| + K + \varepsilon|s - t| + K + \varepsilon|x_2 - x_3| + K + \varepsilon|y_2 - t| \\ &\leq 5K + 3\varepsilon L + 6\varepsilon r \leq 7\varepsilon r\end{aligned}$$

for large enough r. In conclusion, by the last computation, the fact that $\Lambda \cap \Lambda_r^1 \neq \emptyset$ and (4.24), $\varrho_{2,7\varepsilon} \leq L\varepsilon + \delta$. By letting first $\varepsilon \searrow 0$ and then $\delta \searrow 0$, we deduce that $\varrho_2 = 0$ $\mathbb{P}_0$-a.s.

We can use the same construction to go to higher dimensions. More precisely, the element Λ for dimension ρ becomes the element Δ in dimension $\rho + 1$. The base case guarantees that properties (i)-(iv) that appear at the beginning of the proof remain true for $\rho > 1$. This finishes the proof of Lemma 4.8. □

Step 3: Proof of Proposition 4.7. Theorem 4.7 will follows from Corollary 4.6 and Lemma 4.8. Since

$$\begin{aligned}&\inf_y \lambda_1(\{x \in \mathcal{C}_\infty \cap \Lambda_r^1 : |V_G(x,\omega) - V_G(y,\omega)| \geq \varepsilon r\}) \\ &\leq \lambda_1(\{x \in \mathcal{C}_\infty \cap \Lambda_r^1 : |V_G(x,\omega)| \geq \varepsilon r - |V_G(0,\omega)|\}),\end{aligned}$$

and by Corollary 4.6, it holds $\varrho_1 = 0$ for $\mathbb{P}_0$-almost every ω. By changing over to appropriate shifts, we also have $\varrho_1 = 0$ for $\mathbb{P}$-almost every ω. We use Lemma 4.8 repeatedly, which shows that $\varrho_d = 0$ $\mathbb{P}$-a.s. and thus, $\mathbb{P}_0$-a.s. Again by Corollary 4.6, there exists $r_0 = r_0(\omega)$ with $\mathbb{P}_0(r_0 < \infty) = 1$ such that $|V_G(y,\omega)| \le \varepsilon r/2$ for any $r \ge r_0$ and any $y \in \Lambda_r^1 \cap \mathcal{C}_\infty(\omega)$. Therefore,

$$\begin{aligned}
&\lambda_d(\{x \in \mathcal{C}_\infty \cap \Lambda_r^d : |V_G(x,\omega)| \ge \varepsilon r\}) \\
&\quad \le \inf_y \lambda_d(\{x \in \mathcal{C}_\infty \cap \Lambda_r^d : |V_G(x,\omega) - V_G(y,\omega)| \ge \varepsilon r - |V_G(y,\omega)|\}) \\
&\quad \le \inf_y \lambda_d(\{x \in \mathcal{C}_\infty \cap \Lambda_r^d : |V_G(x,\omega) - V_G(y,\omega)| \ge \varepsilon r/2\}),
\end{aligned}$$

and (4.17) holds for any $\varepsilon > 0$. This finishes the proof of Theorem 4.7. □

4.3 Proof of Theorem 4.2. We will prove an equivalent version of Theorem 4.2, namely:

Theorem 4.9. *Fix $d \ge 2$ and $G \in \mathcal{G}_\infty$. Then for $\mathbb{P}_0$-a.e. $\omega \in \Omega_0$,*

$$\lim_{r\to\infty} \sup_{x \in \mathcal{C}_\infty \cap [-r,r]^d} \frac{|V_G(x,\omega)|}{r} = 0.$$

Some preliminary lemmas will be required for the proof of the above result. Before that, let us set some notation that will be useful in the sequel. We will be interested in consider sets on $\mathbb{R}^d \times \mathbb{R}^d$, so we endow this space with the standard product Lebesgue measure, which we denote by $\lambda_d^{\otimes 2}$. The section on the “first"coordinate of a measurable set $A \subset \mathbb{R}^d \times \mathbb{R}^d$ is

$$A^{(x)} := \big\{ y \in \mathbb{R}^d : (x,y) \in A \big\} \quad \forall x \in \mathbb{R}^d. \tag{4.26}$$

Given $a \in (0,1)$ and $r, \delta, \rho > 0$, we also define

$$\begin{aligned}
&C_r(a) := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : ar < |x-y|_\infty < r\}, \\
&D(\rho) = D(\rho,\omega) := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \mathrm{d}_\omega(x,y) \ge \rho |x-y|_\infty ; x,y \in \mathcal{C}_\infty\}, \\
&E(r) := (\mathcal{C}_\infty)^2 \cap ([-r,r]^d)^2.
\end{aligned} \tag{4.27}$$

Lemma 4.10. *For any $a \in (0,1)$, there exists a constant $\rho = \rho(a,d)$ such that for all $\delta > 0$, $\mathbb{P}_0$-a.s. for large enough $n \in \mathbb{N}$, for every $x, y \in \mathcal{C}_\infty \cap [-n,n]^d$ satisfying $a\delta n < |x-y|_\infty < \delta n$, we have $\mathrm{d}_\omega(x,y) \le \rho |x-y|_\infty$.*

Proof. Fix any $\delta' > \delta$ and $\rho < c_0$ (as in (1.14)), and choose $0 < a' < a$ such that for $a'\delta' < a\delta$, so that for

$n \in \mathbb{N}$ large enough, $\delta n + 1 \leq \delta' n$ and $a\delta n - 1 > a'\delta' n$. By **(P4)** and (1.10), we have

$$\begin{aligned}
&\mathbb{P}_0(\exists x, y \in \mathcal{C}_\infty \cap [-n,n]^d, a\delta n < |x-y|_\infty < n\delta, \mathrm{d}_\omega(x,y) \geq \rho|x-y|_\infty) \\
&\leq \frac{1}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \mathbb{P}\Big(\exists x \neq y \in \mathcal{C}_\infty(\omega) \cap [-(n+1), n+1]^d, a'\delta n < |x-y|_\infty < n\delta, \\
&\qquad\qquad \mathrm{d}_\omega(x,y) \geq \rho|x-y|_\infty; 0, x, y \in \mathcal{C}_\infty \Big) \\
&= \frac{1}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \mathbb{E}\Big[\sum_{x,y \in \omega}^{\neq} \mathbb{1}\{(x,y) \in C_{\delta' n}(a') \cap [-(n+1), n+1]^2, \\
&\qquad\qquad \mathrm{d}_\omega(x,y) \geq \rho|x-y|_\infty; 0, x, y \in \mathcal{C}_\infty\}(x,y,\omega) \Big] \\
&= \frac{\zeta^2}{\mathbb{P}(0 \in \mathcal{C}_\infty)} \int_{C_{\delta' n}(a') \cap [-(n+1), n+1]^{2d}} \lambda_d^{\otimes 2}(\mathrm{d}x, \mathrm{d}y)\ \mathbb{P}^{x,y}\Big(0, x, y \in \mathcal{C}_\infty, \mathrm{d}_\omega(x,y) \geq \rho|x-y|_\infty\Big) \\
&\leq C\mathrm{e}^{-C' n}
\end{aligned}$$

for some $C = c(a,d,\rho), C' = C'(a,d,\rho) > 0$, with ζ defined in **(P2)**. The claim follows by the Borel-Cantelli lemma. □

Lemma 4.11. *Let $C \subset \mathbb{R}^d$ be any box of the type $[a_1, b_1] \times [a_2, b_2] \times \cdots [a_d, b_d]$. Then $\mathbb{P}$-a.s.,*

$$\lim_{r\to\infty} \frac{\lambda_d(\mathscr{C}_\infty \cap rC)}{\lambda_d(rC)} = p_\infty.$$

Proof. This is an application of [K02, Theorem 10.14]. □

We are now ready to prove Theorem 4.9 which will also prove Theorem 4.2.

Proof of Theorem 4.9. We consider some $\ell = \ell(d, \mathbb{P}) \in \mathbb{N}$ satisfying

$$p_\infty > \frac{1}{2^{d(\ell-1)}}. \tag{4.28}$$

We claim the proof is complete once we show the following: in a measurable set A such that $\mathbb{P}_0(A) = 1$, for all $\varepsilon > 0$ and $\omega \in A$, there exists some $r_0 = r_0(\omega)$ such that if $r \geq r_0$, for all $x \in [-r,r]^d \cap \mathcal{C}_\infty$ with $|V_G(x,\omega)|_\infty > \varepsilon r$,

$$\lambda_d\left[\left(E(r) \cap C_{\delta r}(2^{-\ell})\right)^x \cap \{|V_G(\cdot,\omega)|_\infty \leq \varepsilon r\}\right] > 0 \tag{4.29}$$

for some $\delta = \delta(d, \mathbb{P}, \varepsilon)$ that vanishes as $\varepsilon \to 0$ (recall the notation (4.26) and (4.27)). Indeed, for any $x, y \in \mathcal{C}_\infty$,

$$|V_G(x,\omega) - V_G(y,\omega)|_\infty \leq \mathrm{d}_\omega(x,y) \operatorname{ess\,sup}_{\mathbb{P}_0} |G(\omega, x)|_\infty. \tag{4.30}$$

For a fixed $x \in \mathcal{C}_\infty \cap [-r,r]^d$, if $|V_G(x,\omega)|_\infty \leq \varepsilon r$ for all $r \geq r_0$, there is nothing else to do. Otherwise, choose $r_1(\omega)$ large enough so that Lemma 4.10 is true for $r \geq r_1$ and $a = 2^{-\ell}$ (of course the lemma is still true if we replace $n \in \mathbb{N}$ by $r \in \mathbb{R}$). Now, if $r \geq r_0 \vee r_1$, by (4.29), for every $x \in [-r,r]^d \cap \mathcal{C}_\infty$ satisfying $|V(x,\omega)|_\infty > \varepsilon r$, we find some $y \in [-r,r]^d \cap \mathcal{C}_\infty$ such that $2^{-\ell}\delta r < |x-y|_\infty \leq \delta r$ and $|V(y,\omega)| \leq \varepsilon r$. In particular, $\mathrm{d}_\omega(x,y) \leq \rho|x-y|_\infty \leq \rho\delta r$. Hence, by (4.30), we deduce that

$$\begin{aligned}|V_G(x,\omega)|_\infty &\leq |V_G(y,\omega)|_\infty + |V_G(x,\omega) - V_G(y,\omega)|_\infty \\ &\leq \varepsilon r + \mathrm{d}_\omega(x,y)\,\mathrm{ess\,sup}_{\mathbb{P}_0}|G(\omega,x)|_\infty \\ &\leq \varepsilon r + \delta\rho r\,\mathrm{ess\,sup}_{\mathbb{P}_0}|G(\omega,x)|_\infty.\end{aligned} \tag{4.31}$$

Since $\delta \to 0$ as $\varepsilon \to 0$, this finishes the proof, once we prove the claim (4.29).

Now we turn to the proof of (4.29). By Theorem 4.7, there is a measurable set A_1 with $\mathbb{P}_0(A_1) = 1$, so that for all $\omega \in A_1$,

$$\limsup_{r\to\infty} \frac{1}{r^d}\int_{\mathcal{C}_\infty\cap[-r,r]^d} \mathbb{1}_{\{|V_G(x,\omega)|_\infty > \varepsilon r\}}\mathrm{d}x = 0. \tag{4.32}$$

On the other hand, by Lemma 4.11 we know that for a fixed box $C = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_d,b_d]$, there exists a measurable set A_C satisfying $\mathbb{P}(A_C) = 1$ and

$$\lim_{r\to\infty}\frac{\lambda_d(\mathcal{C}_\infty\cap rC)}{\lambda_d(rC)} = p_\infty \ \text{ for all } \omega \in A_C. \tag{4.33}$$

Choose any $\kappa = \kappa(d,\mathbb{P}) \in (0,1)$ (which exists due to (4.28)) and $c = (d,\mathbb{P}) > 0$ satisfying

$$1-\kappa > \frac{1}{p_\infty 2^{d(\ell-1)}} \qquad \text{and,} \tag{4.34}$$

$$c\left(1-\kappa-\frac{1}{p_\infty 2^{d(\ell-1)}}\right) > 1. \tag{4.35}$$

Next, for each $\varepsilon > 0$, let

$$\delta := \left(\frac{\varepsilon c}{p_\infty}\right)^{1/d}. \tag{4.36}$$

We can cover $[-1,1]^d$ with finitely many cubes $C_1,\cdots,C_m \subset [-1,1]^d$ of side δ. In particular, for every $x \neq y$ in the same box we will have $|x-y|_\infty < \delta$. By Lemma 4.11 applied to these boxes, we deduce that there exists a measurable set A_2 with $\mathbb{P}_0(A_2) = 1$ such that for all $\omega \in A_2$ and $1 \leq i \leq m$ we have

$$\lim_{r\to\infty}\frac{\lambda_d(\mathcal{C}_\infty\cap rC_i)}{\lambda_d(rC_i)} = p_\infty. \tag{4.37}$$

Let $A := A_1 \cap A_2$. Then for every $\omega \in A$ there exists some $r_0 = r_0(\omega)$ such that for all $r \geq r_0$ and $1 \leq i \leq m$,

$$\begin{aligned}&\lambda_d\big(\mathscr{C}_\infty\cap[-r,r]^d\cap\{|V_G(\cdot,\omega)|_\infty > \varepsilon r\}\big) < \varepsilon r^d, \qquad \text{and} \\ &\lambda_d(\mathcal{C}_\infty\cap rC_i) \geq p_\infty(1-\kappa)\lambda_d(rC_i) = r^d\delta^d p_\infty(1-\kappa) = \varepsilon c(1-\kappa)r^d.\end{aligned} \tag{4.38}$$

For every fixed $x \in [-r,r]^d \cap \mathcal{C}_\infty$ that satisfies $|V_G(x,\omega)|_\infty > \varepsilon r$, we have $x \in rC_i$ for some $1 \leq i \leq m$, so that $x \in \mathcal{C}_\infty \cap rC_i$. We decompose $\lambda_d(\mathcal{C}_\infty \cap rC_i)$ as

$$\lambda_d\big(\mathcal{C}_\infty \cap rC_i\big) = \lambda_d\big(\mathcal{C}_\infty \cap rC_i \cap \{|V_G(\cdot,\omega)|_\infty > \varepsilon r\}\big) + \lambda_d\big(\mathcal{C}_\infty \cap rC_i \cap \{|V_G(\cdot,\omega)|_\infty \le \varepsilon r\}\big).$$

By (4.38), and noting that $rC_i \subset [-r,r]^d$, we know that $\lambda_d(\mathcal{C}_\infty \cap rC_i \cap \{|V_G(\cdot,\omega)|_\infty > \varepsilon r\}) < \varepsilon r^d$. This inequality, combined with the equality above allow us to deduce that

$$\lambda_d\big(\mathcal{C}_\infty \cap rC_i \cap \{|V_G(\cdot,\omega)|_\infty \le \varepsilon r\}\big) \ge \varepsilon r^d\,(c(1-\kappa)-1) > 0, \tag{4.39}$$

and the last inequality holds since $c > \frac{1}{1-\kappa}$ by (4.35). Next, we decompose the Lebesgue measure of $D' := \mathcal{C}_\infty \cap rC_i \cap \{|V_G(\cdot,\omega)|_\infty \le \varepsilon r\}$ as

$$\lambda_d(D') = \lambda_d\big(D' \cap B^\infty_{r\delta/2^\ell}(x)\big) + \lambda_d\big(D' \cap B^\infty_{r\delta/2^\ell}(x)^c\big), \tag{4.40}$$

where $B^\infty_{r\delta/2^\ell}(x)$ is the ball centered at x of radius $r\delta/2^\ell$ with respect to the $|\cdot|_\infty$ norm, which is a cube of side $r\delta/2^{\ell-1}$.

We conclude that

$$\lambda_d\big(D' \cap B^\infty_{r\delta/2^\ell}(x)\big) \le (r\delta/2^{\ell-1})^d = \frac{\varepsilon c r^d}{p_\infty 2^{d(\ell-1)}}. \tag{4.41}$$

Therefore, by (4.39), (4.40) and (4.41),

$$\lambda_d(D' \cap B^\infty_{r\delta/2^\ell}(x)^c) \ge \varepsilon r^d\left(c(1-\kappa) - 1 - \frac{c}{2^{d(\ell-1)}p_\infty}\right). \tag{4.42}$$

By the choice of c in (4.35), we deduce that

$$c(1-\kappa) - 1 - \frac{c}{2^{d(\ell-1)}p_\infty} > 0.$$

Thus,

$$\lambda_d\big(D' \cap B^\infty_{r\delta/2^\ell}(x)^c\big) > 0. \tag{4.43}$$

Finally, recall the notation of $A^{(x)}$ from (4.26) and that of $E(r)$ from (4.27). Then by construction,

$$\big\{D' \cap B^\infty_{r\delta/2^\ell}(x)^c\big\} \subset \big(E(r) \cap C_{\delta r}(2^{-\ell})\big)^{(x)} \cap \big\{|V_G(\cdot,\omega)|_\infty \le \varepsilon r\big\},$$

so by (4.43), the claim (4.29) follows, completing the proof of Theorem 4.9, therefore that of Theorem 4.2. □

5 Entropic variational analysis

Recall the variational formula defined in Theorem 3.8:

$$\overline{H}(\theta) = \sup_{(b,\phi)\in\mathcal{E}} \left(\int \phi d\mathbb{P}_0 \left[\frac{1}{2}\mathrm{div}(a\theta) + \langle\theta, b\rangle_a - L(b,\omega)\right)\right]. \tag{5.1}$$

We also set

$$\overline{\Lambda}(\theta) := \inf_{G\in\mathcal{G}_\delta}\left(\operatorname{ess\,sup}_{\mathbb{P}_0}\left[\frac{1}{2}\operatorname{div}(a(G+\theta)) + H(G+\theta)\right]\right), \tag{5.2}$$

with the class $\mathcal{G}_\delta$ from Definition 4.1. The goal of this section is to show the equivalence of lower bound $\overline{H}(\cdot)$ proved in Section 3 and the upper bound which will be provided by $\overline{\Lambda}(\cdot)$ (see Section 5.3 below). The equivalence is presented in the following theorem.

Theorem 5.1. *Assume **(F1)**, **(F2)** and **(F4)**. Then for any $\theta\in\mathbb{R}^d$,*

$$\overline{H}(\theta) = \overline{\Lambda}(\theta).$$

The proof is divided into several steps. In Section 5.1 is proved the estimate $\overline{H}(\theta)\geq\overline{\Lambda}(\theta)$ in Theorem 5.2 that crucially relies on Proposition 5.5, which is then proved in Section 5.2. In Section 5.3 complete the proof of Theorem 5.1 by establishing the bound $\overline{H}(\theta)\leq\overline{\Lambda}(\theta)$. Finally, Section 5.4 is devoted to the completion of the proof of the main result in Theorem 2.1.

From now on we assume the same hypotheses from Theorem 5.1.

5.1 Proving the lower bound $\overline{H}(\cdot)\geq\overline{\Lambda}(\cdot)$. We will first prove

Theorem 5.2. *Under the assumptions of Theorem 5.1, for any $\theta\in\mathbb{R}^d$,*

$$\overline{H}(\theta)\geq\overline{\Lambda}(\theta).$$

The rest of Section 5.1 and Section 5.2 are devoted to the proof of the above theorem. We set

$$\mathcal{D} := \big\{g : C_c^2(\Omega_0) : g:\Omega_0\to\mathbb{R}\big\} \tag{5.3}$$

to be the linear space of functions on Ω_0 with compact support, such that their first and second weak derivatives (defined in Section 3.2.1) exists and are continuous. For any $g\in\mathcal{D}$, define

$$R_\theta g(\omega) = \frac{1}{2}\operatorname{div}\big(a(\omega)\,(\nabla g(\omega)+\theta)\big), \quad \text{and write} \quad R = R_0. \tag{5.4}$$

As already observed in [KRV06], we have that for any $g\in\mathcal{D}$,

$$\int \mathrm{d}\mathbb{P}_0\phi\big(Rg + \langle b,\nabla g\rangle_a\big)\begin{cases} = 0 \quad \forall g\in\mathcal{D} & \text{if} \quad (b,\phi)\in\mathcal{E},\\ \neq 0 \quad \text{for some } g\in\mathcal{D} & \text{if} \quad (b,\phi)\notin\mathcal{E},\end{cases} \tag{5.5}$$

and hence, by taking constant multiples if $(b,\phi)\notin\mathcal{E}$, we conclude that the infimum over $g\in\mathcal{D}$ in (5.5) is 0 if $(b,\phi)\in\mathcal{E}$, and $-\infty$ otherwise. Therefore,

$$\overline{H}(\theta) = \sup_{\phi\in\Phi}\ \sup_{b\in B_\phi}\ \inf_{g\in\mathcal{D}}\left[\int\mathrm{d}\mathbb{P}_0\phi\bigg(\frac{1}{2}\operatorname{div}(a\theta) + \langle\theta,b\rangle_a - L(b,\omega)) + \big(Rg+\langle b,\nabla g\rangle_a\big)\bigg)\right], \tag{5.6}$$

where

$$\Phi := \left\{\phi\in L_+^1(\mathbb{P}_0) : \int\phi\mathrm{d}\mathbb{P}_0 = 1\right\}. \tag{5.7}$$

Furthermore, for $\phi \in \Phi$,

$$B_\phi := \Big\{ b \in L^1_a(\phi \mathrm{d}\mathbb{P}_0) : \forall \omega \in \Omega_0 : x \mapsto b(\tau_x \omega) \in \mathrm{Lip} \Big\}, \tag{5.8}$$

with $L^1_a(\phi \mathrm{d}\mathbb{P}_0)$ being defined in (3.15). We remark that, for any $\phi \in \Phi$, the set B_ϕ contains constant functions b. First, we will prove

Lemma 5.3. *Let $\overline{H}(\theta)$ be the variational formula defined in* (5.1) *(or equivalently, in* (5.6)*). Then*

$$\overline{H}(\theta) = \sup_{\phi \in \Phi} \inf_{g \in \mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \big(R_\theta g + H(\theta + \nabla g(\omega), \omega) \big) \right]. \tag{5.9}$$

Proof. By (5.6) and (5.4),

$$\overline{H}(\theta) = \sup_{\phi \in \Phi} \sup_{b \in B_\phi} \inf_{g \in \mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega) \big) \right]. \tag{5.10}$$

First, we need to exchange the supremum over b with the infimum over g, for which we would like to apply the min-max theorem from [AE84, Theorem 8, p. 319], the requirements for which are verified as follows. Note that the map

$$b \mapsto \int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega) \big) \qquad \text{is concave and upper semicontinuous,}$$

while the map

$$g \mapsto \int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega) \big) \qquad \text{is convex and lower semicontinuous.}$$

We need to verify the remaining compactness (resp. coercivity): we will show that for a fixed $\phi \in \Phi$ and $g \in \mathcal{D}$, the level sets

$$\begin{aligned} E_c : &= \Big\{ b \in B_\phi : \int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega) \big) \geq c \Big\} \\ &= \Big\{ b \in L^1_a(\phi \mathrm{d}\mathbb{P}_0) : \mathcal{C}_\infty(\omega) \ni x \mapsto b(x, \omega) \in \text{Lip and } \forall\, \omega \in \Omega_0, \\ &\qquad\qquad \int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b \rangle_a + R_\theta g - L(b, \omega) \big) \geq c \Big\} \end{aligned}$$

are weakly compact in $L^1_a(\phi \mathrm{d}\mathbb{P}_0)$. Indeed, by the Eberlein-Šmulian theorem (see [DS58, p.430]), checking the latter condition is equivalent to verifying that the set E_c above is weakly closed and sequentially weakly compact in $L^1_a(\phi \mathrm{d}\mathbb{P}_0)$. For the second condition, it is enough to show that E_c is bounded and uniformly integrable, but both these conditions follow from the coercivity of L. Indeed, recall (2.6) from **(F2)**:

$$c_{10} \|q\|_a^{\alpha'} - c_{11} \leq L(q, \omega) \leq c_{12} \|q\|_a^{\alpha'} + c_{13}, \qquad \alpha' = \frac{\alpha}{\alpha - 1},\ 1 < \alpha < \infty.$$

On the other hand, using that $g \in \mathcal{D}$ has compact support and ∇g is continuous, $|\theta + \nabla g| \leq (|\theta| + \|\nabla g\|_{L^\infty(\mathbb{P}_0)} =: C_1(\theta, g) < \infty$. Moreover, by **(F1)**, $|\mathrm{div}(a)| \leq C$, so we can find a constant $C_2(\theta, g)$ such that by (5.4), $|R_\theta g| \leq C_2(\theta, g)$. Hence,

$$\int \mathrm{d}\mathbb{P}_0 \phi\big(\langle \theta + \nabla g, b\rangle_a + R_\theta g\big) \leq C_2(\theta, g) + C_1(\theta, g) \int \mathrm{d}\mathbb{P}_0 \phi \|b\|_a < \infty \tag{5.11}$$

since $b \in L^1_a(\phi \mathrm{d}\mathbb{P}_0)$, recall (3.15).

Thus, it remains to show that E_c is weakly closed. Since E_c is convex, it suffices to show that E_c is strongly closed. Indeed, suppose that $(b_n)_n \subset E_c$ such that $b_n \to b$ in $L^1_a(\phi d\mathbb{P}_0)$. Passing to a subsequence, since L is lower semicontinuous and by Fatou's lemma, one can easily verify that $\int \mathrm{d}\mathbb{P}_0 \phi\big(\langle \theta + \nabla g, b\rangle_a + R_\theta g - L(b, \omega)\big) \geq c$. We will construct a function $\tilde{b}$ such that $\tilde{b} = b$ $\mathbb{P}_0$-a.s. and for all $\omega \in \Omega_0$, $\mathbb{R}^d \ni x \mapsto \tilde{b}(x, \omega) \in$ Lip. Let Ω_0' with $\mathbb{P}_0(\Omega_0') = 1$ such that $b_n(\omega) \to b(\omega)$ for all $\omega \in \Omega_0'$. For a fixed $\omega \in \Omega_0'$, we know that the family $(b_n(\cdot, \omega))_n$ is uniformly equicontinuous and on any compact set $K \subset \mathbb{R}^d$ and $x \in K$,

$$|b_n(x, \omega)| \leq |x| + |b_n(0, \omega)| \leq \mathrm{diam}(K) + \sup_n |b_n(0, \omega)|.$$

As $b_n(0, \omega) \to b(0, \omega)$, the supremum above is finite. Hence, for fixed ω, the family of continuous functions $(b_n(\cdot, \omega))_n$ is globally uniformly equicontinuous and uniformly bounded on compact sets. By the Arzelà-Ascoli theorem, the sequence $(b_n(\cdot, \omega))$ converges uniformly on compact sets and therefore converges pointwise to some function $\tilde{f}(\cdot, \omega) \in$ Lip. By definition, $\tilde{f}(0, \omega) = b(\omega)$ $\mathbb{P}_0$-a.s. Now let us consider the set

$$\Omega_0'' := \{\omega \in \Omega_0 : \exists x \in \mathbb{R}^d, \omega' \in \Omega_0' : \omega = \tau_x \omega'\}.$$

Then we define

$$\tilde{b}(\omega) := \begin{cases} \tilde{f}(x, \omega') & \text{if } \omega = \tau_x\omega' \text{ for some } x \in \mathbb{R}^d, \omega' \in \Omega_0', \\ 0 & \text{otherwise.} \end{cases} \tag{5.12}$$

Let us first check that $\tilde{b}$ is well-defined. Indeed, suppose that $\omega = \tau_x\omega' = \tau_y\omega''$ for some $x, y \in \mathbb{R}^d$ and $\omega', \omega'' \in \Omega_0'$. Then

$$\begin{aligned} \tilde{f}(x, \omega') &= \lim_{n\to\infty} b_n(x, \omega') = \lim_{n\to\infty} b_n(0, \tau_x\omega') = \lim_{n\to\infty} b_n(0, \tau_y\omega'') \\ &= \lim_{n\to\infty} b_n(y, \omega'') = \tilde{f}(y, \omega''). \end{aligned}$$

Hence, the function $\tilde{b}$ is well defined. Notice that on Ω_0', $\tilde{b}(\omega) = f(0, \omega) = b(\omega)$, so $\tilde{b} = b$ $\mathbb{P}_0$-a.s. Finally, let us check that for all $\omega \in \Omega_0$, $\mathcal{C}_\infty(\omega) \ni x \mapsto \tilde{b}(x, \omega) \in$ Lip. Indeed,

(i) If $\omega \in \Omega_0''$, then $\omega = \tau_z\omega'$ for some $z \in \mathbb{R}^d$ and $\omega' \in \Omega_0$ and for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} |\tilde{b}(x, \omega) - \tilde{b}(y, \omega)| = |\tilde{b}(x, \tau_z\omega') - \tilde{b}(y, \tau_z\omega')| &= |\tilde{f}(x + z, \omega') - \tilde{f}(y + z, \omega')| \\ &\leq |x - y|. \end{aligned}$$

(ii) On the other hand, if $\omega \in \Omega_0 \setminus \Omega_0''$, then the same holds for $\tau_x\omega$ for all $x \in \mathcal{C}_\infty(\omega)$, and the Lipschitz condition is trivially satisfied.

This finishes the proof that E_c is weakly compact in $L^1_a(\phi \mathrm{d}\mathbb{P}_0)$, and therefore, by the aforementioned min-max theorem, we can exchange the $\sup_{b\in B_\phi}$ and $\inf_{g\in\mathcal{D}}$ in (5.10) to obtain

$$\overline{H}(\theta) = \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \sup_{b\in B_\phi} \left[\int \mathrm{d}\mathbb{P}_0 \phi \big(\langle \theta + \nabla g, b\rangle_a + R_\theta g - L(b,\omega) \big) \right].$$

Since the integrand depends locally in b, we can bring the supremum over b inside the integral, and use the duality between H and L to conclude that

$$\begin{aligned} \overline{H}(\theta) &= \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \bigg(R_\theta g + \sup_{b\in B_\phi} \big[\langle \theta + \nabla g, b\rangle_a - L(b,\omega) \big] \bigg) \right] \\ &= \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \big(R_\theta g + H(\theta + \nabla g(\omega), \omega) \big) \right]. \end{aligned} \tag{5.13}$$

In the last equality we used that, for any $\phi \in \Phi$, the set B_ϕ defined in (5.8) contains constants, so that

$$\begin{aligned} \sup_{b\in B_\phi} \big[\langle \theta + \nabla g, b\rangle_a - L(b,\omega) \big] &= \sup_{y\in\mathbb{R}^d} \big[\langle \theta + \nabla g, y\rangle_a - L(y,\omega) \big] \\ &= H(\theta + \nabla g(\omega), \omega). \end{aligned}$$

□

We would like to now swap the order of $\sup_\phi$ and $\inf_g$ in (5.13).

Lemma 5.4. *With R_θ defined in* (5.4)*, let*

$$\mathcal{S}_\theta(g)(\omega) := R_\theta g(\omega) + H(\theta + \nabla g(\omega), \omega)), \quad g \in \mathcal{D}. \tag{5.14}$$

Then for any $\theta \in \mathbb{R}^d$,

$$\overline{H}(\theta) = \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \, \mathcal{S}_\theta(g) \right] \geq \liminf_{\varepsilon\to 0} \inf_{g\in\mathcal{D}} \left[\varepsilon \log \int \mathrm{d}\mathbb{P}_0 \exp\left[\varepsilon^{-1} \mathcal{S}_\theta(g) \right] \right]. \tag{5.15}$$

Proof. For any probability density $\varphi \geq 0$ on Ω_0 (i.e., $\int_{\Omega_0} \varphi \mathrm{d}\mathbb{P}_0 = 1$), let

$$\mathrm{Ent}_{\mathbb{P}_0}(\varphi) = \int \varphi \log \varphi \, \mathrm{d}\mathbb{P}_0 \geq 0$$

be the entropy of φ. Its non-negativity is a consequence of the Jensen's inequality. Moreover, the map $\varphi \mapsto \mathrm{Ent}_{\mathbb{P}_0}(\varphi)$ is convex, weakly lower semicontinuous and has weakly compact sub-level sets, meaning, for any $\ell > 0$, $\{\varphi : \mathrm{Ent}_{\mathbb{P}_0}(\varphi) \leq \ell\}$ is compact in the weak topology. Thus, for any $\varepsilon > 0$ we have the lower bound

$$\overline{H}(\theta) \geq \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \, \mathcal{S}_\theta(g) - \varepsilon \mathrm{Ent}_{\mathbb{P}_0}(\phi) \right] = \sup_{\phi\in\Phi} \inf_{g\in\mathcal{D}} \left[\int \mathrm{d}\mathbb{P}_0 \phi \left(\mathcal{S}_\theta(g) - \varepsilon \log \phi \right) \right].$$

Similarly as in (5.11), we use the fact that $g \in \mathcal{D}$ together with the assumptions **(F1)** to conclude that there is a constant $C_2(\theta, g)$ such that $|R_\theta(g)| \leq C_2(\theta, g)$, so that

$$\int \mathrm{d}\mathbb{P}_0 \phi \mathcal{S}_\theta(g) \leq C_2(\theta, g) + \int \mathrm{d}\mathbb{P}_0 \phi H(\theta + \nabla g) \leq C_2(\theta, g) + c_8 \int \mathrm{d}\mathbb{P}_0 \phi \|\theta + \nabla g\|_a^\alpha + c_9$$
$$\leq C_2(\theta, g) + c_8 C(\alpha, \theta, g) + c_9 < \infty.$$

where for the second inequality we used the upper bound from (2.5) in **(F2)** and for the third inequality we used $\langle a(\omega)x, x\rangle \leq c_5|x|^2$ from **(F1)** and again that $\sup_\omega |\theta + \nabla g(\omega)| \leq |\theta| + \|\nabla g\|_{L^\infty(\mathbb{P}_0)}$ for $g \in \mathcal{D}$. Now, for any fixed $\phi \in \Phi$, the map

$$g \mapsto \int \mathrm{d}\mathbb{P}_0 \phi \left(\mathcal{S}_\theta(g) - \varepsilon \log \phi\right)$$

is convex and continuous, while for any fixed $g \in \mathcal{D}$, the map

$$\varphi \mapsto \int \mathrm{d}\mathbb{P}_0 \varphi \left(\mathcal{S}_\theta(g) - \varepsilon \log \varphi\right)$$

is concave, upper-semicontinuous and has compact superlevel sets in the weak $L^1_+(\mathbb{P}_0)$ topology, we can again use Von-Neumann's minimax theorem to justify changing the order of $\sup_\phi$ and $\inf_{g\in\mathcal{D}}$. This means,

$$\overline{H}(\theta) \geq \inf_{g\in\mathcal{D}} \sup_\phi \left[\int \mathrm{d}\mathbb{P}_0 \phi \left(\mathcal{S}_\theta(g) - \varepsilon \log \phi\right)\right].$$

The above variational problem over ϕ subject to the condition $\int \phi d\mathbb{P}_0 = 1$ can be solved explicitly and the maximizing density is

$$\phi = \frac{\exp[\varepsilon^{-1}\mathcal{S}_\theta(g)]}{\mathbb{P}_0\big[\exp[\varepsilon^{-1}\mathcal{S}_\theta(g)]\big]}.$$

We replace this value of ϕ in the last lower bound for $\overline{H}(\theta)$ to obtain

$$\overline{H}(\theta) \geq \inf_{g\in\mathcal{D}} \left[\varepsilon \log \int \mathrm{d}\mathbb{P}_0 \, \exp\left[\varepsilon^{-1}\mathcal{S}_\theta(g)\right]\right].$$

We let $\varepsilon \to 0$, to deduce the lower bound claimed in (5.15). □

Given the above results, the lower bound in Theorem 5.2 will now be a consequence of the following technical result that will be established in Section 5.2.

Proposition 5.5. *For any given $\varepsilon > 0$, there exists a sequence $\varepsilon_n \to 0$ and a sequence of functions $(g_n)_n \subset \mathcal{D}$ so that*

$$\overline{H}(\theta) \geq \varepsilon_n \log \mathbb{E}_0\left[\mathrm{e}^{\varepsilon_n^{-1}\mathcal{S}_\theta(g_n,\cdot)}\right] - \varepsilon, \tag{5.16}$$

and $G_n(\omega) := \nabla g_n$ converges weakly in $L^{1+\delta}(\mathbb{P}_0)$ (with $\delta > 0$ as in (2.3)*) and in distribution (along a subsequence) to some G. Furthermore, $G \in \mathcal{G}_\delta$, which is defined in Section 4.*

Proof of Theorem 5.2 (assuming Proposition 5.5). : By Proposition 5.5, for $r > 0$, we pick some sequence $\varepsilon_n \to 0$ and $g_n \in \mathcal{D}$ satisfying

$$\overline{H}(\theta) \geq \varepsilon_n \log \mathbb{E}_0\big[\mathrm{e}^{\varepsilon_n^{-1}\mathcal{S}_\theta(g_n,\cdot)}\big] - r.$$

For fixed n, the map $\lambda \in [0,\infty) \to \frac{1}{\lambda}\log \mathbb{E}_0[\mathrm{e}^{\lambda \mathcal{S}_\theta(g_n,\cdot)}]$ is increasing, so for each $\eta, \lambda > 0$, if n is large enough,

$$\begin{aligned}\overline{H}(\theta) &\geq \frac{1}{\lambda}\log \mathbb{E}_0\big[\mathrm{e}^{\lambda \mathcal{S}_\theta(g_n,\cdot)}\big] - r\\ &= \frac{1}{\lambda}\log \mathbb{E}_0\Big[\mathrm{e}^{\lambda\big(\frac{1}{2}\mathrm{div}(a(\omega)(G_n(\omega)+\theta))+H(\theta+\nabla G_n(\omega),\omega)\big)}\Big] - r.\end{aligned}$$

For any $M, \lambda > 0$, the map

$$x \mapsto \mathrm{e}^{\lambda\big(M\wedge\big(\frac{1}{2}\mathrm{div}(a(\omega)(x+\theta))+H(\theta+x,\omega)\big)\big)}$$

is continuous and bounded. Thus, letting $n \to \infty$ and using the fact (from Proposition 5.5) that G_n converges to G in distribution, we conclude from the above bound that

$$\overline{H}(\theta) \geq \frac{1}{\lambda}\log \mathbb{E}_0\Big[\mathrm{e}^{\lambda\big(M\wedge\big(\frac{1}{2}\mathrm{div}(a(\omega)(G(\omega)+\theta))+H(\theta+G(\omega),\omega)\big)\big)}\Big] - r.$$

Now by letting $M \nearrow \infty$ and using monotone convergence theorem, we obtain

$$\overline{H}(\theta) \geq \log \Big\| \mathrm{e}^{\frac{1}{2}\mathrm{div}(a(\omega)(G(\omega)+\theta))+H(\theta+G(\omega),\omega)} \Big\|_{L^\lambda(\mathbb{P}_0)} - r. \tag{5.17}$$

Finally, letting $\lambda \to \infty$, we obtain

$$\begin{aligned}\overline{H}(\theta) &\geq \mathrm{ess\,sup}_{\mathbb{P}_0}\Big[\frac{1}{2}\mathrm{div}(a(G+\theta)) + H(G+\theta)\Big] - r\\ &\geq \inf_{G\in\mathcal{G}_\delta} \overline{\Lambda}(\theta, G) - r.\end{aligned} \tag{5.18}$$

Since $r > 0$ is arbitrary, we are done with the proof of Theorem 5.2. □

5.2 Proof of Proposition 5.5 We divide the proof of Proposition 5.5 into three subsequent lemmas.

Lemma 5.6. *For any given $\varepsilon > 0$, there exists a sequence $\varepsilon_n \to 0$ and a sequence of functions $(g_n)_n \subset \mathcal{D}$ so that* (5.16) *holds, and $G_n(\omega) := \nabla g_n(\omega)$ converges weakly in $L^{1+\delta}(\mathbb{P}_0)$ (with $\delta > 0$ as in* (2.3)*) and in distribution along a subsequence to some random variable $G \in L^{1+\delta}(\mathbb{P}_0)$.*

Proof. We start with the bound (5.15) in Lemma 5.4 which implies that there exist sequences $\varepsilon_n \to 0$ and $(g_n)_n \subset \mathcal{D}$ satisfying

$$\varepsilon_n \log \mathbb{E}_0\big[\mathrm{e}^{\varepsilon_n^{-1}\mathcal{S}_\theta(g_n,\cdot)}\big] \leq \overline{H}(\theta). \tag{5.19}$$

Using this we will first show that

$$\sup_n \|G_n\|_{L^{1+\delta}(\Omega_0)} < \infty. \tag{5.20}$$

In particular, the above bound will imply that G_n converges weakly in $L^{1+\delta}(\mathbb{P}_0)$ along a subsequence to some G.

We now prove (5.20). Note that the map $\lambda \in [0,\infty) \mapsto \frac{1}{\lambda}\log \mathbb{E}_0[\mathrm{e}^{\lambda \mathcal{S}_\theta(g_n,\cdot)}]$ is increasing. Thus, recalling the definition of $\mathcal{S}_\theta$ from (5.14) and using (5.19), we obtain that for n large enough,

$$\log \mathbb{E}_0\left[\mathrm{e}^{R_\theta g_n(\omega)+H(\theta+\nabla g_n(\omega),\omega))}\right] \le \overline{H}(\theta).$$

The lower bound on $H(\cdot,\omega)$ from **(F2)** implies

$$\log \mathbb{E}_0\left[\mathrm{e}^{R_\theta g_n(\omega)+c_6\|\theta+\nabla g_n\|_a^\alpha - c_7}\right] \le \overline{H}(\theta).$$

Set $G_n := \nabla g_n$. Then Jensen's inequality applied to the bound bound and the definition of $R_\theta g_n = \frac{1}{2}\mathrm{div}(a(\nabla g_n+\theta))$ from (5.4) leads to

$$\mathbb{E}_0\left[\frac{1}{2}\mathrm{div}(a(G_n+\theta)) + c_6\|\theta+G_n\|_a^\alpha\right] \le \overline{H}(\theta)+c_7.$$

Since $G_n = \nabla g_n$, we have $\mathbb{E}_0\left[\mathrm{div}(aG_n)\right] = 0$. Thus by **(F1)** we conclude that for some constant $C = C(\theta,\eta)$,

$$\sup_n \mathbb{E}_0[\|G_n\|_a^\alpha] \le C, \qquad \alpha > 1.$$

But by (2.2),

$$\|G_n\|_a^\alpha = \langle a(\omega), G_n, G_n\rangle^{\alpha/2} \ge \xi(\omega)^{\alpha/2}|G_n|^\alpha.$$

Combining the last two displays, we have

$$\sup_n \mathbb{E}_0\left[\xi(\omega)^{\alpha/2}|G_n|^\alpha\right] \le C. \tag{5.21}$$

Hence,

$$\begin{aligned}\mathbb{E}_0[|G_n|^{1+\delta}] &= \mathbb{E}_0\left[|G_n|^{1+\delta}\xi(\omega)^{\frac{1+\delta}{2}}\xi(\omega)^{-\frac{1+\delta}{2}}\right] \\ &\le \mathbb{E}_0\left[|G_n|^\alpha \xi(\omega)^{\alpha/2}\right]^{\frac{1+\delta}{\alpha}} \mathbb{E}_0\left[\xi(\omega)^{-\frac{\alpha(1+\delta)}{2(\alpha-1-\delta)}}\right]^{\frac{\alpha-1-\delta}{\alpha}} < \infty.\end{aligned} \tag{5.22}$$

In the first upper bound we used Hölder's inequality with exponents $\frac{\alpha}{1+\delta} > 1$ (recall that $\alpha > 1+\delta$) and $\frac{\alpha}{\alpha-1-\delta}$, and for the second bound we invoked (5.21) and (2.3) with $\chi = \frac{\alpha}{2}\frac{1+\delta}{\alpha-(1+\delta)}$. Hence,

$$\sup_n \mathbb{E}_0[|G_n\|^{1+\delta}] < \infty,$$

with $\delta > 0$. Consequently, G_n converges weakly in $L^{1+\delta}(\mathbb{P}_0)$ and in distribution along a subsequence to some random variable $G \in L^{1+\delta}(\mathbb{P}_0)$, as claimed. □

Lemma 5.7. *The limit G of G_n from Lemma 5.6 satisfies the closed loop condition defined in* (4.3)*, i.e., for any simple closed path C contained in the infinite cluster $\mathcal{C}_\infty$, we have $\int_{\mathrm{C}} G(\omega,\cdot)\cdot \mathrm{d}r = 0$, almost surely w.r.t. $\mathbb{P}_0$.*

Proof. We first assert that it suffices to prove that for any measurable set $A \subset \Omega_0$,

$$\mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} \int_{\mathrm{C}} G(\omega,\cdot)\cdot \mathrm{d}r\right] = 0 \text{ for each simple closed path } \mathrm{C}\subset\mathcal{C}_\infty. \tag{5.23}$$

Indeed, (5.23) says that for any fixed simple closed path, $\mathbb{P}_0$-a.s., $\mathbb{1}_{C\subset\mathcal{C}_\infty}\int_C G(\omega,\cdot)\cdot \mathrm{d}r = 0$. We want to show that this holds $\mathbb{P}_0$-a.s. uniformly on each closed loop. Since line integrals are independent of the parametrization of the path, for each C, we choose any (but fixed from now) smooth function

$$f_{\mathrm{C}} : [0,1] \to \mathbb{R}^d \qquad \text{satisfying } f(0) = f(1).$$

The space

$$X := \big\{ f \in C^\infty[0,1] : f(0) = f(1) \big\}$$

is separable under the $|\cdot|_\infty$ norm, so there exists some countable dense subset $Y \subset X$. If (5.23) holds, we can show that $\mathbb{P}_0$-a-s., the closed loop condition holds for each curve C such that $f_{\mathrm{C}} \in Y$. To extend this to all simple closed curves in $\mathbb{R}^d$, we can approximate each curve by a sequence C_n such that $f_{\mathrm{C}_n} \in Y$. Since the convergence is uniform, it is easy to deduce that $\mathbb{P}_0$-a.s., $\mathbb{1}_{\mathrm{C}\subset\mathcal{C}_\infty}\int_{\mathrm{C}} G(\omega,\cdot)\cdot r = 0$ for any simple closed curve C. Thus, we only need to show that (5.23) holds for fixed $A \subset \Omega_0$ and simple closed curve C.

Let $f : [0,1] \to \mathbb{R}^d$ be any smooth function that parametrizes C. For each fixed $n \in \mathbb{N}$, we know that $G_n = \nabla g_n$ satisfies the closed loop condition (because it is a gradient). By Fubini's theorem we have

$$\begin{aligned} 0 = \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} \int_{\mathrm{C}} G_n(\omega,\cdot)\cdot \mathrm{d}r\right] &= \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} \int_0^1 G_n(\omega, f(x))\cdot f'(x)\mathrm{d}x\right] \\ &= \int_0^1 f'(x)\cdot \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} G_n(\omega, f(x))\right]\mathrm{d}x. \end{aligned}$$

Since G_n converges weakly to G in $L^{1+\delta}(\mathbb{P}_0)$ (as shown in Lemma 5.6), for fixed $x \in [0,1]$,

$$\lim_{n\to\infty} \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} G_n(\omega, f(x))\right] = \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} G(\omega, f(x))\right].$$

Using that $\sup_n \mathbb{E}_0[G_n^{1+\delta}] < \infty$, and that f' is bounded on $[0,1]$, we can apply dominated convergence theorem to conclude that

$$0 = \lim_{n\to\infty} \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} G_n(\omega, f(x))\right] = \int_0^1 f'(x)\cdot \mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} G(\omega, f(x))\right]\mathrm{d}x.$$

As $G \in L^{1+\delta}(\mathbb{P}_0)$, we can again exchange the order of integration using Fubini's theorem so the right-hand side in the last display is $\mathbb{E}_0\left[\mathbb{1}_{A\cap(\mathrm{C}\subset\mathcal{C}_\infty)} \int_{\mathrm{C}} G(\omega,\cdot)\cdot \mathrm{d}r\right]$. This shows (5.23) and concludes the proof of the lemma. □

The following result will complete the proof of Proposition 5.5.

Lemma 5.8. *The limit G of G_n from Lemma 5.6 belongs to the class $\mathcal{G}_\delta$ from Definition 4.1.*

Proof. We have already proved (4.1). On the other hand, the proof of (4.2) follows from the first inequality in (5.18) (note that for this part we are only using the weak convergence of G_n towards G in $L^{1+\delta}(\mathbb{P}_0)$ and in distribution, which have been established in Lemma 5.6). Also, the closed loop property (4.3) was shown in Lemma 5.7. Thus it remain to check that G satisfies the zero induced mean property $\mathbb{E}_0[V_G(\cdot, \mathfrak{v}_e)] = 0$, recall (4.5).

Let us fix a coordinate unit vector e, and recall the definitions of $\mathfrak{v}_e = n(\omega, e)e$ from (1.13), that of $\tilde{\ell}(\omega) = \tilde{\mathrm{d}}_\omega(0, \mathfrak{v}_e)$ from (4.11) and of the sets $A(x_1, \cdots, x_k)$ from (4.13). Choose $\tilde{A}(x_1, \cdots, x_k) \subset A(x_1, \cdots, x_k)$ so that

$$\{\tilde{\ell} = j\} = \bigsqcup_{k=1}^{3^d j} \bigsqcup_{x_1, \cdots, x_k} \{\tilde{\ell} = j \cap \tilde{A}(x_1, \cdots, x_k)\},$$

where $\bigsqcup$ represents disjoint union. Next, for any $R > 0$ define

$$\eta_R := \mathbb{E}_0[V_G(\omega, \mathfrak{v}_e), \tilde{\ell} \le R]. \tag{5.24}$$

By dominated convergence theorem, the required identity $\mathbb{E}_0[V_g(\cdot, \mathfrak{v}_e)] = 0$ follows once we show that $\eta_R \to 0$ as $R \to \infty$. For this purpose, we further claim that

$$\begin{aligned} \eta_R &= \lim_{n\to\infty} \mathbb{E}_0\bigg[\int_{0 \rightsquigarrow \mathfrak{v}_e} G_n(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} \le R\bigg] \\ &= -\lim_{n\to\infty} \mathbb{E}_0\bigg[\int_{0 \rightsquigarrow \mathfrak{v}_e} G_n(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} > R\bigg]. \end{aligned} \tag{5.25}$$

We observe that the second equality in (5.25) follows from the fact that

$$\mathbb{E}_0\bigg[\int_{0 \rightsquigarrow \mathfrak{v}_e} G_n(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}\bigg] = \mathbb{E}_0\big[g_n(\sigma_e \omega) - g_n(\omega)\big] = 0,$$

because σ_e is measure-preserving under $\mathbb{P}_0$ (recall Proposition 3.5) and for each fixed n, g_n is bounded and continuous. Thus, the only nontrivial claim is the first equality in (5.25). We decompose η_R as (below, $x_0 := 0$)

$$\begin{aligned} \eta_R &= \sum_{j=1}^{R} \sum_{k=1}^{3^d j} \sum_{x_1, \cdots, x_k \in N_j} \mathbb{E}_0\bigg[\int_{0 \rightsquigarrow \mathfrak{v}_e} G(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_k)\bigg] \\ &= \sum_{j=1}^{R} \sum_{k=1}^{3^d j} \sum_{x_1, \cdots, x_k \in N_j} \sum_{i=1}^{k} \mathbb{E}_0\bigg[\int_{x_{i-1} \rightsquigarrow x_i} G(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_k)\bigg]. \end{aligned}$$

On $\tilde{A}(x_1, \cdots, x_j)$, we can always choose the straight line between these two points as a curve. Using that G_n converges to G weakly in $L^{1+\delta}(\mathbb{P}_0)$ (cf. Lemma 5.6) we deduce that

$$\lim_{n\to\infty} \mathbb{E}_0\bigg[\int_{x_{i-1} \rightsquigarrow x_i} G_n(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_j)\bigg] = \mathbb{E}_0\bigg[\int_{x_{i-1} \rightsquigarrow x_i} G(\omega, \cdot) \cdot \mathrm{d}\mathbf{r}, \tilde{\ell} = j, \tilde{A}(x_1, \cdots, x_j)\bigg].$$

Therefore,

$$\eta_R = \sum_{j=1}^{R}\sum_{k=1}^{3^d j}\sum_{x_1,\cdots,x_k\in N_j}\sum_{i=1}^{k}\lim_{n\to\infty}\mathbb{E}_0\Big[\int_{x_{i-1}\rightsquigarrow x_i} G_n(\omega,\cdot)\cdot \mathrm{d}\mathbf{r}, \tilde{\ell}=j, \tilde{A}(x_1,\cdots,x_j)\Big].$$

Finally, we can exchange the limit with the sum over $x_1,\cdots,x_k$ by noting that

$$\mathbb{E}_0\Big[\int_{x_{i-1}\rightsquigarrow x_i} G_n(\omega,\cdot)\cdot \mathrm{d}\mathbf{r}, \tilde{\ell}=j, \tilde{A}(x_1,\cdots,x_j)\Big]$$

is uniformly bounded because $\sup_n \|G_n\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty$. This shows (5.25). To conclude proving that $\eta_R \to 0$ as $R\to\infty$, we use (5.25) to estimate $|\eta_R|$ as

$$|\eta_R| = \lim_{n\to\infty}\Big|\mathbb{E}_0\Big[\int_{0\rightsquigarrow \mathfrak{v}_e} G_n(\omega,\cdot)\cdot \mathrm{d}\mathbf{r}, \tilde{\ell}>R\Big]\Big| \le \limsup_{n\to\infty}\sum_{j=R}^{\infty}\mathbb{E}_0\Big[\int_{0\rightsquigarrow \mathfrak{v}_e} |G_n(\omega,\cdot)|\cdot \mathrm{d}\mathbf{r}, \tilde{\ell}=j\Big].$$

Now, following the arguments exactly as in the proof of Proposition 4.4 and using that $\sup_n \|G_n\|_{L^{1+\delta}(\mathbb{P}_0)} < \infty$, we can show that the last display is bounded above by $C_1 \mathrm{e}^{-C_2 R}$ for some constants $C_1, C_2 > 0$, implying that $|\eta_R|\to 0$, which in turn completes the proof that G satisfies the induced mean zero property. Thus Lemma 5.8 and therefore Proposition 5.5 are proved. □

5.3 Proof of Theorem 5.1. Note that the bound $\overline{\Lambda}(\cdot) \le \overline{H}(\cdot)$ has already been proved in Theorem 5.2. The proof of Theorem 5.1 will be complete once we show the reversed bound

$$\overline{\Lambda}(\cdot) \ge \overline{H}(\cdot). \tag{5.26}$$

The above inequality will follow once we prove an upper bound for Theorem 2.1 w.r.t. a *linear* initial condition $f(x)=\langle\theta,x\rangle$ for any $\theta\in\mathbb{R}^d$ (The upper bound for a general initial condition f satisfying **(F4)** will be provided in Section 5.4 in Theorem 5.10 there).

Proposition 5.9. *Assume **(F1)** and **(F2)**. Let $u_{\varepsilon,\theta}$ be the unique viscosity solution to* (2.10) *with initial condition $f(x)=\langle\theta,x\rangle$. Then*

$$\limsup_{\varepsilon\to 0} u_{\varepsilon,\theta}(t,0,\omega) \;\le\; t\,\overline{\Lambda}(\theta) \quad \mathbb{P}_0\text{-}a.s., \tag{5.27}$$

where $\overline{\Lambda}$ is defined in (5.2).

Assuming the above fact, we can conclude

Proof of Theorem 5.1 (assuming Proposition 5.9): Combining the lower bound from Theorem 3.8 for the particular case $f(x)=\langle\theta,x\rangle$ with Proposition 5.9 we conclude that $\overline{H}(\theta)\le\overline{\Lambda}(\theta)$. The reverse bound has been shown in Theorem 5.2, which proves Theorem 5.1. □

Proof of Proposition 5.9: Let us first sketch the main idea of the proof. To simplify notation, for a fixed $\theta \in \mathbb{R}^d$, we will simply write

$$u_\varepsilon(t,x,\omega) = u_{\varepsilon,\theta}(t,x,\omega).$$

Recall that for a fixed $t > 0$, by Lemma 3.9,

$$u_\varepsilon(t,0,\omega) = \varepsilon \sup_{c\in\mathbf{C}_T^*} E^{P_0^{c,\omega}}\bigg[\langle\theta, X_{t/\varepsilon}\rangle - \int_0^{t/\varepsilon} L(X_s,c(s))\mathrm{d}s\bigg]. \tag{5.28}$$

Next, let us fix any $G \in \mathcal{G}_\delta$ as defined in (4.1), with $V_G(\omega,x) := \int_{0\rightsquigarrow x} G(\omega,\cdot) \ \cdot \mathrm{d}\mathbf{r}$ as defined in (4.4), and set

$$h_G(x) := \langle\theta,x\rangle + V_G(x,\omega)$$

for a fixed $\omega \in \Omega_0$. If V_G were smooth enough, $\nabla h_G = \theta + G$ and by Itô's formula,

$$\begin{aligned}\langle\theta, X_{t/\varepsilon}\rangle + V_G(X_{t/\varepsilon},\omega) = &\int_0^{t/\varepsilon} (\theta + G(X_s))\sigma(X_s)\mathrm{d}\mathcal{B}_s + \frac{1}{2}\int_0^{t/\varepsilon} \mathrm{div}(a(\theta+G))(X_s)\mathrm{d}s\\ &+ \int_0^{t/\varepsilon} \langle c(s), \theta + G(X_s)\rangle_a \mathrm{d}s.\end{aligned}$$

Therefore, we would obtain

$$\begin{aligned}&E^{P_0^{c,\omega}}\bigg[\langle\theta, X_{t/\varepsilon}\rangle - \int_0^{t/\varepsilon} L(X_s,c(s))\mathrm{d}s\bigg] = -E^{P_0^{c,\omega}}[V_G(X_{t/\varepsilon},\omega)]\\ &\quad + E^{P_0^{c,\omega}}\bigg[\int_0^{t/\varepsilon}\Big(\frac{1}{2}\mathrm{div}\,(a(X_s))(\theta+G(X_s)) + \langle c(s),\theta+G(X_s)\rangle_a - L(X_s,c(s))\Big)\mathrm{d}s\bigg]\\ &\leq -E^{P_0^{c,\omega}}\big[V_G(X_{t/\varepsilon},\omega)\big] + E^{P_0^{c,\omega}}\bigg[\int_0^{t/\varepsilon}\frac{1}{2}\mathrm{div}(a(X_s))(\theta+G(X_s)) + H(X_s,\theta+G(X_s))\mathrm{d}s\bigg]\\ &\leq -E^{P_0^{c,\omega}}\big[V_G(X_{t/\varepsilon},\omega)\big] + \frac{t}{\varepsilon}\operatorname{ess\,sup}_{\mathbb{P}_0}\bigg[\frac{1}{2}\mathrm{div}(a(G+\theta)) + H(G+\theta)\bigg].\end{aligned} \tag{5.29}$$

Together with (5.28), we would have the bound

$$u_\varepsilon(t,0,\omega) \leq -\varepsilon \inf_{c\in\mathbf{C}_T^*} E^{P_0^{c,\omega}}[V_G(X_{t/\varepsilon},\omega)] + t\operatorname{ess\,sup}_{\mathbb{P}_0}\bigg[\frac{1}{2}\mathrm{div}(a(G+\theta)) + H(G+\theta)\bigg].$$

If V_G were bounded, we could apply Theorem 4.9 and deduce that $\mathbb{P}_0$-a.s., for all $r > 0$ there exists some $c_r = c_r(\omega)$ such that for all $x \in \mathcal{C}_\infty$, $|V_G(x,\omega)| \leq r|x| + c_r$, leading to

$$u_\varepsilon(t,0,\omega) \leq t\overline{\Lambda}(\theta) + \varepsilon c_r + r\sup_{c\in\mathbf{C}_T^*} E^{P_0^{c,\omega}}[|\varepsilon X_{t/\varepsilon}|].$$

By Lemma 3.9 and the inequalities (3.34)-(3.35), one can deduce (see (5.34) below for details) that $E^{P_0^{c,\omega}}[|\varepsilon X_{t/\varepsilon}|]$ is uniformly bounded over $0 < \varepsilon \leq 1$ and $c \in \mathbf{C}_T^*$. Thus, one simply let first $\varepsilon \to 0$ and then $r \to 0$ to conclude the proof.

However, a priori $G \in \mathcal{G}_{1+\delta}$ is neither smooth enough nor bounded. Nevertheless, we can mollify G to get a smooth and bounded version, so that we can apply the same reasoning as above.

If ρ is any spherically symmetric mollifier with support the unit ball and such that $\int_{\mathbb{R}^d} \rho(y)\mathrm{d}y = 1$ and $r > 0$, we set

$$G_r(\omega) := \int_{\mathbb{R}^d} G(\tau_{ry}\omega)\rho(y)\mathrm{d}y = G * \rho_r, \tag{5.30}$$

where $\rho_r(y) := \delta^{-d}\rho(y/\delta)$. Similarly, define $V_G^r(x,\omega) := \int_{\mathbb{R}^d} V_{G_r}(x+\delta y,\omega)\rho(y)\mathrm{d}y = V_G * \rho_r$. In particular, $\nabla V_G^r = G_r$ and by Young's inequality, $\|G_r\|_\infty \le C_r$ for some constant depending on r. As a consequence, for any $x \in \mathcal{C}_\infty$ and any path $0 \rightsquigarrow x$ inside $\mathcal{C}_\infty$,

$$V_G^r(x,\omega) - V_G^r(0,\omega) = \int_{0\rightsquigarrow x} G_r(\omega, r)\cdot \mathrm{d}\mathbf{r}, \tag{5.31}$$

so that the line integral is independent of the path $0 \rightsquigarrow x$. By Proposition 4.4, it also holds that $\mathbb{E}_0\left[\int_{0\rightsquigarrow \mathfrak{v}_e} |G_r(\omega,\cdot) - G(\omega,\cdot)|\cdot \mathrm{d}\mathbf{r}\right] \to 0$ as $r \to 0$. Therefore, we can replace G_r by $G_r - c_r$ with some constant vector c_r such that $|c_r| \to 0$ as $r \to 0$ to obtain a smooth, bounded element in $\mathcal{G}_\infty$ on which Theorem 4.2 applies. Repeating the arguments in (5.29) with $G_r - c_r$ (equivalently, replacing θ by $\theta - c_r$), we obtain

$$\begin{aligned}
&E^{P_0^{c,\omega}}\left[\langle\theta, \varepsilon X_{t/\varepsilon}\rangle - \varepsilon\int_0^{t/\varepsilon} L(X_s, c(s))\mathrm{d}s\right]\\
&\le -\varepsilon E^{P_0^{c,\omega}}[V_G^r(X_{t/\varepsilon},\omega) - V_G^r(0,\omega) - \langle c_r, X_{t/\varepsilon}\rangle]\\
&+\varepsilon E^{P_0^{c,\omega}}\left[\int_0^{t/\varepsilon}\frac{1}{2}\mathrm{div}\,(a(X_s))(\theta + G_r(X_s) - c_r) + H(X_s, \theta + G_r(X_s) - c_r)\mathrm{d}s\right].
\end{aligned}$$

The first term can be bounded exactly as before. To handle the second and third terms, we use the convexity of H and Jensen's inequality to bound the sum of the second and third expectations above by

$$\begin{aligned}
&\varepsilon E^{P_0^{c,\omega}}\left[\int_0^{t/\varepsilon}\int_{\mathbb{R}^d}\frac{1}{2}\left[\mathrm{div}(a(X_s))(\theta + G(X_s+ry) - c_r)\right]\rho(y)\mathrm{d}y\mathrm{d}s\right]\\
&\quad+\varepsilon E^{P_0^{c,\omega}}\left[\int_0^{t/\varepsilon}\int_{\mathbb{R}^d}\left[H\left(X_s, \theta + G(X_s+ry) - c_r\right)\right]\rho(y)\mathrm{d}y\mathrm{d}s\right]\\
&\le t\,\mathrm{ess\,sup}_{\mathbb{P}_0}\left[\frac{1}{2}\mathrm{div}\big(a(G+\theta-c_r)\big) + H(G+\theta-c_r)\right].
\end{aligned}\tag{5.32}$$

Letting $\varepsilon \to 0$, then $r \to 0$ and using the continuity of the map

$$\theta \mapsto \mathrm{ess\,sup}_{\mathbb{P}_0}\left[\frac{1}{2}\mathrm{div}(a(G+\theta)) + H(G+\theta)\right]$$

we conclude the proof of the proposition. □

5.4 Proof of Theorem 2.1 and Corollary 2.2. In this section we will complete the proof of Theorem 2.1 and Corollary 2.2.

5.4.1 Proof of Theorem 2.1:

Given Theorem 3.8, it remains to show the following result:

Theorem 5.10. *Let $u_\varepsilon(t,x)$ be the solution of* (1.2) *and u_{hom} as in* (3.24). *If **(F1)-(F4)** hold, then $\mathbb{P}_0$-a.s., for any $T,\ell>0$,*

$$\limsup_{\varepsilon\to 0}\sup_{0\le t\le T}\sup_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell}\big(u_\varepsilon(t,x,\omega)-u_{\mathrm{hom}}(t,x)\big)\le 0. \tag{5.33}$$

Proof. For any fixed t,ε,x and ω,

$$\begin{aligned}
&u_\varepsilon(t,x,\omega)-u_{\mathrm{hom}}(t,x)\\
&=\sup_{c\in\mathbf{C}_T^*}\left(E^{P^{c,\omega}_{x/\varepsilon}}\left[f(\varepsilon X_{t/\varepsilon})-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\right]\right)-\sup_{y\in\mathbb{R}^d}\left(f(y)-t\mathcal{I}\Big(\frac{y-x}{t}\Big)\right)\\
&\le\sup_{c\in\mathbf{C}_T^*}E^{P^{c,\omega}_{x/\varepsilon}}\left[t\mathcal{I}\left(\frac{\varepsilon X_{t/\varepsilon}-x}{t}\right)-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\right]\\
&=\sup_{c\in\mathbf{C}_T^*}E^{P^{c,\omega}_{x/\varepsilon}}\left[\sup_{\theta\in\mathbb{R}^d}\langle\theta,\varepsilon X_{t/\varepsilon}-x\rangle-t\overline{H}(\theta)-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\right].
\end{aligned}$$

Since by Lemma 3.9 and Eqs.(3.34)-(3.35)

$$S:=\sup_{c\in\mathbf{C}_T^*}\sup_{0<\varepsilon\le 1}\sup_{0\le t\le T}\sup_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell}E^{P^{c,\omega}_{x/\varepsilon}}\left[|\varepsilon X_{t/\varepsilon}|+\left|\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}x\right|\right]<\infty \tag{5.34}$$

and $\overline{H}$ also satisfies the estimates of (2.5) with the Euclidean norm, we deduce that it is enough to show that for any fixed $\theta\in\mathbb{R}^d$,

$$\limsup_{\varepsilon\to 0}\sup_{c\in\mathbf{C}_T^*}\sup_{0\le t\le T}\sup_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell}E^{P^{c,\omega}_{x/\varepsilon}}\left[\langle\theta,\varepsilon X_{t/\varepsilon}-x\rangle-t\overline{H}(\theta)-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\right]\le 0. \tag{5.35}$$

Following as in the proof of Proposition 5.9, for any $G\in\mathcal{G}_\delta$ and $r>0$, we apply Itô's formula to $\theta+V_r-c_r$ with $|c_r|\to 0$ as $r\to 0$, obtaining

$$\begin{aligned}
&E^{P^{c,\omega}_{x/\varepsilon}}\left[\langle\theta,\varepsilon X_{t/\varepsilon}-x\rangle-\varepsilon\int_0^{t/\varepsilon}L(X_s,c(s))\mathrm{d}s\right]-t\overline{H}(\theta)\\
&\le-\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}\left[V_G^r(X_{t/\varepsilon},\omega)-V_G^r(x/\varepsilon,\omega)-\langle c_r,X_{t/\varepsilon}\rangle\right] &(5.36)\\
&\quad+\varepsilon E^{P_x/\varepsilon^{c,\omega}}\left[\int_0^{t/\varepsilon}\frac{1}{2}\mathrm{div}\big(a(X_s)\big)(\theta+G_r(X_s)-c_r\big)+H(X_s,\theta+G_r(X_s)-c_r)\mathrm{d}s\right]-t\overline{H}(\theta). &(5.37)
\end{aligned}$$

To bound the first term (5.36), we recall that, thanks to Theorem 4.2, $\mathbb{P}_0$-a.s., for all $\tau>0$ there is some $C_\tau=C_\tau(\omega)$ such that for all $x\in\mathcal{C}_\infty$, $|V_G^r(x,\omega)|\le\tau|x|+C_\tau$. Then the first expectation (5.36) is bounded above by

$$2\varepsilon C_\tau+(\tau+|c_r|)E^{P^{c,\omega}_{x/\varepsilon}}[|\varepsilon X_{t/\varepsilon}|+|x|].$$

By (5.34), we deduce that

$$\limsup_{\tau\to 0}\limsup_{\varepsilon\to 0}\sup_{c\in\mathbf{C}_T^*}\sup_{0\le t\le T}\sup_{x\in\varepsilon\mathcal{C}_\infty:|x|\le\ell}\Big(-\varepsilon E^{P^{c,\omega}_{x/\varepsilon}}[V^r_G(X_{t/\varepsilon},\omega)-V^r_G(x/\varepsilon,\omega)-\langle c_r,X_{t/\varepsilon}\rangle]\Big)$$
$$\le S|c_r|.$$

We bound the second term as in (5.32), implying that (5.37) is bounded above by

$$t\,\mathrm{ess\,sup}_{\mathbb{P}_0}\big[\frac{1}{2}\mathrm{div}\big(a(G+\theta-c_r)\big)+H(G+\theta-c_r)\big]-t\overline{H}(\theta).$$

By Theorem 5.1, for any $\varepsilon'>0$, there is some $G\in\mathcal{G}_\delta$ so that the last display is bounded by

$$\varepsilon'+t\overline{H}(\theta-c_r)-t\overline{H}(\theta),$$

so that the final bound is $S|c_r|+\varepsilon'+t\overline{H}(\theta-c_r)-t\overline{H}(\theta)$. As $\overline{H}$ is continuous, letting first $\varepsilon\to 0$, then $\varepsilon'\to 0$ and finally $r\to 0$, we deduce (5.33), concluding thus the proof of Theorem 5.10 and Theorem 2.1. □

5.4.2 Proof of Corollary 2.2.

Let u_ε solve (2.10) for the particular choice (2.14) and initial condition $f(x)=\langle\theta,x\rangle$. We set

$$v(t,x):=\exp\Big\{\frac{u_\varepsilon(\varepsilon t,\varepsilon x)}{\varepsilon}\Big\}, \tag{5.38}$$

then $v(t,x)$ solves

$$\begin{aligned}&\frac{\partial}{\partial t}v(t,x)=(\mathcal{L}^{(b,\omega)}v)(t,x),\qquad v(0,x)=\mathrm{e}^{\langle\theta,x\rangle},\quad\text{where}\\&\mathcal{L}^{(b,\omega)}=\mathrm{div}\big(a(\cdot,\omega)\nabla\cdot\,\big)+\langle b(\cdot,\omega),\nabla\cdot\,\rangle_a\end{aligned} \tag{5.39}$$

is the generator of the $\mathbb{R}^d$-valued diffusion X_t. By Feynman-Kac formula, we have $v(t,x)=E^{b,\omega}_x[\exp\{\langle\theta,X(t)\rangle\}]$ with $E^{b,\omega}_x$ denoting expectation with respect to the diffusion with generator $\mathcal{L}^{(b,\omega)}$ starting at $x\in\mathbb{R}^d$. Since $u_\varepsilon(t,0)=\frac{1}{\varepsilon}\log v(t/\varepsilon,0)$, we have

$$\lim_{\varepsilon\to 0}\varepsilon u_\varepsilon(1,0)=\lim_{t\to\infty}\frac{1}{t}\log v(t,0), \tag{5.40}$$

and the result follows from Theorem 2.1. Note that to prove Corollary 2.2, we only need to prove (5.40) for $x=0$, the lower bound for which follows from Lemma 3.10. Hence, we require (2.3) only for $\chi=\frac{\alpha}{2}\frac{1+\delta}{\alpha-(1+\delta)}$ for some $\alpha>1+\delta$ and $\delta>0$ which is needed for (5.22) to show weak compactness of the gradients and that $G\in\mathcal{G}_\delta$. As explained earlier, the other moment assumption in (2.4) with $\chi=\frac{\alpha\gamma}{\alpha-1}$ with $\gamma>d$ is necessary to deduce Lipschitz estimates, which guarantee locally uniform convergence in the lower bound for Theorem 2.1. □

A Percolation models satisfying assumptions (P1)-(P6)

Let us illustrate some important percolation models that satisfy the assumptions imposed earlier.

The Boolean model: The simplest continuum percolation model, known as the Boolean model, is defined as $\mathcal{C}(\omega) := \bigcup_{x\in\omega} B_{1/2}(x)$, where ω is sampled with respect to a probability measure $\mathbb{P} = \mathbb{P}^\zeta$ that satisfies the following (recall the notation from Section 1.1):

- For any bounded $A \subset \mathbb{R}^d$ and any $n \in \mathbb{N}_0$,

$$\mathbb{P}^\zeta(\#(\omega \cap A) = n) = \frac{(\zeta|A|)^n}{n!} \mathrm{e}^{-\zeta|A|}. \tag{A.1}$$

- For any collection of disjoint, bounded Borel sets $A_1, ..., A_k \subset \mathbb{R}^d$,

$$\mathbb{P}^\zeta\big(\#(\omega \cap A_1) = n_1, ..., \#(\omega \cap A_k) = n_k\big) = \prod_{i=1}^{k} \mathbb{P}(\#(\omega \cap A_i) = n_i). \tag{A.2}$$

The Boolean model satisfies the above assumptions **(P1)**-**(P6)** for ζ large enough.

Proposition A.1. *Fix $d \geq 2$. Then there exists $\zeta_c \in (0,\infty)$ such that for all $\zeta > \zeta_c$, $(\Omega, \mathcal{G}, \mathbb{P}^\zeta)$ satisfies **(P1)**-**(P5)** and also **(P6)** if $d \geq 3$.*

Proof. The proof of these results are well-known: **(P1)** is a consequence of [MR96, Propositions 2.6-2.7]. **(P2)** is consequence of (A.1), **(P3)** follows from [MR96, Theorems 3.5-3.6]. Property **(P4)** can be found in [CGY11, Lemma 3.4], and **(P5)** appears in [MR96, Theorem 2.2]. To verify **(P6)** in $d \geq 3$, given $\zeta > \zeta_c$, let $L > 0$ be large enough so that in $\mathbb{R}^{d-1} \times [0, L]$ there almost surely exists an infinite cluster $\mathfrak{c}_\infty$. That existence is guaranteed by the fact that the critical intensity for $\mathbb{R}^d$ coincides with the limit of the critical intensities for $\mathbb{R}^2 \times [0, L]^{d-2}$ as $L \to \infty$ (see [T93, Theorem 1]). By the uniqueness of the percolation cluster in $\mathbb{R}^d$, the cluster $\mathfrak{c}_\infty$ is almost surely a subset of $\mathcal{C}_\infty$. Now let A_L be the event that at least one element of the set $\{je : j = 1, ..., L\}$ is an element of $\mathfrak{c}_\infty$. Then

$$\{|\mathfrak{v}_e| \geq Lt\} \cap \{0 \in \mathcal{C}_\infty\} \subset \bigcap_{k\leq\lfloor t\rfloor} \tau_{kLe}(A_L^C).$$

Since all the events in the intersection above are independent, with $p_L = \mathbb{P}(A_L)$, we have

$$\mathbb{P}(|\mathfrak{v}_e| \geq Lt, 0 \in \mathcal{C}_\infty) \leq (1 - p_L)^{\lfloor t\rfloor}$$

for all $t > 0$, and **(P6)** follows. □

Continuum random cluster model: Besides the Boolean model, we mention briefly a model which presents long-range correlations, namely the continuum random cluster model (CRCM) [DH15, H18]. We proceed to define the model and give a sketch of the validity of properties **(P1)**-**(P6)**. For $\omega \in \Omega$ and $\Lambda \subset \mathbb{R}^d$, set $\omega_\Lambda := \omega \cap \Lambda$. A probability measure P on $(\Omega, \mathcal{G})$ is a continuum random cluster model with parameters

$q \geq 1$ and $\zeta > 0$ (CRCM(q, ζ)) if it is stationary and for P almost every configuration ω and each bounded set $\Lambda \subset \mathbb{R}^d$, the conditional law of P given ω_{Λ^c} is absolutely continuous with respect to a Poisson point process restricted to Λ, $\mathbb{P}^\zeta_\Lambda$, with density

$$\frac{q^{N^\Lambda_{cc}(\cdot \cup \omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})}.$$

Here, N^Λ_{cc} is the Λ-local number of connected components of a configuration [H18, Definition 2.1] and Z_Λ is the partition function

$$Z_\Lambda(\omega_{\Lambda^c}) := \int_\Omega q^{N^\Lambda_{cc}(\omega'_\Lambda \cup\, \omega_{\Lambda^c})} \mathbb{P}^\zeta_\Lambda(\mathrm{d}\omega'_\Lambda).$$

Equivalently, the DLR equations are satisfied: for every bounded and measurable function f and bounded set $\Lambda \subset \mathbb{R}^d$,

$$\int_\Omega f(\omega)\mathrm{d}P = \int_\Omega \int_\Omega f(\omega'_\Lambda \cup \omega_{\Lambda^c}) \frac{q^{N^\Lambda_{cc}(\omega'_\Lambda \cup\, \omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \mathbb{P}^\zeta_\Lambda(\mathrm{d}\omega'_\Lambda) P(\mathrm{d}\omega).$$

By [DH15, Theorem 1], there exists at least one stationary CRCM(q, ζ), which can be chosen ergodic, so that **(P1)** holds, while **(P2)** is satisfied by construction. Assumption **(P3)** is a consequence of [H18, Theorems 1 and 2].

Next, we sketch the ideas behind the proof of **(P4)** for which we need some definitions.

- Given a finite subset $\Lambda \subset \mathbb{Z}^d$, the outer boundary of Λ is given by $\partial^{\text{out}}\Lambda := \{x \in \Lambda^c : \exists y \in \Lambda, |x - y|_1 = 1\}$.
- If $B := \prod_{i=1}^d [a_i, b_i]$ is a box in $\mathbb{R}^d$, we say a connected component C contained in B is crossing for B if for all $i \in \{1, \cdots, d\}$, there exist vertices $x(i) = (x_1(i), \cdots, x_d(i)) \in C$ and $y(i) = (y_1(i), \cdots, y_d(i)) \in C$ such that $|x_i(i) - a_i| \leq \frac{1}{2}$ and $|y_i(i) - b| \leq \frac{1}{2}$.
- For each $M > 0$, we define a random field $\{X_z : z \in \mathbb{Z}^d\}$ as follows. Let B_z and B_z^+ be concentric cubes centered at z of radius M and $\frac{5M}{4}$ respectively. Define the event

$$\begin{aligned} A_z : = \{&\exists! \text{ crossing component } C_z \text{ in } B_z^+, \text{ each subbox of radius } M/4 \text{ and center} \\ &z + h, h \in \mathbb{Z}^d \cap [-M/2, M/2]^d \text{ contains a unique crossing component, and all} \\ &\text{of these crossing components are connected to } C_z\}, \end{aligned}$$

and set $X_z := \mathbb{1}_{A_z}$. If one can prove that for some integer $k \geq 1$,

$$\lim_{M\to\infty} \sup_{z\in\mathbb{Z}^d} \operatorname{ess\,sup} P(X_z = 1|\sigma(X_y : |y - z|_\infty > k)) = 1, \tag{A.3}$$

then for all $p \in (0, 1)$, if $M = M(p)$ is large enough, the product measure of i.i.d random variables $\{Y_z : z \in \mathbb{Z}^d\}$ such that $Y_z = 1$ with probability p and 0 otherwise, is stochastically dominated by the law of $\{X_z : z \in \mathbb{Z}^d\}$ [LSS97, Theorem 1.3]. Then following as in [CGY11, pp. 160-161] one can conclude the proof of **(P4)**. To check (A.3), see for example [P96, Theorem 3.1] when $d \geq 3$ and [CM04, Theorem 9] when $d = 2$.

Finally, we mention that, at least for ζ sufficiently large (large enough so that it surpasses a *slab critical parameter*), it can shown that **(P6)** is satisfied. Regarding the FKG inequality, we could not find in the literature the version for the CRCM, but following ideas from [G06, Theorem 4.17] and [MR96, Theorem 2.2] we believe it should yield the desired result.

Beyond the models discussed above, there are many other models which exhibit long-range correlations and which satisfy assumptions **(P1)**-**(P6)** – examples of such models include random interlacements, the vacant sets of random interlacements and the level sets of the Gaussian free field in $d \geq 3$, see [Sz10, T09, T09a, CP12, DRS14, PRS15].

B Ergodic properties.

Here we will provide the proof of Proposition 3.5, which is a consequence of the following known result from ergodic theory (see [P89, BB07]). We include it here for the sake of completeness.

Lemma B.1. *Let $(X, \mathscr{F}, \mu)$ be a probability space and let $T : X \to X$ be invertible, measure preserving and ergodic with respect to μ. Let $A \in \mathscr{F}$ with $\mu(A) > 0$. If $n : A \to \mathbb{N} \cup \{\infty\}$ is defined by*

$$n(x) = \min\{k > 0 : T^k(x) \in A\}$$

and $S : A \to A$ by $S(x) = T^{n(x)}(x)$ for $x \in A$, then S is measure preserving and ergodic with respect to $\mu(\cdot|A)$ and almost surely invertible with respect to the same measure.

Proof. We first prove that S is measure preserving. By the Poincaré Theorem, $n(x) < \infty$ almost surely. For any $j \geq 1$ we define $A_j = \{x \in A : n(x) = j\}$. By definition, the A_j are disjoint and as $n(x) < \infty$ almost surely, $\mu(A \setminus \cup_{j\geq 1} A_j) = 0$. As the restriction of S to A_j is T^j and since T^j is measure preserving, S is measure preserving on A_j. We claim that $S(A_i) \cap S(A_j) = \emptyset$. This, together with the fact that S is measure preserving on A_j, proves that S is measure preserving on the disjoint union $\cup_{j\geq 1} A_j$ and therefore on A.

Thus, we only owe the claim $S(A_i) \cap S(A_j) = \emptyset$. We assume that there exists $x \in S(A_i) \cap S(A_j)$ for $1 \leq i < j$. This requires the existence of $y, z \in A$ with $n(y) = i$, $n(z) = j$ and $x = T^i(y) = T^j(z)$. As T is invertible, $y = T^{j-i}(z)$. Thus, $n(z) \leq j - i < j$, which is a contradiction to $n(z) = j$ and the desired claim follows.

Next, we note that T is invertible. Thus, S is almost surely invertible, as the intersection $S^{-1}(\{x\}) \cap \{S \text{ is well defined}\}$ is a one-point set.

We finally want to show that S is ergodic. Let $B \in \mathscr{F}$ such that $B \subseteq A$ is S-invariant. Then if $x \in B$ and $n \geq 1$, it follows that $S^n(x) \notin A \setminus B$. This implies that for any $x \in B$ and $k \geq 1$, if $T^k(x) \in B$, then $T^k(x) \notin A \setminus B$. We conclude that $C = \cup_{k\geq 1} T^k(B)$ is T-invariant and $B \subseteq C \subseteq (X \setminus A) \cup B$. In particular, $\mu(B) \leq \mu(C) \leq 1 + \mu(B) - \mu(A)$. Therefore, ergodicity of T implies $\mu(C) \in \{0, 1\}$, which forces $\mu(B) \in \{0, \mu(A)\}$ and thus, the ergodicity of S with respect to $\mu(\cdot|A)$. □

Proof of Proposition 3.5. The shift τ_e is invertible, measure preserving and ergodic with respect to $\mathbb{P}$. It follows from Lemma B.1 that the induced shift σ_e is $\mathbb{P}_0$-preserving, almost surely invertible and ergodic with respect to $\mathbb{P}_0$. □

C Proof of Proposition 3.3.

We now sketch the ideas behind the proof of existence and uniqueness of solutions of (3.10). While these arguments are well-known, we include them here for the sake of completeness. Since $\omega \in \Omega_0$ is fixed, we omit it from the notation.

Let us first address the existence of solution. Recall from Section 3.1 that we fix a probability space $(\mathscr{X}, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a Brownian motion $(B_t)_{t\geq 0}$ adapted to the filtration. We set $\mathbf{S} := (0,T) \times \mathcal{C}_\infty$, where $T > 0$ is fixed. The controls take values in $U = \mathbb{R}^d$. The diffusion will be governed by controlled functions $b : \mathbf{S} \times U \to \mathbb{R}^d$ and $\sigma : \mathbf{S} \times U \to \mathcal{S}_d$, where σ is defined as in Section 2.1 and $b(t,x,u) := a(x)u + \operatorname{div} a(x)$. Note that both b and σ are independent on t, so we omit such dependence in what follows. Then for each $c \in \mathbf{C}_T$, there exists a unique solution of the SDE

$$\mathrm{d}X_t = b(X_t, c_t)\mathrm{d}t + \sigma(X_t, c_t)\mathrm{d}B_t \tag{C.1}$$

for each initial condition $x \in \mathbb{R}^d$. We follow the set up from [T13], where the cost function $\tilde{J} : [0,T] \times \mathbb{R}^d \times \mathbf{C}_T \to \mathbb{R}$ is defined as

$$\tilde{J}(t,x,c) := E^P \left[f(X_T^{t,x,c}) - \int_t^T L(X_s^{t,x,c}, c_s)\mathrm{d}s \right],$$

with $(X_s^{t,x,c})_{s\geq t}$ being the solution to (C.1) with initial condition $X_t^{t,x,c} = x$. Then the value function $V : \mathbf{S} \to \mathbb{R}$ is defined by

$$V(t,x) := \sup_{c\in\mathbf{C}_T} \tilde{J}(t,x,c).$$

The corresponding Hamiltonian $\tilde{H} : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d \to \mathbb{R}$ is given by

$$\begin{aligned}\tilde{H}(x,p,A) &:= \sup_{u\in\mathbb{R}^d} \left[b(x,u)\cdot p - L(x,u) + \frac{1}{2}\mathrm{Trace}(a(x)A) \right] \\ &= \frac{1}{2}\mathrm{Trace}(a(x)A) + \frac{1}{2}\operatorname{div} a(x)\cdot p + H(x,p),\end{aligned}$$

with H defined in Section 2.1. By [T13, Theorem 7.4], V is a viscosity solution of the (backward) equation

$$\begin{cases} \partial_t V = -\frac{1}{2}\mathrm{Trace}(a(\cdot,)\mathrm{Hess}_x V) - \frac{1}{2}\operatorname{div} a(x)\cdot\nabla V - H(\cdot,\nabla V), & \text{in } [0,T)\times\mathcal{C}_\infty, \\ V(T,x) = f(x), & \text{on } \mathcal{C}_\infty. \end{cases} \tag{C.2}$$

But note that defining $u(t,x) := V(T-t,x)$ transforms (C.2) into (3.10) and $\tilde{J}$ into J from (3.9). This completes the existence proof.

To verify uniqueness, we will appeal to the following comparison principle:

Theorem C.1. *[AT15, Theorem 2.3] Let $U \subset \mathbb{R}^d$ open and $T > 0$. Assume that $u \in \mathrm{USC}([0,T)\times\overline{U})$, $v \in \mathrm{LSC}([0,T)\times\overline{U})$ are of at most linear growth. Suppose that u is a viscosity subsolution and v is a viscosity supersolution of*

$$\partial_t u = \frac{1}{2}\mathrm{Trace}(a(x)\mathrm{Hess}_x u) - H(x,\nabla u), \tag{C.3}$$

in $(0,T)\times U$ such that $u(\cdot,0) \leq v(\cdot,0)$ on $\overline{U}$ and $u \leq v$ on $[0,T)\times\partial U$. Then $u \leq v$ in $[0,T)\times U$.

Note that our equation (3.10) is of the form

$$\partial_t u = \frac{1}{2}\mathrm{Trace}(a(x)\mathrm{Hess}_x u) + H(x, \nabla u). \tag{C.4}$$

But (C.3) and (C.4) are equivalent by mapping u to $-u$ if $p \to H(\cdot, -p)$ is also convex, which we are assuming. This change leaves the hypothesis from **(F2)**-**(F3)** invariant, so that it is enough to obtain a comparison principle for (C.3). The assumptions on a, σ, H in the mentioned articles are similar to ours, except that the coercivity assumption (2.5) in our case is defined with respect to the a-norm instead of the usual Euclidean norm. But this coercivity with respect to the a-norm is enough to use the above comparison principle. Indeed, (2.2)-(2.5) imply that $\mathbb{P}_0$-a.s. $\xi(\omega) > 0$ and for all $x \in \mathcal{C}_\infty$, there exists a neighborhood of x such that for all y in such a neighborhood, $\xi(\tau_y\omega) > \xi(\tau_x\omega)/2$. In particular, for each $x \in \mathcal{C}_\infty$, there is a neighborhood of x such that the lower bound in (2.5) can be replaced by $c(x)|p|^\alpha$. The rest of the argument follows from the above result.

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