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Abstract

We study the homogenization limit on bounded domains for the long-range random conductance model on stationary ergodic point processes on the integer grid. We assume that the conductance between neares neighbors in the point process are always positive and satisfy certain weight conditions. For our proof we use long-range two-scale convergence as well as methods from numerical analysis of finite volume methods.

1 Introduction

We consider a stationary ergodic point process $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^d$. We furthermore consider a random coefficient field

$$\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1] , \qquad (x,y) \mapsto \omega_{x,y} \tag{1.1}$$

where

$$x = y \text{ or } x \notin \mathbf{x} \text{ or } y \notin \mathbf{x} \quad \Rightarrow \quad \omega_{x,y} = 0 . \tag{1.2}$$

Furthermore we demand

$$\omega_{x,y} = \omega_{y,x} , \qquad 0 < \mathbb{E}\left(\sum_{z\in\mathbb{Z}^d} \omega_{0,z} \left|z\right|^2 \right) < \infty . \tag{1.3}$$

Given $\varepsilon > 0$ we consider the sets $\mathbf{x}^\varepsilon$, and the function $\omega^\varepsilon : \varepsilon\mathbb{Z}^d \times \varepsilon\mathbb{Z}^d \to \mathbb{R}$

$$\mathbf{x}^\varepsilon := \varepsilon\mathbf{x} = (\varepsilon x_i)_{i\in\mathbb{N}} = (x_i^\varepsilon)_{i\in\mathbb{N}} , \quad \omega^\varepsilon_{x,y} = \omega_{\frac{x}{\varepsilon},\frac{y}{\varepsilon}} .$$

Introducing the function spaces

$$\mathcal{S}^\varepsilon_{\mathbf{x}} := \{\mathbf{x}^\varepsilon \to \mathbb{R}\} \quad \text{and} \quad \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q}) := \{u \in \mathcal{S}^\varepsilon_{\mathbf{x}} : \ \forall x_i^\varepsilon \in \mathbf{x}^\varepsilon \backslash \boldsymbol{Q} \ \ u(x_i^\varepsilon) = 0\}$$

we write $u_i := u(x_i^\varepsilon)$ for every $u \in \mathcal{S}^\varepsilon_{\mathbf{x}}$ and introduce the linear operator on $\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$:

$$\forall x_i^\varepsilon \in \boldsymbol{Q} \cap \mathbf{x}^\varepsilon : \qquad \left(\mathcal{L}^\varepsilon_{\mathbf{x},\omega} u^\varepsilon\right)_i := \varepsilon^{-2} \sum_{j\neq i} \omega^\varepsilon_{x_j^\varepsilon, x_i^\varepsilon} \left(u_j - u_i\right) .$$

We are particularly interested in the limit behavior of the equation

$$-\mathcal{L}^\varepsilon_{\mathbf{x},\omega} u^\varepsilon = f^\varepsilon , \tag{1.4}$$

where $f^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ is a sequence that converges weakly in a sense to be specify below.

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The homogenization of problem (1.4) has been studied successfully first in [8] for $\mathbf{x} = \mathbb{Z}^d$. Writing $e_1, \dots e_d$ for the canonical basis of $\mathbb{R}^d$, in [8] some additional condition of the type

$$\mathbb{E}\left(\sum_i \omega_{0,e_i}^{-1}\right)^{\frac{d}{2}} < \infty \tag{1.5}$$

is needed. The condition imposed in [8] is more general, but reads similar. Recently, a more general result has been optained in [3] under the condition that for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ it holds

$$\mathbb{E}\left(\sum_{z\in\mathbb{Z}^d} \omega_{0,z} |z|^2\right)^p < \infty \qquad \text{and} \qquad \mathbb{E}\sum_{\substack{z\in\mathbb{Z}^d\\|z|=1}} \omega_{0,z}^{-q} < \infty\,. \tag{1.6}$$

Since all recent results work on $\mathbf{x} = \mathbb{Z}^d$ and with $\omega_{x,y} > 0$ for $|x-y| = 1$, our result is indeed new. Like in our previous work [8] we use stochastic two-scale methods developed in [8] to show that stationary ergodic point processes $\mathbf{x}$ in $\mathbb{Z}^d$ with weights $\omega_{x,y}$ satisfying (1.1)–(1.3) as well as the additional assumptions (1.8)–(1.10) lead to a homogenization result for (1.4). In this context we further use recent results from finite volume analysis [1, 5] to prove our compactness and uniform Poincaré inequalities in $\varepsilon > 0$. Furthermore, the proof that the support of $u \in \mathcal{S}_{\mathbf{x}}^\varepsilon(\boldsymbol{Q})$ – regarded as a function in $L^2(\boldsymbol{Q})$ – lies within a small ball around $\boldsymbol{Q}$ is inspired by recently developed ideas by the author for continuous homogenization [10, 11].

We note at this point that up to now only little is known on long-range interaction besides the two recent work [3, 8]. Another approach in terms of a random resistor network [7] was recently established by Faggionato. It however separates the edges inside $\boldsymbol{Q}$ and accounts only for interaction between points inside $\boldsymbol{Q}$ with thous outside $\boldsymbol{Q}$ but not for "inside –inside", e.g. nearest neighbor interaction.

1.1 Notation

We write $\mathbb{B}_R(x) := \{y \in \mathbb{R}^d : |x-y| < R\}$ for the open ball of radius R around $x \in \mathbb{R}^d$ and more general

$$\mathbb{B}_R(\boldsymbol{Q}) := \left\{y \in \mathbb{R}^d : \exists x \in \boldsymbol{Q} \text{ s.t. } |x-y| < R\right\}.$$

Given $\mathbf{x}^\varepsilon = \left(x_j^\varepsilon\right)_{j\in\mathbb{N}}$ we construct a Voronoi tessellation of cells G_j^ε with center x_j^ε and with mass $m_j^\varepsilon = \left|G_j^\varepsilon\right|$ (the Lebesgue measure) respectively. We furthermore use the following notations

$$\omega_{i,j} := \omega_{x_i,x_j}\,, \qquad \omega_{i,j}^\varepsilon := \omega_{x_i^\varepsilon,x_j^\varepsilon}^\varepsilon\,, \qquad \sum_i := \sum_{i\in\mathbb{N}}\,, \qquad \sum_{i,j} := \sum_{\substack{i,j\in\mathbb{N}\\i\neq j}}\,.$$

The numbers $\omega_{i,j}^\varepsilon$ give rise to the following semi-norm on $\mathcal{S}_{\mathbf{x}}^\varepsilon$:

$$\forall u \in \mathcal{S}_{\mathbf{x}}^\varepsilon : \qquad \lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\omega^\varepsilon} := \left(\varepsilon^{d-p}\sum_{i,j}\omega_{i,j}^\varepsilon\left(u_j - u_i\right)^p\right)^{\frac{1}{p}}$$

We will see below that under certain assumptions, $\lfloor u \rfloor_{2,\mathbf{x}^\varepsilon,\omega^\varepsilon}$ indeed is a norm on $\mathcal{S}_{\mathbf{x}}^\varepsilon(\boldsymbol{Q})$.

For every $\mathbf{x}$ and every $\varepsilon > 0$ as well as for positive numbers $(\alpha_i^\varepsilon)_{i\in\mathbb{N}}$ we find the scalar product $\langle\cdot,\cdot\rangle_{2,\mathbf{x}^\varepsilon,\alpha^\varepsilon}$ and the norm $\|\cdot\|_{p,\mathbf{x}^\varepsilon,\alpha^\varepsilon}$ on $\mathcal{S}_{\mathbf{x}}^\varepsilon$ given by

$$\langle u,v\rangle_{2,\mathbf{x}^\varepsilon,\alpha^\varepsilon} := \varepsilon^d \sum_{x_i\in\mathbf{x}^\varepsilon}\alpha_i^\varepsilon u_i v_i\,, \qquad \|u\|_{p,\mathbf{x}^\varepsilon,\alpha^\varepsilon} := \left(\varepsilon^d\sum_{x_i\in\mathbf{x}^\varepsilon}\alpha_i^\varepsilon u_i^p\right)^{\frac{1}{p}}.$$

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Typical examples are the choices

$$\alpha_i^\varepsilon \equiv 1 \qquad \text{or} \qquad \alpha_i^\varepsilon = m_i^\varepsilon \,. \tag{1.7}$$

When an additional time variable is involved, we write

$$\langle u,v\rangle_{t_1,t_2,2,\mathbf{x}^\varepsilon,\alpha^\varepsilon} := \int_{t_1}^{t_2} \langle u(t),\, v(t)\rangle_{2,\mathbf{x}^\varepsilon,\alpha^\varepsilon}\,\mathrm{d}t\,, \qquad \|u\|_{t_1,t_2,2,\mathbf{x}^\varepsilon,\alpha^\varepsilon} := \langle u,u\rangle^{\frac12}_{t_1,t_2,2,\mathbf{x}^\varepsilon,\alpha^\varepsilon}\,,$$
$$L^2(0,T;\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})) := \left\{u:[0,T]\to\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})\,:\,\|u\|_{t_1,t_2,2,\mathbf{x}^\varepsilon,m^\varepsilon}<\infty\right\}.$$

In order to formulate convergence results of u^ε and U^ε, we need a family of injective maps from $\mathcal{S}^\varepsilon_{\mathbf{x}}$ onto $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. We define $\mathcal{R}_{\varepsilon,\mathbf{x}}:\, L^2_{\mathrm{loc}}(\mathbb{R}^d)\to\mathcal{S}^\varepsilon_{\mathbf{x}}$ and its adjoint $\mathcal{R}^*_{\varepsilon,\mathbf{x}}:\mathcal{S}^\varepsilon_{\mathbf{x}}\to L^2_{\mathrm{loc}}(\mathbb{R}^d)$ through

$$\left(\mathcal{R}_{\varepsilon,\mathbf{x}}\phi\right)_i = |G_i^\varepsilon|^{-1}\int_{G_i^\varepsilon}\phi\,, \qquad \text{and} \qquad \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}u\right)[x] = u(x_i^\varepsilon) \qquad \text{if} \quad x\in G_i^\varepsilon\,.$$

Again, we drop the index $\mathbf{x}$ if no confusion is possible.

The above (semi-) norms as well as $\mathcal{R}^*_{\varepsilon,\mathbf{x}}$ can be restricted to $\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ using $u_{x_i}=0$ for $x_i\notin\mathbf{x}^\varepsilon\cap\boldsymbol{Q}$.

1.2 Our setting

We say that the Voronoi cells G_i^ε and G_j^ε are neighbored if the Hausdorff measure $m^\varepsilon_{i,j} := \left|\partial G_i^\varepsilon\cap\partial G_j^\varepsilon\right|$ is positive. We write $i\sim j$ if the cells G_i^ε and G_j^ε are neighbored and

$$\mathcal{N}(x_i^\varepsilon,\mathbf{x}^\varepsilon) := \left\{x_j^\varepsilon\in\mathbf{x}^\varepsilon\,:\,i\sim j\right\} \quad \text{with} \quad \mathcal{N}(x_i^\varepsilon,\mathbf{x}^\varepsilon) := \varepsilon\mathcal{N}(\frac{x_i^\varepsilon}{\varepsilon},\mathbf{x})\,. \tag{1.8}$$

Two crucial assumptions for the existence of a Rellich-Sobolev inequality and a suitable compact embedding are given by

$$f_{\mathfrak{d}}(R) < R^{-\beta_{\mathfrak{d}}},\ \beta_{\mathfrak{d}} > d+1\,, \qquad \text{where } f_{\mathfrak{d}}(R) := \frac{1}{2d}\mathbb{P}(\mathbb{B}_R(0)\cap\mathbf{x}=\emptyset) \tag{1.9}$$

and for some $p\in\left(\frac{2d}{d+2},\,2\right)$ it holds

$$\mathbb{E}\left(\sum_{z\in\mathcal{N}(0,\mathbf{x})}\left(\omega_{0,z}\,|z|^2\right)^{-\frac{p}{2-p}}\left(m_{0,z}\,|z|\right)^{\frac{2}{2-p}}\;:\;0\in\mathbf{x}\right)<\infty\,. \tag{1.10}$$

Remark 1.1*.* Condition (1.9) is new compared to [8, 3] and is solely due to the fact that $\mathbf{x}\neq\mathbb{Z}^d$.

Example 1.2. If $\mathbb{P}(x\in\mathbf{x}) = p_0\in(0,1)$ is distributed i.i.d. among all $x\in\mathbb{Z}^d$ it is easy to see that

$$f_{\mathfrak{d}}(R) < C\,\exp(-R^d)$$

for some $C>0$ depending on p_0.

Remark 1.3*.* Condition (1.10) implies that (1.10) also holds for the case $p=1$. It will be used to prove strict positivity of the homogenized matrix, while the other case is used to prove a Rellich-type theorem (e.g. Theorem 3.4). Furthermore, condition (1.10) can be understood as a generalization of (1.5): to see this, it is sufficient to remark that $|m_{0,z}| = |z| = 1$ in case $\mathbf{x}=\mathbb{Z}^d$ and to plug in $p=\frac{2d}{d+2}$. However, in this particular case our setting is less general than (1.6) which illustrates that our condition (1.10) still could be improved.

We make a few adjustments to the notation in order to account for the special structure of $\mathbb{Z}^d$ in accordance with the literature [8, 9]. First, we introduce in the following discrete derivatives on $\mathbf{x}$.

Definition 1.4 (Discrete derivatives)**.** For $u\colon \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ we define the ε-forward derivative in the direction $z \in \mathbb{Z}^d$ by

$$\partial^\varepsilon_z u(x) = \varepsilon^{-1}\left(u(x+\varepsilon z) - u(x)\right)\,, \tag{1.11}$$

and the analogous backward derivative,

$$\partial^{\varepsilon-}_z u(x) = \varepsilon^{-1}\left(u(x) - u(x-\varepsilon z)\right)\,. \tag{1.12}$$

Further, we define $\nabla^\varepsilon u(x,z) := \partial^\varepsilon_z u(x)$ and write $\nabla^\varepsilon u(x)$ for the function that maps $z \in \mathbb{Z}^d$ to $\nabla^\varepsilon u(x,z)$. Accordingly, we define $\nabla^{\varepsilon-} u(x,z) := \partial^{\varepsilon-}_z u(x)$ and $\nabla^{\varepsilon-} u(x)$. Moreover, for a function $v : \mathbb{Z}^d_\varepsilon \times \mathbb{Z}^d \to \mathbb{R}$ we define

$$\mathrm{div}^\varepsilon v(x) = \sum_{z\in\mathbb{Z}^d} \partial^{\varepsilon-}_z v(x,z)\,. \tag{1.13}$$

We use this notation to clearly distinguish between ∇^ε, an operator on discrete functions, and ∇, an operator on the Sobolev space $H^1\left(\mathbb{R}^d\right)$. A direct calculation shows that when $A^\varepsilon_{\mathbf{x},\omega}$ maps $v(x,z) \mapsto \omega_{\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z} v(x,z)$, then

$$-\mathcal{L}^\varepsilon_{\mathbf{x},\omega} u^\varepsilon = -\frac{1}{2}\,\mathrm{div}^\varepsilon\left(A^\varepsilon_{\mathbf{x},\omega}\nabla^\varepsilon u^\varepsilon\right)\,. \tag{1.14}$$

Moreover, for $v^\varepsilon\colon \varepsilon\mathbb{Z}^d \to \mathbb{R}$ we observe that

$$\left\langle u^\varepsilon, v^\varepsilon\right\rangle_{\mathbf{x},\varepsilon,\omega} = \frac{\varepsilon^d}{2}\sum_{x\in\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\omega_{x,x+z}\Big(\partial^\varepsilon_z u^\varepsilon(\varepsilon x)\Big)\Big(\partial^\varepsilon_z v^\varepsilon(\varepsilon x)\Big)\,. \tag{1.15}$$

Theorem 1.5. *Let $\mathbf{x}$ a stationary ergodic point process $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ with points solely in $\mathbb{Z}^d$ with ω satisfying* (1.1)–(1.3) *and* (1.9)–(1.10)*. Then almost surely the following properties are satisfied by $\mathbf{x}$, ω and $\mathcal{L}^\varepsilon_{\mathbf{x},\omega}$ and A_{hom} given by* (4.13) *below:*

1 *For some $c > 0$ it holds*

$$\forall x \in \boldsymbol{Q}: \qquad c\left|\xi\right|^2 \le \xi\cdot A_{\mathrm{hom}}(x)\xi \le c^{-1}\left|\xi\right|^2$$

and $\mathcal{L}^\varepsilon_{\mathbf{x},\omega}$ weakly G-converges to $u \mapsto \nabla\cdot A_{\mathrm{hom}}\nabla u$ in the following sense: If $f^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ is a sequence and $f \in L^2(\boldsymbol{Q})$ such that $\mathcal{R}^*_{\varepsilon,\mathbf{x}} f^\varepsilon \rightharpoonup f$ weakly in $L^2(\mathbb{R}^d)$ and if $u^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ is the solution to

$$\forall x^\varepsilon_i \in \boldsymbol{Q}\cap\mathbf{x}^\varepsilon: \qquad -\left(\mathcal{L}^\varepsilon_{\mathbf{x},\omega}u\right)_i = m^\varepsilon_i f^\varepsilon_i\,,$$

then there exists a unique $u \in H^1_0(\boldsymbol{Q})$ such that $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon \to u$ strongly in $L^2(\boldsymbol{Q})$ as $\varepsilon \to 0$ and u is the solution to

$$-\nabla\cdot\left(A_{\mathrm{hom}}\nabla u\right) = f \quad \text{in } \boldsymbol{Q} \qquad \text{with } u|_{\partial\boldsymbol{Q}} \equiv 0\,.$$

2 *There exists $\beta \in (0,1)$ such that for every $u \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ it holds $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}}u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$. Furthermore it holds $\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi \to \phi$ strongly in $L^2(\mathbb{R}^d)$ for every $\phi \in L^2(\boldsymbol{Q})$.*

3 *There exists a constant $C > 0$ such that for every $\varepsilon > 0$ with $\alpha^\varepsilon_i = m^\varepsilon_{x^\varepsilon_i}$ it holds*

$$\forall U \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q}): \qquad \left\|U\right\|_{2,\mathbf{x}^\varepsilon,m^\varepsilon} \le C\left\lfloor U\right\rfloor_{2,\mathbf{x}^\varepsilon,\omega^\varepsilon}$$

and boundedness of $\varepsilon^{d-2}\sum_{j\neq i}\omega^\varepsilon_{i,j}\left(u_j - u_i\right)^2$ implies precompactness of $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u$ in $L^2(\mathbb{R}^d)$.

The proof of Theorem 1.5 will be given at the end of Section 4.4.

Remark 1.6. The most surprising part of Theorem 1.5 is probably part 2., i.e. $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}}u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$ instead of a result $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}}u \subset \mathbb{B}_{C\varepsilon}(\boldsymbol{Q})$ for some $C > 0$. The reason for that is that Voronoi cells G_i^ε in general might become arbitrary large, even for small ε. However, it is "very unlikely" that their diameter becomes larger than $C\varepsilon^\beta$. We highlight at this point that $\beta \in (0,1)$ implies $\varepsilon^\beta \gg \varepsilon$ as $\varepsilon \to 0$ but still $\varepsilon^\beta \to 0$.

2 Ergodic Theorems

By construction, the probability space is given by

$$\Omega = \{0,1\}^{\mathbb{Z}^d} \times [0,+\infty]^{\mathbb{Z}^d\times\mathbb{Z}^d} .$$

and hence Ω with the product topology is a compact metric space (note that $[0,+\infty]$ with the topology of the half circle is compact). Furthermore, we make the following assumption throughout this work.

Assumption 2.1. *The distribution $\mathbb{P}$ on Ω is stationary, that is: there exists a family $(\tau_x)_{x\in\mathbb{Z}^d}$ of measurable bijective mappings $\tau_x : \Omega \mapsto \Omega$, having the properties of a* dynamical system *on $(\Omega, \mathscr{F}, \mathcal{P})$, i.e. they satisfy (i)-(iii):*

(i) $\tau_x \circ \tau_y = \tau_{x+y}$, $\tau_0 = id$ *(Group property)*

(ii) $\mathcal{P}(\tau_{-x}B) = \mathcal{P}(B) \quad \forall x \in \mathbb{R}^d$, $B \subset \Omega$ *measurable (Measure preserving)*

(iii) $A:\ \mathbb{Z}^d \times \Omega \to \Omega \qquad (x,\omega) \mapsto \tau_x\omega$ *is measurable (Measurability of evaluation)*

We finally assume that the system $(\tau_x)_{x\in\mathbb{Z}^d}$ is ergodic. This means that for every measurable function $f : \Omega \to \mathbb{R}$ there holds

$$\left[f(\omega) = f(\tau_x\omega)\ \forall x \in \mathbb{Z}^d,\ a.e.\ \omega \in \Omega\right] \Rightarrow \left[f(\omega) = const \text{ for } \mathcal{P}-a.e.\ \omega \in \Omega\right] . \tag{2.1}$$

The major use of stationary and ergodicity are ergodic theorems:

Theorem 2.2 (Theorem 5.3 of [8])**.** *For every $f \in L^1(\Omega,\mathbb{P})$, for $\mathbb{P}$-almost every $\omega \in \Omega$ the following holds: Let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions from $\varepsilon\mathbb{Z}^d \to \mathbb{R}$ with support in $\boldsymbol{Q}_\varepsilon$ such that $\mathcal{R}^*_{\varepsilon,\mathbb{Z}^d}u^\varepsilon \to u$ pointwise a.e. in $\boldsymbol{Q}$. Furthermore, let $\sup_{\varepsilon>0}\|u^\varepsilon\|_\infty < \infty$. Then $u \in L^\infty(Q)$ and*

$$\lim_{\varepsilon\to0}\varepsilon^d \sum_{x\in\boldsymbol{Q}_\varepsilon} u^\varepsilon(x) f\left(\tau_{\frac{x}{\varepsilon}}\omega\right) = \mathbb{E}[f]\int_{\boldsymbol{Q}} u(x)\mathrm{d}x \tag{2.2}$$

and the Null-set depends on f but not on the sequence u^ε.

The last theorem can be generalized to our setting.

Theorem 2.3. *For every $f \in L^2(\Omega,\mathbb{P})$, for $\mathbb{P}$-almost every $\omega \in \Omega$ the following holds: Let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions from $\mathbf{x}_\varepsilon \to \mathbb{R}$ with support in $\boldsymbol{Q}_\varepsilon$ such that $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon \to u$ in $L^2(\boldsymbol{Q})$. Then*

$$\lim_{\varepsilon\to0}\varepsilon^d \sum_{x\in\boldsymbol{Q}_\varepsilon} u^\varepsilon(x) f\left(\tau_{\frac{x}{\varepsilon}}\omega\right) = \mathbb{E}[f]\int_{\boldsymbol{Q}} u(x)\mathrm{d}x$$

and the Null-set depends on f but not on the sequence u^ε.

Proof. Applying (2.2) to f^2 and $u = 1$ and $u^\varepsilon = 1$ we infer boundedness of $\mathcal{R}^*_{\varepsilon,\mathbf{x}} f\left(\tau_{\frac{x}{\varepsilon}}\omega\right)^2$ in $L^2(\boldsymbol{Q})$. Applying Theorem 2.2 once more to $u \in C_c(\boldsymbol{Q})$ with $u^\varepsilon(x) = u(x)$ and f implies $\mathcal{R}^*_{\varepsilon,\mathbf{x}} f\left(\tau_{\frac{x}{\varepsilon}}\omega\right) \rightharpoonup \mathbb{E}[f]$ weakly in $L^2(\boldsymbol{Q})$. The claim follows from

$$\varepsilon^d \sum_{x\in\boldsymbol{Q}_\varepsilon} u^\varepsilon(x) f\left(\tau_{\frac{x}{\varepsilon}}\omega\right) = \int_{\boldsymbol{Q}} \mathcal{R}^*_{\varepsilon,\mathbf{x}} f\left(\tau_{\frac{x}{\varepsilon}}\omega\right) \mathcal{R}^*_{\varepsilon,\mathbf{x}} u^\varepsilon(x)\,\mathrm{d}x \to \int_{\boldsymbol{Q}} \mathbb{E}[f]\,u\,.$$

□

3 Properties of $\mathcal{S}^\varepsilon_{\mathbf{x}}$

In this section we provide some fundamental properties of functions in $\mathcal{S}^\varepsilon_{\mathbf{x}}$, particularly a Poincaré inequality and a compact embedding result. For this we will use results from numerical analysis. Furthermore, we will show that the support of functions in $\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ lies almost surely within a bounded region around $\boldsymbol{Q}$ while the support decreases towards $\boldsymbol{Q}$ as $\varepsilon \to 0$. This will imply for every $\phi \in L^2(\boldsymbol{Q})$ that $\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi \to \phi$ strongly in $L^2(\boldsymbol{Q})$ as $\varepsilon \to 0$.

3.1 Support of $\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$

Lemma 3.1. *Let $\mathbf{x}$ be a stationary point process in $\mathbb{Z}^d$ with $f_{\mathfrak{d}}$ given in* (1.9)*. Then, if $\boldsymbol{G} := (G_i)_{i\in\mathbb{N}}$ is the Voronoi tessellation for $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ with maximal diameter*

$$\mathfrak{d}(x_i) := \max_{x,y\in G_i} |x-y|\,, \tag{3.1}$$

then

$$\mathbb{P}(\mathfrak{d} > D) < f_{\mathfrak{d}}\left(\frac{1}{6}D\right) \tag{3.2}$$

Proof. We define for a unit vector ν of unit length, $0 < \alpha < \frac{\pi}{2}$ and $R > 0$ the cone

$$\mathbb{C}_{\nu,\alpha,R}(x) := \{z \in \mathbb{B}_R(x)\,:\, z\cdot\nu > |z|\cos\alpha\}\,.$$

Because of the stationarity and because of $\mathbb{P}(A\cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ it holds for $R \in \mathbb{Z}$ and $\mathbf{E} := \{e_1,\dots e_d\} \cup \{-e_1,\dots - e_d\}$ ($\{e_1,\dots e_d\}$ being the canonical basis of $\mathbb{R}^d$)

$$\mathbb{P}(\exists e \in \mathbf{E}\,:\, \mathbb{B}_R(2Re)\cap\mathbf{x} = \emptyset) \le \sum_{i=1}^d\sum_{\pm}\mathbb{P}((\mathbb{B}_R(\pm 2Re_i)\cap\mathbf{x} = \emptyset) \le f_{\mathfrak{d}}(R)\,.$$

In particular, for $\alpha = \arctan\sqrt{1/3} = \frac{\pi}{6}$ we have the smallest opening angle such that $\mathbb{B}_R(2Re)$ lies completely inside $\mathbb{C}_{e,\alpha,3R}(0)$ and we discover

$$\mathbb{P}\Big(\forall e \in \mathbf{E}:\ \mathbf{x}\cap\mathbb{C}_{e,\frac{\pi}{6},3R}(0) \neq \emptyset\Big) \ge 1 - f_{\mathfrak{d}}(R)\,. \tag{3.3}$$

Now we take arbitrary points $x_{\pm j} \in C_{\pm e_j,\alpha,3R}(0)\cap\mathbf{x}$. Then the planes given by the respective equations $\left(x - \frac{1}{2}x_{\pm j}\right)\cdot x_{\pm j} = 0$ define a bounded cell around 0, with a maximal diameter $D(\alpha,R) = CR$ which is proportional to R. The constant $C > 1$ depends solely on the opening angle $\alpha = \frac{\pi}{6}$ of the cones and can be shown from some trigonometric calculations to be smaller than 6. Estimate (3.2) now follows from

$$\mathbb{P}(\mathfrak{d} > D) = \mathbb{P}(\mathfrak{d} > CR) \le \mathbb{P}\left(\exists e \in \mathbf{E}\,:\, \mathbf{x}\cap\mathbb{C}_{e,\alpha,3R}(0) = \emptyset\right) \le f_{\mathfrak{d}}(R) = f_{\mathfrak{d}}(\frac{1}{6}D)\,.$$

□

Lemma 3.2. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f_\mathfrak{d}$ satisfy* (1.9)*. Then for every $\beta \in (0, 1 - \frac{d}{\beta_\mathfrak{d}})$ there exists almost surely $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ and every $u \in \mathcal{S}^\varepsilon_\mathbf{x}(\boldsymbol{Q})$ it holds $\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}}u \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$. Furthermore, for a given bounded Lipschitz domain $\boldsymbol{Q}$ we define*

$$\mathcal{N}(\boldsymbol{Q}, \mathbf{x}^\varepsilon) := \{x \in \mathbf{x}^\varepsilon \backslash \boldsymbol{Q} : \mathcal{N}(x) \cap \boldsymbol{Q} \neq \varnothing\} \, . \tag{3.4}$$

Then there exists almost surely $\varepsilon_0 > 0$ and $\beta \in (0,1)$ such that for every $\varepsilon_0 > \varepsilon$ it holds $\mathcal{N}(\boldsymbol{Q}, \mathbf{x}^\varepsilon) \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q})$.

Proof. Let $u^\varepsilon_1(x) = 1$ if $x \in \boldsymbol{Q} \cap \mathbf{x}^\varepsilon$ and $u^\varepsilon_1(x) = 0$ else. Given $N := \varepsilon^{-1}$, $\beta_0 = 1 - \beta$ the event

$$B_N := \left(\bigcup_{x_i \in \mathbf{x} \cap N\boldsymbol{Q}} G_i \subset \mathbb{B}_{N^{\beta_0}}(N\boldsymbol{Q}) \right)$$

is equivalent with the event

$$\mathrm{supp}\mathcal{R}^*_{\varepsilon,\mathbf{x}} u^\varepsilon_1 \subset \mathbb{B}_{\varepsilon^\beta}(\boldsymbol{Q}) \, .$$

For the complementary event $\neg B_N$ of B_N it holds

$$\begin{aligned} \mathbb{P}(\neg B_N) &\leq \mathbb{P}(\exists x_i \in \mathbf{x} \cap N\boldsymbol{Q} : \mathbb{B}_{\mathfrak{d}_i}(x_i) \not\subset \mathbb{B}_{N^{\beta_0}}(N\boldsymbol{Q})) \\ &\leq \sum_{\mathbb{Z}^d \cap N\boldsymbol{Q}} \mathbb{P}\big(\mathfrak{d} \geq N^{\beta_0}\big) \leq C \, |\boldsymbol{Q}| \, N^d f_\mathfrak{d}\big(N^{\beta_0}\big) \\ &\leq C N^{d - \beta_0 \beta_\mathfrak{d}} \, . \end{aligned}$$

If $\beta_0 \in (\frac{d}{\beta_\mathfrak{d}}, 1)$ it holds $N^{d-\beta_0\beta_\mathfrak{d}} \to 0$ as $N \to \infty$ and hence for almost every ω there exists N_0 such that $\omega \in B_N$ for every $N > N_0$ and the first statement of the lemma holds.

The second statement can be proved similarly taking into account that every $x \in \mathcal{N}(N\boldsymbol{Q}, \mathbf{x})$ satisfies $x \in \mathbb{B}_{2\mathfrak{d}_i}(x_i)$ for some $x_i \in \mathbf{x} \cap N\boldsymbol{Q}$. □

Lemma 3.3. *Let $\boldsymbol{Q} \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f_\mathfrak{d}$ satisfy* (1.9)*. Then almost surely for every $\phi \in L^2(\boldsymbol{Q})$ it holds $\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi \to \phi$ as $\varepsilon \to 0$.*

Proof. Let $\tilde{\boldsymbol{Q}} \supset \mathbb{B}_1(\boldsymbol{Q})$ be a large ball that contains 0. Given $\phi \in C^1_c(\tilde{\boldsymbol{Q}})$ and using the notation (3.1) we find

$$\begin{aligned} \int_{\mathbb{R}^d} \big(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi - \phi\big)^2 &\leq \sum_{x_i \in \mathbf{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \left(\varepsilon \mathfrak{d}(x_i) \, \|\nabla\phi\|_\infty\right)^2 \varepsilon^d \, |G_i| \\ &\leq \varepsilon^{2+d} \, \|\nabla\phi\|^2_\infty \sum_{x_i \in \mathbf{x} \cap \varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(x_i)^2 \, |G_i| \, . \end{aligned}$$

Because of Lemma 3.2 we know that almost surely for ε_0 independent from ϕ and every $\varepsilon < \varepsilon_0$ it holds $\varepsilon\mathfrak{d}(x_i) < \mathrm{diam}\tilde{\boldsymbol{Q}} + 1$ for every $\varepsilon x_i \in \tilde{\boldsymbol{Q}}$. Hence for every $D > 1$ we find

$$\begin{aligned}
\varepsilon^{2+d} \sum_{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(x_i)^2 |G_i| &= \varepsilon^{2+d} \sum_{\substack{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}\\ \mathfrak{d}(x_i)\le D}} \mathfrak{d}(x_i)^2 |G_i| + \varepsilon^d \sum_{\substack{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}\\ \mathfrak{d}(x_i)> D}} |\varepsilon\mathfrak{d}(x_i)|^2 |G_i| \\
&\le \varepsilon^2 \left|\tilde{\boldsymbol{Q}}\right| D^2 + \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 \sum_{k=0}^{\infty} \sum_{\substack{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}\\ D+k<\mathfrak{d}(x_i)<D+k+1}} \mathfrak{d}(x_i)^d \\
&\le \varepsilon^2 \left|\tilde{\boldsymbol{Q}}\right| D^2 + \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 \sum_{k=0}^{\infty} \sum_{\substack{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}\\ D+k<\mathfrak{d}(x_i)<D+k+1}} (D+k+1)^d \\
&\to \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 \sum_{k=0}^{\infty} (D+k+1)^d \mathbb{P}(D+k < \mathfrak{d}(\cdot) < D+k+1) \\
&\le 2^d \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 \sum_{k=0}^{\infty} (D+k)^d f_{\mathfrak{d}}(\frac{1}{6}(D+k)) \\
&\le 2^d \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 \left(\frac{1}{6}\right)^{\beta_{\mathfrak{d}}} \sum_{k=0}^{\infty} (D+k)^{d-\beta_{\mathfrak{d}}} .
\end{aligned}$$

Since $\beta_{\mathfrak{d}}(D) > d+1$ it follows

$$\lim_{\varepsilon\to0} \varepsilon^{2+d} \sum_{x_i\in\mathbf{x}\cap\varepsilon^{-1}\tilde{\boldsymbol{Q}}} \mathfrak{d}(x_i)^2 |G_i| \le 2^d \left(\frac{1}{6}\right)^{\beta_{\mathfrak{d}}} \left(\mathrm{diam}\tilde{\boldsymbol{Q}} + 1\right)^2 D^{d+1-\beta_{\mathfrak{d}}} \to 0$$

as $D \to \infty$ and we obtain that

$$\lim_{\varepsilon\to0} \int_{\mathbb{R}^d} \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi - \phi\right)^2 \le 0 . \tag{3.5}$$

Furthermore for every $\phi \in L^2(\boldsymbol{Q})$ it holds

$$\int_{\mathbb{R}^d} \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi\right)^2 = \sum_i \int_{G_i^\varepsilon} \left(\frac{1}{m_i^\varepsilon}\int_{G_i^\varepsilon}\phi\right)^2 \le \sum_i \int_{G_i^\varepsilon} \frac{1}{m_i^\varepsilon}\int_{G_i^\varepsilon}\phi^2 = \int_{\boldsymbol{Q}} \phi^2 . \tag{3.6}$$

Now let $\phi \in L^2(\boldsymbol{Q})$ and let $(\phi_k)_{k\in\mathbb{N}} \subset C^1_c(\boldsymbol{Q})$ be a sequence with $\|\phi - \phi_k\|_{L^2(\boldsymbol{Q})} < \frac{1}{k}$. Given $\delta > 0$ we find

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi - \phi\right)^2\right)^{\frac12} &\le \|\phi - \phi_k\|_{L^2(\boldsymbol{Q})} + \left\|\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\left(\phi - \phi_k\right)\right\|_{L^2(\boldsymbol{Q})} \\
&\quad + \left(\int_{\mathbb{R}^d} \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi_k - \phi_k\right)^2\right)^{\frac12} .
\end{aligned}$$

We chose $k \in \mathbb{N}$ such that $\|\phi - \phi_k\|_{L^2(\boldsymbol{Q})} \le \frac13\delta$ and with help of (3.5) we choose ε_0 such that for every $\varepsilon < \varepsilon_0$ it holds $\left\|\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi_k - \phi_k\right\|_{L^2(\boldsymbol{Q})} < \frac13\delta$. Due to (3.6) it also holds $\left\|\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\left(\phi - \phi_k\right)\right\|_{L^2(\boldsymbol{Q})} < \frac13\delta$. Then in total for every $\varepsilon < \varepsilon_0$ it holds

$$\left(\int_{\mathbb{R}^d} \left(\mathcal{R}^*_{\varepsilon,\mathbf{x}}\mathcal{R}_{\varepsilon,\mathbf{x}}\phi - \phi\right)^2\right)^{\frac12} < \delta .$$

□

3.2 Rellich-type theorem for discrete functions

We recall the notation introduced in Section 1.1. Furthermore, we write $\sum_{i\sim j}$ for the sum over all pairs of neighbors in $\mathbf{x}^\varepsilon$. Then the (semi-) norms on $\mathcal{S}_{\mathbf{x}}^\varepsilon$ we will work with read

$$\begin{aligned}
\|u\|_{p,\mathbf{x}^\varepsilon,m^\varepsilon} &:= \left(\varepsilon^d \sum_i m_i^\varepsilon |u_i|^p\right)^{\frac{1}{p}} \\
\lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\gamma^\varepsilon} &:= \left(\varepsilon^{d-p} \sum_{i\sim j} \gamma_{i,j}^\varepsilon |u_j - u_i|^p\right)^{\frac{1}{p}}, \qquad \text{where } \gamma_{i,j}^\varepsilon := \frac{\varepsilon^{p-d} m_{i,j}^\varepsilon}{\left|x_i^\varepsilon - x_j^\varepsilon\right|^{p-1}} \\
\|u\|_{p,\mathbf{x}^\varepsilon,m^\varepsilon,\gamma^\varepsilon} &:= \|u\|_{\mathbf{x},0,p,\varepsilon} + \lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\gamma^\varepsilon} .
\end{aligned}$$

Theorem 3.4 (Asymptotic Rellich-Sobolev Theorem for $\mathbf{x}$ and ω)**.** *Let $\mathbf{x}$ a stationary ergodic point process $\mathbf{x} = (x_i)_{i\in\mathbb{N}}$ with points solely in $\mathbb{Z}^d$ with ω satisfying* (1.1)–(1.2) *and* (1.9)–(1.10)*. Then there almost surely exists a constant $C_\omega > 0$ such that for every $\varepsilon > 0$ and every $u^\varepsilon \in \mathcal{S}_{\mathbf{x}}^\varepsilon(\boldsymbol{Q})$ it holds*

$$\|u^\varepsilon\|_{2,\mathbf{x}^\varepsilon,m^\varepsilon} \le C_\omega \lfloor u^\varepsilon \rfloor_{2,\mathbf{x}^\varepsilon,\omega^\varepsilon} . \tag{3.7}$$

Furthermore, any sequence $u_k^{\varepsilon_k} \in \mathcal{S}_{\mathbf{x}}^{\varepsilon_k}(\boldsymbol{Q})$, $k \in \mathbb{N}$, with $\sup_k \lfloor u_k^{\varepsilon_k} \rfloor_{2,\mathbf{x}^{\varepsilon_k},\omega^{\varepsilon_k}} < \infty$ is precompact in the sense that $\mathcal{R}_{\varepsilon_k,\mathbf{x}}^* u_k^{\varepsilon_k}$ is precompact in $L^2(\mathbb{R}^d)$.

Proof. Due to Lemma 3.2 we can assume w.l.o.g that for $\varepsilon > 0$ small enough it holds $\operatorname{supp}\mathcal{R}_{\varepsilon,\mathbf{x}}^* u \subset \mathbb{B}_1(\boldsymbol{Q})$ for every $u \in \mathcal{S}_{\mathbf{x}}^\varepsilon$. Provided $p > \frac{2d}{d+2}$ and $\delta > 0$ small enough we infer from the discrete Sobolev-Poincaré inequality in Theorem 4.3 of [1] that for some constant $C > 0$ depending only on p, d, δ and $\boldsymbol{Q}$

$$\|u\|_{p,\mathbf{x}^\varepsilon,m^\varepsilon} \le \|u\|_{2+\delta,\mathbf{x}^\varepsilon,m^\varepsilon} \le C \lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\gamma^\varepsilon} . \tag{3.8}$$

Using the discrete Gagliardo-Nirenberg-Sobolev inequality of Theorem 3.4 [1] we furthermore infer the existence of $C > 0$ depending only on p, d, δ and $\boldsymbol{Q}$ such that

$$\|u\|_{2,\mathbf{x}^\varepsilon,m^\varepsilon} \le C \|u\|_{p,\mathbf{x}^\varepsilon,m^\varepsilon}^{1-\theta} \|u\|_{p,\mathbf{x}^\varepsilon,m^\varepsilon,\gamma^\varepsilon}^{\theta} \tag{3.9}$$

where $\theta = \frac{1/p-1/2}{1/d} < 1$. Finally, we obtain from Hölders inequality for $\frac{2-p}{2}$

$$\lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\gamma^\varepsilon} \le \left(\sum_{x_i^\varepsilon \in \boldsymbol{Q}\cap\mathbf{x}^\varepsilon} \sum_{i\sim j} \omega_{ij}^{\frac{-p}{2-p}} \left(\frac{m_{ij}}{\left|x_i^\varepsilon - x_j^\varepsilon\right|^{p-1}} \right)^{\frac{2}{2-p}} \right) \lfloor u \rfloor_{2,\mathbf{x}^\varepsilon,\omega^\varepsilon} . \tag{3.10}$$

$$\omega_{ij}^{\frac{-p}{2-p}} \left(\frac{m_{ij}}{\left|x_i^\varepsilon - x_j^\varepsilon\right|^{p-1}} \right)^{\frac{2}{p-2}} = \left(\omega_{ij} \left|x_i^\varepsilon - x_j^\varepsilon\right|^2\right)^{-\frac{p}{2-p}} \left(m_{ij} \left|x_i^\varepsilon - x_j^\varepsilon\right|\right)^{\frac{2}{2-p}}$$

In particular, the left hand side Inequalities (3.8)–(3.10) together with the ergodic theorem, i.e.

$$\sum_{x_i^\varepsilon \in \boldsymbol{Q}\cap\mathbf{x}^\varepsilon} \sum_{i\sim j} \left(\omega_{ij} \left|x_i^\varepsilon - x_j^\varepsilon\right|^2\right)^{-\frac{p}{2-p}} \left(m_{ij} \left|x_i^\varepsilon - x_j^\varepsilon\right|\right)^{\frac{2}{2-p}} \to |\boldsymbol{Q}|\, \mathbb{E} \sum_{z\in\mathcal{N}(0,\mathbf{x})} \left(\omega_{0,z} |z|^2\right)^{-\frac{p}{2-p}} \left(m_{0,z} |z|\right)^{\frac{2}{2-p}}$$

imply (3.7).

Now let $u_k^{\varepsilon_k} \in \mathcal{S}_{\mathbf{x}}^{\varepsilon_k}$, $k \in \mathbb{N}$, be a sequence with

$$\sup_k \lfloor u_k^{\varepsilon_k} \rfloor_{2,\mathbf{x}^{\varepsilon_k},\omega^{\varepsilon_k}} < \infty .$$

Inequalities (3.8) and (3.10) imply

$$\sup_k \left\| u_k^{\varepsilon_k} \right\|_{p,\mathbf{x}^{\varepsilon_k},m^{\varepsilon_k},\gamma^{\varepsilon_k}} < \infty \,. \tag{3.11}$$

From Lemma B.19 of [5] we infer that $\mathcal{R}^*_{\varepsilon_k,\mathbf{x}} u_k^{\varepsilon_k}$ is precompact in $L^p(\boldsymbol{Q})$ and . Hence (3.11) implies also precompactness of $\mathcal{R}^*_{\varepsilon_k,\mathbf{x}} u_k^{\varepsilon_k}$ in $L^2(\boldsymbol{Q})$. □

4 Homogenization

For the rest of this work, it is convenient to modify the notation and to write

$$\varpi_{x,z} := \omega_{x,x+z} \,, \qquad \varpi^\varepsilon_{x,z} := \omega_{\frac{x}{\varepsilon},\frac{x}{\varepsilon}+z} \,.$$

It is obvious that $\varpi_{x,z}$ is stationary in x. We furthermore use the identification

$$\forall u \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q}) : \qquad \lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\varpi^\varepsilon} := \lfloor u \rfloor_{p,\mathbf{x}^\varepsilon,\omega^\varepsilon}$$

4.1 Function spaces

Let $\varpi_{x,z}$ be stationary ergodic in x and satisfy (1.1)–(1.3). Recalling that Ω is metric compact we can make sense of continuity on Ω and directly use the following theory from [8]. A function $\varphi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ is *shift covariant* if it fulfills

$$\varphi(\varpi, x+z) - \varphi(\varpi, x) = \varphi(\tau_x \varpi, z) \tag{4.1}$$

for all $x, z \in \mathbb{Z}^d$ (cf. [2] Eq. (3.14)), which implies that φ fulfills $\varphi(\varpi, 0) = 0$. In particular, (4.1) directly implies that

$$\varphi(\varpi, x) = -\varphi(\tau_x \varpi, -x) \,. \tag{4.2}$$

We define on $\Omega \times \mathbb{Z}^d$ the space $L^2(\Omega \times \mathbb{Z}^d)$ induced by the following scalar product and norm

$$\begin{aligned} \langle \varphi_1, \varphi_2 \rangle_{L^2(\Omega\times\mathbb{Z}^d)} &:= \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \varpi_{0,z} \varphi_1(\varpi, z) \varphi_2(\varpi, z) \right] \\ \|\varphi\|^2_{L^2(\Omega\times\mathbb{Z}^d)} &:= \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \varpi_{0,z} \varphi(\varpi, z)^2 \right] , \end{aligned} \tag{4.3}$$

together with the following subspace

$$\begin{aligned} L^2_{\mathrm{cov}} &:= \left\{ \varphi \in L^2(\Omega \times \mathbb{Z}^d) \;:\; \varphi \text{ satisfies (4.1) and } \|\varphi\|_{L^2_{\mathrm{cov}}} < \infty \right\} , \\ \text{where } \|\varphi\|^2_{L^2_{\mathrm{cov}}} &:= \|\varphi\|^2_{L^2(\Omega\times\mathbb{Z}^d)} = \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \varpi_{0,z} \varphi(\varpi, z)^2 \right] . \end{aligned}$$

Lemma 4.1. *The space L^2_{cov} is a closed subspace of L^2_{skew} and L^2_{sym} and L^2_{skew} are closed orthogonal subspaces of $L^2(\Omega \times \mathbb{Z}^d)$, i.e. $\left(L^2_{\mathrm{sym}}\right)^\perp = L^2_{\mathrm{sym}}$, and where*

$$L^2_{\mathrm{sym}} := \left\{ \varphi \in L^2(\Omega \times \mathbb{Z}^d) \;:\; \varphi(\varpi, x) = \varphi(\tau_x \varpi, -x) \right\} . \tag{4.4}$$

$$L^2_{\mathrm{skew}} := \left\{ \varphi \in L^2(\Omega \times \mathbb{Z}^d) \;:\; \varphi(\varpi, x) = -\varphi(\tau_x \varpi, -x) \right\} . \tag{4.5}$$

Proof. Closednes of L^2_{cov} is immediate since (4.1) is preserved when taking the limit. For the same reason L^2_{sym} and L^2_{cov} are closed. Given $\varphi \in L^2(\Omega\times\mathbb{Z}^d)$ we define

$$\varphi_{\mathrm{sym}}(\varpi,x) := \frac{1}{2}\left(\varphi(\varpi,x)+\varphi(\tau_x\varpi,-x)\right) \quad \text{and} \quad \varphi_{\mathrm{skew}}(\varpi,x) := \frac{1}{2}\left(\varphi(\varpi,x)-\varphi(\tau_x\varpi,-x)\right),$$

with the evident property $\varphi_{\mathrm{skew}}\in L^2_{\mathrm{skew}}$, $\varphi_{\mathrm{sym}}\in L^2_{\mathrm{sym}}$. It remains to verify for $\varphi\in L^2_{\mathrm{skew}}$ and $\psi\in L^2_{\mathrm{sym}}$ that $\langle\varphi,\psi\rangle_{L^2(\Omega\times\mathbb{Z}^d)} = 0$. For this we first use the algebraic properties of L^2_{skew} and L^2_{sym} and then $\varpi_{0,z} = (\tau_z\varpi)_{0,-z}$:

$$\begin{aligned}\langle\varphi,\psi\rangle_{L^2(\Omega\times\mathbb{Z}^d)} &= \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\varphi(\varpi,z)\psi(\varpi,z)\right] = -\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\varphi(\tau_z\varpi,-z)\psi(\tau_z\varpi,-z)\right]\\ &= -\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}(\tau_z\varpi)_{0,-z}\,\varphi(\tau_z\varpi,-z)\psi(\tau_z\varpi,-z)\right].\end{aligned}$$

Stationarity of ϖ now implies for every $z\in\mathbb{Z}^d$:

$$\mathbb{E}\,(\tau_z\varpi)_{0,-z}\,\varphi(\tau_z\varpi,-z)\psi(\tau_z\varpi,-z) = \mathbb{E}\varpi_{0,-z}\varphi(\varpi,-z)\psi(\varpi,-z)$$

and hence using once more the algebraic properties of L^2_{skew} and L^2_{sym}

$$\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\varphi(\varpi,z)\psi(\varpi,z)\right] = -\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\varphi(\varpi,z)\psi(\varpi,z)\right] = 0\,.$$

□

Remark 4.2. It is typically assumed that L^2_{cov} is the right space to work in [2, 8]. However, as we will see, the space L^2_{sym} is more convenient. In fact, the space of solenoidals is larger than $\left(L^2_{\mathrm{pot}}\right)^\perp\cap L^2_{\mathrm{cov}}$, i.e. it is given by $\left(L^2_{\mathrm{pot}}\right)^\perp\cap L^2_{\mathrm{sym}}$, see below.

As observed in [8] $L^2(\Omega\times\mathbb{Z}^d)$ can also be identified with $\bigotimes_{z\in\mathbb{Z}^d} L^2(\Omega,\mu_z)$, where μ_z is the measure on Ω defined by $\mathrm{d}\mu_z(\varpi) = \varpi_{0,z}\mathrm{d}\Pr(\varpi)$. Compactness of Ω implies separability of $L^2(\Omega,\mu_z)$ for all $z\in\mathbb{Z}^d$ and thus also separability of the countable product space $\bigotimes_{z\in\mathbb{Z}^d} L^2(\Omega,\mu_z)$ and its subspaces.

Further, we note that for all $\phi:\Omega\to\mathbb{R}$ it holds that $\mathrm{D}\phi(\varpi,z) := \mathrm{D}_z\phi(\varpi) := \phi(\tau_z\varpi)-\phi(\varpi)$ satisfies $\mathrm{D}\phi(\varpi,x+z)-\mathrm{D}\phi(\varpi,x) = \mathrm{D}\phi(\tau_x\varpi,z)$ and

$$\forall z\in\mathbb{Z}^d\backslash\{0\}: \qquad \mathbb{E}(\mathrm{D}_z\phi) = \mathbb{E}\phi - \mathbb{E}\phi(\tau_z\,\cdot\,) = 0\,. \tag{4.6}$$

Therefore $\mathrm{D}\phi$ is in L^2_{cov}. A *local function* on Ω is a bounded, continuous function that only depends on finitely many coordinates of $[0,\infty]^E$. We define the closed subspaces

$$L^2_{\mathrm{pot}} := \overline{\{\mathrm{D}\phi\,:\,\phi \text{ local}\}}^{L^2_{\mathrm{cov}}}, \qquad L^2_{\mathrm{sol}} := \left(L^2_{\mathrm{pot}}\right)^\perp\cap L^2_{\mathrm{sym}}\,,$$

with the following operator on $L^2(\Omega\times\mathbb{Z}^d)$:

$$\forall b\in L^2(\Omega\times\mathbb{Z}^d): \qquad \operatorname{div}(\varpi b) := \sum_z \varpi_{0,z}\left(b(\varpi,z)-b(\tau_z\varpi,-z)\right)\,. \tag{4.7}$$

It is evident that $\operatorname{div}(\varpi b) = 0$ for every $b\in L^2_{\mathrm{sym}}$. Regarding $b\in L^2_{\mathrm{skew}}$ we have the following lemma.

Lemma 4.3 ([2, Lemma 3.6]). *Let ϖ be given by some ω satisfying* (1.1)–(1.3)*. Then for every $b \in L^2_{\mathrm{skew}}$ it holds*

$$\operatorname{div}(\varpi b) = \sum_z \varpi_{0,z}\left(b(\varpi,z) - b(\tau_z\varpi,-z)\right) = 2\sum_z \varpi_{0,z} b(\varpi,z)\,. \tag{4.8}$$

and moreover

$$\operatorname{div}(\varpi b) = 0 \text{ and } b \in L^2_{\mathrm{skew}} \qquad \text{if and only if} \qquad b \in L^2_{\mathrm{sol}}\,. \tag{4.9}$$

Proof. (4.8) is straight forward.

If $\phi \in C(\Omega)$ is a local function, i.e. $\|\phi\|_\infty < \infty$ we observe for $b \in L^2_{\mathrm{skew}}$ that

$$\begin{aligned}\mathbb{E}(\varpi_{0,z}\phi(\varpi)\left(b(\tau_z\varpi,-z)\right)) &= \mathbb{E}(\varpi_{z,-z}\phi(\tau_{-z}\varpi_z)\left(b(\varpi_z,-z)\right)) \\ &= \mathbb{E}(\varpi_{0,-z}\phi(\tau_{-z}\varpi)\left(b(\varpi,-z)\right))\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}(\phi\operatorname{div}(\varpi b)) &= \mathbb{E}\left(\sum_z \varpi_{0,z}\phi(\varpi)\left(b(\varpi,z) - b(\tau_z\varpi,-z)\right)\right) \\ &= \mathbb{E}\left(\sum_z \varpi_{0,z}\phi(\varpi)\, b(\varpi,z)\right) - \mathbb{E}\left(\sum_z \varpi_{0,-z}\phi(\tau_{-z}\varpi)\left(b(\varpi,-z)\right)\right) \\ &= \mathbb{E}\left(\sum_z \varpi_{0,z}\phi(\varpi)\, b(\varpi,z)\right) - \mathbb{E}\left(\sum_z \varpi_{0,z}\phi(\tau_z\varpi)\left(b(\varpi,z)\right)\right) \\ &= \mathbb{E}\left(\sum_z \varpi_{0,z}\left(\phi(\varpi) - \phi(\tau_z\varpi)\right) b(\varpi,z)\right).\end{aligned}$$

Since the derivatives of local functions are dense in L^2_{pot} and the local functions are dense in $L^2(\Omega)$ we conclude. □

Of particular importance will be the modified $\tilde\varpi_{0,z}$ given in the following way:

$$\tilde\varpi_{0,z} = \begin{cases} m_{0,z}\,|z|^{-1} & \text{if } 0 \in \mathbf{x} \text{ and } z \in \mathcal{N}(0,\mathbf{x}) \\ 0 & \text{else}\end{cases}\,.$$

The parameter $\tilde\varpi$ with corresponding $\tilde\omega$ also satisfies (1.1)–(1.3) and gives rise to the supspaces $\tilde L^2_{\mathrm{pot}}$ and $\tilde L^2_{\mathrm{sol}}$ within the space $\tilde L^2_{\mathrm{cov}}$. Lemma 4.3 allows us to show the following:

Corollary 4.4. *For every $k = 1,\dots,d$ it holds that $p_k : (\varpi,z) \mapsto z_k = z\cdot e_k$ is in $\tilde L^2_{\mathrm{sol}}$, where e_i, $i = 1,\dots,d$, denote the unit base vectors of $\mathbb{R}^d$. Furthermore,*

$$\forall k \neq j : \qquad \mathbb{E}\sum_{z\in\mathcal{N}(0,\mathbf{x})} \tilde\varpi_{0,z} z_k z_j = 0\,, \tag{4.10}$$

$$\forall k \in \{1,\dots d\} : \qquad \sum_{z\in\mathcal{N}(0,\mathbf{x})} \tilde\varpi_{0,z} z_k^2 = m_{0,\mathbf{x}}\,. \tag{4.11}$$

Proof. Given $0 \in \mathbf{x}$ we write $\nu_z = \frac{z}{|z|}$ for the outer normal of G_0 on $\partial G_0 \cap \partial G_z$ and observe that

$$\begin{aligned}\operatorname{div}(\tilde\varpi_0 z_k) &= \sum_{z\in\mathcal{N}(0,\mathbf{x})} m_{0,z}\frac{1}{|z|} z_k = \sum_{z\in\mathcal{N}(0,\mathbf{x})} \int_{\partial G_0\cap\partial G_z} \frac{1}{|z|} z_k \\ &= \sum_{z\in\mathcal{N}(0,\mathbf{x})} \int_{\partial G_0\cap\partial G_z} \nu_z\cdot e_k = \int_{G_0} \operatorname{div} e_k = 0\,.\end{aligned}$$

and hence the first claim by (4.9). To see that (4.10) holds we observe in a similar way

$$\sum_{z\in\mathcal{N}(0,\mathbf{x})} m_{0,z}\frac{1}{|z|}z_k z_j = \sum_{z\in\mathcal{N}(0,\mathbf{x})}\int_{\partial G_0\cap\partial G_z}\frac{1}{|z|}z_k z_j$$
$$= \sum_{z\in\mathcal{N}(0,\mathbf{x})}\int_{\partial G_0\cap\partial G_z}\nu_z\cdot e_k z_j = \int_{G_0}\partial_k z_j = 0\,.$$

Finally (4.11) follows from

$$\sum_{z\in\mathcal{N}(0,\mathbf{x})}\tilde{\varpi}_{0,z}z_k^2 = \sum_{z\in\mathcal{N}(0,\mathbf{x})}\int_{\partial G_0\cap\partial G_z}\nu_z\cdot e_k z_k = \int_{G_0}1 = |G_0|\,.$$

□

Using the above notation, we define $\chi\in\left(L^2_{\mathrm{pot}}\right)^d$ component wise through

$$\chi_j = \operatorname{argmin}\left\{\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}|z_j+\tilde{\chi}(\varpi,z)|^2\right] : \tilde{\chi}\in\left(L^2_{\mathrm{pot}}\right)^d\right\}, \tag{4.12}$$

i.e., χ_j is the orthogonal projection of $z_j\in L^2_{\mathrm{cov}}$ on the space L^2_{pot} with respect to the scalar product defined in (4.3). We will see below that we can write the homogenized matrix as

$$\left(A_{\mathrm{hom}}\right)_{i,j} = \mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(e_i\cdot[z+\chi(\varpi,z)]\right)\left(e_j\cdot[z+\chi(\varpi,z)]\right)\right], \tag{4.13}$$

where the e_i, $i=1,\dots,d$, denote the unit base vectors of $\mathbb{R}^d$. In analogy to [6, Lemma 4.5] we know the following result.

Lemma 4.5. *Suppose that* (1.1)–(1.3) *and for* $p=1$ *also* (1.10)*. Then the matrix* A_{hom} *is strictly positive definite. In particular, the vectorial space spanned by the following vectors*

$$\mathbb{E}\Big[\textstyle\sum_{z\in\mathbb{Z}^d}\varpi(0,z)\,zb(\varpi,z)\Big]\in\mathbb{R}^d,\qquad b\in L^2_{\mathrm{sol}} \tag{4.14}$$

coincides with $\mathbb{R}^d$*.*

Proof. By (4.11) it holds for every $\xi\in\mathbb{R}^d$:

$$\xi_k^2 = \xi_k\mathbb{E}\left(m_0^{-1}\sum_{z\in\mathcal{N}(0,\mathbf{x})}\xi_k\tilde{\varpi}_{0,z}z_k^2\right)$$

We make twice use of Corollary 4.4 (first $\mathbb{E}\sum_{z\in\mathcal{N}(0,\mathbf{x})}\tilde{\varpi}_{0,z}z_k z_i = 0$ then $z_k\in\tilde{L}^2_{\mathrm{sol}}$), and then Hölder's inequality together with $m_0>1$ to find

$$\xi_k^2 = \xi_k\mathbb{E}\left(\sum_{i=1}^d m_0^{-1}\sum_{z\in\mathcal{N}(0,\mathbf{x})}\xi_i\tilde{\varpi}_{0,z}z_k z_i\right)$$
$$= \xi_k\mathbb{E}\left(\sum_{i=1}^d m_0^{-1}\sum_{z\in\mathcal{N}(0,\mathbf{x})}\xi_i\tilde{\varpi}_{0,z}z_k\left(z_i+\chi_i(\varpi,z)\right)\right)$$
$$\le|\xi_k|\left(\mathbb{E}\sum_{z\in\mathcal{N}(0,\mathbf{x})}\frac{\tilde{\varpi}_{0,z}^2}{\varpi_{0,z}}|z|^2\right)^{\frac12}\left(\sum_{i,j=1}^n\xi_i\xi_j A_{\mathrm{hom},ij}\right)^{\frac12}.$$

Summing up over k and taking the square of both sides we find

$$|\xi|^4 \le \left(\sum_k |\xi_k|\right)^2 \left(\mathbb{E} \sum_{z\in\mathcal{N}(0,\mathbf{x})} \frac{\tilde{\varpi}_{0,z}^2}{\varpi_{0,z}} |z|\right) (\xi \cdot A_{\mathrm{hom}}\xi)$$

Due to (1.10) (note that it also holds for $p = 1$ by monotonicity of the exponents in p), the definition of $\tilde{\varpi}_{0,z}$ and the equivalence of norms in $\mathbb{R}^d$ the matrix A_{hom} is strictly positive definite. By following literally the proof of [6, Lemma 4.5] we obtain the claim. □

4.2 Two-scale convergence

We rely on an adaptation [8] of stochastic two-scale convergence by Zhikov and Piatnitsky [12].

Separability of $L^2(\Omega\times\mathbb{Z}^d)$ and L^2_{cov} allows us to chose countable sets $\Phi_{\mathrm{sym}} \subset L^2_{\mathrm{sym}}$, $\Phi_{\mathrm{sol}} \subset L^2_{\mathrm{sol}}$ and $\Phi_{\mathrm{pot}} \subset L^2_{\mathrm{pot}}$ such that

$$\Phi := \Phi_{\mathrm{sym}} \oplus \Phi_{\mathrm{sol}} \oplus \Phi_{\mathrm{pot}} \oplus \{z_1,\dots,z_d\} \oplus \{1\} \tag{4.15}$$

is dense in $L^2(\Omega\times\mathbb{Z}^d)$ and where we can assume that every $\varphi \in \Phi_{\mathrm{pot}}$ is the gradient of a local function. In particular, $\varphi \in \Phi_{\mathrm{pot}}$ has the property $\varphi \in L^\infty(\Omega\times\mathbb{Z}^d)$ and by the characterization (4.4) we can assume $\Phi_{\mathrm{sym}} \subset L^\infty(\Omega\times\mathbb{Z}^d)$. This implies that $\varphi\psi \in L^2(\Omega\times\mathbb{Z}^d)$ for all $\varphi \in \Phi_{\mathrm{pot}}$ and $\psi \in \tilde{\Phi}$. We can iterate this procedure over a finite number of steps, still remaining with a countable family of functions. Hence we set

$$\Phi := \tilde{\Phi} \oplus \left\{ \psi \prod_{i=1}^N \phi_i \,:\, N\in\mathbb{N},\ (\varphi_i)_{i=1,\dots,N} \subset \Phi_{\mathrm{pot}} \text{ and } \psi \in \tilde{\Phi} \right\}.$$

Also there exists a countable subspace $\Psi \subset C_c^\infty(\mathbb{R}^d)$ such that Ψ is dense both in $L^2(\mathbb{R}^d)$ and in $C_c^\infty(\mathbb{R}^d)$. Together this implies that $\Psi\otimes\Phi$ is dense in $L^2(\mathbb{R}^d; L^2_{\mathrm{cov}})$.

Definition 4.6 (Typical realizations)**.** We denote by $\Omega_\Phi \subset \Omega$ the set of all $\varpi \in \Omega$ such that Theorem 2.3 holds

a) for all $f(\varpi) := \sum_{z\in\mathbb{Z}^d} \varpi_{0,z}\varphi(\varpi,z)$, where $\varphi \in \Phi$,

b) for all $f(\varpi) := \sum_{z\in\mathbb{Z}^d} \varpi_{0,z}\left(\varphi_i\varphi_j\right)(\varpi,z)$, where $\varphi_i,\varphi_j \in \Phi$, and

c) and for all $f(\varpi) := \sum_{z\in\mathbb{Z}^d\setminus Z} \varpi_{0,z}|z|^2$, where Z is a finite subset of $\mathbb{Z}^d$,

d) $\mathrm{div}(\varpi b)\circ\tau_x = 2\sum_z \varpi_{x,z} b(\tau_x\varpi, z) = 0$ for all $b \in \Phi_{sol}$ and all $x \in \mathbb{Z}^d$.

Definition 4.7. We call Ω_Φ the set of *typical realizations*.

Remark 4.8. Note that $\mathbb{P}(\Omega_\Phi) = 1$ (compare to [6, Lemma 4.4]).

Definition 4.9 (Two-scale convergence)**.** Let $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d\times\mathbb{Z}^d \to \mathbb{R}$. We say that w_ε *converges weakly in two scales* to $\mathrm{w} \in L^2(\mathbb{R}^d; L^2(\Omega\times\mathbb{Z}^d))$ if

$$\lim_{\varepsilon\to0} \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v(x) \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\mathrm{w}_\varepsilon(x,z)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi,z) = \int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \varpi_{0,z}\mathrm{w}(x,\varpi,z)\varphi(\varpi,z)\right]\mathrm{d}x \tag{4.16}$$

for all $v \in C_c^\infty(\mathbb{R}^d)$ and all $\varphi \in \Phi$. In this case we write $\mathrm{w}_\varepsilon \xrightharpoonup{2s} \mathrm{w}$.

Corollary 4.10. *For all typical realizations $\varpi \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ converges in two-scales to $\mathrm{w} \in L^2(\mathbb{R}^d; L^2(\Omega \times \mathbb{Z}^d))$ and $\phi \in \Phi_{\mathrm{pot}}$ then the function $\tilde{\mathrm{w}}_\varepsilon(x) := \mathrm{w}_\varepsilon(x)\phi(\tau_{\frac{x}{\varepsilon}}\varpi)$ converges in two scales: $\tilde{\mathrm{w}}_\varepsilon \xrightarrow{2s} \mathrm{w}\phi$.*

Proof. If $v \in C_c^\infty(\mathbb{R}^d)$ and all $\varphi \in \Phi$ are test functions, then also $\varphi\phi \in \Phi$ is a test function and from here we conclude by applying the definition of two-scale convergence. □

Proposition 4.1. *For all typical realizations $\varpi \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ and $C < \infty$ are such that*

$$\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z} \mathrm{w}^2_\varepsilon(x,z) \le C \qquad \forall \varepsilon > 0\,, \tag{4.17}$$

then there exists a subsequence $\mathrm{w}_{\varepsilon_k}$ and $\mathrm{w} \in L^2(\mathbb{R}^d; L^2(\Omega \times \mathbb{Z}^d))$ such that

$$\mathrm{w}_{\varepsilon_k} \xrightarrow{2s} \mathrm{w}\,, \qquad \|\mathrm{w}\|_{L^2(\mathbb{R}^d;L^2(\Omega\times\mathbb{Z}^d))} \le \sup_\varepsilon \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z} \mathrm{w}^2_\varepsilon(x,z)\,. \tag{4.18}$$

Proof. The proof is standard and follows (here) exactly the lines of the proof of Proposition 5.10 in [8] □

Proposition 4.2. *For all typical realizations $\varpi \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon : \varepsilon\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ converges in two-scales to $\mathrm{w} \in L^2(\mathbb{R}^d; L^2(\Omega \times \mathbb{Z}^d))$ and if $(\chi_\varepsilon)_\varepsilon$ is sequence of measurable functions with $\sup_\varepsilon \|\chi_\varepsilon\|_{L^\infty(\mathbb{R}^d)} < \infty$ and $\chi_\varepsilon \to \chi$ pointwise a.e. in $\mathbb{R}^d$. Then $\mathrm{w}_\varepsilon\chi_\varepsilon \xrightarrow{2s} \mathrm{w}\chi$.*

Proof. Let $v \in C_c^\infty(\mathbb{R}^d)$ and $\varphi \in \Phi$. By approximation with smooth functions one can show

$$\begin{aligned}\lim_{\varepsilon\to0} \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v(x)\chi(x) \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z} \mathrm{w}_\varepsilon(x,z)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi, z) \\ = \int_{\mathbb{R}^d} v(x)\,\chi(x)\,\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d} \varpi_{0,z}\mathrm{w}(x,\varpi,z)\varphi(\varpi,z)\right] \mathrm{d}x\,.\end{aligned}$$

Furthermore we observe that $(\chi_\varepsilon - \chi)^2 v^2 \to 0$ pointwise and $\left\|(\chi_\varepsilon - \chi)^2 v^2\right\|_{L^\infty} \le 4\|\chi\|^2_{L^\infty}\|v\|^2_{L^\infty}$. Using the Hölder inequality

$$\begin{aligned}&\left|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v(x)\left(\chi(x) - \chi_\varepsilon(x)\right) \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\mathrm{w}_\varepsilon(x,z)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right| \\ &\quad\le \left|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\mathrm{w}^2_\varepsilon(\varepsilon x,z)\right|^{\frac12} \left|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v^2(x)\left(\chi(x)-\chi_\varepsilon(x)\right)^2 \sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\varphi^2(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right|^{\frac12}.\end{aligned}$$

and Theorem 2.2 we eventually get

$$\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v^2(x)\left(\chi(x)-\chi_\varepsilon(x)\right)^2 \sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\varphi^2(\tau_{\frac{x}{\varepsilon}}\varpi,z) \to \int_{\mathbb{R}^d} 0\,\mathrm{d}x\,\mathbb{E}\sum_z \varpi_{0,z}\varphi^2(\tau_{\frac{x}{\varepsilon}}\varpi,z) = 0\,.$$

□

In [8] some more intermediate results on two-scale convergence are demonstrated. They culminate in the following two results.

Corollary 4.11. *For all typical realizations $\varpi \in \Omega_\Phi$ it holds: If $\mathrm{w}_\varepsilon \xrightarrow{2\mathrm{s}} \mathrm{w}$, then*

$$\lim_{\varepsilon\to0}\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\mathrm{w}_\varepsilon(x,z)\partial^\varepsilon_z v(x) = \int_{\mathbb{R}^d}\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\mathrm{w}(x,\varpi,z)\left(\nabla v(x)\cdot z\right)\right]\mathrm{d}x \tag{4.19}$$

for all $v \in C_c^\infty(\mathbb{R}^d)$.

4.3 Two-scale convergence of gradients

We restate the following result from the original Lemma 5.15 in [8]. However, the notion of two-scale convergence in [8] was restricted to the subspace L^2_{cov}. Hence we have to exclude that $v \in L^2_{\mathrm{sym}}$. Furthermore we have to take into account that $\mathcal{R}^*_{\varepsilon',\mathbf{x}}$ is defined with respect to $\mathbf{x}^\varepsilon$ instead of $\varepsilon\mathbb{Z}^d$.

Lemma 4.12. *For all $\varpi \in \Omega_\Phi$ the following holds. If $u^\varepsilon : \varepsilon\mathbb{Z}^d \to \mathbb{R}$ is a family of functions with $\mathrm{supp}(u^\varepsilon) \subseteq Q \cap \varepsilon\mathbb{Z}^d$ for all ε and*

$$\sup_{\varepsilon>0}\left(\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon(x)\right)^2 + \|u^\varepsilon\|_\infty\right) < \infty\,, \tag{4.20}$$

then there exists a subsequence $u^{\varepsilon'}$, $u \in H^1_0(Q)$ and $\mathrm{w} \in L^2(\mathbb{R}^d; L^2_{\mathrm{pot}})$ such that

$$\mathcal{R}^*_{\varepsilon',\mathbf{x}} u^{\varepsilon'} \to u \text{ in } L^2(\mathbb{R}^d)\,, \qquad \partial^{\varepsilon'}_z u^{\varepsilon'}(x) \xrightarrow{2s} \nabla u(x)\cdot z + \mathrm{w}(x,\varpi,z) \quad \text{as } \varepsilon' \to 0\,. \tag{4.21}$$

Proof. We take $v \in C^\infty_c(\mathbb{R}^d)$ and $\varphi \in \Phi_{\mathrm{sym}}$ (i.e. $\varphi \in L^\infty(\Omega\times\mathbb{Z}^d)$) as testfunctions in the two-scale formula (4.16). Since $\partial^\varepsilon_z u^\varepsilon(x) = -\partial^\varepsilon_{-z} u^\varepsilon(x+\varepsilon z)$ and $\varphi(\tau_{\frac{x}{\varepsilon}}\varpi, z) = \varphi(\tau_{\frac{x}{\varepsilon}+z}\varpi, -z)$ we find

$$\begin{aligned}
&\lim_{\varepsilon\to0}\left|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v(x) \sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon(x)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right| \\
&\le \lim_{\varepsilon\to0}\varepsilon^d\frac12 \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left|\partial^\varepsilon_z u^\varepsilon(x)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right|\left|v(x)-v(x+\varepsilon z)\right| \\
&\le \lim_{\varepsilon\to0}\varepsilon^d\frac12\|\varphi\|_\infty\left(\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left|\partial^\varepsilon_z u^\varepsilon(x)\right|^2\right)^{\frac12}\varepsilon\|\nabla v\|_\infty\left(\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}z^2\right)^{\frac12} \\
&= 0\,.
\end{aligned}$$

In particular, $\partial^\varepsilon_z u^\varepsilon(x) \xrightarrow{2s} \mathrm{w}$ implies $\mathrm{w} \in L^2(\boldsymbol{Q}; \left(L^2_{\mathrm{sym}}\right)^\perp)$. From here we can mostly proceed as in Lemma 5.15 in [8]: We skip all (rather algebraic) calculations which are not affected by the change from $\varepsilon\mathbb{Z}^d$ to $\mathbf{x}^\varepsilon$. We choose $b \in \Phi_{\mathrm{sol}}$ and $v \in C^\infty_c(\mathbb{R}^d)$ and calculate the discrete product rule

$$\partial^\varepsilon_z(vu^\varepsilon)(\varepsilon x) = v(\varepsilon x)\partial^\varepsilon_z u^\varepsilon(\varepsilon x) + u^\varepsilon(\varepsilon x + \varepsilon z)\partial^\varepsilon_z v(\varepsilon x)$$

to obtain that

$$0 = \varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\Big(v(x)\partial^\varepsilon_z u^\varepsilon(x) + u^\varepsilon(x+\varepsilon z)\partial^\varepsilon_z v(x)\Big)b(\tau_{\frac{x}{\varepsilon}}\varpi,z)\,. \tag{4.22}$$

For the first term on the RHS of (4.22), we obtain from the two-scale convergence of $\nabla^\varepsilon u^\varepsilon$ that

$$\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d} v(x)\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon(x)b(\tau_{\frac{x}{\varepsilon}}\varpi,z) \to \int_{\mathbb{R}^d} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\mathrm{w}(x,\varpi,z)b(\varpi,z)\right]\mathrm{d}x\,. \tag{4.23}$$

For the second term on the RHS of (4.22), it follows similar as in Lemma 5.15 in [8] that

$$\begin{aligned}
&\lim_{\varepsilon\to0}\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}u^\varepsilon(x+\varepsilon z)\partial^\varepsilon_z v(x)b(\tau_{\frac{x}{\varepsilon}}\varpi,z) \\
&\qquad = \lim_{\varepsilon\to0}\sum_{x\in\mathbf{x}^\varepsilon} m^\varepsilon_x u^\varepsilon(x)\nabla v(x)\cdot\left(\sum_{z\in\mathbb{Z}^d} z\,\frac{\varpi^\varepsilon_{x,z}}{m_{\frac{x}{\varepsilon}}}b(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right)\,,
\end{aligned}$$

where we use that $\varpi^\varepsilon_{x,z}=0$ if $x\notin\mathbf{x}^\varepsilon$. By the assumptions on ϖ and b, the last bracket on the above RHS is in $L^1(\Omega,\mathbb{P})$. By Theorem 3.4 and Lemma 3.2 it holds that the subsequence $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon\to u$ in $L^2(\mathbb{R}^d)$, where $u\in H^1_0(\boldsymbol{Q})$, there exists a further subsequence, which we still index by $\varepsilon\to0$, where $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon$ converges pointwise a.e. in $\boldsymbol{Q}$ [4, Theorem 4.9]. Moreover, for ε small enough, u^ε has support in $\mathbb{B}_1(\boldsymbol{Q})$ and $\sup_{\varepsilon>0}\|u^\varepsilon\|_\infty<\infty$ by assumption. It follows that we can apply Theorem 2.3 along the above subsequence and obtain that

$$\sum_{x\in\mathbf{x}^\varepsilon}m^\varepsilon_x u^\varepsilon(x)\nabla v(x)\cdot\left(\sum_{z\in\mathbb{Z}^d}z\,\frac{\varpi^\varepsilon_{x,z}}{m_{\frac{x}{\varepsilon}}}b(\tau_{\frac{x}{\varepsilon}}\varpi,z)\right)\to\int_{\mathbb{R}^d}u(x)\nabla v(x)\cdot\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}z\frac{\varpi_{0,z}}{m_0}b(\varpi,z)\right]\mathrm{d}x\,. \tag{4.24}$$

From here we conclude exactly as in Lemma 5.15 in [8]. □

Lemma 4.13. *Let* (1.1)–(1.3) *and* (1.9)–(1.10) *hold. Then for all* $\varpi\in\Omega_\Phi$ *the following holds. If* $u^\varepsilon\in\mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ *is a family of functions and*

$$\sup_{\varepsilon>0}\lfloor u\rfloor_{p,\mathbf{x}^\varepsilon,\varpi^\varepsilon}<\infty\,, \tag{4.25}$$

then there exists a subsequence $u^{\varepsilon'}$ *and* $u\in H^1_0(Q)$ *and* $\mathrm{w}\in L^2(\mathbb{R}^d;L^2_{\mathrm{pot}})$ *such that*

$$\mathcal{R}^*_{\varepsilon',\mathbf{x}}u^{\varepsilon'}\to u\text{ in }L^2(\mathbb{R}^d)\,,\qquad\partial^{\varepsilon'}_z u^{\varepsilon'}(x)\overset{2s}{\rightharpoonup}\nabla u(x)\cdot z+\mathrm{w}(x,\varpi,z)\quad\text{as }\varepsilon'\to0\,. \tag{4.26}$$

Proof. Given $M\in\mathbb{N}$ we define the Lipschitz continuous functions

$$F_M(u):=\max\{-M,\min\{M,u\}\}$$

and set $u^\varepsilon_M:=F_M(u^\varepsilon)$ with $-M\le u^\varepsilon\le M$. Due to (4.25) it holds for some $C_0<\infty$

$$\sup_{\varepsilon>0}\lfloor u\rfloor_{p,\mathbf{x}^\varepsilon,\varpi^\varepsilon}+\sup_{\varepsilon>0}\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon_M(x)\right)^2\le C_0\,, \tag{4.27}$$

Step 1: By Lemma 4.12 for every $M\in\mathbb{N}$ there exists $u_M\in H^1_0(\boldsymbol{Q})$, $\mathrm{w}_M\in L^2(\mathbb{R}^d;L^2_{\mathrm{pot}})$ such that

$$\mathcal{R}^*_{\varepsilon',\mathbf{x}}u^{\varepsilon'}_M\to u_M\text{ in }L^2(\mathbb{R}^d)\,,\qquad\partial^{\varepsilon'}_z u^{\varepsilon'}_M(x)\overset{2s}{\rightharpoonup}\nabla u_M(x)\cdot z+\mathrm{w}_M(x,\varpi,z)\quad\text{as }\varepsilon'\to0\,. \tag{4.28}$$

The existence of $u_M\in H^1_0(\boldsymbol{Q})$ such that $\mathcal{R}^*_{\varepsilon',\mathbf{x}}u^{\varepsilon'}_M\to u_M$ strongly in $L^2(\mathbb{R}^d)$ follows from Theorem 3.4 and Lemma 3.2.

Step 2: By a Cantor diagonal argument the sequence ε' in (4.28) is independent from M. In a similar way, Theorem 3.4 and Lemma 3.2 imply $\mathcal{R}^*_{\varepsilon',\mathbf{x}}u^{\varepsilon'}\to u$ strongly in $L^2(\boldsymbol{Q})$ for some $u\in H^1_0(\boldsymbol{Q})$. For readability, we furtheron write ε instead of ε'. Since F_M is Lipschitz, it holds

$$u_M\leftarrow\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon_M=F_M(\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon)\to F_M(u)\,.$$

We further know

$$\|\nabla u_M(x)\cdot z+\mathrm{w}_M(x,\varpi,z)\|_{L^2(\mathbb{R}^d;L^2_{\mathrm{cov}})}\le\sup_{\varepsilon>0}\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon_M(x)\right)^2$$

and by (4.6) it holds for every canonical basis vector e_i that $\mathbb{E}(\nabla u_M(x)\cdot e_i+\mathrm{w}_M(x,\varpi,e_i))=0$. This implies also

$$\sup_M\|\nabla u_M(x)\|_{L^2(\boldsymbol{Q})}\le\sup_{\varepsilon>0}\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon_M(x)\right)^2\,,$$

and hence $\nabla u_M \rightharpoonup \nabla u$ as $M \to \infty$. Thus w_M is bounded in $L^2(\boldsymbol{Q}; L^2_{\mathrm{pot}})$ and $\mathrm{w}_M(x,\varpi,z) \rightharpoonup \mathrm{w}$ weakly for some $\mathrm{w} \in L^2(\boldsymbol{Q}; L^2_{\mathrm{pot}})$.

Step 3: Now we observe for every $v \in C_c^\infty(\mathbb{R}^d)$ and every $\varphi \in \Phi$ that

$$I^\varepsilon \le I_1^{M,\varepsilon} + I_2^M + I_3^{M,\varepsilon}\,,$$

where

$$\begin{aligned}
I^\varepsilon &:= \Bigg|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon(x)\, v(x)\varphi\big(\tau_{\frac{x}{\varepsilon}}\varpi\big) \\
&\qquad - \int_{\boldsymbol{Q}} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u + \mathrm{w}(x,\varpi,z)\right)\varphi(\varpi,z)\right]\mathrm{d}x\Bigg| \\
I_1^{M,\varepsilon} &:= \left|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon_M(x) - \partial^\varepsilon_z u^\varepsilon(x)\right)\, v(x)\varphi\big(\tau_{\frac{x}{\varepsilon}}\varpi, z\big)\right| \\
I_2^M &:= \left|\int_{\boldsymbol{Q}} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u - \nabla u_M + \mathrm{w}(x,\varpi,z) - \mathrm{w}_M(x,\varpi,z)\right)\varphi(\varpi,z)\right]\mathrm{d}x\right| \\
I_3^{M,\varepsilon} &:= \Bigg|\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon_M(x)\, v(x)\varphi\big(\tau_{\frac{x}{\varepsilon}}\varpi\big) \\
&\qquad - \int_{\boldsymbol{Q}} v(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u_M + \mathrm{w}_M(x,\varpi,z)\right)\varphi(\varpi,z)\right]\mathrm{d}x\Bigg|
\end{aligned}$$

Two-scale convergence of $\partial^\varepsilon_z u^\varepsilon_M$ implies $I_3^{M,\varepsilon} \to 0$ as $\varepsilon \to 0$. Furthermore, weak convergence implies for every $\delta > 0$ that there exists $M_\delta \in \mathbb{N}$ such that $I_2^M \le \delta$ for every $M \ge M_\delta$.

In order to derive an estimate on $I_1^{M,\varepsilon}$, we define

$$\forall u \in L^2(\mathbb{R}^d):\qquad \chi_M[u](x) := \begin{cases} 1 & F_M(u)(x) - u(x) \ne 0 \\ 0 & \text{else}\end{cases}\,,$$

and write for simplicity $\chi_{\varepsilon,M}(x) := \chi_M[\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon](x)$ and $\chi_M(x) := \chi_M[u](x)$. Since if $u^\varepsilon_M(x) - u^\varepsilon(x) = 0$ and $\chi_{\varepsilon,M}(x) := 1$ if $u^\varepsilon_M(x) - u^\varepsilon(x) \ne 0$. Because $\mathcal{R}^*_{\varepsilon,\mathbf{x}}u^\varepsilon \to u$ strongly in $L^2(\mathbb{R}^d)$ and F_M is Lipschitz continuous we conclude that along a subsequence $\chi_{\varepsilon,M}(x) \to \chi_M(x)$ strongly in $L^p(\boldsymbol{Q})$, $1 \le p < \infty$, and pointwise. Since

$$\partial^\varepsilon_z u^\varepsilon_M(x) - \partial^\varepsilon_z u^\varepsilon(x) \ne 0 \qquad\Rightarrow\qquad x, x+\varepsilon z \in \mathbf{x}^\varepsilon \text{ and } \left[\,\chi_{\varepsilon,M}(x) = 1 \text{ or } \chi_{\varepsilon,M}(x+\varepsilon z) = 1\,\right]$$

we find

$$I_1^{M,\varepsilon} \le \varepsilon^d \sum_{\substack{x\in\mathbf{x}^\varepsilon\\ \chi_{\varepsilon,M}(x)=1}}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z} 2\left|\partial^\varepsilon_z u^\varepsilon(x)\right| 2\left\|v\right\|_\infty \left|\varphi\big(\tau_{\frac{x}{\varepsilon}}\varpi, z\big)\right|.$$

However, since

$$\sup_\varepsilon \sum_{x\in\mathbf{x}^\varepsilon\cap\boldsymbol{Q}}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\left(\left|\partial^\varepsilon_z u^\varepsilon(x)\right|\left\|v\right\|_\infty\right)^2 < \infty$$

we infer that $\psi^\varepsilon(x,z) := 4\left|\partial^\varepsilon_z u^\varepsilon(x)\right|\left\|v\right\|_\infty$ converges weakly in two scales to $\psi \in L^2(\boldsymbol{Q}; L^2(\Omega\times\mathbb{Z}^d))$. Moreover, Proposition 4.2 implies that $\chi_{\varepsilon,M}(x)\psi^\varepsilon(x,z)$ converges weakly in two scales to $\chi_M\psi$. Hence

$$\limsup_{\varepsilon\to0} I_1^{M,\varepsilon} \le \left|\int_{\boldsymbol{Q}} \chi_M \mathbb{E}\sum_z \omega_{0,z}\psi(\omega,z)\,\left|\phi(\omega,z)\right|\right|.$$

Again, there exists $M_1 \geq M_0$ such that

$$\forall M > M_1 : \qquad \left| \int_{\boldsymbol{Q}} \chi_M \mathbb{E} \sum_z \omega_{0,z} \psi(\omega, z)\ |\phi(\omega, z)| \right| < \delta\,.$$

Together we find for every $M \geq M_1$ that

$$\lim_{\varepsilon \to 0} I^\varepsilon \leq 2\delta\,,$$

which implies the claim. □

4.4 Convergence of solutions

Lemma 4.14. *Let $f^\varepsilon : Q \cap \mathbb{Z}^d_\varepsilon \to \mathbb{R}$ be a sequence of functions such that $\mathcal{R}^*_\varepsilon f^\varepsilon \rightharpoonup f$ weakly in $L^2(Q)$ for some $f \in L^2(Q)$ and such that $\sup_\varepsilon \|f^\varepsilon\|_\infty < \infty$. Then for almost all $\varpi \in \Omega$ it holds: The sequence of solutions $u^\varepsilon \in \mathcal{S}^\varepsilon_{\mathbf{x}}(\boldsymbol{Q})$ to the problem*

$$\forall x_i \in \boldsymbol{Q} \cap \mathbf{x}^\varepsilon : \qquad -\left(\mathcal{L}^\varepsilon_{\mathbf{x},\varpi} u^\varepsilon\right)_i = m^\varepsilon_i f^\varepsilon_i \tag{4.29}$$

satisfies $\mathcal{R}^*_\varepsilon u^\varepsilon \to u$ strongly in $L^2(Q)$, where $u \in H^1_0(Q) \cap H^2(Q)$ solves the limit problem

$$-\nabla \cdot (A_{\mathrm{hom}} \nabla u) = f\,. \tag{4.30}$$

Proof. We test Equation (4.29) with an arbitrary test function $g^\varepsilon : \mathbf{x}^\varepsilon \to \mathbb{R}$ with $\operatorname{supp} g^\varepsilon \subseteq Q \cap \varepsilon\mathbb{Z}^d$ and obtain by (1.15) that

$$\langle -\mathcal{L}^\varepsilon_\varpi u^\varepsilon, g^\varepsilon \rangle_{2,\mathbf{x}^\varepsilon,1} = \varepsilon^d \sum_{x \in \mathbf{x}^\varepsilon} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \partial^\varepsilon_z u^\varepsilon(x) \partial^\varepsilon_z g^\varepsilon(x) = \langle f^\varepsilon, g^\varepsilon \rangle_{2,\mathbf{x}^\varepsilon,m^\varepsilon}\,. \tag{4.31}$$

We now choose $g^\varepsilon = u^\varepsilon$ and apply (3.7) and Cauchy-Schwarz to obtain that

$$\|u^\varepsilon\|^2_{2,\mathbf{x}^\varepsilon,m^\varepsilon} \leq C\varepsilon^d \sum_{x \in \mathbf{x}^\varepsilon} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \left(\partial^\varepsilon_z u^\varepsilon(x)\right)^2 \leq 2C \|u^\varepsilon\|_{2,\mathbf{x}^\varepsilon,m^\varepsilon} \|f^\varepsilon\|_{2,\mathbf{x}^\varepsilon,m^\varepsilon}\,. \tag{4.32}$$

It follows that by virtue of Theorem 3.4 and Lemma 4.13, there exists $u \in H^1_0(Q)$, $v \in L^2(Q; L^2_{\mathrm{pot}})$ and a subsequence, which we still index by ε, such that

$$\mathcal{R}^*_{\varepsilon,\mathbf{x}} u^\varepsilon \to u\,, \text{ strongly in } L^2(Q) \quad \text{and} \quad \partial^\varepsilon_z u^\varepsilon(x) \overset{2s}{\rightharpoonup} \nabla u(x) \cdot z + v(x, \varpi, z) \text{ as } \varepsilon \to 0 \tag{4.33}$$

for all $\varpi \in \Omega_\Phi$.

Let us choose $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with $\operatorname{supp} v \in Q$ and $\varphi \in \Phi_{\mathrm{pot}}$ with $\varphi = D\tilde{\varphi}$ for some bounded local function $\tilde{\varphi}$. When we insert $g^\varepsilon = \varepsilon\psi\tilde{\varphi}$ into (4.31), then we observe for all $\varepsilon > 0$ that

$$\begin{aligned}
&\sum_{x \in \mathbf{x}^\varepsilon} 2m^\varepsilon_x f^\varepsilon(\varepsilon x) \left(\varepsilon\psi(\varepsilon x)\tilde{\varphi}(\tau_x \varpi)\right) \\
&\quad = \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \partial^\varepsilon_z u^\varepsilon(x) \left(\psi(x + \varepsilon z)\tilde{\varphi}(\tau_{\frac{x}{\varepsilon}+z}\varpi) - \psi(x)\tilde{\varphi}(\tau_{\frac{x}{\varepsilon}}\varpi)\right) \\
&\quad = \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \partial^\varepsilon_z u^\varepsilon(x) \left[\psi(x)\left(\tilde{\varphi}(\tau_{\frac{x}{\varepsilon}+z}\varpi) - \tilde{\varphi}(\tau_{\frac{x}{\varepsilon}}\varpi)\right) + \varepsilon\tilde{\varphi}(\tau_{\frac{x}{\varepsilon}+z}\varpi)\partial^\varepsilon_z\psi(x)\right] \\
&\quad = \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \partial^\varepsilon_z u^\varepsilon(x)\, \psi(x)\varphi(\tau_{\frac{x}{\varepsilon}}\varpi, z) \\
&\qquad\quad + \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} \varpi^\varepsilon_{x,z} \partial^\varepsilon_z u^\varepsilon(x)\, \varepsilon\tilde{\varphi}(\tau_{\frac{x}{\varepsilon}+z}\varpi)\partial^\varepsilon_z\psi(x)\,.
\end{aligned} \tag{4.34}$$

The second summand on the above RHS vanishes as $\varepsilon \to 0$ since

$$\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d} \varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon(x)\,\varepsilon\tilde\varphi(\tau_{\frac{x}{\varepsilon}+z}\varpi)\partial^\varepsilon_z\psi(x) \le \varepsilon^{d+1}\|\tilde\varphi\|_\infty \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_z\psi(x)$$
$$\le \varepsilon\|\tilde\varphi\|_\infty \left(\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z u^\varepsilon(x)\right)^2\right)^{1/2}\left(\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\left(\partial^\varepsilon_z \psi(x)\right)^2\right)^{1/2} \tag{4.35}$$

By assumption $\|\tilde\varphi\|_\infty$ is bounded. The second factor is bounded due to (4.32) and the third factor is bounded due to (1.3) and the Lipschitz regularity of ψ. Since the LHS of (4.34) vanishes as well, (4.33) and (4.34) imply that in the limit $\varepsilon \to 0$ and along the chosen subsequence we obtain

$$\int_Q \psi(x)\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u(x)\cdot z + v(x,\varpi,z)\right)\varphi(\varpi,z)\right] = 0\,. \tag{4.36}$$

Since Φ_{pot} is dense in L^2_{pot} and Ψ is dense in $H^1_0(Q)$, Equation (4.36) holds for all $\varphi \in L^2_{\mathrm{pot}}$ and all $v \in H^1_0(Q)$.

Let $\chi \in \left(L^2_{\mathrm{pot}}\right)^d$ be given through (4.12). Since $u \in H^1_0(Q)$ is given, the function $v(x,\tilde\varpi,z) := \nabla u(x)\cdot\chi(\tilde\varpi,z)$ is the unique solution to (4.36). We have thus identified v.

Now we observe that if we test (4.31) by an arbitrary $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with support in Q, we obtain that

$$\varepsilon^d \sum_{x\in\varepsilon\mathbb{Z}^d}\sum_{z\in\mathbb{Z}^d}\varpi^\varepsilon_{x,z}\,\partial^\varepsilon_z u^\varepsilon(x)\,\partial^\varepsilon_z g(x) = \sum_{x\in\mathbf{x}^\varepsilon} 2m^\varepsilon_x f^\varepsilon(x)g(x)\,.$$

Passing to the limit, we obtain by virtue of Corollary 4.11 and $v(x,\varpi,z) = \nabla u(x)\cdot\chi(\varpi,z)$ that

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u(x)\cdot(z+\chi)\right)\left(\nabla g(x)\cdot z\right)\right]\mathrm{d}x = \int_{\mathbb{R}^d} 2f(x)g(x)\,\mathrm{d}x\,. \tag{4.37}$$

When we now insert $\psi = \partial_i g$ and $\varphi = \chi_i$ for $i = 1,\dots,d$ into (4.36) and add the resulting equations to (4.37), then we obtain that

$$\int_{\mathbb{R}^d}\mathbb{E}\left[\sum_{z\in\mathbb{Z}^d}\varpi_{0,z}\left(\nabla u(x)\cdot(z+\chi)\right)\left(\nabla g(x)\cdot(z+\chi)\right)\right]\mathrm{d}x = \int_{\mathbb{R}^d} 2f(x)g(x)\,\mathrm{d}x\,. \tag{4.38}$$

A comparison with the definition of A_{hom} in (4.13) finally yields that u solves

$$\int_Q \nabla u\cdot\left(A_{\mathrm{hom}}\nabla g\right) = \int_Q 2f\,g \qquad \text{for all } g\in C^\infty_{\mathrm{c}}(\mathbb{R}^d) \text{ with } \mathrm{supp} g\subseteq Q\,. \tag{4.39}$$

Since A_{hom} is non-degenerate, we find that (4.39) is the weak formulation of (4.30). Hence, from elliptic regularity theory, we obtain that $u \in H^2(Q)\cap H^1_0(Q)$.

Since the solution u of (4.30) is unique, it follows that (4.33) holds for the entire sequence. □

4.5 Proof of Theorem 1.5

Part 1 of the theorem follows almost surely from Lemma 4.14.

Part 2 follows from on one hand from Lemmas 3.2 and 3.3.

On the other hand Part 3 of of the theorem is proved by Theorem 3.4.

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