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Abstract

We study linear evolution equations in separable Hilbert spaces defined by a bounded linear operator. We answer the question which of these equations can be written as a gradient flow, namely those for which the operator is real diagonalisable. The proof is constructive, from which we also derive geodesic lambda-convexity.

1 Introduction

A great amount of literature is dedicated to proving that certain evolution equations can be written as a gradient flow, see for example [JKO98, Maa11, Mie13, CMN19, BCF12]. Such results are important since a gradient flow structure shows that the equation is consistent with thermodynamic principles [Ött05], it facilitates the use of additional tools to prove existence, stability and convergence to equilibrium [AGS08], and it opens up the possibility of using certain numerical schemes [CMW21].

In [BCF12], it is proven that every ordinary differential equation on a finite-dimensional manifold is a gradient flow if it possesses a strict Lyapunov function. Here in this paper, we answer the question of which linear equations of the form

$$\dot{x}(t) = Ax(t) \tag{1.1}$$

in a possibly infinite-dimensional separable Hilbert space H can be written as a gradient flow if the evolution operator $A : H \rightarrow H$ is linear and bounded.

Gradient systems. In this section, we give a short introduction to gradient systems in Hilbert spaces, where we follow the notation of [Mie13]. For a basic and more extensive introduction to this topic, we refer to [CF10].

Since our results will be based on spectral theory we restrict our analysis to separable Hilbert spaces; for brevity we identify H^* with H . Throughout this paper a *gradient system* will be a triple (H, K, \mathcal{F}) where H is a separable Hilbert space, $K(x) : H \rightarrow H$ is a positive definite symmetric operator and $\mathcal{F} : H \rightarrow \mathbb{R}$. For ease of presentation we implicitly assume that K is continuous Gâteaux-differentiable and \mathcal{F} is twice continuous Gâteaux-differentiable. Such K and \mathcal{F} satisfying these properties will be called an *Onsager operator* and a *free energy*, respectively. The aim is to find, given operator A and Hilbert space H , a gradient system (H, K, \mathcal{F}) so that $Ax = -K(x)D\mathcal{F}(x)$ for all $x \in H$. For such a gradient system one can rewrite (1.1) as $\dot{x}(t) = -K(x(t))D\mathcal{F}(x(t)) = -\text{grad}_{x(t)} \mathcal{F}(x(t))$ on the manifold defined by the corresponding metric:

$$d(x_1, x_2)^2 := \inf_{\gamma \in \Gamma(x_1, x_2)} \left\{ \int_0^1 \left[\sup_{\xi \in H} \langle \xi, 2\dot{\gamma}(s) - K(x)\xi \rangle_H \right] ds \right\}, \tag{1.2}$$

with $\Gamma(x_1, x_2)$ denoting the space of all curves $\gamma \in C^1([0, 1], H)$ connecting x_1 and x_2 .

In the special case where the Onsager operator $K(x) \equiv K$ is independent of x and the driving functional is quadratic, i.e. $\mathcal{F}(x) = \frac{1}{2}\langle B(x - \pi), x - \pi \rangle_H$ for some symmetric bounded linear operator $B : H \rightarrow H$ and equilibrium point $\pi \in H$, we shall call (H, K, \mathcal{F}) a *canonical gradient system*.

More generally, we also allow the quadratic form $\frac{1}{2}\langle \xi, K(x)\xi \rangle$ to be replaced by a more general $\Psi^*(x, \xi)$ that is strictly convex in the second argument and satisfies $\inf_{\xi \in H} \Psi^*(x, \xi) = \Psi^*(x, 0) = 0$. Such Ψ^* is called a *dissipation potential*, and we implicitly assume that Ψ^* is continuous Gâteaux-differentiable in the first argument and twice in the second. We then call (H, Ψ^*, \mathcal{F}) a *generalised gradient system* if $\Psi^* : H \times H \rightarrow \mathbb{R}$ is a dissipation potential and $\mathcal{F} : H \rightarrow \mathbb{R}$ is a free energy. Similar as before, we shall assume that besides Ψ^* also \mathcal{F} is twice continuous Gâteaux-differentiable. The evolution equation corresponding to the flow of (H, Ψ^*, \mathcal{F}) is $\dot{x}(t) = D_\xi \Psi^*(x(t), -D\mathcal{F}(x(t)))$, thus giving rise to a possible nonlinear relation between ‘driving forces’ and ‘velocities’. These structures go back at least to [dDvR36], and arise in a large variety of physical and mathematical problems, see [MPR14, PRST20] and the references therein.

There are good reasons to believe that symmetry of A (i.e. reversibility of generators in the setting of Markov processes) is a sufficient condition for the evolution (1.1) to be a gradient flow equation: the foundations of this argument trace back to Boltzmann’s H-Theorem [Bol72] and Onsager’s reciprocity relations [Ons31], but can also be motivated in relation to large-deviation theory [MPR14]. Perhaps surprisingly, it turns out that the class of equations that are the flow of a gradient system is much bigger, namely those equations for which A is real diagonalisable. We illustrate this result in the following paragraph.

Finite-state Markov chains. Consider the Kolmogorov equation of a finite-state continuous-time Markov chain; in that case $A \in \mathbb{R}^{d \times d}$ is a (transposed) generator matrix such that the evolution (1.1) preserves non-negativity and total probability. The discovery of [Maa11] and [Mie13] showed that reversible Markov chains are the flow of a system driven by the relative entropy with respect to the invariant measure. For example if we define

$$A := \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad \mathcal{F}(x) := \sum_{i=1}^3 x_i \log 3x_i,$$

$$K(x) := \frac{1}{3} \begin{bmatrix} \sum_{j \neq 1} \Lambda(3x_1, 3x_j) & -\Lambda(3x_1, 3x_2) & -\Lambda(3x_1, 3x_3) \\ -\Lambda(3x_1, 3x_2) & \sum_{j \neq 2} \Lambda(3x_2, 3x_j) & -\Lambda(3x_1, 3x_3) \\ -\Lambda(3x_1, 3x_3) & -\Lambda(3x_2, 3x_2) & \sum_{j \neq 3} \Lambda(3x_3, 3x_j) \end{bmatrix},$$

where $\Lambda(a, b) := (a - b)/(\log a - \log b)$ is the logarithmic mean, then $K(x)$ is positive definite and symmetric, and indeed $Ax = -K(x)D\mathcal{F}(x)$.

However, [Die15] showed that reversibility is only a necessary assumption *if* the driving functional \mathcal{F} is the relative entropy; when allowing more general functionals, then the existence of a gradient system whose flow is the Kolmogorov forward equation (1.1) is equivalent to real diagonalisability of A . As an illustration, the Markov chain with the following transposed generator is not reversible but

real diagonalisable, and it is the flow of the following (canonical) gradient system:

$$A := \begin{bmatrix} -2 & 0 & 2 \\ 1 & -3 & 2 \\ 1 & 3 & -4 \end{bmatrix}, \quad \mathcal{F}(x) := \frac{1}{6} x^\top \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} x,$$

$$K(x) := \begin{bmatrix} 3 & 3/2 & -3/2 \\ 3/2 & 9/4 & -3/4 \\ -3/2 & -3/4 & 21/4 \end{bmatrix}.$$

Bounded linear operators on a separable Hilbert space. In this work we generalise the result of [Die15] to linear equations (1.1) on a separable Hilbert space, not necessarily conserving non-negativity and total mass. We further generalise by allowing generalised gradient structures as well. Apart from assuming boundedness of the operator A we also require surjectivity. The surjectivity is motivated by the fact that solutions to the linear equation (1.1) are contained in $\text{Ran}(A)$ modulo the initial condition. Hence if A is not surjective we can always restrict it – without too much loss of generality – to the Hilbert space $\text{Ran}(A)$.

Our first main result is the following:

Theorem 1.1. *Let H be a separable Hilbert space, and $A : H \rightarrow H$ a surjective bounded linear operator. Then the following statements are equivalent:*

- (i) $Ax = D_\xi \Psi^*(x, -D\mathcal{F}(x))$ for some generalised gradient system (H, Ψ^*, \mathcal{F}) with some equilibrium $\pi \in H$, i.e. $D\mathcal{F}(\pi) = 0$,
- (ii) $Ax = -K(x)D\mathcal{F}(x)$ for some gradient system (H, K, \mathcal{F}) with some equilibrium $\pi \in H$,
- (iii) $Ax = -KD\mathcal{F}(x)$ for some canonical gradient system (H, K, \mathcal{F}) with some equilibrium $\pi \in H$,
- (iv) A is real diagonalisable, i.e. there exist:

- (1) a measure space $(\Omega, \mathcal{A}, \mu)$,
- (2) an essentially bounded, real-valued $f \in L^\infty_\mu(\Omega)$ with corresponding bounded linear multiplication operator $M_f : L^2_\mu(\Omega) \rightarrow L^2_\mu(\Omega)$ defined by

$$M_f(\varphi)(\omega) := \varphi(\omega)f(\omega) \quad \text{for } \mu\text{-a.e. } \omega \in \Omega,$$

- (3) an invertible bounded linear $V : H \rightarrow L^2_\mu(\Omega)$ so that¹

$$A = V^{-1}M_fV.$$

We postpone the proof of this result to Section 2. Let us mention here that the proof of (iv) \implies (iii) will be based on the explicit construction

$$\mathcal{F}(x) := -\frac{1}{2}(Vx, M_fVx)_{L^2_\mu(\Omega)} \quad \text{and} \quad K := V^{-1}V^{-\top}, \quad (1.3)$$

¹Note that V does not need to be unitary, as often required in the literature [Kub12, Sec. 3.6].

where $V^\top : L^2_\mu(\Omega) \rightarrow H$ is the adjoint of V , and $V^{-\top}$ is the adjoint of the inverse or equivalently the inverse of the adjoint.

Recall from the spectral theorem that any symmetric bounded operator A is real diagonalisable. Hence symmetry is indeed a sufficient condition for the evolution equation (1.1) to be a gradient flow, which is consistent with Onsager theory as mentioned above. However, our Theorem 1.1 shows that the necessary condition is not symmetry but real diagonalisability.

As a byproduct of Theorem 1.1, we derive geodesic λ -convexity of the constructed gradient structure (1.3). The relevance of this property lies in the fact – among other facts – that it implicates a Lipschitz continuous dependence of the solutions on their initial data by the inequality, using the metric (1.2), see for example [AGS08, Th. 4.0.4], [DS14, Th. 2.6]

$$d(x_1(t), x_2(t)) \leq e^{-\lambda t} d(x_1(0), x_2(0)), \quad \text{for all } t \geq 0.$$

When $\lambda \geq 0$, this yields a contraction semigroup, and in the special case $\lambda > 0$, one even gains exponential decay towards the unique equilibrium state π . This provides a new perspective to the classical argument that convergence speed to equilibrium is related to the spectral gap of the operator.

Theorem 1.2. *Let H be a separable Hilbert space and $A : H \rightarrow H$ a surjective, bounded, linear and real diagonalisable operator $A = V^{-1}M_fV$, and let (H, K, \mathcal{F}) be the canonical gradient system (1.3). Then \mathcal{F} is geodesically λ -convex w.r.t. d with*

$$\lambda := - \operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) c_V,$$

where the constant c_V is given by

$$c_V := \begin{cases} \|V^{-1}\|_{L(L^2_\mu(\Omega), H)}^2 \|V\|_{L(H, L^2_\mu(\Omega))}^2, & \text{if } \operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) \geq 0, \\ \|V^{-1}\|_{L(L^2_\mu(\Omega), H)}^{-2} \|V\|_{L(H, L^2_\mu(\Omega))}^{-2}, & \text{else.} \end{cases}$$

If we additionally assume that the measure space $(\Omega, \mathcal{A}, \mu)$ associated to the diagonalisability of A is finite and the spectrum satisfies $\sigma(A) \subset (-\infty, 0]$ then $f \leq 0$ μ -a.e. and so $\lambda \geq 0$.

We give a proof of this theorem in Section 3. After the two proof sections we conclude with a discussion in Section 4.

2 Gradient systems and diagonalisability

This section is devoted our main Theorem 1.1, which is proven in three separate lemmas.

Lemma 2.1. *Statements (i), (ii) and (iii) from Theorem 1.1 are equivalent.*

Proof. Clearly a canonical gradient system (H, K, \mathcal{F}) is also a gradient system, which in turn is a generalised gradient system if we define $\Psi^*(x, \xi) := \frac{1}{2} \langle \xi, K\xi \rangle$. Hence, we only need to show (i) \implies (iii).

Differentiating $Ax = D_\xi \Psi^*(x, -D\mathcal{F}(x))$ in the equilibrium point $\pi \in H$ and using that A is linear yield:

$$\begin{aligned} A &= D_x D_\xi \Psi^*(\pi, -D\mathcal{F}(\pi)) - D^2\mathcal{F}(\pi) D_\xi^2 \Psi^*(\pi, -D\mathcal{F}(\pi)) \\ &= -D_\xi^2 \Psi^*(\pi, 0) D^2\mathcal{F}(\pi), \end{aligned}$$

since $D\mathcal{F}(\pi) = 0$ and $D_\xi \Psi^*(x, 0) \equiv 0$. Define the canonical gradient system by

$$\hat{\mathcal{F}}(x) := \frac{1}{2} \langle D^2\mathcal{F}(\pi)x, x \rangle_H \quad \text{and} \quad \hat{K} := D_\xi^2 \Psi^*(\pi, 0),$$

the latter being positive definite since Ψ^* is assumed to be strictly convex. It follows that for all $x \in H$,

$$Ax = -D_\xi^2 \Psi^*(\pi, 0) D^2\mathcal{F}(\pi)x = -\hat{K} D\hat{\mathcal{F}}(x).$$

□

Lemma 2.2. *Statement (iv) implies (iii) in Theorem 1.1.*

Proof. Let A be bounded, linear and real diagonalisable, see Theorem 1.1, and define \mathcal{F} and K by (1.3). It is easily seen that (H, K, \mathcal{F}) is a canonical gradient system. Indeed, K is independent of x and therefore smooth, it is symmetric due to $K^\top = (V^{-1}V^{-\top})^\top = K$, and positive definite because:

$$\langle \xi, K\xi \rangle_H = \langle \xi, V^{-1}V^{-\top}\xi \rangle_H = (V^{-\top}\xi, V^{-\top}\xi)_{L^2_\mu(\Omega)} > 0 \quad \text{for all } 0 \neq \xi \in H.$$

Further, \mathcal{F} is quadratic since M_f is self-adjoint whenever f is real-valued. In particular we get $D\mathcal{F}(x) = -V^{-\top}M_f Vx \in H$. Now we have

$$Ax = V^{-1}M_f V = V^{-1}V^{-\top}V^\top M_f Vx = -K D\mathcal{F}(x),$$

which was to be demonstrated. □

Lemma 2.3. *Statement (iii) implies (iv) in Theorem 1.1.*

Proof. We are given a canonical gradient system (H, K, \mathcal{F}) with $\mathcal{F}(x) = \frac{1}{2} \langle Bx, x \rangle_H$ so that $Ax = -K D\mathcal{F}(x) = -KBx$ for all $x \in H$. Since K is symmetric positive definite it has a unique symmetric positive definite bounded square root $\sqrt{K} : H \rightarrow H$ such that $K = \sqrt{K}^\top \sqrt{K} = \sqrt{K} \sqrt{K}$ [RSN90, VII.104, Th. 4].

We show that this linear operator \sqrt{K} is invertible. For the injectivity, pick an arbitrary $x \in H$ for which $\sqrt{K}x = 0$. Then

$$\langle Kx, x \rangle_H = \left\langle \sqrt{K}x, \sqrt{K}x \right\rangle_H = 0,$$

and so $x = 0$ by the positive definiteness of K . For the surjectivity, note that $A = -\sqrt{K} \sqrt{K} B$, and so by the assumed surjectivity of A we have $H = \text{Ran}(A) \subset \text{Ran}(\sqrt{K}) \subset H$.

Define the following two bounded linear operators:

$$\bar{A} := \sqrt{K}^{-1} A \sqrt{K} \quad \text{and} \quad \bar{B} := \sqrt{K} B \sqrt{K}.$$

Clearly, \bar{B} is symmetric since B is symmetric. From $A = -KB$ it follows that $\bar{A} = -\bar{B}$, and so \bar{A} is also symmetric.

We now invoke the spectral theorem for symmetric bounded linear operators [Kub12, Th. 3.11] and obtain the existence of a finite measure space $(\Omega, \mathcal{A}, \mu)$, a unitary operator $U : H \rightarrow L^2_\mu(\Omega)$ and a real-valued $f \in L^\infty_\mu(\Omega)$ with corresponding multiplication operator M_f such that $\bar{A} = U^\top M_f U$. After defining the invertible bounded linear operator $V := U\sqrt{K}^{-1} : H \rightarrow L^2_\mu(\Omega)$, we conclude that

$$A = \sqrt{K}\bar{A}\sqrt{K}^{-1} = V^{-1}M_fV.$$

□

Remark 2.4. *Our arguments imply that A is complex diagonalisable if and only if the linear equation can be written as a “complex-valued gradient flow”. It is unclear to us whether such gradient structures have any practical relevance.*

3 Geodesic λ -convexity

The aim of this section is to prove Theorem 1.2: the geodesic λ -convexity of the function \mathcal{F} with respect to the metric d . Throughout this section we assume that A is surjective, bounded, linear and real diagonalisable, and (H, K, \mathcal{F}) is the corresponding canonical gradient system (1.3). It follows from the construction that the inverse $G := K^{-1} : H \rightarrow H$ exists, and so

$$d(x_1, x_2)^2 = \inf_{\gamma \in \Gamma(x_1, x_2)} \left\{ \int_0^1 \langle G\dot{\gamma}(s), \dot{\gamma}(s) \rangle_H ds \right\}. \quad (3.1)$$

Recall [DS14] that geodesic λ -convexity means that the mapping $s \mapsto \mathcal{F}(\gamma(s))$ is λ -convex for all arc-length parametrised (constant speed) geodesics $\gamma : [0, 1] \rightarrow H$, i.e. for all $\theta \in [0, 1]$ we have

$$\mathcal{F}(\gamma(\theta)) \leq (1 - \theta)\mathcal{F}(\gamma(0)) + \theta\mathcal{F}(\gamma(1)) - \lambda \frac{\theta(1 - \theta)}{2} d(\gamma(0), \gamma(1))^2.$$

Before we prove this inequality, we show the following lemma. For brevity we shall omit the indices from the norms $\|\cdot\|_{L(L^2_\mu(\Omega), H)}$ and $\|\cdot\|_{L(H, L^2_\mu(\Omega))}$.

Lemma 3.1. *The free energy functional \mathcal{F} from (1.3) is $\tilde{\lambda}$ -convex in the Hilbert space H , where*

$$\tilde{\lambda} := -\operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) \tilde{c}_V, \quad \text{and} \quad \tilde{c}_V := \begin{cases} \|V\|^2, & \text{if } \operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) \geq 0, \\ \|V^{-1}\|^{-2}, & \text{otherwise.} \end{cases}$$

Proof. It holds

$$\begin{aligned} \langle D\mathcal{F}(x_1) - D\mathcal{F}(x_2), x_1 - x_2 \rangle_H &= -\langle V^\top M_f V(x_1 - x_2), x_1 - x_2 \rangle_H \\ &= -(M_f V(x_1 - x_2), V(x_1 - x_2))_{L^2_\mu(\Omega)} = -\int_\Omega f(\omega) (V(x - y)(\omega))^2 d\mu(\omega) \\ &\geq -\operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) \|V(x_1 - x_2)\|_{L^2_\mu(\Omega)}^2 \geq -\operatorname{ess\,sup}_{\omega \in \Omega} f(\omega) \tilde{c}_V \|x_1 - x_2\|_H^2, \end{aligned}$$

which is equivalent to $\tilde{\lambda}$ -convexity since \mathcal{F} is Gâteaux-differentiable (see [Pey15, Prop. 3.12]). □

Next, we show that in our case geodesic λ -convexity can be obtained from $\tilde{\lambda}$ -convexity in H . More precisely, we have the following lemma:

Lemma 3.2. *Let d be the metric (3.1) with metric tensor $G(x) := K^{-1}(x) = V^T V$, and let $\mathcal{F} : H \rightarrow \mathbb{R}$ be a $\tilde{\lambda}$ -convex functional in H . Then \mathcal{F} is geodesically λ -convex with respect to the metric d , where*

$$\lambda := \begin{cases} \tilde{\lambda} \|V\|^{-2}, & \text{if } \tilde{\lambda} > 0, \\ \tilde{\lambda} \|V^{-1}\|^2, & \text{if } \tilde{\lambda} \leq 0. \end{cases}$$

Proof. Let $x_1, x_2 \in H$ be arbitrary and let $\gamma \in \Gamma(x_1, x_2)$ be a geodesic connecting x_1 and x_2 . As K^{-1} is independent of x , the metric space is flat, and the geodesic between x_1 and x_2 is a straight line $\gamma(s) = (1-s)x_1 + sx_2$. Therefore

$$d(x_1, x_2)^2 \stackrel{(3.1)}{=} \int_0^1 \langle V^T V \dot{\gamma}(s), \dot{\gamma}(s) \rangle_H ds = \|V(x_2 - x_1)\|_{L_\mu^2}^2.$$

This yields the estimates

$$\frac{1}{\|V^{-1}\|} \|x_2 - x_1\|_H \leq d(x_1, x_2) \leq \|V\| \|x_2 - x_1\|_H. \quad (3.2)$$

First consider the case $\tilde{\lambda} \leq 0$. Exploiting the $\tilde{\lambda}$ -convexity of \mathcal{F} , we can bound for arbitrary $\theta \in [0, 1]$,

$$\begin{aligned} \mathcal{F}(\gamma(\theta)) &= \mathcal{F}((1-\theta)x_1 + \theta x_2) \\ &\leq (1-\theta)\mathcal{F}(x_1) + \theta\mathcal{F}(x_2) - \frac{\tilde{\lambda}(1-\theta)\theta}{2} \|x_2 - x_1\|_H^2 \\ &\stackrel{(3.2)}{\leq} (1-\theta)\mathcal{F}(x_1) + \theta\mathcal{F}(x_2) - \frac{\tilde{\lambda}\|V^{-1}\|^2(1-\theta)\theta}{2} d(x_1, x_2)^2 \\ &= (1-\theta)\mathcal{F}(\gamma(0)) + \theta\mathcal{F}(\gamma(1)) - \frac{\lambda(1-\theta)\theta}{2} d(\gamma(0), \gamma(1))^2. \end{aligned}$$

Similarly for $\tilde{\lambda} > 0$,

$$\begin{aligned} \mathcal{F}(\gamma(\theta)) &\stackrel{(3.2)}{\leq} (1-\theta)\mathcal{F}(x_1) + \theta\mathcal{F}(x_2) - \frac{\tilde{\lambda}(1-\theta)\theta}{2\|V\|^2} d(x_1, x_2)^2 \\ &= (1-\theta)\mathcal{F}(\gamma(0)) + \theta\mathcal{F}(\gamma(1)) - \frac{\lambda(1-\theta)\theta}{2} d(\gamma(0), \gamma(1))^2. \end{aligned}$$

□

Putting the two lemmas together, we proved the first part of Theorem 1.2, the geodesically λ -convexity of \mathcal{F} in the metric space (H, d) . The last claim is stated in the following lemma:

Lemma 3.3. *Under the additional assumption that the measure space $(\Omega, \mathcal{A}, \mu)$ associated to the diagonalisability of A is finite and the spectrum satisfies $\sigma(A) \subset (-\infty, 0]$ then $f \leq 0$ μ -a.e..*

Proof. First, as the measure space $(\Omega, \mathcal{A}, \mu)$ is finite, the spectrum of M_f is given by the essential range of f (see [Kna08, Lemma 4.55] for a proof). Second, we use that similar operators have the same spectrum, and obtain

$$\text{ess ran}(f) = \sigma(M_f) = \sigma(V^{-1}AV) = \sigma(A) \subset (-\infty, 0].$$

□

4 Discussion

We end our paper with a discussion about our assumptions and results.

First, our work is restricted to separable Hilbert spaces H because our arguments are based on spectral theory. Further, the assumed boundedness of the operator A can be generalised to a large extent. However, it turns out that for unbounded operators the free energy \mathcal{F} generally fails to be Gâteaux differentiable. Our assumptions that both \mathcal{F} and $\Psi^*(x, \cdot)$ need to be twice Gâteaux differentiable can probably be generalised by approximation arguments, but the once differentiability of \mathcal{F} is fundamental in our definition of a gradient flow. This hints that possible workarounds to allow for unbounded operators may be found by using more general definitions of gradient flows that do not require differentiability, like “curves of maximal slope” and “minimising movements” [AGS08]. This is beyond the scope of the current paper.

Let us also mention that our restriction to linear equations is essential, again because we couple gradient systems to spectral theory, but also because we use a linearisation around the equilibrium to simplify any (generalised) gradient system to a canonical gradient system. Therefore we expect our results to be approximately true for nonlinear equations near the equilibrium state.

Finally, the gradient system that we construct for real diagonalisable operators could be perceived as rather unnatural. After all, canonical gradient systems have quadratic free energy and do not allow for entropic expressions. In this regard, we stress that our main Theorem 1.1 is an existence result without any claim of uniqueness.

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