Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

On the Oseen-type resolvent problem associated with time-periodic flow past a rotating body

Thomas Eiter

submitted: November 2, 2021

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: thomas.eiter@wias-berlin.de

No. 2888
Berlin 2021

2020 *Mathematics Subject Classification.* 76D07, 76U05, 47A10, 35B10, 76D05, 35Q30, 35A01.

Key words and phrases. Oseen flow, rotating obstacle, resolvent problem, time-periodic solutions.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

On the Oseen-type resolvent problem associated with time-periodic flow past a rotating body

Thomas Eiter

Abstract

Consider the time-periodic flow of an incompressible viscous fluid past a body performing a rigid motion with non-zero translational and rotational velocity. We introduce a framework of homogeneous Sobolev spaces that renders the resolvent problem of the associated linear problem well posed on the whole imaginary axis. In contrast to the cases without translation or rotation, the resolvent estimates are merely uniform under additional restrictions, and the existence of time-periodic solutions depends on the ratio of the rotational velocity of the body motion to the angular velocity associated with the time period. Provided that this ratio is a rational number, time-periodic solutions to both the linear and, under suitable smallness conditions, the nonlinear problem can be established. If this ratio is irrational, a counterexample shows that in a special case there is no uniform resolvent estimate and solutions to the time-periodic linear problem do not exist.

1 Introduction

Consider the system

$$\begin{cases} isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v) - \Delta v - \lambda \partial_1 v + \nabla p = g & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a three-dimensional exterior domain, and $\lambda,\ \omega > 0$ and $s \in \mathbb{R}$ are given parameters. The function $g\colon \Omega \to \mathbb{R}^3$ is a given vector field, and the unknown solution (v, p) consists of the vector field $v\colon \Omega \to \mathbb{R}^3$ and the scalar field $p\colon \Omega \to \mathbb{R}$. Problem (1.1) can be regarded as a resolvent problem with a purely imaginary resolvent parameter $is,\ s \in \mathbb{R}$. In this article we provide function classes where the existence of a unique solution to (1.1) can be established. We further investigate the availability of uniform resolvent estimates, which allow to establish solutions to the associated time-periodic linear problem

$$\begin{cases} \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u) - \Delta u - \lambda \partial_1 u + \nabla \mathfrak{p} = f & \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \Omega, \\ u = 0 & \text{on } \mathbb{T} \times \partial\Omega. \end{cases} \tag{1.2}$$

This linear theory can then be applied to show existence of solutions to the nonlinear problem

$$\begin{cases} \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u) - \lambda \partial_1 u + u \cdot \nabla u = f + \Delta u - \nabla \mathfrak{p} & \text{in } \mathbb{T} \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \Omega, \\ u = \lambda \,\mathrm{e}_1 + \omega \,\mathrm{e}_1 \wedge x & \text{on } \mathbb{T} \times \partial\Omega, \\ \lim_{|x| \to \infty} u(t, x) = 0 & \text{for } t \in \mathbb{R}, \end{cases} \tag{1.3}$$

DOI 10.20347/WIAS.PREPRINT.2888 Berlin 2021

which describes the time-periodic flow of an incompressible viscous fluid past a rigid body that moves with (non-vanishing, time-independent) translational and rotational velocities $\lambda\,\mathrm{e}_1$ and $\omega\,\mathrm{e}_1$ for $\lambda,\,\omega>0$. Here $\Omega\subset\mathbb{R}^3$ is the exterior domain surrounding the rigid body, and to indicate that all occurring functions are time periodic, the time axis is given by the torus group $\mathbb{T}:=\mathbb{R}/\mathcal{T}\mathbb{Z}$ for a prescribed time period $\mathcal{T}>0$. The functions $u\colon\,\mathbb{T}\times\Omega\to\mathbb{R}^3$ and $\mathfrak{p}\colon\,\mathbb{T}\times\Omega\to\mathbb{R}$ denote velocity and pressure fields of the fluid, expressed in a frame attached to the body, and $f\colon\,\mathbb{T}\times\Omega\to\mathbb{R}^3$ is an external body force. Here, viscosity and density constants are set equal to 1, and the fluid is assumed to be attached to the boundary of the body. Moreover, (1.3)$_4$ indicates that the flow is at rest at infinity. Since this condition will later be included in the definition of the function spaces in a generalized sense, we omitted the corresponding equation in (1.1) and (1.2). Our analysis of the time-periodic problems will be based on the study of the resolvent problem (1.1). Note that if u is a $\mathcal{T}$-periodic solution to (1.2), then the Fourier coefficient of order $k\in\mathbb{Z}$ satisfies (1.1) with $s=\frac{2\pi}{\mathcal{T}}k$. This also explains why we only consider purely imaginary resolvent parameters is, $s\in\mathbb{R}$, in this article.

The analysis of solutions to the nonlinear time-periodic problem (1.3) was initiated by Galdi and Silvestre [15], who derived the existence of weak solutions when the body performs a general time-periodic rigid motion. However, the established L^2 framework was not appropriate to capture the spatial asymptotic properties of the flow. This issue was addressed in a recent article by Galdi [11], who showed the existence of regular solutions subject to pointwise decay estimates. An alternative approach that reflects the asymptotic behavior away from the body is based on the fundamental work by Yamazaki [21], who considered the time-periodic flow around a body at rest, that is, system (1.3) for $\lambda=\omega=0$. He established solutions in $\mathrm{L}^{3,\infty}(\Omega)$, also known as weak-$\mathrm{L}^3(\Omega)$, by exploiting well-known L^p-L^q smoothing estimates for the Stokes semigroup. Similar estimates were derived by Shibata [20] for the semigroup in the case $\lambda,\,\omega>0$. Using these estimates, Yamazaki's method leads to time-periodic solutions to (1.3) in $\mathrm{L}^{3,\infty}(\Omega)$, as was later shown by Geissert, Hieber and Nguyen [16], who developed a general approach to time-periodic problems based on semigroup theory. Recently, Eiter and Kyed [7] used a different method based on a direct analysis of the linear problem (1.2) without relying on the associated semigroup, and existence of solutions to (1.3) was shown such that the velocity field belongs to $\mathrm{L}^q(\Omega)$, $q\in(2,\infty)$, under the assumption that the rotational velocity ω and the time period $\mathcal{T}$ are related by $\omega=\frac{2\pi}{\mathcal{T}}$. This severe restriction already appears in the existence theory for the linear problem (1.2) derived in [7]. In contrast, in the cases without translation ($\lambda=0$) or without rotation ($\omega=0$), existence of time-periodic solutions to (1.2) can be shown without further restrictions on the time-period $\mathcal{T}$; see [5, 14]. A leading question of this article is whether the condition $\omega=\frac{2\pi}{\mathcal{T}}$ is necessary for the existence of time-periodic solutions to (1.2) and (1.3) if $\lambda,\,\omega>0$, and in how far it can be weakened.

Since the additional term $\omega(\mathrm{e}_1\wedge u-\mathrm{e}_1\wedge x\cdot\nabla u)$ for $\omega>0$ may be regarded as a differential operator with unbounded coefficient, the linear problem (1.2) cannot be treated as a lower-order perturbation of the case $\omega=0$. Instead, we shall handle this term by a method recently developed by Galdi and Kyed [12, 13] to investigate steady-state solutions to (1.2), that is, solutions to (1.1) for $s=0$. This method was successfully applied to the time-periodic problem (1.2) for $\lambda=0$ in [5] and for $\lambda>0$ in [7]. As mentioned above, while the linear theory in [5] holds for all $\mathcal{T},\,\omega>0$, in [7] the assumption $\frac{2\pi}{\mathcal{T}}=\omega$ was imposed. This restriction makes it possible to absorb the term $\omega(\mathrm{e}_1\wedge u-\mathrm{e}_1\wedge x\cdot\nabla u)$ into the time derivative $\partial_t u$ by a suitable transformation when $\Omega=\mathbb{R}^3$, and to reduce the problem to the case $\omega=0$. Note that if $\omega\neq\frac{2\pi}{\mathcal{T}}$, then the associated transformation is not an isomorphism between $\mathcal{T}$-periodic functions. To circumvent this problem, we proceed as in [5], and adapt this method to first derive well-posedness of the resolvent problem (1.1) by a reduction to the auxiliary problem

$$\begin{cases} isu + \partial_t u - \Delta u - \lambda\partial_1 u + \nabla\mathfrak{p} = f & \text{in } \mathbb{T}\times\mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T}\times\mathbb{R}^3. \end{cases} \tag{1.4}$$

This system may be regarded as the mixture of the classical Oseen resolvent problem and the time-periodic Oseen problem. Since the relevant differential operator in (1.4) has constant coefficients, a solution formula can directly be deduced in terms of a Fourier multiplier in the group setting $\mathbb{T}\times\mathbb{R}^3$, and we can derive associated *a priori* estimates by L^q multiplier theorems, which lead to resolvent estimates for (1.1) that are uniform for all $s \in \mathbb{R}$ satisfying $\operatorname{dist}(s, \omega\mathbb{Z}\setminus\{s\}) > \delta$ for some fixed $\delta > 0$. This result differs from the case $\lambda = 0$, where uniform resolvent estimates for all $s \in \mathbb{R}$ are available (see [5]). This observation parallels the known results for the non-rotating case $\omega = 0$, that is, for the resolvent problem

$$\begin{cases} isv - \Delta v - \lambda\partial_1 v + \nabla p = g & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

In the Stokes case ($\lambda = 0$), there exists a constant $C > 0$ such that for all $s \in \mathbb{R}\setminus\{0\}$ and $g \in \mathrm{L}^q(\Omega)^3$, $q \in (1,\infty)$, the velocity field v of the (unique) solution (v,p) to (1.5) satisfies

$$|s|\|v\|_q \leq C\|g\|_q. \tag{1.6}$$

In contrast, in the Oseen case ($\lambda > 0$), an analogous statement can only be shown with a uniform constant C as long as $|s| \geq \delta$ for some $\delta > 0$. Moreover, one cannot expect the validity of a uniform estimate as $s \to 0$, as was pointed out by Deuring and Varnhorn [3], who constructed a counterexample in the special case $q = 2$ and $\Omega = \mathbb{R}^3$. Our results below show that in the rotating case $\omega > 0$ the situation becomes even more involved, and we cannot derive a uniform estimate if s approaches $\omega\mathbb{Z}$. Similarly to [3], we further construct a counterexample showing that uniform resolvent estimates cannot exist in the case $q = 2$ and $\Omega = \mathbb{R}^3$.

The described phenomenon is in accordance with the following observation: For $\omega \geq 0$ we can understand (1.1) as the resolvent problem $(is - A_\omega)v = g$ of a closed operator $A_\omega\colon D(A_\omega) \to \mathrm{L}^q_\sigma(\Omega)$ with domain $D(A_\omega) \subset \mathrm{L}^q_\sigma(\Omega)$, where $\mathrm{L}^q_\sigma(\Omega)$ is the class of all solenoidal functions in $\mathrm{L}^q(\Omega)^3$. Then the (essential) spectra of A_ω and A_0 are related by

$$\sigma_{\text{ess}}(A_\omega) = \sigma_{\text{ess}}(A_0) + i\omega\mathbb{Z};$$

see [8]. Therefore, one would expect that the singular behavior of problem (1.5) at $s = 0$ can be observed in a similar fashion for solutions to (1.1) at any $s \in \omega\mathbb{Z}$. Indeed, this is what our findings indicate. Moreover, since $0 \in \sigma_{\text{ess}}(A_0)$, problem (1.1) is ill-posed for all $s \in \omega\mathbb{Z}$ in this functional framework of closed operators. Therefore, we introduce a different framework and show that (1.1) can be rendered well-posed within homogeneous Sobolev spaces. In particular, we shall not derive an estimate of the form (1.6), which would contradict $i\omega\mathbb{Z} \subset \sigma_{\text{ess}}(A_\omega)$, but the non-standard resolvent estimate

$$\|isv + \omega(\mathrm{e}_1\wedge v - \mathrm{e}_1\wedge x\cdot\nabla v)\|_q \leq C\|g\|_q. \tag{1.7}$$

Clearly, in the case $\omega = 0$, estimate (1.7) reduces to (1.6).

Based on the analysis of the resolvent problem (1.1), we then investigate the time-periodic linear problem (1.2). Provided that we have a uniform resolvent estimate for the relevant resolvent parameters, that is, that $\operatorname{dist}(s, \omega\mathbb{Z} \setminus \{s\}) > \delta$ for all $s \in \frac{2\pi}{\mathcal{T}}\mathbb{Z}$, we then show existence of solutions to (1.2) in a framework of absolutely convergent Fourier series. We see below that this assumption is satisfied if and only if $\frac{2\pi}{\mathcal{T}}/\omega$ is a rational number. Under this condition and suitable smallness assumptions on the data f, λ and ω, we further establish existence of a solution to the nonlinear problem (1.3). Moreover, the aforementioned counterexample for the resolvent problem (1.1) enables us to derive a result on the non-existence of a time-periodic solution to the linear problem (1.2) in $\mathrm{L}^2(\mathbb{R}^3)$ if $\frac{2\pi}{\mathcal{T}}/\omega \notin \mathbb{Q}$. This suggests that the restriction to the case $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$ may also be necessary for existence in the general setting $\mathrm{L}^q(\Omega)$.

We first introduce the basic notation in Section 2, which allows us to formulate the main results of this article in Section 3. In Section 4 we prepare some preliminary results. Section 5 and Section 6 focus on the well-posedness of the resolvent problem (1.1) in the whole space $\Omega = \mathbb{R}^3$ and in an exterior domain $\Omega \subset \mathbb{R}^3$, respectively. In Section 7 we prove the existence results for the time-periodic problems (1.2) and (1.3). Finally, in Section 8 we construct the counterexample to the uniformity of the resolvent estimate (1.6) and conclude the non-existence result for the linear time-periodic problem (1.2).

2 Notation

For a fixed period $\mathcal{T} > 0$, the symbol $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ denotes the associated torus group. Occasionally, we identify elements of $\mathbb{T}$ with their unique representatives in $[0, \mathcal{T})$. Points $(t, x) \in \mathbb{T} \times \mathbb{R}^3$, consist of a time variable $t \in \mathbb{T}$ and a space variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We write $|x|$ for the Euclidean norm of x, and $x \cdot y$, $x \wedge y$ and $x \otimes y$ denote the scalar, vector and tensor products of $x, y \in \mathbb{R}^3$, We further use the notation $x \wedge y \cdot z := (x \wedge y) \cdot z$ for $x, y, z \in \mathbb{R}^3$.

For derivatives in time and space we write ∂_t and $\partial_j := \partial_{x_j}$, $j = 1, 2, 3$, respectively, and ∇, div and Δ denote (spatial) gradient, divergence and Laplace operator. By $\nabla^2 u$ we denote the collection of all second-order spatial derivatives of a function u.

We use the symbol C to denote a generic positive constant that may change from line to line. When we want to emphasize that C depends on a specific set of quantities $\{a, b, \dots\}$, we write $C = C(a, b, \dots)$.

Unless stated otherwise, $\Omega \subset \mathbb{R}^3$ always denotes a three-dimensional exterior domain, that is, Ω is a domain that is the complement of a compact nonempty set. We let $B_R \subset \mathbb{R}^3$ denote the ball of radius $R > 0$ centered at 0, and we set $\Omega_R := \Omega \cap B_R$.

Classical Lebesgue and Sobolev spaces are denoted by $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{k,q}(\Omega)$ for $q \in [1, \infty]$ and $k \in \mathbb{N}$, and we write $\|\cdot\|_{q;\Omega}$ and $\|\cdot\|_{k,q;\Omega}$ for the associated norms. If the underlying domain is clear from the context, we simply write $\|\cdot\|_q$ and $\|\cdot\|_{k,q}$. The same convention is used for the norm $\|\cdot\|_{q,\mathbb{T}\times\Omega}$ of the Lebesgue space $\mathrm{L}^q(\mathbb{T} \times \Omega)$ in space and time. We further define $\mathrm{W}^{1,q}_0(\Omega)$ as the closure of $\mathrm{C}^\infty_0(\Omega)$ in $\mathrm{W}^{1,q}(\Omega)$, where $\mathrm{C}^\infty_0(\Omega)$ is the set of all smooth functions with compact support in Ω, and $\mathrm{W}^{-1,q'}(\Omega)$ is the dual space of $\mathrm{W}^{1,q}_0(\Omega)$, where $1/q + 1/q' = 1$. We denote the norm of $\mathrm{W}^{-1,q'}(\Omega)$ by $\|\cdot\|_{-1,q'}$. The classes $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$ and $\mathrm{W}^{k,q}_{\mathrm{loc}}(\Omega)$ consist of all functions that locally belong to $\mathrm{L}^q(\Omega)$ and $\mathrm{W}^{k,q}(\Omega)$, respectively.

When clear from the context, we often do not distinguish between a space X and its n-fold Cartesian product X^n, $n \in \mathbb{N}$. Moreover, $\|\cdot\|_X$ denotes the norm of a normed vector space X. The symbol $\mathrm{L}^q(\mathbb{T}; X)$ denotes the Bochner–Lebesgue space for $q \in [1, \infty]$, and we define $\mathrm{W}^{1,q}(\mathbb{T}; X) := \big\{u \in \mathrm{L}^q(\mathbb{T}; X) : \partial_t u \in \mathrm{L}^q(\mathbb{T}; X)\big\}$. Here the torus group $\mathbb{T}$ is always equipped with the normalized

Haar measure such that

$$\forall f \in \mathrm{C}(\mathbb{T}): \quad \int_{\mathbb{T}} f(t)\,\mathrm{d}t := \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} f(t')\,\mathrm{d}t',$$

where $\mathrm{C}(\mathbb{T})$ denotes the class of continuous functions on $\mathbb{T}$.

In the case $\Omega = \mathbb{R}^3$, the space-time domain is given by the locally compact abelian group $G := \mathbb{T} \times \mathbb{R}^3$. The dual group of G can be identified with $\widehat{G} := \mathbb{Z} \times \mathbb{R}^3$. By $\mathscr{S}(G)$ we denote the associated Schwartz–Bruhat space, and $\mathscr{S}'(G)$ is its dual space, the space of tempered distributions. Both were first introduced by Bruhat [1], see also [6] for more details. We define the Fourier transform $\mathscr{F}_G$ on G by

$$\begin{aligned}
&\mathscr{F}_G \colon \mathscr{S}(G) \to \mathscr{S}(\widehat{G}), \quad \mathscr{F}_G[u](k,\xi) := \int_{\mathbb{T}} \int_{\mathbb{R}^3} u(t,x)\, \mathrm{e}^{-ix\cdot\xi - ikt}\,\mathrm{d}x\mathrm{d}t,\\
&\mathscr{F}_G^{-1} \colon \mathscr{S}(\widehat{G}) \to \mathscr{S}(G), \quad \mathscr{F}_G^{-1}[w](t,x) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^3} w(k,\xi)\, \mathrm{e}^{ix\cdot\xi + ikt}\,\mathrm{d}\xi.
\end{aligned}$$

Then $\mathscr{F}_G$ is an isomorphism with inverse $\mathscr{F}_G^{-1}$ provided that the Lebesgue measure $\mathrm{d}\xi$ is suitably normalized. By duality, the Fourier transform also becomes an isomorphism between the corresponding spaces of tempered distributions. By analogy, we define the Fourier transform on the groups $\mathbb{T}$ and $\mathbb{R}^3$ as

$$\begin{aligned}
&\mathscr{F}_{\mathbb{T}} \colon \mathscr{S}(\mathbb{T}) \to \mathscr{S}(\mathbb{Z}), && \mathscr{F}_{\mathbb{T}}[u](k) := \int_{\mathbb{T}} u(t)\,\mathrm{e}^{-ikt}\,\mathrm{d}t,\\
&\mathscr{F}_{\mathbb{T}}^{-1} \colon \mathscr{S}(\mathbb{Z}) \to \mathscr{S}(\mathbb{T}), && \mathscr{F}_{\mathbb{T}}^{-1}[w](t) := \sum_{k\in\mathbb{Z}} w(k)\,\mathrm{e}^{ikt},\\
&\mathscr{F}_{\mathbb{R}^3} \colon \mathscr{S}(\mathbb{R}^3) \to \mathscr{S}(\mathbb{R}^3), && \mathscr{F}_{\mathbb{R}^3}[u](\xi) := \int_{\mathbb{R}^3} u(x)\,\mathrm{e}^{-ix\cdot\xi}\,\mathrm{d}x,\\
&\mathscr{F}_{\mathbb{R}^3}^{-1} \colon \mathscr{S}(\mathbb{R}^3) \to \mathscr{S}(\mathbb{R}^3), && \mathscr{F}_{\mathbb{R}^3}^{-1}[w](x) := \int_{\mathbb{R}^3} w(\xi)\,\mathrm{e}^{ix\cdot\xi}\,\mathrm{d}\xi.
\end{aligned}$$

For the investigation of the time-periodic problem (1.2), we also work within the space of absolutely convergent X-valued Fourier series given by

$$\begin{aligned}
&\mathrm{A}(\mathbb{T};X) := \Big\{ f \colon \mathbb{T} \to X \;:\; f(t) = \sum_{k\in\mathbb{Z}} f_k\,\mathrm{e}^{ikt},\; f_k \in X,\; \sum_{k\in\mathbb{Z}} \|f_k\|_X < \infty \Big\},\\
&\|f\|_{\mathrm{A}(\mathbb{T};X)} := \sum_{k\in\mathbb{Z}} \|f_k\|_X
\end{aligned} \tag{2.1}$$

for a normed space X. When X is a Banach space, then $\mathrm{A}(\mathbb{T};X)$ coincides with the Banach space $\mathscr{F}_{\mathbb{T}}^{-1}\big[\ell^1(\mathbb{Z};X)\big]$, whence $\mathrm{A}(\mathbb{T};X) \hookrightarrow \mathrm{C}(\mathbb{T};X)$. Since useful inequalities can directly be transferred from spaces X to the corresponding spaces $\mathrm{A}(\mathbb{T};X)$ (see [7, Prop. 3.1 and 3.2] for example), these spaces also provide a useful framework for the treatment of nonlinear time-periodic problems. For simplicity, we also write $u \in \mathrm{A}(\mathbb{T};\mathrm{W}^{k,q}_{\mathrm{loc}}(\Omega))$ if $u \in \mathrm{A}(\mathbb{T};\mathrm{W}^{k,q}(K))$ for all compact sets $K \subset \Omega$.

Next we formulate the function spaces for solutions to the time-periodic problems (1.2) and (1.3). For fixed $\mathcal{T},\ \lambda,\ \omega > 0$ and $q \in (1,2)$, we define the space for the time-periodic velocity field by

$$\begin{aligned}
\mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega) := \big\{ u \in{} &\mathrm{A}(\mathbb{T};\mathrm{W}^{2,q}_{\mathrm{loc}}(\Omega)^3) :\\
&\nabla^2 u,\ \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u),\ \partial_1 u \in \mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega)),\\
&\quad u \in \mathrm{A}(\mathbb{T};\mathrm{L}^{2q/(2-q)}(\Omega)),\ \nabla u \in \mathrm{A}(\mathbb{T};\mathrm{L}^{4q/(4-q)}(\Omega)) \big\}.
\end{aligned}$$

The function class for the pressure is independent of $\lambda,\ \omega>0$ and given by

$$\mathcal{Y}^q(\mathbb{T}\times\Omega):=\big\{\mathfrak{p}\in \mathrm{A}(\mathbb{T};\mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)) : \nabla\mathfrak{p}\in\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega)),\ \mathfrak{p}\in\mathrm{A}(\mathbb{T};\mathrm{L}^{3q/(3-q)}(\Omega))\big\}.$$

For the analysis of the resolvent problem (1.1), the function class for the velocity fields additionally depends on $s\in\mathbb{R}$ and is defined by

$$\begin{aligned}\mathrm{X}^q_{\lambda,\omega,s}(\Omega):=\big\{v\in\mathrm{W}^{2,q}_{\mathrm{loc}}(\Omega)^3 : \nabla^2 v, isv+\omega(\mathrm{e}_1\wedge v-\mathrm{e}_1\wedge x\cdot\nabla v),\ \partial_1 v\in\mathrm{L}^q(\Omega),\\ v\in\mathrm{L}^{2q/(2-q)}(\Omega),\ \nabla v\in\mathrm{L}^{4q/(4-q)}(\Omega)\big\},\end{aligned}$$

and the corresponding pressure is characterized by

$$\mathrm{Y}^q(\Omega):=\big\{p\in\mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega) : \nabla p\in\mathrm{L}^q(\Omega),\ p\in\mathrm{L}^{3q/(3-q)}(\Omega)\big\}.$$

These spaces are constructed in such a way that if a $\mathcal{T}$-time-periodic function u belongs to $\mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$, then its k-th Fourier coefficient $u_k:=\mathscr{F}_{\mathbb{T}}[u](k),\ k\in\mathbb{Z}$, belongs to $\mathrm{X}^q_{\lambda,\omega,s}(\Omega)$ for $s=\frac{2\pi}{\mathcal{T}}k$. Similarly, Fourier coefficients of elements $\mathfrak{p}\in\mathcal{Y}^q(\mathbb{T}\times\Omega)$ belong to $\mathrm{Y}^q(\Omega)$.

3 Main Results

Here we collect our main results on the well-posedness of the resolvent problem (1.1) and the time-periodic problems (1.2) and (1.3). For the whole section, let $\Omega=\mathbb{R}^3$ or $\Omega\subset\mathbb{R}^3$ be an exterior domain with C^3-boundary. At first, we address the resolvent problem (1.1).

Theorem 3.1. *Let $\lambda>0$, $s\in\mathbb{R}$ and $0<\omega\leq\omega_0$, and let $q\in(1,2)$ and $g\in\mathrm{L}^q(\Omega)^3$. Then there exists a unique solution $(v,p)\in\mathrm{X}^q_{\lambda,\omega,s}(\Omega)\times\mathrm{Y}^q(\Omega)$ to* (1.1)*, which obeys the estimate*

$$\begin{aligned}\|\mathrm{dist}(s,\omega\mathbb{Z})\,v\|_q+\|isv+\omega(\mathrm{e}_1\wedge v-\mathrm{e}_1\wedge x\cdot\nabla v)\|_q+\|\nabla^2 v\|_q+\lambda\|\partial_1 v\|_q\\ +\|\nabla p\|_q+\lambda^{1/4}\|\nabla v\|_{4q/(4-q)}+\lambda^{1/2}\|v\|_{2q/(2-q)}+\|p\|_{3q/(3-q)}\leq C\|g\|_q\end{aligned}\tag{3.1}$$

for a constant $C=C(\Omega,q,\lambda,\omega,s)>0$. If $\theta>0$ such that

$$\lambda^2\leq\theta\min\big\{|s-\omega k| : k\in\mathbb{Z},\ s\neq\omega k\big\},\tag{3.2}$$

then $C=C(\Omega,q,\lambda,\omega_0,\theta)>0$, that is, C is independent of s and ω. If additionally $q\in(1,3/2)$, then C can be chosen uniformly in s, ω and λ, that is, such that $C=C(\Omega,q,\omega_0,\theta)>0$.

Observe that we may reformulate (3.2) as

$$\lambda^2\leq\theta\,\mathrm{dist}(s,\omega\mathbb{Z}\setminus\{s\})=\begin{cases}\theta\omega & \text{if } s\in\omega\mathbb{Z},\\ \theta\,\mathrm{dist}(s,\omega\mathbb{Z}) & \text{if } s\notin\omega\mathbb{Z}.\end{cases}$$

One readily sees that there is no $\theta>0$ such that this condition is satisfied for all $s\in\mathbb{R}$ at the same time. Therefore, Theorem 3.1 does not allow to choose the same constant C for all $s\in\mathbb{R}$. As explained in the introduction, this lack of a uniform constant is not surprising since the same phenomenon occurs for (1.5), the Oseen resolvent problem without rotation, as $s\to 0$.

To obtain a $\mathcal{T}$-periodic solution to (1.2) in terms of a Fourier series, a uniform constant for all $s\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}$ is necessary. As follows from Proposition 4.2 below, condition (3.2) can only be satisfied for all $s\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}$ if the quotient $\frac{2\pi}{\mathcal{T}}/\omega$ is a rational number. This observation leads to the following existence result for the time-periodic problem (1.2).

Theorem 3.2. *Let $\mathcal{T}$, $\lambda > 0$ and $0 < \omega \leq \omega_0$, and let $q \in (1,2)$ and $f \in \mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega)^3)$. If $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$, then there exists a unique $\mathcal{T}$-periodic solution $(u,\mathfrak{p}) \in \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)\times\mathcal{Y}^q(\mathbb{T}\times\Omega)$ to* (1.2)*, which obeys the estimate*

$$\begin{aligned}\|\partial_t u + \omega(\mathrm{e}_1\wedge u - \mathrm{e}_1\wedge x\cdot\nabla u)\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} + \|\nabla^2 u\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} + \lambda\|\partial_1 u\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))}\\ + \|\nabla\mathfrak{p}\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} + \lambda^{1/4}\|\nabla u\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^{4q/(4-q)}(\Omega))} + \lambda^{1/2}\|u\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^{2q/(2-q)}(\Omega))}\\ + \|\mathfrak{p}\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^{3q/(3-q)}(\Omega))} \leq C\|f\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))}\end{aligned} \tag{3.3}$$

for a constant $C = C(\Omega,q,\lambda,\omega,\mathcal{T}) > 0$. If $\theta > 0$ such that

$$\lambda^2 \leq \theta\min\big\{a>0 : a\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}+\omega\mathbb{Z}\big\}, \tag{3.4}$$

then $C = C(\Omega,q,\lambda,\omega_0,\theta) > 0$, that is, C is independent of $\mathcal{T}$ and ω. If additionally $q\in(1,3/2)$, then C can be chosen independently of ω, $\mathcal{T}$ and λ, that is, such that $C = C(\Omega,q,\omega_0,\theta) > 0$.

Observe that, due to the linear structure of $\frac{2\pi}{\mathcal{T}}\mathbb{Z}+\omega\mathbb{Z}$, the condition (3.4) can be reformulated as

$$\lambda^2 \leq \theta\min\big\{|a-b| : a,b\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}+\omega\mathbb{Z},\ a\neq b\big\}$$

or

$$\lambda^2 \leq \theta\min\big\{|s-\omega k| : s\in\frac{2\pi}{\mathcal{T}}\mathbb{Z},\ k\in\mathbb{Z},\ s\neq\omega k\big\}.$$

This shows that (3.4) is directly obtained from (3.2) by taking the minimum over all $s\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}$. As follows from Proposition 4.2 below, existence and positivity of this minimum are ensured by the restriction to $\frac{2\pi}{\mathcal{T}}/\omega\in\mathbb{Q}$. As explained above, this restriction is due to the lack of a uniform estimate for the resolvent problem (1.1) as s approaches $\omega\mathbb{Z}$.

Remark 3.3. In contrast to the other terms on the left-hand side of (3.1), the term $\|\mathrm{dist}(s,\omega\mathbb{Z})\,v\|_q$ does not directly correspond to any of the terms in (3.3). However, if we let $A_1 := \{k\in\mathbb{Z} : \frac{2\pi}{\mathcal{T}}k\in\omega\mathbb{Z}\}$ and $A_2 := \{k\in\mathbb{Z} : \frac{2\pi}{\mathcal{T}}k\notin\omega\mathbb{Z}\}$ and decompose the velocity field u as

$$u = u^{(1)} + u^{(2)},\qquad u^{(1)} := \sum_{k\in A_1} u_k\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt},\qquad u^{(2)} := \sum_{k\in A_2} u_k\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt},$$

then our proof below also yields the estimate

$$\min\big\{a>0 : a\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}+\omega\mathbb{Z}\big\}\big\|u^{(2)}\big\|_q \leq C\|f\|_q.$$

As mentioned above, the existence and positivity of the minimum is due to $\frac{2\pi}{\mathcal{T}}/\omega\in\mathbb{Q}$.

For the treatment of the nonlinear problem (1.3), first observe that λ and ω appear as data on the right-hand side of $(1.3)_3$. Therefore, to obtain a solution to (1.3) for "small" data, it is important that the constant C in the *a priori* estimate (3.3) can be controlled as $\lambda,\omega\to 0$. By Theorem 3.2, this is the case if (3.4) holds and $q<3/2$. Under these conditions and suitable smallness assumptions, we can derive existence of solutions to the nonlinear problem (1.3).

Theorem 3.4. *Let $q\in[\frac{12}{11},\frac{6}{5}]$, $\rho\in(\frac{3q-3}{q},1]$ and $\mathcal{T},\theta,\kappa>0$. Then there exists $\lambda_0>0$ such that for all $\lambda,\omega>0$ with $\frac{2\pi}{\mathcal{T}}/\omega\in\mathbb{Q}$ and*

$$\lambda\leq\lambda_0,\qquad \omega\leq\kappa\lambda^\rho,\qquad \lambda^2\leq\theta\min\big\{a>0 : a\in\frac{2\pi}{\mathcal{T}}\mathbb{Z}+\omega\mathbb{Z}\big\}, \tag{3.5}$$

there exists $\varepsilon>0$ such that for all $f\in\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega)^3)$ with $\|f\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega)^3)}<\varepsilon$ there is a solution $(u,\mathfrak{p})\in\mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)\times\mathcal{Y}^q(\mathbb{T}\times\Omega)$ to (1.3)*.*

For the proof we will proceed as in [7], where the case $\omega = \frac{2\pi}{\mathcal{T}}$ was treated, and combine the linear theory from Theorem 3.2 with a fixed-point argument. Note that by following [7], one could also allow for a time-dependent translational velocity, where λ in (1.3) is replaced with a time-periodic function $\alpha\colon \mathbb{T} \to \mathbb{R}$ that satisfies suitable smallness conditions and has a non-zero mean value $\lambda := \int_{\mathbb{T}} \alpha \,\mathrm{d}t > 0$.

As explained above, (3.5) cannot be satisfied for $\lambda > 0$ if $\frac{2\pi}{\mathcal{T}}/\omega \notin \mathbb{Q}$, since this implies that $\inf\{a > 0 : a \in \frac{2\pi}{\mathcal{T}}\mathbb{Z} + \omega\mathbb{Z}\} = 0$. In contrast, if $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$, one can find suitable parameters λ, ω to satisfy (3.5). Indeed, if $\frac{2\pi}{\mathcal{T}}/\omega = c/d$ with $c, d \in \mathbb{N}$ coprime, then

$$\min\{a > 0 : a \in \frac{2\pi}{\mathcal{T}}\mathbb{Z} + \omega\mathbb{Z}\} = \frac{\omega}{d} \min\{a > 0 : a \in c\mathbb{Z} + d\mathbb{Z}\} = \frac{\omega}{d} = \frac{2\pi}{\mathcal{T}c}.$$

To satisfy (3.5) for given $\mathcal{T} > 0$, one can thus fix $d \in \mathbb{N}$ and decide ω by choosing a number $c \in \mathbb{N}$ coprime to d and so large that $\lambda^2/\theta \leq \frac{2\pi}{\mathcal{T}}/c = \omega/d \leq \kappa\lambda^\rho/d$, which is possible if $\lambda > 0$ is sufficiently small.

Now a natural question is what happens for $\frac{2\pi}{\mathcal{T}}/\omega \notin \mathbb{Q}$ and whether the exclusion of this case in Theorem 3.2 and Theorem 3.4 is only a remnant of our proof or a necessary condition for the existence of a $\mathcal{T}$-time-periodic solution. The conjecture that it might be necessary is supported by the following result. It shows the existence of a right-hand side f such that a time-periodic solution to the linear problem (1.2) satisfying an estimate of the form (3.3) cannot exist in the case $q = 2$ and $\Omega = \mathbb{R}^3$. Actually, we show two non-existence results, one in the setting of absolutely convergent Fourier series of the previous theorems, and one in the more general setting of $\mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)$ functions.

Theorem 3.5. *Let $\Omega = \mathbb{R}^3$, $q = 2$ and $\lambda, \omega, \mathcal{T} > 0$ such that $\frac{2\pi}{\mathcal{T}}/\omega \notin \mathbb{Q}$.*

i. There is $f \in \mathrm{A}(\mathbb{T}; \mathrm{L}^2(\mathbb{R}^3)^3)$ such that there exists no time-periodic solution $(u, \mathfrak{p}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R}^3)^{3+1}$ to (1.2) with $\partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u),\ \nabla^2 u,\ \partial_1 u \in \mathrm{A}(\mathbb{T}; \mathrm{L}^2(\mathbb{R}^3)^3)$.

ii. There is $f \in \mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)^3$ such that there exists no time-periodic solution $(u, \mathfrak{p}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R}^3)^{3+1}$ to (1.2) with $\partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u),\ \nabla^2 u,\ \partial_1 u \in \mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)^3$.

In order to prove these non-existence results, we construct a counterexample that shows that there is no uniform resolvent estimate if $\lambda > 0$, $q = 2$ and $\Omega = \mathbb{R}^3$. More precisely, we construct a sequence of resolvent parameters, right-hand sides and corresponding solutions to the resolvent problem (1.1) that violates the existence of a uniform constant in estimate (1.7) with $q = 2$.

Theorem 3.6. *Let $\Omega = \mathbb{R}^3$, $q = 2$ and $\lambda, \omega > 0$. Then there exist sequences $(s_n) \subset \mathbb{R}$ and $(g_n) \subset \mathrm{L}^2(\mathbb{R}^3)^3$ and a sequence of solutions $(v_n, p_n) \subset \mathrm{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^3)^3 \times \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$ to (1.1) (with $s = s_n$ and $g = g_n$) such that $\nabla^2 v_n,\ i s v_n + \omega(\mathrm{e}_1 \wedge v_n - \mathrm{e}_1 \wedge x \cdot \nabla v_n),\ \partial_1 v_n \in \mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)$ and*

$$\|i s_n v_n + \omega(\mathrm{e}_1 \wedge v_n - \mathrm{e}_1 \wedge x \cdot \nabla v_n)\|_2 \geq C n^{1/2} \|g_n\|_2 \tag{3.6}$$

for a constant C independent of n. Moreover, for any $\alpha > 0$ such that $\alpha/\omega \notin \mathbb{Q}$, we can choose $(s_n) \subset \alpha\mathbb{Z}$ with $\operatorname{dist}(s_n, \omega\mathbb{Z}) \to 0$ as $n \to \infty$.

4 Preliminaries

Before we start with the proofs of the main theorems, we collect some auxiliary results in this section. We begin with the following estimates, which may be regarded as anisotropic versions of Sobolev's inequality. For the sake of generality, we consider the n-dimensional case.

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $q \in (1,\infty)$. Then there exists a constant $C = C(n,q) > 0$ such that for all $\lambda > 0$ and all $v \in \mathrm{W}^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\nabla^2 v,\ \partial_1 v \in \mathrm{L}^q(\mathbb{R}^n)$ and $v \in \mathrm{L}^r(\mathbb{R}^n)$ for some $r \in [1,\infty)$ it holds*

$$\lambda^{1/(n+1)} \|\nabla v\|_{s_1} \leq C \|\Delta v + \lambda \partial_1 v\|_q \qquad \text{if } q \in (1, n+1), \tag{4.1}$$

$$\lambda^{2/(n+1)} \|v\|_{s_2} \leq C \|\Delta v + \lambda \partial_1 v\|_q \qquad \text{if } q \in (1, \frac{n+1}{2}), \tag{4.2}$$

where

$$s_1 := \frac{(n+1)q}{n+1-q}, \qquad s_2 := \frac{(n+1)q}{n+1-2q}.$$

Proof. At first, assume that $v \in \mathscr{S}(\mathbb{R}^n)$. Then we can express v and $\partial_j v$, $j = 1,\dots,n$, by means of the Fourier transform such that

$$\lambda^{1/(n+1)} \partial_j v = \mathscr{F}^{-1}_{\mathbb{R}^n}\big[m_1\, \mathscr{F}_{\mathbb{R}^n}[\Delta v + \lambda \partial_1 v]\big], \quad \lambda^{2/(n+1)} v = \mathscr{F}^{-1}_{\mathbb{R}^n}\big[m_2\, \mathscr{F}_{\mathbb{R}^n}[\Delta v + \lambda \partial_1 v]\big]$$

with

$$m_1(\xi) := \frac{\lambda^{1/(n+1)} i\xi_j}{i\lambda\xi_1 - |\xi|^2}, \qquad m_2(\xi) := \frac{\lambda^{2/(n+1)}}{i\lambda\xi_1 - |\xi|^2}.$$

As in the proof of [10, Lemma VII.4.2], one readily deduces from Lizorkin's multiplier theorem that m_1 and m_2 are Fourier multipliers such that

$$\begin{aligned} &\forall g \in \mathscr{S}(\mathbb{R}^n): \qquad \big\|\mathscr{F}^{-1}_{\mathbb{R}^n}\big[m_1\, \mathscr{F}_{\mathbb{R}^n}[g]\big]\big\|_{s_1} \leq C\|g\|_q \quad \text{if } q < n+1, \\ &\forall g \in \mathscr{S}(\mathbb{R}^n): \qquad \big\|\mathscr{F}^{-1}_{\mathbb{R}^n}\big[m_2\, \mathscr{F}_{\mathbb{R}^n}[g]\big]\big\|_{s_2} \leq C\|g\|_q \quad \text{if } q < (n+1)/2, \end{aligned}$$

where C is independent of λ. Together with the above representation formulas, this property directly implies (4.1) and (4.2) for $v \in \mathscr{S}(\mathbb{R}^n)$. For general $v \in \mathrm{L}^r(\mathbb{R}^n)$ with $\nabla^2 v,\ \partial_1 v \in \mathrm{L}^q(\mathbb{R}^n)$, the inequalities now follow from an approximation by functions from $\mathscr{S}(\mathbb{R}^n)$. $\square$

The following elementary result will explain how the restriction $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$ comes into play in the existence result of Theorem 3.2.

Proposition 4.2. *Let $\alpha, \omega > 0$ and define $M := \{k\alpha + \ell\omega : k, \ell \in \mathbb{Z}\}$. Then M is discrete in $\mathbb{R}$ if and only if $\alpha/\omega \in \mathbb{Q}$, and M is dense in $\mathbb{R}$ if and only if $\alpha/\omega \notin \mathbb{Q}$. In particular,*

$$\inf\{a > 0 : a \in M\} = \begin{cases} \min\{a > 0 : a \in M\} > 0 & \text{if } \alpha/\omega \in \mathbb{Q}, \\ 0 & \text{if } \alpha/\omega \notin \mathbb{Q}. \end{cases} \tag{4.3}$$

Proof. It is well known that $N := \{k\beta + \ell : k, \ell \in \mathbb{Z}\} \subset \mathbb{R}$ is discrete if $\beta \in \mathbb{Q}$, and N is dense if $\beta \notin \mathbb{Q}$. Choosing $\beta = \alpha/\omega$, we have $M = \omega N$, and the whole statement follows directly. $\square$

For the construction of the sequence of resolvent parameters in Theorem 3.6, we use that we still obtain a dense set after restricting to odd multiples of ω if $\alpha/\omega \notin \mathbb{Q}$.

Corollary 4.3. *Let $\alpha, \omega > 0$ with $\alpha/\omega \notin \mathbb{Q}$. Then $\alpha\mathbb{Z} + \omega(2\mathbb{Z}+1)$ is dense in $\mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$. Since $(2\alpha)/(4\omega) \in \mathbb{R} \setminus \mathbb{Q}$, Proposition 4.2 shows existence of sequences $(b_n), (c_n) \subset 2\alpha\mathbb{Z} + 4\omega\mathbb{Z}$ such that $b_n \to a$ and $c_n \to a - 2\omega$ as $n \to \infty$. We define $a_n := \frac{1}{2}(b_n + c_n + 2\omega)$. Then $(a_n) \subset \alpha\mathbb{Z} + 2\omega\mathbb{Z} + \omega = \alpha\mathbb{Z} + \omega(2\mathbb{Z}+1)$ and $a_n \to a$ as $n \to \infty$. This completes the proof. $\square$

5 The resolvent problem in the whole space

We begin to study the resolvent problem (1.1) in the case $\Omega = \mathbb{R}^3$, where it reduces to

$$\begin{cases} isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v) - \Delta v - \lambda\partial_1 v + \nabla p = g & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{5.1}$$

This investigation prepares the analysis of the exterior-domain problem (1.1) in the subsequent section. Moreover, it allows us to establish solutions to the associated time-periodic problem (1.2) for $\Omega = \mathbb{R}^3$. For this purpose, it is important to keep track of the constants in the resolvent estimates for (5.1). The main result of this section reads as follows.

Theorem 5.1. *Let $q \in (1, \infty)$, and let $\lambda > 0$, $\omega > 0$ and $s \in \mathbb{R}$. For each $g \in \mathrm{L}^q(\mathbb{R}^3)^3$ there exists a solution $(v, p) \in \mathrm{W}^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)^3 \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\mathbb{R}^3)$ to* (5.1) *that satisfies*

$$\begin{aligned} &\|\operatorname{dist}(s, \omega\mathbb{Z})\, v\|_q + \|isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v)\|_q \\ &\qquad + \|\nabla^2 v\|_q + \lambda\|\partial_1 v\|_q + \|\nabla p\|_q \leq C\|g\|_q \end{aligned} \tag{5.2}$$

as well as

$$\lambda^{1/4}\|\nabla v\|_{4q/(4-q)} \leq C\|g\|_q \qquad \text{if } q < 4, \tag{5.3}$$

$$\|\nabla v\|_{3q/(3-q)} + \|p\|_{3q/(3-q)} \leq C\|g\|_q \qquad \text{if } q < 3, \tag{5.4}$$

$$\lambda^{1/2}\|v\|_{2q/(2-q)} \leq C\|g\|_q \qquad \text{if } q < 2, \tag{5.5}$$

$$\|v\|_{3q/(3-2q)} \leq C\|g\|_q \qquad \text{if } q < 3/2 \tag{5.6}$$

for a constant $C = C(q, \lambda, \omega, s) > 0$ given by

$$C = \begin{cases} C_0 P\big(\lambda^2/\omega\big) & \text{if } s \in \omega\mathbb{Z}, \\ C_0 P\big(\lambda^2/\operatorname{dist}(s, \omega\mathbb{Z})\big) & \text{if } s \notin \omega\mathbb{Z}, \end{cases} \tag{5.7}$$

where $C_0 = C_0(q) > 0$ is a constant only depending on q, and $P(\theta) := (1 + \theta)^3$. Moreover, if $(w, \mathfrak{q}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3)^{3+1}$ is another distributional solution to (5.1)*, then the following holds:*

i. If $\nabla^2 w,\ isw + \omega(\mathrm{e}_1 \wedge w - \mathrm{e}_1 \wedge x \cdot \nabla w),\ \partial_1 w \in \mathrm{L}^q(\mathbb{R}^3)$, then

$$\begin{gathered} isw + \omega(\mathrm{e}_1 \wedge w - \mathrm{e}_1 \wedge x \cdot \nabla w) = isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v), \\ \nabla^2 w = \nabla^2 v, \qquad \partial_1 w = \partial_1 v, \qquad \nabla \mathfrak{q} = \nabla p. \end{gathered}$$

ii. If $q < 2$ or $s \notin \omega\mathbb{Z}$, and if $w \in \mathrm{L}^r(\mathbb{R}^3)^3$ for some $r \in [1, \infty)$, then $w = v$ and $\mathfrak{q} = p + c$ for some constant $c \in \mathbb{R}$.

For the analysis of (5.1) we first study the auxiliary problem

$$\begin{cases} isu + \partial_t u - \Delta u - \lambda\partial_1 u + \nabla \mathfrak{p} = f & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^3. \end{cases} \tag{5.8}$$

As before, the torus group $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ indicates time periodicity of all functions with a given period $\mathcal{T} > 0$. Problem (5.8) has the advantage that, in contrast to the original time-periodic problem (1.2), the associated differential operator has constant coefficients. Therefore, one can readily derive a representation formula for its solution in terms of Fourier multipliers. Since problem (5.8) is formulated in the locally compact abelian group $G := \mathbb{T} \times \mathbb{R}^3$, we also use multiplier expressions in G. One tool to derive L^q multiplier estimates in this framework is the so-called transference principle, which goes back to de Leeuw [2] and, in a generalized form, to Edwards and Gaudry [4, Theorem B.2.1]. For an introduction how to employ this theory in the context of the Navier–Stokes equations, we refer to [6]. Here we use the following version of the transference principle.

Theorem 5.2. *Let $G := \mathbb{T} \times \mathbb{R}^3$ and $H := \mathbb{R} \times \mathbb{R}^3$. For each $q \in (1, \infty)$ there exists a constant $C_q > 0$ with the following property: If a continuous function $M \colon H \to \mathbb{C}$ is an $\mathrm{L}^q(H)$ multiplier, that is,*

$$\forall h \in \mathscr{S}(H): \quad \big\| \mathscr{F}_H^{-1}\big[M\, \mathscr{F}_H[h]\big] \big\|_{\mathrm{L}^q(H)} \leq C_M \|h\|_{\mathrm{L}^q(H)}$$

for some $C_M > 0$, then the restriction $m := M\big|_{\mathbb{Z}\times\mathbb{R}^3}$ is an $\mathrm{L}^q(G)$ multiplier such that

$$\forall g \in \mathscr{S}(G): \quad \big\| \mathscr{F}_G^{-1}\big[m\, \mathscr{F}_G[g]\big] \big\|_{\mathrm{L}^q(G)} \leq C_q C_M \|g\|_{\mathrm{L}^q(G)}.$$

With this theorem one can reduce Fourier multipliers in the group $G = \mathbb{T} \times \mathbb{R}^3$ to Fourier multipliers in the Euclidean setting $H = \mathbb{R} \times \mathbb{R}^3$, where tools like the multiplier theorems by Mikhlin and Marcinkiewicz are available. We use this method to derive L^q estimates of solutions to (5.8) in the context of the following existence theorem.

Theorem 5.3. *Let $q \in (1, \infty)$, let $\lambda,\ \mathcal{T} > 0$ and $s \in \mathbb{R}$, and set $\omega := \frac{2\pi}{\mathcal{T}}$. For each $f \in \mathrm{L}^q(\mathbb{T}\times\mathbb{R}^3)^3$ there exists a solution $(u, \mathfrak{p})$ with*

$$u \in \mathrm{W}^{1,q}(\mathbb{T}; \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^3)^3) \cap \mathrm{L}^q(\mathbb{T}; \mathrm{W}^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)^3), \qquad \mathfrak{p} \in \mathrm{L}^q(\mathbb{T}; \mathrm{W}^{1,q}_{\mathrm{loc}}(\mathbb{R}^3))$$

to (5.8) *that satisfies*

$$\|\mathrm{dist}(s, \omega\mathbb{Z})\, u\|_q + \|isu + \partial_t u\|_q + \|\nabla^2 u\|_q + \lambda\|\partial_1 u\|_q + \|\nabla\mathfrak{p}\|_q \leq C\|f\|_q \tag{5.9}$$

as well as

$$\lambda^{1/4}\|\nabla u\|_{\mathrm{L}^q(\mathbb{T};\mathrm{L}^{4q/(4-q)}(\mathbb{R}^3))} \leq C\|f\|_q \qquad \text{if } q < 4, \tag{5.10}$$

$$\|\nabla u\|_{\mathrm{L}^q(\mathbb{T};\mathrm{L}^{3q/(3-q)}(\mathbb{R}^3))} + \|\mathfrak{p}\|_{\mathrm{L}^q(\mathbb{T};\mathrm{L}^{3q/(3-q)}(\mathbb{R}^3))} \leq C\|f\|_q \qquad \text{if } q < 3, \tag{5.11}$$

$$\lambda^{1/2}\|u\|_{\mathrm{L}^q(\mathbb{T};\mathrm{L}^{2q/(2-q)}(\mathbb{R}^3))} \leq C\|f\|_q \qquad \text{if } q < 2, \tag{5.12}$$

$$\|u\|_{\mathrm{L}^q(\mathbb{T};\mathrm{L}^{3q/(3-2q)}(\mathbb{R}^3))} \leq C\|f\|_q \qquad \text{if } q < 3/2, \tag{5.13}$$

for the constant $C > 0$ from (5.7)*. Moreover, if $(w, \mathfrak{q}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R}^3)^{3+1}$ is another distributional solution to* (5.8)*, then the following holds:*

i. If $\nabla^2 w,\ isw + \partial_t w,\ \partial_1 w \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)$, then

$$isw + \partial_t w = isu + \partial_t u, \qquad \nabla^2 w = \nabla^2 u, \qquad \partial_1 w = \partial_1 u \qquad \nabla\mathfrak{q} = \nabla\mathfrak{p}.$$

ii. If $q < 2$ or $s \notin \omega\mathbb{Z}$, and if $w \in \mathrm{L}^1(\mathbb{T}; \mathrm{L}^r(\mathbb{R}^3)^3)$ for some $r \in [1, \infty)$, then $u = w$ and $\mathfrak{p} = \mathfrak{q} + d$ for a (space-independent) function $d \colon \mathbb{T} \to \mathbb{R}$.

Proof. We mainly follow the proof of [5, Theorem 4.1], where a similar existence result was shown for (5.8) in the Stokes case $\lambda = 0$. However, the derivation of the estimates (5.9)–(5.13) is more involved in the present case $\lambda > 0$.

At first, let $s \in \mathbb{R}$ and consider $\ell \in \mathbb{Z}$ such that $|s - \omega\ell| \leq \omega/2$. We set $\widetilde{s} = s - \omega\ell$ and $\widetilde{f}(t,x) = f(t,x)\,\mathrm{e}^{i\omega\ell t}$, and assume that $(\widetilde{u}, \widetilde{\mathfrak{p}})$ is a solution to (5.8) satisfying (5.9)–(5.13) with s and f replaced with $\widetilde{s}$ and $\widetilde{f}$, respectively. Then $(u, \mathfrak{p})$ with $u(t,x) := \widetilde{u}(t,x)\,\mathrm{e}^{-i\omega\ell t}$ and $\mathfrak{p}(t,x) := \widetilde{\mathfrak{p}}(t,x)\,\mathrm{e}^{-i\omega\ell t}$ satisfies the original problem (5.8) and the corresponding estimates (5.9)–(5.13). This shows that it suffices to only consider $s \in \mathbb{R}$ with $|s| \leq \omega/2$.

For $0 < |s| \leq \omega/2$, we first consider $f \in \mathscr{S}(G)^3$, where $G := \mathbb{T} \times \mathbb{R}^3$. To derive a representation formula for $\mathfrak{p}$, we compute the divergence of $(5.8)_1$, which leads to $\Delta\mathfrak{p} = \operatorname{div} f$ and, by means of the Fourier transform $\mathscr{F}_G$, the identity $-|\xi|^2\mathscr{F}_G[\mathfrak{p}] = i\xi \cdot \mathscr{F}_G[f]$. This yields

$$\mathfrak{p} = \mathscr{F}_G^{-1}\Big[\frac{-i\xi}{|\xi|^2}\mathscr{F}_G[f]\Big], \qquad \nabla\mathfrak{p} = \mathscr{F}_G^{-1}\Big[\frac{\xi\otimes\xi}{|\xi|^2}\mathscr{F}_G[f]\Big]. \tag{5.14}$$

In particular, $\mathfrak{p}$ is well defined as a distribution in $\mathscr{S}'(G)$ in this way, and the continuity of the Riesz transforms $\mathrm{L}^q(\mathbb{R}^3) \to \mathrm{L}^q(\mathbb{R}^3)$ implies

$$\|\nabla\mathfrak{p}\|_q \leq C\|f\|_q. \tag{5.15}$$

Next we apply the Fourier transform to $(5.8)_1$ and conclude

$$u = \mathscr{F}_G^{-1}\big[m\,\mathscr{F}_G[f - \nabla\mathfrak{p}]\big] = \mathscr{F}_G^{-1}\Big[m\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_G[f]\Big] \tag{5.16}$$

with

$$m\colon \mathbb{Z}\times\mathbb{R}^3 \to \mathbb{R}, \qquad m(k,\xi) := \frac{1}{is + i\omega k - i\lambda\xi_1 + |\xi|^2}.$$

Since $0 < |s| < \omega/2$, the denominator of m has no zeros $(k,\xi) \in \mathbb{Z}\times\mathbb{R}^3$ and is bounded, so that u is a well-defined distribution in $\mathscr{S}'(G)$. Moreover, we derive the formulas

$$\begin{aligned}
isu &= \mathscr{F}_G^{-1}\Big[m_0\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_G[f]\Big], &\qquad m_0(k,\xi) &:= \frac{is}{is + i\omega k - i\lambda\xi_1 + |\xi|^2},\\
\partial_t u &= \mathscr{F}_G^{-1}\Big[m_1\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_G[f]\Big], & m_1(k,\xi) &:= \frac{i\omega k}{is + i\omega k - i\lambda\xi_1 + |\xi|^2},\\
\partial_j\partial_\ell u &= \mathscr{F}_G^{-1}\Big[m_{j\ell}\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_G[f]\Big], & m_{j\ell}(k,\xi) &:= \frac{-\xi_j\xi_\ell}{is + i\omega k - i\lambda\xi_1 + |\xi|^2},\\
\lambda\partial_1 u &= \mathscr{F}_G^{-1}\Big[m_2\Big(I - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathscr{F}_G[f]\Big], & m_2(k,\xi) &:= \frac{i\lambda\xi}{is + i\omega k - i\lambda\xi_1 + |\xi|^2}.
\end{aligned}$$

For the L^q estimates, we now employ the transference principle from Theorem 5.2. We focus on m_2 in the following, the other multipliers can be treated in a similar fashion. We introduce $\chi \in \mathrm{C}^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$ and such that $\chi(x) = 0$ for $|x| \leq 1/2$ and $\chi(x) = 1$ for $|x| \geq 1$, and we define $M_2\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ by

$$M_2(\eta,\xi) := \frac{i\lambda\xi_1\,\chi\big(1 + \frac{\omega\eta}{s}\big)}{N(\eta,\xi)}, \qquad N(\eta,\xi) := is + i\omega\eta - i\lambda\xi + |\xi|^2.$$

The numerator vanishes in a neighborhood of the only zero $(\eta,\xi)=(-s/\omega,0)$ of the denominator N. Hence M_2 is a well-defined continuous function with $M_2\big|_{\mathbb{Z}\times\mathbb{R}^3}=m_2$. Moreover, M_2 vanishes for $|s+\omega\eta|\leq|s|/2$, and for $|s+\omega\eta|\geq|s|/2$ we have

$$|s+\omega\eta|\geq 2\lambda|\xi_1| \implies \big|N(\eta,\xi)\big|\geq|s+\omega\eta-\lambda\xi_1|\geq\frac{1}{2}|s+\omega\eta|,$$
$$|s+\omega\eta|\leq 2\lambda|\xi_1| \implies \big|N(\eta,\xi)\big|\geq|\xi|^2\geq\frac{1}{4\lambda^2}|s+\omega\eta|^2\geq\frac{|s|}{8\lambda^2}|s+\omega\eta|,$$

so that

$$\frac{|s+\omega\eta|}{|N(\eta,\xi)|}\leq C\big(1+\frac{\lambda^2}{|s|}\big).$$

Using this estimate, for $|s+\omega\eta|\geq|s|/2$ we further obtain

$$\frac{|s|}{|N(\eta,\xi)|}+\frac{|\omega\eta|}{|N(\eta,\xi)|}+\frac{|\lambda\xi_1|}{|N(\eta,\xi)|}\leq C\big(1+\frac{\lambda^2}{|s|}\big).$$

With these estimates at hand, one shows that

$$\sup\big\{\big|\eta^\alpha\xi^\beta\partial_\eta^\alpha\partial_\xi^\beta M_2(\eta,\xi)\big| : \alpha\in\{0,1\},\, \beta\in\{0,1\}^3,\, (\eta,\xi)\in\mathbb{R}\times\mathbb{R}^3\big\}\leq C\left(1+\frac{\lambda^2}{|s|}\right)^3.$$

By the Marcinkiewicz multiplier theorem (see [17, Corollary 5.2.5] for example) we thus conclude that M_2 is an $\mathrm{L}^q(\mathbb{R}\times\mathbb{R}^3)$ multiplier, and the transference principle (Theorem 5.2) implies that m_2 is an $\mathrm{L}^q(G)$ multiplier with

$$\|\mathscr{F}_G^{-1}\big[m_2\mathscr{F}_G[g]\big]\|_q\leq C\left(1+\frac{\lambda^2}{|s|}\right)^3\|g\|_q$$

for all $g\in\mathscr{S}(G)$, where $C=C(q)$. In the same way, we show that m_0, m_1 and $m_{j\ell}$ are multipliers with associated norms bounded by $C(1+\lambda^2/|s|)^3$. By combining these estimates with the continuity of the Riesz transforms, the above representation formulas yield

$$\|isu\|_q+\|\partial_t u\|_q+\|\nabla^2 u\|_q+\lambda\|\partial_1 u\|_q\leq C_0\left(1+\frac{\lambda^2}{|s|}\right)^3\|f\|_q$$

with a constant $C_0=C_0(q)$. Since $0<|s|<\omega/2$, the combination of this estimate with (5.15) yields (5.9). Finally, (5.11) and (5.13) follow from Sobolev's inequality, and (5.10) and (5.12) are implied by Lemma 4.1. In summary, for $f\in\mathscr{S}(G)$ we have now constructed a solution to (5.8) with the desired properties. A classical approximation argument based on the estimates (5.9)–(5.13) finally yields the existence of a solution for any $f\in\mathrm{L}^q(G)$.

In the case $s=0$, problem (5.8) reduces to the classical time-periodic Oseen system. Solutions $(u,\mathfrak{p})$ to this problem were established in [19], and the validity of the *a priori* estimate (5.9) with a constant $C=C_0P(\lambda^2/\omega)$ was derived in the proof of [7, Theorem 5.1], where P is a polynomial. Arguing as above, one can show that P can be chosen in the claimed form.

For the uniqueness assertion, we let $(\widetilde{u},\widetilde{\mathfrak{p}}):=(u-w,\mathfrak{p}-\mathfrak{q})\in\mathrm{L}^1_{\mathrm{loc}}(G)^{3+1}$, which is a solution to (5.8) with $f=0$. Computing the divergence of both sides of $(5.8)_1$, we conclude $\Delta\widetilde{\mathfrak{p}}=0$ and, in particular, $\operatorname{supp}\mathscr{F}_G[\widetilde{\mathfrak{p}}]\subset\mathbb{Z}\times\{0\}$. An application of $\mathscr{F}_G$ to $(5.8)_1$ thus leads to $(is+i\omega k+|\xi|^2-i\lambda\xi_1)\mathscr{F}_G[\widetilde{u}]=-i\xi\mathscr{F}_G[\widetilde{\mathfrak{p}}]$, and we deduce

$$\operatorname{supp}\big[(is+i\omega k+|\xi|^2-i\lambda\xi_1)\mathscr{F}_G[\widetilde{u}]\big]\subset\mathbb{Z}\times\{0\}.$$

Since $is + i\omega k + |\xi|^2 - i\lambda\xi_1$ can only vanish for $\xi = 0$, we conclude $\operatorname{supp} \mathscr{F}_G[\widetilde{u}] \subset \mathbb{Z} \times \{0\}$, so that $\operatorname{supp} \mathscr{F}_{\mathbb{R}^3}[\widetilde{u}](t,\cdot) \subset \{0\}$ for a.a. $t \in \mathbb{T}$ due to $\widetilde{u} \in \mathrm{L}^1_{\mathrm{loc}}(G)$. Hence, $\widetilde{u}(t,\cdot)$ is a polynomial for a.a. $t \in \mathbb{T}$. In the same way we show that $\widetilde{\mathfrak{p}}(t,\cdot)$ is a polynomial for a.a. $t \in \mathbb{T}$. In case i. we additionally have $\nabla^2\widetilde{u},\ \partial_1\widetilde{u},\ \nabla\widetilde{\mathfrak{p}} \in \mathrm{L}^q(G)$, which is only possible if $\nabla^2\widetilde{u} = 0$ and $\partial_1\widetilde{u} = \nabla\widetilde{\mathfrak{p}} = 0$. In virtue of $(5.8)_1$, this also implies $is\widetilde{u} + \partial_t\widetilde{u} = 0$. This shows the statement in case i. In case ii. we have that $\widetilde{u}$ is a polynomial with $\widetilde{u} \in \mathrm{L}^1(\mathbb{T}; \mathrm{L}^{r_0}(\mathbb{R}^3)^3 + \mathrm{L}^r(\mathbb{R}^3)^3)$ where $r_0 = 2q/(2-q)$ if $q < 2$, and $r_0 = q$ if $s \notin \omega\mathbb{Z}$. This is only possible if $\widetilde{u} = 0$, and returning to $(5.8)_1$, we also conclude $\nabla\widetilde{\mathfrak{p}} = 0$. In total, this completes the proof. ☐

Next we introduce the rotation term and study the problem

$$\begin{cases} isu + \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u) - \Delta u - \lambda\partial_1 u + \nabla\mathfrak{p} = f & \text{in } \mathbb{T} \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{T} \times \mathbb{R}^3 \end{cases} \tag{5.17}$$

in the case that $\omega = \frac{2\pi}{\mathcal{T}}$. By means of a suitable coordinate transform, we derive well-posedness of (5.17) by a reduction to problem (5.8).

Theorem 5.4. *Let $q \in (1,\infty)$ and $\lambda,\ \mathcal{T} > 0$, and let $\omega = \frac{2\pi}{\mathcal{T}}$. For each $f \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)^3$ there exists a solution $(u,\mathfrak{p})$ with*

$$u \in \mathrm{W}^{1,q}(\mathbb{T}; \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^3)^3) \cap \mathrm{L}^q(\mathbb{T}; \mathrm{W}^{2,q}_{\mathrm{loc}}(\mathbb{R}^3)^3), \qquad \mathfrak{p} \in \mathrm{L}^q(\mathbb{T}; \mathrm{W}^{1,q}_{\mathrm{loc}}(\mathbb{R}^3))$$

to (5.17) *that satisfies*

$$\begin{aligned} \|\operatorname{dist}(s,\omega\mathbb{Z})\, u\|_q + \|isu + \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u)\|_q \\ + \|\nabla^2 u\|_q + \lambda\|\partial_1 u\|_q + \|\nabla\mathfrak{p}\|_q \leq C\|f\|_q \end{aligned} \tag{5.18}$$

as well as (5.10)–(5.13) *for the constant $C > 0$ from* (5.7). *Moreover, if $(w,\mathfrak{q}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R}^3)^{3+1}$ is another distributional solution to* (5.17)*, then the following holds:*

i. If $\nabla^2 w,\ isw + \partial_t w + \omega(\mathrm{e}_1 \wedge w - \mathrm{e}_1 \wedge x \cdot \nabla w),\ \partial_1 w \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)$, then

$$\begin{gathered} isw + \partial_t w + \omega(\mathrm{e}_1 \wedge w - \mathrm{e}_1 \wedge x \cdot \nabla w) = isu + \partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u), \\ \nabla^2 w = \nabla^2 u, \qquad \partial_1 w = \partial_1 u \qquad \nabla\mathfrak{q} = \nabla\mathfrak{p}. \end{gathered}$$

ii. If $q < 2$ or $s \notin \omega\mathbb{Z}$, and if $w \in \mathrm{L}^1(\mathbb{T}; \mathrm{L}^r(\mathbb{R}^3)^3)$ for some $r \in [1,\infty)$, then $u = w$ and $\mathfrak{p} = \mathfrak{q} + d$ for a (space-independent) function $d \colon \mathbb{T} \to \mathbb{R}$.

Proof. The proof is based on the idea to absorb the rotational term $\omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u)$ into the time derivative by the coordinate transform arising from the rotation matrix

$$Q_\omega(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & -\sin(\omega t) \\ 0 & \sin(\omega t) & \cos(\omega t) \end{pmatrix}. \tag{5.19}$$

Let $f \in \mathrm{L}^q(\mathbb{T} \times \mathbb{R}^3)^3$ and define the vector field $\widetilde{f}$ by

$$\widetilde{f}(t,x) := Q_\omega(t) f(t, Q_\omega(t)^\top x).$$

Then $\widetilde{f} \in \mathrm{L}^q(\mathbb{T}\times\mathbb{R}^3)^3$ since $\mathbb{T} = \mathbb{R}/\mathcal{T}\mathbb{Z}$ with $\mathcal{T} = \frac{2\pi}{\omega}$. By Theorem 5.3 there exists a solution $(\widetilde{u}, \widetilde{\mathfrak{p}})$ to (5.8) (with f replaced by $\widetilde{f}$), which satisfies the estimates (5.9)–(5.13). We now define the $\mathcal{T}$-time-periodic functions

$$u(t,x) := Q_\omega(t)^\top \widetilde{u}(t, Q_\omega(t)x), \qquad \mathfrak{p}(t,x) := \widetilde{\mathfrak{p}}(t, Q_\omega(t)x).$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}[Q_\omega(t)x] = \omega\,\mathrm{e}_1 \wedge [Q_\omega(t)x] = Q_\omega(t)[\omega\,\mathrm{e}_1 \wedge x]$ for any $x \in \mathbb{R}^3$, a direct computation shows that $(u, \mathfrak{p})$ is a solution to (5.17) and obeys the estimates (5.18) and (5.10)–(5.13).

For uniqueness, we use the above transformation to obtain solutions $(\widetilde{u}, \widetilde{\mathfrak{p}})$ and $(\widetilde{w}, \widetilde{\pi})$ to (5.8) with the same right-hand side $\widetilde{f}$. The statement is then a consequence of follows from Theorem 5.3. □

Observe that, by simply considering $s = 0$ in (5.17), we would obtain the original time-periodic problem (1.2), and Theorem 5.4 yields existence of a unique solution. However, in Theorem 5.4 we required $\omega = \frac{2\pi}{\mathcal{T}}$, and the presented proof does not allow to choose ω and $\mathcal{T}$ independently. To make this possible, we first consider time-independent solutions to (5.17), which are solutions to the resolvent problem (5.1). In this way, we conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. Set $\mathbb{T} := \mathbb{R}/\mathcal{T}\mathbb{Z}$ with $\mathcal{T} = \frac{2\pi}{\omega}$, let $g \in \mathrm{L}^q(\mathbb{R}^3)^3$ and define $f(t,x) := g(x)$. Then $f \in \mathrm{L}^q(\mathbb{T}\times\mathbb{R}^3)^3$, and there exists a solution $(u, \mathfrak{p})$ to (5.17) by Theorem 5.4. Computing the time means

$$v(x) := \int_{\mathbb{T}} u(t,x)\,\mathrm{d}t, \qquad p(x) := \int_{\mathbb{T}} \mathfrak{p}(t,x)\,\mathrm{d}t,$$

we obtain a solution (v,p) to (5.1), and estimates (5.2)–(5.6) follow directly from (5.18), (5.10)–(5.13). Since every solution to (5.1) is a (time-independent) solution to (5.17), the uniqueness statement follows directly from Theorem 5.4. □

6 The resolvent problem in an exterior domain

After having established well-posedness of the resolvent problem (5.1) in $\mathbb{R}^3$, we next consider the corresponding problem in an exterior domain $\Omega \subset \mathbb{R}^3$, that is, the resolvent problem (1.1) with a purely imaginary resolvent parameter is, $s \in \mathbb{R}$. We first address the question of uniqueness of solutions.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $\mathrm{C}^{1,1}$. Let $\lambda \geq 0$, $\omega > 0$, $s \in \mathbb{R}$, and let (v,p) be a distributional solution to* (1.1) *with $g = 0$ and $\nabla^2 v,\ \partial_1 v,\ \nabla p \in \mathrm{L}^q(\Omega)$ for some $q \in (1,\infty)$ and $v \in \mathrm{L}^r(\Omega)$ for some $r \in (1,\infty)$. Then $v = 0$ and p is constant.*

Proof. For $\lambda = 0$, the statement was shown in [5, Lemma 5.1]. For $\lambda > 0$ one can follow the proof of [7, Lemma 5.6], which treats the case $s \in \omega\mathbb{Z}$. Therefore, we only sketch the main arguments here. One first employs a cut-off argument that leads to a Stokes problem in a bounded domain and to the resolvent problem (5.1) in the whole space, both with error terms on the right-hand side. Using elliptic regularity of the Stokes problem and regularity properties for (5.1) established in Theorem 5.1, one can show that

$$\begin{aligned}
&\forall r \in (1,2): \quad isv + \mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v,\ \nabla^2 v,\ \nabla p \in \mathrm{L}^r(\Omega),\\
&\forall r \in \Big(\frac{3}{2}, 6\Big): \quad \nabla v \in \mathrm{L}^r(\Omega),\\
&\forall r \in (3,\infty): \quad v \in \mathrm{L}^r(\Omega).
\end{aligned}$$

DOI 10.20347/WIAS.PREPRINT.2888

These regularities allow us to multiply $(1.1)_1$ the complex conjugate of v, to integrate the resulting identity over Ω_R, and to pass to the limit $R \to \infty$. This leads to

$$0 = is \int_\Omega |v|^2 \,\mathrm{d}x + \int_\Omega |\nabla v|^2 \,\mathrm{d}x,$$

so that $\nabla v = 0$, which implies $v = 0$ in view of $(1.1)_3$. From $(1.1)_1$ we finally conclude $\nabla p = 0$. □

Suitable *a priori* estimates for solutions to (1.1) can be derived by a similar cut-off procedure, which first leads to the following intermediate result. For simplicity, we only consider the case $q < 2$, where estimates of v and ∇v are available.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class $\mathrm{C}^{1,1}$, and $\lambda, \omega > 0$ and $s \in \mathbb{R}$. Let $q \in (1, 2)$ and $g \in \mathrm{L}^q(\Omega)^3$. Consider a solution (v, p) to* (1.1) *that satisfies*

$$isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v),\ \nabla^2 v,\ \partial_1 v,\ \nabla p \in \mathrm{L}^q(\Omega)^3$$

and $v \in \mathrm{L}^r(\Omega)^3$, $p \in \mathrm{L}^{\overline{r}}(\Omega)$ for some $r, \overline{r} \in (1, \infty)$. Fix $R > 0$ such that $\partial\Omega \subset B_R$. Then (v, p) satisfies the estimate

$$\begin{aligned} &\|\mathrm{dist}(s, \omega\mathbb{Z})\, v\|_q + \|isv + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v)\|_q + \|\nabla^2 v\|_q + \lambda\|\partial_1 v\|_q + \|\nabla p\|_q \\ &\qquad + \lambda^{1/4}\|\nabla v\|_{4q/(4-q)} + \lambda^{1/2}\|v\|_{2q/(2-q)} + \|p\|_{3q/(3-q)} \\ &\qquad\quad \leq C\big(\|g\|_q + (1 + \lambda + \omega)\|v\|_{1,q;\Omega_R} + \|p\|_{q;\Omega_R} + |s|\|v\|_{-1,q;\Omega_R}\big) \end{aligned} \tag{6.1}$$

for a constant $C > 0$ given by (5.7) *with $C_0 = C_0(q, \Omega, R) > 0$ and $P(\theta) := (1 + \theta)^3$.*

Proof. The estimate can be shown by a classical cut-off procedure. We skip the details here and refer to [7, Lemma 5.7], where the special case $s \in \omega\mathbb{Z}$ was considered. In the present situation one may proceed in the very same way by invoking estimates (5.2), (5.3) and (5.5) as well as the uniqueness result from Lemma 6.1. □

Next we show how to omit the error terms on the right-hand side of (6.1) by means of a compactness argument in order to derive estimate (3.1). Again we want to keep track of the constant in this *a priori* estimate and its dependence on the various parameters. Observe that the dependence on λ can only be avoided if $q < 3/2$.

Lemma 6.3. *In the situation of Lemma 6.2, the solution (v, p) satisfies the estimate* (3.1) *for a constant $C = C(q, \Omega, \omega, s, \lambda) > 0$. If $\omega \in (0, \omega_0]$ for some $\omega_0 > 0$, and if*

$$\lambda^2 \leq \theta \min\big\{|s - \omega k| : k \in \mathbb{Z},\ s \neq \omega k\big\},$$

for some $\theta > 0$, then we can choose C such that $C = C(q, \Omega, \omega_0, \theta, \lambda) > 0$. If additionally $q \in (1, 3/2)$, then C can be chosen such that $C = C(q, \Omega, \omega_0, \theta) > 0$. In particular, then C is independent of ω, λ and s.

Proof. We perform a contradiction argument. Consider the case $q \in (1, 3/2)$ at first, and assume that there is no constant C with the claimed properties. Then there exist sequences $(s_j) \subset \mathbb{R}$, $(\omega_j) \subset (0, \omega_0]$, $(\lambda_j) \subset (0, \infty)$ with

$$\lambda_j^2 \leq \theta \min\big\{|s_j - \omega_j k| : k \in \mathbb{Z},\ s_j \neq \omega_j k\big\} \tag{6.2}$$

and sequences $(v_j) \subset \mathrm{W}^{2,q}_{\mathrm{loc}}(\Omega)^3$, $(p_j) \subset \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$, $(g_j) \subset \mathrm{L}^q(\Omega)^3$ such that

$$\begin{cases} is_j v_j + \omega_j(\mathrm{e}_1 \wedge v_j - \mathrm{e}_1 \wedge x \cdot \nabla v_j) - \Delta v_j - \lambda_j \partial_1 v_j + \nabla p_j = g_j & \text{in } \Omega, \\ \operatorname{div} v_j = 0 & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega \end{cases} \tag{6.3}$$

as well as

$$\begin{aligned} \|\operatorname{dist}(s_j, \omega_j\mathbb{Z})\, v_j\|_q + \|is_j v_j + \omega_j(\mathrm{e}_1 \wedge v_j - \mathrm{e}_1 \wedge x \cdot \nabla v_j)\|_q + \|\nabla^2 v_j\|_q + \lambda_j \|\partial_1 v_j\|_q \\ + \|\nabla p_j\|_q + \lambda^{1/4}\|\nabla v_j\|_{4q/(4-q)} + \lambda^{1/2}\|v_j\|_{2q/(2-q)} + \|p_j\|_{3q/(3-q)} = 1 \end{aligned} \tag{6.4}$$

and

$$\lim_{j\to\infty} \|g_j\|_q = 0.$$

Moreover, there are sequences (r_j), $(\overline{r}_j) \subset (1,\infty)$ such that $v_j \in \mathrm{L}^{r_j}(\Omega)^3$, $p_j \in \mathrm{L}^{\overline{r}_j}(\Omega)$. Here we used that the left-hand side of (6.4) is finite due to Lemma 6.2, so that it can be normalized to 1. Note that (6.2) implies $\lambda_j^2 \leq \theta\omega_j^2 \leq \theta\omega_0^2$ for all $j \in \mathbb{N}$. Upon the choice of a subsequence, we thus conclude the convergence of the sequences $\omega_j \to \omega \in [0,\omega_0]$, $\lambda_j \to \lambda \in [0, \sqrt{\theta\omega}]$, $s_j \to s \in [-\infty,\infty]$ and $\operatorname{dist}(s_j, \omega_j\mathbb{Z}) \to \delta \in [-\omega/2, \omega/2]$ as $j \to \infty$. For the moment, fix $R > 0$ with $\partial\Omega \subset B_R$. In virtue of (6.4) and the estimate

$$\begin{aligned} &\|is_j v_j\|_{q;\Omega_R} \\ &\quad\leq \|is_j v_j + \omega_j(\mathrm{e}_1 \wedge v_j - \mathrm{e}_1 \wedge x \cdot \nabla v_j)\|_{q;\Omega_R} + \|\omega_j(\mathrm{e}_1 \wedge v_j - \mathrm{e}_1 \wedge x \cdot \nabla v_j)\|_{q;\Omega_R} \\ &\quad\leq \|is_j v_j + \omega_j(\mathrm{e}_1 \wedge v_j - \mathrm{e}_1 \wedge x \cdot \nabla v_j)\|_q + \omega_0\big(\|v_j\|_{q;\Omega_R} + R\|\nabla v_j\|_{q;\Omega_R}\big), \end{aligned} \tag{6.5}$$

the sequences $(isv_j\big|_{\Omega_R})$, $(v_j\big|_{\Omega_R})$ and $(p_j\big|_{\Omega_R})$ are bounded in $\mathrm{L}^q(\Omega_R)$, $\mathrm{W}^{2,q}(\Omega_R)$ and $\mathrm{W}^{1,q}(\Omega_R)$, respectively. Upon selecting suitable subsequences, we thus infer the existence of $w \in \mathrm{L}^q_{\mathrm{loc}}(\Omega)^3$, $v \in \mathrm{W}^{2,q}_{\mathrm{loc}}(\Omega)^3$ and $p \in \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ such that

$$is_j v_j \rightharpoonup w \quad \text{in } \mathrm{L}^q(\Omega_R), \qquad v_j \rightharpoonup v \quad \text{in } \mathrm{W}^{2,q}(\Omega_R), \qquad p_j \rightharpoonup p \quad \text{in } \mathrm{W}^{1,q}(\Omega_R).$$

By a Cantor diagonalization argument one can find the limit functions w, v, p that are independent of the radius $R > 0$. Moreover, due to the uniform bounds in (6.4), we can assume weak convergence in the norms on the left-hand side of (6.4), which yields

$$\begin{aligned} \|\delta v\|_q + \|w + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v)\|_q + \|\nabla^2 v\|_q + \lambda\|\partial_1 v\|_q + \|\nabla p\|_q \\ + \lambda^{1/4}\|\nabla v\|_{4q/(4-q)} + \lambda^{1/2}\|v\|_{2q/(2-q)} + \|p\|_{3q/(3-q)} \leq 1, \end{aligned}$$

and Sobolev's inequality yields $v \in \mathrm{L}^{3q/(3-2q)}(\Omega)$ since $q < 3/2$. Moreover, we can pass to the limit $j \to \infty$ in (6.3) and conclude

$$\begin{cases} w + \omega(\mathrm{e}_1 \wedge v - \mathrm{e}_1 \wedge x \cdot \nabla v) - \Delta v - \lambda\partial_1 v + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases} \tag{6.6}$$

We now distinguish the following cases:

i. If $|s| < \infty$ and $\omega = 0$, then $w = isv$ and $\lambda = 0$ and (6.6) simplifies to the classical Stokes resolvent problem with resolvent parameter is. If $s \neq 0$, this yields $isv = \Delta v - \nabla p \in \mathrm{L}^q(\Omega)$, so that $v \in \mathrm{W}^{2,q}(\Omega)$. Uniqueness in this functional framework is well known, so that $v = \nabla p = 0$; see [9] for example. If $s = 0$, then (6.6) coincides with the steady-state Stokes problem, and $v \in \mathrm{L}^{3q/(3-2q)}(\Omega)$ implies $v = \nabla p = 0$; see [10, Theorem V.4.6] for example.

ii. If $|s| < \infty$ and $\omega > 0$, then $w = isv$ and (6.6) coincides with (1.1) with $g = 0$ and $\lambda \geq 0$. From Lemma 6.1 and $v \in \mathrm{L}^{3q/(3-2q)}(\Omega)$ we now conclude $v = \nabla p = 0$.

iii. If $|s| = \infty$, we note that estimates (6.4) and (6.5) imply $\|v_j\|_{q;\Omega_R} \leq C/|s_j|$ for some R-dependent constant C. Passing to the limit $j \to \infty$ and employing that R was arbitrary, we deduce $v = 0$ and (6.6) reduces to $w + \nabla p = 0$, which, in particular, yields $w \in \mathrm{L}^q(\Omega)$. Since we also have $\operatorname{div} w = 0$ and $w\big|_{\partial\Omega} = 0$, this equality corresponds to the Helmholtz decomposition in $\mathrm{L}^q(\Omega)$ of the zero function, which is unique, so that $w = \nabla p = 0$.

All three cases lead to $w = v = \nabla p = 0$, and $p \in \mathrm{L}^{3q/(3-q)}(\Omega)$ further implies $p = 0$. By Lemma 6.2 we further have (6.1) with v, p, g replaced with v_j, p_j, g_j and the constant C given by $C = C_0(1+\theta)^3$. In particular, C is uniform in $j \in \mathbb{N}$. Employing (6.4) and passing to the limit $j \to \infty$ in this inequality, we thus deduce

$$1 \leq C\big((1 + \lambda + \omega)\|v\|_{1,q;\Omega_R} + \|p\|_{q;\Omega_R} + \|w\|_{-1,q;\Omega_R}\big) = 0,$$

which is a contradiction. This finishes the proof for $q \in (1, 3/2)$.

In the case $q \in (1, 2)$, where the constant C may depend on λ, the proof follows nearly the same lines, but we consider a fixed value $\lambda > 0$ instead of a sequence (λ_j). Therefore, the case $\lambda = 0$ cannot occur in the limit system (6.6) and we have $v \in \mathrm{L}^{2q/(2-q)}(\Omega)$, which is sufficient to deduce $w = v = \nabla p = 0$ as before and to conclude the contradiction argument. $\square$

We next show the existence of a solution to the resolvent problem (1.1) for a smooth right-hand side g.

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class C^3. Let $\lambda,\ \omega > 0$, $s \in \mathbb{R}$ and $g \in \mathrm{C}_0^\infty(\Omega)^3$. Then there exists a solution (v, p) to* (1.1) *with*

$$\forall q \in (1, 2):\ (v, p) \in \mathrm{X}^q_{\lambda,\omega,s}(\Omega) \times \mathrm{Y}^q(\Omega).$$

Proof. In the case $s \in \omega\mathbb{Z}$, existence was shown in [7, Lemma 5.11] in full detail based on energy estimates and an "invading domains" technique as well as the L^q estimates (3.1), which we established in Lemma 6.3. The proof for general $s \in \mathbb{R}$ follows along the same lines, and we only give a rough sketch here.

First of all, we choose $R > 0$ such that $\partial\Omega \subset B_R$. For $m \in \mathbb{N}$ with $m > R$ we consider the resolvent problem (1.1) in the bounded domain $\Omega_m = \Omega \cap B_m$, which is given by

$$\begin{cases} isv_m + \omega(\mathrm{e}_1 \wedge v_m - \mathrm{e}_1 \wedge x \cdot \nabla v_m) - \Delta v_m - \lambda\partial_1 v_m + \nabla p_m = g & \text{in } \Omega_m, \\ \operatorname{div} v_m = 0 & \text{in } \Omega_m, \\ v_m = 0 & \text{on } \partial\Omega_m. \end{cases}$$

By formally testing with the complex conjugates of v_m and $\mathcal{P}_{\Omega_m}\Delta v_m$, where $\mathcal{P}_{\Omega_m}$ is the Helmholtz projection in $\mathrm{L}^2(\Omega_m)$, one can then derive the *a priori* estimates

$$\begin{aligned} \|v_m\|_{6;\Omega_m} + \|\nabla v_m\|_{2;\Omega_m} &\leq C\|g\|_{6/5}, \\ \|\mathcal{P}_{\Omega_m}\Delta v_m\|_{2;\Omega_m} &\leq C\big(\|g\|_{6/5} + \|g\|_2\big), \end{aligned}$$

where the constant $C > 0$ is independent of m. In order to derive a uniform estimate on the full second-order norm, we employ the inequality

$$\|\nabla^2 w\|_{2;\Omega_m} \leq C(\|\mathcal{P}_{\Omega_m}\Delta w\|_{2;\Omega_m} + \|\nabla w\|_{2;\Omega_m})$$

for all $w \in \mathrm{W}^{1,2}_0(\Omega_m)^3 \cap \mathrm{W}^{2,2}(\Omega_m)^3$ with $\operatorname{div} w = 0$. Since we assumed $\partial\Omega \in \mathrm{C}^3$, the constant C can be chosen independently of m; see [18, Lemma 1]. Using these formal *a priori* estimates, one can then apply a Galerkin method, based on a basis of eigenfunctions of the Stokes operator on the bounded domain Ω_m, to conclude the existence of a solution (v_m, p_m), which satisfies the *a priori* estimate

$$\|v_m\|_{6;\Omega_m} + \|\nabla v_m\|_{1,2;\Omega_m} \leq C(\|g\|_{6/5} + \|g\|_2),$$

where C is independent of m. After multiplication with suitable cut-off functions, one can then pass to the limit $m \to \infty$, which leads to a solution (v, p) to the original resolvent problem (1.1). Finally, another cut-off argument that uses the uniqueness properties from Lemma 6.1 reveals that $(v, p) \in \mathrm{X}^q_{\lambda,\omega,s}(\Omega) \times \mathrm{Y}^q(\Omega)$ for all $q \in (1, 2)$. □

Now we can combine the previous lemmas to conclude the proof Theorem 3.1 by a final approximation argument.

Proof of Theorem 3.1. In the case $\Omega = \mathbb{R}^3$ the statement follows from Theorem 5.1 above. If $\Omega \subset \mathbb{R}^3$ is an exterior domain, the uniqueness statement is a consequence of Lemma 6.1, and estimate (3.1) was shown in Lemma 6.3. For existence, let $g \in \mathrm{L}^q(\Omega)^3$ and consider a sequence $(g_j) \subset \mathrm{C}^\infty_0(\Omega)^3$ converging to g in $\mathrm{L}^q(\Omega)$. By Lemma 6.4 there exists a solution $(v_j, p_j) \in \mathrm{X}^q_{\lambda,\omega,s}(\Omega) \times \mathrm{Y}^q(\Omega)$ to (1.1) with $g = g_j$ for each $j \in \mathbb{N}$. From Lemma 6.3 we infer that (v_j, p_j) is a Cauchy sequence in $\mathrm{X}^q_{\lambda,\omega,s}(\Omega) \times \mathrm{Y}^q(\Omega)$. Since this is a Banach space, there exists a unique limit $(v, p) \in \mathrm{X}^q_{\lambda,\omega,s}(\Omega) \times \mathrm{Y}^q(\Omega)$, which is a solution to (1.1). This completes the proof. □

7 Existence of time-periodic solutions

Now we consider the linear and nonlinear time-periodic problems (1.2) and (1.3), and we prove the existence results from Theorem 3.2 and Theorem 3.4. We begin with the well-posedness of the linear problem (1.2).

Proof of Theorem 3.2. Let $f \in \mathrm{A}(\mathbb{T}; \mathrm{L}^q(\Omega)^3)$, and let $f_k \in \mathrm{L}^q(\Omega)^3$, $k \in \mathbb{Z}$, such that

$$f(t,x) = \sum_{k\in\mathbb{Z}} f_k(x)\, \mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}.$$

By Theorem 3.1, for each $k \in \mathbb{Z}$ there is a solution $(u_k, \mathfrak{p}_k) \in \mathrm{X}^q_{\lambda,\omega,\frac{2\pi}{\mathcal{T}}k}(\Omega) \times \mathrm{Y}^q(\Omega)$ to

$$\begin{cases} i\frac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k) - \Delta u_k - \lambda\partial_1 u_k + \nabla \mathfrak{p}_k = f_k & \text{in } \Omega, \\ \operatorname{div} u_k = 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

which satisfies

$$\begin{aligned} \|i\tfrac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k)\|_q + \|\nabla^2 u_k\|_q + \|\nabla u_k\|_q \\ + \|\nabla u_k\|_{3q/(3-q)} + \|u_k\|_{3q/(3-2q)} + \|\mathfrak{p}_k\|_{3q/(3-q)} \leq C\|f_k\|_q. \end{aligned}$$

Since $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$, Proposition 4.2 implies that condition (3.4) can always be satisfied for some $\theta > 0$. Moreover, (3.4) implies (3.2) for all $s \in \frac{2\pi}{\mathcal{T}}\mathbb{Z}$, so that the constant C can be chosen independently of $k \in \mathbb{Z}$. Therefore, the series

$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(x)\, \mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}, \qquad \mathfrak{p}(t,x) = \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\, \mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt},$$

define a pair $(u, \mathfrak{p}) \in \mathcal{X}^q_{0,\omega}(\mathbb{T}\times\Omega)\times\mathcal{Y}^q(\mathbb{T}\times\Omega)$, which satisfies estimate (3.3) with the same constant C and solves the time-periodic problem (1.2).

To show the uniqueness statement, consider a solution $(u, \mathfrak{p}) \in \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)\times\mathcal{Y}^q(\mathbb{T}\times\Omega)$ to (1.2) with $f = 0$. Then the Fourier coefficients $(u_k, \mathfrak{p}_k)$, $k \in \mathbb{Z}$, are elements of $\mathrm{X}^q_{\lambda,\omega,\frac{2\pi}{\mathcal{T}}k}(\Omega)\times\mathrm{Y}^q(\Omega)$ and satisfy (7.1) with $f_k = 0$. From Theorem 3.1 we thus conclude $(u_k, \mathfrak{p}_k) = (0, 0)$ for all $k \in \mathbb{Z}$, so that $(u, \mathfrak{p}) = (0, 0)$. This shows uniqueness of the solution and completes the proof. □

Next we provide the proof of Theorem 3.4 on the existence of a solution to the nonlinear problem (1.3) for "small" data, which is based on a fixed-point argument.

Proof of Theorem 3.4. The proof largely follows that of [7, Theorem 2.3], where existence of a solution to (1.3) was shown for $\frac{2\pi}{\mathcal{T}} = \omega$, which is why we skip the details here.

Let $R > 0$ such that $\partial\Omega \subset B_R$, and let $\varphi \in \mathrm{C}^\infty_0(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq R$, and $\varphi(x) = 0$ for $|x| \geq 2R$. Define $U(x) := \frac{1}{2}\operatorname{rot}[(\lambda\,\mathrm{e}_1 \wedge x - \omega\,\mathrm{e}_1\,|x|^2)\varphi(x)]$. Then $(u, \mathfrak{p})$ is a solution to (1.3) if and only if $(w, \mathfrak{q}) := (u - U, \mathfrak{p})$ is a solution to

$$\begin{cases} \partial_t w + \omega(\mathrm{e}_1 \wedge w - \mathrm{e}_1 \wedge x \cdot \nabla w) - \lambda\partial_1 w - \Delta w + \nabla\mathfrak{q} = f + \mathcal{N}(w) & \text{in } \mathbb{T}\times\Omega, \\ \operatorname{div} w = 0 & \text{in } \mathbb{T}\times\Omega, \\ w = 0 & \text{on } \mathbb{T}\times\partial\Omega, \\ \lim_{|x|\to\infty} w(t,x) = 0 & \text{for } t \in \mathbb{R}, \end{cases} \tag{7.2}$$

where

$$\mathcal{N}(w) := -\omega(\mathrm{e}_1 \wedge U - \mathrm{e}_1 \wedge x \cdot \nabla U) + \lambda\partial_1 U + \Delta U - (w + U)\cdot\nabla(w + U).$$

It now suffices to show existence of a solution $(w, \mathfrak{q})$ to (7.2).

On the space $\mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$ we define the norm $\|\cdot\|_{q,\lambda,\omega}$ by

$$\begin{aligned} \|z\|_{q,\lambda,\omega} := \|\nabla^2 z\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} + \|\partial_t z + \omega(\mathrm{e}_1 \wedge z - \mathrm{e}_1 \wedge x \cdot \nabla z)\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} + \lambda\|\partial_1 z\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} \\ + \lambda^{1/2}\|z\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^{2q/(2-q)}(\Omega))} + \lambda^{1/4}\|\nabla z\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^{4q/(4-q)}(\Omega))}. \end{aligned}$$

As in [7, Proof of Theorem 2.3], one then derives the estimates

$$\begin{aligned} \|\mathcal{N}(z)\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq C\big((\lambda+\omega)(1+\lambda+\omega+\|z\|_{q,\lambda,\omega}) + \lambda^{-\frac{3q-3}{q}}\|z\|^2_{q,\lambda,\omega}\big), \\ \|\mathcal{N}(z_1) - \mathcal{N}(z_2)\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} &\leq C\big(\lambda+\omega+\lambda^{-\frac{3q-3}{q}}(\|z_1\|_{q,\lambda,\omega} + \|z_2\|_{q,\lambda,\omega})\big)\|z_1 - z_2\|_{q,\lambda,\omega} \end{aligned}$$

for all $z, z_1, z_2 \in \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$ and $q \in [\frac{12}{11}, \frac{6}{5}]$, where C is independent of λ and ω.

Since we assume $\frac{2\pi}{\mathcal{T}}/\omega \in \mathbb{Q}$, Theorem 3.2 provides the existence of a solution operator $\mathcal{S}\colon \mathrm{L}^q(\mathbb{T}\times\Omega)^3 \to \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$ that maps a right-hand side $f \in \mathrm{L}^q(\mathbb{T}\times\Omega)^3$ onto the velocity field $u \in \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$ of a solution $(u, \mathfrak{p})$ to (1.2). We further introduce the set $A_\delta := \{z \in \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega) : \|z\|_{q,\lambda,\omega} \leq \delta\}$. Then $(w, \mathfrak{q})$ is a solution to (7.2) if w is a fixed point of the mapping $\mathcal{F}\colon A_\delta \to \mathcal{X}^q_{\lambda,\omega}(\mathbb{T}\times\Omega)$, $z \mapsto \mathcal{S}(f + \mathcal{N}(z))$. Now let $\lambda_0 > 0$ and $0 < \omega \leq \omega_0 := \kappa\lambda_0^\rho$ and assume that (3.5) as well as $\|f\|_{\mathrm{A}(\mathbb{T};\mathrm{L}^q(\Omega))} < \varepsilon$ are satisfied. In virtue of the previous estimates and Theorem 3.2, we then obtain a constant C, independent of λ and ω, such that

$$\begin{aligned} \|\mathcal{F}(z)\|_{q,\lambda,\omega} &\leq C\big(\varepsilon + (\lambda+\kappa\lambda^\rho)(1+\lambda+\kappa\lambda^\rho+\delta) + \lambda^{-\frac{3q-3}{q}}\delta^2\big), \\ \|\mathcal{F}(z_1) - \mathcal{F}(z_2)\|_{q,\lambda,\omega} &\leq C\big(\lambda+\kappa\lambda^\rho+\lambda^{-\frac{3q-3}{q}}\delta\big)\|z_1 - z_2\|_{q,\lambda,\omega} \end{aligned}$$

for C independent of λ and ω and for all $z, z_1, z_2 \in A_\delta$. From these estimates one readily derives that $\mathcal{F}\colon A_\delta \to A_\delta$ is a contractive self-mapping if we choose $\varepsilon = \lambda^2$, $\delta = \lambda^\mu$ for some $\mu \in (\frac{3q-3}{q}, \rho)$, and $0 < \lambda \leq \lambda_0$ sufficiently small. The contraction mapping principle then yields the existence of a unique fixed point $w \in A_\delta$, which finally shows the existence of a solution $(u, \mathfrak{p})$ to (1.3). □

8 Construction of the counterexamples

In this section we shall prove Theorem 3.5 and Theorem 3.6. We first address the latter and construct an explicit counterexample, which is a modification of the sequence from the proof of [3, Theorem 3.1], where the Oseen problem (1.5) was considered.

Proof of Theorem 3.6. Fix $\lambda > 0$ and consider a sequence $(\sigma_n) \subset \mathbb{R}$ with $\frac{1}{n} \leq \sigma_n \leq \frac{2}{n}$. Since we assume that $\alpha/\omega \notin \mathbb{Q}$, Corollary 4.3 enables us to choose $\sigma_n = \alpha k_n + \omega \ell_n$ for $k_n,\ \ell_n \in \mathbb{Z}$ with ℓ_n odd. Moreover, we have that $s_n := \alpha k_n$ satisfies $|s_n| \geq \frac{1}{n}$ for all $n \in \mathbb{N}$ sufficiently large. Indeed, if $|s_n| < \frac{1}{n}$, then $0 < \omega \ell_n < \frac{3}{n}$, which is impossible for large $n \in \mathbb{N}$ and $\ell_n \in \mathbb{Z}$.

For simplicity, we denote the spatial Fourier transform of a function $\varphi \in \mathrm{L}^2(\mathbb{R}^3)$ by $\widehat{\varphi} := \mathscr{F}_{\mathbb{R}^3}[\varphi]$. By Plancherel's theorem, we can define a sequence $(G_n) \subset \mathrm{L}^2(\mathbb{R}^3)^3$ by

$$\widehat{G}_n(\xi) := n^{3/2} \mathbb{1}_{I_n}(\xi) \frac{(0, \xi_3, -\xi_2)}{|\xi'|},$$

where $\xi' := (\xi_2, \xi_3)$ and

$$I_n := (\tfrac{1}{2\lambda n}, \tfrac{1}{\lambda n}) \times A_n, \qquad A_n := \big\{\xi' = (\xi_2, \xi_3) \in \mathbb{R}^2 : |\xi'| < \tfrac{1}{n},\, \xi_3 > 0\big\}.$$

We have $|A_n| = \frac{\pi}{2n^2}$ and $|I_n| = \frac{\pi}{4\lambda n^3}$, which yields

$$\|G_n\|_2^2 = \|\widehat{G}_n\|_2^2 = n^3 \big|I_n\big| = \tfrac{\pi}{4\lambda}. \tag{8.1}$$

Now set

$$\widehat{V}_n(\xi) := \frac{1}{i\sigma_n + |\xi|^2 - i\lambda\xi_1} \widehat{G}_n(\xi).$$

Since $\sigma_n \neq 0$, this defines a sequence $(V_n) \subset \mathrm{L}^2(\mathbb{R}^3)^3$ by Plancherel's theorem. With a similar argument, one shows $\nabla^2 V_n \in \mathrm{L}^2(\mathbb{R}^3)$. Additionally, V_n satisfies

$$i\sigma_n V_n - \Delta V_n - \lambda \partial_1 V_n = G_n, \quad \operatorname{div} V_n = 0 \qquad \text{in } \mathbb{R}^3, \tag{8.2}$$

where we used $\xi \cdot \widehat{G}_n(\xi) = 0$. In particular, we also have $\partial_1 V_n \in \mathrm{L}^2(\mathbb{R}^3)$, and V_n satisfies a classical Oseen resolvent problem (with pressure $P_n \equiv 0$). As in our approach in Section 5, we now transform this solution into a solution to the resolvent problem (5.1). To this end, we recall the rotation matrix Q_ω from (5.19), and we let $\mathcal{T}_\omega = \frac{2\pi}{\omega}$ and $\mathbb{T} = \mathbb{R}/\mathcal{T}_\omega\mathbb{Z}$ denote the associated time period and torus group. Set

$$\begin{aligned}
U_n(t,x) &:= V_n(x)\, \mathrm{e}^{i\omega\ell_n t}, & F_n(t,x) &:= G_n(x)\, \mathrm{e}^{i\omega\ell_n t},\\
u_n(t,x) &:= Q_\omega(t)^\top U_n(t, Q_\omega(t)x), & f_n(t,x) &:= Q_\omega(t)^\top F_n(t, Q_\omega(t)x),\\
v_n(x) &:= \int_{\mathbb{T}} u_n(t,x)\,\mathrm{d}t, & g_n(x) &:= \int_{\mathbb{T}} f_n(t,x)\,\mathrm{d}t.
\end{aligned}$$

First of all, we have $\partial_t U_n = i\omega\ell_n V_n = (s_n - i\alpha k_n)V_n$, so that $i\alpha k_n U_n + \partial_t U_n$, $\nabla^2 U_n$, $\partial_1 U_n \in \mathrm{L}^2(\mathbb{T}\times\mathbb{R}^3)$ and

$$i\alpha k_n U_n + \partial_t U_n - \Delta U_n - \lambda\partial_1 U_n = F_n, \quad \operatorname{div} U_n = 0 \qquad \text{in } \mathbb{T}\times\mathbb{R}^3.$$

Mimicking the calculations from the proof of Theorem 5.4, we can further deduce that we have $i\alpha k_n u_n + \partial_t u_n + \omega(\mathrm{e}_1\wedge u_n - \mathrm{e}_1\wedge x\cdot\nabla u_n)$, $\nabla^2 u_n$, $\partial_1 u_n \in \mathrm{L}^2(\mathbb{T}\times\mathbb{R}^3)$ and

$$i\alpha k_n u_n + \partial_t u_n + \omega(\mathrm{e}_1\wedge u_n - \mathrm{e}_1\wedge x\cdot\nabla u_n) - \Delta u_n - \lambda\partial_1 u_n = f_n, \quad \operatorname{div} u_n = 0 \qquad \text{in } \mathbb{T}\times\mathbb{R}^3.$$

Computing the time means over $\mathbb{T}$, we now deduce $i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)$, $\nabla^2 v_n$, $\partial_1 v_n \in \mathrm{L}^2(\mathbb{R}^3)$ and

$$i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n) - \Delta v_n - \lambda\partial_1 v_n = g_n, \quad \operatorname{div} v_n = 0 \qquad \text{in } \mathbb{R}^3.$$

Hence, $(v,p) = (v_n, 0)$ is a solution to the resolvent problem (5.1) with $g = g_n$ and $s = s_n = \alpha k_n \in \alpha\mathbb{Z}$, and it belongs to the asserted function class. It thus remains to show inequality (3.6). At first, let us address the right-hand side. Using that $\mathscr{F}_{\mathbb{R}^3}$ is an isomorphism $\mathrm{L}^2(\mathbb{R}^3)\to\mathrm{L}^2(\mathbb{R}^3)$ and commutes with the transformation induced by $Q_\omega(t)$, we obtain

$$\|g_n\|_2^2 = \int_{\mathbb{R}^3}\Big|\int_{\mathbb{T}} Q_\omega(t)^\top \widehat{G}_n(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2\mathrm{d}\xi = n^3\int_{\mathbb{R}^3}\Big|\int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2\mathrm{d}\xi.$$

To first compute the integral over $\mathbb{T}$, we fix $\xi\in\mathbb{R}^3$ for the moment, let $\eta = (\xi_1, |\xi'|, 0)$ and choose $\varphi\in[0,\mathcal{T}_\omega)$ such that $Q_\omega(\varphi)\eta = \xi$. Employing the equivalence

$$\begin{aligned} Q_\omega(\psi)\eta\in I_n &\iff (\eta_1, |\eta'|, \psi)\in(\tfrac{1}{2\lambda n}, \tfrac{1}{\lambda n})\times(0,\tfrac{1}{n})\times(0,\tfrac{\pi}{\omega}) \\ &\iff (\xi_1, |\xi'|, \psi)\in(\tfrac{1}{2\lambda n}, \tfrac{1}{\lambda n})\times(0,\tfrac{1}{n})\times(0,\tfrac{\pi}{\omega}), \end{aligned}$$

we infer

$$\begin{aligned} \int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t &= \int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t+\varphi)\eta)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t = \mathrm{e}^{-i\omega\ell_n\varphi}\int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t)\eta)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t \\ &= \mathrm{e}^{-i\omega\ell_n\varphi}\;\mathbb{1}_{\left(\frac{1}{2\lambda n},\frac{1}{\lambda n}\right)}(\xi_1)\mathbb{1}_{[0,\frac{1}{n})}(|\xi'|)\,\frac{1}{\mathcal{T}}\int_0^{\frac{\pi}{\omega}} \mathrm{e}^{i\omega\ell_n t}\;\mathrm{d}t \\ &= \frac{1}{2\pi i\ell_n}\,\mathrm{e}^{-i\omega\ell_n\varphi}\left(\mathrm{e}^{i\pi\ell_n}-1\right)\mathbb{1}_{\left(\frac{1}{2\lambda n},\frac{1}{\lambda n}\right)}(\xi_1)\,\mathbb{1}_{\left(0,\frac{1}{n}\right)}(|\xi'|). \end{aligned}$$

Since ℓ_n is an odd number, we have $\mathrm{e}^{i\pi\ell_n} = -1$ and conclude

$$\Big|\int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\;\mathrm{d}t\Big|^2 = \frac{1}{\pi^2\ell_n^2}\,\mathbb{1}_{\left(\frac{1}{2\lambda n},\frac{1}{\lambda n}\right)}(\xi_1)\,\mathbb{1}_{\left(0,\frac{1}{n}\right)}(|\xi'|), \tag{8.3}$$

which leads to

$$\|g_n\|_2^2 = \frac{n^3}{\pi^2\ell_n^2}\int_{\mathbb{R}^3}\mathbb{1}_{\left(\frac{1}{2\lambda n},\frac{1}{\lambda n}\right)}(\xi_1)\,\mathbb{1}_{\left(0,\frac{1}{n}\right)}(|\xi'|)\,\mathrm{d}\xi = \frac{1}{2\lambda\pi\ell_n^2}. \tag{8.4}$$

Now let us turn to the left-hand side of (3.6). Arguing as above, we obtain

$$\|i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)\|_2^2 = s_n^2\int_{\mathbb{R}^3}\Big|\int_{\mathbb{T}} Q_\omega(t)^\top\widehat{V}_n(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2\mathrm{d}\xi. \tag{8.5}$$

Again, we focus on the integral over $\mathbb{T}$ at first. From the definition of $\widehat{V}_n$ and the invariance properties $|Q_\omega(t)\xi| = |\xi|$ and $Q_\omega(t)\,\mathrm{e}_1 = \mathrm{e}_1$, we deduce the identity

$$\begin{aligned}
\Big|\int_{\mathbb{T}} Q_\omega(t)^\top \widehat{V}_n(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2 &= \Big|\int_{\mathbb{T}} \frac{1}{is_n + |\xi|^2 - i\lambda\xi_1} Q_\omega(t)^\top \widehat{G}_n(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2 \\
&= \frac{n^3}{\big|is_n + |\xi|^2 - i\lambda\xi_1\big|^2} \Big|\int_{\mathbb{T}} \mathbb{1}_{I_n}(Q_\omega(t)\xi)\,\mathrm{e}^{i\omega\ell_n t}\,\mathrm{d}t\Big|^2 \\
&= \frac{n^3}{\pi^2\ell_n^2} \frac{1}{\big|is_n + |\xi|^2 - i\lambda\xi_1\big|^2}\, \mathbb{1}_{\left(\frac{1}{2\lambda n},\frac{1}{\lambda n}\right)}(\xi_1)\,\mathbb{1}_{\left(0,\frac{1}{n}\right)}(|\xi'|),
\end{aligned}$$

where we invoked (8.3) for the last equality. For $\frac{1}{2\lambda n} < \xi_1 < \frac{1}{\lambda n}$ and $|\xi'| < \frac{1}{n}$ we have $s_n - \lambda\xi_1 > \frac{1}{n} - \lambda\xi_1 \geq 0$, and we can estimate

$$\begin{aligned}
\frac{1}{\big||\xi|^2 + i(s_n - \lambda\xi_1)\big|^2} &\geq \frac{1}{(\xi_1^2 + \frac{1}{n^2})^2 + (\frac{1}{n} - \lambda\xi_1)^2} \geq \frac{1}{2\xi_1^4 + \frac{2}{n^4} + (\frac{1}{n} - \lambda\xi_1)^2} \\
&\geq \frac{1}{\big(\frac{\sqrt{2n^4\xi_1^4+2}}{n^2} + \frac{1}{n} - \lambda\xi_1\big)^2},
\end{aligned}$$

where the relation $a^2 + b^2 \leq (a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in [0,\infty)$ was employed in the last two estimates. Returning to identity (8.5), we have thus shown

$$\begin{aligned}
\|i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)\|_2^2 &\geq \frac{s_n^2 n^3}{\pi^2\ell_n^2}\int_{\frac{1}{2\lambda n}}^{\frac{1}{\lambda n}}\int_{|\xi'|<\frac{1}{n}} \frac{1}{\big(\frac{\sqrt{2n^4\xi_1^4+2}}{n^2} + \frac{1}{n} - \lambda\xi_1\big)^2}\,\mathrm{d}\xi'\mathrm{d}\xi_1 \\
&= \frac{s_n^2 n}{\pi\ell_n^2}\int_{\frac{1}{2\lambda n}}^{\frac{1}{\lambda n}} \frac{1}{\big(\frac{\sqrt{2n^4\xi_1^4+2}}{n^2} + \frac{1}{n} - \lambda\xi_1\big)^2}\,\mathrm{d}\xi_1.
\end{aligned}$$

With the transformation $\xi_1 = \frac{\theta}{\lambda n}$ we further obtain

$$\begin{aligned}
&\|i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)\|_2^2 = \frac{s_n^2 n^2}{\lambda\pi\ell_n^2}\int_{\frac{1}{2}}^{1} \frac{1}{\left(\frac{1}{n}\sqrt{2+\frac{2\theta^4}{\lambda^4}} + 1 - \theta\right)^2}\,\mathrm{d}\theta \\
&\geq \frac{s_n^2 n^2}{\lambda\pi\ell_n^2}\int_{\frac{1}{2}}^{1} \frac{1}{\left(\frac{1}{n}\sqrt{2+\frac{2}{\lambda^4}} + 1 - \theta\right)^2}\,\mathrm{d}\theta = \frac{s_n^2 n^2}{\lambda\pi\ell_n^2}\left(\frac{n}{\sqrt{2+\frac{2}{\lambda^4}}} - \frac{1}{\frac{1}{n}\sqrt{2+\frac{2}{\lambda^4}} + \frac{1}{2}}\right) \\
&\geq \frac{s_n^2 n^2}{\lambda\pi\ell_n^2}\left(\frac{n}{\sqrt{2+\frac{2}{\lambda^4}}} - 2\right)
\end{aligned}$$

If we choose $n \in \mathbb{N}$ large enough, that is, such that $n \geq 4\sqrt{2+\frac{2}{\lambda^4}}$, then

$$\|i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)\|_2^2 \geq \frac{s_n^2 n^2}{\lambda\pi\ell_n^2}\frac{n}{2\sqrt{2+\frac{2}{\lambda^4}}} = \frac{s_n^2 n^3}{2\lambda\pi\ell_n^2\sqrt{2+\frac{2}{\lambda^4}}}.$$

With $|s_n| \geq \frac{1}{n}$ and (8.4), this yields

$$\|i\alpha k_n v_n + \omega(\mathrm{e}_1\wedge v_n - \mathrm{e}_1\wedge x\cdot\nabla v_n)\|_2^2 \geq \frac{n}{\sqrt{2+\frac{2}{\lambda^4}}}\|g_n\|_2^2,$$

which corresponds to (3.6) with $C = (2 + 2\lambda^{-4})^{-1/4}$. □

To conclude Theorem 3.5, we construct a suitable right-hand side f as a Fourier series defined by means of the sequence (g_n), and we show that the corresponding solutions (v_n, p_n) cannot constitute a Fourier series that is a solution to (1.2) with the desired properties.

Proof of Theorem 3.5. Let us focus on case i. at first. Consider the sequences $(s_n) \subset \frac{2\pi}{\mathcal{T}}\mathbb{Z}$ and $(g_n) \subset \mathrm{L}^2(\mathbb{R}^3)^3$ from Theorem 3.6 and the corresponding sequence of velocity fields (v_n). We may assume that $s_n \neq s_m$ if $n \neq m$ and, by a renormalization, that $\|g_n\|_2 = n^{-3/2}$. Then we can define $f \in \mathrm{A}(\mathbb{T}; \mathrm{L}^2(\mathbb{R}^3)^3)$ by

$$f(t,x) = \sum_{k\in\mathbb{Z}} f_k(x)\,\mathrm{e}^{i\frac{2\pi}{\mathcal{T}}kt}, \qquad f_k = \begin{cases} g_n & \text{if } \frac{2\pi}{\mathcal{T}}k = s_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \frac{2\pi}{\mathcal{T}}k \neq s_n \text{ for all } n \in \mathbb{N}. \end{cases}$$

Now assume that there exists a solution $(u, \mathfrak{p}) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R}^3)^{3+1}$ to (1.2) such that $\partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u)$, $\nabla^2 u$, $\partial_1 u \in \mathrm{A}(\mathbb{T}; \mathrm{L}^2(\mathbb{R}^3)^3)$. Let $(u_k, \mathfrak{p}_k) := (\mathscr{F}_{\mathbb{T}}[u](k), \mathscr{F}_{\mathbb{T}}[\mathfrak{p}](k))$, $k \in \mathbb{Z}$, be the Fourier coefficients of $(u, \mathfrak{p})$. Then

$$i\frac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k) - \Delta u_k + \nabla \mathfrak{p}_k = f_k, \quad \operatorname{div} u_k = 0 \qquad \text{in } \mathbb{R}^3$$

and $i\frac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k)$, $\nabla^2 u_k$, $\partial_1 u_k \in \mathrm{L}^2(\mathbb{R}^3)$. The uniqueness statement from Theorem 5.1 now implies that $i\frac{2\pi}{\mathcal{T}}u_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k) = 0$ if $\frac{2\pi}{\mathcal{T}}k \neq s_n$ for all $n \in \mathbb{N}$, and that

$$i\frac{2\pi}{\mathcal{T}}u_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k) = is_n v_n + \omega(\mathrm{e}_1 \wedge v_n - \mathrm{e}_1 \wedge x \cdot \nabla v_n)$$

if $\frac{2\pi}{\mathcal{T}}k = s_n$ for some $n \in \mathbb{N}$. In virtue of inequality (3.6), we thus conclude

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|i\frac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k)\|_2 &= \sum_{n=1}^{\infty} \|is_n v_n + \omega(\mathrm{e}_1 \wedge v_n - \mathrm{e}_1 \wedge x \cdot \nabla v_n)\|_2 \\ &\geq C \sum_{n=1}^{\infty} n^{1/2}\|g_n\|_2 = \infty. \end{aligned}$$

Hence, the left-hand side is not summable and $\partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u) \notin \mathrm{A}(\mathbb{T}; \mathrm{L}^2(\mathbb{R}^3)^3)$, which contradicts the assumption and completes the proof of i.

For ii. we proceed in the same way but renormalize g_n such that $\|g_n\|_2 = n^{-1}$. By Plancherel's theorem we then obtain $f \in \mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)^3$ and

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|i\frac{2\pi}{\mathcal{T}}ku_k + \omega(\mathrm{e}_1 \wedge u_k - \mathrm{e}_1 \wedge x \cdot \nabla u_k)\|_2^2 &= \sum_{n=1}^{\infty} \|is_n v_n + \omega(\mathrm{e}_1 \wedge v_n - \mathrm{e}_1 \wedge x \cdot \nabla v_n)\|_2^2 \\ &\geq C \sum_{n=1}^{\infty} n\|g_n\|_2^2 = \infty. \end{aligned}$$

This shows $\partial_t u + \omega(\mathrm{e}_1 \wedge u - \mathrm{e}_1 \wedge x \cdot \nabla u) \notin \mathrm{L}^2(\mathbb{T} \times \mathbb{R}^3)^3$ and completes the proof. □

References

[1] F. Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques. *Bull. Soc. Math. Fr.*, 89:43–75, 1961.

[2] K. de Leeuw. On L_p multipliers. *Ann. Math. (2)*, 81:364–379, 1965.

[3] P. Deuring and W. Varnhorn. On Oseen resolvent estimates. *Differential Integral Equations*, 23(11-12):1139–1149, 2010.

[4] R. Edwards and G. Gaudry. *Littlewood-Paley and multiplier theory.* Berlin-Heidelberg-New York: Springer-Verlag, 1977.

[5] T. Eiter. On the Stokes-type resolvent problem associated with time-periodic flow around a rotating obstacle. arXiv:2109.07949, 2021.

[6] T. Eiter and M. Kyed. Time-periodic Linearized Navier–Stokes equations: An Approach based on Fourier Multipliers. In *Particles in flows*, Adv. Math. Fluid Mech., pages 77–137. Birkhäuser/Springer, Cham, 2017.

[7] T. Eiter and M. Kyed. Viscous Flow Around a Rigid Body Performing a Time-periodic Motion. *J. Math. Fluid Mech.*, 23(1):28, 2021.

[8] R. Farwig and J. Neustupa. Spectral properties in L^q of an Oseen operator modelling fluid flow past a rotating body. *Tohoku Math. J. (2)*, 62(2):287–309, 2010.

[9] R. Farwig and H. Sohr. Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. In *Theory of the Navier-Stokes equations*, volume 47 of *Ser. Adv. Math. Appl. Sci.*, pages 11–30. World Sci. Publ., River Edge, NJ, 1998.

[10] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. 2nd ed.* New York: Springer, 2011.

[11] G. P. Galdi. Existence, Uniqueness and Asymptotic Behavior of Regular Time-Periodic Viscous Flow around a Moving Body: Rotational Case. arXiv:2006.03469, 2020.

[12] G. P. Galdi and M. Kyed. A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions. *Proc. Amer. Math. Soc.*, 141(2):573–583, 2013.

[13] G. P. Galdi and M. Kyed. A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part II: Weak solutions. *Proc. Amer. Math. Soc.*, 141(4):1313–1322, 2013.

[14] G. P. Galdi and M. Kyed. Time-periodic flow of a viscous liquid past a body. In *Partial differential equations in fluid mechanics*, volume 452 of *London Math. Soc. Lecture Note Ser.*, pages 20–49. Cambridge Univ. Press, Cambridge, 2018.

[15] G. P. Galdi and A. L. Silvestre. Existence of time-periodic solutions to the Navier-Stokes equations around a moving body. *Pac. J. Math.*, 223(2):251–267, 2006.

[16] M. Geissert, M. Hieber, and T. H. Nguyen. A general approach to time periodic incompressible viscous fluid flow problems. *Arch. Ration. Mech. Anal.*, 220(3):1095–1118, 2016.

[17] L. Grafakos. *Classical Fourier analysis. 2nd ed.* New York, NY: Springer, 2008.

[18] J. G. Heywood. The Navier-Stokes equations: On the existence, regularity and decay of solutions. *Indiana Univ. Math. J.*, 29:639–681, 1980.

[19] M. Kyed. Maximal regularity of the time-periodic linearized Navier-Stokes system. *J. Math. Fluid Mech.*, 16(3):523–538, 2014.

[20] Y. Shibata. On the Oseen semigroup with rotating effect. In *Functional analysis and evolution equations*, pages 595–611. Birkhäuser, Basel, 2008.

[21] M. Yamazaki. The Navier-Stokes equations in the weak-L^n space with time-dependent external force. *Math. Ann.*, 317(4):635–675, 2000.