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Abstract

Based on a recent work that exposed the lack of uniformly bounded $W^{1,p} \to W^{1,p}$ extension operators on randomly perforated domains, we study stochastic homogenization of nonlinear elasticity on such structures using instead the extension operators constructed in [11]. We thereby introduce two-scale convergence methods on such random domains under the intrinsic loss of regularity and prove some generally useful calculus theorems on the probability space Ω, e.g. abstract Gauss theorems.

1 Introduction

Homogenization of elasticity problems has a long history with a first stochastic result provided in [5] for pure bulk homognenization of linear elasticity. Further work in this direction have been published in between and we refer to the recent work [11] for an overview.

In this work, we consider homogenization of p-elasticity with nonlinear bulk terms on perforated domains and with nonlinear Robin conditions on the microscale. More precisely, let $\mathbf{P}(\omega) \subset \mathbb{R}^d$ be a stationary random Lipschitz domain and let $\varepsilon > 0$ be the smallness parameter and assume w.o.l.g $\mathbf{P}(\omega)$ is almost surely connected and has locally Lipschitz boundary.

For a bounded domain $\mathbf{Q} \subset \mathbb{R}^d$, we consider $\mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega) := \mathbf{Q} \cap \varepsilon\mathbf{P}(\omega)$ and $\Gamma^{\varepsilon}(\omega) := \mathbf{Q} \cap \varepsilon\partial\mathbf{P}(\omega)$ with outer normal $\nu_{\Gamma^{\varepsilon}(\omega)}$. For $u^{\varepsilon}:\ \mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega) \to \mathbb{R}^d$ we then consider

$$\begin{aligned} -\operatorname{div}\left(a\left|\nabla^{s} u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon}\right) &= g(u^{\varepsilon}) && \text{on } \mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega)\,, \\ u &= 0 && \text{on } \partial\mathbf{Q}\,, \\ \left|\nabla^{s} u^{\varepsilon}\right|^{p-2} \nabla u^{\varepsilon} \cdot \nu_{\Gamma^{\varepsilon}(\omega)} &= \varepsilon f(u^{\varepsilon}) && \text{on } \Gamma^{\varepsilon}(\omega)\,, \end{aligned} \tag{1}$$

where $\nabla^{s} u := \frac{1}{2}\left(\nabla u + (\nabla u)^{\top}\right)$ is the symmetrized gradient. The parameter a might be a random variable but this is not relevant for our investigation and we assume $a \equiv 1$ for simplicity.

As well known, problem (1) can be recast into a variational problem, i.e. solutions of (1) are local minimizers of the energy functional

$$\mathcal{E}_{\varepsilon,\omega}(u) = \int_{\mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega)} \left(\left|\nabla^{s} u\right|^{p} - G(u)\right) + \varepsilon \int_{\Gamma^{\varepsilon}(\omega)} F(u)\,,$$

where $F' = pf$ and $G' = pg$. If F and G both are Hölder continuous functions, there exist minimizers of $\mathcal{E}_{\varepsilon,\omega}$ for every $\varepsilon > 0$ in the space

$$\boldsymbol{W}_{0,\partial\mathbf{Q}}^{1,p}(\mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega)) := \left\{u \in \boldsymbol{W}^{1,p}(\mathbf{Q}_{\mathbf{P}}^{\varepsilon}(\omega))\ :\ u|_{\partial\mathbf{Q}\cap(\varepsilon\mathbf{P}(\omega))} \equiv 0\right\}\,,$$

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where the bold symbol $\boldsymbol{W}$ indicates $\mathbb{R}^d$-valued functions and normal symbols like $W^{1,p}$ will indicate $\mathbb{R}$-valued functions, if used.

In periodic homogenization, a lot of effort has been made to prove the existence of an extension operator, which have properties that - transfered to the case of stochastic homogenization - would read as follows: there exists an extension operator $\mathcal{U}_{\varepsilon,\omega}\,:\,W^{1,p}_{\rm loc}(\varepsilon\tilde{\mathbf{P}}(\omega))\to W^{1,p}_{\rm loc}(\mathbb{R}^d)$ such that $\mathrm{supp}\mathcal{U}_{\varepsilon,\omega}u\subset\mathbb{B}_\varepsilon(\mathbf{Q})$ for every $u\in\boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$ and for some constant $C(\omega)>0$ independent from ε it holds

$$\forall u\in\boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)):\quad \left\|\nabla^s\left(\mathcal{U}_{\varepsilon,\omega}u\right)\right\|_{L^p(\mathbb{R}^d)}\leq C(\omega)\left\|\nabla^s u\right\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}\,,\quad \mathrm{supp}\mathcal{U}_\varepsilon u\subset\mathbb{B}_\varepsilon(\mathbf{Q})\,. \tag{2}$$

Together with the classical Poincaré and Korn inequality, (2) establishes

$$\|u\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}\leq C\left\|\nabla^s\mathcal{U}_{\varepsilon,\omega}u\right\|_{L^p(\mathbb{R}^d)}\leq C(\omega)\left\|\nabla^s u\right\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}\,, \tag{3}$$

uniformly in ε and (conceptually this) will then allow to perform homogenization by Γ-convergence in the space $\boldsymbol{W}^{1,p}(\mathbf{Q})$ with a limit functional $\mathcal{E}_{\rm hom}(u)$ similar to the one established in Theorem 1.3 below.

However, in a recent work [11] the author has shown that (2) *has to fail* for general random geometries. This is because random geometries can have arbitrary bad local Lipschitz regularity thereby violating to be uniformly John regular. However, as can be seen from [4, 14] a uniform John property is necessary in order for (2) to hold.

On the other hand, in the same paper it was shown there is still hope to find $\mathcal{U}_\varepsilon\,:\,W^{1,p}_{\rm loc}(\varepsilon\tilde{\mathbf{P}}(\omega))\to W^{1,p}_{\rm loc}(\mathbb{R}^d)$ satisfying the strong symmetric (r,p)-extension property, $1\leq r<p$, as introduced in the following definition.

Definition 1.1. A stationary random geometry has the *weak* (r,p)-extension property if there almost surely exists $C>0$ and an extension operator $\mathcal{U}_\varepsilon\,:\,W^{1,p}_{\rm loc}(\varepsilon\tilde{\mathbf{P}}(\omega))\to W^{1,p}(\mathbb{R}^d)$ such that for every bounded domain $\mathbf{Q}\subset\mathbb{R}^d$ and every $u\in W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\varepsilon\mathbf{P}(\omega))$ it holds

$$\|\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}+\|\varepsilon\nabla\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C\left(\|u\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\varepsilon\mathbf{P}(\omega))}+\|\varepsilon\nabla u\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\varepsilon\mathbf{P}(\omega))}\right)\,.$$

A stationary random geometry has the *strong* (symmetric) (r,p)-extension property if additionally there almost surely exists $\beta\in(0,1)$, $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and for every $u\in\boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}\cap\varepsilon\tilde{\mathbf{P}}(\omega))$ the support of $\mathcal{U}_\varepsilon u$ lies within $\mathbb{B}_{\varepsilon^\beta}(\mathbf{Q})$ and it holds

$$\|\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C\,\|u\|_{L^p(\mathbf{Q}\cap\varepsilon\tilde{\mathbf{P}}(\omega))}$$

with $\left(\mathcal{U}_\varepsilon u\right)|_{\mathbb{R}^d\backslash\mathbb{B}_\varepsilon(\mathbf{Q})}\equiv 0$ and either

$$\|\nabla\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C\,\|\nabla u\|_{L^p(\mathbf{Q}\cap\varepsilon\tilde{\mathbf{P}}(\omega))}\,,$$
$$\left(\text{resp.}\quad\|\nabla^s\mathcal{U}_\varepsilon u\|_{L^r(\mathbf{Q})}\leq C\,\|\nabla^s u\|_{L^p(\mathbf{Q}\cap\varepsilon\tilde{\mathbf{P}}(\omega))}\right)\,.$$

We emphasize that [11] yields also the following concept for traces

Definition 1.2. A stationary random geometry has the (r,p)-trace property if for almost every ω there exists $C_\omega>0$ such that the trace operator $\mathcal{T}\,:\,W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))\to L^r(\mathbf{Q}\cap\varepsilon\Gamma(\omega))$ satisfies

$$\varepsilon\,\|\mathcal{T}u\|^r_{L^r(\mathbf{Q}\cap\varepsilon\Gamma(\omega))}\leq C_\omega\left(\|u\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}+\varepsilon\,\|\nabla u\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}\right)^{\frac{r}{p}} \tag{4}$$

and $\mathbb{E}(C_\omega^{\frac{p}{p-r}})<\infty$.

In [11] the above inequalities for weak and strong (r,p)-extensions as well as the trace property have been verified in an unscaled form for a pipe model and a Boolean model.

It turns out there is a further property which has to be verified in order to guaranty regularity properties of solutions and may - additionally - be important in some other applications beyond the scope of this work. Using the notation [11] this property is the following:

$$\forall i=1,\dots,d: \qquad \operatorname{dist}\left(\mathbf{e}_i, L^2_{\mathrm{pot}}(\mathbf{P})\right) > 0\,. \tag{5}$$

We will close this introduction with our main theorem and some final explanation. For the underlying notation of $\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ we refer to Definition 2.6.

Theorem 1.3. *Let F and G be Hölder continuous and bounded from below, let $p>1$ and let $\mathbf{P}$ be a stationary ergodic random connected open set. Then for every $\varepsilon>0$ the functional $\mathcal{E}_{\varepsilon,\omega}$ has a unique minimizer $u^\varepsilon \in \boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$. If $\mathbf{P}$ has the symmetric (r,p) extension property, $1<r\leq p$ and the (s,r) trace property $1\leq s\leq r$, then $u^\varepsilon \rightharpoonup |\mathbf{P}|\,u$ weakly in $L^r(\mathbf{Q})$ as $\varepsilon\to 0$, where $u\in \boldsymbol{W}^{1,r}_0(\mathbf{Q})$ is a minimizer of the functional*

$$\mathcal{E}_{\mathrm{hom}}(u) := \inf_{v\in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}))} \int_{\mathbf{Q}}\int_{\mathbf{P}} a\left|\nabla^{\mathfrak{s}}u + v^{\mathfrak{s}}\right|^p - \int_{\mathbf{Q}}\int_{\mathbf{P}} G(u) + \int_{\mathbf{Q}}\int_{\Gamma} F(u)\mathrm{d}\mu_{\Gamma,\mathcal{P}}$$

in the space $u\in \boldsymbol{W}^{1,r}_0(\mathbf{Q})$. Furthermore, if (5) holds then $u\in \boldsymbol{W}^{1,p}_0(\mathbf{Q})$ is a minimizer of $\mathcal{E}_{\mathrm{hom}}$ in this space.

Proof. This is a coarse reformulation of Theorem 3.14 below. □

We first observe that the above limit functional is exactly what we would expect from the "classical" results. However, it is not trivial: It is not clear at all that the apriori bound on the sequence of symmetric gradients implies that 1. u^ε converges in any strong sense at all (which requires some kind of Korn and Poincaré property) to a limit function $\tilde{u}$ and 2. that the limit function u is a minimizer of the expected limit functional, i.e. $u=\tilde{u}$, because on the way there we necessarily loose some order of integrability. Furthermore, the L^p-regularity of ∇u can be inferred from boundedness of $\mathcal{E}_{\mathrm{hom}}$ only if (5) holds true.

2 Sobolev Spaces on the Probability Space $(\Omega,\mathbb{P})$

Assumption 2.1. *Let Ω be a precompact metric space with Borel sigma-algebra σ and a probability measure $\mathbb{P}$. Assume there is a family $(\tau_x)_{x\in\mathbb{R}^d}$ (called a dynamical system) of measurable bijective mappings $\tau_x:\Omega\mapsto\Omega$ satisfying (i)-(iii):*

(i) $\tau_x\circ\tau_y = \tau_{x+y}$, $\tau_0 = id$ *(Group property)*

(ii) $\mathbb{P}(\tau_{-x}B)=\mathbb{P}(B) \quad \forall x\in\mathbb{R}^d,\ B\in\mathscr{F}$ *(Measure preserving)*

(iii) $A:\ \mathbb{R}^d\times\Omega\to\Omega \qquad (x,\omega)\mapsto\tau_x\omega$ *is continuous (Continuity of evaluation)*

Definition 2.2. The dynamical system τ called is ergodic if $\mathbb{P}\left((A\cup\tau_x A)\setminus(A\cap\tau_x A)\right)=0$ implies $\mathbb{P}(A)\in\{0,1\}$. If X is a measurable space and $f:\Omega\times\mathbb{R}^d\to X$, then f is called (weakly) stationary if $f(\omega,x)=f(\tau_x\omega,0)$ for (almost) every x.

It has been shown in the recent work [10] that the assumption 2.1 is met by many coefficient fields that relate to applications. Furthermore, it allows to use the common results in the literature, i.e. [6, 7, 17, 19, 18] and to draw some conclusions on functions spaces which we summarize in the following.

We find $C(\overline{\Omega})$ to be separable and dense in $L^p(\Omega;\mu)$, $1\leq p<\infty$, μ a Borel measure on Ω and every such $L^p(\Omega;\mu)$ hence is separable. For $f:\Omega\to X$, X a metric space, and $\omega\in\Omega$ we define the *realization* f_ω of f as

$$f_\omega:\mathbb{R}^d\to X\,,\qquad x\mapsto f(\tau_x\omega)\,.$$

If $f\in L^p(\Omega)$ for $1\leq p\leq\infty$, then for almost every $\omega\in\Omega$ and for every bounded domain $\mathbf{Q}$ it holds $f_\omega\in L^p(\mathbf{Q})$ [19]. Given the canonical basis $(\mathbf{e}_i)_{i=1,\dots,d}$ of $\mathbb{R}^d$, we define the operators

$$\mathrm{D}_i f(\omega)=\lim_{t\to 0}\frac{f(\tau_{t\mathbf{e}_i}\omega)-f(\omega)}{t}$$

and $\mathrm{D}_i f$ is called i-th derivative of f having the property

$$\int_\Omega g\mathrm{D}_i f\mathrm{d}\mathbb{P}=-\int_\Omega f\mathrm{D}_i g\mathrm{d}\mathbb{P}\,.$$

The joint domain of all D_i equiped with the operator norm in $L^p(\Omega)$ is a Banach space

$$W^{1,p}(\Omega):=\{f\in L^p(\Omega)\,|\,\forall i=1,\dots,d\;:\;\mathrm{D}_i f\in L^p(\Omega)\}\,,$$
$$\|f\|_{W^{1,p}(\Omega)}:=\|f\|_{L^p(\Omega)}+\sum_{i=1}^d\|\mathrm{D}_i f\|_{L^p(\Omega)}\,.$$

We finally denote $\mathrm{D}_\omega f:=(\mathrm{D}_1 f,\dots,\mathrm{D}_d f)^T$ the gradient with respect to ω and by $-\mathrm{div}_\omega$ the adjoint of D_ω. Sometimes we write $\nabla_\omega f:=\mathrm{D}_\omega f$ to underline the gradient aspect. We further denote

$$C^1(\overline{\Omega}):=\left\{f\in C(\overline{\Omega})\,:\;\mathrm{D}_\omega f\in C(\overline{\Omega};\mathbb{R}^d)\right\}$$

and note that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $1\leq p<\infty$. We define

$$\mathcal{V}^p_{\mathrm{pot}}(\Omega)=\mathrm{closure}_{L^p}\left\{\mathrm{D}u\,|\,u\in W^{1,p}(\Omega)\right\}\,,\tag{6}$$

and observe that for

$$L^p_{\mathrm{pot,loc}}(\mathbb{R}^d):=\left\{u\in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\;|\;\forall\mathbf{U}\text{ bounded domain, }\exists\varphi\in W^{1,p}(\mathbf{U})\;:\;u=\nabla\varphi\right\}\,,$$

we find the characterization

$$\mathcal{V}^p_{\mathrm{pot}}(\Omega)=\left\{u\in L^p(\Omega;\mathbb{R}^d)\backslash\mathbb{R}^d\;:\;u_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)\text{ for }\mathbb{P}-\text{a.e. }\omega\in\Omega\right\}\,.$$

2.1 Random Sets, Random Measures and Palm Theory

A random set is a random variable with values in the space of Radon measures in $\mathbb{R}^d$. More precisely, $\omega\mapsto\mu_\omega$ is a random measure if for every bounded Borel set $A\subset\mathbb{R}^d$ or alternatively for every $f\in C_c(\mathbb{R}^d)$ the following respective maps are measureable

$$\omega\mapsto\mu_\omega(A)\,,\qquad\text{or}\qquad\omega\mapsto\int f\,\mathrm{d}\mu_\omega\,.$$

If for every bounded $A \subset \mathbb{R}^d$ the distribution of $\mu_\omega(A)$ is invariant under translations of A we call μ_ω stationary. By Mecke's theorem (see [15, 2]) the measure

$$\mu_{\mathcal{P}}(A) = \int_\Omega \int_{\mathbb{R}^d} g(s)\, \chi_A(\tau_s\omega)\, \mathrm{d}\mu_\omega(s)\, \mathrm{d}\mathbb{P}(\omega)$$

can be defined on Ω for every positive $g \in L^1(\mathbb{R}^d)$ with compact support and is called Palm measure. $\mu_{\mathcal{P}}$ is independent from g and in case $\mu_\omega = \mathcal{L}$ we find $\mu_{\mathcal{P}} = \mathbb{P}$. The Campbell formula for $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Omega)$-measurable non negative functions f reads

$$\int_\Omega \int_{\mathbb{R}^d} f(x, \tau_x\omega)\, \mathrm{d}\mu_\omega(x)\, \mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d} \int_\Omega f(x,\omega)\, \mathrm{d}\mu_{\mathcal{P}}(\omega)\, \mathrm{d}x\,,$$

and we say μ_ω has finite intensity if $\mu_{\mathcal{P}}(\Omega) < +\infty$.

Theorem 2.3 (General Ergodic Theorem for the Lebesgue measure)**.** *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathbf{Q} \subset \mathbb{R}^d$ be a bounded open set with $0 \in \mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f \in L^p(\Omega; \mu_{\mathcal{P}})$ and $\varphi \in L^q(\mathbf{Q})$, where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\mathbb{P}$-almost all $\omega \in \Omega$ it holds*

$$n^{-d} \int_{n\mathbf{Q}} \varphi(\frac{x}{n}) f(\tau_x\omega)\, \mathrm{d}\mu_\omega(x) \to \int_\Gamma \int_{\mathbf{Q}} f\varphi\, \mathrm{d}x\, \mathrm{d}\mu_{\Gamma,\mathcal{P}}\,. \tag{7}$$

The relation between random closed sets and random measures as well as the importance for homogenization theory have been outlined in many places before [7, 8]and we will not go into detail on this. For this work, the most important is the following:

If $\mathbf{P}(\omega)$ is a random open set with boundary $\Gamma(\omega) := \partial\mathbf{P}(\Omega)$ then the measures

$$\mu_\omega(A) := \mathcal{L}(A \cap \mathbf{P}(\omega))\,, \qquad \mu_{\Gamma(\omega)}(A) := \mathcal{H}^{d-1}(A \cap \Gamma(\omega))$$

are random measures, where $\mathcal{L}$ is the Lebesgue measure and $\mathcal{H}^{d-1}$ is the $d-1$ dimensional Hausdorff measure. The respective Palm measures will be denoted by $\mu_{\mathcal{P}}$ and $\mu_{\Gamma,\mathcal{P}}$.

An important observation made in [7] is the following.

Lemma 2.4. *There exists $\Gamma \subset \Omega$ and $\mathbf{P} \subset \Omega$ with characteristic functions $\chi_\Gamma(\omega)$ and $\chi_{\mathbf{P}}(\omega)$ such that the following holds: For almost every ω it holds $\chi_{\mathbf{P}(\omega)}(x) = \chi_{\mathbf{P}}(\tau_x\omega)$ in the sense of Lebesgue and $\chi_{\Gamma(\omega)}(x) = \chi_\Gamma(\tau_x\omega)$ in the Hausdorff sense. Furthermore, $\mathbb{P}(\chi_\Gamma) = 0$, $\mu_{\Gamma,\mathcal{P}}(\Omega\backslash\Gamma) = 0$ and $\mu_{\Gamma,\mathcal{P}}(\Gamma) = \mathbb{E}\big(\mu_{\Gamma(\omega)}(0,1)^d\big)$.*

2.2 Traces and Extensions

Assumption 2.5. *Under the Assumption 2.1 let $\mathbf{P}(\omega)$ be a random open set with boundary $\Gamma(\omega) := \partial\mathbf{P}(\omega)$ such that $\Gamma(\omega)$ is a random closed set. The corresponding prototypes $\mathbf{P}, \Gamma \subset \Omega$ in the sense of Section 2.1 have Palm measures $\chi_{\mathbf{P}}\mathbb{P}$ and $\mu_{\Gamma,\mathcal{P}}$ respectively.*

Definition 2.6. Under the Assumption 2.5 we introduce for $1 \leq p \leq \infty$ the space

$$\begin{aligned} W^{1,p}(\mathbf{P}) &:= \mathrm{closure}_{\|\cdot\|_{W^{1,p}(\mathbf{P})}} \left\{\chi_{\mathbf{P}} u:\ u \in C^1(\overline{\Omega})\right\} \\ \|u\|_{W^{1,p}(\mathbf{P})} &:= \|u\|_{L^p(\mathbf{P})} + \|\mathrm{D}u\|_{L^p(\mathbf{P})}\,. \end{aligned}$$

Furthermore, we define for $r \leq p$

$$\begin{aligned} W^{1,r,p}(\Omega,\mathbf{P}) &:= \left\{ u \in W^{1,r}(\Omega):\ u|_{\mathbf{P}} \in L^p(\mathbf{P}),\ \mathrm{D}_\omega u \in L^p(\mathbf{P};\mathbb{R}^d) \right\} \,, \\ \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}) &:= \mathrm{closure}_{L^p} \left\{ \mathrm{D}u \mid u \in W^{1,p}(\mathbf{P}) \right\} \,, \\ \mathcal{V}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P}) &:= \left\{ \mathrm{D}u \in \mathcal{V}^r_{\mathrm{pot}}(\Omega) \mid \mathrm{D}u \in \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}) \right\} \,. \end{aligned}$$

Similarly we define $\boldsymbol{W}^{1,p}(\mathbf{P})$ and $\boldsymbol{W}^{1,r,p}(\Omega,\mathbf{P})$ as well as $\boldsymbol{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P})$ and $\boldsymbol{\mathcal{V}}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})$ for vector valued functions. For $\upsilon \in \boldsymbol{\mathcal{V}}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})^d$ we define $\upsilon^{\mathfrak{s}} := \frac{1}{2}\left(\upsilon + \upsilon^\top\right)$ and

$$\boldsymbol{\mathcal{V}}^{r,p}_{\mathrm{pot},\mathfrak{s}}(\Omega,\mathbf{P}) := \left\{ \upsilon^{\mathfrak{s}} :\ \upsilon \in \boldsymbol{\mathcal{V}}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})^d \right\}$$

and similar $\boldsymbol{\mathcal{V}}^p_{\mathrm{pot},\mathfrak{s}}(\mathbf{P})$.

We observe that $C^1(\overline{\Omega})$ is dense in $\boldsymbol{W}^{1,p}(\mathbf{P})$ [10] and hence $C^1(\overline{\Omega})$ is dense in $W^{1,r,p}(\Omega,\mathbf{P})$ because of $W^{1,r}(\Omega) \supset W^{1,r,p}(\Omega,\mathbf{P}) \supset W^{1,p}(\Omega)$.

For $u \in C^1(\overline{\Omega})$ we can define $\mathcal{T}_\Omega[u] := u|_\Gamma$ and observe that (4) implies for every $R > 1$

$$\left\| (\mathcal{T}_\Omega[u])_\omega \right\|_{L^r(\varepsilon\Gamma(\omega)\cap\mathbb{B}_1(0))} \leq C \left(\|u_\omega\|_{L^p(\varepsilon\mathbf{P}(\omega)\cap\mathbb{B}_{1+\varepsilon}(0))} + \|\nabla u_\omega\|_{L^p(\varepsilon\mathbf{P}(\omega)\cap\mathbb{B}_{1+\varepsilon}(0))} \right) \,,$$

which yields by the ergodic theorem

$$\left\| \mathcal{T}_\Omega[u] \right\|_{L^r(\Gamma)} \leq C \left\| u \right\|_{W^{1,p}(\mathbf{P})}$$

and the operator $\mathcal{T}_\Omega$ can be extended to $W^{1,p}(\mathbf{P})$ by density for every $1 \leq p < \infty$. We furthermore find the following properties.

Theorem 2.7. *Let Assumption 2.5 hold and let $\mathbf{P}(\omega)$ have the weak (r,p)-extension property. Then there exists a continuous linear operator $\mathcal{U}_\Omega : W^{1,p}(\mathbf{P}) \to W^{1,r}(\Omega)$ such that $(\mathcal{U}_\Omega u)|_{\mathbf{P}} = u$.*

Theorem 2.8. *Let Assumption 2.1 hold and let $\mathbf{P}(\omega)$ have the strong (r,p)-extension property. Then there exists a continuous linear operator $\mathcal{U}_\Omega : W^{1,p}(\mathbf{P}) \to W^{1,r}(\Omega)$ such that $(\mathcal{U}_\Omega u)|_{\mathbf{P}} = u$ and such that*

$$\left\| \mathrm{D}_\omega \mathcal{U}_\Omega u \right\|_{L^r(\Omega)} \leq C \left\| \mathrm{D}_\omega u \right\|_{L^p(\Omega)} \,.$$

Furthermore, the operator $\mathcal{U}_\Omega$ can be extended to a continuous operator $\mathcal{U}_\Omega :\ \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}) \to \mathcal{V}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})$. More precisely we can identify $\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ with

$$\tilde{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P}) = \mathrm{closure}_{L^{r,p}(\Omega,\mathbb{P})} \left\{ \mathcal{U}_\Omega \mathrm{D}_\omega u :\ u \in W^{1,p}(\Omega) \right\} \,, \tag{8}$$

$$= \mathrm{closure}_{L^{r,p}(\Omega,\mathbb{P})} \left\{ \mathcal{U}_\Omega \mathrm{D}_\omega u :\ u \in W^{1,r,p}(\Omega;\mathbf{P}) \right\} \,, \tag{9}$$

$$\|\xi\|_{L^{r,p}(\Omega,\mathbb{P})} = \|\xi\|_{L^r(\Omega)} + \|\xi\|_{L^p(\mathbf{P})} \,.$$

This means that for $\phi \in \mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ and $\tilde{\phi} \in \tilde{\mathcal{V}}^p_{\mathrm{pot}}(\mathbf{P})$ it holds $\tilde{\phi}|_{\mathbf{P}} = \phi$ iff $\tilde{\phi} = \mathcal{U}_\Omega \phi$.

If $\mathbf{P}(\omega)$ has the strong symmetric *(r,p)-extension property, then there exists a continuous linear operator $\mathcal{U}_\Omega : \boldsymbol{W}^{1,p}(\mathbf{P}) \to \boldsymbol{W}^{1,r}(\Omega)$ such that $(\mathcal{U}_\Omega u)|_{\mathbf{P}} = u$ and such that*

$$\left\| \mathrm{D}^{\mathfrak{s}}_\omega \mathcal{U}_\Omega u \right\|_{L^r(\Omega)} \leq C \left\| \mathrm{D}^{\mathfrak{s}}_\omega u \right\|_{L^p(\Omega)} \,,$$

with $\mathrm{D}^{\mathfrak{s}}_\omega u := \frac{1}{2}\left(\mathrm{D}_\omega u + (\mathrm{D}_\omega u)^\top\right)$ and (8) and (9) hold also in this case.

We will prove Theorems 2.7 and 2.8 in Section 3.1 using homogenization theory.

2.3 The Outer Normal Field of $\mathbf{P}$

The following result has been proved in [10] for $r = p$. However, the argumentation remains valid in the following setting.

Theorem 2.9. *Let Assumption 2.5 hold and let $\Gamma(\omega)$ have the (r,p)-trace property for $1 < r < p$. Let τ be ergodic, let $\Gamma(\omega)$ be almost surely locally Lipschitz and let $\nu_{\Gamma(\omega)}$ be the outer normal of $\mathbf{P}(\omega)$ on $\Gamma(\omega)$. Then there exists a measurable function $\nu_\Gamma : \Gamma \to \mathbb{S}^{d-1}$ such that almost surely $\nu_{\Gamma(\omega)}(x) = \nu_\Gamma(\tau_x\omega)$. Furthermore, for $f \in C^1(\overline{\Omega};\mathbb{R}^d)$ and $\phi \in C^1(\overline{\Omega})$ it holds*

$$\int_{\mathbf{P}} \mathrm{div}_\omega(f\phi)\,\mathrm{d}\mathbb{P} = \int_\Gamma \phi f \cdot \nu_\Gamma \,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,. \tag{10}$$

If Γ satisfies the weak $(1,p)$-extension property, the equation (10) extends to $\phi \in W^{1,1,p}(\Omega,\mathbf{P})$ and $f \in C^1(\overline{\Omega};\mathbb{R}^d)$ or to $f \in W^{1,1,p}(\Omega,\mathbf{P})^d$ and $\phi \in C^1(\overline{\Omega})$.

Definition 2.10. Let $\Gamma(\omega)$ have the (r,p)-Trace property for $1 < r < p$ and the weak $(1,p)$-extension property. We say that $f \in L^p(\mathbf{P};\mathbb{R}^d)$ has the weak normal trace $f_\nu \in L^r(\Gamma)$ and weak divergence $\mathrm{div}_\omega f \in L^1(\mathbf{P})$ if for all $\phi \in C_b^1(\Omega)$

$$\int_{\mathbf{P}} \left(\phi \mathrm{div}_\omega f + f \cdot \nabla_\omega \phi\right)\,\mathrm{d}\mathbb{P} = \int_\Gamma \phi f_\nu \,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.$$

Theorem 2.11. *Let Assumption 2.5 hold and for some $r \in (1,2)$ let Γ have the $(r,2)$-trace property and the weak $(r,2)$-extension property. Let $\Gamma(\omega)$ be almost surely locally Lipschitz and let $\nu_{\Gamma(\omega)}$ be the outer normal of $\mathbf{P}(\omega)$ on $\Gamma(\omega)$. Then there exists $u_\Omega \in W^{1,r}(\Omega) \cap W^{1,2}(\mathbf{P};\mathbb{R}^d)$, such that $\nabla_\omega u_\Omega$ has a weak normal trace $f_\nu \in L^1(\Gamma)$ and weak divergence u_Ω, i.e.*

$$\forall \phi \in C_b^1(\omega):\quad \int_{\mathbf{P}} \left(\phi u_\Omega + \nabla u_\Omega \cdot \nabla_\omega \phi\right)\,\mathrm{d}\mathbb{P} = \int_\Gamma \phi f_\nu \,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.$$

The last theorem is less trivial than one might think. In particular, we lack a Poincaré-type inequality on Ω, which is typically used to prove corresponding results in $\mathbb{R}^d$. We shift the proof to Section 3.1.

3 Homogenization of Elasticity

In this section we provide the main homogenization result. We will use stochastic two-scale convergence in a modified version [10] of the original approach by Zhikov and Piatnitsky [20].

For the rest of this work, we consider a stationary random measure $\omega \to \mu_\omega$ with Palm measure $\mu_\mathcal{P}$ and we define

$$\mu_\omega^\varepsilon(A) := \varepsilon^d \mu_\omega\big(\varepsilon^{-1}A\big)\,. \tag{11}$$

For the corresponding Lebesgue spaces we write $L^p(\Omega;\mu_\mathcal{P})$ or $L^p(\mathbf{Q};\mu_\omega^\varepsilon)$, where $\mathbf{Q} \subset \mathbb{R}^d$ is a convex domain with C^1-boundary. If $\mu_\omega = \mathcal{L}$, i.e. $\mu_\mathcal{P} = \mathbb{P}$, or $\mu_\omega = \chi_{\mathbf{P}(\omega)}\mathcal{L}$ we omit the notion of μ_ω^ε and $\mu_\mathcal{P}$.

In our applications, either

$$\mathrm{d}\mu_\omega = \begin{cases} \mathrm{d}\mathcal{L} \\ \chi_{\mathbf{P}(\omega)}\mathrm{d}\mathcal{L} \\ \mathrm{d}\mu_{\Gamma(\omega)} := \chi_{\Gamma(\omega)}\mathrm{d}\mathcal{H}^{d-1} \end{cases} \qquad \text{with Palm measure } \mathrm{d}\mu_\mathcal{P} = \begin{cases} \mathrm{d}\mathbb{P} \\ \chi_{\mathbf{P}}\mathrm{d}\mathbb{P} \\ \mathrm{d}\mu_{\Gamma,\mathcal{P}} \end{cases}\,.$$

Moreover, in view of (11), we write

$$\mu^{\varepsilon}_{\Gamma(\omega)}(A) = \varepsilon^{d}\mu_{\Gamma(\omega)}\big(\varepsilon^{-1}A\big) = \varepsilon\mathcal{H}^{d-1}(A\cap\varepsilon\Gamma(\omega))$$

In case of $\mu_\omega = \chi_{\mathbf{P}(\omega)}\mathcal{L}$, we drop the notation $\mu^{\varepsilon}_{\omega}$.

Definition 3.1. We say that $\omega\in\Omega$ is typical if for every $f\in C(\overline{\Omega})$ and both random measures μ_ω it holds

$$n^{-d}\int_{\mathbb{B}_n(0)} f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x) \to \int_\Gamma\int_{\mathbb{B}_1(0)} f\mathrm{d}x\,\mathrm{d}\mu_{\mathcal{P}}\,.$$

According to [10] the set of typical ω has full measure.

Definition 3.2. Let ω be trypical and let $u^\varepsilon\in L^p(\mathbf{Q};\mu^\varepsilon_\omega)$ for all $\varepsilon>0$. We say that (u^ε) converges (weakly) in two scales to $u\in L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))$ and write $u^\varepsilon\overset{2s}{\rightharpoonup}u$ if $\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q};\mu^\varepsilon_\omega)}<\infty$ and if for every $\psi\in C(\overline{\Omega})$, $\varphi\in C(\overline{\mathbf{Q}})$ there holds with $\phi_{\omega,\varepsilon}(x):=\varphi(x)\psi(\tau_{\frac{x}{\varepsilon}}\omega)$

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}} u^\varepsilon(x)\phi_{\omega,\varepsilon}(x)\mathrm{d}\mu^\varepsilon_\omega(x) = \int_{\mathbf{Q}}\int_\Omega u(x,\tilde\omega)\varphi(x)\psi(\tilde\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\tilde\omega)\,\mathrm{d}x\,.$$

Lemma 3.3 ([8] Lemma 4.4-1.). *Let $\omega\in\Omega$ be typical and $u^\varepsilon\in L^p(\mathbf{Q};\mu^\varepsilon_\omega)$ be a sequence of functions such that $\|u^\varepsilon\|_{L^p(\mathbf{Q};\mu^\varepsilon_\omega)}\le C$ for some $C>0$ independent of ε. Then there exists a subsequence of $(u^{\varepsilon'})_{\varepsilon'\to0}$ and $u\in L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))$ such that $u^{\varepsilon'}\overset{2s}{\rightharpoonup}u$ and*

$$\|u\|_{L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))}\le\liminf_{\varepsilon'\to0}\left\|u^{\varepsilon'}\right\|_{L^p(\mathbf{Q};\mu^\varepsilon_\omega)}\,. \tag{12}$$

Furthermore, we will need the following result on the lower estimate in homogenization of convex functionals using two-scale convergence, which was obtained in [12].

Lemma 3.4. *Let μ_ω be a random measure. Let $f:\mathbf{Q}\times\Omega\times\mathbb{R}^N\to\mathbb{R}$ be a convex functional in $\mathbb{R}^d$. For almost all $\omega\in\Omega_{\Phi_p}$ the following holds: Let $u^\varepsilon\in L^q(\mathbf{Q};\mu^\varepsilon_\omega)$ be a sequence such that $\|u^\varepsilon\|_{L^q(\mathbf{Q};\mu^\varepsilon_\omega)}\le C$ for some $0<C<\infty$ and such that $u^\varepsilon\overset{2s}{\rightharpoonup}u\in L^q(\mathbf{Q}\times\Omega;\mathcal{L}\otimes\mu_{\mathcal{P}})$. Then, it holds*

$$\int_{\mathbf{Q}}\int_\Omega f(x,\tilde\omega,u(x,\tilde\omega))\,\mathrm{d}\mu_{\mathcal{P}}(\tilde\omega)\,dx\le\liminf_{\varepsilon\to0}\int_{\mathbf{Q}} f(x,\tau_{\frac{x}{\varepsilon}}\omega,u^\varepsilon(x))\,\mathrm{d}\mu^\varepsilon_\omega(x)\,.$$

The following result has been proven in various work under various assumptions, see e.g. [1] for the periodic case and [20, 16, 8] in the stochastic case.

Theorem 3.5. *For almost every typical $\omega\in\Omega$ the following holds: If $u^\varepsilon\in W^{1,p}(\mathbf{Q};\mathbb{R}^d)$ for all ε and if there exists $0<C_u<\infty$ with*

$$\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q})}+\varepsilon^\gamma\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q})}<C_u$$

Then there exists $u\in L^p(\mathbf{Q}L^p(\Omega;\mathbb{P}))$ such that $u^\varepsilon\overset{2s}{\rightharpoonup}u$. Depending on the choice of γ, the following holds:

1. *If $\gamma=0$, then $u\in W^{1,p}(\mathbf{Q})$ with $u^\varepsilon\rightharpoonup u$ weakly in $W^{1,p}(\mathbf{Q})$ and there exists $\boldsymbol{\upsilon}_1\in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$ such that $\nabla u^\varepsilon\overset{2s}{\rightharpoonup}\nabla_x u+\boldsymbol{\upsilon}_1$weakly in two scales.*

2 If $\gamma \in (0,1)$ then $\varepsilon^\gamma \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \boldsymbol{\upsilon}_1$ for some $\boldsymbol{\upsilon}_1 \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$.

3 If $\gamma = 1$ then $u \in L^p(\mathbf{Q};W^{1,p}(\Omega))$ and $\varepsilon \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \mathrm{D}_\omega u$.

4 If $\gamma > 1$ then $\varepsilon^\gamma \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} 0$.

Important in the context of Γ-convergence is also the following recovery lemma, obtained in [13, Section 2.3] for the L^2-case. The general case can be proved similarly [9].

Lemma 3.6. *Let $\upsilon \in \mathcal{V}^p_{\mathrm{pot}}(\Omega)$, $1 < p < \infty$ and let $\mathbf{Q}$ be a bounded convex domain. For almost every ω there exists $C > 0$ such that the following holds: For every $\varepsilon > 0$ there exists a unique $V^\omega_\varepsilon \in W^{1,p}(\mathbf{Q})$ with $\nabla V^\omega_\varepsilon(x) = \upsilon(\tau_{\frac{x}{\varepsilon}}\omega)$, $\int_{\mathbf{Q}} V^\omega_\varepsilon = 0$ and $\|V_\varepsilon\|_{W^{1,p}(\mathbf{Q})} \leq C\|\upsilon\|_{L^p_{\mathrm{pot}}(\Omega)}$ for all $\varepsilon > 0$. Furthermore,*

$$\lim_{\varepsilon\to 0} \|V^\omega_\varepsilon\|_{L^p(\mathbf{Q})} = 0\,.$$

3.1 Homogenization on Domains with Holes

Lemma 3.7. *Let $\mathbf{P}(\omega)$ be a random open domain with the weak (r,p)-extension property on $\mathbf{Q}$ for $1 < r < p < \infty$. Then for almost every $\omega \in \Omega$ the following holds: If $u^\varepsilon \in W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega);\mathbb{R}^d)$ for all ε with*

$$\sup_\varepsilon \left(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\right) < C$$

for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and a function $u \in L^p(\mathbf{Q};W^{1,r}(\Omega)) \cap L^p(\mathbf{Q};W^{1,p}(\mathbf{P}))$ such that

$$\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \text{and} \quad \varepsilon'\nabla\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \nabla_\omega u \tag{13}$$

as well as

$$u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \text{and} \quad \varepsilon'\nabla u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}}\nabla_\omega u \tag{14}$$

as $\varepsilon' \to 0$.

Proof. We find

$$\begin{aligned}\sup_\varepsilon &\left(\|\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\|\nabla\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}\right) \\ &\leq C\sup_\varepsilon \left(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\right)\end{aligned} \tag{15}$$

Theorem 3.5 implies for some $u \in L^r(\mathbf{Q};W^{1,r}(\Omega))$ that (13) and (14) hold. The $L^p(\mathbf{Q};W^{1,p}(\mathbf{P}))$-regularity of u follows from the bounds on $u^{\varepsilon'}$. □

Proof of Theorem 2.7. $W^{1,p}(\mathbf{P})$ is a closed subspace of $L^p(\mathbf{P}) \times L^p(\mathbf{P})^d$, hence separable. If $(u_k)_{k\in\mathbb{N}}$ is a countable dense subset of $W^{1,p}(\mathbf{P})$, we find a set of full measure $\tilde{\Omega} \subset \Omega$ such that for every $k \in \mathbb{N}$ and every $\omega \in \tilde{\Omega}$ the realizations $u_{k,\omega}$ are well defined elements of $W^{1,p}(\mathbf{P}(\omega))$.

Given such ω and $k\in\mathbb{N}$, we define $u^\varepsilon(x):=u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ and by Lemma 3.7 we find $\tilde u\in L^p(\mathbf{Q};W^{1,r}(\Omega))\cap L^p(\mathbf{Q}\times\mathbf{P})$ such that $\mathcal{U}_\varepsilon u^\varepsilon \overset{2s}{\rightharpoonup} \tilde u_k$ and $\varepsilon\nabla\mathcal{U}_\varepsilon u^\varepsilon \overset{2s}{\rightharpoonup} \nabla_\omega\tilde u_k$ and such that

$$\begin{aligned}\|\tilde u_k\|_{L^r(\mathbf{Q}\times\Omega)}+\|\nabla_\omega\tilde u_k\|_{L^r(\mathbf{Q}\times\Omega)}&\le\liminf_{\varepsilon\to0}\Big(\|\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q})}+\varepsilon\,\|\nabla\mathcal{U}_\varepsilon u^\varepsilon\|_{L^r(\mathbf{Q})}\Big)\\&\le C\liminf_{\varepsilon\to0}\Big(\|u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}+\varepsilon\,\|\nabla u^\varepsilon\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\Big)\\&=C\Big(\|u_k\|_{L^p(\mathbf{Q}\times\Omega)}+\|\nabla_\omega u_k\|_{L^p(\mathbf{Q}\times\Omega)}\Big)\,.\end{aligned}$$

Since the operator $u_k\to\tilde u_k$ is linear and bounded, it can be extended to the whole of $W^{1,p}(\mathbf{P})$. □

Proof of Theorem 2.11. For every $\varepsilon>0$ and $f^\varepsilon_{\nu,\omega}(x):=f_\nu\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ there exists a unique u^ε that solves

$$\begin{aligned}-\varepsilon^2\Delta u^\varepsilon+u^\varepsilon&=0 &&\text{on } \mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\Omega)\,,\\-\varepsilon\nabla u^\varepsilon\cdot\nu_{\Gamma^\varepsilon(\omega)}&=f^\varepsilon_{\nu,\omega} &&\text{on } \Gamma^\varepsilon(\omega)\cap\mathbf{Q}\,,\\u^\varepsilon&=0 &&\text{on } \partial\mathbf{Q}\,.\end{aligned}$$

Deriving apriori estimates in the usual way, for some $C>0$ independent from ε it holds

$$\varepsilon\,\|\nabla u^\varepsilon\|_{L^2(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\Omega))}+\|u^\varepsilon\|_{L^2(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\Omega))}\le C$$

and thus according to Lemma 3.7 we find $u\in L^r(\mathbf{Q};W^{1,r}(\Omega))\cap L^p(\mathbf{Q}\times\mathbf{P})$ such that

$$\mathcal{U}_{\varepsilon'}u^{\varepsilon'}\overset{2s}{\rightharpoonup}u\quad\text{and}\quad\varepsilon'\nabla\mathcal{U}_{\varepsilon'}u^{\varepsilon'}\overset{2s}{\rightharpoonup}\nabla_\omega u$$

along a subsequence $u^{\varepsilon'}$ which we again denote u^ε in the following. But then for $\phi\in C^1(\overline{\Omega})$ and $\psi\in C^1_c(\mathbf{Q})$ it follows

$$\begin{aligned}&\varepsilon\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)}f_{\nu,\omega}\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)=-\varepsilon^2\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)}\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\nabla u^\varepsilon(x)\cdot\nu_{\Gamma(\omega)}(\tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}\mathcal{H}^{d-1}(x)\\&=\int_{\mathbf{Q}\cap\mathbf{P}_\varepsilon(\omega)}\varepsilon\nabla u^\varepsilon\cdot\big(\nabla_\omega\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)+\varepsilon\phi(\tau_{\frac{x}{\varepsilon}}\omega)\nabla\psi(x)\big)\,\mathrm{d}x+\int_{\mathbf{Q}\cap\mathbf{P}_\varepsilon(\omega)}u^\varepsilon\phi(\tau_{\frac{x}{\varepsilon}}\omega)\psi(x)\,\mathrm{d}x\\&\quad\to\int_{\mathbf{Q}}\int_{\mathbf{P}}\left(\nabla_\omega u\cdot\nabla_\omega\phi\psi+u\phi\psi\right)\,.\end{aligned}$$

Since the left hand side of the above calculation converges to $\int_{\mathbf{Q}}\int_\Gamma f_\nu\phi\psi\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}$ and ψ was arbitrary, we conclude. □

Lemma 3.8. *Let $\mathbf{P}(\omega)$ be a random open domain with strong (r,p)-extension property for $1<r<p<\infty$. Then for almost every $\omega\in\Omega$ the following holds:*

1 *If $u^\varepsilon\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega);\mathbb{R}^d)$ for all ε with $\sup_\varepsilon\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q}\cap\mathbf{P}_\varepsilon(\omega))}<C$ for C independent from $\varepsilon>0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and functions $u\in W^{1,r}_0(\mathbf{Q};\mathbb{R}^d)$ and $\upsilon\in L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega))$ such that*

$$u^{\varepsilon'}\overset{2s}{\rightharpoonup}\chi_{\mathbf{P}}u\quad\textit{and}\quad\nabla u^{\varepsilon'}\overset{2s}{\rightharpoonup}\chi_{\mathbf{P}}\nabla u+\chi_{\mathbf{P}}\upsilon\qquad\textit{as }\varepsilon\to0\,,\tag{16}$$

$$\mathcal{U}_{\varepsilon'}u^{\varepsilon'}\overset{2s}{\rightharpoonup}u\quad\textit{and}\quad\nabla\mathcal{U}_{\varepsilon'}u^{\varepsilon'}\overset{2s}{\rightharpoonup}\nabla u+\upsilon\qquad\textit{as }\varepsilon\to0\,.\tag{17}$$

Furthermore, $\mathcal{U}_{\varepsilon'}u^{\varepsilon'}\rightharpoonup u$ weakly in $W^{1,r}(\mathbf{Q})$).

2 *If $p \geq 2$ and the Assumptions of Theorem 2.11 are satisfied and $\Gamma^\varepsilon(\omega)$ additionally has the (s,p)-trace property for some $s > 1$ then*

$$\mathcal{T}_{\varepsilon'} u^{\varepsilon'} \stackrel{2s}{\rightharpoonup} u \quad \text{in } L^s(\Gamma^\varepsilon \cap \mathbf{Q}; \mu^\varepsilon_{\Gamma(\omega)})\,.$$

If, even further, $\Gamma^\varepsilon(\omega)$ has the (s,r)-trace property with r from Part 1, then

$$\lim_{\varepsilon\to 0} \left\| \mathcal{T}_{\varepsilon'} u^{\varepsilon'} - \mathcal{T}_{\varepsilon'} u \right\|_{L^s(\Gamma^{\varepsilon'}\cap\mathbf{Q};\mu^{\varepsilon'}_{\Gamma(\omega)})} \to 0\,. \tag{18}$$

Remark 3.9. For the reader familiar to the field it may be astonishing, even unsatisfactory, that the limit function $u \in W^{1,r}_0(\mathbf{Q};\mathbb{R}^d)$ loses integrability compared to u^ε. However, let us stress once more that the extension of $W^{1,p}$ functions to $W^{1,p}$-functions really is an intrinsic property of the geometry which in general is not satisfied uniformly on random domains. This regularity also cannot be recoverd from the improved L^p-regularity of $\chi_{\mathbf{P}}\nabla u + \chi_{\mathbf{P}}\upsilon$. To understand this in more detail, take $f \in L^q(\mathbf{Q};\mathbb{R}^d)$, $\frac{1}{q}+\frac{1}{p}=1$ and observe

$$|\mathbf{P}| \left| \int_{\mathbf{Q}} f\cdot\nabla u \right| \leq \left| \lim_{\varepsilon\to0} \int_{\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega)} f\cdot\nabla u^\varepsilon \right| + \left| \int_{\mathbf{Q}}\int_{\mathbf{P}} f\cdot\upsilon \right|\,.$$

Now, the limit $\varepsilon\to 0$ provides $\chi_{\mathbf{P}}\nabla u + \chi_{\mathbf{P}}\upsilon \in L^p(\mathbf{Q}\times\mathbf{P})$ but not $\chi_{\mathbf{P}}\upsilon \in L^p(\mathbf{Q}\times\mathbf{P})$. Hence, we rely on $\int_{\mathbf{Q}} f\cdot\int_{\mathbf{P}}\upsilon = 0$, a property that holds for $\mathbf{P}=\Omega$, and maybe in more generality, but we currently lack a proof.

Proof. In what follows, convergences always hold along subsequences of u^ε, which we always relabel by u^ε.

Proof of 1: Let $\frac{1}{r}+\frac{1}{q}=1$. Then Theorem 3.5 yields for some $u \in W^{1,r}(\mathbf{Q};\mathbb{R}^d)$ and $\upsilon \in L^r(\mathbf{Q};L^r_{\text{pot}}(\Omega))$ that (17) holds. Due to the decreasing support of $\mathcal{U}_\varepsilon u^\varepsilon$ we find $u \in W^{1,r}_0(\mathbf{Q};\mathbb{R}^d)$. (16) follows from using $\chi_{\mathbf{P}}$ as a testfunction.

Proof of 2: Now let $p\geq 2$ and let the Assumptions of Theorem 2.11 be satisfied and let $\Gamma^\varepsilon(\omega)$ additionally have the (s,p)-trace property for some $s>1$. If u_Ω is the function from Theorem 2.11 for $f_\nu = 1$ we observe for $u^\varepsilon_\Omega(x) := u_\Omega\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ for every $\psi \in C^\infty_c(\mathbf{Q})$ and $\phi \in C^1(\overline{\Omega})$ with $\phi^\varepsilon(x) := \phi\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ that

$$\begin{aligned}
\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)} u^\varepsilon\psi\phi^\varepsilon\,\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)} &= \varepsilon\int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)} u^\varepsilon\psi\phi^\varepsilon\varepsilon\nabla_\omega u^\varepsilon_\Omega\cdot\nu_{\Gamma^\varepsilon(\omega)}\,\mathrm{d}\mathcal{H}^{d-1}\\
&= \int_{\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega)} \left(u^\varepsilon\psi\phi^\varepsilon u^\varepsilon_\Omega + \varepsilon\nabla u^\varepsilon_\Omega\cdot\left(u^\varepsilon\phi^\varepsilon\varepsilon\nabla\psi + \psi\phi^\varepsilon\varepsilon\nabla u^\varepsilon + \psi u^\varepsilon\varepsilon\nabla\phi^\varepsilon\right)\right)\\
&\to \int_{\mathbf{Q}}\int_{\mathbf{P}} \left(u\psi\phi u_\Omega + \psi u\nabla_\omega u_\Omega\cdot\nabla_\omega\phi\right)\\
&= \int_{\mathbf{Q}}\int_{\Gamma} u\psi\phi\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.
\end{aligned}$$

In order to show (18) note that

$$\left\|\mathcal{T}_\varepsilon u^\varepsilon - \mathcal{T}_\varepsilon u\right\|_{L^s(\Gamma^\varepsilon\cap\mathbf{Q};\mu^\varepsilon_{\Gamma(\omega)})} \leq \left\|u^\varepsilon - u\right\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\left\|\nabla\left(u^\varepsilon - u\right)\right\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\,.$$

Since the first term on the right hand side converges to zero and $\left\|\nabla\left(u^\varepsilon-u\right)\right\|_{L^r(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}$ is bounded, the claim follows. □

Proof of Theorem 2.8. For $u\in W^{1,p}(\mathbf{P})$ with $u^\varepsilon(x):=u\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ we find for almost every ω that $\mathcal{U}_\varepsilon$ satisfies

$$\begin{aligned}\varepsilon\left\|\nabla\mathcal{U}_\varepsilon u^\varepsilon\right\|_{L^r(\mathbf{Q})}&\leq C\left(\varepsilon\left\|\nabla u^\varepsilon\right\|_{L^p(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}\right)\\ \left\|\mathcal{U}_\varepsilon u^\varepsilon\right\|_{L^r(\mathbf{Q})}&\leq C\left\|u^\varepsilon\right\|_{L^p(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega))}\end{aligned} \tag{19}$$

As $\varepsilon\to0$, Lemma 3.7 yields $u^\varepsilon\stackrel{2s}{\rightharpoonup}\tilde u$, $\nabla\mathcal{U}_\varepsilon u^\varepsilon\stackrel{2s}{\rightharpoonup}\mathrm{D}_\omega\tilde u$, where $\tilde u\in L^p(\mathbf{Q};W^{1,r,p}(\Omega,\mathbf{P}))$. Moreover, inequality (19) implies in the limit that

$$\left\|\mathrm{D}_\omega\tilde u\right\|_{L^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P})}\leq C\left\|\mathrm{D}_\omega u\right\|_{L^p_{\mathrm{pot}}(\mathbf{P})}\,.$$

Hence we can set $\mathcal{U}_\Omega\mathrm{D}_\omega u:=\int_{\mathbf{Q}}\mathrm{D}_\omega\tilde u$. By density, this operator extends to $\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$. □

3.2 Homogenization of p-Laplace Equations

Assumption 3.10. *For the rest of this work, let the assumptions of Theorem 1.3 hold.*

For

$$\begin{aligned}\mathcal{E}:\ &\boldsymbol{W}^{1,r}(\mathbf{Q})\times L^r(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}))\to\mathbb{R}\\ &(u,\upsilon)\mapsto\int_{\mathbf{Q}}\int_{\mathbf{P}}a\left|\nabla^{\mathfrak{s}}u+\upsilon^{\mathfrak{s}}\right|^p-\int_{\mathbf{Q}}\int_{\mathbf{P}}G(u)+\int_{\mathbf{Q}}\int_{\Gamma}F(u)\mathrm{d}\mu_{\Gamma,\mathcal{P}}\end{aligned}$$

it holds

$$\mathcal{E}_{\mathrm{hom}}(u):=\inf_{\upsilon\in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P}))}\mathcal{E}(u,\upsilon)\,.$$

We start with two observations. The first is a direct consequence of the lower bound on F and G.

Lemma 3.11. *Let Assumption of Theorem 1.3 hold. Then there exists $C>0$ such that for every $u^\varepsilon\in\boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$ it holds*

$$\left\|\nabla^{\mathfrak{s}}u^\varepsilon\right\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}\leq\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)+C\,. \tag{20}$$

Lemma 3.12. *Let Assumption of Theorem 1.3 hold. Then almost surely every sequence of functions $u^\varepsilon\in\boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$ with $\sup_\varepsilon\left\|\nabla^{\mathfrak{s}}u^\varepsilon\right\|_{L^p(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))}<\infty$ and $u^\varepsilon\rightharpoonup u$ weakly in $L^r(\mathbf{Q})$ satisfies*

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}}G(u^\varepsilon)=\int_{\mathbf{Q}}\int_{\mathbf{P}}G(u)\,\mathrm{d}\mathbb{P}\,\mathrm{d}x\,, \tag{21}$$

$$\lim_{\varepsilon\to0}\int_{\Gamma^\varepsilon}F(u^\varepsilon)\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}=\int_{\mathbf{Q}}\int_{\Gamma}F(u)\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,\mathrm{d}x\,, \tag{22}$$

with equality in case of Hölder continuity of F.

Proof. According to Lemma it holds $u\in\boldsymbol{W}^{1,r}_0(\mathbf{Q})$ and $\mathcal{U}_\varepsilon u^\varepsilon\to u$ strongly in $L^r(\mathbf{Q})$. In the first case, F is Hölder and the (s,r)-trace property implies

$$\begin{aligned}\int_{\Gamma^\varepsilon}\left|F(u^\varepsilon)-F(u)\right|\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}&\leq C\int_{\Gamma^\varepsilon}\left|u^\varepsilon-u\right|^s\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}\\ &\leq C\left(\left\|u^\varepsilon-u\right\|_{L^r(\mathbf{Q}^\varepsilon_{\mathbf{P}})}+\varepsilon\left\|\nabla\mathcal{U}_\varepsilon u^\varepsilon-\nabla u\right\|_{L^r(\mathbf{Q})}\right)\,.\end{aligned} \tag{23}$$

The convergence (21) follows accordingly. □

Theorem 3.13. *Let Assumption of Theorem 1.3 hold. Then, for almost every $\omega \in \Omega$ we find $\mathcal{E}_{\varepsilon,\omega} \stackrel{2s\Gamma}{\longrightarrow} \mathcal{E}$ in the following sense*

1 *For $u^\varepsilon \rightharpoonup u$ weakly in $L^r(\mathbf{Q})$, $u^\varepsilon \in \boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$ with $\sup_\varepsilon \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon) < \infty$, there holds $u \in \boldsymbol{W}^{1,r}_0(\mathbf{Q})$ and there exists $\upsilon \in L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot},\mathfrak{s}}(\Omega, \mathbf{P}))$ such that $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}} \cdot (\nabla u + \upsilon)$ and*

$$\mathcal{E}(u, \upsilon) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,. \tag{24}$$

2 *For each pair $(u, \upsilon) \in \boldsymbol{W}^{1,r}_0(\mathbf{Q}) \times L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot}}(\Omega))$ with $\mathcal{E}(u, \upsilon) < +\infty$ there exists a sequence $u^\varepsilon \in \boldsymbol{W}^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))$ such that $\mathcal{U}_\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $\boldsymbol{W}^{1,r}(\mathbf{Q})$ and $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \chi_{\mathbf{P}} \cdot (\nabla u + \upsilon)$ weakly in two scales and*

$$\mathcal{E}(u, \upsilon) = \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,. \tag{25}$$

Proof. 1. We find

$$\left|\int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}} G(u^\varepsilon)\right| \leq C \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}} |u^\varepsilon|^r \leq C \int_{\mathbf{Q}} |\mathcal{U}_\varepsilon u^\varepsilon|^r \leq C \left(\int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}} |\nabla u^\varepsilon|^p\right)^{\frac{r}{p}}$$

with a similar estimate for $\int_{\Gamma^\varepsilon} F(u^\varepsilon)\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)}$ in case of Hölder continuous F and exploiting the lower bound of F otherwise. Then because of (20)

$$\int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} \frac{1}{p} |\nabla^{\mathfrak{s}} u^\varepsilon|^p \leq \mathcal{E}_{\varepsilon,\omega}(u^\varepsilon) + C$$

for C independent from ε. Hence the statement follows from Lemmas 3.8 and 3.12.

2. Step a: Let $(u_k)_{k\in\mathbb{N}} \subset C^1(\overline{\Omega})$ be a countable dense family in $\boldsymbol{W}^{1,p}(\Omega)$ and $(\phi_j)_{j\in\mathbb{N}} \subset C^\infty_c(\mathbf{Q})$ be dense in $W^{1,p}_0(\mathbf{Q})$. Then the span of the functions $\phi_j \nabla_\omega u_k$ is dense in $L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot}}(\Omega))$. Writing $S = \mathrm{span}\phi_j \nabla_\omega u_k$ we show statement 2. for $(u, \upsilon) \in (\phi_j)_{j\in\mathbb{N}} \times S$. However, for such (u, υ) we find $V \in \mathrm{span}\phi_j u_k$ such that $\upsilon = \nabla_\omega V$ and $V^\varepsilon(x) := V(x, \tau_{\frac{x}{\varepsilon}}\omega)$ is well defined and measurable for every ω. For simplicity of notation, we assume $V = \phi_j u_k$

In particular, we have for $u^\varepsilon = u + \varepsilon V^\varepsilon$ that $u^\varepsilon \stackrel{2s}{\rightharpoonup} u$ and $\nabla u^\varepsilon = \nabla u + \varepsilon \nabla\phi_j\, u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big) + \phi_j \nabla_\omega u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ and hence $u^\varepsilon \rightharpoonup u$ weakly in $\boldsymbol{W}^{1,p}(\mathbf{Q})$ and $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla u + \phi_j \nabla_\omega u_k$. Using essential boundedness of $\nabla\phi_j\, u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$, the ergodic theorem now yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} |\nabla^{\mathfrak{s}} u^\varepsilon|^p &= \lim_{\varepsilon\to 0} \int_{\mathbf{Q}} \chi_{\mathbf{P}}\big(\tau_{\frac{x}{\varepsilon}}\omega\big) \left|\nabla^{\mathfrak{s}} u + \phi_j \nabla^{\mathfrak{s}}_\omega u_k\big(\tau_{\frac{x}{\varepsilon}}\omega\big)\right|^p \\ &= \int_{\mathbf{Q}} \int_{\mathbf{P}} |(\nabla^{\mathfrak{s}} u + \upsilon^{\mathfrak{s}})|^p\,.\end{aligned}$$

We obtain $\int_{\mathbf{Q}^\varepsilon(\omega)} G(u^\varepsilon) \to \int_{\mathbf{Q}} \int_{\mathbf{P}} G(u)$ and $\int_{\Gamma^\varepsilon} F(u^\varepsilon)\mathrm{d}\mu^\varepsilon_{\Gamma(\omega)} \to \int_{\mathbf{Q}} \int_\Gamma F(u)\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,\mathrm{d}x$ from Lemma 3.12. This implies (25) for the above sequence u^ε.

Step b: We pick up an idea of [3], Proposition 6.2. For general $(u, \upsilon) \in W^{1,r}_0(\mathbf{Q}) \times L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot}}(\Omega))$ with $\mathcal{E}(u, \upsilon) < +\infty$ let $(u_n, \upsilon_n) \in (\phi_j)_{j\in\mathbb{N}} \times S$ with

$$\|(u, \upsilon) - (u_n, \upsilon_n)\|_{W^{1,r}_0(\mathbf{Q}) \times L^r(\mathbf{Q}; \mathcal{V}^r_{\mathrm{pot}}(\Omega))} \leq \frac{1}{n} \tag{26}$$

and

$$|\mathcal{E}(u,\upsilon)-\mathcal{E}(u_n,\upsilon_n)|\leq\frac{1}{n}\,. \tag{27}$$

We achieve (27) in the following way: due to Hölder continuity, there exists $C>0$ such that $|F(u)|+|G(u)|\leq C(|u|+1)$. For $M>0$ we write $u_M:=\max\{-M,\min\{u,M\}\}$ and set $\upsilon_M(x,\omega)=\chi_{(-M,M)}(u(x))\,\upsilon(x,\omega)$, i.e. $u_M=M$ implies $\upsilon=0$. Then u_M and υ_M are still in the same respective spaces. Furthermore, as $M\to\infty$ we find $\mathcal{E}(u_M,\upsilon_M)\to\mathcal{E}(u,\upsilon)$ by the Lebesgue dominated convergence theorem. Next, we approximate (u_M,υ_M) in $W^{1,p}(\mathbf{Q})\times\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$ by elements $(u_{M,\delta},\upsilon_{M,\delta})\in(\phi_j)_{j\in\mathbb{N}}\times S$ and again by the Lebesgue dominated convergence theorem we get convergence $\mathcal{E}(u_{M,\delta},\upsilon_{M,\delta})\to\mathcal{E}(u_M,\upsilon_M)$. Successively choosing M and δ, we find $(u_n,\upsilon_n)\in(\phi_j)_{j\in\mathbb{N}}\times S$ satisfying 26–27.

Starting from 26–27 we set $\varepsilon_0(\omega)=1$ and for each $(u_n,\upsilon_n)\in(\phi_j)_{j\in\mathbb{N}}\times S$ we find by Steps a and b for almost every ω some $\varepsilon_n(\omega)\leq\frac{1}{2}\varepsilon_{n-1}(\omega)$ such that for $\varepsilon<\varepsilon_n(\omega)$ and $u^\varepsilon_{n,\omega}=u_n(x)+\varepsilon V_n(x,\tau_{\frac{x}{\varepsilon}}\omega)$ it holds

$$\left|\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{n,\omega})-\mathcal{E}(u_n,\upsilon_n)\right|\leq\frac{1}{n}\,.$$

The set $\tilde\Omega\subset\Omega$ such that all $\varepsilon_n(\omega)$ are well defined has measure 1. For such ω we choose $u^\varepsilon=u^\varepsilon_{n,\omega}$ if $\varepsilon\in(\varepsilon_{n+1},\varepsilon_n)$. Then

$$|\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)-\mathcal{E}(u,\upsilon)|\leq\frac{2}{n}\qquad\text{for }\varepsilon<\varepsilon_n\,.$$

which implies the claim. □

Theorem 3.14. *Let Assumption 3.10 hold. Then for almost every ω the following holds: For every $\varepsilon>0$ let $u^\varepsilon_{\min}\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ be a global minimizer of $\mathcal{E}_{\varepsilon,\omega}$. Then*

$$\sup_{\varepsilon>0}\|u^\varepsilon_{\min}\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))}+\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{\min})\leq\infty$$

and for every subsequence such that $\mathcal{U}_\varepsilon u^\varepsilon_{\min}\rightharpoonup u$ weakly in $L^p(\mathbf{Q})$ and weakly in $W^{1,r}(\mathbf{Q})$ with $\upsilon\in L^r(\mathbf{Q};\mathcal{V}^{r,p}_{\mathrm{pot}}(\Omega,\mathbf{P}))$ such that $\nabla u^\varepsilon_{\min}\overset{2s}{\rightharpoonup}\nabla u+\upsilon$ it further holds $u\in W^{1,r}_0(\mathbf{Q})$ and (u,υ) is a global minimizer of $\mathcal{E}$ in $W^{1,r}_0(\mathbf{Q})\times\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$. Finally, in case (5) holds, we find

$$(u,\upsilon)\in W^{1,p}_0(\mathbf{Q})\times\mathcal{V}^p_{\mathrm{pot}}(\mathbf{P})$$

Proof. In what follows, we denote

$$W_r:=W^{1,r}_0(\mathbf{Q}),\quad\mathcal{V}_r:=\mathcal{V}^r_{\mathrm{pot}}(\Omega)\,,$$

and note that every of the following countable steps works for almost every ω.

Step 1: Since $W_p\times\mathcal{V}_p\subset W_r\times\mathcal{V}_r$ the functional $\mathcal{E}$ has a at least one local minimizer (u_R,υ_R) on every closed ball of sufficiently large radius R in $W_r\times\mathcal{V}_r$

$$\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_R(0):=\left\{(u,\upsilon)\in W_r\times\mathcal{V}_r:\ \|u\|_{W_r}+\|\upsilon\|_{\mathcal{V}_r}\leq R\right\}\,.$$

By Theorem 3.13.2 there exists a recovery sequence $u^\varepsilon\in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))$ such that $\mathcal{U}_\varepsilon u^\varepsilon\rightharpoonup u_R$ weakly in $W^{1,r}(\mathbf{Q})$ and $\nabla u^\varepsilon\overset{2s}{\rightharpoonup}\chi_{\mathbf{P}}\cdot(\nabla u_R+\upsilon_R)$ weakly in two scales and

$$\mathcal{E}(u_R,\upsilon_R)=\lim_{\varepsilon\to0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)\,.$$

Step 2: We conclude for the minimizers

$$\liminf_{\varepsilon\to0}\|u^\varepsilon_{\min}\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))}\leq\liminf_{\varepsilon\to0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon_{\min})+C\leq\liminf_{\varepsilon\to0}\mathcal{E}_{\varepsilon,\omega}(u^\varepsilon)+C\leq\mathcal{E}(u_R,\upsilon_R)+C\,,$$

which at the same time implies by Theorem 3.13.1 that $\mathcal{U}_\varepsilon u^\varepsilon \rightharpoonup u$ weakly in $W^{1,r}(\mathbf{Q})$ and there exists $\upsilon\in L^r(\mathbf{Q};\mathcal{V}^r_{\mathrm{pot}}(\Omega,\mathbf{P}))$ such that $\nabla u^\varepsilon \xrightharpoonup{2s} \chi_{\mathbf{P}}\cdot(\nabla u+\upsilon)$ and with (12)

$$\begin{aligned}\|u\|_{W_r}+\|\upsilon\|_{\mathcal{V}_r}&\leq C\left(\mathcal{E}(u_R,\upsilon_R)+1\right)\,,\\ \mathcal{E}(u,\upsilon)&\leq\mathcal{E}(u_R,\upsilon_R)\,,\end{aligned}\tag{28}$$

with C independent from (u_R,υ_R). This implies that the theorem holds if there exists a global minimizer of $\mathcal{E}$

Since also $\|u^\varepsilon\|_{W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}^\varepsilon(\omega))}\leq\mathcal{E}(u_R,\upsilon_R)$, we conclude

$$\|u_R\|_{W_r}+\|\upsilon_R\|_{\mathcal{V}_r}\leq C\,\left(\mathcal{E}(u_R,\upsilon_R)+1\right)\,,$$

Step 3: Similarly, if (u_{R^*},υ_{R^*}) is a further minimizer on any ball $\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_{R^*}(0)$ with $\mathcal{E}(u_{R^*},\upsilon_{R^*})\leq\mathcal{E}(u_R,\upsilon_R)$ we can conclude

$$\|u_{R^*}\|_{W_r}+\|\upsilon_{R^*}\|_{\mathcal{V}_r}\leq C\,\left(\mathcal{E}(u_R,\upsilon_R)+1\right)$$

from the argument of Step 2 and a suitable recovery sequence.

Step 4: Hence, repeating Step 1 among the local minimizers, there exists a global minimizer $(\bar u,\bar\upsilon)\in\overline{\mathbb{B}}^{W_r\times\mathcal{V}_r}_{C\,\mathcal{E}(u_R,\upsilon_R)}(0)$. □

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