

The geometry of controlled rough paths

Mazyar Ghani Varzaneh¹, Sebastian Riedel², Alexander Schmeding³,

Nikolas Tapia^{1,4}

submitted: March 16, 2022

¹ Mathematics Institute
Technische Universität Berlin
Str. des 17. Juni 136
10586 Berlin
Germany
E-Mail: mazyarghani69@gmail.com

² Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
Germany
E-Mail: riedel@math.uni-hannover.de

³ Nord universitet
Høgskoleveien 27
7601 Levanger
Norway
E-Mail: alexander.schmeding@nord.no

⁴ Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: nikolasesteban.tapiamunoz@wias-berlin.de

No. 2926
Berlin 2022



2020 *Mathematics Subject Classification.* 34K50, 37H10, 37H15, 60H99, 60G15.

Key words and phrases. Continuous fields of Banach spaces, rough paths, controlled rough paths.

SR acknowledges financial support by the DFG via Research Unit FOR 2402. NT is supported by the DFG MATH⁺ Excellence Cluster.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: <http://www.wias-berlin.de/>

The geometry of controlled rough paths

Mazyar Ghani Varzaneh, Sebastian Riedel, Alexander Schmeding,
Nikolas Tapia

ABSTRACT. We prove that the spaces of controlled (branched) rough paths of arbitrary order form a continuous field of Banach spaces. This structure has many similarities to an (infinite-dimensional) vector bundle and allows to define a topology on the total space, the collection of all controlled path spaces, which turns out to be Polish in the geometric case. The construction is intrinsic and based on a new approximation result for controlled rough paths. This framework turns well-known maps such as the rough integration map and the Itô-Lyons map into continuous (structure preserving) mappings. Moreover, it is compatible with previous constructions of interest in the stability theory for rough integration.

1. INTRODUCTION AND STATEMENT OF RESULTS

One of the key insights in rough path theory is that there is no canonical integration theory that allows to integrate two arbitrary paths of low regularity against one another¹. Lyons' fundamental observation was that paths have to be augmented with higher order objects which play the role of iterated integrals in order to build a robust theory of controlled ordinary differential equations [29], and he called these augmented paths *rough paths*. Later, Gubinelli realized that given a reference rough path, there is a canonical notion of an integral that allows to integrate paths that “look like the reference path” on small time scales, known as *controlled rough paths* [18, 19]. These principles were carried over from the world of paths to the world of distributions (or generalized functions) by Hairer [21] and Gubinelli-Imkeller-Perkowski [20]. In Hairer's theory of regularity structures, the reference distribution has to be augmented with products of itself and its derivatives, and multiplying distributions is explained when they are *modelled* after the reference distribution.

Although these concepts are nowadays used extensively in the theory of stochastic ordinary and partial differential equations, the mathematical structure which is formed by rough paths and their controlled paths is still not very well understood. Since for every reference α -rough path \mathbf{X} , the set of all controlled paths $\mathcal{D}_{\mathbf{X}}^{\alpha}$ constitutes a linear (Banach) space, it is natural to suspect that they form some sort of “infinite-dimensional vector bundle” [24, p. 123]. However, making these ideas precise, one encounters several difficulties, starting with the fact that the space of rough paths does not carry any known (infinite-dimensional) manifold structure. Surprisingly, it turns out that there is still a non-canonical homeomorphism transforming the collection of spaces of controlled paths to a trivial infinite-dimensional vector bundle [12, Remark 4.8]. However, this map is highly non-explicit since it uses the Lyons-Victoir extension theorem [28], and the applicability of this result in practice is unclear. Furthermore, note that the existence of this homeomorphism does not imply that the spaces of controlled rough paths form a smooth (Banach) vector bundle.

One motivation for this article is an observation two of us made in [16]. In this work, we considered the solution map induced by linear stochastic delay differential equations (SDDE) driven by a Brownian motion, which turns out to be a linear map between spaces of controlled paths. Since these spaces were random, we faced serious measurability issues when considering, for example, the operator norm for this map. To overcome these issues, we proved that the spaces of controlled paths form a *measurable field of Banach spaces* [16, Definition 3.13]. It

¹This fact is visible e.g. in stochastic analysis: there is no exclusive notion of a stochastic integral, both Itô and Stratonovich integral have their justification.

turns out that this structure is extremely useful and a perfect infinite-dimensional substitute for what is called a *measurable bundle* [1, 1.9.2 Definition]. In fact, we could prove that SDEs induce random dynamical systems (RDS) on this measurable field of Banach spaces, and that important theorems like the multiplicative ergodic theorem hold for these RDSs [17]. Note, however, that we proved the existence of a measurable field only for Brownian-type rough paths, i.e. in the regime of an Hölder-index close to $\frac{1}{2}$. One goal of the present article is to generalize this important result to rough paths of arbitrary regularity.²

In the present paper, we first construct a novel approximation of controlled rough paths by rough integrals of smooth functions and smooth remainders. We do this for paths which are controlled by branched rough paths of arbitrary order. It will turn out that once the algebra underlying the process has been worked out, it is straightforward to construct a dense subset of controlled rough paths depending continuously on the underlying controlling path. Our results here are a far reaching generalisation of our earlier results in [16]. Based on this novel approximation result, we are able to construct a finer structure for the spaces of controlled rough paths. Indeed we prove that the spaces of controlled rough paths form a so-called *continuous field of Banach spaces* (see Definition 4.1). This structure is well-known from the representation theory of C^* -algebras (see e.g. [8, 9]). It will turn out that our field of Banach spaces sits in the middle between the two trivial bundles, namely for $\alpha \in (1/3, 1/2)$ we have

$$\mathcal{C}^\alpha \times (C^\infty \times C^\infty) \subseteq \mathcal{C}^\alpha \times \mathcal{D}^\alpha \subseteq \mathcal{C}^\alpha \times (C^{2\alpha} \times C^\alpha)$$

where the trivial bundle on the right is the one from [12] and the trivial bundle on the left forms a “dense” subset of the field of Banach spaces (we will say more about the topology we use here below).

In more detail, our results subsume the following theorem:

Theorem 1.1. *Fix $\alpha \in (0, 1)$ and let \mathcal{C}^α denote the space of α -branched rough paths. For $\mathbf{X} \in \mathcal{C}^\alpha$, denote by $\mathcal{D}_{\mathbf{X}}^\alpha$ the associated space of controlled paths. Then the family $(\mathcal{D}_{\mathbf{X}}^\alpha)_{\mathbf{X} \in \mathcal{C}^\alpha}$ forms a continuous field of Banach spaces (FoBS) over \mathcal{C}^α .*

Theorem 1.1 has several implications from which we will list a few here:

- Every continuous field of Banach spaces allows to define an intrinsic topology (that we call the *tube topology*) on the total space, i.e., in our case on the product space

$$\prod_{\mathbf{X} \in \mathcal{C}^\alpha} \mathcal{D}_{\mathbf{X}}^\alpha.$$

It turns out (Section 4.2) that this topology is compatible with rough integration, therefore making the Itô-Lyons map a *continuous section* of the bundle. This also gives a nice interpretation of the “metric” that compares two controlled paths that live in different fibres (see [12, p. 74]): adding the distance of the reference rough paths to it, we can show that this metric is exactly the one inducing the topology on the total space. This also implies that the total space is a Polish space when restricting to geometric rough paths³ (Theorem 4.18).

- The continuous field of Banach spaces induces a so-called *Banach bundle* of controlled paths, cf. Section 4.1. It turns out that well-known mappings in rough paths theory such as the rough integration map and the Itô-Lyons map are structure-preserving mappings

²We note that extending these results to arbitrary regularity is of interest, for example, for financial applications in the context of rough volatility models [3, 15, 30] where the typical regularity of the driving noise is of the order of $\alpha = \frac{1}{10}$.

³See [12, Section 2.4] for a precise definition of geometric rough path.

on this Banach bundle. We would like to mention here that for proving these statement, we had to establish sharp bounds for the rough integral (Proposition 2.19) and local Lipschitz continuity of the Itô-Lyons map (Proposition 4.21) *in full generality*, i.e. for rough integrals and rough differential equations driven by branched rough paths of any order. Although these bounds are widely accepted in the rough paths community, we could not find them explicitly worked out anywhere in the literature. Having closed this gap is another contribution of the present work.

- In [12, Remark 4.9], it is claimed that “the notion of ‘controlled rough paths’ (...) does not come with a natural approximation theory”. As a remedy, one can construct non-canonical approximations using the Lyons-Victoir extension theorem [12, Exercise 4.8], but this is not trivial as the reader can easily convince herself. Furthermore, the construction in the cited reference only works in the case $\frac{1}{3} < \alpha < \frac{1}{2}$. We will show that Theorem 1.1 immediately yield *canonical* approximations for controlled rough paths of any order (Remark 3.11). In principle, this approximation result should yield, as particular cases, results concerning convergence and stability properties of random dynamical systems [16], rough differential equations [12, 14], and numerical schemes for stochastic (partial) differential equations [2, 27], but we do not explore this further and content ourselves with setting up the general framework underlying these results.
- Continuous fields of Banach spaces are also measurable fields. The results in this article allow to study dynamical properties of SDDE as in [16] driven by more general processes than Brownian motion, e.g. fractional Brownian motion with lower Hurst parameter.

The remainder of the article is organized as follows: in Section 2 we recall the definition of branched rough paths and set up basic notation and results necessary for the next sections. In Section 3 we introduce the main construction of the paper, namely how to build a controlled rough path out of a collection of smooth functions. Next, in Section 3.1 we use this construction to show how to locally approximate any given controlled path by piecewise-linear controlled paths in the Banach norm, and then extend this approximation to the full time interval in Section 3.2. In Section 4.1 we recall the main definition of a continuous field of Banach spaces and show that the bundle of controlled paths satisfies this definition with the space of branched rough paths as base space, and use this result to construct the so-called tube topology. We end by showing several continuity results under this particular topology for well-known objects associated to rough paths, such as the Itô-Lyons map, in Section 4.2.

2. BRANCHED ROUGH PATHS

Let us first recall the Hopf algebraic framework for branched and controlled rough paths. In what follows, we write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. All linear structures are defined over the field of real numbers, denoted by \mathbb{R} .

2.1. Let A be a finite, non-empty set, and denote by \mathcal{T} the linear span of all rooted trees decorated with labels from A . We recall that the *decorated Connes–Kreimer Hopf algebra* \mathcal{H} is the polynomial algebra $\mathcal{H} = \mathcal{S}(\mathcal{T})$, which can be identified with the vector space spanned by all forests of decorated trees; the basis will be denoted by \mathcal{F} . We grade \mathcal{H} by the number of nodes, and we denote the *degree* of $h \in \mathcal{H}$ by $|h|$. There is a unique forest of degree zero, called the *empty forest* and denoted by $\mathbf{1}$. We also set, for each $n \geq 0$,

$$\mathcal{F}_n := \{h \in \mathcal{F} : |h| = n\}, \quad \mathcal{F}_{(n)} := \{h \in \mathcal{F} : |h| \leq n\}, \quad \mathcal{F}_{(n)}^< := \mathcal{F}_{(n)} \setminus \mathcal{F}_n$$

and we let \mathcal{H}_n and $\mathcal{H}_{(n)}$ denote the corresponding linear spans of \mathcal{F}_n and $\mathcal{F}_{(n)}$, respectively. We finally introduce the set $\mathcal{F}^+ := \mathcal{F} \setminus \{\mathbf{1}\}$ and its linear span

$$\mathcal{H}^+ = \bigoplus_{n>0} \mathcal{H}_n.$$

The space \mathcal{H} comes with a collection of maps $([\cdot]_a, a \in A)$, where $[\tau_1 \cdots \tau_n]_a$ is obtained by grafting the trees τ_1, \dots, τ_n to a new root labeled by a . Observe that by definition, $||[h]_a| = |h| + 1$. Moreover, for each tree $\tau \in \mathcal{T}$ with $|\tau| = n + 1$ there is a unique label $a \in A$ and a unique forest h such that $[h]_a = \tau$. These maps define uniquely a *coproduct* $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ via the relations $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$, $\Delta(h_1 \cdots h_n) = \Delta h_1 \cdots \Delta h_n$, and

$$\Delta[h]_a = (\text{id} \otimes [\cdot]_a)\Delta h + [h]_a \otimes \mathbf{1} \quad (2.1)$$

for all $a \in A$ and $h, h_1, \dots, h_n \in \mathcal{H}$. One can immediately check from this definition that Δ is coassociative, i.e., the identity $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ holds. Therefore, the triple $(\mathcal{H}, \cdot, \Delta)$ defines a bialgebra.

It can be shown that for trees $\tau \in \mathcal{T}$, the image $\Delta\tau$ admits a representation in terms of *admissible cuts* of τ [7], but we refrain from providing further details since we will not make use of it in the following. We shall instead use sumless Sweedler's notation

$$\Delta h = h_{(1)} \otimes h_{(2)}, \quad \Delta' h = h' \otimes h''$$

where $\Delta' h := \Delta h - h \otimes \mathbf{1} - \mathbf{1} \otimes h$ denotes the *reduced coproduct*. Since product and coproduct respect the grading we obtain a graded and connected bialgebra of finite-type, i.e. disassembling degreewise we obtain $\mathcal{H} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{H}_n$ where $\mathcal{H}_0 = \mathbb{R}\mathbf{1}$ and $\dim \mathcal{H}_n < \infty$. Note that any graded connected bialgebra is automatically a Hopf algebra whose antipode S can be computed recursively (whence we do not discuss it here, but see [31]).

We introduce the *n-fold iterated coproduct* $\Delta^{(n)}: \mathcal{H} \rightarrow \mathcal{H}^{\otimes(n+1)}$ inductively by $\Delta^{(0)} = \text{id}$, and $\Delta^{(n)} = (\Delta^{(n-1)} \otimes \text{id}) \circ \Delta$ for all $n \geq 1$. Note that these are indeed well defined by coassociativity of Δ . An iterated reduced coproduct can be defined in a similar way. In particular, we can write the image of an element $h \in \mathcal{H}$ using Sweedler's notation as

$$\Delta^{(n)} h = h_{(1)} \otimes \cdots \otimes h_{(n+1)}, \quad (\Delta')^{(n)} h = h^{(1)} \otimes \cdots \otimes h^{(n+1)}.$$

Finally, we remark that the *n-fold tensor product* $\mathcal{H}^{\otimes n}$ is also graded, with graded components

$$(\mathcal{H}^{\otimes n})_k = \bigoplus_{j_1 + \cdots + j_n = k} \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n},$$

and so, since Δ is a graded map, we see that if $|h| = k$ then $|h_1| + \cdots + |h_{n+1}| = k$ for all $n \geq 0$.

2.2. Denote by \mathcal{H}^* the dual space of \mathcal{H} , i.e. the space of all linear mappings $\mathcal{H} \rightarrow \mathbb{R}$ and let $m_{\mathbb{R}}$ be the multiplication map of real numbers. We write $\langle \phi, h \rangle$ for the duality pairing between \mathcal{H}^* and \mathcal{H} given by evaluation. The convolution product

$$\phi \star \psi := m_{\mathbb{R}} \circ (\phi \otimes \psi) \circ \Delta$$

turns \mathcal{H}^* into a unital algebra (the unit being the counit ε of \mathcal{H}). Then the *character group* $(G(\mathcal{H}, \mathbb{R}), \star)$ of \mathcal{H} is defined as

$$G(\mathcal{H}, \mathbb{R}) := \{\phi \in \mathcal{H}^* : \langle \phi, ab \rangle = \langle \phi, a \rangle \langle \phi, b \rangle, \forall a, b \in \mathcal{H} \text{ and } \langle \phi, 1_{\mathcal{H}} \rangle = 1\}.$$

Inversion in $G(\mathcal{H}, \mathbb{R})$ is induced by precomposition with the antipode, i.e. for a character $\iota(\phi) = \phi^{-1} = \phi \circ S$. The counit ε of \mathcal{H} thus becomes the unit of the character group. One

can show, [5] that the character group carries a natural Lie group structure whose Lie algebra is given by the *Lie algebra of infinitesimal characters* $(\mathfrak{g}(\mathcal{H}, \mathbb{R}), [\cdot, \cdot])$;

$$\mathfrak{g}(\mathcal{H}, \mathbb{R}) := \{\psi \in \text{Hom}(\mathcal{H}, \mathbb{R}) : \langle \psi, ab \rangle = \langle \psi, a \rangle \langle \varepsilon, b \rangle + \langle \varepsilon, a \rangle \langle \psi, b \rangle, \forall a, b \in \mathcal{H}\}$$

with the Lie bracket $[\psi, \kappa] := \psi \star \kappa - \kappa \star \psi$.

There is an injection $\mathcal{H} \hookrightarrow \mathcal{H}^*$ mapping $h \in \mathcal{H}$ to the linear functional $h' \mapsto \delta_{h,h'}$ (the Kronecker delta mapping h to 1). In the sequel we will just identify $h \in \mathcal{H}$ with this functional. Using this identification, we write dual elements $X \in \mathcal{H}^*$ as formal forest series, i.e.,

$$X = \sum_{h \in \mathcal{F}} X^h h$$

where $X^h = \langle X, h \rangle \in \mathbb{R}$.

2.3. An important subspace is the space of *primitive elements*

$$\text{Prim} := \{h \in \mathcal{H} : \Delta' h = 0\}.$$

Let h and h' be forests. We define the *natural growth operation* $h \top h'$ as the sum of all forests obtained by grafting h to every node of h' , normalized by $|h'|$. This definition is extended bilinearly to the natural growth operator $\top : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$. For example

$$\begin{aligned} \bullet \bullet \top \bullet &= \frac{1}{2} \left(\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \right) \\ \bullet \top \bullet \bullet &= \frac{1}{2} \left(\begin{array}{c} \bullet \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \bullet \end{array} \right) = \begin{array}{c} \bullet \\ \bullet \end{array} \end{aligned}$$

We observe that this operator is neither associative nor commutative. Given a collection $h_1, \dots, h_n \in \mathcal{H}$, we set

$$\top(h_1, \dots, h_n) = (\dots((h_1 \top h_2) \top h_3) \dots) \top h_n.$$

The following results are due to Foissy:

Lemma 2.4. [11, Lemma 4.3 and Theorem 9.6]

1 If $p_1, \dots, p_n \in \text{Prim}$, then

$$\Delta' \top(p_1, \dots, p_n) = \sum_{j=1}^{n-1} \top(p_1, \dots, p_j) \otimes \top(p_{j+1}, \dots, p_n).$$

2 Let $\mathcal{P} = \{p_i : i \geq 1\}$ be a basis for Prim . Then

$$\mathcal{P}^\top := \{\top(p_{i_1}, \dots, p_{i_k}) : i_1, \dots, i_k \geq 1, k \geq 1\}$$

is a basis for \mathcal{H} .

As before, for $n \in \mathbb{N}$ we define

$$\mathcal{P}_{(n)} := \mathcal{P} \cap \mathcal{H}_{(n)}, \quad \mathcal{P}_n := \mathcal{P} \cap \mathcal{H}_n, \quad \mathcal{P}_{(n)}^< := \mathcal{P}_{(n)} \setminus \mathcal{P}_n,$$

and similarly for $\mathcal{P}_{(n)}^\top$ and so on.

As a consequence, we have that for every forest $h \in \mathcal{F}$, there are coefficients $(c_\rho(h) : \rho \in \mathcal{P}^\top)$ with only finitely many being non zero, such that

$$h = \sum_{\rho \in \mathcal{P}^\top} c_\rho(h) \rho.$$

In particular, every element (where only finitely many coefficients Z^h are non-zero)

$$Z = \sum_{h \in \mathcal{H}} Z^h h \in \mathcal{H}$$

can be rewritten as

$$Z = \sum_{\rho \in \mathcal{P}^\top} Z^\rho \rho,$$

with

$$Z^\rho = \sum_{h \in \mathcal{H}} c_\rho(h) Z^h.$$

Denote by $\{f_{i_1, \dots, i_k}\}$ the basis of the graded dual \mathcal{H}^{gr} , dual to \mathcal{P}^\top , i.e., such that $\langle f_{i_1, \dots, i_k}, \top(\rho_{j_1}, \dots, \rho_{j_l}) \rangle = 1$ if and only if $k = l$ and $i_1 = j_1, \dots, i_k = j_k$. We will also use the notation $\rho^* := f_{i_1, \dots, i_k}$ whenever $\rho = \top(\rho_{i_1}, \dots, \rho_{i_k}) \in \mathcal{P}^\top$. In particular, we identify the change of basis coefficients as $c_\rho(h) := \langle \rho^*, h \rangle$, and we have the identity

$$\sum_{h \in \mathcal{H}} c_\rho(h) h = \rho^*.$$

Therefore, any dual forest series

$$X = \sum_{h \in \mathcal{H}} X^h h \in \mathcal{H}^*$$

can be re-expanded in the new basis as

$$X = \sum_{\rho \in \mathcal{P}^\top} X^\rho \rho^*$$

where

$$X^\rho = \sum_{h \in \mathcal{H}} c_\rho(h) X^h.$$

2.5. Given a rooted tree τ and a vertex $v \in V(\tau)$, let τ_v denote the subtree of τ with v as root. For $v \in V(\tau)$, let $SG(\tau, v)$ be the group of permutations of identical branches out of v , i.e., if $\{v_1, \dots, v_k\}$ are the children of v , then $SG(\tau, v)$ is the group generated by the permutations that exchange τ_{v_i} and τ_{v_j} when they are isomorphic rooted trees. The *symmetry group of τ* is the direct product

$$SG(\tau) := \prod_{v \in V(\tau)} SG(\tau, v),$$

and the *symmetry factor* $\Sigma(\tau)$ of τ is defined to be the order of $SG(\tau)$ [23]. For a forest $h \in \mathcal{H}$, we let $\Sigma(h) := \Sigma(\mathfrak{A}_j(h))$ for some $j \in A$. It is not hard to see that in fact this definition is independent of the choice of $j \in A$.

Letting $\zeta_h(h') = \Sigma(h) \delta_{h, h'}$, one can show that $\{\zeta_h : h \in \mathcal{H}\}$ is a basis of \mathcal{H}^* dual to the forest basis [23, Proposition 4.4], and in particular $\zeta_1 = \varepsilon$.

2.6. Recall that a (right) pre-Lie algebra is a vector space V with a bilinear operator $\triangleleft : V \otimes V \rightarrow V$ such that the associator $a_\triangleleft(x, y, z) := (x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z)$ is symmetric in the last two variables, i.e.,

$$a_\triangleleft(x, y, z) = a_\triangleleft(x, z, y)$$

for all $x, y, z \in V$.

There is a left pre-Lie structure on \mathcal{T} given by grafting of trees, denoted by $\curvearrowright : \mathcal{T} \otimes \mathcal{T} \rightarrow \mathcal{T}$.

Example 2.7.

$$\begin{aligned} \bullet \curvearrowright \vee &= \vee \\ \vee \curvearrowright \bullet &= 2\vee + \vee \end{aligned}$$

◇

This is in fact the free pre-Lie algebra on d generators [6].

Grafting can be extended to an operator $\curvearrowright: \mathcal{T} \otimes \mathcal{H} \rightarrow \mathcal{T}$ by grafting every forest on the right to some node of the tree on the left.

Example 2.8.

$$\begin{aligned} \bullet \curvearrowright \bullet\bullet &= \vee, \\ \vee \curvearrowright \bullet\bullet &= \vee + 2\vee + \vee \end{aligned}$$

◇

Let V be a vector space. Let us recall that the symmetric algebra $S(V)$ carries a coproduct Δ_* defined by $\Delta_* v = v \otimes 1 + 1 \otimes v$ for all $v \in V$. We stick to the sumless Sweedler's notation for this coproduct as well. A *symmetric brace algebra* [25] is a vector space V equipped with a brace $V \otimes S(V) \rightarrow V, x \otimes a \mapsto x\{a\}$ such that

$$\begin{aligned} x\{1\} &= x \\ x\{y_1 \cdots y_n\}\{a\} &= x\{y_1\{a_{(1)}\} \cdots y_n\{a_{(n)}\} a_{(n+1)}\}. \end{aligned}$$

It can be shown that grafting endows \mathcal{T} with the structure of a symmetric brace [32].⁴ In particular, the identity

$$(\tau \curvearrowright \bar{h}_1) \curvearrowright \bar{h}_2 = \tau \curvearrowright (\bar{h}_1 \star \bar{h}_2) \tag{2.2}$$

holds for any $\tau \in \mathcal{T}$ and $\bar{h}_1, \bar{h}_2 \in \mathcal{F}$.

Finally, we extend \curvearrowright to $\mathcal{H} \otimes \mathcal{H}$ via

$$h_1 h_2 \curvearrowright \bar{h} = (h_1 \curvearrowright \bar{h}_{(1)}) (h_2 \curvearrowright \bar{h}_{(2)}).$$

Example 2.9. Since \bullet is primitive and

$$\Delta_* \bullet\bullet = \bullet\bullet \otimes 1 + 2\bullet \otimes \bullet + 1 \otimes \bullet\bullet$$

we have

$$\begin{aligned} \bullet\bullet \curvearrowright \bullet &= 2\bullet, \\ \bullet\bullet \curvearrowright \bullet\bullet &= 2\vee + 2\bullet. \end{aligned}$$

◇

Furthermore, the \star product admits the following description:

$$\zeta_{\bar{h}} \star \zeta_h = \zeta_{(h \curvearrowright \bar{h}_{(1)}) \bar{h}_{(2)}}.$$

More concretely, we have the formula

$$\zeta_{\bar{h}} \star \zeta_{\tau_1 \cdots \tau_n} = \zeta_{(\tau_1 \curvearrowright \bar{h}_{(1)}) \cdots (\tau_n \curvearrowright \bar{h}_{(n)}) \bar{h}_{(n+1)}}.$$

⁴In general, this holds for any pre-Lie algebra, i.e., if (V, \triangleleft) is pre-Lie, then $x\{a\} = x \triangleleft a$ is a symmetric brace for some suitable extension of \triangleleft to $S(V)$ on the right.

Example 2.10. We compute all products of forests up to degree 3:

$$\begin{aligned} \zeta_{\bullet} \star \zeta_{\bullet} &= \zeta_{\bullet\bullet} + \zeta_{\bullet\downarrow} \\ \zeta_{\bullet\bullet} \star \zeta_{\bullet} &= \zeta_{\bullet\bullet\bullet} + 2\zeta_{\bullet\bullet\downarrow} + \zeta_{\bullet\downarrow\downarrow} & \zeta_{\bullet} \star \zeta_{\bullet\bullet} &= \zeta_{\bullet\bullet\bullet} + 2\zeta_{\bullet\bullet\downarrow} \\ \zeta_{\bullet\downarrow} \star \zeta_{\bullet} &= \zeta_{\bullet\downarrow\bullet} + \zeta_{\bullet\downarrow\downarrow} & \zeta_{\bullet} \star \zeta_{\bullet\downarrow} &= \zeta_{\bullet\downarrow\bullet} + \zeta_{\bullet\downarrow\downarrow} + \zeta_{\bullet\downarrow\downarrow\downarrow}. \end{aligned}$$

In particular we see that the \star product is not commutative. ◇

Remark 2.11. When written in terms of the “pure basis” \mathcal{F} , there are some non-trivial factors in front of each term in the \star product. For example

$$\bullet \star \bullet = 2\bullet\bullet + \bullet\downarrow, \quad \bullet\bullet \star \bullet = 3\bullet\bullet\bullet + \bullet\downarrow\bullet + \bullet\downarrow\downarrow.$$

This becomes even more apparent in the case of decorated trees, as

$$\bullet_i \star \bullet_j = (1 + \delta_{i,j})\bullet_i\bullet_j + \bullet_i\downarrow_j.$$

The formula in terms of the ζ_h basis stays, however, the same in both cases.

Lemma 2.12. Let $h, \bar{h} \in \mathcal{H}$ and denote by $\Pi: \mathcal{F} \rightarrow \mathcal{T}$ the projection onto trees. The identity

$$\Pi(\bar{h} \star [h]_a) = [\bar{h} \star h]_a$$

holds.

Proof. Let us start by noticing that $[h]_a = \bullet_a \frown h$. Now, from the formula for the product in terms of grafting we see that

$$\begin{aligned} \zeta_{\bar{h} \star [h]_a} &= \zeta_{([\bar{h}]_a \frown \bar{h}_{(1)}) \bar{h}_{(2)}} \\ &= \zeta_{((\bullet_a \frown h) \frown \bar{h}_{(1)}) \bar{h}_{(2)}}. \end{aligned}$$

Therefore, by eq. (2.2),

$$\begin{aligned} \zeta_{\Pi(\bar{h} \star [h]_a)} &= \zeta_{(\bullet_a \frown h) \frown \bar{h}} \\ &= \zeta_{\bullet_a \frown (\bar{h} \star h)} \\ &= \zeta_{[\bar{h} \star h]_a}. \end{aligned}$$

The proof is finished by noting that the map $[\cdot]_a$ leaves the symmetry factor invariant. □

2.13. Let $n \in \mathbb{N}$, then we denote by C_n the vector space of continuous scalar functions on $[0, 1]^n$, vanishing whenever two contiguous arguments coincide. More precisely, C_n consists of functions $f := [0, 1]^n \rightarrow \mathbb{R}$ such that $f_{t_1 \dots t_n} = 0$ when $t_i = t_{i+1}$ for some index $i \in \{1, \dots, n-1\}$. As a convention, we set $C_0 = \mathbb{R}$.

Given $f \in C_n$, we define $\delta f \in C_{n+1}$ by setting

$$\delta f_{t_1 \dots t_{n+1}} = \sum_{k=1}^{n+1} (-1)^k f_{t_1 \dots \hat{t}_k \dots t_{n+1}}$$

where \hat{t}_k means that this argument is omitted. For instance, if $f \in C_1$ then $\delta f_{s,t} = f_t - f_s$; and if $f \in C_2$ then $\delta f_{s,u,t} = f_{s,t} - f_{s,u} - f_{u,t}$ and so on. It is a known fact that if $f \in C_2$ is such that $\delta f = 0$ then $f = \delta g$ for some $g \in C_1$.

Given $f \in C_2$ and $\alpha > 0$, we define

$$\|f\|_{\alpha} := \sup_{s \neq t} \frac{|f_{s,t}|}{|t-s|^{\alpha}}$$

and we set $C_2^{\alpha} := \{f \in C_2 : \|f\|_{\alpha} < \infty\}$.

Before we continue let us fix some useful notation (which has the unfortunate sideeffect of identifying the dual and primal Hopf algebra structures).

2.14. Recall that $\mathcal{H} \rightarrow \mathcal{H}^*$ is an injection defined on $h \in \mathcal{F}$ by the linear functional δ_h (and we suppress the identification in the notation), hence it makes sense to use both the evaluation $\mathbf{X}_{s,t}^h := \langle \mathbf{X}_{s,t}, h \rangle$ and $\mathbf{X}_{s,t} \star h := \mathbf{X}_{s,t} \star \delta_h$.

2.15. For $\alpha \in (0, 1)$, an α -Hölder branched rough path is a family of characters $(\mathbf{X}_{s,t} : s, t \in [0, T])$ over \mathcal{H} such that

$$|\langle \mathbf{X}_{s,t}, h \rangle| \lesssim |t - s|^{\alpha|h|}$$

for all $h \in \mathcal{H}$, and $\mathbf{X}_{s,u} \star \mathbf{X}_{u,t} = \mathbf{X}_{s,t}$ for all $s, u, t \in [0, T]$.

With the help of the notation introduced in the previous paragraph, we remark that Chen's identity can be rewritten as

$$\delta \mathbf{X}_{s,u,t}^h = \langle \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}, \Delta' h \rangle = \mathbf{X}_{s,u}^{h'} \mathbf{X}_{u,t}^{h''} \quad (2.3)$$

This identity together with eq. (2.1) implies that for all labels $a \in A$ we have

$$\delta \mathbf{X}_{s,u,t}^{[h]_a} = \mathbf{X}_{s,u}^h \mathbf{X}_{u,t}^{\bullet a} + \mathbf{X}_{s,u}^{h'} \mathbf{X}_{u,t}^{[h'']_a}.$$

In particular, if $h \in \text{Prim}(\mathcal{H})$, there exists a path $\Gamma^h \in C_1$ such that $\mathbf{X}_{s,t}^h = \delta \Gamma_{s,t}^h$. In the case where $h = \bullet a$ for some $a \in A$, we just write X^a . We observe that this path is in general **not** unique, but there is a canonical choice with $\Gamma_0^h = 0$. We also note that by definition, Γ^h is $\alpha|h|$ -Hölder continuous.

We will denote by \mathcal{E}^α the set of α -Hölder branched rough paths. We also set $N := \lfloor \alpha^{-1} \rfloor$ so that $N\alpha \leq 1 < (N+1)\alpha$.

We endow the set \mathcal{E}^α with the distance

$$\rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) := \max_{h \in \mathcal{F}_{(N)}^+} \|\mathbf{X}^h - \tilde{\mathbf{X}}^h\|_{|h|\alpha}$$

and we define

$$\|\mathbf{X}\|_\alpha := \rho_\alpha(\mathbf{1}, \mathbf{X}) = \max_{h \in \mathcal{F}_{(N)}^+} \|\mathbf{X}^h\|_{|h|\alpha}.$$

2.16. Let $\mathbf{X} \in \mathcal{E}^\alpha$ for $\alpha \neq 1/n$ for any $n \in \mathbb{N}$.⁵ A path controlled by \mathbf{X} is a path $\mathbf{Z} : [0, T] \rightarrow \mathcal{H}_{(N)}^{<}$ such that

$$|\langle h, \mathbf{Z}_t \rangle - \langle \mathbf{X}_{s,t} \star h, \mathbf{Z}_s \rangle| \lesssim |t - s|^{(N-|h|)\alpha}$$

for all $h \in \mathcal{F}_{(N)}^{<}$. We set $R_{s,t}^h := \langle h, \mathbf{Z}_t \rangle - \langle \mathbf{X}_{s,t} \star h, \mathbf{Z}_s \rangle$ and we note that this condition is equivalent to requiring that $R^h \in C_2^{(N-|h|)\alpha}$ for all $h \in \mathcal{F}_{(N-1)}$. We denote the space of controlled paths by $\mathcal{D}_\mathbf{X}^\alpha$. It is a Banach space when endowed with the norm

$$\|\mathbf{Z}\|_\alpha := \sum_{h \in \mathcal{F}_{(N)}^{<}} \left(|\langle h, \mathbf{Z}_0 \rangle| + \|R^h\|_{(N-|h|)\alpha} \right).$$

Given $\beta < \alpha$, we let $\mathcal{D}_\mathbf{X}^{\alpha,\beta} := \overline{\mathcal{D}_\mathbf{X}^\alpha}^{\|\cdot\|_\beta}$.

If $\mathbf{Z} \in \mathcal{D}_\mathbf{X}^\alpha$ and $\tilde{\mathbf{Z}} \in \mathcal{D}_{\tilde{\mathbf{X}}}^\alpha$, we define

$$\|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\alpha := \sum_{h \in \mathcal{F}_{(N)}^{<}} \left(|\langle h, \mathbf{Z}_0 - \tilde{\mathbf{Z}}_0 \rangle| + \|R^h - \tilde{R}^h\|_{(N-|h|)\alpha} \right) \quad (2.4)$$

where $\tilde{R}_{s,t}^h := \langle h, \tilde{\mathbf{Z}}_t \rangle - \langle \tilde{\mathbf{X}}_{s,t} \star h, \tilde{\mathbf{Z}}_s \rangle$.

⁵This technical restriction on α is standard, cf. [28] and we will require it from now on.

Unraveling the definition of controlledness in coordinates, we see that the path $Z_t^h := \langle h, \mathbf{Z}_t \rangle$ has to satisfy for all $h \in \mathcal{F}_{(N)}^<$ and the product from Paragraph 2.6 the following relation to be controlled:

$$\delta Z_{s,t}^h = \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \langle \bar{h} \star h, \mathbf{Z}_s \rangle \mathbf{X}_{s,t}^{\bar{h}} + R_{s,t}^h \quad (2.5)$$

This coincides with Gubinelli's definition in [19].

Example 2.17. Suppose $\alpha \in (\frac{1}{5}, \frac{1}{4})$. A path $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$ satisfies (cf. [19, Example 8.2]),

$$\begin{aligned} \delta Z_{s,t}^1 &= Z_s^\bullet \mathbf{X}_{s,t}^\bullet + Z_s^{\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet\bullet} + R_{s,t}^1, \\ \delta Z_{s,t}^\bullet &= \left(2Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \right) \mathbf{X}_{s,t}^\bullet + \left(3Z_s^{\bullet\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + \left(Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet\bullet} + R_{s,t}^\bullet, \\ \delta Z_{s,t}^{\bullet\bullet} &= \left(3Z_s^{\bullet\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + R_{s,t}^{\bullet\bullet}, \\ \delta Z_{s,t}^{\bullet\bullet\bullet} &= \left(Z_s^{\bullet\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet\bullet} + Z_s^{\bullet\bullet\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet\bullet} + R_{s,t}^{\bullet\bullet\bullet} \end{aligned}$$

with $Z_s^{\bullet\bullet\bullet\bullet}, Z_s^{\bullet\bullet\bullet\bullet}, Z_s^{\bullet\bullet\bullet\bullet}, Z_s^{\bullet\bullet\bullet\bullet} \in C^\alpha$, $R_s^{\bullet\bullet}, R_s^{\bullet\bullet\bullet} \in C_{2}^{2\alpha}$, $R_s^\bullet \in C_{2}^{3\alpha}$ and $R^1 \in C_{2}^{4\alpha}$, where we have used the computations in Example 2.10. \diamond

Thanks to Paragraph 2.3 the controlledness condition can be rewritten in term of the basis \mathcal{P}^\top relative to a basis \mathcal{P} of Prim. Indeed, for any $h \in \mathcal{F}_{(N)}^<$ the remainder rewrites as

$$R_{s,t}^h = \delta Z_{s,t}^h - \sum_{\rho \in \mathcal{P}_{(N-|h|)}^{\top, <}} Z^{\rho^* \star h} \mathbf{X}_{s,t}^\rho.$$

2.18. Controlled paths are ‘‘good integrands’’ for rough paths, in the sense that if $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$ then for any label $a \in A$,

$$\int_0^t Z_s d\mathbf{X}_s^a := \lim_{|\pi| \rightarrow 0} \sum_{[u,v] \in \pi} \sum_{h \in \mathcal{F}_{(N)}^<} Z_u^h \mathbf{X}_{u,v}^{[h]_a}$$

exists, and defines what is known as the *rough integral of \mathbf{Z} against \mathbf{X}^a* . It satisfies the fundamental inequality

$$\left| \int_s^t Z_u d\mathbf{X}_u^a - \sum_{h \in \mathcal{F}_{(N)}^<} Z_s^h \mathbf{X}_{s,t}^{[h]_a} \right| \leq C \|\mathbf{Z}\|_\alpha \|\mathbf{X}\|_\alpha |t - s|^{(N+1)\alpha} \quad (2.6)$$

Moreover, it defines an element $\mathfrak{I}_{\mathbf{X}}^a(\mathbf{Z}) \in \mathcal{H}_{(N)}$ with components

$$\langle 1, \mathfrak{I}_{\mathbf{X}}^a(\mathbf{Z}) \rangle = \int_0^t Z_u d\mathbf{X}_u^a, \quad \langle [h]_a, \mathfrak{I}_{\mathbf{X}}^a(\mathbf{Z}) \rangle = Z_t^h$$

and zero otherwise.

Proposition 2.19. *The map $\mathfrak{I}_{\mathbf{X}}^a: \mathcal{D}_{\mathbf{X}}^\alpha \rightarrow \mathcal{D}_{\mathbf{X}}^\alpha$ is bounded, i.e., there is a constant $C = C(\alpha)$ such that*

$$\|\mathfrak{I}_{\mathbf{X}}^a(\mathbf{Z})\|_\alpha \leq C(1 + T^\alpha)(1 + \|\mathbf{X}\|_\alpha) \|\mathbf{Z}\|_\alpha.$$

Proof. For all $h \in \mathcal{F}_{(N-1)}$ set

$$\mathfrak{R}_{s,t}^h := \delta \mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})_{s,t} - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}}.$$

We first show that for all $h \in \mathcal{F}_{(N-2)}$ we have

$$\mathfrak{R}_{s,t}^{[h]_a} = R_{s,t}^h + \sum_{\bar{h} \in \mathcal{F}_{N-|h|-1}} Z_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}}$$

and zero otherwise. Indeed, recall that for any $h \in \mathcal{F}_{(N-1)}$, we have that

$$\mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})_s^{[h]_a} = Z_s^h$$

We note that by Lemma 2.12, the identity

$$\langle \bar{h} \star [h]_a, \mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})_s \rangle = \langle [\bar{h} \star h]_a, \mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})_s \rangle = Z_s^{\bar{h} \star h}$$

holds. Hence

$$\begin{aligned} \mathfrak{R}_{s,t}^{[h]_a} &= \delta Z_{s,t}^h - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-2)}^+} Z_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}} \\ &= R_{s,t}^h + \sum_{\bar{h} \in \mathcal{F}_{N-|h|-1}} Z_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}}. \end{aligned}$$

Now, given $h \in \mathcal{F}_{(N-2)}$, the remainder satisfies

$$\|\mathfrak{R}^{[h]_a}\|_{(N-|h|-1)\alpha} \leq \|R^h\|_{(N-|h|-1)\alpha} T^\alpha + \|\mathbf{X}\|_\alpha \sum_{\bar{h} \in \mathcal{F}_{N-|h|}} \|Z^{\bar{h} \star h}\|_\infty.$$

Moreover, from the fundamental estimate (2.6) we see that

$$\|\mathfrak{R}^1\|_{N\alpha} \leq C \|\mathbf{Z}\|_\alpha \|\mathbf{X}\|_\alpha T^\alpha + \|\mathbf{X}\|_\alpha \sum_{h \in \mathcal{F}_{N-1}} \|Z^h\|_\infty.$$

For any $h \in \mathcal{F}_{N-1}$ we have that

$$\|Z^h\|_\infty \leq |Z_0^h| + T^\alpha \|Z^h\|_\alpha.$$

Finally, the norm can be bounded:

$$\begin{aligned} \|\mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z})\|_\alpha &= \sum_{h \in \mathcal{F}_{(N)}^<} \|\mathfrak{R}^h\|_{(N-|h|)\alpha} \\ &= \|\mathfrak{R}^1\|_{N\alpha} + \sum_{h \in \mathcal{F}_{(N-1)}^<} \|\mathfrak{R}^{[h]_a}\|_{(N-|h|-1)\alpha} \\ &\leq C \|\mathbf{Z}\|_\alpha \|\mathbf{X}\|_\alpha T^\alpha + \|\mathbf{Z}\|_\alpha T^\alpha + \|\mathbf{X}\|_\alpha \sum_{h \in \mathcal{F}_{(N)}^<} (|Z_0^h| + \|Z^h\|_\alpha) \\ &\leq C(1 + \|\mathbf{X}\|_\alpha)(1 + T^\alpha) \|\mathbf{Z}\|_\alpha. \quad \square \end{aligned}$$

Proposition 2.20. *Let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{C}^\alpha$ such that $\|\mathbf{X}\|_\alpha \vee \|\tilde{\mathbf{X}}\|_\alpha \leq M$ and let $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$, $\tilde{\mathbf{Z}} \in \mathcal{D}_{\tilde{\mathbf{X}}}^\alpha$ be such that $\|\mathbf{Z}\|_\alpha \vee \|\tilde{\mathbf{Z}}\|_\alpha \leq M$. Then there is a constant $C = C(\alpha, M)$ such that for all $a \in A$, the bound*

$$\|\mathfrak{S}_{\mathbf{X}}^a(\mathbf{Z}); \mathfrak{S}_{\tilde{\mathbf{X}}}^a(\tilde{\mathbf{Z}})\|_\alpha \leq C(\|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\alpha + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}))$$

holds uniformly.

Proof. By considering the germ

$$\Xi_{s,t} = \sum_{h \in \mathcal{F}_{(N)}^<} Z_s^h \mathbf{X}_{s,t}^{[h]_a} - \sum_{h \in \mathcal{F}_{(N)}^<} \tilde{Z}_s^h \tilde{\mathbf{X}}_{s,t}^{[h]_a}$$

it is possible to show, using the Sewing Lemma [12, Lemma 4.2], that (using the same notations as in Proposition 2.19)

$$\|\mathfrak{R}^1 - \tilde{\mathfrak{R}}^1\|_{N\alpha} \leq CT^\alpha (\|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\alpha + \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}})) + M\rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + M \sum_{h \in \mathcal{F}_{N-1}} \|Z^h - \tilde{Z}^h\|_\infty.$$

Proceeding in a similar way to the proof of Proposition 2.19, we see that

$$\mathfrak{R}_{s,t}^{[h]_a} - \tilde{\mathfrak{R}}_{s,t}^{[h]_a} = R_{s,t}^h - \tilde{R}_{s,t}^h + \sum_{\tilde{h} \in \mathcal{F}_{N-|h|-1}} Z_s^{\tilde{h}*h} \mathbf{X}_{s,t}^{\tilde{h}} - \sum_{\tilde{h} \in \mathcal{F}_{N-|h|-1}} \tilde{Z}_s^{\tilde{h}*h} \tilde{\mathbf{X}}_{s,t}^{\tilde{h}}$$

from where the bound

$$\|\mathfrak{R}^{[h]_a} - \tilde{\mathfrak{R}}^{[h]_a}\|_{(N-|h|-1)\alpha} \leq \|R^h - \tilde{R}^h\|_{(N-|h|-1)\alpha} T^\alpha + M\rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + M \sum_{\tilde{h} \in \mathcal{F}_{N-|h|-1}} \|Z^{\tilde{h}*h} - \tilde{Z}^{\tilde{h}*h}\|_\infty$$

follows. Summing over $h \in \mathcal{F}_{(N)}^<$ yields the desired bound. \square

3. MAIN APPROXIMATION RESULT

In this section, we generalise a key result obtained in [16]. There it was shown in [16, Theorem 3.10] that for a fixed α -rough path \mathbf{X} , if $\beta < \alpha < 1/2$ are sufficiently close to $1/2$, the set

$$\left\{ (\psi, \psi') \mid \psi_{s,t} = \int_s^t f_r d\mathbf{X}_r + \delta g_{s,t}, \psi'_s = f_s \text{ where } f, g \in C_1^\infty \right\},$$

is dense in $\mathcal{D}_{\mathbf{X}}^{\alpha,\beta}$ (where the integral is understood in the Young sense). As a consequence, if $\mathbf{X}(\omega)$ is a random rough path of α -regularity (e.g. the lift of a Brownian motion), the spaces $\{\mathcal{D}_{\mathbf{X}(\omega)}^{\alpha,\beta}\}_{\omega \in \Omega}$ turned out to be a measurable field of Banach spaces [16, Proposition 3.15]. Our aim is to generalise and strengthen these results. We shall see that the field is indeed a *continuous* field, we will remove the cumbersome conditions imposed on α, β in loc.cit. and obtain the result for arbitrary $\beta < \alpha$. To this end, we will construct a dense subset of smooth functions for every order:

Definition 3.1. Fix $\varepsilon \in (0, 1 - N\alpha)$ (where we recall that $N\alpha < 1$ due to our assumption that $\alpha \neq 1/n, n \in \mathbb{N}$) and $C^{0,1-\varepsilon}$ denote the closure of C^∞ under the $(1 - \varepsilon)$ -Hölder norm.⁶ We recall that this space is a separable Banach space, and the following inclusions hold: let $\text{PL} \subset \text{Lip}$ denote the space of piecewise linear functions on $[0, T]$, then

$$\overline{\text{PL}}^{\|\cdot\|^{1-\varepsilon}} = C^{0,1-\varepsilon} \subset C^{1-\varepsilon}.$$

Moreover, since $1 - \varepsilon > N\alpha > (N - 1)\alpha > \dots > \alpha$ we get that

$$C^{0,1-\varepsilon} \subset C^{N\alpha} \subset \dots \subset C^\alpha$$

and $\|f\|_{k\alpha} \leq \|f\|_{1-\varepsilon} T^{1-\varepsilon-k\alpha}$ for all $k \in \{1, \dots, N\}$.

Denote by $\Omega \subset C^{0,1-\varepsilon}$ a countable dense subset and define for $N \in \mathbb{N}$ the sets

$$\mathcal{S}_N := \bigoplus_{h \in \mathcal{F}_{(N-1)}} C^{0,1-\varepsilon}, \quad \mathcal{S}_N^0 := \bigoplus_{h \in \mathcal{F}_{(N-1)}} \Omega.$$

⁶These spaces are also known as the little Lipschitz spaces, see [33, Chapter 4] and cf. [13, Theorem 5.33]

We endow \mathcal{S}_N with the norm

$$\|f\| := \max_{h \in \mathcal{F}_{(N-1)}} \|f^h\|_{1-\varepsilon}.$$

It will turn out that (rough) integration of elements in \mathcal{S}_N leads to a dense subset of the Banach space of controlled paths over a given rough path \mathbf{X} . To establish this result, let us define a map $\Gamma_{\mathbf{X}}: \mathcal{S}_N \rightarrow \mathcal{D}_{\mathbf{X}}^\alpha$ with the property that for all $h \in \mathcal{F}_{(N-1)}$, the remainder

$$R_{s,t}^h = \delta\Gamma_{\mathbf{X}}(f)_{s,t}^h - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \Gamma_{\mathbf{X}}(f)_s^{\bar{h}*h} \mathbf{X}_{s,t}^{\bar{h}} \quad (3.1)$$

satisfies $|R_{s,t}^h| \leq C(1 + \|\mathbf{X}\|_\alpha)^{N-1-|h|} \|f\| |t-s|^{1-\varepsilon}$. The definition of $\Gamma_{\mathbf{X}}(f)$ is recursive:

$$\Gamma_{\mathbf{X}}(f)_t^h := f_t^h \text{ for every } h \in \mathcal{F}_{N-1}. \quad (3.2)$$

Clearly,

$$\|R^h\|_{1-\varepsilon} = \|f\|_{1-\varepsilon} \leq \|f\|$$

and in particular $R^h \in C_2^\alpha$ with $\|R^h\|_\alpha \leq \|f\| T^{1-\varepsilon-\alpha}$.

Given $n < N-1$, let $h \in \mathcal{F}_n$, and suppose that we have defined $\Gamma_{\mathbf{X}}(f)^{\bar{h}}$ for all forests $\bar{h} \in \mathcal{F}$ with $n < |\bar{h}| \leq N-1$, in a way such that $|R_{s,t}^{\bar{h}}| \leq C(1 + \|\mathbf{X}\|_\alpha)^{N-1-|\bar{h}|} \|f\| |t-s|^{1-\varepsilon}$.

Recall that every $\rho \in \mathcal{P}^\top$ is of the form $\rho = \top(\rho_{i_1}, \dots, \rho_{i_k})$ for some integer $k \geq 1$ and primitive elements $\rho_{i_1}, \dots, \rho_{i_k} \in \mathcal{P}$. Given $\rho \in \mathcal{P}^\top$ we define $\rho^* = f_{i_1, \dots, i_k}$ to be its dual basis element.

Lemma 3.2. *Let $h \in \mathcal{F}_n$ and $\rho \in \mathcal{P}$ be a primitive element with $|\rho| < N-n$. Then the rough integral*

$$\int_0^t \Gamma_{\mathbf{X}}(f)_r^{\rho^* * h} d\mathbf{X}_r^\rho := \lim_{|\pi| \rightarrow 0} \sum_{[a,b] \in \pi} \sum_{\rho \in \mathcal{P}_{(N-n-|\rho|)}^{\top, <}} \Gamma_{\mathbf{X}}(f)_a^{\rho^* * \rho^* * h} \mathbf{X}_{a,b}^{\rho \top \rho}$$

exists along any sequence of partitions, and it is independent of any choice. Moreover,

$$\left| \int_s^t \Gamma_{\mathbf{X}}(f)_r^{\rho^* * h} d\mathbf{X}_r^\rho - \sum_{\rho \in \mathcal{P}_{(N-n-|\rho|)}^{\top, <}} \Gamma_{\mathbf{X}}(f)_s^{\rho^* * \rho^* * h} \mathbf{X}_{s,t}^{\rho \top \rho} \right| \leq C \|f\| (1 + \|\mathbf{X}\|_\alpha)^{N-2-n} \|\mathbf{X}\|_\alpha |t-s|^{1-\varepsilon+|\rho|\alpha}.$$

Proof. Let

$$\Xi_{s,t} := \sum_{\rho \in \mathcal{P}_{(N-n-k)}^{\top, <}} \Gamma_{\mathbf{X}}(f)_s^{\rho^* * \rho^* * h} \mathbf{X}_{s,t}^{\rho \top \rho}.$$

Observe that

$$\delta\Xi_{s,u,t} = \sum_{\rho \in \mathcal{P}_{(N-n-k)}^{\top, <}} \left(\Gamma_{\mathbf{X}}(f)_s^{\rho^* * \rho^* * h} \delta\mathbf{X}_{s,u,t}^{\rho \top \rho} - \delta\Gamma_{\mathbf{X}}(f)_{s,u}^{\rho^* * \rho^* * h} \mathbf{X}_{u,t}^{\rho \top \rho} \right).$$

By recalling that

$$\delta\mathbf{X}_{s,u,t}^{\rho \top \rho} = \mathbf{X}_{s,u}^\rho \mathbf{X}_{u,t}^\rho + \sum_{(\rho)} \mathbf{X}_{s,u}^{\rho'} \mathbf{X}_{u,t}^{\rho'' \top \rho},$$

a standard computation gives

$$\delta\Xi_{s,u,t} = - \sum_{\rho \in \mathcal{P}_{(N-n-|\rho|)}^{\top, <}} R_{s,u}^{\rho^* * \rho^* * h} \mathbf{X}_{u,t}^{\rho \top \rho}.$$

Therefore

$$|\delta \Xi_{s,u,t}| \leq C \|f\| (1 + \|\mathbf{X}\|_\alpha)^{N-2-n} \|\mathbf{X}\|_\alpha \sum_{\rho \in \mathcal{P}_{(N-n-|\rho|)}^{\top, <}} |u-s|^{1-\varepsilon} |t-u|^{(|\rho|+|\rho|)\alpha}.$$

The result then follows from the Sewing Lemma since $1 - \varepsilon + |\rho|\alpha > (N + |\rho|)\alpha > (N + 1)\alpha > 1$. \square

We then set for every $h \in \mathcal{F}_n, 0 < n < N - 1$

$$\Gamma_{\mathbf{X}}(f)_t^h := \sum_{\substack{\rho \in \mathcal{P} \\ |\rho| < N-|h|}} \int_0^t \Gamma_{\mathbf{X}}(f)_r^{\rho^* \star h} d\mathbf{X}_r^\rho + f_t^h. \quad (3.3)$$

Lemma 3.3. *The map $\Gamma_{\mathbf{X}}: \mathcal{S}_N \rightarrow \mathcal{D}_{\mathbf{X}}^\alpha$ sending f to $\Gamma_{\mathbf{X}}(f)$ as in eqs. (3.2) and (3.3) is well-defined, linear and bounded, with*

$$\|\Gamma_{\mathbf{X}}(f)\|_\alpha \leq C \frac{e^{N\|\mathbf{X}\|_\alpha} - 1}{\|\mathbf{X}\|_\alpha} \|f\| \quad (3.4)$$

for all $f \in \mathcal{S}_N$.

Proof. The proof is by induction. Linearity clearly holds when $|h| = N - 1$. Indeed,

$$\Gamma_{\mathbf{X}}(f + \lambda g)_t^h = (f + \lambda g)_t^h = f_t^h + \lambda g_t^h = \Gamma_{\mathbf{X}}(f)_t^h + \lambda \Gamma_{\mathbf{X}}(g)_t^h.$$

Now pick $h \in \mathcal{F}_n$, and suppose that linearity holds for all forests \bar{h} with $|\bar{h}| > n$. Then

$$\begin{aligned} \Gamma_{\mathbf{X}}(f + \lambda g)_t^h &= \sum_{\substack{\rho \in \mathcal{P} \\ |\rho| < N-n}} \int_0^t \Gamma_{\mathbf{X}}(f + \lambda g)_r^{\rho^* \star h} d\mathbf{X}_r^\rho + (f + \lambda g)_t^h \\ &= \sum_{\substack{\rho \in \mathcal{P} \\ |\rho| < N-n}} \int_0^t \left\{ \Gamma_{\mathbf{X}}(f)_r^{\rho^* \star h} + \lambda \Gamma_{\mathbf{X}}(g)_r^{\rho^* \star h} \right\} d\mathbf{X}_r^\rho + f_t^h + \lambda g_t^h \\ &= \Gamma_{\mathbf{X}}(f)_t^h + \lambda \Gamma_{\mathbf{X}}(g)_t^h. \end{aligned}$$

From the previous sections, we have that $\|R^h\|_\alpha \leq C \|f\| T^{1-\varepsilon-\alpha}$ for all $h \in \mathcal{F}_{N-1}$. Take $h \in \mathcal{F}_n$, and note that

$$\begin{aligned} R_{s,t}^h &= \delta \Gamma_{\mathbf{X}}(f)_{s,t}^h - \sum_{\bar{h} \in \mathcal{P}_{(N-n-1)}^\top} \Gamma_{\mathbf{X}}(f)_s^{\rho^* \star h} \mathbf{X}_{s,t}^\rho \\ &= \sum_{\rho \in \mathcal{P}_{(N-n-1)}} \int_s^t \Gamma_{\mathbf{X}}(f)_u^{\rho^* \star h} d\mathbf{X}_u^\rho - \sum_{\rho \in \mathcal{P}_{(N-n-1)}^\top} \Gamma_{\mathbf{X}}(f)_s^{\rho^* \star h} \mathbf{X}_{s,t}^\rho + \delta f_{s,t}^h \\ &= \sum_{\rho \in \mathcal{P}_{(N-n-1)}} \left(\int_s^t \Gamma_{\mathbf{X}}(f)_u^{\rho^* \star h} d\mathbf{X}_u^\rho - \sum_{\rho \in \mathcal{P}_{(N-n-|\rho|)}^\top} \Gamma_{\mathbf{X}}(f)^{\rho^* \star \rho^* \star h} \mathbf{X}_{s,t}^{\rho \top \rho} \right) + \delta f_{s,t}^h. \end{aligned}$$

Therefore, by Lemma 3.2 we see that

$$|R_{s,t}^h| \leq C \|f\| (1 + \|\mathbf{X}\|_\alpha)^{N-2-|h|} \|\mathbf{X}\|_\alpha \sum_{\rho \in \mathcal{P}_{(N-n-1)}} |t-s|^{1-\varepsilon+|\rho|\alpha} + \|f^h\|_{1-\varepsilon} |t-s|^{1-\varepsilon},$$

so that

$$\|R^h\|_{(N-n)\alpha} \leq C_T (1 + \|\mathbf{X}\|_\alpha)^{N-1-|h|} \|f\|. \quad (3.5)$$

since $1 - \varepsilon - (N - n + k)\alpha > (n - k)\alpha > (N - 1)\alpha > 0$ for all $k \in \{0, \dots, N - n - 1\}$. In particular, $R^h \in C_2^{(N-|h|)\alpha}$ for all $h \in \mathcal{F}_{(N-1)}$, i.e., $\Gamma_{\mathbf{X}}(f) \in \mathcal{D}_{\mathbf{X}}^\alpha$, and the required bound holds after summation over $h \in \mathcal{F}_{(N)}^<$. \square

Theorem 3.4. *Let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{C}^\alpha$ such that $\|\mathbf{X}\|_\alpha \vee \|\tilde{\mathbf{X}}\|_\alpha \leq M$ and $f \in \mathcal{S}_N$. Then,*

$$\|\Gamma_{\mathbf{X}}(f); \Gamma_{\tilde{\mathbf{X}}}(f)\|_\alpha \leq C \|f\| \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}).$$

Proof. For $h \in \mathcal{F}_{(N-1)}$ and $p \in \mathcal{P}$, let us define

$$Q_{s,t}^{h,p} := \int_s^t \Gamma_{\mathbf{X}}(f)_u^{\rho^{*\star h}} d\mathbf{X}_u^p - \sum_{\rho \in \mathcal{P}_{(N-|h|-|p|)}^{T,<}} \Gamma_{\mathbf{X}}(f)_s^{\rho^{*\star h}} \mathbf{X}_{s,t}^{\rho T p}.$$

Using the Sewing Lemma and proceeding as in the proof of Lemma 3.2 one can inductively show that

$$|Q_{s,t}^{h,p} - \tilde{Q}_{s,t}^{h,p}| \leq C \|f\| \rho_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) |t - s|^{1-\varepsilon+|p|\alpha}.$$

Then, arguing as in the proof of Lemma 3.3, we see that

$$|R_{s,t}^h - \tilde{R}_{s,t}^h| \leq C \sum_{p \in \mathcal{P}_{(N-|h|-1)}} |Q_{s,t}^{h,p} - \tilde{Q}_{s,t}^{h,p}|,$$

and the required bound is obtained after summation over $h \in \mathcal{F}_{(N)}^<$. \square

In order to make the presentation more amenable to the reader, the construction of our approximation will be split in two parts. First we show how to approximate a given controlled path $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$ in the $\|\cdot\|_\beta$ norm over a small interval $[0, T]$, for any $\beta < \alpha$. Then, we extend this approximation to an arbitrary interval by patching together each of the individual approximations.

3.1. Short-time approximations. Given $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$, we start by defining a controlled path of the form $\Gamma_{\mathbf{X}}(f): [0, T] \rightarrow \mathcal{F}_{(N-1)}$ on a given time interval $[0, T]$ such that $\|\mathbf{Z}; \Gamma_{\mathbf{X}}(f)\|_\beta$ is arbitrarily small for small $T > 0$ and any $\beta < \alpha$.

We do this by a backward procedure: The highest order functions are defined as affine functions. That is, for every $h \in \mathcal{F}_{N-1}$, we set

$$f_t^h = Z_0^h + \frac{t}{T} (Z_T^h - Z_0^h), \quad t \in [0, T]. \tag{3.6}$$

The idea is to define all lower order terms by integration against the rough path.

Assume that we have defined f^h for all $h \in \mathcal{F}$ with $n < |h| \leq N - 1$. Given $h \in \mathcal{F}_n$ we set

$$f_t^h = Z_0^h + \frac{t}{T} \left[\delta Z_{0,T}^h - \sum_{p \in \mathcal{P}_{(N-|h|-1)}} \int_0^T \Gamma_{\mathbf{X}}(f)_u^{\rho^{*\star h}} d\mathbf{X}_u^p \right]. \tag{3.7}$$

where the integral is defined by Lemma 3.2. Note that by construction we have

$$\|f^h\|_{1-\varepsilon} \leq C \|R^h\|_{(N-|h|)\alpha} T^{\alpha(N-|h|)+\varepsilon-1} \tag{3.8}$$

for some $C > 0$.

The controlled path $\Gamma_{\mathbf{X}}(f)$ so constructed can serve as a first-order approximation to a given controlled path \mathbf{Z} . This is summarized in the next lemma.

Lemma 3.5. *Let \mathbf{Z} be a controlled path. Define $\tilde{\mathbf{Z}} := \Gamma_{\mathbf{X}}(f)$ with f as in eqs. (3.6) and (3.7) and \tilde{R}^h as in (3.1). Then $\tilde{\mathbf{Z}}$ has the following properties:*

- (i) $\mathbf{Z}_0 = \tilde{\mathbf{Z}}_0$ and $\mathbf{Z}_T = \tilde{\mathbf{Z}}_T$.
- (ii) $\tilde{R}_{0,T}^h = R_{0,T}^h$ for every $h \in \mathcal{F}_{(N-1)}$.
- (iii) For every $\beta < \alpha$, there is a constant $C > 0$ such that

$$\|R^h - \tilde{R}^h\|_{(N-|h|)\beta} \leq CT^{(N-|h|)(\alpha-\beta)}.$$

In particular,

$$\|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\beta \leq C \frac{T^{N(\alpha-\beta)} - T^{\alpha-\beta}}{T^{\alpha-\beta} - 1}.$$

Proof. (i) follows by definition. Next,

$$\begin{aligned} \tilde{R}_{0,T}^h &= \delta \tilde{\mathbf{Z}}_{0,T}^h - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{\mathbf{Z}}_0^{\bar{h} \star h} \mathbf{X}_{0,T}^{\bar{h}} \\ &= \delta \mathbf{Z}_{0,T}^h - \sum_{\bar{h} \in \mathcal{P}_{(N-|h|-1)}^+} \mathbf{Z}_0^{\bar{h} \star h} \mathbf{X}_{0,T}^{\bar{h}} \\ &= R_{0,T}^h \end{aligned} \quad (3.9)$$

and property (ii) is shown. Finally, in view of (3.8) we obtain an estimate for $\|f\|$ in terms of $\|\mathbf{Z}\|$. Thus property (iii) follows from the estimate (3.5) and the triangle inequality. \square

3.2. From local to global. Having constructed affine controlled paths on a given interval, we will glue them together now. Fix a dissection $\pi = \{0 = t_0 < t_1 < \dots < t_N < t_{N+1} = T\}$ of $[0, T]$ with mesh $\theta := \max_k |t_{k+1} - t_k|$. Set $I_k := [t_k, t_{k+1}]$. Assume that we are given controlled paths

$$\tilde{\mathbf{Z}}^k: I_k \rightarrow \mathcal{H}$$

for which $\tilde{\mathbf{Z}}_{t_{k+1}}^k = \tilde{\mathbf{Z}}_{t_{k+1}}^{k+1}$ and with remainders satisfying $|\tilde{R}_{s,t}^{k;h}| \leq C|t-s|(t_{k+1}-t_k)^{(N-|h|)\alpha-1}$ for every $s, t \in [t_k, t_{k+1}]$. We define $\tilde{\mathbf{Z}}: [0, T] \rightarrow \mathcal{H}$ by

$$\tilde{\mathbf{Z}}_t := \sum_{k=0}^N \tilde{\mathbf{Z}}_t^k 1_{I_k}(t) \quad (3.10)$$

and set

$$\tilde{R}_{s,t}^h := \delta \tilde{\mathbf{Z}}_{s,t}^h - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{\mathbf{Z}}_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}}.$$

Lemma 3.6. For all $h \in \mathcal{F}_{(N-1)}$ we have

$$\delta \tilde{R}_{s,u,t}^h = \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{R}_{s,u}^{\bar{h} \star h} \mathbf{X}_{u,t}^{\bar{h}}.$$

If $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$, the same identity holds with \tilde{R} replaced by the remainder R of \mathbf{Z} .

Proof. By definition we have that

$$\tilde{R}_{s,t}^h = \langle h, \delta \tilde{\mathbf{Z}}_{s,t} \rangle - \langle \mathbf{X}_{s,t} \otimes h, \Delta' \tilde{\mathbf{Z}}_s \rangle$$

Applying δ to both sides of this equation and recalling that if $f \in C_1$, $g \in C_2$ and $F_{s,t} := f_s g_{s,t} \in C_2$ then

$$\delta F_{s,u,t} = f_s \delta g_{s,u,t} - \delta f_{s,u} g_{u,t}$$

we see that

$$\delta \tilde{R}_{s,u,t}^h = -\langle \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} \otimes h, (\Delta' \otimes \text{id}) \Delta' \tilde{\mathbf{Z}}_s \rangle + \langle \mathbf{X}_{u,t} \otimes h, \Delta' \delta \tilde{\mathbf{Z}}_{s,u} \rangle$$

Now, we have that for any $f \in \mathcal{F}_{(N-|h|-1)}$,

$$\langle f \otimes h, \Delta' \delta \tilde{\mathbf{Z}}_{s,u} \rangle = \sum_{\bar{h} \in \mathcal{F}^0} \langle f \otimes h, \Delta' \bar{h} \rangle \tilde{R}_{s,u}^{\bar{h}} + \langle \mathbf{X}_{s,u} \otimes f \otimes h, (\text{id} \otimes \Delta') \Delta' \tilde{\mathbf{Z}}_s \rangle$$

and therefore

$$\begin{aligned} \delta \tilde{R}_{s,u,t}^h &= -\langle \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} \otimes h, (\Delta' \otimes \text{id}) \Delta' \tilde{\mathbf{Z}}_s \rangle + \langle \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} \otimes h, (\text{id} \otimes \Delta') \Delta' \tilde{\mathbf{Z}}_s \rangle + \sum_{\bar{h} \in \mathcal{F}^0} \langle \mathbf{X}_{u,t} \otimes h, \Delta' \bar{h} \rangle \tilde{R}_{s,u}^{\bar{h}} \\ &= \sum_{\bar{h} \in \mathcal{F}^0} \langle \mathbf{X}_{u,t} \otimes h, \Delta' \bar{h} \rangle \tilde{R}_{s,u}^{\bar{h}} \end{aligned}$$

where the first two terms cancel by coassociativity of Δ' .

Finally, we have

$$\begin{aligned} \delta \tilde{R}_{s,u,t}^h &= \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \sum_{f \in \mathcal{F}_{(N-|h|-1)}^+} \langle f \otimes h, \Delta' \bar{h} \rangle \tilde{R}_{s,u}^{\bar{h}} \mathbf{X}_{u,t}^f \\ &= \sum_{f \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{R}_{s,u}^{f \star h} \mathbf{X}_{s,u}^f \quad \square \end{aligned}$$

Proposition 3.7. *The path $\tilde{\mathbf{Z}}$ defined in (3.10) is an α -controlled path.*

Proof. We have to show that $\|\tilde{R}^h\|_{(N-|h|)\alpha} < \infty$ for every $h \in \mathcal{F}_{(N-1)}$. Fix $s < t$ with $s \in [t_k, t_{k+1}]$ and $t \in [t_l, t_{l+1}]$. Using Lemma 3.6 twice shows that

$$\tilde{R}_{s,t}^h = \tilde{R}_{s,t_{k+1}}^{k;h} + \tilde{R}_{t_{k+1},t_l}^h + \tilde{R}_{t_l,t}^{l;h} + \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{R}_{s,t_{k+1}}^{k;\bar{h} \star h} \mathbf{X}_{t_{k+1},t}^{\bar{h}} + \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} \tilde{R}_{t_{k+1},t_l}^{\bar{h} \star h} \mathbf{X}_{t_l,t}^{\bar{h}}.$$

By the triangle inequality, it suffices to show that $\tilde{R}_{s,t}^h \leq C|t-s|^{(N-|h|)\alpha}$ for every $s, t \in \pi$. This follows by induction over the length of $|t-s|$ using again Lemma 3.6 for the induction step. \square

Proposition 3.8. *Let $\mathbf{Z} \in \mathcal{D}_X^\alpha$. Fix a dissection $\pi_N = \{0 = t_0 < t_1 < \dots < t_N < t_{N+1} = T\}$ of $[0, T]$ with mesh $\theta := \max_k |t_{k+1} - t_k|$. On every interval $I_k = [t_k, t_{k+1}]$, we define $\tilde{\mathbf{Z}}^k: I_k \rightarrow \mathcal{H}$ as in Lemma 3.5 and $\tilde{\mathbf{Z}}$ as in (3.10). Then*

$$\|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\beta \rightarrow 0 \quad \text{as } \theta \rightarrow 0$$

for every $\beta < \alpha$.

Proof. By construction, $\tilde{\mathbf{Z}}$ satisfies $\tilde{\mathbf{Z}}_{t_k} = \mathbf{Z}_{t_k}$ and $\tilde{R}_{t_k, t_{k+1}}^h = R_{t_k, t_{k+1}}^h$ for all $k = 0, \dots, N$. Moreover, for all $h \in \mathcal{F}_{(N)}^+$ the remainders satisfy the bounds

$$\sup_{k=0, \dots, N} \sup_{s, t \in I_k} \frac{|\tilde{R}_{s,t}^h|}{|t-s|} \leq C\theta^{(N-|h|)\alpha-1}, \quad \sup_{k=0, \dots, N} \|R^h - \tilde{R}^h\|_{(N-|h|)\beta; I_k} \leq C\theta^{(N-|h|)(\alpha-\beta)}.$$

Lemma 3.6 implies that for any pair of contiguous mesh points $t_k < t_{k+1}$ and any $t > t_{k+1}$ we have

$$\delta R_{t_k, t_{k+1}, t}^h = \delta \tilde{R}_{t_k, t_{k+1}, t}^h.$$

Inductively, this implies that $\tilde{R}_{t_j, t_k}^h = R_{t_j, t_k}^h$ for all $0 \leq j < k \leq N$, and so

$$\delta R_{t_j, t_k, t}^h = \delta \tilde{R}_{t_j, t_k, t}^h$$

as well. Consider now any two points $s < t \in [0, T]$, and suppose that $s \in I_j$, $t \in I_k$ for some $0 \leq j < k \leq N$. Note that

$$\begin{aligned} R_{s,t}^h - \tilde{R}_{s,t}^h &= R_{s,t_{j+1}}^h - \tilde{R}_{s,t_{j+1}}^h + R_{t_{j+1},t}^h - \tilde{R}_{t_{j+1},t}^h + \delta R_{s,t_{j+1},t}^h - \delta \tilde{R}_{s,t_{j+1},t}^h \\ &= R_{s,t_{j+1}}^h - \tilde{R}_{s,t_{j+1}}^h + R_{t_k,t}^h - \tilde{R}_{t_k,t}^h + \delta R_{s,t_{j+1},t}^h - \delta \tilde{R}_{s,t_{j+1},t}^h, \end{aligned}$$

hence, since both $|t_{j+1} - s| \leq |t - s|$ and $|t - t_k| \leq |t - s|$,

$$\begin{aligned} |R_{s,t}^h - \tilde{R}_{s,t}^h| &\leq C\theta^{(N-|h|)(\alpha-\beta)}|t_{j+1} - s|^{(N-|h|)\beta} + C\theta^{(N-|h|)(\alpha-\beta)}|t - t_k|^{(N-|h|)\beta} \\ &\quad + \sum_{\tilde{h} \in \mathcal{F}_{(N-|h|-1)}^+} |R_{s,t_{j+1}}^{\tilde{h}\star h} - \tilde{R}_{s,t_{j+1}}^{\tilde{h}\star h}| |\mathbf{X}_{t_{j+1},t}^{\tilde{h}}| \\ &\leq 2C\theta^{(N-|h|)(\alpha-\beta)}|t - s|^{(N-|h|)\beta} \\ &\quad + C \sum_{\tilde{h} \in \mathcal{F}_{(N-|h|-1)}^+} \theta^{(N-|h|-\tilde{h})(\alpha-\beta)}|t_{k+1} - s|^{(N-|h|-\tilde{h})\beta}|t_{j+1} - t|^{|\tilde{h}|\alpha} \\ &\leq 3C\theta^{(N-|h|)(\alpha-\beta)}|t - s|^{(N-|h|)\beta}. \end{aligned}$$

Therefore, for all $h \in \mathcal{F}_{(N-1)}$ and $\beta \leq \alpha$,

$$\|R^h - \tilde{R}^h\|_{(N-|h|)\beta} \leq C\theta^{(N-|h|)(\alpha-\beta)}.$$

Moreover,

$$\begin{aligned} \|\mathbf{Z}; \tilde{\mathbf{Z}}\|_\beta &= \sum_{h \in \mathcal{F}_{(N)}^+} \|R^h - \tilde{R}^h\|_{(N-|h|)\beta} \\ &\leq C \sum_{h \in \mathcal{F}_{(N)}^+} \theta^{(N-|h|)(\alpha-\beta)} \\ &= C \sum_{k=1}^N \theta^k(\alpha-\beta) \rightarrow 0 \end{aligned}$$

as $\theta \rightarrow 0$. □

Corollary 3.9. For every $\mathbf{X} \in C^\alpha$, the linear subspace $\Gamma_{\mathbf{X}}(\mathcal{S}_N)$ is dense in $\mathcal{D}_{\mathbf{X}}^{\alpha,\beta}$ under $\|\cdot\|_\beta$ for $\beta < \alpha$.

Theorem 3.10. Let $N = \lfloor 1/\alpha \rfloor$. Define the following sets of sections

$$\Gamma := \{\mathbf{X} \mapsto \Gamma_{\mathbf{X}}(f) : f \in \mathcal{S}_N\}, \quad \Gamma_0 := \{\mathbf{X} \mapsto \Gamma_{\mathbf{X}}(f) : f \in \mathcal{S}_N^0\}$$

Then the following holds:

- (i) Γ is a linear subspace of $\prod_{\mathbf{X}} \mathcal{D}_{\mathbf{X}}^{\alpha,\beta}$
- (ii) For every \mathbf{X} , $\{\gamma(\mathbf{X}) : \gamma \in \Gamma_0\}$ is a countable dense subset of $\mathcal{D}_{\mathbf{X}}^{\alpha,\beta}$.
- (iii) For every $\gamma \in \Gamma$ with $\gamma(\mathbf{X}) = \Gamma_{\mathbf{X}}(f)$, the function $\mathbf{X} \mapsto \|\gamma(\mathbf{X})\|_\alpha = \|\Gamma_{\mathbf{X}}(f)\|_\alpha$ is continuous.

Proof. Part (i) follows from Lemma 3.3 applied fiberwise, and part (ii) follows from Proposition 3.8 together with the observation that Γ_0 is countable as \mathcal{S}_N^0 is.

Finally, we have to check continuity of the mapping $\mathbf{X} \mapsto \|\gamma(\mathbf{X})\|_\alpha = \|\Gamma_{\mathbf{X}}(f)\|_\alpha$. Since we are working in a metric space, it suffices to pick a sequence $\mathbf{X}_n \rightarrow \mathbf{X}$ as α -Hölder rough paths. Then we apply Theorem 3.4 to obtain

$$|\|\Gamma_{\mathbf{X}_n}(f)\|_\alpha - \|\Gamma_{\mathbf{X}}(f)\|_\alpha| \leq \|\Gamma_{\mathbf{X}_n}(f); \Gamma_{\mathbf{X}}(f)\|_\alpha \leq C\|f\|_{\rho_\alpha(\mathbf{X}_n, \mathbf{X})}.$$

Since \mathbf{X}_n converges to \mathbf{X} in the α -Hölder topology, we see that the map $\mathbf{X} \mapsto \|\gamma(\mathbf{X})\|_\alpha$ is continuous. \square

It is important to note that Γ can be completed to yield a structure which is known as a *continuous field of Banach spaces*, [8]. We will review the general theory of these structures in the next section.

Remark 3.11 (Approximation of controlled paths). We will show now that our results immediately yield canonical approximations for controlled rough paths. Let \mathbf{X} be an α -rough path and assume that there are smooth rough paths \mathbf{X}_ε (i.e. canonical lifts of smooth paths) such that $\mathbf{X}_\varepsilon \rightarrow \mathbf{X}$ as $\varepsilon \rightarrow 0$. Note that this is always the case when \mathbf{X} is geometric. Let $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^{\alpha, \beta}$. We aim to construct smooth $\mathbf{Z}_\varepsilon \in \mathcal{D}_{\mathbf{X}_\varepsilon}^{\alpha, \beta}$ such that $\|\mathbf{Z}; \mathbf{Z}_\varepsilon\|_\beta \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let $\delta > 0$. From property (ii) in Theorem 3.10, we can choose $f \in \mathcal{S}_N$ such that

$$\|\mathbf{Z}; \Gamma_{\mathbf{X}}(f)\|_\beta = \|\mathbf{Z} - \Gamma_{\mathbf{X}}(f)\|_\beta \leq \delta/2.$$

From Theorem 3.4, we can choose $\varepsilon > 0$ sufficiently small to obtain

$$\|\Gamma_{\mathbf{X}}(f); \Gamma_{\mathbf{X}_\varepsilon}(f)\|_\beta \leq \delta/2.$$

Since \mathbf{X}_ε is smooth, $\Gamma_{\mathbf{X}_\varepsilon}(f)$ is also smooth by construction. Therefore we can set $\mathbf{Z}_\varepsilon = \Gamma_{\mathbf{X}_\varepsilon}(f)$ and conclude with the triangle inequality that $\|\mathbf{Z}; \mathbf{Z}_\varepsilon\|_\beta \leq \delta$.

4. BANACH BUNDLES OF CONTROLLED ROUGH PATHS

In this section we construct Banach bundles of controlled rough paths. We recall first some general definitions and results on Banach bundles and continuous fields of Banach spaces.

4.1. Continuous fields and Banach bundles. Let us first review two closely related structures which arose in conjunction with C^* -algebras and provide a convenient framework for our investigation of spaces of controlled rough paths. We shall now present these frameworks and discuss the pertinent examples from the theory of rough paths thereafter.

Definition 4.1. Let T be a topological space. A *continuous field* Γ of Banach spaces over T is a family $(E_t)_{t \in T}$ of Banach spaces, together with a set $\hat{\Gamma} \subseteq \prod_{t \in T} E_t$ (where the elements of $\hat{\Gamma}$ are thought of as functions $\gamma: T \rightarrow \prod_t E_t$ with $\gamma(t) \in E_t, \forall t \in T$), such that:

- (i) $\hat{\Gamma}$ is a linear subspace of $\prod_{t \in T} E_t$,
- (ii) for every $t \in T$ the set $\hat{\Gamma}(t) := \{\gamma(t) \mid \gamma \in \hat{\Gamma}\}$ is dense in E_t ,
- (iii) For every $\gamma \in \hat{\Gamma}$, the function $T \rightarrow \mathbb{R}, t \mapsto \|\gamma(t)\|$ is continuous,
- (iv) Let $\tilde{\gamma} \in \prod_{t \in T} E_t$. If for every $t \in T$ and every $\varepsilon > 0$, there exists an $\gamma_t \in \hat{\Gamma}$ such that $\|\tilde{\gamma}(x) - \gamma_t(x)\| \leq \varepsilon$ on some neighborhood of t in T , then $\tilde{\gamma} \in \hat{\Gamma}$.

If moreover $\hat{\Gamma}$ contains a countable subset Λ such that $\Lambda(t) = \{\gamma(t) \mid \gamma \in \Lambda\}$ is a dense subset in E_t for every $t \in T$, then $\hat{\Gamma}$ is called a *separable continuous field* of Banach spaces over T .

In the previous section we have constructed an example of a family of sections of the spaces of controlled rough paths which is almost a continuous field of Banach spaces. Indeed our example satisfies properties (i)-(iii) but not (iv) of Definition 4.1. Let us agree to call such a collection a *continuous pre-field* of Banach spaces. The reason the notion of pre-field does not exist as an independent object of study is that every pre-field can uniquely be completed to a continuous field. Namely, [8, Proposition 10.2.3] yields

Proposition 4.2. *If T is a topological space and Γ a pre-field of Banach spaces over T . Then there exists a unique continuous field $\hat{\Gamma}$ of Banach spaces over T such that $\Gamma \subseteq \hat{\Gamma}$. Furthermore, if Γ admits a countable subset Λ such that $\Lambda(t) = \{\gamma(t) \mid \gamma \in \Lambda\}$ is a dense subset of the Banach space over t , for all $t \in T$, then $\hat{\Gamma}$ is a separable continuous field of Banach spaces.*

Proof. In view of the cited proposition, we only need to establish the separability. This is however trivial since we have assumed that there is a countable subset Λ of $\Gamma \subseteq \hat{\Gamma}$ which yields the (countable) dense set $\Lambda(t)$ in each fibre for $t \in T$. Thus $\hat{\Gamma}$ is separable. \square

The concept of a continuous fields is a convenient framework in which one can speak about continuous sections on a topological space with values in a collection of Banach spaces. The main point here is that the union of fibres $E := \sqcup_{t \in T} E_t$ does not need to carry a topology while the sections are still continuous in an appropriate sense. If one considered $E \rightarrow T$ as a bundle in the usual sense, it would be natural to define a topology on E making the elements of a continuous field continuous. Indeed there exists a canonical way to introduce such a topology on E from a given continuous field.

Definition 4.3. Let Γ be a continuous (pre-)field of Banach spaces on a topological space E and denote by $(E_t)_{t \in T}$ the family of Banach spaces over T . Define $E := \sqcup_{t \in T} E_t$ and let $p: E \rightarrow T$ be the canonical projection $p(e) = t$ if $e \in E_t$. Then we consider for $\gamma \in \Gamma$, $U \subseteq T$ open and $\varepsilon > 0$ the *tube*

$$W(\gamma, U, \varepsilon) = \{b \in E \mid p(b) \in U, \|b - \gamma(p(b))\| < \varepsilon\} \quad (4.1)$$

From now on we endow E with the topology generated by the base

$$\{W(\gamma, U, \varepsilon) \mid \gamma \in \Gamma, U \subseteq T \text{ open, and } \varepsilon > 0\}.$$

The point is that the tubes define a base of a Hausdorff topology on E . Before we discuss this, let us recall following concept.

Definition 4.4. Let E, T be Hausdorff topological spaces and $p: E \rightarrow T$ be a continuous and open surjective map. We call the triple $\mathcal{E} = (E, p, T)$ a *Banach bundle* over T if each fibre $E_t := p^{-1}(t)$, $t \in T$ is a Banach space and the following conditions hold

- (i) the map $E \rightarrow \mathbb{R}, s \mapsto \|s\|$ is continuous.
- (ii) $+$: $E_t \times E_t \rightarrow E$, $(a_1, a_2) \mapsto a_1 + a_2$ is continuous for all $t \in T$,
- (iii) For each $\lambda \in \mathbb{R}$, the map $E \rightarrow E, a \mapsto \lambda a$ is continuous,
- (iv) if $t \in T$ and $\{a_j\}_{j \in I}$ is any net of elements⁷ in E such that $\|a_j\| \rightarrow 0$ and $p(a_j) \rightarrow t$ in T , then $a_j \rightarrow 0_t$ in T .

Note that in the notation we usually suppress the index for the vector space operations and norms if the fibre is clear from the context. A continuous map $\sigma: T \rightarrow E$ such that $p \circ \sigma = \text{id}_T$ is called *continuous crossection*.

Remark 4.5. A Banach bundle in the above sense is a special type of fibre bundle whose fibres are Banach spaces. It is more general than a vector bundle whose fibres are Banach spaces (and which is also often called a Banach bundle in the literature). Note that contrary to vector bundles, Banach bundles do in general not admit a local trivialisation by a bundle trivialisation. Consequently, we do not claim that a canonical trivialisation exists for the Banach bundle B .

⁷While the general definition of a Banach bundle is naturally phrased using nets, we will only encounter Banach bundles which are metrisable topological spaces. Thus all nets we have to care about are countable, i.e. sequences.

Banach bundles are (up to some technicalities) equivalent to continuous fields of Banach spaces as the next result shows:

Proposition 4.6. *Let Γ be a continuous pre-field of Banach spaces over a metric space (T, d) . Let again $(E_t)_{t \in T}$ be the family of Banach fibres and endow $E := \sqcup_{t \in T} E_t$ with the topology from Definition 4.3. Then the canonical projection $p: E \rightarrow T$ turns (E, p, T) into the unique Banach bundle such that the following holds:*

- 1 *Every element $\sigma \in \Gamma$ is a continuous cross section of the Banach bundle.*
- 2 *The Banach bundle has enough cross sections, i.e. for every $b \in E$ there exists a continuous cross section σ_b with $\sigma_b(p(\sigma)) = b$.*
- 3 *If the completion $\hat{\Gamma}$ is a separable continuous field of Banach spaces, E is a metric space. Moreover, E is separable if and only if T is separable.*

Proof. By definition of a continuous pre-field of Banach spaces the set $\Gamma(t) = \{\gamma(t) \mid t \in T\}$ is a dense linear subspace of E_t for each $t \in T$. Moreover, for every $\gamma \in \Gamma$, the function $t \mapsto \|\gamma(t)\|$ is continuous. Hence, the prerequisites of [9, 13.18 Theorem] are verified. Applying the theorem, we see that the tubes (4.1) form the basis of a Hausdorff topology on E . This topology turns (E, p, T) into a Banach bundle such that every element σ of Γ becomes a continuous cross section of this bundle. Furthermore this topology is unique and satisfies (1) and has enough cross sections by [9, Remark 13.19], i.e. (2) holds.

We are left to prove (3) and note first that by uniqueness of the Banach bundle topology, we may replace Γ with its completion $\hat{\Gamma}$. Hence without loss of generality we may assume now that Γ is a separable continuous field of Banach spaces with a countable subset Λ such that $\Lambda(t) = \{\gamma(t) \mid \gamma \in \Lambda\}$ is a dense subset of E_t for each $t \in T$. To see that E is metrisable we note first that, as a metric space, T is paracompact. Hence we can apply [26, Proposition A.1] and deduce that E is a completely regular topological space. In particular, E is a regular topological space. Further, [10, Corollary 4.4.4] shows that T admits a σ -finite topological base \mathcal{B} , i.e. a topological base which is a countable union of locally finite families of open sets $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$.

Let us construct a σ -finite topological base for E . For this let $\mathcal{B}_n = \{U_i\}_{i \in S_n}$, $n \in \mathbb{N}$ for the locally finite families of open sets U_i comprising \mathcal{B} . Consider the family \mathcal{T} of tubes defined via

$$W(\gamma, U_i, r), \gamma \in \Lambda, i \in S_n, r \in \mathbb{Q} \cap (0, 1). \quad (4.2)$$

By construction \mathcal{T} consists of open neighborhoods in E and if $W(\gamma, U_i, r) \cap W(\gamma', U'_i, s) \neq \emptyset$ we must have $U_i \cap U'_i \neq \emptyset$. As Λ is countable, the set

$$\mathcal{T} = \bigcup_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{Q} \cap (0, 1)} \bigcup_{\gamma \in \Lambda} \{W(\gamma, U_i, r)\}_{i \in S_n}$$

is again σ -finite. We are left to prove that it is a topological base. Pick $W(\eta, U, r)$, for $\eta \in \Gamma, U \subseteq T$ open and some $r > 0$. For any fixed $b \in W(\eta, U, r)$ we construct a tube of the form (4.2) contained in $W(\eta, U, r)$ which contains b . Pick a rational number $0 < \varepsilon < \|b - \eta(p(b))\|/2$. Exploiting that the image $\Lambda(t)$ of Λ is dense in E_t for every $t \in T$, we can pick $\gamma_b \in \Gamma_0$ such that $\|b - \gamma_b(p(b))\| < \varepsilon$. Applying the triangle inequality we see that $\gamma_b(p(b)) \in W(\eta, U, r)$. Moreover, $t \mapsto \|\gamma_b(t) - \eta(t)\|$ is continuous, whence there is a small neighborhood O of $p(b)$ in U such that $W(\gamma_b, O, \varepsilon) \subseteq W(\eta, U, r)$. As \mathcal{B} is a topological base of T , we can pick $i \in S_n$ for some $n \in \mathbb{N}$ such that $p(b) \in U_i \subseteq O$. This implies that $b \in W(\gamma_b, U_i, \varepsilon) \subseteq W(\eta, U, r)$. We conclude that \mathcal{T} is a σ -finite topological base.

Summing up, E is a regular topological space with a σ -finite topological base, hence the Nagata-Smirnov metrisation theorem [10, 4.4.7] establishes that E is metrisable. To establish

the claim on separability, we recall that (E, ρ, T) has enough cross sections and T is a metric space. Metric spaces are second countable if and only if they are separable. Since $\Lambda(t)$ is dense in every E_t , [9, Proposition 13.21] shows that the metric space E is second countable (equivalently separable) if and only if T is so. This establishes (3) and finishes the proof. \square

Remark 4.7. Note that most of the statements in Proposition 4.6 are easy corollaries from well known results on Banach bundles. Hence they are hardly surprising. However, we were unable to locate the statement on the metrisability of the total space in the literature. (A metrisability statement for the total space under the more restrictive assumption that T is separable can be found in [26].)

The uniqueness assertion in Proposition 4.6 shows that a continuous (pre-)field of Banach spaces uniquely determines a Banach bundle with enough sections. A deep result due to A. Douady, L. dal Soglio Herault and K.H. Hofmann shows that this can be reversed. Indeed over paracompact topological spaces, there is a one-to-one correspondence between continuous fields of Banach spaces and Banach bundles which admit continuous cross sections through every point (see [9, Appendix C]). However, for later use (see Remark 4.16) it is important to note that the general construction only allows one to recover the continuous field, not the pre-field from which the field might have arisen by completion.

We conclude this section with a brief discussion of the construction of the metric on the total space of the Banach bundle in Proposition 4.6. This is essentially just a recap of the proof of the Nagata-Smirnov metrisation theorem. However, we need the concrete form of the metric constructed in the next section to study the Banach bundle of controlled rough paths.

4.1.1. The Nagata-Smirnov metric on the Banach bundle. Let (E, ρ, T) be a Banach bundle over a metric space (T, ρ) constructed from a separable continuous Banach bundle Γ with countable dense subset Λ via Proposition 4.6. To understand convergence in the metric space (E, d_E) we explicitly construct d_E by revisit the Nagata-Smirnov metrisation theorem, [10, 4.4.7]. Its proof shows that the construction hinges on two choices:

- 1 the σ -finite topological base $U_i, i \in S_n, n \in \mathbb{N}$. We note that since (T, ρ) is a metric space, we can choose every $U_i = B_{r_i}(t_i)$ as a metric ball in T with $t_i \in T$, and $0 < r_i < 1$ (cf. [10, Theorem 4.4.3]).
- 2 For every tube in the basis of the topology one picks

$$W(\gamma, B_{r_i}(t_i), \delta), i \in S_n, \gamma \in \Lambda, n \in \mathbb{N}, \delta \in \mathbb{Q} \cap (0, 1)$$

a continuous function $f_{(\gamma, i, \delta)}: E \rightarrow [0, 1]$ such that $W(\gamma, B_{r_i}(t_i), \delta) = f_{(\gamma, i, \delta)}^{-1}([0, \infty[)$.

Note that also here $\delta < 1$ is enough since the sections in Λ have dense image in every fibre. In the situation at hand, the following functions are valid choices

$$f_{(\gamma, i, \delta)}(b) := \begin{cases} (r_i - \rho(p(b), t_i))(\delta - \|b - \gamma(p(b))\|), & \text{if } b \in W(\gamma, B_{r_i}(t_i), \delta) \\ 0, & \text{else} \end{cases}. \quad (4.3)$$

For later use we remark that both factors in the product in (4.3) are smaller than 1 by choice of r_i and δ . By construction $|f_{(\gamma, i, \delta)}(x) - f_{(\gamma, i, \delta)}(y)| = 0$ if $(x, y) \notin W(\gamma, U_i, \delta) \times E \cup E \times W(\gamma, U_i, \delta)$. To ease notation, let us agree on the following definition:

Definition 4.8. For $\gamma \in \Gamma, i \in S_n, n \in \mathbb{N}$ such that $U_i = B_{r_i}(t_i)$ for some $r_i > 0$ and rational $\delta > 0$, we define a pseudometric on E

$$d_{(\gamma, i, \delta)}(x, y) = |f_{(\gamma, i, \delta)}(x) - f_{(\gamma, i, \delta)}(y)|.$$

Pick now an enumeration $m \mapsto (n_m, \gamma_m, \delta_m)$ of the countable set $\mathbb{N} \times \Lambda \times (\mathbb{Q} \cap]0, \infty[)$ and recall that the open sets $U_i = B_{r_i}(t_i)$ for $i \in S_{n_m}$ form a locally finite family. Thus we obtain a well-defined continuous function

$$d_m: E \times E \rightarrow [0, \infty[, \quad (x, y) \mapsto \sum_{i \in S_{n_m}} d_{(\gamma_m, i, \delta_m)}(x, y).$$

Cutting off at 1 and summing up, this yields the Nagata-Smirnov metric on E :

$$d_E: E \times E \rightarrow \mathbb{R}, \quad d_E(x, y) := \sum_{m \in \mathbb{N}} 2^{-m} \min\{1, d_m(x, y)\}.$$

Note that convergence $x_n \rightarrow x$ in the metric d_E implies $d_m(x_n, x) \rightarrow 0$ for all $m \in \mathbb{N}$. As a direct consequence of the construction of the d_m we thus obtain the following.

Lemma 4.9. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in (E, d_E) . Then $x_n \rightarrow_{d_E} x$ if and only if*

$$\lim_{k \rightarrow \infty} d_{(\gamma_m, i, \delta_m)}(x_k, x) = 0 \text{ for all } m \in \mathbb{N}, i \in S_{n_m}.$$

Finally, we note that if $x, y \in W(\gamma, B_{r_i}(t_i), \delta)$ then an elementary estimate yields

$$\begin{aligned} d_{(\gamma, i, \delta)}(x, y) &\leq \underbrace{(r_i - \rho(\rho(x), t_i))}_{\in(0,1)} (\|x - \gamma(\rho(x))\| - \|y - \gamma(\rho(y))\|) \\ &\quad + \underbrace{(\delta - \|y - \gamma(\rho(y))\|)}_{\in(0,1)} |\rho(\rho(y), t_i) - \rho(\rho(x), t_i)| \end{aligned}$$

Hence the pseudometric $d_{(\gamma, i, \delta)}(x, y)$ is dominated by the sum of the two terms

$$\| \|x - \gamma(\rho(x))\| - \|y - \gamma(\rho(y))\| \| \tag{4.4}$$

$$|\rho(\rho(y), t_i) - \rho(\rho(x), t_i)| \leq \rho(\rho(y), \rho(x)) \tag{4.5}$$

The expression (4.4) can only be simplified using the reverse triangle inequality (as we did in (4.5)) if $\rho(x) = \rho(y)$, i.e. if both points are contained in the same fibre.

For a converging sequence $x_n \rightarrow_{d_E} x$, where x lies in the tube $W(\gamma, B_{r_i}(t_i), \delta)$, the above terms control convergence if enforced for all tubes in the base in which x is contained. By Lemma 4.9 all terms (4.4) and (4.5) which are not cut off need to converge to 0. Besides the estimate (4.5), continuity of ρ with respect to the tube topology yields $\rho(x_n) \rightarrow_{\rho} \rho(x)$ if $x_n \rightarrow_{d_E} x$.

In the next section, we shall consider the Nagata-Smirnov metric for the Banach bundle of controlled rough paths. Our preparation here enables us to prove that it is equivalent to a metric constructed extrinsically on the union of the Banach fibres. This metric has been used extensively in the literature (see e.g. [12]) to establish stability of rough integration.

4.2. Properties of the Banach bundles of controlled rough paths. We have seen in Theorem 3.10 that the spaces of controlled rough paths form a continuous pre-field of Banach spaces over the space of branched rough paths. Moreover, we deduce from the result and Proposition 4.2 the following:

Corollary 4.10. *The continuous pre-field Γ from Theorem 3.10 can be uniquely completed to a separable continuous field of Banach spaces $\hat{\Gamma}$ over the space of branched α -rough paths.*

We will now leverage the theory recalled in the last section and consider two Banach bundles of controlled rough paths. For this let us fix some notation

Definition 4.11. We denote by \mathcal{C}^α the space of branched rough paths for some $0 < \alpha < 1$ and recall that it is a metric space with respect to the α -rough path metric ρ_α . Further, we let \mathcal{C}_g^α be the closed subspace of geometric rough paths endowed with the metric induced by the space of branched rough paths. For brevity we will also write ρ_α for the metric on this space.

For any $0 < \beta < \alpha$ we define now the bundles

$$\rho: E^{\alpha,\beta} := \bigsqcup_{\mathbf{x} \in \mathcal{C}^\alpha} \mathcal{D}_{\mathbf{x}}^{\alpha,\beta} \rightarrow \mathcal{C}^\alpha \quad (4.6)$$

$$\rho: E_g^{\alpha,\beta} := \bigsqcup_{\mathbf{x} \in \mathcal{C}_g^\alpha} \mathcal{D}_{\mathbf{x}}^{\alpha,\beta} \rightarrow \mathcal{C}_g^\alpha \quad (4.7)$$

where ρ is the canonical projection of a controlled rough path onto the controlling rough path.

Combining now Corollary 4.10 with Proposition 4.6 we see that the field of controlled rough paths induces a unique Banach bundle structure on the bundles (4.6) and (4.7). Moreover, we obtain the following result.

Proposition 4.12. *The total spaces of the Banach bundles (4.6) and (4.7) are separable metric spaces with the Nagata-Smirnov metric. Moreover, $E_g^{\alpha,\beta}$ is separable while $E^{\alpha,\beta}$ is not.*

Having obtained the Banach bundles of controlled rough paths over the branched and geometric rough paths, let us introduce another metric which has been used in stability analysis of rough integrals (see e.g. [12])

Definition 4.13. Let E be either $E^{\alpha,\beta}$ or $E_g^{\alpha,\beta}$. We define a map $d_\alpha^b: E \times E \rightarrow \mathbb{R}$ with the help of (2.4). Namely for $x, y \in E$, we set

$$d_\alpha^b(x, y) := \rho_\alpha(p(x), p(y)) + \|x; y\|_\alpha.$$

In the following, whenever we derive results on d_α^b which hold for both Banach bundles we consider we shall denote their total spaces as E .

Note that $\|x; y\|_\alpha$ makes sense for arbitrary elements $x, y \in E$ but it only constitutes a norm for elements belonging to the same fibre of the total space. The trick is that we can still compute a Hölder distance of the remainders as they take their values in the same Banach space. From the point of view of the Banach bundle this is however an extrinsic construction which does not reflect the geometric structure of the bundle (whence we chose to call this metric the *flat* metric, as it exploits an extrinsic embedding into a flat space).

Lemma 4.14. *The map d_α^b is a metric on the total space E , called the flat metric.*

Proof. We only have to prove that $d_\alpha^b(x, y) = 0$ implies $x = y$. To see this, we first note that $d_\alpha^b(x, y) = 0$ implies $p(x) = p(y)$. In this case, $\|x; y\|_\alpha = \|x - y\|_\alpha$ and since $\|\cdot\|_\alpha$ is a norm on the space of paths controlled by $p(x)$, the claim follows. \square

Proposition 4.15. *The flat metric d_α^b and the metric d_E are topologically equivalent. In other words, for a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ we have $d_\alpha^b(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ if and only if $d_E(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Assume first that $\lim_{n \rightarrow \infty} d_\alpha^b(x_n, x) = 0$. By definition this implies $\rho_\alpha(p(x_n), p(x)) \rightarrow 0$ and the term (4.5) converges to 0. For (4.4) we find with the negative triangle inequality

$$\| \|x - \gamma(p(x))\| - \|x_n - \gamma(p(x_n))\| \| \leq |d_\alpha^b(x, \gamma(p(x))) - d_\alpha^b(x_n, \gamma(p(x_n)))| \leq d_\alpha^b(x, x_n).$$

Hence convergence in d_α^b implies convergence of (4.4) and (4.5) for all choices of $d_{(\gamma,i,\delta)}$. Thus Lemma 4.9 shows that convergence with respect to d_α^b implies convergence in d_B .

For the converse, assume that $d_E(x_n, x) \rightarrow 0$. Since the bundle projection p is a continuous map, this implies $\rho_\alpha(p(x_n), p(x)) \rightarrow 0$ as $n \rightarrow \infty$.

Now let $\varepsilon > 0$. For every $\gamma \in \Gamma$ (note that for the bundle over the geometric rough paths, we replace Γ by the restrictions of its elements to the subspace of geometric rough paths),

$$\|x; x_n\|_\alpha \leq \|x - \gamma(p(x))\|_\alpha + \|\gamma(p(x)); \gamma(p(x_n))\|_\alpha + \|\gamma(p(x_n)) - x_n\|_\alpha.$$

We can choose $\gamma \in \Gamma_0$ such that $\|x - \gamma(p(x))\|_\alpha \leq \varepsilon$. Continuity of the rough integral implies that we can find $M_1 > 0$ such that $\|\gamma(p(x)); \gamma(p(x_n))\|_\alpha \leq \varepsilon$ for every $n \geq M_1$. Choose an open neighborhood $U_i = B_{r_i}(\mathbf{X}_i)$ as in the definition of the Nagata-Smirnov metric (where $0 < r_i < 1$ and \mathbf{X} is a branched rough path for the bundle over the branched rough paths, otherwise we can pick a geometric rough path) such that $p(x) \in B_{r_i/2}(\mathbf{X}_i)$ and $p(x_n) \in B_{r_i}(\mathbf{X}_i)$ for every $n \in \mathbb{N}$ large enough. Since $x \in \mathcal{W}(\gamma, B_{r_i}(\mathbf{X}_i), 2\varepsilon)$, we must have $x_n \in \mathcal{W}(\gamma, B_{r_i}(\mathbf{X}_i), 2\varepsilon)$ for all $n \geq M_2$ and some M_2 . We already know that $\rho_\alpha(p(x_n), p(x)) \rightarrow 0$ as $n \rightarrow \infty$ which implies that

$$(r - \rho_\alpha(p(x_n), \mathbf{X})) \rightarrow (r - \rho_\alpha(p(x), \mathbf{X}))$$

as $n \rightarrow \infty$. Since $d_{(\gamma,i,\delta)}(x, x_n) \rightarrow 0$ for every rational $\delta > 0$ as $n \rightarrow \infty$, this implies that

$$\|x_n - \gamma(p(x_n))\|_\alpha \rightarrow \|x - \gamma(p(x))\|_\alpha$$

as $n \rightarrow \infty$. Since $\|x - \gamma(p(x))\|_\alpha \leq \varepsilon$, we can conclude that $\|x_n - \gamma(p(x_n))\|_\alpha \leq 2\varepsilon$ for $n \geq M_3$. Therefore, we have shown that for $n \geq M := \max\{M_1, M_2, M_3\}$,

$$\|x; x_n\|_\alpha \leq 4\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof. \square

Remark 4.16. In light of the equivalence of the Nagata-Smirnov metric and the flat metric, it is not hard to see that the Banach bundle structure can also be established using the flat metric. Hence from the flat metric, we could have constructed the Banach bundle and the associated continuous field. Now the reader may wonder whether the constructions in [12] (for level 2-rough paths) would not lead to a direct proof of the articles results so far. While we can recover the continuous field by the general result, the same is not true for the pre-field of Banach spaces constructed through the approximation arguments. The point is that by working with the algebraic structure and the smaller set of sections from the pre-field our arguments yield estimates and control which can not (without further arguments) be deduced from the flat metric and the (non-canonical) homeomorphism with a trivial Banach bundle introduced in [12].

Note that the equivalence of the metrics is quite weak as they only induce the same topology. There are stronger notions of metric equivalence which ensure that properties established for one metric also carry over to the other. We will briefly discuss this now.

Remark 4.17. Topologically equivalent metrics do not need to have the same Cauchy-sequences (whence completeness of one metric is not automatically inherited by a topologically equivalent metric). For this, *uniform equivalence of metrics* is sufficient. This means that the identity map $\text{id}_X: (X, d_1) \rightarrow (X, d_2)$ and $\text{id}_X: (X, d_2) \rightarrow (X, d_1)$ is uniformly continuous (and not just continuous as in topological equivalence). Unfortunately, without additional properties on the open cover uniform estimates of one metric against the other seem to be out of reach. The problem lies in the construction of the functions d_m which are defined as sums of pseudometrics. Locally, every d_m is a finite sum of such pseudometrics. However, since the

number of summands can potentially be unbounded over the whole space (this depends on the chosen cover of the space of rough paths), this makes a uniform estimate impossible without further knowledge.

Consequently, we were only able to establish the completeness for the flat metric (completeness for the Nagata-Smirnov metric runs into a conceptually similar problem as a proof of the uniform equivalence). Note however, that the proof leverages the topological equivalence with the Nagata-Smirnov metric.

Theorem 4.18. *Let (E, d_α^b) be the total space of the Banach bundle (4.6) or (4.7), then E is complete. In particular, the space $(E_g^{\alpha,\beta}, d_\alpha^b)$ is a Polish space while $(E^{\alpha,\beta}, d_\alpha^b)$ is a complete metric space.*

Proof. In view of Proposition 4.12, we know that the topological space is separable only for the bundle $(E_g^{\alpha,\beta}, d_E)$. Now the Nagata-Smirnov metric and the flat metric are topologically equivalent by Proposition 4.15, whence the (non-)separability is inherited by the flat metric. Thus we only have to establish completeness of the space (E, d_α^b) . For this we consider a Cauchy-sequence $(x_n)_{n \in \mathbb{N}}$ with respect to the flat metric. Recall from Definition 4.13 that being Cauchy in the flat metric implies that the sequence $(\rho(x_n))_{n \in \mathbb{N}}$ of basepoints must be a Cauchy-sequence with respect to the metric ρ_α . Now the space of branched (respectively geometric) rough paths is complete with respect to ρ_α , whence there exists a rough path \mathbf{X} such that $\lim_{n \rightarrow \infty} \rho(x_n) = \mathbf{X}$. Furthermore, Definition 4.13 implies that for every $h \in \mathcal{F}_{(N)}^<$, the series $(\langle h, x_n \rangle)_n$ is a Cauchy sequence with respect to the uniform topology. By completeness, there exist continuous paths x^h such that $\langle h, x_n \rangle \rightarrow x^h$ uniformly. Define $\mathbf{Z}: [0, T] \rightarrow \mathcal{H}_{(N)}^<$ by $\langle h, \mathbf{Z}_t \rangle = x_t^h$. We show that \mathbf{Z} is controlled by \mathbf{X} . Define

$$\begin{aligned} R_{s,t}^{h;n} &:= \langle h, x_n(t) \rangle - \langle \rho(x_n)_{s,t} \star h, x_n(s) \rangle \quad \text{and} \\ R_{s,t}^h &:= \langle h, \mathbf{Z}_t \rangle - \langle \mathbf{X}_{s,t} \star h, \mathbf{Z}_s \rangle. \end{aligned}$$

Let $\varepsilon > 0$. Since $(x_n)_n$ is a Cauchy sequence w.r.t. d_α^b , there is an M such that

$$\frac{|R_{s,t}^{h;n} - R_{s,t}^{h;m}|}{|t - s|^{(N-|h|)\alpha}} \leq \varepsilon$$

for every $s \neq t$ and every $n, m \geq M$. Letting $m \rightarrow \infty$, pointwise convergence implies that

$$\|R^{h;n} - R^h\|_{(N-|h|)\alpha} \leq \varepsilon$$

for $n \geq M$. This implies that \mathbf{Z} is indeed controlled by \mathbf{X} and that $d_\alpha^b(x_n, \mathbf{Z}) \rightarrow 0$ as $n \rightarrow \infty$ which proves completeness. \square

It is now easy to recast stability results from the theory of rough paths in the language of Banach bundles (or equivalently continuous fields of Banach spaces).

Proposition 4.19. *Let again E be the total space of the Banach bundle (4.6) or (4.7). Then the integration map*

$$\mathfrak{I}: E \rightarrow E, \mathbb{Y} \mapsto \int \mathbb{Y} d\mathbf{X}, \quad \text{where } \mathbf{X} = \rho(\mathbb{Y}).$$

is a morphism of Banach bundles over the identity, i.e. \mathfrak{I} is continuous, fibrewise linear and projects via the bundle projection down to the identity.

Proof. Let us note first that due to Paragraph 2.18 the integral of a controlled path is controlled again, whence the integration map makes sense and respect the fibres of the bundle. Since

integration is linear in the integrand, we see that \mathfrak{I} is fibre-wise linear. Now continuity with respect to the Banach bundle topology is equivalent to continuity with respect to d_α^b and this follows immediately from Proposition 2.20. \square

Having established the integration map as a morphism of Banach bundles, we turn to the other prominent mapping in the setting, the Itô-Lyons map which assigns to a rough differential equation its solution as a controlled rough path.

For this let us recall first that the definition depends on the choice of vector fields $f_1, \dots, f_d \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Following [22], given $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $v_1, \dots, v_m \in \mathbb{R}^n$ we denote

$$D^m \varphi(y) : (v_1, \dots, v_m) := \sum_{\alpha_1, \dots, \alpha_m=1}^n \frac{\partial^m}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_m}} \varphi(y) v_1^{\alpha_1} \cdots v_m^{\alpha_m}.$$

Define the *elementary differentials* $f_\tau \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ for $\tau \in \mathcal{T}$ recursively by $f_1(y) = y$ and

$$f_{[\tau_1 \cdots \tau_m]_i} := D^m f_i : (f_{\tau_1}, \dots, f_{\tau_m}).$$

Lemma 4.20. *Let $\tau, \rho_1, \dots, \rho_n \in \mathcal{T}$, $n \geq 1$. Then*

$$D^n f_\tau : (f_{\rho_1}, \dots, f_{\rho_n}) = f_{\tau \curvearrowright \rho_1 \cdots \rho_n}$$

Proof. We first show the statement for the case $n = 1$ by induction on $|\tau|$. The case when $|\tau| = 1$, i.e., when $\tau = \bullet_i$ follows from the definition. Indeed,

$$Df_i(y) f_\rho(y) = f_{[\rho]_i}(y) = f_{\bullet_i \curvearrowright \rho}(y).$$

Now, if $|\tau| = n + 1$ there are $\tau_1, \dots, \tau_m \in \mathcal{T}$ with $|\tau_1| + \cdots + |\tau_m| = n$ and some label $i \in \{1, \dots, d\}$, such that $\tau = [\tau_1 \cdots \tau_m]_i = \bullet_i \curvearrowright \tau_1 \cdots \tau_m$. Hence, by the chain rule

$$\begin{aligned} Df_\tau : f_\rho &= D^{m+1} f_i : (f_{\tau_1}, \dots, f_{\tau_m}, f_\rho) + \sum_{k=1}^m D^m f_i : (f_{\tau_1}, \dots, Df_{\tau_k} f_\rho, \dots, f_{\tau_m}) \\ &= f_{[\tau_1 \cdots \tau_m \rho]_i} + \sum_{k=1}^m f_{[\tau_1 \cdots (\tau_k \curvearrowright \rho) \cdots \tau_m]_i}. \end{aligned}$$

Whence we conclude by noting that, at the level of trees, the identity

$$[\tau_1 \cdots \tau_m \rho]_i + \sum_{k=1}^m [\tau_1 \cdots (\tau_k \curvearrowright \rho) \cdots \tau_m]_i = \bullet_i \curvearrowright (\tau_1 \cdots \tau_m \star \rho) = [\tau_1 \cdots \tau_m]_i \curvearrowright \rho$$

holds.

Now, for the case $n > 1$, we note that by eq. (2.2)

$$\begin{aligned} D^{n+1} f_\tau : (f_{\rho_1}, \dots, f_{\rho_n}, f_\rho) &= Df_{\tau \curvearrowright \rho_1 \cdots \rho_n} : f_\rho - \sum_{k=1}^n D^n f_\tau : (f_{\rho_1}, \dots, Df_{\rho_k} f_\rho, \dots, f_{\rho_n}) \\ &= f_{(\tau \curvearrowright \rho_1 \cdots \rho_n) \curvearrowright \rho} - \sum_{k=1}^n f_{\tau \curvearrowright (\rho_1 \cdots (\rho_k \curvearrowright \rho) \cdots \rho_n)} \\ &= f_{\tau \curvearrowright \rho_1 \cdots \rho_n \rho}. \end{aligned} \quad \square$$

Given a branched rough path \mathbf{X} and vector fields $f_1, \dots, f_d: \mathbb{R}^n \rightarrow \mathbb{R}^n$, we say that Y solves the Rough Differential Equation (RDE)

$$dY_t = \sum_{i=1}^d f_i(Y_t) dX_t^i$$

if it has the local expansion

$$\delta Y_{s,t} = \sum_{\tau \in \mathcal{T}(N)} \frac{1}{\sigma(\tau)} f_{\tau}(Y_s) \mathbf{X}_{s,t}^{\tau} + r_{s,t}, \quad (4.8)$$

with $r \in C_2^{(N+1)\alpha}$, where we recall that $N = \lfloor \alpha^{-1} \rfloor$ so that $N\alpha \leq 1 < (N+1)\alpha$, and the path

$$\mathbf{Y}_t := \sum_{\tau \in \mathcal{T}(N) \cup \{1\}} \frac{1}{\sigma(\tau)} f_{\tau}(Y_t) \tau$$

belongs to $\mathcal{D}_{\mathbf{X}}^{\alpha}$.

It can be shown [19] that if $f_i \in C_b^{N+1}$ then for each $\xi \in \mathbb{R}^n$, the RDE has a unique solution $\mathbf{Y} := \Phi(\xi, \mathbf{X})$ starting from $Y_0 = \xi$. The map $\Phi: \mathbb{R}^n \times \mathcal{C}^{\alpha} \rightarrow \bigsqcup_{\mathbf{X} \in \mathcal{C}^{\alpha}} \mathcal{D}_{\mathbf{X}}^{\alpha}$, $(\xi, \mathbf{X}) \mapsto \Phi(\xi, \mathbf{X})$ is known as the Itô-Lyons map.

Proposition 4.21. *The Itô-Lyons map satisfies the bound*

$$\|\Phi(\xi, \mathbf{X}); \Phi(\tilde{\xi}, \tilde{\mathbf{X}})\|_{\alpha} \leq C(|\xi - \tilde{\xi}| + \rho_{\alpha}(\mathbf{X}, \tilde{\mathbf{X}})). \quad (4.9)$$

Hence, the Itô-Lyons map is a parameter dependent continuous crosssection of the Banach bundle $(E^{\alpha, \beta}, \rho, \mathcal{C}^{\alpha})$

Before proving the theorem, we recall the following result from [19, 22],

Lemma 4.22. *Let $\mathbf{X} \in \mathcal{C}^{\alpha}$ and $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^{\alpha}$. If $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a function of class C^N then the path $\varphi(\mathbf{Z}): [0, T] \rightarrow \mathcal{H}_{(N)}^{\leq}$ defined by $\langle \mathbf{1}, \varphi(\mathbf{Z})_t \rangle := \varphi(Z_t)$ and*

$$\langle h, \varphi(\mathbf{Z})_t \rangle := \sum_{k=1}^{N-1} \sum_{\substack{f_1, \dots, f_k \in \mathcal{F} \\ f_1 \cdots f_k = h}} \frac{1}{k!} D^k \varphi(Z_t)(Z_t^{f_1}, \dots, Z_t^{f_k})$$

also belongs to $\mathcal{D}_{\mathbf{X}}^{\alpha}$.

Proof of Proposition 4.21. Note that by definition $\Phi: \mathbb{R}^d \times \mathcal{C}^{\alpha} \rightarrow E^{\alpha, \beta}$ satisfies $\Phi(\cdot, \mathbf{X}) \in \mathcal{D}_{\mathbf{X}}^{\beta, \alpha}$. Hence we only need to establish continuity of Φ . For simplicity we only deal with the single noise case, i.e., $|A| = 1$ since the general case differs from this case only in notation. The proof follows ideas present in [4], in the setting of discrete rough paths. First, note that the controlled path $\mathbf{Y} = \Phi(\xi, \mathbf{X})$ solves the fixed-point equation

$$\mathbf{Y}_t = \mathbf{Y}_0 + \mathfrak{I}_{\mathbf{X}}(f(\mathbf{Y}))_t$$

in $\mathcal{D}_{\mathbf{X}}^{\alpha}$. Therefore, by Proposition 2.20 we immediately obtain the bound

$$\|\Phi(\xi, \mathbf{X}); \Phi(\tilde{\xi}, \tilde{\mathbf{X}})\|_{\alpha} \leq C(\|f(\mathbf{Y}); f(\tilde{\mathbf{Y}})\|_{\alpha} + \rho_{\alpha}(\mathbf{X}, \tilde{\mathbf{X}})).$$

From Lemma 4.22 we see that for any forest $h = \tau_1 \cdots \tau_k \in \mathcal{F}_{(N)}^{\leq}$ we have

$$\begin{aligned} \langle h, f(\mathbf{Y})_t \rangle &= D^k f(Y_t)(\mathbf{Y}^{\tau_1}, \dots, \mathbf{Y}^{\tau_k}) \\ &= D^k f(Y_t)(f_{\tau_1}(Y_t), \dots, f_{\tau_k}(Y_t)) \\ &= f_{[\tau_1 \cdots \tau_k]}(Y_t) \end{aligned}$$

Therefore, the remainder term can be expressed as

$$\begin{aligned} R_{s,t}^h &= \delta f(\mathbf{Y})_{s,t}^h - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} f(\mathbf{Y})_s^{\bar{h} \star h} \mathbf{X}_{s,t}^{\bar{h}} \\ &= f_{[h]}(Y_t) - f_{[h]}(Y_s) - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} f_{[\bar{h} \star h]}(Y_s) \mathbf{X}_{s,t}^{\bar{h}}. \end{aligned}$$

Now, we can perform a Taylor expansion on $f_{[h]}$ and see that

$$f_{[h]}(Y_t) - f_{[h]}(Y_s) = \sum_{k=1}^{N-|h|-1} \frac{1}{k!} D^k f_{[h]}(Y_s) (\delta Y_{s,t})^{\otimes k} + T_{s,t}^h$$

where

$$T_{s,t}^h := \int_0^1 \frac{1}{(N-|h|)!} D^{(N-|h|)} f_{[h]}(Y_s) (Y_s + \theta \delta Y_{s,t})^{\otimes (N-|h|)} (1-\theta)^{N-|h|-1} d\theta.$$

Thus, the remainder may be rewritten as

$$R_{s,t}^h = T_{s,t}^h + \sum_{k=1}^{N-|h|-1} \frac{1}{k!} D^k f_{[h]}(Y_s) (\delta Y_{s,t})^{\otimes k} - \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} f_{[\bar{h} \star h]}(Y_s) \mathbf{X}_{s,t}^{\bar{h}}.$$

Now, fix $1 \leq k < N - |h|$. Replacing eq. (4.8) into $D^k f_{[h]}(Y_s) (\delta Y_{s,t})^{\otimes k}$ we obtain

$$\begin{aligned} D^k f_{[h]}(Y_s) (\delta Y_{s,t})^{\otimes k} &= D^k f_{[h]}(Y_s) \left(\sum_{\tau \in \mathcal{T}_{(N)}} Y_s^\tau \mathbf{X}_{s,t}^\tau + r_{s,t} \right)^{\otimes k} \\ &= k! \sum_{\bar{h} \in \mathcal{F}_{(N-|h|-1)}^+} D^k f_{[h] \curvearrowright \bar{h}}(Y_s) \mathbf{X}_{s,t}^{\bar{h}} + B_{s,t}^h, \end{aligned}$$

where B^h contains terms of the form $D^k f_{[h]} : (f_{\rho_1}, \dots, f_{\rho_\ell}, R_{s,t}^{h_1}, \dots, R_{s,t}^{h_m})$.

Proceeding similarly with the remainder corresponding to $\tilde{\mathbf{Y}}$, we obtain that

$$|R_{s,t}^h - \tilde{R}_{s,t}^h| \leq |T_{s,t}^h - \tilde{T}_{s,t}^h| + |B_{s,t}^h - \tilde{B}_{s,t}^h|.$$

A straightforward bound gives

$$|T_{s,t}^h - \tilde{T}_{s,t}^h| \lesssim \left(\|Y - \tilde{Y}\|_\alpha P_{N-|h|}(\|Y\|_\alpha, \|\tilde{Y}\|_\alpha) + \|Y - \tilde{Y}\|_\infty \|\tilde{Y}\|_\alpha^{(N-|h|)} \right) |t - s|^{(N-|h|)\alpha}$$

with

$$P_m(x, y) := \sum_{j=1}^m x^j y^{m-j}.$$

The difference $B^h - \tilde{B}^h$ can be estimated in terms of the difference of remainders $R^{\bar{h}} - \tilde{R}^{\bar{h}}$ and the argument is closed recursively.

Due to the bound (4.9) we see that Φ is continuous as a mapping into $(E^{\alpha,\beta}, d_\alpha^b)$. Now the flat metric generates the topology of the Banach bundle $E^{\alpha,\beta}$, Proposition 4.15, whence the continuity of Φ follows. \square

APPENDIX A. PRIMITIVE ELEMENTS AND TREE BASES

This section presents tables of the primitive elements forests up to level four. We use these elements to generate a new basis by applying the natural growth operator (cf. Paragraph 2.3). At the end, we apply our approximation procedure to an example. For calculating the primitive elements, we use the following operator defined in [11, Theorem 9.6],

$$\pi_1 : \mathcal{H} \rightarrow \text{Prim},$$

$$\pi_1(h) = h - \sum_{(h)} h^1 \top \pi_1(h^2),$$

where $\Delta' h = \sum_{(h)} h^1 \otimes h^2$ and $\pi_1(\bullet) = \bullet$. By [11, Theorem 9.6], this operator is surjective. Accordingly, since this is also explicitly defined, we can quickly generate a basis for the space of primitive elements. The following vectors constitute a basis for this space up to level five.

Level (1)	$\pi_1(\bullet) = \bullet$	
Level (2)	$\pi_1(\bullet\bullet) = \bullet\bullet - 2\begin{array}{c} \bullet \\ \\ \bullet \end{array}$	
Level (3)	$\pi_1(\bullet\bullet\bullet) = \bullet\bullet\bullet - 3\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 3\begin{array}{c} \bullet \\ \\ \bullet \end{array}$	
Level (4)	$\pi_1(\bullet\bullet\bullet\bullet) = \bullet\bullet\bullet\bullet - 4\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 4\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - 4\begin{array}{c} \bullet \\ \\ \bullet \end{array} + 4\begin{array}{c} \bullet \\ \\ \bullet \end{array} - 2\begin{array}{c} \bullet \\ \\ \bullet \end{array} + 2\begin{array}{c} \bullet \\ \\ \bullet \end{array} - 2\begin{array}{c} \bullet \\ \\ \bullet \end{array}$	$\pi_1(\bullet\bullet\bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - 2\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$

We can now apply to Lemma 2.4 and find a new basis for \mathcal{H} . Here are the remaining vectors we need to add to construct a basis for the space of forests up to level four.

Level (2)	$\bullet \top \bullet = \begin{array}{c} \bullet \\ \\ \bullet \end{array}$		
Level (3)	$\top(\bullet, \bullet, \bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	$\pi_1(\bullet\bullet) \top \bullet = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - 2\begin{array}{c} \bullet \\ \\ \bullet \end{array}$	$\bullet \top \pi_1(\bullet\bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$

For the level (4),

$\top(\bullet, \bullet, \pi_1(\bullet\bullet)) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	$\top(\bullet, \pi_1(\bullet\bullet), \bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	$\top(\pi_1(\bullet\bullet), \bullet, \bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - 2\begin{array}{c} \bullet \\ \\ \bullet \end{array}$
$\top(\pi_1(\bullet\bullet), \pi_1(\bullet\bullet)) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 2\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 2\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 2\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	$\pi_1(\bullet\bullet\bullet) \top \bullet = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - 3\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + 3\begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	$\bullet \top \pi_1(\bullet\bullet\bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$
	$\top(\bullet, \bullet, \bullet, \bullet) = \begin{array}{c} \bullet \\ \\ \bullet \\ \\ \bullet \\ \\ \bullet \end{array}$	

We now apply our approximation scheme to the Example 2.17.

Example A.1. Recall from the example that we consider a path $\mathbf{Z} \in \mathcal{D}_{\mathbf{X}}^\alpha$ for $\alpha \in (\frac{1}{5}, \frac{1}{4})$. We can rewrite the controlledness condition of \mathbf{Z} with respect to \mathbf{X} in terms of the primitive basis

as shown below:

$$\begin{aligned} \delta Z_{s,t}^1 &= Z_s^\bullet \mathbf{X}_{s,t}^\bullet + Z_s^{\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet-2\bullet} + \left(2Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + Z_s^{\bullet\bullet\bullet} \mathbf{X}_{s,t}^{\bullet\bullet\bullet-3\bullet+3\bullet} \\ &\quad + \left(3Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet-2\bullet} + \left(3Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet-2\bullet-\bullet} \\ &\quad + \left(6Z_s^{\bullet\bullet\bullet} + 3Z_s^{\bullet\bullet} + 2Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + R_{s,t}^1 \\ \delta Z_{s,t}^\bullet &= \left(2Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^\bullet + \left(3Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet-2\bullet} \\ &\quad + \left(6Z_s^{\bullet\bullet\bullet} + 3Z_s^{\bullet\bullet} + 2Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + R_{s,t}^\bullet \\ \delta Z_{s,t}^{\bullet\bullet} &= \left(3Z_s^{\bullet\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + R_{s,t}^{\bullet\bullet}, \\ \delta Z_{s,t}^{\bullet\bullet} &= \left(Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} + Z_s^{\bullet\bullet} \right) \mathbf{X}_{s,t}^{\bullet\bullet} + R_{s,t}^{\bullet\bullet}. \end{aligned}$$

Let us to fix a dissection $\pi = \{0 = t_0 < t_1 < \dots < t_N < t_{N+1} = T\}$ of $[0, T]$ and set $I_k := [t_k, t_{k+1}]$. For $s \in I_k$, we define

$$\begin{aligned} \tilde{Z}_s^{\bullet\bullet} &= Z_{t_k}^{\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} Z_{t_k, t_{k+1}}^{\bullet\bullet}, & \tilde{Z}_s^{\bullet\bullet} &= Z_{t_k}^{\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} Z_{t_k, t_{k+1}}^{\bullet\bullet} \\ \tilde{Z}_s^{\bullet\bullet} &= Z_{t_k}^{\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} Z_{t_k, t_{k+1}}^{\bullet\bullet}, & \tilde{Z}_s^{\bullet\bullet\bullet} &= Z_{t_k}^{\bullet\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} Z_{t_k, t_{k+1}}^{\bullet\bullet\bullet}. \end{aligned}$$

We then define

$$\tilde{Z}_s^{\bullet\bullet\bullet} = Z_{t_k}^{\bullet\bullet\bullet} + \int_{t_k}^s (3\tilde{Z}_\tau^{\bullet\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} \left[Z_{t_k, t_{k+1}}^{\bullet\bullet\bullet} - \int_{t_k}^{t_{k+1}} (3\tilde{Z}_\tau^{\bullet\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet} \right],$$

and

$$\tilde{Z}_s^{\bullet\bullet} = Z_{t_k}^{\bullet\bullet} + \int_{t_k}^s (\tilde{Z}_\tau^{\bullet\bullet} + 2\tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet} + \frac{s - t_k}{t_{k+1} - t_k} \left[Z_{t_k, t_{k+1}}^{\bullet\bullet} - \int_{t_k}^{t_{k+1}} (\tilde{Z}_\tau^{\bullet\bullet} + 2\tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet} \right].$$

Finally, set

$$\begin{aligned} \tilde{Z}_s^\bullet &= Z_{t_k}^\bullet + \int_{t_k}^s (2\tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^\bullet + \int_{t_k}^s (3\tilde{Z}_\tau^{\bullet\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet-2\bullet} + \\ &\quad \frac{s - t_k}{t_{k+1} - t_k} \left[Z_{t_k, t_{k+1}}^\bullet - \int_{t_k}^{t_{k+1}} (2\tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^\bullet - \int_{t_k}^{t_{k+1}} (3\tilde{Z}_\tau^{\bullet\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet} + \tilde{Z}_\tau^{\bullet\bullet}) d\mathbf{X}_\tau^{\bullet\bullet-2\bullet} \right], \end{aligned}$$

and

$$\begin{aligned} \tilde{Z}_s^1 &= Z_{t_k}^1 + \int_{t_k}^s \tilde{Z}_\tau^{\bullet\bullet} d\mathbf{X}_\tau^\bullet + \int_{t_k}^s \tilde{Z}_\tau^{\bullet\bullet\bullet} d\mathbf{X}_\tau^{\bullet\bullet-2\bullet} + \int_{t_k}^s \tilde{Z}_\tau^{\bullet\bullet\bullet} d\mathbf{X}_\tau^{\bullet\bullet\bullet-3\bullet+3\bullet} + \\ &\quad \frac{s - t_k}{t_{k+1} - t_k} \left[Z_{t_k, t_{k+1}}^1 - \int_{t_k}^{t_{k+1}} \tilde{Z}_\tau^{\bullet\bullet} d\mathbf{X}_\tau^\bullet - \int_{t_k}^{t_{k+1}} \tilde{Z}_\tau^{\bullet\bullet\bullet} d\mathbf{X}_\tau^{\bullet\bullet-2\bullet} - \int_{t_k}^{t_{k+1}} \tilde{Z}_\tau^{\bullet\bullet\bullet} d\mathbf{X}_\tau^{\bullet\bullet\bullet-3\bullet+3\bullet} \right]. \end{aligned}$$

REFERENCES

- [1] L. Arnold, *Random dynamical systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
- [2] C. Bayer, D. Belomestny, M. Redmann, S. Riedel, and J. Schoenmakers, *Solving linear parabolic rough partial differential equations*, J. Math. Anal. Appl. **490** (2020), no. 1, 44 (English), Id/No 124236.
- [3] C. Bayer, P. Friz, and J. Gatheral, *Pricing under rough volatility*, Quant. Finance **16** (2016), no. 6, 887–904.
- [4] C. Bayer, P. Friz, and N. Tapia, *Stability of deep neural networks via discrete rough paths*, 2020, WIAS Preprint:2732.
- [5] G. Bogfjellmo, R. Dahmen, and A. Schmeding, *Character groups of Hopf algebras as infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 5, 2101–2155.
- [6] F. Chapoton and M. Livernet, *Pre-Lie algebras and the rooted trees operad*, Internat. Math. Res. Notices (2001), no. 8, 395–408.
- [7] A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys. **199** (1998), no. 1, 203–242.
- [8] J. Dixmier, *C*-algebras. Translated by Francis Jellet*, vol. 15, Elsevier (North-Holland), Amsterdam, 1977 (English).
- [9] R. S. Doran and J. M. G. Fell, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1: Basic representation theory of groups and algebras*, Boston, MA etc.: Academic Press, Inc., 1988 (English).
- [10] R. Engelking, *General topology.*, vol. 6, Berlin: Heldermann Verlag, 1989 (English).
- [11] L. Foissy, *Finite-dimensional comodules over the Hopf algebra of rooted trees*, J. Algebra **255** (2002), no. 1, 89–120.
- [12] P. K. Friz and M. Hairer, *A course on rough paths. With an introduction to regularity structures. 2nd updated edition*, 2nd updated edition ed., Cham: Springer, 2020 (English).
- [13] P. K. Friz and N. B. Victoir, *Multidimensional stochastic processes as rough paths*, Cambridge Studies in Advanced Mathematics, vol. 120, Cambridge University Press, Cambridge, 2010, Theory and applications.
- [14] P. K. Friz and H. Zhang, *Differential equations driven by rough paths with jumps*, J. Differ. Equations **264** (2018), no. 10, 6226–6301 (English).
- [15] J. Gatheral, T. Jaisson, and M. Rosenbaum, *Volatility is rough*, Quant. Finance **18** (2018), no. 6, 933–949.
- [16] M. Ghani Varzaneh, S. Riedel, and M. Scheutzow, *A dynamical theory for singular stochastic delay differential equations i: Linear equations and a multiplicative ergodic theorem on fields of banach spaces*, SIAM Journal on Applied Dynamical Systems **21** (2022), no. 1, 542–587.
- [17] M. Ghani Varzaneh and S. Riedel, *Oseledets splitting and invariant manifolds on fields of Banach spaces*, J. Dyn. Diff. Equat. (2021).
- [18] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140 (English).
- [19] M. Gubinelli, *Ramification of rough paths*, J. Differential Equations **248** (2010), no. 4, 693–721.
- [20] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75.
- [21] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), no. 2, 269–504.
- [22] M. Hairer and D. Kelly, *Geometric versus non-geometric rough paths*, Ann. Inst. Henri Poincaré Probab. Stat. **51** (2015), no. 1, 207–251.
- [23] M. E. Hoffman, *Combinatorics of rooted trees and Hopf algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 9, 3795–3811.
- [24] Y. Inahama, *Rough path theory and stochastic calculus*, Sugaku Expo. **32** (2019), no. 1, 113–136 (English).
- [25] T. Lada and M. Markl, *Symmetric brace algebras*, Appl. Categ. Struct. **13** (2005), no. 4, 351–370 (English).
- [26] A. J. Lazar, *A selection theorem for Banach bundles and applications*, J. Math. Anal. Appl. **462** (2018), no. 1, 448–470 (English).
- [27] Y. Liu and S. Tindel, *First-order Euler scheme for SDEs driven by fractional Brownian motions: the rough case*, Ann. Appl. Probab. **29** (2019), no. 2, 758–826.
- [28] T. Lyons and N. Victoir, *An extension theorem to rough paths*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire **24** (2007), no. 5, 835–847 (English).

- [29] T. J. Lyons, *Differential equations driven by rough signals.*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310 (English).
- [30] J. Ma and H. Wu, *A fast algorithm for simulation of rough volatility models*, Quantitative Finance **0** (2021), no. 0, 1–16.
- [31] D. Manchon, *Hopf algebras in renormalisation*, Handbook of algebra. Volume 5, Amsterdam: Elsevier/Noth-Holland, 2008, pp. 365–427 (English).
- [32] J.-M. Oudom and D. Guin, *On the Lie enveloping algebra of a pre-Lie algebra*, J. K-Theory **2** (2008), no. 1, 147–167.
- [33] N. Weaver, *Lipschitz algebras*, Hackensack, NJ: World Scientific, 2018 (English).