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Optimal control of a nonconserved phase field model of Caginalp type with thermal memory and double obstacle potential

Dedicated to the memory of Professor Gunduz Caginalp

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Abstract

In this paper, we investigate optimal control problems for a nonlinear state system which constitutes a version of the Caginalp phase field system modeling nonisothermal phase transitions with a nonconserved order parameter that takes thermal memory into account. The state system, which is a first-order approximation of a thermodynamically consistent system, is inspired by the theories developed by Green and Naghdi. It consists of two nonlinearly coupled partial differential equations that govern the phase dynamics and the universal balance law for internal energy, written in terms of the phase variable and the so-called thermal displacement, i.e., a primitive with respect to time of temperature. We extend recent results obtained for optimal control problems in which the free energy governing the phase transition was differentiable (i.e., of regular or logarithmic type) to the nonsmooth case of a double obstacle potential. As is well known, in this nondifferentiable case standard methods to establish the existence of appropriate Lagrange multipliers fail. This difficulty is overcome utilizing of the so-called *deep quench* approach. Namely, the double obstacle potential is approximated by a family of (differentiable) logarithmic ones for which the existence of optimal controls and first-order necessary conditions of optimality in terms of the adjoint state variables and a variational inequality are known. By proving appropriate bounds for the adjoint states of the approximating systems, we can pass to the limit in the corresponding first-order necessary conditions, thereby establishing meaningful first-order necessary optimality conditions also for the case of the double obstacle potential.

1 Introduction

Suppose that $\Omega \subset \mathbb{R}^3$ (the two-dimensional case can be treated in the same fashion) is a bounded, open, and connected set having a smooth boundary $\Gamma = \partial\Omega$ with unit outward normal field $\mathbf{n}$ and associated normal derivative $\partial_{\mathbf{n}}$, and let, for some given final time $T > 0$,

$$Q_t := \Omega \times (0,t) \text{ for } t \in (0,T), \quad Q := Q_T, \text{ and } \Sigma := \Gamma \times (0,T).$$

We consider in this paper the following optimal control problem:

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$(\mathbf{CP})$ Minimize the cost functional

$$\begin{aligned}\mathcal{J}((\varphi,w),u) := \frac{k_1}{2}\|\varphi-\varphi_Q\|^2_{L^2(Q)} + \frac{k_2}{2}\|\varphi(T)-\varphi_\Omega\|^2_{L^2(\Omega)} + \frac{k_3}{2}\|w-w_Q\|^2_{L^2(Q)} \\ + \frac{k_4}{2}\|w(T)-w_\Omega\|^2_{L^2(\Omega)} + \frac{k_5}{2}\|\partial_t w - w_Q'\|^2_{L^2(Q)} + \frac{k_6}{2}\|\partial_t w(T) - w_\Omega'\|^2_{L^2(\Omega)} \\ + \ell\,\|u\|^2_{L^2(Q)} \end{aligned} \tag{1.1}$$

subject to the state system

$$\partial_t\varphi - \Delta\varphi + F_1'(\varphi) + \tfrac{2}{\theta_c}F_2'(\varphi) - \tfrac{1}{\theta_c^2}\partial_t w\, F_2'(\varphi) = 0 \quad \text{in } Q\,, \tag{1.2}$$

$$\partial_{tt} w - \Delta(\alpha\partial_t w + \beta w) + F_2'(\varphi)\partial_t\varphi = u \quad \text{in } Q\,, \tag{1.3}$$

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}(\alpha\partial_t w + \beta w) = 0 \quad \text{on } \Sigma, \tag{1.4}$$

$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = v_0 \quad \text{in } \Omega, \tag{1.5}$$

and the control constraint

$$u \in \mathcal{U}_{\rm ad},$$

with the control space $\mathcal{U} := L^\infty(Q)$ and

$$\mathcal{U}_{\rm ad} := \big\{ u \in \mathcal{U} : u_* \le u \le u^* \text{ a.e. in } Q \big\}. \tag{1.6}$$

Above, the symbols $k_1, \dots, k_6$, and ℓ denote nonnegative constants which are not all zero, $\varphi_Q, w_Q, w_Q' \in L^2(Q)$ and $\varphi_\Omega, w_\Omega, w_\Omega' \in L^2(\Omega)$ denote some prescribed targets, and $\theta_c > 0$ denotes a critical temperature. As for the set of admissible controls $\mathcal{U}_{\rm ad}$, we assume that u_*, u^* with $u_* \le u^*$ (to be intended pointwise) are prescribed threshold functions in $L^\infty(Q)$. Notice that the control variable u has the physical meaning of a distributed heat source.

The state system (1.2)–(1.5) constitutes an extension of the phase field model for nonisothermal phase transitions with nonconserved order parameter taking place in the container Ω which was introduced by G. Caginalp in his seminal paper [1]. The primary variables of the system are φ, the order parameter of the phase transition, and w, the so-called *thermal displacement* or *freezing index*. The latter is directly connected to the absolute temperature θ of the system through the relation

$$w(\cdot,t) = w_0 + \int_0^t \theta(\cdot,s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{1.7}$$

In the recent paper [13], the system (1.2)–(1.5) was derived from the general principles of thermodynamics, where the specific free energy governing the evolution was (up to some physical constants that here are assumed to equal unity for simplicity) of the form

$$F(\theta,\varphi) = \theta(1-\ln(\theta/\theta_1)) + \theta\, F_1(\varphi) + F_2(\varphi) + \frac{\theta}{2}\,|\nabla\varphi|^2\,. \tag{1.8}$$

Here, $\theta_1 > 0$ is a reference temperature. In this framework, equation (1.2) describes the dynamics of the phase evolution, while equation (1.3) is the universal balance of internal energy, in which the heat flux is in place of the standard Fourier law assumed in the Green–Naghdi form (see, e.g., [16, 17, 18, 23])

$$\mathbf{q} = -\alpha\nabla(\partial_t w) - \beta\nabla w\,, \quad \text{with positive constants } \alpha \text{ and } \beta\,, \tag{1.9}$$

which models the presence of a thermal memory in the system.

For the nonlinearities driving the phase transformation, we assume that F_2 is differentiable with a globally Lipschitz continuous derivative F_2' on $\mathbb{R}$ (typically, F_2 is a concave function), while for F_1 we consider the convex functions

$$F_{1,\log}(r) = \begin{cases} (1+r)\,\ln(1+r) + (1-r)\,\ln(1-r) & \text{for } r \in (-1,1) \\ 2\ln 2 & \text{for } r \in \{-1,1\}\,, \\ +\infty & \text{for } r \notin [-1,1] \end{cases} \tag{1.10}$$

$$I_{[-1,1]}(r) = \begin{cases} 0 & \text{for } r \in [-1,1] \\ +\infty & \text{for } r \notin [-1,1] \end{cases}\,. \tag{1.11}$$

We assume that $I_{[-1,1]} + F_2$ is a double-well potential. This is actually the case if $F_2(r) = k(1-r^2)$, where $k > 0$; the function $I_{[-1,1]} + F_2$ is then referred to as a *double obstacle* potential. Note also that $F_{1,\log}'(r)$ becomes unbounded as $r \searrow -1$ and $r \nearrow 1$, and that in the case of (1.11) the first equation (1.2) has to be interpreted as a differential inclusion, where $F_1'(\varphi)$ is understood in the sense of subdifferentials. Namely, (1.2) has to be written as

$$\partial_t\varphi - \Delta\varphi + \xi + \frac{2}{\theta_c}F_2'(\varphi) - \frac{1}{\theta_c^2}\partial_t w\, F_2'(\varphi) = 0\,, \quad \xi \in \partial I_{[-1,1]}(\varphi). \tag{1.12}$$

We also notice that the equation (1.2) is of Allen–Cahn type and is suited for the case of nonconserved order parameters (while the case of a conserved order parameter would require a Cahn–Hilliard structure).

As far as well-posedness is concerned, the above model was already treated in [13] (see Theorem 2.1 and Theorem 2.4 below). A discussion of a simpler problem for (1.2)–(1.5) was already given in [22]. The papers [2, 3] dealt with well-posedness issues and asymptotic analyses with respect to the positive coefficients α and β as one of them approaches zero. Other results for this class of systems may be found in [14, 15]. About optimal control problems for phase field systems, in particular of Caginalp type, we can quote the pioneering work [19]; one may also see the specific sections in the monograph [26]. For other contributions, we mention the article [20] dedicated to a thermodynamically consistent version of the phase field system described above, and the more recent papers [7] and [11], where the interested reader can find a list of related references.

The optimal control problem $(\mathbf{CP})$ has been treated in [13] for the case of regular and logarithmic nonlinearities F_1. For such nonlinearities, differentiability properties of the control-to-state mapping, the existence of optimal controls, as well as first-order necessary optimality conditions could be established. Actually, in [13] a more general cost functional was considered which involved the initial temperature υ_0 as a second control variable. In this paper, we focus on the nondifferentiable case when $F_1 = I_{[-1,1]}$. While a well-posedness result for system (1.2)–(1.5) was proved in [13] also for this case, in which (1.2) has to be replaced by the inclusion (1.12), the corresponding optimal control problem has not yet been treated. While the existence of optimal controls is not too difficult to show, the derivation of necessary optimality is challenging since standard constraint qualifications to establish the existence of suitable Lagrange multipliers are not available. In order to overcome this difficulty, we employ the so-called *deep quench approximation* which has proven to be a useful tool in a number of optimal control problems for Allen–Cahn and Cahn–Hilliard systems involving double obstacle potentials: see, e.g., the papers [4, 5, 8, 9, 10, 12, 24].

In all of these works, the starting point was that the optimal control problem had been successfully treated (by proving Fréchet differentiability of the control-to-state operator and establishing first-order necessary optimality conditions in terms of a variational inequality and the adjoint state system) for the case when in the state system (1.2)–(1.5) the nonlinearity F_1 is, for $\gamma > 0$, given by

$$F_{1,\gamma} := \gamma\, F_{1,\log}.$$

We obviously have that

$$0 \le F_{1,\gamma_1}(r) \le F_{1,\gamma_2}(r) \quad \forall\, r \in \mathbb{R}, \quad \text{if } 0 < \gamma_1 < \gamma_2, \tag{1.13}$$

$$\lim_{\gamma \searrow 0} F_{1,\gamma}(r) = I_{[-1,1]}(r) \quad \forall\, r \in \mathbb{R}. \tag{1.14}$$

In addition, we note that $F'_{1,\log}(r) = \ln\left(\frac{1+r}{1-r}\right)$ and $F''_{1,\log}(r) = \frac{2}{1-r^2} > 0$ for $r \in (-1,1)$, and thus, in particular,

$$\lim_{\gamma \searrow 0} F'_{1,\gamma}(r) = \lim_{\gamma \searrow 0} \gamma\, F'_{1,\log}(r) = 0 \quad \text{for } r \in (-1,1),$$
$$\lim_{\gamma \searrow 0}\Bigl(\lim_{r \searrow -1} F'_{1,\gamma}(r)\Bigr) = -\infty, \quad \lim_{\gamma \searrow 0}\Bigl(\lim_{r \nearrow 1} F'_{1,\gamma}(r)\Bigr) = +\infty.$$

We may therefore regard the graphs of the single-valued functions

$$F'_{1,\gamma}(r) = \gamma\, F'_{1,\log}(r), \quad \text{for} \quad r \in (-1,1) \quad \text{and} \quad \gamma > 0,$$

as approximations to the graph of the multi-valued subdifferential $\partial I_{[-1,1]}$ from the interior of $(-1,1)$.

For both $F_1 = I_{[-1,1]}$ (in which case (1.2) has to be replaced by the inclusion (1.12)) and $F_1 = F_{1,\gamma}$ (where $\gamma > 0$), the well-posedness results from [13] yield the existence of a unique solution (φ, w) and $(\varphi_\gamma, w_\gamma)$ to the state system (1.2)–(1.5). It is natural to expect that $(\varphi_\gamma, w_\gamma) \to (\varphi, w)$ as $\gamma \searrow 0$ in a suitable topology. Below (cf. Theorem 3.1), we will show that this is actually true. Owing to the above construction and the singularity of $F_{1,\gamma}$, the approximating functions φ_γ automatically attain their values in the domain of $I_{[-1,1]}$; that is, we have $\|\varphi_\gamma\|_{L^\infty(Q)} \le 1$ for all $\gamma > 0$ which shows that the order parameter φ_γ is limited to range in the physical interval $[-1,1]$.

In the following, the optimal control problem $(\mathbf{CP})$ will be denoted by $(\mathbf{CP_0})$ if $F_1 = I_{[-1,1]}$ and by $(\mathbf{CP_\gamma})$ if $F_1 = F_{1,\gamma}$, $\gamma \in (0,1]$. The general strategy is to derive uniform (with respect to $\gamma \in (0,1]$) a priori estimates for the state and adjoint state variables of an "adapted" version of $(\mathbf{CP_\gamma})$ that are sufficiently strong as to permit a passage to the limit as $\gamma \searrow 0$ in order to derive meaningful first-order necessary optimality conditions also for $(\mathbf{CP_0})$. It turns out that this strategy succeeds.

The remainder of the paper is organized as follows: Section 2 is devoted to the mathematical analysis of the state system (1.2)–(1.5), where we cite results obtained in [13] and derive a qualitative estimate for the difference between the solutions for different values of γ. The subsequent Section 3 then brings a discussion of the deep quench approximation and its properties. In the final Section 4, the first-order necessary optimality conditions for the problem $(\mathbf{CP_0})$ will be derived.

At this point, we fix some notation we are going to employ throughout the paper. Given a Banach space X, we denote by $\|\cdot\|_X$ the corresponding norm, by X^* its dual space, and by $\langle\cdot,\cdot\rangle_X$ the related duality pairing between X^* and X. The standard Lebesgue and Sobolev spaces defined on Ω, for every $1 \le p \le \infty$ and $k \ge 0$, are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, and the associated norms by $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. For the special case $p = 2$, these spaces become Hilbert spaces, and we denote by $\|\cdot\| = \|\cdot\|_2$ the norm of $L^2(\Omega)$ and employ the usual notation $H^k(\Omega) := W^{k,2}(\Omega)$.

For convenience, we also introduce the shorthands

$$H := L^2(\Omega)\,, \quad V := H^1(\Omega)\,, \quad W := \{v \in H^2(\Omega) :\ \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}. \tag{1.15}$$

Besides, for Banach spaces X and Y, we introduce the linear space $X \cap Y$, which becomes a Banach space when equipped with its natural norm $\|v\|_{X\cap Y} := \|v\|_X + \|v\|_Y$, for $v \in X \cap Y$. To conclude, for a normed space X and $v \in L^1(0,T;X)$, we introduce the convolution products

$$(1 * v)(t) := \int_0^t v(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{1.16}$$

$$(1 \circledast v)(t) := \int_t^T v(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{1.17}$$

2 Properties of the state system

The following structural assumptions are postulated throughout this paper.

(A1) α, β, and θ_c are fixed positive constants.

(A2) $F_2 \in C^3(\mathbb{R})$, and F_2' is a Lipschitz continuous function on $\mathbb{R}$.

(A3) $\varphi_0 \in V,\ w_0 \in V,\ v_0 \in H,$ and there are constants r_*, r^* such that $-1 < r_* \le \varphi_0(x) \le r^* < 1$ for almost every $x \in \Omega$.

The first result concerns the existence of weak solutions. Here, let us incidentally notice that conditions **(A2)** and **(A3)** may in fact be weakened if we are merely interested in the existence of weak solutions to system (1.2)–(1.5). However, as will be clarified later on, the optimal control problem we aim at solving requires sufficient regularity properties for the state system to derive the corresponding first-order necessary conditions. For this reason, we immediately assume **(A2)** and **(A3)** to hold in the current form.

Theorem 2.1. *Assume that* **(A1)–(A3)** *hold, and assume that either $F_1 = F_{1,\gamma}$ for some $\gamma \in (0,1]$ or $F_1 = I_{[-1,1]}$. Then the state system* (1.2)–(1.5) *has for every $u \in L^2(0,T;H)$ a unique weak solution (φ, w, ξ) in the sense that*

$$\begin{aligned}
&\varphi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W),\\
&w \in H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap H^1(0,T;V),\\
&\xi \in L^2(0,T;H), \quad \textit{and} \quad \xi \in \partial F_1(\varphi) \ \textit{a.e. in } Q,
\end{aligned}$$

and that the variational equalities

$$\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega \xi v + \frac{2}{\theta_c}\int_\Omega F_2'(\varphi)v - \frac{1}{\theta_c^2}\int_\Omega \partial_t w\, F_2'(\varphi)v = 0\,, \tag{2.1}$$

$$\langle \partial_{tt} w, v\rangle_V + \alpha\int_\Omega \nabla(\partial_t w)\cdot\nabla v + \beta\int_\Omega \nabla w\cdot\nabla v + \int_\Omega F_2'(\varphi)\partial_t\varphi\, v = \int_\Omega uv\,, \tag{2.2}$$

are satisfied for every $v \in V$ and almost every $t \in (0,T)$. Moreover, it holds that

$$\varphi(0) = \varphi_0, \quad w(0) = w_0, \quad \partial_t w(0) = v_0.$$

Furthermore, there exists a constant $K_1 > 0$, which depends only on $\Omega, T, \alpha, \beta, \theta_c$ and the data of the system, such that

$$\|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \|F_1(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\ + \|w\|_{H^2(0,T;V^*)\cap W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \le K_1 \,. \tag{2.3}$$

Remark 2.2. In the single-valued situation, when $F = F_{1,\gamma}$, the inclusion $\xi \in \partial F_{1,\gamma}(\varphi)$ becomes the identity $\xi = F'_{1,\gamma}(\varphi)$. Note also that the initial conditions are meaningful at least in H, since, in particular, $\varphi \in C^0([0,T];V)$ and $w \in C^1([0,T];H)$ by interpolation. Besides, we point out that (2.3) implies that $-1 \le \varphi \le 1$ a.e. in Q, which entails that φ is also uniformly bounded in $L^\infty(Q)$.

Proof of Theorem 2.1 The existence and uniqueness results follow as a special case of Theorem 2.1 and Theorem 2.2 in [13]. Moreover, it follows from **(A3)** that $I_{[-1,1]}(\varphi_0) = 0$ and that, for all $\gamma \in (0,1]$, we have $\|F_{1,\gamma}(\varphi_0)\|_1 = \gamma \|F_{1,\log}(\varphi_0)\|_1 \le 2\,\ln(2)\,|\Omega|$, where $|\Omega|$ denotes the Lebesgue measure of Ω. Therefore, the estimates performed in the proof of [13, Thm. 2.1] apply for both $F_1 = I_{[-1,1]}$ and $F_1 = F_{1,\gamma}$, $\gamma \in (0,1]$, which proves the validity of (2.3). □

By virtue of Theorem 2.1, the control-to-state operators $\mathcal{S}_0 : u \mapsto (\varphi, w, \xi)$ and $\mathcal{S}_\gamma : u \mapsto (\varphi, w, F'_{1,\gamma}(\varphi))$ corresponding to the choices $F_1 = I_{[-1,1]}$ and $F_1 = F_{1,\gamma}$ for $\gamma > 0$, respectively, are well defined. Obviously, the above result tacitly defines the space where the solution operators $\mathcal{S}$ and $\mathcal{S}_\gamma$ map. Namely, it shows that both the operators $\mathcal{S}$ and $\mathcal{S}_\gamma$ have to be intended as mappings from the control space $\mathcal{U}$ into the regularity space $\mathcal{Y}$ defined by

$$\mathcal{Y} := \big(H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)\big) \\ \times \big(H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap H^1(0,T;V)\big) \times L^2(0,T;H).$$

We will see in the forthcoming Theorem 2.4 that, upon requiring stronger assumptions for the initial data (cf. (2.14)), the solution operators $\mathcal{S}_\gamma$ can also be interpreted as mappings from $\mathcal{U}$ into a smaller space than $\mathcal{Y}$ defined by the regularity properties (2.15)–(2.17). The following result provides a qualitative comparison between the solutions associated with $F_1 = F_{1,\gamma}$ for different values of $\gamma \in (0,1]$.

Theorem 2.3. *Suppose that* **(A1)**–**(A3)** *hold, and let, for fixed $0 < \gamma_1 < \gamma_2 \le 1$, controls $u_{\gamma_1}, u_{\gamma_2} \in \mathcal{U}_{\rm ad}$ be given. Then the corresponding solutions $(\varphi_{\gamma_i}, w_{\gamma_i}, F'_{1,\gamma_i}(\varphi_{\gamma_i})) = \mathcal{S}_{\gamma_i}(u_{\gamma_i})$ to* (1.2)–(1.5) *associated with $F_1 = F_{1,\gamma_i}$ and $u = u_{\gamma_i}$, $i = 1,2$, satisfy the estimate*

$$\|\varphi_{\gamma_1} - \varphi_{\gamma_2}\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|w_{\gamma_1} - w_{\gamma_2}\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\ \le K_2\big(\gamma_2 - \gamma_1\big)^{1/2} + K_2\,\|1 * (u_{\gamma_1} - u_{\gamma_2})\|_{L^2(0,T;H)}\,, \tag{2.4}$$

with a positive constant K_2 that depends only on $\Omega, T, \alpha, \beta, \theta_c$ and the data of the system.

Proof. We set, for convenience,

$$u := u_{\gamma_1} - u_{\gamma_2}, \quad \varphi := \varphi_{\gamma_1} - \varphi_{\gamma_2}, \quad w := w_{\gamma_1} - w_{\gamma_2}, \\ \rho_i := F'_2(\varphi_{\gamma_i}) \quad \text{for } i = 1,2, \quad \rho := \rho_1 - \rho_2\,. \tag{2.5}$$

Using this notation, we take the difference of the weak formulation (2.1)–(2.2) written for $(\varphi_{\gamma_i}, w_{\gamma_i}, \xi_{\gamma_i})$, where $\xi_{\gamma_i} = F'_{1,\gamma_i}(\varphi_{\gamma_i})$, $i = 1, 2$, which yields the system

$$\begin{aligned}
&\int_\Omega \partial_t\varphi\, v + \int_\Omega \nabla\varphi\cdot\nabla v + \int_\Omega \big(F'_{1,\gamma_1}(\varphi_{\gamma_1}) - F'_{1,\gamma_2}(\varphi_{\gamma_2})\big)v + \frac{2}{\theta_c}\int_\Omega \rho\, v \\
&\quad = \frac{1}{\theta_c^2}\int_\Omega \partial_t w\, \rho_1 v + \frac{1}{\theta_c^2}\int_\Omega \partial_t w_2\, \rho\, v\,, \qquad (2.6)\\
&\langle \partial_{tt} w, v\rangle_V + \alpha\int_\Omega \nabla(\partial_t w)\cdot\nabla v + \beta\int_\Omega \nabla w\cdot\nabla v + \int_\Omega \partial_t(F_2(\varphi_{\gamma_1}) - F_2(\varphi_{\gamma_2}))v \\
&\quad = \int_\Omega uv\,, \qquad (2.7)
\end{aligned}$$

for all $v \in V$ and almost everywhere in $(0, T)$. Of course, we also have the initial conditions

$$\varphi(0) = 0, \quad w(0) = 0, \quad \partial_t w(0) = 0 \quad \text{a.e. in } \Omega. \qquad (2.8)$$

We now first add the term $\int_\Omega \varphi\, v$ to both sides of (2.6), then take $v = \varphi$ and integrate with respect to time, which leads to the identity

$$\begin{aligned}
&\frac12\|\varphi(t)\|^2 + \int_0^t \|\varphi(s)\|_V^2\,\mathrm{d}s + \int_{Q_t}\big(F'_{1,\gamma_1}(\varphi_{\gamma_1}) - F'_{1,\gamma_1}(\varphi_{\gamma_2})\big)\,\varphi \\
&\quad = \int_{Q_t}\Big(\varphi - \frac{2}{\theta_c}\rho\Big)\varphi + \frac{1}{\theta_c^2}\int_{Q_t}\partial_t w\,\rho_1\,\varphi + \frac{1}{\theta_c^2}\int_{Q_t}\partial_t w_2\,\rho\,\varphi \\
&\qquad - \int_{Q_t}\big(F'_{1,\gamma_1}(\varphi_{\gamma_2}) - F'_{1,\gamma_2}(\varphi_{\gamma_2})\big)\varphi\,, \qquad (2.9)
\end{aligned}$$

for all $t \in [0, T]$. Due to the monotonicity of F'_{1,γ_1}, we immediately conclude that the third term on the left-hand side is nonnegative. The integrals on the right-hand side have to be estimated individually, where in the following $C > 0$ denotes generic constants that may depend on the data of the system but not on γ_1 and γ_2.

At first, using the Lipschitz continuity of F'_2 along with the fact that, owing to Theorem 2.1, $\partial_t w_i \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\varphi_i \in H^1(0, T; H) \cap L^\infty(0, T; V)$, $i = 1, 2$, we infer that

$$\int_{Q_t}\Big(\varphi - \frac{2}{\theta_c}\rho\Big)\varphi \le C\int_{Q_t}|\varphi|^2\,,$$

and, with the help of Hölder's inequality and the continuous embedding $V \subset L^4(\Omega)$,

$$\begin{aligned}
&\frac{1}{\theta_c^2}\int_{Q_t}\partial_t w\,\rho_1\,\varphi \;\le\; C\int_0^t \|\partial_t w\|\Big(\|\varphi_1\|_4 + 1\Big)\,\|\varphi\|_4\,\mathrm{d}s \\
&\quad \le\; C\Big(\|\varphi_1\|_{L^\infty(0,T;V)} + 1\Big)\int_0^t \|\partial_t w\|\,\|\varphi\|_V\,\mathrm{d}s \;\le\; \frac14\int_0^t \|\varphi\|_V^2\,\mathrm{d}s \;+\; D_1\int_{Q_t}|\partial_t w|^2\,,
\end{aligned}$$

where D_1 is a computable and by now fixed constant. Moreover, we have that

$$\begin{aligned}
&\frac{1}{\theta_c^2}\int_{Q_t}\partial_t w_2\,\rho\,\varphi \;\le\; C\int_0^t \|\partial_t w_2\|_4\,\|\varphi\|\,\|\varphi\|_4\,\mathrm{d}s \\
&\quad \le\; C\int_0^t \|\partial_t w_2\|_V\,\|\varphi\|\,\|\varphi\|_V\,\mathrm{d}s \;\le\; \frac14\int_0^t \|\varphi\|_V^2\,\mathrm{d}s \;+\; C\int_0^t \|\partial_t w_2\|_V^2\,\|\varphi\|^2\,\mathrm{d}s\,,
\end{aligned}$$

where, owing to Theorem 2.1, the function $t \mapsto \|\partial_t w_2(t)\|_V^2$ belongs to $L^1(0,T)$.

Finally, we recall that $F_{1,\gamma_i} = \gamma_i \, F_{1,\log}$, $i = 1,2$, and it follows from the convexity of $F_{1,\log}$ on $[-1,1]$ and the fact that φ_{γ_1}, φ_{γ_2} attain their values in $[-1,1]$ almost everywhere in Q that, a.e. in Q,

$$\begin{aligned} &- \bigl(F'_{1,\gamma_1}(\varphi_{\gamma_2}) - F'_{1,\gamma_2}(\varphi_{\gamma_2})\bigr)\, \varphi \;=\; (\gamma_2 - \gamma_1)\, F'_{1,\log}(\varphi_{\gamma_2})\, \varphi \\ &\quad \le\; (\gamma_2 - \gamma_1)\bigl(F_{1,\log}(\varphi_{\gamma_1}) - F_{1,\log}(\varphi_{\gamma_2})\bigr) \;\le\; (\gamma_2 - \gamma_1)\, 2\, \ln 2 \,. \end{aligned}$$

Therefore, collecting the above estimates, it follows from (2.9) that

$$\begin{aligned} &\frac{1}{2}\|\varphi(t)\|^2 + \frac{1}{2}\int_0^t \|\varphi(s)\|_V^2 \,\mathrm{d}s \\ &\quad \le C\,(\gamma_2 - \gamma_1) \,+\, C\int_0^t \Bigl(1 + \|\partial_t w_2\|_V^2\Bigr)\, \|\varphi\|^2 \,\mathrm{d}s + D_1 \int_{Q_t} |\partial_t w|^2 \,. \end{aligned} \tag{2.10}$$

Next, we integrate (2.7) with respect to time using (2.8), then take $v = \partial_t w$, and integrate once more over $(0,t)$ for an arbitrary $t \in [0,T]$. Adding to both sides the terms $\frac{\alpha}{2}\,\|w(t)\|^2 \;=\; \alpha \int_{Q_t} w\, \partial_t w$ (recall that now $w(0) = 0$ from (2.8)), we obtain that

$$\begin{aligned} &\int_{Q_t} |\partial_t w|^2 + \frac{\alpha}{2}\|w(t)\|_V^2 \;=\; -\beta \int_{Q_t} (1 * \nabla w)\cdot \nabla(\partial_t w) \\ &\quad - \int_{Q_t} (F_2(\varphi_{\gamma_1}) - F_2(\varphi_{\gamma_2}))\partial_t w + \int_{Q_t} (1 * u)\partial_t w + \alpha \int_{Q_t} w\, \partial_t w. \end{aligned} \tag{2.11}$$

We estimate each term on the right-hand side individually. At first, using the identity

$$\int_{Q_t} (1 * \nabla w)\cdot \nabla(\partial_t w) = \int_\Omega (1 * \nabla w(t))\cdot \nabla w(t) - \int_{Q_t} |\nabla w|^2,$$

the fact that $\|1 * \nabla w(t)\|^2 \le \Bigl(\int_0^t \|\nabla w\| \,\mathrm{d}s\Bigr)^2 \le T \int_{Q_t} |\nabla w|^2$, as well as Young's inequality, we infer that

$$-\beta \int_{Q_t} (1 * \nabla w)\cdot \nabla(\partial_t w) \;\le\; \frac{\alpha}{4}\, \|\nabla w(t)\|^2 + C\int_{Q_t} |\nabla w|^2.$$

Then, we recall that the mean value theorem and the Lipschitz continuity of F_2' yield the existence of some $\widehat{C} > 0$ such that

$$|F_2(r) - F_2(s)| \;\le\; \widehat{C}(|r| + |s| + 1)|r - s| \quad \text{for all } r,\, s \in \mathbb{R}. \tag{2.12}$$

Hence, by virtue of the continuous and compact embedding $V \subset L^p(\Omega)$, $1 \le p < 6$, we deduce from Ehrling's lemma (see, e.g., [21, Lemme 5.1, p. 58]), along with the Hölder and Young inequalities, and invoking (2.3), that the second term on the right-hand side can be estimated as follows:

$$\begin{aligned} &- \int_{Q_t} \bigl(F_2(\varphi_{\gamma_1}) - F_2(\varphi_{\gamma_2})\bigr)\partial_t w \le C\int_0^t \bigl\||\varphi_{\gamma_1}| + |\varphi_{\gamma_2}| + 1\bigr\|_4 \, \|\varphi_{\gamma_1} - \varphi_{\gamma_2}\|_4 \, \|\partial_t w\| \,\mathrm{d}s \\ &\le \frac{1}{4}\int_{Q_t} |\partial_t w|^2 + C\bigl(\|\varphi_{\gamma_1}\|^2_{L^\infty(0,T;V)} \,+\, \|\varphi_{\gamma_2}\|^2_{L^\infty(0,T;V)} + 1\bigr) \int_0^t \|\varphi\|_4^2 \,\mathrm{d}s \\ &\le \frac{1}{4}\int_{Q_t} |\partial_t w|^2 + \delta \int_0^t \|\varphi\|_V^2 \,\mathrm{d}s + C_\delta \int_{Q_t} |\varphi|^2 \,, \end{aligned}$$

for any positive coefficient δ (yet to be chosen). Finally, Young's inequality easily yields that

$$\int_{Q_t}(1*u)\partial_t w+\alpha\int_{Q_t} w\,\partial_t w\le\frac14\int_{Q_t}|\partial_t w|^2+C\int_{Q_t}|1*u|^2+C\int_{Q_t}|w|^2.$$

Thus, in view of (2.11), upon collecting the above estimates, we realize that

$$\begin{aligned}&\frac12\int_{Q_t}|\partial_t w|^2+\frac{\alpha}{4}\,\|w(t)\|_V^2\\&\le\ \delta\int_0^t\|\varphi\|_V^2\,\mathrm{d}s+C_\delta\int_{Q_t}|\varphi|^2+C\int_{Q_t}|1*u|^2+C\int_0^t\|w\|_V^2\,\mathrm{d}s\,.\end{aligned}\tag{2.13}$$

At this point, we multiply (2.13) by $4D_1$ and add it to (2.10); then, fixing $\delta>0$ such that $4\,D_1\,\delta<1/2$, and applying the Gronwall lemma, we obtain the estimate

$$\begin{aligned}&\|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}+\|w\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\&\quad\le C\big((\gamma_2-\gamma_1)^{1/2}\ +\ \|1*u\|_{L^2(0,T;H)}\big)\,,\end{aligned}$$

which finishes the proof of the assertion. □

We now derive better regularity and boundedness results for the weak solutions to the state system that correspond to the logarithmic potentials $F_1=F_{1,\gamma}$, $\gamma\in(0,1]$.

Theorem 2.4. *Assume that* **(A1)–(A3)** *are fulfilled. Moreover, assume that the following condition is satisfied:*

$$\varphi_0\in W,\quad v_0\in V\cap L^\infty(\Omega),\quad w_0\in V\cap L^\infty(\Omega).\tag{2.14}$$

Then the state system (1.2)–(1.5) *with* $F_1=F_{1,\gamma}$ *has for every* $\gamma\in(0,1]$ *and every* $u\in\mathcal{U}$ *a unique strong solution* $(\varphi_\gamma,w_\gamma)$ *with the regularity*

$$\varphi_\gamma\in W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W),\tag{2.15}$$

$$w_\gamma\in H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;W),\tag{2.16}$$

$$\partial_t w_\gamma\in L^\infty(Q),\tag{2.17}$$

such that the equations (1.2)–(1.5) *are fulfilled almost everywhere in* Q*, on* Σ*, or in* Ω*, respectively. In addition, the phase variable* φ_γ *enjoys the so-called separation property, i.e., there exist two values* $r_-(\gamma)\in(-1,r_*]$, $r_+(\gamma)\in[r^*,1)$*, which depend only on* $\Omega,T,\alpha,\beta,\theta_c$ *and the data of the system, such that*

$$-1<r_-(\gamma)\le\varphi_\gamma\le r_+(\gamma)<1\quad\textit{a.e. in } Q.\tag{2.18}$$

Moreover, there exists a constant $K_3>0$ *such that, for all* $\gamma\in(0,1]$*,*

$$\begin{aligned}&\|\varphi_\gamma\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;H^2(\Omega))}\ +\ \|F'_{1,\gamma}(\varphi_\gamma)\|_{L^\infty(0,T;H)}\\&\quad+\|w_\gamma\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)\cap H^1(0,T;H^2(\Omega))}+\|\partial_t w_\gamma\|_{L^\infty(Q)}\le K_3\,.\end{aligned}\tag{2.19}$$

Proof. We want to apply [13, Thm. 2.3]. To this end, we observe the following facts: first, it obviously holds that $\Delta\varphi_0-F'_{1,\gamma}(\varphi_0)-\frac{2}{\theta_c}F'_2(\varphi_0)+\frac{1}{\theta_c^2}v_0F'_2(\varphi_0)\in H$. Moreover, the restriction of $F_{1,\gamma}$ to the interval $(-1,1)$ belongs to $C^2(-1,1)$ and satisfies the conditions $\lim_{r\searrow-1}F'_{1,\gamma}(r)=-\infty$ and $\lim_{r\nearrow1}F'_{1,\gamma}(r)=+\infty$. Therefore, all of the prerequisites for arguing along the lines of the proof of [13, Thm. 2.3] are fulfilled, from which we conclude the validity of the assertion. We only remark that the global estimate (2.19) is a consequence of the special form of $F_{1,\gamma}=\gamma F_{1,\log}$ and of the fact that we only admit parameters γ in the bounded interval $(0,1]$. □

Remark 2.5. It cannot be excluded that for $\gamma \searrow 0$ we have $r_-(\gamma) \to -1$ and/or $r_+(\gamma) \to +1$. Therefore, a global (for $\gamma \in (0,1]$) bound for the $L^\infty(Q)$-norm of $F'_{1,\gamma}(\varphi_\gamma)$ cannot be guaranteed.

3 Deep quench approximation of the state system

In this section, we discuss the deep quench approximation of the state system (1.2)–(1.5), where we generally assume that the conditions **(A1)–(A3)** and (2.14) are fulfilled. As in the previous section, we consider the state system for the cases $F_1 = I_{[-1,1]}$ and $F_1 = F_{1,\gamma}$ ($\gamma \in (0,1]$), respectively. By Theorem 2.4, we have for every $u \in \mathcal{U}$ and $F_1 = F_{1,\gamma}$, $\gamma \in (0,1]$, a unique strong solution $(\varphi_\gamma, w_\gamma, \xi_\gamma) = \mathcal{S}_\gamma(u)$ with the regularity specified by (2.15)–(2.17) and with $\xi_\gamma := F'_{1,\gamma}(\varphi_\gamma) \in L^\infty(0,T;H)$, while Theorem 2.1 implies the existence of a unique weak solution $(\varphi^0, w^0, \xi^0) = \mathcal{S}_0(u)$ to the weak form (2.1)–(2.2) of the state system for $F_1 = I_{[-1,1]}$ that enjoys the regularity specified in Theorem 2.1. Clearly, we must have

$$-1 \le \varphi_\gamma \le 1 \text{ a.e. in } Q, \text{ for all } \gamma \in (0,1], \text{ and } -1 \le \varphi^0 \le 1 \text{ a.e. in } Q. \tag{3.1}$$

We are now going to investigate the behavior of the family $\{(\varphi_\gamma, w_\gamma)\}_{\gamma>0}$ of deep quench approximations as $\gamma \searrow 0$. We expect that the solution operator $\mathcal{S}_\gamma$ yields an approximation of $\mathcal{S}_0$ as $\gamma \searrow 0$. This is made rigorous through the following result.

Theorem 3.1. *Suppose that the assumptions* **(A1)–(A3)** *and* (2.14) *are fulfilled, and let sequences* $\{\gamma_n\} \subset (0,1]$ *and* $\{u_{\gamma_n}\} \subset \mathcal{U}_{\rm ad}$ *be given such that* $\gamma_n \searrow 0$ *and* $u_{\gamma_n} \to u$ *weakly-star in* $\mathcal{U}$ *as* $n \to \infty$ *for some* $u \in \mathcal{U}_{\rm ad}$. *Moreover, let* $(\varphi_{\gamma_n}, w_{\gamma_n}, \xi_{\gamma_n}) = \mathcal{S}_{\gamma_n}(u_n)$, $n \in \mathbb{N}$, *and* $(\varphi^0, w^0, \xi^0) = \mathcal{S}_0(u)$. *Then, as* $n \to \infty$, *we have that*

$$\begin{aligned}
\varphi_{\gamma_n} \to \varphi^0 \quad & \textit{weakly-star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \\
& \textit{and strongly in } C^0(\overline{Q}), && (3.2)\\
\xi_{\gamma_n} := F'_{1,\gamma_n}(\varphi_{\gamma_n}) \to \xi^0 \quad & \textit{weakly-star in } L^\infty(0,T;H), && (3.3)\\
w_{\gamma_n} \to w^0 \quad & \textit{weakly-star in } H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W) \\
& \textit{and strongly in } C^0(\overline{Q}), && (3.4)\\
\partial_t w_{\gamma_n} \to \partial_t w^0 \quad & \textit{weakly-star in } L^\infty(Q). && (3.5)
\end{aligned}$$

Proof. By virtue of the global estimate (2.19), it follows the existence of a subsequence, which we label again by $n \in \mathbb{N}$, and of limits (φ, w, ξ) such that the convergence statements (3.2)–(3.5) hold true with (φ^0, w^0, ξ^0) replaced by (φ, w, ξ). In this connection, the strong convergence results in (3.2) and (3.4) follow from standard compactness results (cf., e.g., [25, Sect. 8, Cor. 4]). Observe that the strong convergence in (3.2) along with the Lipschitz continuity of F'_2 implies that $F'_2(\varphi_{\gamma_n}) \to F'_2(\varphi)$ strongly in $C^0(\overline{Q})$.

We then need to show that $(\varphi, w, \xi) = (\varphi^0, w^0, \xi^0)$. To this end, consider the time-integrated version of the system (2.1)–(2.2) with test functions $v \in L^2(0,T;V)$ for $F_1 = F_{1,\gamma_n}$ and control $u = u_{\gamma_n}$ for $n \in \mathbb{N}$. Passage to the limit as $n \to \infty$ then shows that (φ, w, ξ) satisfies the initial conditions and is a solution to the time-integrated version of (2.1)–(2.2) for the control u, which is equivalent to (2.1)–(2.2). In order to conclude the proof, it remains to show that $\xi \in \partial I_{[-1,1]}(\varphi)$ almost everywhere in Q. Indeed, if this is the case, then (φ, w, ξ) is the (uniquely determined) solution to the weak form of the state system for $F_1 = I_{[-1,1]}$ and control u and thus coincides with (φ^0, w^0, ξ^0). Once this is shown, the unicity of the limit point also entails that the convergence properties (3.2)–(3.5) are actually valid for the whole sequence $\{\gamma_n\}$ and not just for a subsequence.

Now define on $L^2(Q)$ the convex functional

$$\Phi(v) = \int_Q I_{[-1,1]}(v), \quad \text{if } I_{[-1,1]}(v) \in L^1(Q), \ \text{ and } \Phi(v) = +\infty \, , \text{ otherwise.}$$

It then suffices to show that ξ belongs to the subdifferential of Φ at φ, i.e., that

$$\Phi(v) - \Phi(\varphi) \ \geq \ \int_Q \xi\,(v-\varphi) \quad \forall\, v \in L^2(Q). \tag{3.6}$$

At this point, we recall that $\varphi_{\gamma_n}(x,t) \in [-1,1]$ in $\overline{Q}$. Hence, by (3.2), also $\varphi(x,t) \in [-1,1]$ in $\overline{Q}$, and thus $\Phi(\varphi) = 0$. Now observe that in case that $I_{[-1,1]}(v) \notin L^1(Q)$ the inequality (3.6) holds true since its left-hand side is infinite. If, however, $I_{[-1,1]}(v) \in L^1(Q)$, then obviously $v(x,t) \in [-1,1]$ almost everywhere in Q, and by virtue of (1.13) and (1.14) it follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_Q F_{1,\gamma_n}(v) = \Phi(v) = 0.$$

Now, by the convexity of F_{1,γ_n}, and since $F_{1,\gamma_n}(\varphi_{\gamma_n})$ is nonnegative, we have for all $v \in L^2(Q)$ that

$$F'_{1,\gamma_n}(\varphi_{\gamma_n})(v-\varphi_{\gamma_n}) \ \leq \ F_{1,\gamma_n}(v) - F_{1,\gamma_n}(\varphi_{\gamma_n}) \ \leq \ F_{1,\gamma_n}(v) \quad \text{a.e. in } Q.$$

Using (3.2) and (3.3), we thus obtain the following chain of (in)equalities:

$$\begin{aligned} \int_Q \xi(v-\varphi) \ &= \ \lim_{n\to\infty} \int_Q F'_{1,\gamma_n}(\varphi_{\gamma_n})(v-\varphi_{\gamma_n}) \ \leq \ \limsup_{n\to\infty} \int_Q \Big(F_{1,\gamma_n}(v) - F_{1,\gamma_n}(\varphi_{\gamma_n}) \Big) \\ &\leq \ \lim_{n\to\infty} \int_Q F_{1,\gamma_n}(v) \ = \ \Phi(v) \ = \ \Phi(v) - \Phi(\varphi), \end{aligned}$$

which shows the validity of (3.6). This concludes the proof of the assertion. □

Remark 3.2. Note that the stronger conditions on the data required by (2.14) yield additional regularity for the solution also in the case $F_1 = I_{[-1,1]}$ with respect to the one obtained from Theorem 2.1. Indeed, we have

$$\begin{aligned} &\varphi^0 \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \quad \xi^0 \in L^\infty(0,T;H), \\ &w^0 \in H^2(0,T;H) \cap W^{1,\infty}(0,T;V) \cap H^1(0,T;W), \quad \partial_t w^0 \in L^\infty(Q). \end{aligned}$$

The following result provides a qualitative comparison between the solutions associated with $F_1 = I_{[-1,1]}$ and $F_1 = F_{1,\gamma}$ for $\gamma \in (0,1]$.

Theorem 3.3. *Suppose that* **(A1)–(A3)** *hold, and let* $(\varphi^0, w^0, \xi^0) = \mathcal{S}_0(u)$ *and, for any* $\gamma \in (0,1]$, $(\varphi_\gamma, w_\gamma, F'_{1,\gamma}(\varphi_\gamma)) = \mathcal{S}_\gamma(u)$. *Then it holds the estimate*

$$\|\varphi_\gamma - \varphi^0\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|w_\gamma - w^0\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \ \leq K_2\, \gamma^{1/2}\,,$$

with the positive constant K_2 *introduced in Theorem 2.3.*

Proof. The result follows immediately from Theorem 2.3 and the semicontinuity properties of norms if we set $u_{\gamma_1} = u_{\gamma_2} = u$ and $\gamma_2 = \gamma$ in the estimate (2.4) and take the limit as $\gamma_1 \searrow 0$. □

4 Existence and approximation of optimal controls

Beginning with this section, we investigate the optimal control problem $(\mathbf{CP_0})$ of minimizing the cost functional (1.1) over the admissible set $\mathcal{U}_{\rm ad}$ subject to the state system (1.2)–(1.5) in the form (2.1),(2.2),(1.5) for $F_1 = I_{[-1,1]}$ under the following additional assumptions on the data of the cost functional:

(A4) The constants $k_1, \ldots, k_6, \ell$ are nonnegative and not all equal to zero.

(A5) $\varphi_\Omega, w_\Omega, w'_\Omega \in L^2(\Omega)$ and $\varphi_Q, w_Q, w'_Q \in L^2(Q)$.

(A6) $u_*, u^* \in L^\infty(Q)$ and satisfy $u_* \le u^*$ a.e. in Q.

In comparison with $(\mathbf{CP_0})$, we consider for $\gamma > 0$ the following control problem:

$(\mathbf{CP_\gamma})$ Minimize $\mathcal{J}((\varphi, w), u)$ for $u \in \mathcal{U}_{\rm ad}$, where φ, w denote the components of the solution $\mathcal{S}_\gamma(u)$ to the state system.

We expect that the minimizers of $(\mathbf{CP_\gamma})$ are for $\gamma \searrow 0$ related to minimizers of $(\mathbf{CP_0})$. Prior to giving an affirmative answer to this conjecture, we first recall that $(\mathbf{CP_\gamma})$ has, by virtue of [13, Thm. 3.1], for every $\gamma > 0$ a solution; a corresponding result for $(\mathbf{CP_0})$ is not yet known and will be shown below. We begin our analysis with the following result.

Proposition 4.1. *Suppose that* **(A1)–(A6)** *and* (2.14) *are satisfied, and let sequences* $\{\gamma_n\} \subset (0,1]$ *and* $\{u_n\} \subset \mathcal{U}_{\rm ad}$ *be given such that, as* $n \to \infty$*,* $\gamma_n \searrow 0$ *and* $u_n \to u$ *weakly-star in* $\mathcal{U}$ *for some* $u \in \mathcal{U}_{\rm ad}$*. Then it can be shown that*

$$\mathcal{J}(\mathcal{S}_0(u), u) \le \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(u_n), u_n), \tag{4.1}$$

$$\mathcal{J}(\mathcal{S}_0(v), v) = \lim_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(v), v) \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{4.2}$$

Proof. Theorem 3.1 yields that $(\varphi_{\gamma_n}, w_{\gamma_n}, F'_{1,\gamma_n}(\varphi_{\gamma_n})) = \mathcal{S}_{\gamma_n}(u_n)$ fulfills the convergence relations (3.2)–(3.4). The validity of (4.1) is then a direct consequence of the semicontinuity properties of the cost functional $\mathcal{J}$.

Now suppose that $v \in \mathcal{U}_{\rm ad}$ is arbitrarily chosen, and put $(\varphi_{\gamma_n}, w_{\gamma_n}, F'_{1,\gamma_n}(\varphi_{\gamma_n})) := \mathcal{S}_{\gamma_n}(v)$ for all $n \in \mathbb{N}$, as well as $(\varphi^0, w^0, \xi^0) := \mathcal{S}_0(v)$. Applying Theorem 3.1 with the constant sequence $u_n = v$, $n \in \mathbb{N}$, we see that (3.2)–(3.5) are valid once more. Since the first six summands of the cost functional are continuous with respect to the strong topology of $C^0([0,T]; H)$, we conclude the validity of (4.2). $\square$

We are now in a position to prove the existence of minimizers for the control problem $(\mathbf{CP_0})$. We have the following result.

Corollary 4.2. *Suppose that* **(A1)–(A6)** *and* (2.14) *are fulfilled. Then the optimal control problem* $(\mathbf{CP_0})$ *has at least one solution.*

Proof. Pick an arbitrary sequence $\{\gamma_n\} \subset (0,1]$ such that $\gamma_n \searrow 0$ as $n \to \infty$. Then the optimal control problem $(\mathbf{CP_{\gamma_n}})$ has for every $n \in \mathbb{N}$ a solution $((\varphi_{\gamma_n}, w_{\gamma_n}), u_{\gamma_n})$, where $(\varphi_{\gamma_n}, w_{\gamma_n}, F'_{1,\gamma_n}(\varphi_{\gamma_n}))$

$= \mathcal{S}_{\gamma_n}(u_{\gamma_n})$ for $n \in \mathbb{N}$. Since $\mathcal{U}_{\mathrm{ad}}$ is bounded in $\mathcal{U}$, we may without loss of generality assume that $u_{\gamma_n} \to u$ weakly-star in $\mathcal{U}$ for some $u \in \mathcal{U}_{\mathrm{ad}}$, the latter being a consequence of the convexity and the strong closedness of $\mathcal{U}_{\mathrm{ad}}$. We then obtain from Theorem 3.1 that (3.2)–(3.5) hold true with $(\varphi^0, w^0, \xi^0) = \mathcal{S}_0(u)$. Invoking the optimality of $((\varphi_{\gamma_n}, w_{\gamma_n}), u_{\gamma_n})$ for $(\mathcal{CP}_{\gamma_n})$, we then find from Proposition 4.1 for every $v \in \mathcal{U}_{\mathrm{ad}}$ the chain of (in)equalities

$$\mathcal{J}(\mathcal{S}_0(u), u) \leq \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) \leq \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(v), v) = \mathcal{J}(\mathcal{S}_0(v), v),$$

which yields that $(\mathcal{S}_0(u), u)$ is an optimal pair for $(\mathbf{CP_0})$. The assertion is thus proved. □

Theorem 3.1 and the proof of Corollary 4.2 indicate that optimal controls of $(\mathbf{CP_\gamma})$ are "close" to optimal controls of $(\mathbf{CP_0})$ as γ approaches zero. However, they do not yield any information on whether every optimal control of $(\mathbf{CP_0})$ can be approximated in this way. In fact, such a global result cannot be expected to hold true. Nevertheless, a local answer can be given by employing a well-known trick. To this end, let $\overline{u} \in \mathcal{U}_{\mathrm{ad}}$ be an optimal control for $(\mathbf{CP_0})$ with the associated state $\mathcal{S}_0(\overline{u})$. We associate with this optimal control the *adapted cost functional*

$$\widetilde{\mathcal{J}}((\varphi, w), u) := \mathcal{J}((\varphi, w), u) + \frac{1}{2}\|u - \overline{u}\|^2_{L^2(Q)} \tag{4.3}$$

and a corresponding *adapted optimal control problem* for $\gamma > 0$, namely:

$(\widetilde{\mathbf{CP}}_\gamma)$ Minimize $\widetilde{\mathcal{J}}((\varphi, w), u)$ for $u \in \mathcal{U}_{\mathrm{ad}}$ subject to $(\varphi, w) = \mathcal{S}_\gamma(u)$.

With essentially the same proof as that of [13, Thm. 3.1] (which needs no repetition here), we can show that the adapted optimal control problem $(\widetilde{\mathbf{CP}}_\gamma)$ has for every $\gamma > 0$ at least one solution. The following result gives a partial answer to the question raised above concerning the approximation of optimal controls for $(\mathbf{CP_0})$ by the approximating problem $(\widetilde{\mathbf{CP}}_\gamma)$.

Theorem 4.3. *Let the assumptions of Proposition 4.1 be fulfilled, suppose that $\overline{u} \in \mathcal{U}_{\mathrm{ad}}$ is an arbitrary optimal control of $(\mathbf{CP_0})$ with associated state $(\overline{\varphi}, \overline{w}, \overline{\xi}) = \mathcal{S}_0(\overline{u})$, and let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0, 1]$ be any sequence such that $\gamma_k \searrow 0$ as $k \to \infty$. Then, for any $k \in \mathbb{N}$ there exists an optimal control $u_{\gamma_k} \in \mathcal{U}_{\mathrm{ad}}$ of the adapted problem $(\widetilde{\mathbf{CP}}_{\gamma_k})$ with associated state $(\varphi_{\gamma_k}, w_{\gamma_k}, \xi_{\gamma_k}) = \mathcal{S}_{\gamma_k}(u_{\gamma_k})$, such that, as $k \to \infty$,*

$$u_{\gamma_k} \to \overline{u} \quad \text{strongly in } L^2(Q), \tag{4.4}$$

and such that (3.2)–(3.5) *hold true with (φ^0, w^0, ξ^0) replaced by $(\overline{\varphi}, \overline{w}, \overline{\xi})$. Moreover, we have*

$$\lim_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) = \mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u}). \tag{4.5}$$

Proof. For any $k \in \mathbb{N}$, we pick an optimal control $u_{\gamma_k} \in \mathcal{U}_{\mathrm{ad}}$ for the adapted problem $(\widetilde{\mathbf{CP}}_{\gamma_k})$ and denote by $(\varphi_{\gamma_k}, w_{\gamma_k}, \xi_{\gamma_k}) = \mathcal{S}_{\gamma_k}(u_{\gamma_k})$ the associated strong solution to the state system (1.2)–(1.5). By the boundedness of $\mathcal{U}_{\mathrm{ad}}$ in $\mathcal{U}$, there is some subsequence $\{\gamma_n\}$ of $\{\gamma_k\}$ such that

$$u_{\gamma_n} \to u \quad \text{weakly-star in } \mathcal{U} \quad \text{as } n \to \infty, \tag{4.6}$$

for some $u \in \mathcal{U}_{\mathrm{ad}}$. Thanks to Theorem 3.1, the convergence properties (3.2)–(3.5) hold true correspondingly for the triple $(\varphi^0, w^0, \xi^0) = \mathcal{S}_0(u)$. In addition, the pair $(\mathcal{S}_0(u), u)$ is admissible for $(\mathbf{CP_0})$.

We now aim at showing that $u = \overline{u}$. Once this is shown, it follows from the unique solvability of the state system that also $(\varphi^0, w^0, \xi^0) = (\overline{\varphi}, \overline{w}, \overline{\xi})$. Now observe that, owing to the weak sequential lower semicontinuity properties of $\widetilde{\mathcal{J}}$, and in view of the optimality property of $(\mathcal{S}_0(\overline{u}), \overline{u})$ for problem $(\mathbf{CP_0})$,

$$\begin{aligned}\liminf_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) &\geq \mathcal{J}(\mathcal{S}_0(u), u) + \frac{1}{2}\|u - \overline{u}\|^2_{L^2(Q)} \\ &\geq \mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u}) + \frac{1}{2}\|u - \overline{u}\|^2_{L^2(Q)}\,. \end{aligned} \tag{4.7}$$

On the other hand, the optimality property of $(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n})$ for problem $(\widetilde{\mathbf{CP}}_{\boldsymbol{\gamma_n}})$ yields that for any $n \in \mathbb{N}$ we have

$$\widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) \leq \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(\overline{u}), \overline{u}) = \mathcal{J}(\mathcal{S}_{\gamma_n}(\overline{u}), \overline{u})\,, \tag{4.8}$$

whence, taking the limit superior as $n \to \infty$ on both sides and invoking (4.2) in Proposition 4.1,

$$\begin{aligned}\limsup_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) &\leq \limsup_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(\overline{u}), \overline{u}) \\ &= \limsup_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(\overline{u}), \overline{u}) = \mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u})\,. \end{aligned} \tag{4.9}$$

Combining (4.7) with (4.9), we have thus shown that $\frac{1}{2}\|u - \overline{u}\|^2_{L^2(Q)} = 0$, so that $u = \overline{u}$ and thus also $(\overline{\varphi}, \overline{w}, \overline{\xi}) = (\varphi^0, w^0, \xi^0)$. Moreover, (4.7) and (4.9) also imply that

$$\begin{aligned}\mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u}) &= \widetilde{\mathcal{J}}(\mathcal{S}_0(\overline{u}), \overline{u}) = \liminf_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) \\ &= \limsup_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) = \lim_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n})\,, \end{aligned}$$

which proves (4.5). Moreover, the convergence properties (3.2)–(3.5) are satisfied. On the other hand, we have that

$$\begin{aligned}\mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u}) &\leq \liminf_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) \leq \limsup_{n\to\infty} \mathcal{J}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) \\ &\leq \limsup_{n\to\infty} \widetilde{\mathcal{J}}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n}) = \mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u}), \end{aligned}$$

so that also $\mathcal{J}(\mathcal{S}_{\gamma_n}(u_{\gamma_n}), u_{\gamma_n})$ converges to $\mathcal{J}(\mathcal{S}_0(\overline{u}), \overline{u})$ as $n \to \infty$, and the relation in (4.3) enables us to infer the strong convergence in (4.4) for the subsequence $\{u_{\gamma_n}\}$.

We now claim that (4.4) holds true even for the entire sequence, due to the complete identification of the limit u as $\overline{u}$. Assume that (4.4) does not hold true. Then there exist some $\varepsilon > 0$ and a subsequence $\{\gamma_j\}$ of $\{\gamma_k\}$ such that

$$\|u_{\gamma_j} - \overline{u}\|_{L^2(Q)} \geq \varepsilon \quad \forall\, j \in \mathbb{N}.$$

However, by the boundedness of $\mathcal{U}_{\mathrm{ad}}$, there is some subsequence $\{\gamma_{j_n}\}$ of $\{\gamma_j\}$ such that, with some $\tilde{u} \in \mathcal{U}_{\mathrm{ad}}$,

$$u_{\gamma_{j_n}} \to \tilde{u} \quad \text{weakly-star in } \mathcal{U} \quad \text{as } n \to \infty\,.$$

Arguing as above, it then turns out that $\tilde{u} = \overline{u}$ and that (4.4) holds for the subsequence $\{u_{\gamma_{j_n}}\}$ as well, which contradicts the obvious fact that $\{u_{\gamma_j}\}$ cannot have a subsequence which converges strongly to $\overline{u}$ in $L^2(Q)$. $\square$

5 First-order necessary optimality conditions

We now derive first-order necessary optimality conditions for the control problem $(\mathbf{CP_0})$, using the corresponding conditions for $(\widetilde{\mathbf{CP}}_\gamma)$ as approximations. To this end, we generally assume that the conditions **(A1)**–**(A6)** and (2.14) are fulfilled. Moreover, we need an additional assumption:

(A7) At least one of the conditions $k_6 = 0$ or $w'_\Omega \in V$ is satisfied.

Now let $\overline{u} \in \mathcal{U}_{\rm ad}$ be any fixed optimal control for $(\mathbf{CP_0})$ with associated state $(\overline{\varphi}, \overline{w}, \overline{\xi}) = \mathcal{S}_0(\overline{u})$, and assume that $\gamma \in (0,1]$ is fixed. Moreover, suppose that $\overline{u}_\gamma \in \mathcal{U}_{\rm ad}$ is an optimal control for $(\widetilde{\mathbf{CP}}_\gamma)$ with corresponding state $(\overline{\varphi}_\gamma, \overline{w}_\gamma) = \mathcal{S}_\gamma(\overline{u}_\gamma)$. The corresponding adjoint problem is given, in its strong form for simplicity, by

$$
\begin{aligned}
&- \partial_t p_\gamma - F_2'(\overline{\varphi}_\gamma)\,\partial_t q_\gamma - \Delta p_\gamma + F''_{1,\gamma}(\overline{\varphi}_\gamma)\,p_\gamma + \tfrac{2}{\theta_c} F_2''(\overline{\varphi}_\gamma)\,p_\gamma - \tfrac{1}{\theta_c^2}\partial_t \overline{w}_\gamma\, F_2''(\overline{\varphi}_\gamma)\,p_\gamma \\
&\qquad = k_1(\overline{\varphi}_\gamma - \varphi_Q) && \text{in } Q, \quad (5.1)\\
&- \partial_t q_\gamma - \alpha\Delta q_\gamma + \beta\Delta(1 \circledast q_\gamma) - \tfrac{1}{\theta_c^2} F_2'(\overline{\varphi}_\gamma)\,p_\gamma \\
&\qquad = k_3(1 \circledast (\overline{w}_\gamma - w_Q)) + k_5(\partial_t \overline{w}_\gamma - w_Q') + k_4(\overline{w}_\gamma(T) - w_\Omega) && \text{in } Q, \quad (5.2)\\
&\partial_{\mathbf{n}} p_\gamma = \partial_{\mathbf{n}} q_\gamma = 0 && \text{on } \Sigma, \quad (5.3)\\
&p_\gamma(T) = k_2(\overline{\varphi}_\gamma(T) - \varphi_\Omega) - k_6 F_2'(\overline{\varphi}_\gamma(T))(\partial_t \overline{w}_\gamma(T) - w'_\Omega), \\
&\quad q_\gamma(T) = k_6(\partial_t \overline{w}_\gamma(T) - w'_\Omega) && \text{in } \Omega, \quad (5.4)
\end{aligned}
$$

where the product $\circledast$ is defined in (1.17). Let us, for convenience, denote by f_{q_γ} the source term in (5.2), that is,

$$
f_{q_\gamma} := k_3(1 \circledast (\overline{w}_\gamma - w_Q)) + k_5(\partial_t \overline{w}_\gamma - w_Q') + k_4(\overline{w}_\gamma(T) - w_\Omega). \tag{5.5}
$$

According to [13, Thm. 3.6], the adjoint system has under the assumptions **(A1)**–**(A7)** a unique weak solution

$$
\begin{aligned}
p_\gamma &\in H^1(0,T;V^*) \cap L^\infty(0,T;H) \cap L^2(0,T;V), && (5.6)\\
q_\gamma &\in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), && (5.7)
\end{aligned}
$$

that satisfies the weak variational form

$$
\begin{aligned}
&- \langle \partial_t p_\gamma, v\rangle_V - \int_\Omega F_2'(\overline{\varphi}_\gamma)\,\partial_t q_\gamma\, v + \int_\Omega \nabla p_\gamma \cdot \nabla v + \int_\Omega F''_{1,\gamma}(\overline{\varphi}_\gamma)\,p_\gamma\, v \\
&\quad + \frac{2}{\theta_c}\int_\Omega F_2''(\overline{\varphi}_\gamma)\,p_\gamma\, v - \frac{1}{\theta_c^2}\int_\Omega \partial_t \overline{w}_\gamma\, F_2''(\overline{\varphi}_\gamma)\,p_\gamma\, v = k_1 \int_\Omega (\overline{\varphi}_\gamma - \varphi_Q)\, v, && (5.8)\\
&- \int_\Omega \partial_t q_\gamma\, v + \alpha \int_\Omega \nabla q_\gamma \cdot \nabla v - \beta \int_\Omega \nabla(1 \circledast q_\gamma)\cdot\nabla v - \frac{1}{\theta_c^2}\int_\Omega F_2'(\overline{\varphi}_\gamma) p_\gamma\, v = \int_\Omega f_{q_\gamma}\, v && (5.9)
\end{aligned}
$$

for every $v \in V$, almost everywhere in $(0,T)$, and the final conditions

$$
\begin{aligned}
p_\gamma(T) &= k_2(\overline{\varphi}_\gamma(T) - \varphi_\Omega) - k_6 F_2'(\overline{\varphi}_\gamma(T))(\partial_t \overline{w}_\gamma(T) - w'_\Omega) && \text{in } \Omega, && (5.10)\\
q_\gamma(T) &= k_6(\partial_t \overline{w}_\gamma(T) - w'_\Omega) && \text{in } \Omega. && (5.11)
\end{aligned}
$$

The variational inequality representing the first-order necessary optimality condition for $(\widetilde{\mathbf{CP}}_\gamma)$ then takes the form

$$\int_Q \big(q_\gamma + \ell\,\overline{u}_\gamma + (\overline{u}_\gamma - \overline{u})\big)(v - \overline{u}_\gamma) \ \geq\ 0 \quad \forall\, v \in \mathcal{U}_{\rm ad}. \tag{5.12}$$

In the following, we now derive some a priori bounds for the adjoint state variables, where we denote by C_i, $i \in \mathbb{N}$, constants that depend only on the data of the problem and not on $\gamma \in (0,1]$. At first, we conclude from the general assumptions **(A1)–(A7)** and (2.14) and from the global bound (2.19) that the following holds true:

$$\begin{aligned}&\|F_2'(\overline{\varphi}_\gamma)\|_{C^0(\overline{Q})} \ +\ \|F_2''(\overline{\varphi}_\gamma)\|_{C^0(\overline{Q})} \ +\ \|k_1(\overline{\varphi}_\gamma - \varphi_Q)\|_{L^2(Q)} \ +\ \|f_{q_\gamma}\|_{L^2(Q)} \\ &+\ \|p_\gamma(T)\| \ +\ \|q_\gamma(T)\|_V \ \leq\ C_1 \quad \forall\, \gamma \in (0,1]\,. \end{aligned} \tag{5.13}$$

First estimate: We take $v = p_\gamma$ in (5.8), $v = -\theta_c^2 \partial_t q_\gamma$ in (5.9), add the resulting equalities and note that two terms cancel out. Then, we integrate over (t,T) and by parts. Putting $Q_t^T := \Omega \times (t,T)$, we obtain the identity

$$\begin{aligned}&\frac{1}{2}\|p_\gamma(t)\|^2 + \int_{Q_t^T} |\nabla p_\gamma|^2 + \int_{Q_t^T} F_{1,\gamma}''(\overline{\varphi}_\gamma)|p_\gamma|^2 + \theta_c^2 \int_{Q_t^T} |\partial_t q_\gamma|^2 + \frac{\alpha\theta_c^2}{2}\|\nabla q_\gamma(t)\|^2 \\ &= \frac{1}{2}\|p_\gamma(T)\|^2 + \frac{\alpha\theta_c^2}{2}\|\nabla q_\gamma(T)\|^2 + k_1 \int_{Q_t^T} (\overline{\varphi}_\gamma - \varphi_Q) p_\gamma - \frac{2}{\theta_c}\int_{Q_t^T} F_2''(\overline{\varphi}_\gamma)\, p_\gamma^2 \\ &\quad + \frac{1}{\theta_c^2}\int_{Q_t^T} \partial_t \overline{w}_\gamma\, F_2''(\overline{\varphi}_\gamma)\, p_\gamma^2 - \beta\theta_c^2 \int_{Q_t^T} \nabla(1 \circledast q_\gamma)\cdot\nabla(\partial_t q_\gamma) - \theta_c^2 \int_{Q_t^T} f_{q_\gamma}\, \partial_t q_\gamma. \end{aligned} \tag{5.14}$$

Notice that the third term on the left-hand side is nonnegative since $F_{1,\gamma}'' \geq 0$. As for the sixth term on the right-hand side, we note that $(1 \circledast q_\gamma)(T) = 0$ in Ω, thus the Young and Hölder inequalities allow us to deduce that

$$\begin{aligned}&-\beta\theta_c^2 \int_{Q_t^T} \nabla(1 \circledast q_\gamma)\cdot\nabla(\partial_t q_\gamma) \\ &= \beta\theta_c^2 \int_\Omega \nabla(1 \circledast q_\gamma)(t)\cdot\nabla q_\gamma(t) - \beta\theta_c^2 \int_{Q_t^T} |\nabla q_\gamma|^2 \\ &\leq \frac{\alpha\theta_c^2}{4}\|\nabla q_\gamma(t)\|^2 + C_2 \int_{Q_t^T} |\nabla q_\gamma|^2. \end{aligned}$$

For third and last terms on the right-hand side, we infer from (5.13) and Young's inequality that

$$k_1 \int_{Q_t^T} (\overline{\varphi}_\gamma - \varphi_Q) p_\gamma - \theta_c^2 \int_{Q_t^T} f_{q_\gamma}\, \partial_t q_\gamma \leq \frac{\theta_c^2}{2}\int_{Q_t^T} |\partial_t q_\gamma|^2 + C_3 \int_{Q_t^T} (|p_\gamma|^2 + 1)\,.$$

Moreover, (5.13) implies that the terms involving the terminal conditions are bounded by a constant $C_4 > 0$. Finally, we invoke (5.13) and the fact that $\partial_t \overline{w}_\gamma$ is uniformly bounded in $L^\infty(Q)$ to deduce the estimate

$$-\frac{2}{\theta_c}\int_{Q_t^T} F_2''(\overline{\varphi}_\gamma)\, p_\gamma^2 + \frac{1}{\theta_c^2}\int_{Q_t^T} \partial_t \overline{w}_\gamma\, F_2''(\overline{\varphi}_\gamma)\, p_\gamma^2 \leq C_5 \int_{Q_t^T} |p_\gamma|^2.$$

Collecting the above computations, and applying Gronwall's lemma, we infer that

$$\|p_\gamma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|q_\gamma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq C_6 \quad \forall\, \gamma \in (0,1]. \tag{5.15}$$

Second estimate: Next, we proceed with comparison in equation (5.2) to deduce that

$$\big\|\Delta\big(\alpha q_\gamma - \beta(1 \circledast q_\gamma)\big)\big\|_{L^2(0,T;H)} \le C_7 \quad \forall\, \gamma \in (0,1].$$

Then, setting $g_\gamma = \alpha q_\gamma - \beta(1 \circledast q_\gamma)$, the elliptic regularity theory entails that $\|g_\gamma\|_{L^2(0,T;W)} \le C_8$. Hence, solving the equation $\alpha q_\gamma - \beta(1 \circledast q_\gamma) = g_\gamma$ with respect to $1 \circledast q_\gamma$, we eventually obtain that

$$\|1 \circledast q_\gamma\|_{L^2(0,T;W)} + \|q_\gamma\|_{L^2(0,T;W)} \le C_9 \quad \forall\, \gamma \in (0,1]. \tag{5.16}$$

Third estimate: For the next estimate, we introduce the space

$$\mathcal{Q} = \{v \in H^1(0,T;V^*) \cap L^2(0,T;V) :\ v(0) = 0\}, \tag{5.17}$$

which is a closed subspace of $H^1(0,T;V^*) \cap L^2(0,T;V)$ and thus a Hilbert space. As is well known, $\mathcal{Q}$ is continuously embedded in $C^0([0,T];H)$, and we have the dense and continuous embeddings $\mathcal{Q} \subset L^2(0,T;H) \subset \mathcal{Q}^*$, where it is understood that

$$\langle v, w\rangle_{\mathcal{Q}} = \int_0^T (v(t), w(t))\, dt \quad \text{for all } w \in \mathcal{Q} \text{ and } v \in L^2(0,T;H). \tag{5.18}$$

Next, we recall the well-known integration-by-parts formula for functions in $H^1(0,T;V^*) \cap L^2(0,T;V)$, which yields that for all $v \in \mathcal{Q}$ it holds the estimate

$$\begin{aligned}
&\Big|\int_0^T \langle \partial_t p_\gamma(t), v(t)\rangle_V\, dt\Big| \le \Big|\int_0^T \langle \partial_t v(t), p_\gamma(t)\rangle_V\, dt\Big| + \Big|\int_\Omega p_\gamma(T)\, v(T) - \int_\Omega p_\gamma(0)\, v(0)\Big| \\
&\quad \le \|p_\gamma\|_{L^2(0,T;V)}\, \|\partial_t v\|_{L^2(0,T;V^*)} + \|p_\gamma(T)\|\, \|v(T)\| \\
&\quad \le C_{10}\, \|v\|_{H^1(0,T;V^*)} + C_{11}\, \|v\|_{C^0([0,T];H)} \le C_{12}\, \|v\|_{\mathcal{Q}},
\end{aligned} \tag{5.19}$$

where we used (5.13) and (5.15). This actually means that

$$\|\partial_t p_\gamma\|_{\mathcal{Q}^*} \le C_{13} \quad \forall\, \gamma \in (0,1]. \tag{5.20}$$

At this point, we can conclude from the estimates (5.13), (5.15), (5.16), and (5.20), using a comparison argument in (5.8), that the linear and continuous mapping

$$\Lambda_\gamma : \mathcal{Q} \to \mathbb{R}, \quad \langle \Lambda_\gamma, v\rangle_{\mathcal{Q}} := \int_Q F''_{1,\gamma}(\overline{\varphi}_\gamma)\, p_\gamma\, v\,, \tag{5.21}$$

satisfies

$$\|\Lambda_\gamma\|_{\mathcal{Q}^*} \le C_{14} \quad \forall\, \gamma \in (0,1]. \tag{5.22}$$

Consider now any sequence $\gamma_n \searrow 0$. According to Theorem 3.1 and Theorem 4.3, we may without loss of generality assume that the sequence $\{\overline{u}_{\gamma_n}\}$ converges strongly in $L^2(Q)$ to $\overline{u}$ and that the convergence properties (3.2)–(3.5) are satisfied with $(\varphi_{\gamma_n}, w_{\gamma_n})$ and (φ^0, w^0, ξ^0) replaced by $(\overline{\varphi}_{\gamma_n}, \overline{w}_{\gamma_n})$ and $(\overline{\varphi}, \overline{w}, \overline{\xi})$, respectively. By virtue of the estimates (5.15), (5.16), (5.20), and (5.22), we may also assume without loss of generality that, as $n \to \infty$,

$$p_{\gamma_n} \to p \quad \text{weakly-star in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{5.23}$$

$$q_{\gamma_n} \to q \quad \text{weakly-star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W), \tag{5.24}$$

$$1 \circledast q_{\gamma_n} \to 1 \circledast q \quad \text{weakly in } L^2(0,T;W), \tag{5.25}$$

$$\partial_t p_{\gamma_n} \to \partial_t p \quad \text{weakly in } \mathcal{Q}^*, \tag{5.26}$$

$$\Lambda_{\gamma_n} \to \Lambda \quad \text{weakly in } \mathcal{Q}^*, \tag{5.27}$$

for suitable limit points p, q, and Λ.

Then we perform a passage to the limit as $n \to \infty$ in the adjoint system (5.8)–(5.11), written for $\gamma = \gamma_n$ and $(p,q) = (p_{\gamma_n}, q_{\gamma_n})$, for $n \in \mathbb{N}$. At first, we recall that by (3.2) we have that $\overline{\varphi}_{\gamma_n} \to \overline{\varphi}$ strongly in $C^0(\overline{Q})$, and **(A2)** implies that, as $n \to \infty$,

$$F_2'(\overline{\varphi}_{\gamma_n}) \to F_2'(\overline{\varphi}) \quad \text{and} \quad F_2''(\overline{\varphi}_{\gamma_n}) \to F_2''(\overline{\varphi}), \quad \text{both strongly in } C^0(\overline{Q}). \tag{5.28}$$

From the convergence results stated above it is then readily seen that, as $n \to \infty$,

$$F_2'(\overline{\varphi}_{\gamma_n})\,\partial_t q_{\gamma_n} \to F_2'(\overline{\varphi})\,\partial_t q \quad \text{weakly in } L^2(Q), \tag{5.29}$$

$$F_2''(\overline{\varphi}_{\gamma_n})\,p_{\gamma_n} \to F_2''(\overline{\varphi})\,p \quad \text{weakly in } L^2(Q), \tag{5.30}$$

$$F_2'(\overline{\varphi}_{\gamma_n})\,p_{\gamma_n} \to F_2'(\overline{\varphi})\,p \quad \text{weakly in } L^2(Q). \tag{5.31}$$

Next, observe that by virtue of (3.4) and (3.5) we have that $\partial_t \overline{w}_{\gamma_n} \to \partial_t \overline{w}$ weakly-star in $H^1(0,T;H)$ $\cap\, L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q)$ and thus, by continuous embedding, also

$$\partial_t \overline{w}_{\gamma_n} \to \partial_t \overline{w} \quad \text{weakly in } C^0([0,T];V). \tag{5.32}$$

In addition, [25, Sect. 8, Cor. 4] implies that we also may assume that

$$\partial_t \overline{w}_{\gamma_n} \to \partial_t \overline{w} \quad \text{strongly in } C^0([0,T];L^\sigma(\Omega)) \quad \text{for } 1 \le \sigma < 6. \tag{5.33}$$

It then easily follows from (5.5), (5.10), and (5.11), that

$$f_{q_{\gamma_n}} \to f_q := k_3(1 \circledast (\overline{w} - w_Q)) + k_5(\partial_t \overline{w} - w_Q') + k_4(\overline{w}(T) - w_\Omega) \quad \text{weakly in } L^2(Q), \tag{5.34}$$

$$p_{\gamma_n}(T) \to p(T) = k_2(\overline{\varphi}(T) - \varphi_\Omega) - k_6 F_2'(\overline{\varphi}(T))(\partial_t \overline{w}(T) - w_\Omega') \quad \text{weakly in } L^2(\Omega), \tag{5.35}$$

$$q_{\gamma_n}(T) \to q(T) = k_6(\partial_t \overline{w}(T) - w_\Omega') \quad \text{weakly in } V. \tag{5.36}$$

Finally, we claim that also

$$\partial_t \overline{w}_{\gamma_n}\, F_2''(\overline{\varphi}_{\gamma_n})\, p_{\gamma_n} \to \partial_t \overline{w}\, F_2''(\overline{\varphi})\, p \quad \text{weakly in } L^2(Q). \tag{5.37}$$

Indeed, we have for every $v \in L^2(Q)$ the identity

$$\begin{aligned}
&\int_Q \big(\partial_t \overline{w}_{\gamma_n}\, F_2''(\overline{\varphi}_{\gamma_n})\, p_{\gamma_n} - \partial_t \overline{w}\, F_2''(\overline{\varphi})\, p\big)\, v \\
&= \int_Q \big(\partial_t \overline{w}_{\gamma_n} - \partial_t \overline{w}\big)\, F_2''(\overline{\varphi})\, p_{\gamma_n}\, v + \int_Q \partial_t \overline{w}_{\gamma_n} \big(F_2''(\overline{\varphi}_{\gamma_n}) - F_2''(\overline{\varphi})\big)\, p_{\gamma_n}\, v \\
&\quad + \int_Q \partial_t \overline{w}\, F_2''(\overline{\varphi})\, (p_{\gamma_n} - p)\, v \;=:\; I_{1n} + I_{2n} + I_{3n}\,,
\end{aligned}$$

with obvious notation. Since $\partial_t \overline{w}\, F_2''(\overline{\varphi})\, v \in L^2(Q)$, we have that $I_{3n} \to 0$ as $n \to \infty$. Moreover, the sequence $\{\partial_t \overline{w}_{\gamma_n}\, p_{\gamma_n}\, v\}$ is bounded in $L^1(Q)$ so that (5.28) implies that $I_{2n} \to 0$ as $n \to \infty$. Finally, using (5.13), (5.15), (5.33) with $\sigma = 4$, Hölder's inequality, and the continuous embedding $V \subset L^4(\Omega)$, we see that

$$\begin{aligned}
|I_{1n}| &\le C_{15} \int_0^T \|p_{\gamma_n}(t)\|_4\, \|\partial_t \overline{w}_{\gamma_n}(t) - \partial_t \overline{w}(t)\|_4\, \|v(t)\|\, dt \\
&\le C_{16}\, \|\partial_t \overline{w}_{\gamma_n} - \partial_t \overline{w}\|_{C^0([0,T];L^4(\Omega))}\, \|p_{\gamma_n}\|_{L^2(0,T;V)}\, \|v\|_{L^2(0,T;H)} \;\to\; 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which proves the validity of the claim (5.37).

Besides, for every $v \in \mathcal{Q}$,

$$\begin{aligned}\langle \partial_t p, v\rangle_{\mathcal{Q}} \,&=\, \lim_{n\to\infty} \langle \partial_t p_{\gamma_n}, v\rangle_{\mathcal{Q}} \,=\, \lim_{n\to\infty} \int_0^T \langle \partial_t p_{\gamma_n}(t), v(t)\rangle_V \, dt \\ &=\, \lim_{n\to\infty} \Big(\int_\Omega p_{\gamma_n}(T) v(T) \,-\, \int_0^T \langle \partial_t v(t), p_{\gamma_n}(t)\rangle_V \, dt \Big) \\ &=\, \int_\Omega p(T) v(T) \,-\, \int_Q p\, \partial_t v \,. \end{aligned} \tag{5.38}$$

At this point, we may pass to the limit as $n \to \infty$ in the adjoint system (5.8)–(5.11) to arrive at the following limit system:

$$\begin{aligned}\langle \Lambda, v\rangle_{\mathcal{Q}} \,&=\, -\int_Q p\, \partial_t v \,+\, \int_\Omega p(T)\, v(T) \,+\, \int_Q F_2'(\overline{\varphi})\, \partial_t q\, v - \int_Q \nabla p \cdot \nabla v \\ &\quad - \frac{2}{\theta_c} \int_Q F_2''(\overline{\varphi})\, p\, v \,+\, \frac{1}{\theta_c^2} \int_Q \partial_t \overline{w}\, F_2''(\overline{\varphi})\, p\, v \,+\, \int_Q k_1(\overline{\varphi} - \varphi_Q) \quad \text{for all } v \in \mathcal{Q}\,, \end{aligned} \tag{5.39}$$

$$\begin{aligned}&- \int_\Omega \partial_t q(t)\, v \,+\, \alpha \int_\Omega \nabla q(t) \cdot \nabla v \,-\, \int_\Omega \nabla (1 \circledast q(t)) \cdot \nabla v \\ &\quad - \frac{1}{\theta_c^2} \int_\Omega F_2'(\overline{\varphi}(t))\, p(t)\, v \,=\, \int_\Omega f_q(t)\, v \qquad \text{for all } v \in V \text{ and a.e. } t \in (0,T), \end{aligned} \tag{5.40}$$

$$p(T) \,=\, k_2(\overline{\varphi}(T) - \varphi_\Omega) - k_6 F_2'(\overline{\varphi}(T))(\partial_t \overline{w}(T) - w_\Omega') \quad \text{in } \Omega, \tag{5.41}$$

$$q(T) \,=\, k_6(\partial_t \overline{w}(T) - w_\Omega') \quad \text{in } \Omega, \tag{5.42}$$

where f_q is defined in (5.34).

Finally, we consider the variational inequality (5.12) for $\gamma = \gamma_n$, $n \in \mathbb{N}$. Passage to the limit as $n \to \infty$, using the above convergence results, yields that

$$\int_Q (q + \ell\, \overline{u})(v - \overline{u}) \,\geq\, 0 \quad \forall\, v \in \mathcal{U}_{\mathrm{ad}}. \tag{5.43}$$

Summarizing the above considerations, we have proved the following first-order necessary optimality conditions for the optimal control problem $(\mathbf{CP_0})$.

Theorem 5.1. *Suppose that the conditions* **(A1)–(A7)** *and* (2.14) *are fulfilled, and let* $\overline{u} \in \mathcal{U}_{\mathrm{ad}}$ *be a minimizer of the optimal control problem* $(\mathbf{CP_0})$ *with associate state* $(\overline{\varphi}, \overline{w}, \overline{\xi}) = \mathcal{S}_0(\overline{u})$. *Then there exist* p, q, *and* Λ *such that the following holds true:*

(i) $p \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $q \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W)$, $\Lambda \in \mathcal{Q}^*$.

(ii) *The adjoint system* (5.39)–(5.42) *and the variational inequality* (5.43) *are satisfied.*

Remark 5.2. (i) Observe that the adjoint state (p,q) and the Lagrange multiplier Λ are not unique. However, all possible choices satisfy (5.43), i.e., $\overline{u}$ is for $\ell > 0$ the $L^2(Q)$-orthogonal projection of $-\ell^{-1} q$ onto the closed and convex set $\{u \in L^\infty(Q) : u_* \leq u \leq u^* \text{ a.e. in } Q\}$, and

$$\overline{u}(x,t) = \max\big\{u_*(x,t), \min\{u^*(x,t), -\ell^{-1} q(x,t)\}\big\} \quad \text{for a.a. } (x,t) \in Q.$$

(ii) We have, for every $n \in \mathbb{N}$, the complementarity slackness condition (cf. (5.21))

$$\Lambda_{\gamma_n}(p_{\gamma_n}) = \int_Q F''_{1,\gamma_n}(\overline{\varphi}_{\gamma_n})\,|p_{\gamma_n}|^2 = \int_Q \frac{2\gamma_n}{1-\overline{\varphi}^2_{\gamma_n}}\,|p_{\gamma_n}|^2 \,\geq\, 0.$$

Unfortunately, our convergence properties for $\{\overline{\varphi}_{\gamma_n}\}$ and $\{p_{\gamma_n}\}$ do not permit a passage to the limit in this inequality to derive a corresponding result for $(\mathbf{CP_0})$.

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