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Michael Hintermüller,Tobias Keil

ABSTRACT. This paper is concerned with the distributed optimal control of a time-discrete Cahn–Hilliard/Navier–Stokes system with variable densities. It focuses on the double-obstacle potential which yields an optimal control problem for a variational inequality of fourth order and the Navier–Stokes equation. The existence of solutions to the primal system and of optimal controls is established. The Lipschitz continuity of the constraint mapping is derived and used to characterize the directional derivative of the constraint mapping via a system of variational inequalities and partial differential equations. Finally, strong stationarity conditions are presented following an approach from Mignot and Puel.

1. INTRODUCTION

The goal of this paper is to derive strong stationarity conditions for an optimal control problem subject to the semi-discrete (in time) Cahn-Hilliard-Navier-Stokes system.

The Cahn-Hilliard-Navier-Stokes system describes the behavior of immiscible multiphase fluids, where the Cahn-Hilliard system models the evolution of the interface between the fluid components and the Navier-Stokes equations capture the hydrodynamics of the system. The first model, which combined the hydrodynamic effects and the phase seperation process of multiphase flows was given by Hohenberg and Halperin in [26]. Their basic model for immiscible, viscous two-phase flows, the so-called 'model H' is, however, restricted to the case where the two fluids possess nearly identical densities, i.e., matched densities. Recently, Abels, Garcke and Grün [2] derived the following diffuse interface model for two-phase flows with non-matched densities:

$$\partial_t\varphi + v\nabla\varphi - \mathrm{div}(m(\varphi)\nabla\mu) = 0, \tag{1a}$$

$$-\Delta\varphi + \partial\Psi_0(\varphi) - \mu - \kappa\varphi \ni 0, \tag{1b}$$

$$\begin{aligned}\partial_t(\rho(\varphi)v) + \mathrm{div}(v\otimes\rho(\varphi)v) - \mathrm{div}(2\eta(\varphi)\epsilon(v)) + \nabla p&\\ +\mathrm{div}(v\otimes J) - \mu\nabla\varphi &= 0,\end{aligned} \tag{1c}$$

$$\mathrm{div} v = 0, \tag{1d}$$

$$v_{|\partial\Omega} = 0, \tag{1e}$$

$$\partial_n\varphi_{|\partial\Omega} = \partial_n\mu_{|\partial\Omega} = 0, \tag{1f}$$

$$(v,\varphi)_{|t=0} = (v_{in},\varphi_{in}), \tag{1g}$$

where $\Omega\times(0,\infty)$ is the space-time cylinder and $\partial\Omega$ denotes the boundary of Ω. Moreover, $\partial\Psi_0$ represents the subdifferential of convex analysis. Whenever Ψ_0 is nonsmooth at φ (i.e., continuous but not differentiable, then $\partial\Psi_0$ at φ is typically set-valued. In the above model, v represents the velocity of the fluid and p describes the fluid pressure. The symmetric gradient of v is defined by $\epsilon(v) := \frac{1}{2}(\nabla v + \nabla v^\top)$. The density ρ of the mixture of the fluids depends on the order parameter φ, which reflects the mass concentration of the fluid phases. More precisely,

$$\rho(\varphi) = \frac{\rho_1+\rho_2}{2} + \frac{\rho_2-\rho_1}{2}\varphi, \tag{2}$$

where φ ranges in the interval $[-1,1]$, and $0<\rho_1\le\rho_2$ are the given densities of the two fluids under consideration. The relative flux $J := -\frac{\rho_2-\rho_1}{2}m(\varphi)\nabla\mu$, which corresponds to the diffusion of the two phases, involves the gradient of the chemical potential μ. The viscosity and mobility coefficients of the system, η and m, depend on the actual concentration of the two fluids at each point in time and space. The initial states are given by v_{in} and φ_{in}, and $\kappa>0$ is a positive constant.

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Furthermore, Ψ_0 represents the convex part of the homogeneous free energy density Ψ associated with the Ginzburg-Landau energy. It restricts the order parameter φ to the physically meaningful interval $[-1, 1]$ and captures the spinodal decomposition of the phases. For this reason, it is typically non-convex and maintains two local minima near or at -1 and 1. Depending on the underlying applications, different choices have been investigated for Ψ_0 in the literature. However, in this work we focus on the double-obstacle potential $\Psi_0 \equiv I_{[-1,1]}(\varphi)$, where $I_{[-1,1]}$ denotes the indicator function of the interval $[-1, 1]$, which was proven to be the best modeling choice in the case of deep quenches of, e.g., binary alloys, cf. [36]. A similar observation appears to be true in the case of polymeric membrane formation under rapid wall hardening. We point out that the presence of a non-smooth homogeneous free energy density gives rise to a variational inequality in (1b) which complicates the analytical and numerical treatment of the overall model.

The model (1) is thermodynamically consistent in the sense that it allows for the derivation of local entropy or free energy inequalities. In addition, phase field models are appreciated for their ability to overcome both, analytical difficulties associated with topological changes, such as, e.g., droplet break-ups or the coalescence of interfaces, as well as numerical challenges in capturing the interface dynamics. This is one of the reasons why the Cahn-Hilliard-Navier-Stokes system is used to model a variety of situations. These range from the aforementioned solidification process of liquid metal alloys, cf. [11], the simulation of bubble dynamics, as in Taylor flows [4], or the pinch-offs of liquid-liquid jets [28], to the formation of polymeric membranes [45] and protein crystallization, see e.g. [29] and references within. Furthermore, the model can be easily adapted to include the effects of surfactants such as colloid particles at fluid-fluid interfaces in gels and emulsions used in food, pharmaceutical, cosmetic, or petroleum industries [5, 38]. In many of these situations an optimal control context is desirable in order to influence the system in such a way that a prescribed system behavior is guaranteed.

This motivates the investigation of optimal control problems for the Cahn-Hilliard-Navier-Stokes (CHNS) system in this paper. Due to the non-smooth homogeneous free energy density, the constraint system of the optimal control problem is in general degenerate which poses severe problems in the derivation of stationarity conditions for characterizing solutions. More precisely, classical constraint qualifications (see, e.g., [46]) fail in the optimal control context, preventing the application of the Karush-Kuhn-Tucker (KKT) theory in Banach spaces for a primal-dual first-order characterization of an optimal solution. In fact, it is known [19, 23] that the optimal control problem with nonsmooth Ψ_0 falls into the realm of mathematical programs with equilibrium constraints (MPECs) in function spaces. A problem class, which is well-known for its constraint degeneracy [31, 37] even in finite dimensions. As a result, stationarity conditions are no longer unique (in contrast to KKT conditions); compare [19, 20] in function space and, e.g., [40] in finite dimensions. Rather they depend on the underlying problem structure and/or on the chosen analytical approach. In the present work we aim at deriving the most selective notion of stationarity known for this problem class. The so-called *strong stationarity* is a primal-dual description.

Although the control-to-state map of the Cahn-Hilliard-Navier-Stokes system is generally not differentiable in the sense of, e.g., Gâteau or Fréchet, this paper proves that a so-called conical derivative, is available; see, e.g., [33, 41]. Utilizing a proper characterization of this conical derivative a methodology for deriving strong stationarity conditions similar to [20, 33, 34] is provided. When it is associated with the primal notion of B(ouligand)-differentiability it can be exploited to create efficient numerical methods for detecting an approximate solution. We emphasize that strong stationarity constitutes a notable improvement over stationarity conditions which have been derived earlier for the optimal control problem of the semi-discrete Cahn-Hilliard-Navier-Stokes system in [23, 18]. In this latter references, employing a Yosida regularization technique with a subsequent passage to the limit results in the weaker primal-dual notion of C-stationarity. Concerning our strategy for characterizing the conical derivative we note that a similar idea has been employed for the differentiable sensitivity of an elastic contact problem including a viscous membrane in [27].

In the literature, the classical case of two-phase flows of liquids with matched densities is well investigated, see e.g. [26]. Concerning the modeling of fluids with different densities, the literature presents various approaches ranging from quasi-incompressible models with non-divergence free velocity fields to possibly thermodynamically inconsistent models with solenoidal fluid velocities, see e.g. [1, 6, 7, 10, 13, 15, 30]. Optimal control problems associated to the Cahn-Hilliard-Navier-Stokes system with a non-smooth homogeneous free energy density (double-obstacle potential) have been previously studied by the authors of this work in [23, 18]. We also refer to the recent articles [12, 25, 42], with [12, 42] treating the CHNS system in two dimensions. In addition, there are numerous publications concerning the optimal control of the phase separation process itself, i.e. the sole Cahn-Hilliard system, see e.g. [8, 9, 17, 22, 43, 44].

The remainder of the paper is organized as follows. In Section 2 the semi-discrete Cahn-Hilliard-Navier-Stokes system is introduced and assigned to the corresponding optimal control problem. Section 3 establishes the existence of feasible points and globally optimal solutions by transferring the results of [18]. The Lipschitz continuity of the solution operator of the Cahn-Hilliard-Navier-Stokes is shown in Section 4. Then Section 5 characterizes its directional derivative via a system of variational inequalities and partial differential equations. In Section 6, we finally present the strong stationarity system for the optimal control problem.

At the end of this introduction we fix some notation used below. Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a bounded, convex domain with smooth boundary $\partial\Omega \in C^2$. We define the Sobolev spaces $H^1_{0,\sigma}(\Omega;\mathbb{R}^N) = \{f \in H^1_0(\Omega;\mathbb{R}^N) : \operatorname{div} f = 0, \text{ a.e. on } \Omega\}$ and $H^2_{\partial_n}(\Omega) = \{f \in H^2(\Omega) : \partial_n f_{|\partial\Omega} = 0 \text{ a.e. on } \partial\Omega\}$, where 'a.e.' stands for 'almost everywhere'. Here, $H^k(\Omega)$ and $H^k_0(\Omega)$ denote the usual Sobolev spaces, see [3]. Furthermore, $C^\alpha(\overline{\Omega})$ denotes the space of Hölder continuous functions on $\overline{\Omega}$ with Hölder coefficient $0 < \alpha \leq 1$. By $(\cdot,\cdot)$ we denote the L^2-inner product, $\|\cdot\|$ is the induced norm, and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)^*$. For a Banach space W, we denote by W^* its topological dual. In our notation for norms, we do not distinguish between scalar- or vector-valued functions. The inner product of vectors is denoted by '$\cdot$' and the vector product is represented by '$\otimes$'. We note that in what follows, C, C_1 and C_2 denote generic non-negative constants which may take different values at different occasions.

2. Problem formulation

In this section, we present the optimization problem, which constitutes the main subject of this work, along with some important assumptions and remarks.

In order to formulate the state system, i.e. the discretization of the Cahn-Hilliard-Navier-Stokes system, we invoke the following assumptions on the given data.

Assumption 2.1. 1 *The coefficient functions $m, \eta \in C^2(\mathbb{R})$ are bounded such that there exist constants C_1, C_2 with $\min_{x\in\mathbb{R}}\{m(x), \eta(x)\} \geq C_1 > 0$ and*

$$\max_{x\in\mathbb{R}}\{m(x), \eta(x), |m'(x)|, |\eta'(x)|, |m''(x)|, |\eta''(x)|\} \leq C_2.$$

2 *The density ρ depends on the order parameter φ via*

$$\rho(\varphi) = \max\{\frac{\rho_1+\rho_2}{2} + \frac{\rho_2-\rho_1}{2}(\varphi + \overline{\varphi_{-1}}), 0\}.$$

3 *The initial data satisfies $\varphi_{-1}, \varphi_{-2} \in H^2_{\partial_n}(\Omega) \cap \mathbb{K}$, $\mu_{-1} \in H^2_{\partial_n}(\Omega)$ and $v_{-1} \in H^2_{0,\sigma}(\Omega;\mathbb{R}^N)$, where $\mathbb{K}$ denotes the constraint set*

$$\mathbb{K} := \left\{\phi \in H^1(\Omega) : -1 \leq \phi \leq 1 \text{ a.e. in } \Omega\right\}.$$

In addition, it holds that $-1 < \overline{\varphi_{-1}} := \frac{1}{|\Omega|}\int_\Omega \varphi_{-1}(x)dx < 1$*, where* $|\Omega|$ *is the Lebesgue measure of* Ω*.*

4 *The tuple* $(\varphi_{-1}, \varphi_{-2}, \mu_{-1}, v_{-1})$ *satisfies*

$$\frac{\varphi_{-1} - \varphi_{-2}}{\tau} + v_{-1}\nabla\varphi_{-2} - \mathrm{div}(m(\varphi_{-1})\nabla\mu_{-1}) = 0. \tag{3}$$

In the sequel, we use the variables $\psi_1 := -1$ and $\psi_2 := 1$ for the lower bound and the upper bound of the constraint set, repsectively. Note that by Assumption 2.1(*iii*) we exclude the simplified case $|\overline{\varphi_{-1}}| = 1$, where only one of the two fluids is present. Furthermore, by Assumption 2.1(*iv*) the initial data is consistent in the sense that it solves the Cahn-Hilliard equation (4) introduced below.

With the help of Assumption 2.1 we define the semi-discrete Cahn-Hilliard-Navier-Stokes system, which characterizes the feasible set of our optimal control problem.

Definition 2.2 (Semi-discrete CHNS-system)**.** *Let* $\Psi_0 : H^1(\Omega) \to \mathbb{R}$ *be a convex functional with subdifferential* $\partial\Psi_0$ *and let* $\tau > 0$ *be a given time step size. We say that a triple* $(\varphi, \mu, v) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega; \mathbb{R}^N)$ *solves the semi-discrete CHNS system with respect to a given control* $u \in H^1_{0,\sigma}(\Omega; \mathbb{R}^N)^*$*, denoted by* $(\varphi, \mu, v) \in S_\Psi(u)$*, if it holds for all* $\phi \in H^1(\Omega)$ *and* $\psi \in H^1_{0,\sigma}(\Omega; \mathbb{R}^N)$ *that*

$$\left\langle \frac{\varphi - \varphi_{-1}}{\tau}, \phi \right\rangle + \langle v\nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla\mu, \nabla\phi) = 0, \tag{4}$$

$$(\nabla\varphi, \nabla\phi) + \langle\partial\Psi_0(\varphi), \phi\rangle - \langle\mu, \phi\rangle - \langle\kappa\varphi_{-1}, \phi\rangle \ni 0, \tag{5}$$

$$\begin{aligned}
&\left\langle \frac{\rho(\varphi_{-1})v - \rho(\varphi_{-2})v_{-1}}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (v \otimes \rho(\varphi_{-2})v_{-1}, \nabla\psi) \\
&+ \left(v \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}, \nabla\psi \right) + (2\eta(\varphi_{-1})\epsilon(v), \epsilon(\psi)) \\
&\qquad - \langle\mu\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle u, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} .
\end{aligned} \tag{6}$$

As discussed above, the free energy density Ψ_0 can be chosen differently depending on the applications. In this work, we primarily focus on the double-obstacle potential introduced below.

Definition 2.3 (Double-obstacle potential)**.** *The functional* $\overline{\Psi}_0 : H^1(\Omega) \to \mathbb{R}$ *is given by* $\overline{\Psi}_0(\varphi) := \int_\Omega i_{[\psi_1,\psi_2]}(\varphi(x))dx$*, where* $i_{[\psi_1,\psi_2]} : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ *represents the indicator function of the set* $[\psi_1, \psi_2]$*, i.e.,*

$$i_{[\psi_1,\psi_2]}(x) := \begin{cases} +\infty & \text{if } x < \psi_1, \\ 0 & \text{if } \psi_1 \le x \le \psi_2, \\ +\infty & \text{if } x > \psi_2. \end{cases}$$

In order to employ the existence and stationarity results derived in [18], we additionally introduce the subsequent approximating double-well type potentials.

Definition 2.4 (Double-well type potentials)**.** *Let a mollifier* $\zeta \in C^1(\mathbb{R})$ *with* $\operatorname{supp}\zeta \subset [-1, 1]$, $\int_\mathbb{R} \zeta = 1$ *and* $0 \le \zeta \le 1$ *a.e. on* $\mathbb{R}$*, and a non-decreasing function* $\theta : \mathbb{R}^+ \to \mathbb{R}^+$*, with* $\theta(1) = 1$, $\theta(\alpha) > 0$ *and* $\frac{\theta(\alpha)}{\alpha} \to 0$ *as* $\alpha \to 0$*, be given. For* $\alpha > 0$ *we define the double-well type potentials* $\Psi_{0,\alpha}$ *via*

$$\zeta_\alpha(s) := \frac{1}{\alpha}\zeta\Big(\frac{s}{\alpha}\Big), \;\; \psi^{(\alpha)}(s) := \int_0^s \gamma_\alpha * \zeta_{\theta(\alpha)}(t)\, dt, \;\; \Psi_{0,\alpha}(x) := \int_\Omega \psi^{(\alpha)}(\varphi(x))\, dx,$$

where γ_α *denotes the Yosida approximation with parameter* $\alpha > 0$ *of* $\partial i_{[\psi_1,\psi_2]} \subset \mathbb{R} \times \mathbb{R}$*.*

It is easy to check that the above potentials possess the following properties. For more details we refer the reader to [18], in particular, Assumption 2.2, Assumption 3.1, Definition 7.1 and the corresponding remarks.

Remark 2.5. *The functionals defined in Definition 2.3 and Definition 2.4 are convex, proper and lower-semicontinuous.*

In case of the double-well type potentials, it additionally holds that

1. *For every $\varphi \in \mathbb{K}$ and $k \in \mathbb{N}$ it holds that $\Psi_{0,\frac{1}{k}}(\varphi) \leq 1$.*
2. *For every sequence $\left\{(x^{(k)}, y^{(k)})\right\}_{k\in\mathbb{N}} \subset H^1(\Omega) \times H^1(\Omega)^*$ with $y^{(k)} = \Psi_{0,\frac{1}{k}}{}'(x^{(k)})$ and $(x^{(k)}, y^{(k)}) \to (x^{(\infty)}, y^{(\infty)})$ strongly in $H^1(\Omega) \times H^1(\Omega)^*$, it holds that $y^{(\infty)} \in \partial\overline{\Psi}_0(x^{(\infty)})$.*
3. *$\Psi_{0,\frac{1}{k}}$ is Fréchet differentiable for every $\varphi \in H^1(\Omega)$, and for every $C \in \mathbb{R}$ there exists a constant $C_1 \in \mathbb{R}$ such that the Fréchet derivative satisfies*

$$\Psi_{0,\frac{1}{k}}(\varphi) < C \Rightarrow \|\varphi\| + \|\Psi_{0,\frac{1}{k}}{}'(\varphi)\| \leq C_1. \tag{7}$$

4. *The free energy density $\Psi_{\frac{1}{k}}(\varphi) := \Psi_{0,\frac{1}{k}}(\varphi) - \int_\Omega \frac{\kappa}{2}\varphi(x)^2 dx$ is bounded from below by a constant $C \in \mathbb{R}$ if $k > \kappa$ is sufficiently large.*

   In the following, we always assume that this is the case, since we focus on smooth potentials approximating the double obstacle potential.

Next, we define the corresponding optimal control problem, where we consider a tracking-type objective functional $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ with

$$\mathcal{X} := H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega; \mathbb{R}^N) \times L^2(\Omega; \mathbb{R}^N).$$

Definition 2.6. *The optimal control problem is given by*

$$\begin{aligned} &\min \mathcal{J}(\varphi, \mu, v, u) = \frac{1}{2}\|\varphi - \varphi_d\|^2 + \frac{a}{2}\|u\|^2 \text{ over } (\varphi, \mu, v, u) \in \mathcal{X} \\ &\text{s.t. } (\varphi, \mu, v) \in S_\Psi(u), \end{aligned} \tag{P_Ψ}$$

where $\varphi_d \in H^1(\Omega)$ denotes the desired state and $a > 0$ is a positive constant.

Note that the control problem is considered in $L^2(\Omega; \mathbb{R}^N)$, which allows for a pointwise interpretation of the controls $u \in L^2(\Omega; \mathbb{R}^N)$ in practice. Nevertheless, we point out that the subsequent existence theory is handled in the more general space $H^1_{0,\sigma}(\Omega; \mathbb{R}^N)^*$.

Since the dependencies of the operators, the corresponding solutions, and their regularity on the interfaces and the previous time steps make this problem very challenging without further restrictive assumptions or additional constraints, we employ an instantaneous control approach inspired by finite horizon model predictive control, see e.g., [14, 24], where a formal discussion and definition can be found.

3. Existence of solutions and C-stationarity

A slight modification of the optimal control problem (P_Ψ) has been already studied by the same authors in [18]. Therein, the existence of feasible points as well as globally optimal solutions was verified, if the mean values of the order parameter φ and the chemical potential μ are restricted to zero. Moreover, locally optimal points were characterized via $\mathcal{E}$-almost C-stationarity conditions. In what follows, we briefly show how these results can be transferred to our current setting.

We start with the existence theorem for double-well type potentials. The first part of the proof is similar to the proof of [18, Theorem 3.2]. However, in our setting, it is necessary to employ additional arguments in order to bound the mean values of φ and μ, respectively.

Theorem 3.1 (Existence of solutions to the CHNS system for smooth potentials)**.** *In the system (4)-(6) let $\Psi_{0,\frac{1}{k}}$ denote the potential function defined in Definition 2.4 for $k \in \mathbb{N}$ and $u \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$. Then the system (4)-(6) has a solution (φ,μ,v) which is contained in $H^2_{\partial_n}(\Omega) \times H^2_{\partial_n}(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$. Moreover, there exists a constant $C = C(u,N,\Omega,b_1,b_2,\tau,\kappa) > 0$ such that*

$$\|\varphi\|_{H^2} + \|\mu\|_{H^2} + \|v\|_{H^1} \le C(\|\varphi\| + \|\mu\| + \|\varphi_{-1}\| + \|v\|_{H^1}\|\varphi_{-1}\|_{H^2}). \tag{8}$$

If u is contained in $L^2(\Omega;\mathbb{R}^N)$, then $v \in H^2_{0,\sigma}(\Omega;\mathbb{R}^N)$.

Proof. In order to keep the notation short, we set ν to be

$$\nu := \rho(\varphi_{-2})v_{-1} - \frac{\rho_2-\rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}. \tag{9}$$

Then we start the proof by defining the spaces

$$X := H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N), \tag{10}$$

$$Y := H^1(\Omega)^* \times H^1(\Omega)^* \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*, \tag{11}$$

and the operators $\mathcal{G}_1 : H^1(\Omega) \to H^1(\Omega)^*$, $\mathcal{G}_2 : H^1(\Omega) \to H^1(\Omega)^*$, $\mathcal{G}_3 : H^1_{0,\sigma}(\Omega;\mathbb{R}^N) \to H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$, $\mathcal{G} : X \to Y$, $\mathcal{G} := (\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)^\top$ and $\mathcal{F} : X \to Y, \mathcal{F} := (\mathcal{F}_1,\ \mathcal{F}_2,\mathcal{F}_3)^\top$ via

$$\begin{aligned}
\langle \mathcal{G}_1(\mu),\phi\rangle &:= (m(\varphi_{-1})\nabla\mu,\nabla\phi) + (\mu,\phi)\,,\\
\langle \mathcal{G}_2(\varphi),\phi\rangle &:= (\nabla\varphi,\nabla\phi) + (\varphi,\phi) + \left\langle {\Psi_{0,\frac{1}{k}}}'(\varphi),\phi\right\rangle,\\
\langle \mathcal{G}_3(v),\psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} &:= (2\eta(\varphi_{-1})\epsilon(v),\epsilon(\psi)) - \langle u,\psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}}\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_1(\varphi,\mu,v) &:= -\frac{\varphi-\varphi_{-1}}{\tau} - v\nabla\varphi_{-1} + \mu,\ \mathcal{F}_2(\varphi,\mu,v) := \mu + \kappa\varphi_{-1} + \varphi,\\
\langle \mathcal{F}_3(\varphi,\mu,v),\psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} &:= \left\langle -\frac{\rho(\varphi_{-1})v - \rho(\varphi_{-2})v_{-1}}{\tau} + \mu\nabla\varphi_{-1},\psi\right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}}\\
&\quad + (v\otimes\nu,\nabla\psi).
\end{aligned}$$

The system (4)-(6) can be rewritten as

$$0 = \mathcal{G}(\varphi,\mu,v) - \mathcal{F}(\varphi,\mu,v) \subset Y. \tag{12}$$

By standard arguments, the mappings $\mathcal{G}_1$ and $\mathcal{G}_3$ are invertible and the respective inverse mapping is continuous. Since the operator '$-\Delta + id$' is invertible from $H^1(\Omega)$ to $H^1(\Omega)^*$ and the Fréchet derivative ${\Psi_{0,\frac{1}{k}}}'$ is maximal monotone (cf. [39, Theorem A]), $\mathcal{G}_2$ is invertible. Moreover, the continuity of $\mathcal{G}_2^{-1}$ follows analogously to [18].

Due to the compact embedding of the space $\overline{Y} := L^{\frac{3}{2}}(\Omega) \times L^{\frac{3}{2}}(\Omega) \times L^{\frac{3}{2}}(\Omega;\mathbb{R}^N)$, into Y, the inverse of $\mathcal{G}$ is a compact operator from $\overline{Y}$ to X. Furthermore, the continuous operator $\mathcal{F} : X \to \overline{Y}$ maps bounded sets into bounded sets. Hence, the operator $\mathcal{F}\circ\mathcal{G}^{-1} : \overline{Y} \to \overline{Y}$ is compact.

In what follows, we show the existence of a solution δ^* to the fixed point equation

$$\delta^* - \mathcal{F}\circ\mathcal{G}^{-1}(\delta^*) = 0 \in \overline{Y}, \tag{13}$$

which ensures that $\mathcal{G}^{-1}(\delta^*)$ constitutes a solution to the system (4)-(6).

In order to apply Schaefer's theorem with respect to the operator $\mathcal{F}\circ\mathcal{G}^{-1}$ we verify that the set $\mathcal{D} := \bigcup_{0\le\iota\le 1}\left\{\delta \in \overline{Y} | \delta = \iota\mathcal{F}\circ\mathcal{G}^{-1}(\delta)\right\}$ is bounded, cf., e.g., [16]. For this purpose, assume that $\delta \in \overline{Y}$ and $\iota \in [0,1]$ satisfy

$$\delta = \iota\mathcal{F}\circ\mathcal{G}^{-1}(\delta), \tag{14}$$

and define $(\varphi, \mu, v) := \mathcal{G}^{-1}(\delta) \in X$. Thus, (14) can be rewritten as

$$\mathcal{G}(\varphi, \mu, v) - \iota\mathcal{F}(\varphi, \mu, v) = 0, \tag{15}$$

which is equivalent to the following system of equations

(16)
$$\langle (1-\iota)\mu, \phi\rangle + \left\langle \iota\frac{\varphi - \varphi_{-1}}{\tau}, \phi \right\rangle + \langle \iota v \nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla\mu, \nabla\phi) = 0,\ \forall \phi \in H^1(\Omega),$$

$$\langle (1-\iota)\varphi, \phi\rangle + (\nabla\varphi, \nabla\phi) + \left\langle {\Psi_{0,\frac{1}{k}}}'(\varphi)), \phi \right\rangle - \langle \iota\mu, \phi\rangle - \langle \iota\kappa\varphi_{-1}, \phi\rangle = 0,\ \forall \phi \in H^1(\Omega), \tag{17}$$

$$\begin{aligned}\iota\left\langle \frac{\rho(\varphi_{-1})v - \rho(\varphi_{-2})v_{-1}}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} &- \iota\left(v \otimes \nu, \nabla\psi\right) + \left(2\eta(\varphi_{-1})\epsilon(v), \epsilon(\psi)\right) \\ &- \iota\left\langle \mu\nabla\varphi_{-1}, \psi\right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - \langle u, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = 0,\ \forall \psi \in H^1_{0,\sigma}(\Omega; \mathbb{R}^N).\end{aligned} \tag{18}$$

We test this system by μ, $\frac{\varphi - \varphi_{-1}}{\tau}$ and v, respectively, and sum up the resulting equations to derive

$$\begin{aligned}0 =&\iota\int_\Omega \frac{\rho(\varphi_{-1})\,|v|^2 - \rho(\varphi_{-2})\,|v_{-1}|^2}{2\tau}dx + \iota\int_\Omega \rho(\varphi_{-2})\frac{|v - v_{-1}|^2}{2\tau}dx \\ &+ \int_\Omega 2\eta(\varphi_{-1})\,|\epsilon(v)|^2\,dx + \int_\Omega m(\varphi_{-1})\,|\nabla\mu|^2\,dx + \frac{1}{\tau}\int_\Omega {\Psi_{0,\frac{1}{k}}}'(\varphi)(\varphi - \varphi_{-1})dx \\ &- \iota\kappa\int_\Omega \varphi_{-1}\frac{\varphi - \varphi_{-1}}{\tau}dx + \frac{1}{\tau}\int_\Omega \nabla\varphi(\nabla\varphi - \nabla\varphi_{-1})dx - \langle u, v\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} \\ &+ (1-\iota)\int_\Omega |\mu|^2 dx + (1-\iota)\int_\Omega \varphi\frac{\varphi - \varphi_{-1}}{\tau}dx,\end{aligned} \tag{19}$$

where we additionally used (3) in combination with the fact that

$$\begin{aligned}(\mathrm{div}(v\otimes\nu), v) &= ((\mathrm{div}\nu)v + (\nu\cdot\nabla)v, v) \\ &= \int_\Omega ((\mathrm{div}\nu)\frac{v}{2} + (\nu\cdot\nabla)v)v dx + \int_\Omega (\mathrm{div}\nu)\frac{v}{2}v dx \\ &= \int_\Omega \mathrm{div}\left(\nu\frac{|v|^2}{2}\right) + (\mathrm{div}\nu)\frac{|v|^2}{2}dx = \int_\Omega (\mathrm{div}\nu)\frac{|v|^2}{2}dx.\end{aligned} \tag{20}$$

We point out that due to the regularity of the involved quantities, the duality pairings of H^1, $H^1_{0,\sigma}$ and their respective dual spaces equal the inner product of the respective L^2-spaces and can be interpreted as integrals over Ω, cf. [3]. Rearranging these terms leads to

$$\begin{aligned}\int_\Omega 2\eta(\varphi_{-1})\,|\epsilon(v)|^2\,dx + \int_\Omega m(\varphi_{-1})\,|\nabla\mu|^2\,dx + \frac{1}{\tau}\Psi_{\frac{1}{k}}(\varphi) + \frac{1}{\tau}\int_\Omega |\nabla\varphi|^2\,dx& \\ - \langle u, v\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} + (1-\iota)\int_\Omega |\mu|^2 dx + (1-\iota)\int_\Omega \frac{|\varphi|^2}{2}dx& \\ \leq \iota\int_\Omega \frac{\rho(\varphi_{-2})\,|v_{-1}|^2}{2\tau}dx + \frac{1}{\tau}\int_\Omega |\nabla\varphi_{-1}|^2\,dx + \frac{1}{\tau}\Psi_{\frac{1}{k}}(\varphi_{-1}) + (1-\iota)\int_\Omega \frac{|\varphi_{-1}|^2}{2}dx.&\end{aligned} \tag{21}$$

With the help of Remark 2.5(*i*) the right-hand side of (21) can be bounded by a constant $C := C(N, \Omega, \tau, \varphi_{-1}, \varphi_{-2}, v_{-1}) > 0$ which is independent of ι. Hence

$$\begin{aligned}\int_\Omega 2\eta(\varphi_{-1})\,|\epsilon(v)|^2\,dx + \int_\Omega m(\varphi_{-1})\,|\nabla\mu|^2\,dx + \frac{1}{\tau}\Psi_{\frac{1}{k}}(\varphi) + \frac{1}{\tau}\int_\Omega |\nabla\varphi|^2\,dx& \\ + (1-\iota)\int_\Omega |\mu|^2 dx \leq C + \langle u, v\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}.&\end{aligned} \tag{22}$$

Since $\Psi_{\frac{1}{k}}$ is bounded from below, cf. Remark 2.5(*iv*), we can rely on Assumption 2.1(i), the boundary condition of v, and Korn's inequality to conclude that $\|v\|_{H^1}$ is bounded by a constant C_1 which depends only on C and u. Employing Poincaré's inequality and (7), we further infer $\|\varphi\|_{H^1} \leq C$.

If $0 \leq \iota \leq \frac{1}{2}$, then the inequality (22) - in combination with Assumption 2.1(i) - additionally yields $\|\mu\|_{H^1} \leq C$. Otherwise, we conclude from (17), that

$$\iota \int_\Omega \mu dx = (1-\iota)\int_\Omega \varphi dx - \iota\kappa \int_\Omega \varphi_{-1} dx + \int_\Omega \Psi_{0,\frac{1}{k}}{}'(\varphi)dx \tag{23}$$

$$\leq C(\|\varphi\| + \|\varphi_{-1}\| + \|\Psi_{0,\frac{1}{k}}{}'(\varphi)\|) \leq C_1, \tag{24}$$

where we used (7) and the fact that $\int_\Omega \varphi dx$ is bounded by the L^2-norm of φ. In combination with (22) and Poincaré's inequality, this ensures $\|\mu\|_{H^1} \leq C$.

In summary, we verified the boundedness of (φ, μ, v) in X. Next, we derive bounds for $\mathcal{F}$. In fact, we have

$$\begin{aligned}
&\|\mathcal{F}_1(\varphi,\mu,v)\|_{L^{3/2}} \leq C(\|\varphi\| + \|\varphi_{-1}\| + \|v\|_{H^1}\|\varphi_{-1}\|_{H^1} + \|\mu\|),\\
&\|\mathcal{F}_2(\varphi,\mu,v)\|_{L^{3/2}} \leq C(\|\mu\| + \|\varphi_{-1}\| + \|\varphi\|),\\
&\|\mathcal{F}_3(\varphi,\mu,v)\|_{L^{3/2}} \leq C(\|v\|_{H^1} + \|v\|_{H^1}\|\nu\|_{H^1} + \|\mu\|\,\|\varphi_{-1}\|_{H^1} + \|v_{-1}\|_{H^1}).
\end{aligned}$$

Since φ_{-1}, v_{-1} and ν are fixed and δ and ι were chosen arbitrarily, this ensures the boundedness of $\mathcal{D}$ in $\overline{Y}$. Hence Schaefer's theorem is applicable implying that equation (13) admits a fixed point $\delta^* \in \overline{Y}$. Then $\mathcal{G}^{-1}(\delta^*)$ solves the system (4)-(6).

The additional regularity and the boundedness of the solution follow analogously to the proof of [18, Lemma 3.3]. □

With the help pf Theorem 3.1 we can show the existence of a solution to the state system (4)-(6) for the double-obstacle potential by considering an approximating sequence of double-well type potentials and passing to the limit for $k \to \infty$.

Theorem 3.2 (Existence of solutions to the CHNS system)**.** *In the system (4)-(6) let $\overline{\Psi}_0$ be the double-obstacle potential given in Definition 2.3 and $u \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$. Then the system* (4)-(6) *has a solution $(\varphi,\mu,v) \in H^2_{\partial_n}(\Omega) \times H^2_{\partial_n}(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$. If u is contained in $L^2(\Omega;\mathbb{R}^N)$, then $v \in H^2_{0,\sigma}(\Omega;\mathbb{R}^N)$.*

Proof. Let $\left\{\Psi_{0,\frac{1}{k}}\right\}_{k\in\mathbb{N}}$ be a sequence of potentials satisfying Definition 2.4. Due to Theorem 3.1, there exists a bounded sequence $\{(\varphi^{(k)},\mu^{(k)},v^{(k)})\}_{k\in\mathbb{N}} \subset H^2_{\partial_n}(\Omega)\times H^2_{\partial_n}(\Omega)\times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ such that $(\varphi^{(k)},\mu^{(k)},v^{(k)})$ solves (4)-(6) with respect to $\Psi_{0,\frac{1}{k}}$ for every $k \in \mathbb{N}$. Hence there is a weakly convergent sequence $\left\{(\varphi^{(k_l)},\mu^{(k_l)},v^{(k_l)})\right\}_{l\in\mathbb{N}}$ with limit point $(\overline{\varphi},\overline{\mu},\overline{v}) \in H^2_{\partial_n}(\Omega) \times H^2_{\partial_n}(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$.

Based on the convergence property (*ii*) from Remark 2.5, it is shown that the limit point satisfies the system (4)-(6) with respect to $\overline{\Psi}_0$. Since the proof follows the exact same line of argumentation as the proofs of [18, Theorem 4.1] and [18, Theorem 5.1], we omit it at this point and refer the reader to [18].

Moreover, the additional regularity of v is shown analogously to the proof of [18, Lemma 3.3] □

Having guaranteed the existence of feasible points for the optimal control problem (P_Ψ), it is straightforward to verify the existence of global solutions. The details of the proof can be found in [18, Theorem 4.1].

Theorem 3.3 (Existence of global solutions)**.** *The optimization problem (P_Ψ) admits a global solution.*

Using the techniques of [18], an $\mathcal{E}$-almost C-stationarity system is obtained for the problem (P_Ψ). In the subsequent theorem we only use the respective adjoint system, since a more restrictive stationarity system for (P_Ψ) is derived in the course of this paper.

Theorem 3.4 (Adjoint system)**.** *In the problem* (P_Ψ) *let* $\overline{\Psi}_0$ *be the double-obstacle potential given in Definition 2.3. If* $\hat{u}$ *is an optimal control of* (P_Ψ)*, then* $\hat{u} \in H^1(\Omega;\mathbb{R}^N)$ *and there exists an adjoint state* $(p,r,\chi) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ *and* $\lambda \in H^1(\Omega)^*$ *such that for all* $\phi \in H^1(\Omega)$ *and* $\psi \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ *it holds that*

$$\left\langle D_\varphi \mathcal{J}[\hat{z}] + \frac{r}{\tau}, \phi \right\rangle + (\nabla p, \nabla \phi) + \langle \lambda, \phi \rangle = 0, \tag{25}$$

$$(m(\varphi_{-1})\nabla r, \nabla \phi) - \langle p, \phi \rangle - \langle \chi \nabla \varphi_{-1}, \phi \rangle = 0, \tag{26}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})}{\tau}\chi, \psi \right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle \nabla\chi\nu, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} \\ &+ \langle 2\eta(\varphi_{-1})\epsilon(\chi), \epsilon(\psi)\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle r\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0 \end{aligned} \tag{27}$$

$$\langle -\chi, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} + \langle D_u\mathcal{J}[\hat{z}], \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0, \tag{28}$$

where $\hat{z} := (\hat{\varphi}, \hat{\mu}, \hat{v}, \hat{u}) := (S_\Psi(\hat{u}), \hat{u})$.

Proof. Once more, we can apply the same techniques as in [18] (more precisely, as in the proof of Theorem 6.4), if we ensure the boundedness of the corresponding mean values.

This is achieved with the help of the following result.

Lemma 3.5. *For* $k \in \mathbb{N}$ *let* $\Psi_{0,\frac{1}{k}}$ *be given as in Definition 2.4. Let* $\hat{z}$ *be given as in the previous theorem. If* $(p,r,\chi) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ *satisfies*

$$\left\langle D_\varphi \mathcal{J}[\hat{z}] + \frac{r}{\tau}, \phi \right\rangle + (\nabla p, \nabla \phi) + \left\langle {\Psi_{0,\frac{1}{k}}}''(\hat{\varphi})^* p, \phi \right\rangle = 0, \tag{29}$$

$$(m(\varphi_{-1})\nabla r, \nabla \phi) - \langle p, \phi \rangle - \langle \chi \nabla \varphi_{-1}, \phi \rangle = 0, \tag{30}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})}{\tau}\chi, \psi \right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle \nabla\chi\nu, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} \\ &+ \langle 2\eta(\varphi_{-1})\epsilon(\chi), \epsilon(\psi)\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle r\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0, \end{aligned} \tag{31}$$

$$\langle -\chi, \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} + \langle D_u\mathcal{J}[\hat{z}], \psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0, \tag{32}$$

then $(p,r,\chi) \in H^1(\Omega)\times H^1(\Omega)\times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ *is bounded by a positive constant* $C = C(N,\Omega,b_1,b_2,\tau,\hat{\varphi},\varphi_d)$.

Proof. Since $\chi \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$, seting $\phi \equiv 1$ in (30) yields

$$\int_\Omega p(x)dx = 0. \tag{33}$$

Next, we test the equations (29)-(31) with τp, r and χ, respectively, and sum up to obtain

$$\begin{aligned} &\langle D_\varphi\mathcal{J}[\hat{z}], p\rangle + \tau\,(\nabla p, \nabla p) + \tau\left\langle {\Psi_{0,\frac{1}{k}}}''(\hat{\varphi})^* p, p\right\rangle + (m(\varphi_{-1})\nabla r, \nabla r) \\ &+ \left\langle \frac{\rho(\varphi_{-1})}{\tau}\chi, \chi \right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle \nabla\chi\nu, \chi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} + \langle 2\eta(\varphi_{-1})\epsilon(\chi), \epsilon(\chi)\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0. \end{aligned} \tag{34}$$

Due to the definition of ν in (9), it holds that

$$\left\langle \frac{\rho(\varphi_{-1})}{\tau}\chi, \chi \right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle \nabla\chi\nu, \chi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} \geq 0 \tag{35}$$

Since $\psi^{(\frac{1}{k})}$ is convex, cf. Definition 2.4, we further observe that

$$\tau \left\langle \Psi_{0,\frac{1}{k}}{}''(\hat{\varphi})^* p, p \right\rangle = \tau \int_\Omega \psi^{(\frac{1}{k})}{}''(\hat{\varphi})|p|^2 dx \geq 0. \tag{36}$$

By omitting the non-negative terms in (34) and applying Korn's inequality and Poincaré's inequality in combination with the boundary condition for χ and (33), we infer the existence of a constant $C = C(N, \Omega, b_1, b_2, \tau) > 0$ such that

$$\|p\|_{H^1}^2 + \|\nabla r\|^2 + \|\chi\|_{H^1}^2 \leq C \|D_\varphi \mathcal{J}[\hat{z}]\| \|p\|, \tag{37}$$

where we also took Assumption 2.1(i) into account.

It remains to show that the mean value $c_r := \frac{1}{|\Omega|}\int_\Omega r(x)dx$ is bounded.

We set $w_r := r - c_r$. Due to Assumption 2.1(iii), there exists a $\delta > 0$ such that $|\Omega_r| := \{|\hat{\varphi}(x)| < 1-\delta\} > 0$. By definition, the mean value of w_r is equal to zero and we can apply Poincaré's inequality to infer

$$\int_{\Omega_r} w_r dx \leq \int_{\Omega_r} |w_r| dx \leq \|w_r\|_{L^1} \leq C\|\nabla w_r\| = C\|\nabla r\|. \tag{38}$$

Moreover, since $\Psi_{0,\frac{1}{k}}{}''(\hat{\varphi})^* p = \psi^{(\frac{1}{k})}{}''(\hat{\varphi})p$ is contained in $L^2(\Omega)$, standard regularity theory yields that $p \in H^2(\Omega)$, cf. e.g. [32, Theorem 2.3.6]. Thus, equation (29) leads to

$$r = -\tau\left(\hat{\varphi} - \varphi_d\right) - \Psi_{0,\frac{1}{k}}{}''(\hat{\varphi})^* p + \tau\Delta p = -\tau\left(\hat{\varphi} - \varphi_d\right) + \tau\Delta p \text{ a.e. on } \Omega_r. \tag{39}$$

Consequently, it holds that

$$\begin{aligned}
|c_r|\,|\Omega_r| = \left|\int_{\Omega_r} c_r dx\right| &= \left|-\tau\int_{\Omega_r}\left(\hat{\varphi} - \varphi_d\right)dx + \tau\int_{\Omega_r}\Delta p dx - \int_{\Omega_r} w_r dx\right| && (40)\\
&= \left|-\tau\int_{\Omega_r}\left(\hat{\varphi} - \varphi_d\right)dx - \tau\int_{\partial\Omega_r}\nabla p\cdot\vec{n} dx - \int_{\Omega_r} w_r dx\right| && (41)\\
&\leq C(\|\varphi\| + \|p\|_{H^1} + \|\nabla r\| + C_1). && (42)
\end{aligned}$$

Hence, the mean value of r is bounded with respect to $\|\nabla r\|$, which - in combination with (37) - proves the assertion. ☐

As noted above the assertion of Theorem 3.4 is now verified following the same line of argumentation as in [18, Theorem 6.4]. ☐

An important consequence of the above theorem is the additional regularity of an optimal solution $\hat{u}$ in contrast to the L^2-regularity required by the control problem. This allows to exploit a structure result concerning the directional derivative of the control-to-state map in order to strengthen the stationarity result in Section 6.

4. Lipschitz continuity of the constraint mapping

For the purpose of deriving strong stationarity conditions, we take a closer look at the constraint mapping S_Ψ. More precisely, it is our intention to characterize its directional derivative DS_Ψ.

As a first step towards this goal, we verify the Lipschitz continuity of S_Ψ. We start by reformulating the system (4)-(6) with the help of the slack variable $\Lambda \in \partial\Psi_0(\varphi)$. Since $\varphi \in H^2(\Omega)$, cf. Lemma 3.2,

Λ is contained in $L^2(\Omega)$. Moreover, we set $\mu := \mu + \kappa\varphi_{-1} \in H^1(\Omega)$ such that the system (4)-(6) transforms into

$$\left\langle \frac{\varphi - \varphi_{-1}}{\tau}, \phi \right\rangle + \langle v\nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla\mu, \nabla\phi) - \kappa\,(m(\varphi_{-1})\nabla\varphi_{-1}, \nabla\phi) = 0, \tag{43}$$

$$(\nabla\varphi, \nabla\phi) - \langle \mu, \phi\rangle + \langle \Lambda, \phi\rangle = 0, \tag{44}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})v - \rho(\varphi_{-2})v_{-1}}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (v \otimes \rho(\varphi_{-2})v_{-1}, \nabla\psi) \\ &+ \left(v \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}, \nabla\psi \right) + (2\eta(\varphi_{-1})\epsilon(v), \epsilon(\psi)) \\ &- \langle \mu\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} + \langle \kappa\varphi_{-1}\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle u, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}. \end{aligned} \tag{45}$$

Clearly, any solution of the system (43)-(45) can be transformed into a solution of (4)-(6) by adding/subtracting $\kappa\varphi_{-1}$ to μ and vice versa. Slightly abusing notation, we subsequently refer to the solution operator of the system (43)-(45) by S_Ψ and maintain the same notation.

Theorem 4.1 (Lipschitz continuity of S_Ψ). *The mapping $S_\Psi : H^1_{0,\sigma}(\Omega; \mathbb{R}^N)^* \to H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega; \mathbb{R}^N)$ is Lipschitz continuous.*

Proof. For $i = 1, 2$ let $(\varphi_i, \mu_i, v_i) \in S_\Psi(u_i)$ and $\Lambda_i \in \partial\Psi_0(\varphi_i)$ be the assoicated slack variable. We easily verify that

$$\left\langle \frac{\varphi_1 - \varphi_2}{\tau}, \phi \right\rangle + \langle (v_1 - v_2)\nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla(\mu_1 - \mu_2), \nabla\phi) = 0, \tag{46}$$

$$(\nabla(\varphi_1 - \varphi_2), \nabla\phi) + \langle \Lambda_1 - \Lambda_2, \phi\rangle - \langle \mu_1 - \mu_2, \phi\rangle = 0, \tag{47}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})(v_1 - v_2)}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - ((v_1 - v_2) \otimes \rho(\varphi_{-2})v_{-1}, \nabla\psi) \\ &+ \left((v_1 - v_2) \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}, \nabla\psi \right) + (2\eta(\varphi_{-1})\epsilon(v_1 - v_2), \epsilon(\psi)) \\ &- \langle (\mu_1 - \mu_2)\nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle u_1 - u_2, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}. \end{aligned} \tag{48}$$

Testing with $\tau(\mu_1 - \mu_2)$, $\varphi_1 - \varphi_2$ and $\tau(v_1 - v_2)$, respectively, and summing up, yields

$$\begin{aligned} &\int_\Omega \rho(\varphi_{-1})|v_1 - v_2|^2 dx + \tau \int_\Omega \mathrm{div}\nu \frac{|v_1 - v_2|^2}{2} dx + 2\tau \int_\Omega \eta(\varphi_{-1})|\epsilon(v_1 - v_2)|^2 dx \\ &+\tau \int_\Omega m(\varphi_{-1})|\nabla(\mu_1 - \mu_2)|^2 dx + \int_\Omega |\nabla(\varphi_1 - \varphi_2)|^2 dx + \langle \Lambda_1 - \Lambda_2, \varphi_1 - \varphi_2\rangle \\ &= \tau\,\langle u_1 - u_2, v_1 - v_2\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}. \end{aligned}$$

Here we also employed the relations (9) and (20). Using Assumption 2.1(*iv*) we conclude that

$$\begin{aligned} &\int_\Omega \frac{\rho(\varphi_{-1}) + \rho(\varphi_{-2})}{2} |v_1 - v_2|^2 dx + 2\tau \int_\Omega \eta(\varphi_{-1})|\epsilon(v_1 - v_2)|^2 dx \\ &+\tau \int_\Omega m(\varphi_{-1})|\nabla(\mu_1 - \mu_2)|^2 dx + \int_\Omega |\nabla(\varphi_1 - \varphi_2)|^2 dx + \langle \Lambda_1 - \Lambda_2, \varphi_1 - \varphi_2\rangle \\ &= \tau\,\langle u_1 - u_2, v_1 - v_2\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}. \end{aligned}$$

Due to the monotonicity of $\partial\Psi$, the non-negativity of the density ρ, and the boundedness of the coefficients m and η this leads to

$$C_1\|\nabla(v_1-v_2)\|_{L^2}^2+C_2\|\nabla(\mu_1-\mu_2)\|_{L^2}^2+\|\nabla(\varphi_1-\varphi_2)\|_{L^2}^2 \tag{49}$$

$$\leq \tau\|u_1-u_2\|_{H^{-1}_{0,\sigma}}\|v_1-v_2\|_{H^1}, \tag{50}$$

where $C_1, C_2 > 0$ are positive constants depending on τ, η, and m. Since $v_1 - v_2 \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$, testing (46) with $\phi \equiv 1$ yields

$$0 = \frac{1}{\tau}\int_\Omega \varphi_1(x)-\varphi_2(x)dx + \int_\Omega (v_1(x)-v_2(x))\nabla\varphi_{-1}(x)dx \tag{51}$$

$$= \frac{1}{\tau}\left(\int_\Omega \varphi_1(x)-\varphi_2(x)dx\right). \tag{52}$$

In combination with Poincaré's inequality, Korn's inequality and (50) this verifies the existence of a constant C such that

$$\|v_1-v_2\|_{H^1}^2+\|\nabla(\mu_1-\mu_2)\|_{L^2}^2+\|\varphi_1-\varphi_2\|_{H^1}^2 \leq C\|u_1-u_2\|_{H^{-1}_{0,\sigma}}\|v_1-v_2\|_{H^1}. \tag{53}$$

This already ensures that $\|\varphi_1-\varphi_2\|_{H^1} \leq C\|u_1-u_2\|_{H^{-1}}$.

By Sobolev's imbedding theorem $\varphi_1, \varphi_2 \in H^2(\Omega)$ are contained in the Hölder space $C^\alpha(\overline{\Omega})$ for some $\alpha > 0$. In the following, we show that there exists a constant C_1 such that

$$\|\varphi_1-\varphi_2\|_C := \max_{x\in\overline{\Omega}}\{|\varphi_1(x)-\varphi_2(x)|\} \leq C_1\|u_1-u_2\|_{H^{-1}}. \tag{54}$$

For this purpose, we set

$$\|\varphi_1-\varphi_2\|_C =: \varphi_1(x_{max})-\varphi_2(x_{max}) =: \delta_{max}. \tag{55}$$

Since $\varphi_1-\varphi_2$ is continuous, there exists a neighborhood $x_{max} \in \Omega_{x_{max}} \subset \overline{\Omega}$ with positive measure such that $\varphi_1(x)-\varphi_2(x) > \frac{\delta_{max}}{2}$ for all $x \in \Omega_{x_{max}}$. Hence, (54) is satisfied, since it holds that

$$\|\varphi_1-\varphi_2\|_C\frac{|\Omega_{x_{max}}|}{2} = \frac{\delta_{max}}{2}|\Omega_{x_{max}}| \leq \|\varphi_1-\varphi_2\|_{H^1} \leq C\|u_1-u_2\|_{H^{-1}}. \tag{56}$$

By Assumption 2.1(*iii*) there exists a $\delta > 0$ and a subset $\Omega_\delta \subset \Omega$ with positive measure such that $-1+\delta < \varphi_1(x) < 1-\delta$ a.e. on Ω_δ. If $\|u_1-u_2\|_{H^{-1}}$ is sufficiently small, it holds that $\|\varphi_1-\varphi_2\|_C \leq \delta$ and therefore $-1 < \varphi_2(x) < 1$ a.e. on Ω_δ. Using the characterization of the subdifferential $\partial\Psi_0$ of the double-obstacle potential, we infer that $\Lambda_1 = \Lambda_2 = 0$ a.e. on Ω_δ.

Hence, by (47), it holds that

$$\mu_1-\mu_2 = -\Delta(\varphi_1-\varphi_2)+\Lambda_1-\Lambda_2 = -\Delta(\varphi_1-\varphi_2), \text{ a.e. on } \Omega_\delta. \tag{57}$$

In order to estimate the mean value of $\mu_1 - \mu_2$, we further define $c_\mu := \frac{1}{|\Omega|}\int_\Omega \mu_1 - \mu_2 dx$ and $w_\mu := \mu_1-\mu_2-c_\mu$. By definition, the mean value of w_μ is equal to zero and we can apply Poincaré's inequality to infer

$$\int_{\Omega_\delta} w_\mu dx \leq \int_{\Omega_\delta}|w_\mu|dx \leq \|w_\mu\|_{L^1} \leq C\|\nabla w_\mu\| = C\|\nabla(\mu_1-\mu_2)\|. \tag{58}$$

Using the divergence theorem, we derive the existence of a constant C such that

$$|c_\mu|\,|\Omega_\delta| = \left|\int_{\Omega_\delta} c_\mu dx\right| = \left|\int_{\Omega_\delta} -\Delta(\varphi_1-\varphi_2)dx - \int_{\Omega_\delta} w_\mu dx\right| \tag{59}$$

$$= \left|\int_{\partial\Omega_\delta} \nabla(\varphi_1-\varphi_2)\vec{n}dx - \int_{\Omega_\delta} w_\mu dx\right| \tag{60}$$

$$\leq C(\|\varphi_1-\varphi_2\|_{H^1}+\|\nabla(\mu_1-\mu_2)\|). \tag{61}$$

Then the assertion follows from inequality (53). □

We point out that an immediate consequence of the above theorem is that the solutions to the constraint system are uniquely determined by the control u.

5. Directional derivative

In order to show the directional differentiability of the constraint mapping S_Ψ, we reformulate the variational inequality (44) as a complementarity problem with the help of the slack variables $\Lambda^+,\ \Lambda^- \in H^1(\Omega)^*$ with $\Lambda = \Lambda^+ + \Lambda^-$. More precisely, (44) is reformulated as follows

$$\left\langle \Lambda^+ + \Lambda^-, \phi \right\rangle = \langle \nabla\varphi, \nabla\phi\rangle - \langle \mu, \phi\rangle\,,\ \forall \phi \in H^1(\Omega) \tag{62}$$

$$\left\langle \Lambda^-, \varphi - \psi_2 \right\rangle = 0,\ \left\langle \Lambda^+, \varphi - \psi_1 \right\rangle = 0, \tag{63}$$

$$\left\langle \Lambda^+, \phi_2 \right\rangle \geq 0,\ \left\langle \Lambda^-, \phi_2 \right\rangle \leq 0,\ \forall \phi_2 \in H^1(\Omega) : \phi_2 \geq 0 \text{ a.e. on } \Omega. \tag{64}$$

The directional derivative DS_Ψ is characterized by the following theorem.

Theorem 5.1. *The directional derivative of S_Ψ at $u_0 \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$ with $S_\Psi(u_0) = (\varphi_0, \mu_0, v_0)$ in direction $h \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$ is given by $DS_\Psi[u_0](h) = (q, w, \zeta)$, where $(q, w, \zeta) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ is the unique solution to the variational system*

$$q \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp, \tag{65}$$

$$\langle -\Delta q - w, \phi - q\rangle \geq 0,\ \forall \phi \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp, \tag{66}$$

$$\left\langle \frac{q}{\tau}, \phi \right\rangle + \langle \zeta \nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla w, \nabla\phi) = 0, \tag{67}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})\zeta}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta \otimes \rho(\varphi_{-2}) v_{-1}, \nabla\psi) \\ &+ \left(\zeta \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2}) \nabla\mu_{-1}, \nabla\psi \right) + (2\eta(\varphi_{-1})\epsilon(\zeta), \epsilon(\psi)) \\ &- \langle w \nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle h, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}\,. \end{aligned} \tag{68}$$

Here, $T_{\mathbb{K}}(\varphi_0)$ is the tangent cone of $\mathbb{K}$ at φ_0 and $\Lambda^\perp := \{\phi \in H^1(\Omega) : \langle \phi, \Lambda\rangle = 0\}$.

Proof. For $\theta > 0$ let $u_\theta := u_0 + \theta h \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$, $(\varphi_\theta, \mu_\theta, v_\theta) \in S_\Psi(u_\theta)$, and $\Lambda_\theta := \Lambda_\theta^+ + \Lambda_\theta^-$ be the associated slack variable(s) as introduced above.

By the Lipschitz continuity of S_Ψ the sets $\{\frac{\varphi_\theta - \varphi_0}{\theta} : 0 < \theta \leq 1\}$, $\{\frac{\mu_\theta - \mu_0}{\theta} : 0 < \theta \leq 1\}$, $\{\frac{v_\theta - v_0}{\theta} : 0 < \theta \leq 1\}$, $\{\frac{\Lambda_\theta^+ - \Lambda_0^+}{\theta} : 0 < \theta \leq 1\}$ and $\{\frac{\Lambda_\theta^- - \Lambda_0^-}{\theta} : 0 < \theta \leq 1\}$ are bounded in $H^1(\Omega)$, $H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ and $H^{-1}(\Omega)$, respectively.

As a consequence, we can construct a sequence $\theta_k \to 0$ such that each of the sequences $\frac{\varphi_{\theta_k} - \varphi_0}{\theta_k} \rightharpoonup q$, $\frac{\mu_{\theta_k} - \mu_0}{\theta_k} \rightharpoonup w$, $\frac{v_{\theta_k} - v_0}{\theta_k} \rightharpoonup \zeta$ $\frac{\Lambda_{\theta_k}^+ - \Lambda_0^+}{\theta_k} \rightharpoonup \Xi^+$, and $\frac{\Lambda_{\theta_k}^- - \Lambda_0^-}{\theta_k} \rightharpoonup \Xi^-$ converges weakly, where q, w, ζ, Ξ^+ and Ξ^- denote the respecive weak limit points. In addition, we define $\Xi := \Xi^+ + \Xi^-$ and note that $\frac{\Lambda_{\theta_k} - \Lambda_0}{\theta_k} \rightharpoonup \Xi$.

Clearly, the triple $(\frac{\varphi_{\theta_k}-\varphi_0}{\theta_k}, \frac{\mu_{\theta_k}-\mu_0}{\theta_k}, \frac{v_{\theta_k}-v_0}{\theta_k})$ along with the slack variable $\frac{\Lambda_{\theta_k}-\Lambda_0}{\theta_k}$ satisfies the linear system (43)-(45) for every $k \in \mathbb{N}$. By passing to the limit for $\theta_k \to 0$, we verify that

$$\langle \nabla q, \nabla \phi \rangle - \langle w, \phi \rangle = \langle \Xi, \phi \rangle \,, \tag{69}$$

$$\left\langle \frac{q}{\tau}, \phi \right\rangle + \langle \zeta \nabla \varphi_{-1}, \phi \rangle + (m(\varphi_{-1}) \nabla w, \nabla \phi) = 0, \tag{70}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})\zeta}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta \otimes \rho(\varphi_{-2}) v_{-1}, \nabla \psi) \\ &+ \left(\zeta \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2}) \nabla \mu_{-1}, \nabla \psi \right) + (2\eta(\varphi_{-1}) \epsilon(\zeta), \epsilon(\psi)) \\ &- \langle w \nabla \varphi_{-1}, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle h, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} \,, \end{aligned} \tag{71}$$

where we used the same imbedding arguments as in the proof of [18, Theorem 4.1, Theorem 5.1]).

In order to show that the variational inequality (65),(66) is satsified, we split the rest of the proof into two separate lemmata.

Lemma 5.2. *Let q, Ξ^+ and Ξ^- denote weak limit points of $\frac{\varphi_\theta-\varphi_0}{\theta}$, $\frac{\Lambda^+_\theta-\Lambda^+_0}{\theta}$, and $\frac{\Lambda^-_\theta-\Lambda^-_0}{\theta}$, respectively, for $\theta \to 0$. Then $q \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda^+_0\}^\perp \cap \{\Lambda^-_0\}^\perp$ and for all $\phi \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda^+_0\}^\perp \cap \{\Lambda^-_0\}^\perp$ it holds that*

$$\left\langle \Xi^+ + \Xi^-, \phi \right\rangle \geq 0. \tag{72}$$

Proof. Clearly, $q \in T_{\mathbb{K}}(\varphi_0)$ due to the definition of the tangent cone. Employing (63) further yields

$$\left\langle \Lambda^-_0, q \right\rangle = \lim_{\theta \to 0} \left\langle \Lambda^-_0, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle = \lim_{\theta \to 0} \left\langle \Lambda^-_0, \frac{\varphi_\theta - \psi_2}{\theta} \right\rangle \geq 0. \tag{73}$$

Moreover, we observe that

$$\lim_{\theta \to 0} \left\langle \Lambda^-_0 - \Lambda^-_\theta, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle = 0.$$

Hence,

$$\begin{aligned} \left\langle \Lambda^-_0, q \right\rangle &= \lim_{\theta \to 0} \left\langle \Lambda^-_0, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle && (74) \\ &= \lim_{\theta \to 0} \left(\left\langle \Lambda^-_\theta, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle + \left\langle \Lambda^-_0 - \Lambda^-_\theta, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle \right) && (75) \\ &= \lim_{\theta \to 0} \left\langle \Lambda^-_\theta, \frac{\varphi_\theta - \varphi_0}{\theta} \right\rangle = \lim_{\theta \to 0} \left\langle \Lambda^-_\theta, \frac{-(\varphi_0 - \psi_2)}{\theta} \right\rangle \leq 0. && (76) \end{aligned}$$

In combination with (73), this leads to

$$\left\langle \Lambda^-_0, q \right\rangle = 0. \tag{77}$$

Analogously, we derive $\left\langle \Lambda^+_0, q \right\rangle = 0$. In summary, q is contained in the cone $T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda^+_0\}^\perp \cap \{\Lambda^-_0\}^\perp$.

In order to show (72), let $\theta_1, \theta_2 \geq 0$ be arbitrarily chosen. By definition $(\varphi_{\theta_i}, \mu_{\theta_i}, \Lambda^+_{\theta_i}, \Lambda^-_{\theta_i})$ solves the complementarity system (62)-(64) for $i = 1, 2$ and hence $\left\langle -\Lambda^-_{\theta_2}, (\varphi_{\theta_1} - \psi_2) \right\rangle \leq 0$ and $\left\langle \Lambda^-_{\theta_1}, -(\varphi_{\theta_2} - \psi_2) \right\rangle \leq 0$. This yields

$$\begin{aligned} \left\langle \Lambda^-_{\theta_1} - \Lambda^-_{\theta_2}, \varphi_{\theta_1} - \varphi_{\theta_2} \right\rangle &= \left\langle \Lambda^-_{\theta_1} - \Lambda^-_{\theta_2}, (\varphi_{\theta_1} - \psi_2) - (\varphi_{\theta_2} - \psi_2) \right\rangle \\ &= \left\langle \Lambda^-_{\theta_1}, -(\varphi_{\theta_2} - \psi_2) \right\rangle + \left\langle -\Lambda^-_{\theta_2}, (\varphi_{\theta_1} - \psi_2) \right\rangle \leq 0. \end{aligned}$$

By taking advantage of the Lipschitz continuity of S_Ψ we derive

$$C|\theta_1-\theta_2|^2 \geq |\langle \Lambda_{\theta_1}-\Lambda_{\theta_2}, \varphi_{\theta_1}-\varphi_{\theta_2}\rangle| \geq |\langle \Lambda^-_{\theta_1}-\Lambda^-_{\theta_2}, \varphi_{\theta_1}-\varphi_{\theta_2}\rangle|$$
$$\geq |\langle \Lambda^-_{\theta_1}, -(\varphi_{\theta_2}-\psi_2)\rangle|.$$

Employing this estimate with $\theta_1 := \theta$ and $\theta_2 = 0$, we infer

(78) $$\langle \Xi^-, \varphi_0-\psi_2\rangle = \lim_{\theta\to 0}\left\langle \frac{\Lambda^-_\theta-\Lambda^-_0}{\theta}, \varphi_0-\psi_2\right\rangle = \lim_{\theta\to 0}\left\langle \frac{\Lambda^-_\theta}{\theta}, \varphi_0-\psi_2\right\rangle = 0.$$

Now let $\phi\in H^1(\Omega)$ be an arbitrary element of the intersection $C_{\mathbb{K}}(\varphi_0)\cap\{\Lambda^-_0\}^\perp$, where $C_{\mathbb{K}}(\varphi_0) := \{\phi\in H^1(\Omega) : \exists_{t>0}\varphi_0+t\phi\in\mathbb{K}\}$. Then there exists a $t>0$ such that $\varphi_0+t\phi-\psi_2\leq 0$ a.e. on Ω. Equation (63) further implies $\varphi_0+t\phi-\psi_2\in\{\Lambda^-_0\}^\perp$. Thus

$$\langle \Xi^-, \varphi_0+t\phi-\psi_2\rangle = \lim_{\theta\to 0}\left\langle \frac{\Lambda^-_\theta-\Lambda^-_0}{\theta}, \varphi_0+t\phi-\psi_2\right\rangle = \lim_{\theta\to 0}\left\langle \frac{\Lambda^-_\theta}{\theta}, \varphi_0+t\phi-\psi_2\right\rangle \geq 0,$$

where we additionally used (64). In combination with (78), this leads to

(79) $$\langle \Xi^-, \phi\rangle \geq 0.$$

Analogously, we verify that every $\phi\in C_{\mathbb{K}}(\varphi_0)\cap\{\Lambda^+_0\}^\perp$ satisfies

(80) $$\langle \Xi^+, \phi\rangle \geq 0.$$

Due to the polyhedricity of $\mathbb{K}$, cf. e.g. [33], the intersection $C_{\mathbb{K}}(\varphi_0)\cap\{\Lambda^+_0\}^\perp\cap\{\Lambda^-_0\}^\perp$ is dense in $T_{\mathbb{K}}(\varphi_0)\cap\{\Lambda^+_0\}^\perp\cap\{\Lambda^-_0\}^\perp$. Consequently, we can extend the inequalities (79) and (80) to hold for every $\phi\in T_{\mathbb{K}}(\varphi_0)\cap\{\Lambda^+_0\}^\perp\cap\{\Lambda^-_0\}^\perp$, which proves the assertion. ☐

Lemma 5.3. *Let q and Ξ be weak limit points of $\frac{\varphi_\theta-\varphi_0}{\theta}$ and $\frac{\Lambda_\theta-\Lambda_0}{\theta}$, respectively, for $\theta\to 0$. Then it holds that*

(81) $$\langle \Xi, q\rangle = 0.$$

Proof. We define the linear operator $D : H^1(\Omega)\times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)\to H^1(\Omega)^*\times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$ by

$$D(\mu,v) := \begin{pmatrix} v\nabla\varphi_{-1} - \operatorname{div}(m(\varphi_{-1})\nabla\mu) \\ \frac{\rho(\varphi_{-1})v}{\tau} + \operatorname{div}(v\otimes\nu) - \operatorname{div}(2\eta(\varphi_{-1})\epsilon(v)) - \mu\nabla\varphi_{-1}\end{pmatrix},$$

where ν is given by (9). The invertibility of D follows by Schaefer's Theorem and the same arguments as in the proof of Theorem 3.1. For an arbitrary φ we set $\mu := [D^{-1}(-\frac{\varphi}{\tau},0)]_1$ and $v := [D^{-1}(-\frac{\varphi}{\tau},0)]_2$, where $[D^{-1}(x)]_i$ denotes the i-th component of the linear operator D^{-1} at x. Then it holds for all $\phi\in H^1(\Omega)$ and $\psi\in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ that

(82) $$\langle v\nabla\varphi_{-1},\phi\rangle + \langle m(\varphi_{-1})\nabla\mu,\nabla\phi\rangle = \left\langle -\frac{\varphi}{\tau},\phi\right\rangle,$$

$$\left\langle \frac{\rho(\varphi_{-1})v}{\tau},\psi\right\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} - \langle (v\otimes\nu),\nabla\psi\rangle + \langle 2\eta(\varphi_{-1})\epsilon(v),\epsilon(\psi)\rangle$$

(83) $$-\langle \mu\nabla\varphi_{-1},\psi\rangle_{H^{-1}_{0,\sigma},H^1_{0,\sigma}} = 0.$$

Testing (82),(83) with $\tau\mu$ and τv, respectively, summing up and using assumption (3) leads to

$$0 \leq \int_\Omega \frac{\rho(\varphi_{-1})+\rho(\varphi_{-2})}{2}|v|^2dx + 2\tau\int_\Omega \eta(\varphi_{-1})|\epsilon(v)|^2dx + \tau\int_\Omega m(\varphi_{-1})|\nabla(\mu)|^2dx$$

(84) $$= -\langle\mu,\varphi\rangle = -\left\langle [D^{-1}(-\frac{\varphi}{\tau},0)]_1,\varphi\right\rangle.$$

Next, we define the linear operator $A : H^1(\Omega) \to H^1(\Omega)^*$ by

$$A(\varphi) := -\Delta\varphi - [D^{-1}(-\frac{\varphi}{\tau}, 0)]_1, \tag{85}$$

where the Laplace operator is understood in the weak sense. Due to (84), A is coercive in the sense that

$$\forall \varphi \in H^1(\Omega) \quad \langle A(\varphi), \varphi\rangle = \|\nabla\varphi\|_{L^2}^2 - \left\langle [D^{-1}(-\frac{\varphi}{\tau}, 0)]_1, \varphi \right\rangle \geq 0.$$

Note that the equations (43) and (45) imply

$$D(\mu_\theta, v_\theta) = \begin{pmatrix} -\frac{\varphi_\theta - \varphi_{-1}}{\tau} - \kappa \mathrm{div}(m(\varphi_{-1})\nabla\varphi_{-1}) \\ \frac{\rho(\varphi_{-2})v_{-1}}{\tau} + u_\theta - \kappa\varphi_{-1}\nabla\varphi_{-1} \end{pmatrix}, \tag{86}$$

which yields

$$(\mu_\theta, v_\theta) - D^{-1}(-\frac{\varphi_\theta}{\tau}, 0) = D^{-1} \begin{pmatrix} \frac{\varphi_{-1}}{\tau} - \kappa \mathrm{div}(m(\varphi_{-1})\nabla\varphi_{-1}) \\ \frac{\rho(\varphi_{-2})v_{-1}}{\tau} + u_\theta - \kappa\varphi_{-1}\nabla\varphi_{-1} \end{pmatrix}. \tag{87}$$

Thus, we obtain

$$A(\varphi_\theta) = \Lambda_\theta + \mu_\theta - [D^{-1}(-\frac{\varphi_\theta}{\tau}, 0)]_1 \tag{88}$$

$$= \Lambda_\theta + \left[D^{-1} \begin{pmatrix} \frac{\varphi_{-1}}{\tau} - \kappa \mathrm{div}(m(\varphi_{-1})\nabla\varphi_{-1}) \\ \frac{\rho(\varphi_{-2})v_{-1}}{\tau} + u_\theta - \kappa\varphi_{-1}\nabla\varphi_{-1} \end{pmatrix} \right]_1, \tag{89}$$

where we additionally employed (62). Consequently,

$$A(\frac{\varphi_\theta - \varphi_0}{\theta}) = \frac{1}{\theta}\left(\Lambda_\theta - \Lambda_0 + \left[D^{-1} \begin{pmatrix} 0 \\ u_\theta - u_0 \end{pmatrix} \right]_1 \right) \tag{90}$$

$$= \frac{1}{\theta}(\Lambda_\theta - \Lambda_0) + \left[D^{-1} \begin{pmatrix} 0 \\ h \end{pmatrix} \right]_1. \tag{91}$$

To simplify the notation we introduce the linear operator $C(h) := \left[D^{-1} \begin{pmatrix} 0 \\ h \end{pmatrix} \right]_1$.

Since A is coercive, the following inequality holds true for every $y, z \in H^1(\Omega)$

$$\frac{1}{2}(\langle A(z), y\rangle + \langle A(y), z\rangle) \leq \langle A(z), z\rangle^{\frac{1}{2}} \langle A(y), y\rangle^{\frac{1}{2}}. \tag{92}$$

Setting $z := \frac{\varphi_{\theta_1} - \varphi_0}{\theta_1}$ and $y := \frac{\varphi_{\theta_2} - \varphi_0}{\theta_2}$ for arbitrary $\theta_1, \theta_2 > 0$, we rewrite (92) as follows

$$\frac{1}{2}\left(\left\langle \frac{1}{\theta_1}(\Lambda_{\theta_1} - \Lambda_0) + C(h), \frac{\varphi_{\theta_2} - \varphi_0}{\theta_2} \right\rangle + \left\langle \frac{1}{\theta_2}(\Lambda_{\theta_2} - \Lambda_0) + C(h), \frac{\varphi_{\theta_1} - \varphi_0}{\theta_1} \right\rangle \right)$$

$$\leq \left\langle \frac{1}{\theta_1}(\Lambda_{\theta_1} - \Lambda_0) + C(h), \frac{\varphi_{\theta_1} - \varphi_0}{\theta_1} \right\rangle^{\frac{1}{2}} \left\langle \frac{1}{\theta_2}(\Lambda_{\theta_2} - \Lambda_0) + C(h), \frac{\varphi_{\theta_2} - \varphi_0}{\theta_2} \right\rangle^{\frac{1}{2}} \tag{93}$$

$$\leq \left\langle C(h), \frac{\varphi_{\theta_1} - \varphi_0}{\theta_1} \right\rangle^{\frac{1}{2}} \left\langle C(h), \frac{\varphi_{\theta_2} - \varphi_0}{\theta_2} \right\rangle^{\frac{1}{2}}, \tag{94}$$

where the last inequality follows directly from the monotonicity of $\partial\Psi_0$

Now we consider an arbitrary sequence $\theta_1^k \to 0$ such that $\frac{\varphi_{\theta_1^k} - \varphi_0}{\theta_1^k}$ and $\frac{\Lambda_{\theta_1^k} - \Lambda_0}{\theta_1^k}$ converge weakly to the weak limit points q_1 and Ξ_1, respectively. By setting $\theta_1 := \theta_1^k$ and passing to the limit for $\theta_1^k \to 0$, we obtain

$$\frac{1}{2}\left(\left\langle \Xi_1 + C(h), \frac{\varphi_{\theta_2}-\varphi_0}{\theta_2}\right\rangle + \left\langle \frac{\Lambda_{\theta_2}-a_0}{\theta_2} + C(h), q_1\right\rangle\right) \tag{95}$$
$$\leq \langle C(h), q_1\rangle^{\frac{1}{2}} \left\langle C(h), \frac{\varphi_{\theta_2}-\varphi_0}{\theta_2}\right\rangle^{\frac{1}{2}}.$$

The subsequent limiting process for a sequence $\theta_2^k \to 0$ with the corresponding weak limit points q_2 and Ξ_2 yields

$$\langle C(h), q_1\rangle^{\frac{1}{2}} \langle C(h), q_2\rangle^{\frac{1}{2}} \geq \frac{1}{2}(\langle \Xi_1 + C(h), q_2\rangle + \langle \Xi_2 + C(h), q_1\rangle). \tag{96}$$

Employing Lemma 5.2, we further infer

$$\begin{aligned}\langle C(h), q_1\rangle^{\frac{1}{2}} \langle C(h), q_2\rangle^{\frac{1}{2}} &\geq \frac{1}{2}(\langle \Xi_1 + C(h), q_2\rangle + \langle \Xi_2 + C(h), q_1\rangle)\\ &\geq \frac{1}{2}(\langle C(h), q_2\rangle + \langle C(h), q_1\rangle) \geq \langle C(h), q_1\rangle^{\frac{1}{2}} \langle C(h), q_2\rangle^{\frac{1}{2}}.\end{aligned}$$

Consequently, the above inequalities hold as equations and it holds that.

$$\langle \Xi_2, q_1\rangle = 0 \tag{97}$$

Since θ_1^k, θ_2^k were chosen aribitrarily, this holds for all weak limit points q_1, q_2 and Ξ_1, Ξ_2 of the sequences $\frac{\varphi_\theta-\varphi_0}{\theta}$ and $\frac{\Lambda_\theta-\Lambda_0}{\theta}$, respectively. ☐

Combining Lemma 5.2 and Lemma 5.3, as well as taking (69) into account, we conclude that each weak limit point q of $\frac{\varphi_\theta-\varphi_0}{\theta}$ satisfies the variational inequality

$$\begin{gathered} q \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp,\\ \langle -\Delta q - w, \phi - q\rangle \geq 0,\ \forall \phi \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp. \end{gathered} \tag{98}$$

Since the variational inequality has a unique solution, this concludes the proof of Theorem 5.1. ☐

The above theorem characterizes the directional derivative of the control-to-state operator as a solution to a system of a variational inequality coupled to partial differential equations. The constraint set associated to the variational inequality is represented by a convex cone, which is also known as the critical cone, see, e.g., [35].

6. Stationarity conditions

In this section, we finally derive strong stationarity conditions for the optimal control problem (P_Ψ), which provides a more restrictive stationarity system than the ϵ-almost C-stationarity system from [18] depicted in Theorem 3.4.

Theorem 6.1. *If $z_0 := (\varphi_0, \mu_0, v_0, u_0)$ is an optimal point of (P_Ψ), then there exists an adjoint state $(p, r, \chi) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega; \mathbb{R}^N)$ and $\lambda \in H^1(\Omega)^*$ such that for all $\phi \in H^1(\Omega)$ and*

$\psi \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ it holds that

$$\left\langle D_\varphi \mathcal{J}[z_0] + \frac{r}{\tau}, \phi \right\rangle + (\nabla p, \nabla \phi) + \langle \lambda, \phi \rangle = 0, \tag{99}$$

$$(m(\varphi_{-1})\nabla r, \nabla \phi) - \langle p, \phi \rangle - \langle \chi \nabla \varphi_{-1}, \phi \rangle = 0, \tag{100}$$

$$\begin{aligned} &\left\langle \frac{\rho(\varphi_{-1})}{\tau}\chi, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - \langle \nabla \chi \nu, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} \\ &+ \langle 2\eta(\varphi_{-1})\epsilon(\chi), \epsilon(\psi) \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - \langle r \nabla \varphi_{-1}, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = 0, \end{aligned} \tag{101}$$

$$\langle -\chi, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} + \langle D_u \mathcal{J}[\hat{z}], \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = 0, \tag{102}$$

$$\lambda \in \left(T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp \right)^0, \tag{103}$$

$$\chi \in \left(\left[D\left(\left(T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp \right)^0 \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N) \right) \right]_2 \right)^0, \tag{104}$$

where the subscript K^0 signifies the polar cone of a cone K.

Proof. The existence of an adjoint state $(p, r, \chi) \in H^1(\Omega) \times H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$ and $\lambda \in H^1(\Omega)^*$ such that (99)-(102) are satisfied follows directly from Theorem 3.4.

Now we reformulate the optimal control problem with the help of the reduced objective functional $\overline{\mathcal{J}} : L^2(\Omega;\mathbb{R}^N) \to \mathbb{R}$

$$\min_{u \in L^2(\Omega;\mathbb{R}^N)} \overline{\mathcal{J}}(u) := \min_{u \in L^2(\Omega;\mathbb{R}^N)} \mathcal{J}(S_\Psi(u), u),$$

Due to $(\varphi_0, \mu_0, v_0, u_0)$ being an optimal point, it holds that

$$D\overline{\mathcal{J}}[u_0](h) \geq 0, \ \forall h \in L^2(\Omega;\mathbb{R}^N). \tag{105}$$

Since u_0 is contained in $H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$, due to Theorem 3.4, and $L^2(\Omega;\mathbb{R}^N)$ is dense in $H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$, this yields

$$D\overline{\mathcal{J}}[u_0](h) \geq 0, \ \forall h \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*, \tag{106}$$

cf. e.g. [34, Lemma 3.1]. Using the characterization of the directional derivative $(q, w, \zeta) = DS_\Psi[u_0](h)$ provided by Theorem 5.1, we infer

$$0 \leq D\overline{\mathcal{J}}[u_0](h) = \langle D_\varphi \mathcal{J}[z_0], D_\varphi S_\Psi[u_0](h) \rangle + \langle D_u \mathcal{J}[z_0], h \rangle \tag{107}$$

$$= \langle D_\varphi \mathcal{J}[z_0], q \rangle + \langle D_u \mathcal{J}[z_0], h \rangle, \tag{108}$$

where (q, w, ζ) denotes the unique solution of the system (65)-(68).

In order to show the inclusion (104) we consider an arbitrary element (w^*, ζ^*) of the subsequent cone

$$(w^*, \zeta^*) \in \left(T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp \right)^0 \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N). \tag{109}$$

and define $h^* \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$ by $h^* := [D(w^*, \zeta^*)]_2$ such that the following equation holds true

$$\begin{aligned} \langle h^*, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} := &\left\langle \frac{\rho(\varphi_{-1})\zeta^*}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta^* \otimes \nu, \nabla \psi) \\ &+ (2\eta(\varphi_{-1})\epsilon(\zeta^*), \epsilon(\psi)) - \langle w^* \nabla \varphi_{-1}, \psi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}. \end{aligned} \tag{110}$$

Consequently, the triple $(0, w^*, \zeta^*)$ satisfies the following system

$$\begin{aligned}
0 \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp,\\
\langle -w^*, \phi\rangle \geq 0,\ \forall \phi \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp,\\
\langle \zeta^* \nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla w^*, \nabla\phi) = 0,\\
\left\langle \frac{\rho(\varphi_{-1})\zeta^*}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta^* \otimes \nu, \nabla\psi) + (2\eta(\varphi_{-1})\epsilon(\zeta^*), \epsilon(\psi))\\
- \langle w^* \nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - \langle h^*, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = 0.
\end{aligned}$$

Hence, $(0, w^*, \zeta^*) = DS_\Psi\,[u_0]\,(h^*)$ due to Theorem 5.1.

With the help of (108) and (102) we infer

$$0 \leq \langle D_u\mathcal{J}[z_0], h^*\rangle = \langle \chi, h^*\rangle\,,$$

which validates the inclusion (104).

To verify inclusion (103) we consider an arbitrary $q^* \in T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp$. As above we define $h^* \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)^*$ and

$$(w^*, \zeta^*) := D^{-1}\left(-\frac{q^*}{\tau}, h^*\right) \in H^1(\Omega) \times H^1_{0,\sigma}(\Omega;\mathbb{R}^N) \tag{111}$$

such that the following system is satisfied for all $\phi \in H^1(\Omega)$ and $\psi \in H^1_{0,\sigma}(\Omega;\mathbb{R}^N)$

$$(\nabla q^*, \nabla\phi) - \langle w^*, \phi\rangle = 0, \tag{112}$$

$$\left\langle \frac{q^*}{\tau}, \phi \right\rangle + \langle \zeta^* \nabla\varphi_{-1}, \phi\rangle + (m(\varphi_{-1})\nabla w^*, \nabla\phi) = 0, \tag{113}$$

$$\begin{aligned}
\left\langle \frac{\rho(\varphi_{-1})\zeta^*}{\tau}, \psi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta^* \otimes \rho(\varphi_{-2})v_{-1}, \nabla\psi)\\
+ \left(\zeta^* \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}, \nabla\psi\right) + (2\eta(\varphi_{-1})\epsilon(\zeta^*), \epsilon(\psi))\\
- \langle w^* \nabla\varphi_{-1}, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} = \langle h, \psi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}\,,
\end{aligned} \tag{114}$$

and therefore $(q^*, w^*, \zeta^*) = DS_\Psi\,[u_0]\,(h^*)$. Employing inequality (108) and (114), we derive

$$\begin{aligned}
0 &\leq (D_\varphi\mathcal{J}[z_0], q) + (D_u\mathcal{J}[z_0], h)\\
&= (D_\varphi\mathcal{J}[z_0], q) + \left\langle \frac{\rho(\varphi_{-1})\zeta^*}{\tau}, D_u\mathcal{J}[z_0] \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta^* \otimes \rho(\varphi_{-2})v_{-1}, \nabla D_u\mathcal{J}[z_0])\\
&\quad + \left(\zeta^* \otimes \frac{\rho_2 - \rho_1}{2} m(\varphi_{-2})\nabla\mu_{-1}, \nabla D_u\mathcal{J}[z_0]\right) + (2\eta(\varphi_{-1})\epsilon(\zeta^*), \epsilon(D_u\mathcal{J}[z_0]))\\
&\quad - \langle w^* \nabla\varphi_{-1}, D_u\mathcal{J}[z_0]\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}\\
&= (D_\varphi\mathcal{J}[z_0], q) + \left\langle \frac{\rho(\varphi_{-1})\zeta^*}{\tau}, \chi \right\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}} - (\zeta^* \otimes \nu, \nabla\chi)\\
&\quad + (2\eta(\varphi_{-1})\epsilon(\zeta^*), \epsilon(\chi)) - \langle w^* \nabla\varphi_{-1}, \chi\rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}\,,
\end{aligned}$$

where the last equality follows from (28). Taking advantage of (27) and (113), we infer

$$0 \leq (D_\varphi \mathcal{J}[z_0], q) + \langle r \nabla \varphi_{-1}, \zeta^* \rangle - \langle w^* \nabla \varphi_{-1}, \chi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}$$
$$= (D_\varphi \mathcal{J}[z_0], q) - \left\langle \frac{q^*}{\tau}, r \right\rangle - (m(\varphi_{-1}) \nabla w^*, \nabla r) - \langle w^* \nabla \varphi_{-1}, \chi \rangle_{H^{-1}_{0,\sigma}, H^1_{0,\sigma}}.$$

In combination with (26),(112) and (25), this yields

$$0 \leq (D_\varphi \mathcal{J}[z_0], q) - \left\langle \frac{q^*}{\tau}, r \right\rangle - \langle p, w^* \rangle = (D_\varphi \mathcal{J}[z_0], q) - \left\langle \frac{q^*}{\tau}, r \right\rangle - \langle \nabla p, \nabla q^* \rangle$$
$$= - \langle \lambda, q^* \rangle .$$

Consequently, λ is an element of the polar cone $\left(T_{\mathbb{K}}(\varphi_0) \cap \{\Lambda_0^+\}^\perp \cap \{\Lambda_0^-\}^\perp\right)^0$, which completes the proof of inclusion (103). ☐

In summary, we extended the results from [18] to the case of function spaces with arbitrary mean value. More importantly, we established strong stationarity conditions for the optimal control problem (P_Ψ), which is ruled by a degenerate constraint system with the overall problem falling into the realm of mathematical programs with equilibrium constraints (MPECs). These conditions replace the C-stationarity conditions which were the most (and, to the best of our knowledge, only) selective stationarity system available up to this point. The strong stationarity system is a considerable step towards the application and development of more advanced numerical methods, such as, e.g., in [21], to detect an approximate solution of the optimal control problem associated to the Cahn-Hilliard-Navier-Stokes system.

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