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Global algebraic Poincaré–Bendixson annulus for van der Pol systems

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Abstract

By means of planar polynomial systems topologically equivalent to the van der Pol system we demonstrate an approach to construct algebraic transversal ovals forming a parameter depending Poincaré-Bendixson annulus which contains a unique limit cycle for the full parameter domain. The inner boundary consists of the zero-level set of a special Dulac-Cherkas function which implies the uniqueness of the limit cycle. For the construction of the outer boundary we present a corresponding procedure.

1 Introduction

In the qualitative theory of planar autonomous systems

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \tag{1.1}$$

a special class of ω-limit sets, the so-called limit cycles, plays a fundamental role. A general approach to prove the existence of at least one limit cycle consists in the construction of an annulus $\mathcal{A}$ in the phase plane which contains no equilibrium and whose boundaries are simple closed curves (in what follows called ovals) with the property that if any trajectory of system (1.1) meeting the boundary of $\mathcal{A}$ will enter $\mathcal{A}$ either for increasing or for decreasing t (see e.g.[10]). We call such curves transversal curves and the corresponding annulus a Poincaré-Bendixson annulus since the application of the Poincaré-Bendixson theorem [2, 9] to that annulus provides the existence of at least one limit cycle of system (1.1) in $\mathcal{A}$. The crucial problem in that approach is: how to construct the transversal ovals? There is no general procedure to construct such ovals, even in the case of polynomial systems. In the past, Poincaré-Bendixson annuli have been constructed especially for Liénard-type systems where the transversal ovals consist of piecewise smooth curves constructed in a sophisticated way, see e.g. [3, 2, 9, 10, 11, 13]. Recently, two papers have been published [7, 4] devoted to the construction of smooth transversal ovals as boundaries of a Poincaré-Bendixson annulus for polynomial systems (1.1). For both papers it is characteristic that each transversal oval is constructed by means of the approximation of an orbit of system (1.1). In the paper by H. Giacomini and M. Grau [7], for the construction of the inner and of the outer boundary of $\mathcal{A}$ the approximation of two different orbits are required. In the paper by A. Gasull, H. Giacomini and M. Grau [4], only the approximation of one orbit is required, but this orbit is assumed to be a limit cycle.

In what follows we present an approach for the construction of a Poincaré-Bendixson annulus which does not require the approximation of any orbit. We consider polynomial systems (1.1) in a simply connected region $\mathcal{G}$ with a unique equilibrium point and assume that there exists a Dulac-Cherkas function for system (1.1) whose zero-level set consists of a unique oval surrounding the equilibrium. This assumption implies that system (1.1) has at most one limit cycle in $\mathcal{G}$ and that this oval can be

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used as interior boundary of a possible Poincaré-Bendixson annulus in $\mathcal{G}$. Our main goal is to present a procedure for the construction of an outer boundary as an algebraic oval.

The paper is organized as follows: In the Section 2 we introduce two systems which are topologically equivalent to the van der Pol equation for $\lambda > 0$. Section 3 deals with the construction of the interior boundary for a Poincaré-Bendixson annulus by using a Dulac-Cherkas function whose zero-level set consists of a unique oval. In the Section 4 we describe in details a new procedure to construct an outer boundary as an algebraic oval. Section 5 presents the global algebraic Poincaré-Bendixson annuli for two topologically equivalent van der Pol systems including a singularly perturbed system. The Appendix contains the main facts and key properties about Dulac-Cherkas functions used in this paper.

2 Van der Pol system and two equivalent systems

The scalar second-order autonomous differential equation

$$\frac{d^2\bar{x}}{dt^2} + \lambda(\bar{x}^2 - 1)\frac{d\bar{x}}{dt} + \bar{x} = 0, \tag{2.1}$$

where λ is a scalar parameter, has been introduced by Balthasar van der Pol [12] in 1926 to describe self-oscillations in a triod circuit. If we replace t by $-t$ and λ by $-\lambda$, then equation (2.1) remains invariant. Thus, the study of the the phase portrait of the system

$$\begin{aligned}\frac{d\bar{x}}{dt} &= -\bar{y},\\ \frac{d\bar{y}}{dt} &= \bar{x} - \lambda(\bar{x}^2 - 1)\bar{y},\end{aligned} \tag{2.2}$$

which corresponds to equation (2.1), can be restricted to the case $\lambda \geq 0$.
It is well-known (see e.g. [9]) that system (2.2) has to any $\lambda \neq 0$ a unique limit cycle $\overline{\Gamma}_\lambda$. For small λ, equation (2.1) can be considered as a perturbation of the harmonic oscillator, thus the shape of $\overline{\Gamma}_\lambda$ looks like a circle. For increasing λ, the amplitude of the limit cycle $\overline{\Gamma}_\lambda$ increases unbounded. Therefore, to study the van der Pol equation for large λ it is usual to apply the Liénard transformation to system (2.2) such that the limit cycles of the transformed system stay for all λ in a bounded region. For defining the Liénard transformation for $\lambda > 0$ we introduce a new time τ by $t = \lambda\tau$ and a new parameter ε by $\varepsilon = \frac{1}{\lambda^2}$. By this way, equation (2.1) can be rewritten in the form

$$\frac{d}{d\tau}\Big[\varepsilon\,\frac{d\bar{x}}{d\tau} - \bar{x} + \frac{\bar{x}^3}{3}\Big] + \bar{x} = 0. \tag{2.3}$$

Applying the nonlinear Liénard transformation

$$\eta = \bar{x}, \quad \xi = -\sqrt{\varepsilon}\bar{y} - \bar{x} + \frac{\bar{x}^3}{3} = \varepsilon\frac{d\bar{x}}{d\tau} - \bar{x} + \frac{\bar{x}^3}{3},$$

equation (2.3) is equivalent to the system

$$\begin{aligned}\frac{d\xi}{d\tau} &= -\eta,\\ \varepsilon\,\frac{d\eta}{d\tau} &= \xi + \eta - \frac{\eta^3}{3}\end{aligned} \tag{2.4}$$

which represents a singularly perturbed system for small $\varepsilon > 0$. A.D. Flanders and J.J. Stoker [3] constructed in 1946 for sufficiently small ε a Poincaré-Bendixson annulus containing the unique limit cycle Γ_ε of system (2.4).

In a recent paper [11] we applied for $\lambda > 0$ the linear time scaling $\sigma = \lambda\, t$ to the van der Pol equation (2.1). Using the notation $\varepsilon = 1/\lambda^2$ we get the equation

$$\frac{d^2x}{d\sigma^2} + (x^2 - 1)\frac{dx}{d\sigma} + \varepsilon x = 0 \tag{2.5}$$

which is equivalent to the system

$$\begin{aligned} \frac{dx}{d\sigma} &= -y, \\ \frac{dy}{d\sigma} &= \varepsilon x - (x^2 - 1)y. \end{aligned} \tag{2.6}$$

For sufficiently small ε, we constructed in [11] a local Poincaré-Bendixson annulus $\mathcal{A}_\varepsilon$ with the following properties:

(i). The interior boundary is for any ε a transversal curve and is defined by the zero-level set of a Dulac-Cherkas function for system (2.6). Since this set consists of a unique oval we could immediately conclude that the annulus $\mathcal{A}_\varepsilon$ contains a unique limit cycle which is orbitally stable. Basic information about Dulac-Cherkas functions can be found in the Appendix.

(ii). For the construction of the outer boundary we used special techniques, especially the theory of rotated vector fields, and it is only for sufficiently small ε a transversal curve.

In what follows now we present a new approach to construct transversal algebraic ovals for system (2.6) which form an algebraic Poincaré-Bendixson annulus $\mathcal{A}_\varepsilon$ for **any** $\varepsilon > 0$, called global algebraic Poincaré-Bendixson annulus.

3 Construction of the interior boundary

System (2.6) has for all ε the origin as unique equilibrium representing a focus. Thus, any limit cycle of (2.6) must surround the origin. Our goal is to construct two transversal ovals $\mathcal{I}_\varepsilon$ and Ω_ε surrounding the origin which can be used to form an annulus $\mathcal{A}_\varepsilon$ for system (2.6). We suppose that $\mathcal{I}_\varepsilon$ is the interior boundary. One possibility to construct transversal curves is to use the zero-level set $\mathcal{W}$ of a Dulac-Cherkas function (see Definition 6.1 in the Appendix) of system (2.6). Under the additional assumption that this set consists of a unique oval we can conclude that system (2.6) has at most one limit cycle and that it surrounds $\mathcal{W}$ (see e.g. [8]). In [11] the following result has been proved

Lemma 3.1. *The function*

$$\Psi(x, y, \varepsilon) := \varepsilon x^2 + y^2 - \varepsilon$$

is a Dulac-Cherkas function of system (2.6) for $\varepsilon > 0$ in $\mathbb{R}^2$ whose zero-level set $\mathcal{W}$ is the ellipse

$$\mathcal{I}_\varepsilon := \{(x, y) \in \mathbb{R}^2 : \varepsilon x^2 + y^2 = \varepsilon\}. \tag{3.1}$$

The derivative of Ψ along system (2.6) on $\mathcal{I}_\varepsilon$ is positive except at the points $(-1, 0)$ and $(1, 0)$ where this derivative vanishes.

Using Theorem 6.6 and Theorem 6.9 in the appendix this lemma implies

Lemma 3.2. *System (2.6) has at most one limit cycle Γ_ε. If Γ_ε exists, it is hyperbolic and orbitally stable.*

Lemma 3.3. *The ellipse $\mathcal{I}_\varepsilon$ can be used as interior boundary for a Poincaré-Bendixson annulus $\mathcal{A}_\varepsilon$ of system (2.6) for any $\varepsilon > 0$.*

4 Construction of the outer boundary

In what follows we present a new approach to construct the outer boundary of $\mathcal{A}_\varepsilon$ as an algebraic oval defined by the equation

$$O(x,y,\varepsilon) := O_0(x,\varepsilon) + O_1(x,\varepsilon)y + O_2(x,\varepsilon)y^2 = 0, \tag{4.1}$$

where the functions $O_i(x,\varepsilon)$ are polynomials in x. Our idea is to determine the function O_i in such a way that
(i). The set $\mathcal{N}_\varepsilon$, where the derivative of O with respect to system (2.6) vanishes, contains a subset $\mathcal{C}_\varepsilon$, where the derivative of O with respect to system (2.6) changes sign. $\mathcal{C}_\varepsilon$ is an oval surrounding the interior boundary $\mathcal{I}_\varepsilon$.
(ii). The zero-level set Ω_ε of the polynomial $O(x,y,\varepsilon)$ is an oval surrounding $\mathcal{C}_\varepsilon$.
Then we can conclude that Ω_ε is a transversal curve with respect to the trajectories of system (2.6).

Differentiating O with respect to system (2.6) yields

$$\begin{aligned}\frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)} =& O_1(x,\varepsilon)\varepsilon x + \Big(-\frac{dO_0(x,\varepsilon)}{dx} - O_1(x,\varepsilon)(x^2-1) + O_2(x,\varepsilon)2\varepsilon x\Big)y \\ &- \Big(\frac{dO_1(x,\varepsilon)}{dx} + O_2(x,\varepsilon)2(x^2-1)\Big)y^2 - \frac{dO_2(x,\varepsilon)}{dx}y^3.\end{aligned} \tag{4.2}$$

In order to render that the set

$$\mathcal{N}_\varepsilon := \{(x,y)\in\mathbb{R}^2 : \frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)} = 0\} \tag{4.3}$$

can have an oval $\mathcal{C}_\varepsilon$ in the phase plane where $\frac{dO(x,y,\varepsilon)}{d\sigma}$ changes sign we require

$$\frac{dO_2(x,\varepsilon)}{dx} \equiv 0, \quad -\frac{dO_0(x,\varepsilon)}{dx} - O_1(x,\varepsilon)(x^2-1) + O_2(x,\varepsilon)2\varepsilon x \equiv 0, \tag{4.4}$$

and that $O_1(x,\varepsilon)x$ is an even function in x.
For O_1 we make the ansatz

$$O_1(x,\varepsilon) \equiv c_1(\varepsilon)x + c_3(\varepsilon)x^3, \tag{4.5}$$

moreover, we have by (4.4)

$$O_2(x,\varepsilon) \equiv c_2(\varepsilon). \tag{4.6}$$

Taking into account (4.4) – (4.6), the relation (4.2) reads

$$\frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)} = \varepsilon\big(c_1(\varepsilon)x^2 + c_3(\varepsilon)x^4\big) + \big[-\big(c_1(\varepsilon) + 3c_3(\varepsilon)x^2\big) - 2c_2(\varepsilon)(x^2-1)\big]y^2. \tag{4.7}$$

Further we want to guarantee that the sign of $-\big(c_1(\varepsilon) + 3c_3x^2\big) - 2c_2(\varepsilon)(x^2-1)$ does not depend on x. For this purpose we put

$$c_1(\varepsilon) = 2c_2(\varepsilon) \tag{4.8}$$

such that we have

$$\frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)} = x^2\Big[\varepsilon c_3(\varepsilon)x^2 + \varepsilon 2c_2(\varepsilon) - \big(3c_3(\varepsilon) + 2c_2(\varepsilon)\big)y^2\Big]. \tag{4.9}$$

If we require $c_3(\varepsilon) < 0, c_2(\varepsilon) > 0, 3c_3(\varepsilon) + 2c_2(\varepsilon) > 0$, then $\mathcal{N}_\varepsilon$ contains an ellipse $\mathcal{C}_\varepsilon$ where $\frac{dO(x,y,\varepsilon)}{dt}$ changes sign. Finally we set

$$c_2(\varepsilon) = -3c_3(\varepsilon) \tag{4.10}$$

such that it holds

$$\frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)} = 6x^2 c_3(\varepsilon)\Big[\frac{\varepsilon x^2}{6} + \frac{y^2}{2} - \varepsilon\Big]. \tag{4.11}$$

Thus, we have the result

Lemma 4.1. *The derivative $\frac{dO(x,y,\varepsilon)}{d\sigma}_{|(2.6)}$ vanishes on the y-axis, it has positive (negative) sign at all other points of the region located in the interior (exterior) of the ellipse*

$$\mathcal{C}_\varepsilon := \{(x,y) \in \mathbb{R}^2 : \frac{\varepsilon x^2}{6} + \frac{y^2}{2} = \varepsilon\}.$$

The following lemma is obvious

Lemma 4.2. *The ellipse $\mathcal{C}_\varepsilon$ surrounds the ellipse $\mathcal{I}_\varepsilon$ for any ε.*

In the final step we have to ensure that the outer boundary Ω_ε surrounds the ellipse $\mathcal{C}_\varepsilon$. From (4.5), (4.8) and (4.10) we get

$$O_1(x,\varepsilon) = c_3(\varepsilon)(-6x + x^3), \quad O_2(x,\varepsilon) = -3c_3(\varepsilon). \tag{4.12}$$

Using these functions we obtain from (4.4) for $O_0(x,\varepsilon)$ the differential equation

$$\frac{dO_0(x,\varepsilon)}{dx} = -c_3(\varepsilon)\big[(x^2-1)(-6x+x^3) + 6\varepsilon x\big] \tag{4.13}$$

which has the first integral

$$O_0(x,\varepsilon) = -3c_3(\varepsilon)\Big(\frac{1}{18}x^6 - \frac{7}{12}x^4 + (1+\varepsilon)x^2 + c_0(\varepsilon)\Big). \tag{4.14}$$

Thus, it holds

$$O(x,y,\varepsilon) = -3c_3(\varepsilon)\Big[y^2 + yx(2 - \frac{x^2}{3}) + (1+\varepsilon)x^2 - \frac{7}{12}x^4 + \frac{1}{18}x^6 + c_0(\varepsilon)\Big]. \tag{4.15}$$

If we introduce the polynomial

$$P(x,y,\varepsilon) := y^2 + yx\Big(2 - \frac{x^2}{3}\Big) + (1+\varepsilon)x^2 - \frac{7}{12}x^4 + \frac{1}{18}x^6 + c_0(\varepsilon), \tag{4.16}$$

then the zero-level set Ω_ε of the polynomial $O(x,y,\varepsilon)$ coincides with the zero-level set $\P_\varepsilon$ of the polynomial $P(x,y,\varepsilon)$. In what follows we prove that there is a function $c_0 : \mathbb{R}^+ \to \mathbb{R}^-$ such that $\P_\varepsilon$ is an oval surrounding $\mathcal{C}_\varepsilon$.

Theorem 4.3. *For the continuous function $c_0 : \mathbb{R}^+ \to \mathbb{R}^-$ defined by*

$$c_0(\varepsilon) := -18 - 8\varepsilon - 3\sqrt{\varepsilon} \tag{4.17}$$

$\P_\varepsilon$ represents an oval which is centrosymmetric and surrounds $\mathcal{N}_\varepsilon$.

Proof. First we note that the curve ¶$_\varepsilon$ is invariant under the transformation $(x, y) \to (-x, -y)$, that is, ¶$_\varepsilon$ is centrosymmetric. From (4.16) and (4.17) it follows that ¶$_\varepsilon$ is an oval surrounding the origin. To prove that ¶$_\varepsilon$ surrounds $\mathcal{C}_\varepsilon$ we first note that $\mathcal{C}_\varepsilon$ intersects the x-axis at the points $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$. Since both curves are centrosymmetric, it suffices to prove

$$P\Big(x, \sqrt{\frac{\varepsilon}{3}(6-x^2)}, \varepsilon\Big) < 0 \quad \text{for} \quad -\sqrt{6} \le x \le \sqrt{6}$$

in order to guarantee that ¶$_\varepsilon$ surrounds $\mathcal{C}_\varepsilon$.

$$P\Big(x, \sqrt{\frac{\varepsilon}{3}(6-x^2)}, \varepsilon\Big) = \frac{\varepsilon}{3}(6-x^2) + \sqrt{\frac{\varepsilon}{3}(6-x^2)}\Big(2x - \frac{x^3}{3}\Big) + (1+\varepsilon)x^2 - \frac{7}{12}x^4 + \frac{1}{18}x^6 + c_0(\varepsilon)$$

Using

$$max_{0 \le x \le \sqrt{6}}\Big(2x - \frac{x^3}{3}\Big)\sqrt{\frac{6-x^2}{3}} < 3$$

we get

$$P\Big(x, \sqrt{\frac{\varepsilon}{3}(6-x^2)}, \varepsilon\Big) < 2\varepsilon + 3\sqrt{\varepsilon} + (1+\varepsilon)6 + 12 + c_0(\varepsilon) < 18 + 8\varepsilon + 3\sqrt{\varepsilon} + c_0(\varepsilon) = 0.$$

The same inequality is valid for $-\sqrt{6} \le x < 0$. □

From Lemma 4.1 we can conclude that the derivative $\frac{dO}{dt}_{|(2.6)}$ takes on Ω_ε negative values except at the two points on the axis $x = 0$ where the derivative vanishes. This fact implies that any trajectory of system (2.6) which meets Ω_ε will cross it. Using Lemma 3.3, we can conclude that the annulus $\mathcal{A}_\varepsilon$ bounded by the affine algebraic ovales $\mathcal{I}_\varepsilon$ and Ω_ε is a region for system (2.6) to which the Poincaré-Bendixson theorem can be applied. Taking into account Lemma 3.2, we get

Theorem 4.4. *For any $\varepsilon > 0$, the annulus $\mathcal{A}_\varepsilon$ contains a unique limit cycle Γ_ε of the van der Pol equivalent system (2.6). Γ_ε is hyperbolic and stable.*

Annulus $\mathcal{A}_\varepsilon$ bounded by the ovals $\mathcal{I}_\varepsilon$ and Ω_ε together with limit cycle Γ_ε of system (2.6) for cases $\varepsilon = 0.1$ and $\varepsilon = 10$ is presented on the Figure 1.

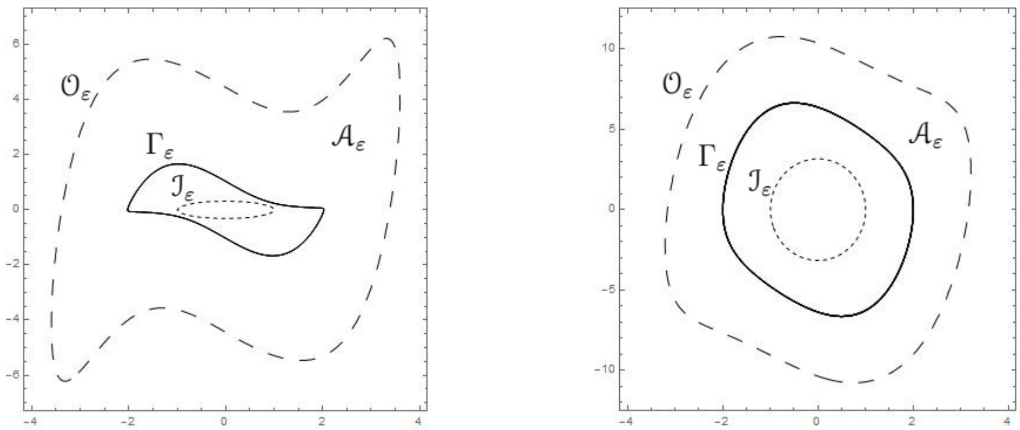

Fig. 1. Annulus $\mathcal{A}_\varepsilon$ with limit cycle Γ_ε of system (2.6) for $\varepsilon = 0.1$ (left) and $\varepsilon = 10$ (right).

5 Global Algebraic Poincaré-Bendixson annulus for equivalent van der Pol systems

In the sections 3 and 4 we have constructed a global algebraic Poincaré-Bendixson annulus for system (2.6). The same procedure we used for system (2.6) could be applied to system (2.2). The question we study now reads: is there another (simpler) way to construct a global algebraic Poincaré-Bendixson annulus for system (2.2)? It is clear that system (2.6) is topologically equivalent to the original van der Pol system (2.2). But if we compare their vector fields by considering the corresponding differential equations

$$\frac{d\bar{y}}{d\bar{x}} = -\frac{\bar{x}}{\bar{y}} + \lambda(\bar{x}^2 - 1) \quad \text{for system (2.2)} \tag{5.1}$$

and

$$\frac{dy}{dx} = -\frac{1}{\lambda^2}\frac{x}{y} + (x^2 - 1) \quad \text{for system (2.6)} \tag{5.2}$$

it is obvious that these fields are different. Thus, the Poincaré-Bendixson annulus $\mathcal{A}_\varepsilon$ constructed for system (2.6) is not a Poincaré-Bendixson annulus for system (2.2). To find a transformation which maps the Poincaré-Bendixson annulus $\mathcal{A}_\varepsilon$ for system (2.6) into a Poincaré-Bendixson annulus for system (2.2) we look for a transformation which maps the vector field defined by system (2.6) onto the vector field defined by system (2.2). If we multiply equation (5.2) by λ and apply the transformation

$$x = \bar{x}, \quad \lambda y = \bar{y}$$

then the differential equation (5.2) will be mapped into the differential equation (5.1). Hence, we have the result

Theorem 5.1. *The algebraic ovals*

$$\bar{x}^2 + \bar{y}^2 = 1$$

and

$$\bar{y}^2 + \lambda\bar{y}\bar{x}\Big(2 - \frac{\bar{x}^2}{3}\Big) + (1+\lambda^2)\bar{x}^2 - \frac{7\lambda^2}{12}\bar{x}^4 + \frac{\lambda^2}{18}\bar{x}^6 - 8 - 3\lambda - 18\lambda^2 = 0$$

form a global algebraic Poincaré-Bendixson annulus for system (2.2).

If we scale for $\lambda > 0$ in system (2.2) the state variables $\bar{x}$ and $\bar{y}$ by the transformation

$$u = \sqrt{\lambda}\bar{x}, v = \sqrt{\lambda}\bar{y}, \tag{5.3}$$

then system (2.2) takes the form

$$\begin{aligned} \frac{du}{dt} &= -v, \\ \frac{dv}{dt} &= u + \lambda v - u^2 v. \end{aligned} \tag{5.4}$$

System (2.2) and system (5.4) are topologically equivalent for $\lambda > 0$, but not for $\lambda = 0$: system (2.2) is linear while system (5.4) is nonlinear, passing λ the value 0 is connected in system (2.2) with the bifurcation of a limit cycle from a circle centered at the origin, while in system (5.4) Hopf bifurcation takes places.

A peculiarity of system (5.4) is that it represent a rotated vector field, that means the limit cycle $\bar{\Gamma}_\lambda$ bifurcating from the origin uniformly expands with λ: $\bar{\Gamma}_{\bar{\lambda}}$ is located in the interior of $\bar{\Gamma}_\lambda$ for $\lambda > \bar{\lambda}$. The question whether $\bar{\Gamma}_\lambda$ exists for all $\lambda > 0$ or if there exists a blow up for a finite value $\hat{\lambda}$ can be answered by constructing a Poincaré-Bendixson annulus where the outer boundary is a bounded curve for any $\lambda > 0$. It is trivial that the vector field defined by system (2.2) will be mapped onto the vector field of system (5.4) by means of the transformation (5.3). Thus we have the result

Theorem 5.2. *The algebraic ovals*

$$u^2 + v^2 = \lambda$$

and

$$v^2 + vu\Big(2\lambda - \frac{u^2}{3}\Big) + (1+\lambda^2)u^2 - \frac{7\lambda}{12}u^4 + \frac{u^6}{18} - 8\lambda - 3\lambda^2 - 18\lambda^3 = 0$$

form a global algebraic Poincaré-Bendixson annulus for system (5.4).

In order to get another topologically equivalent system especially for large λ we use the scaling $t = \lambda\tau$, the notation $\varepsilon = 1/\lambda^2$, and introduce new variables by $\bar{\xi} = \bar{x}$, $\bar{\eta} = \bar{y}/\sqrt{\varepsilon}$. Then system (2.2) can be written in the form

$$\begin{aligned} \frac{d\bar{\xi}}{d\tau} &= -\bar{\eta}, \\ \varepsilon\,\frac{d\bar{\eta}}{d\tau} &= \bar{\xi} - (\bar{\xi}^2 - 1)\bar{\eta}, \end{aligned} \tag{5.5}$$

which represents for small ε, that is for large λ, a singularly perturbed system. This system is equivalent to the differential equation

$$\frac{1}{\lambda}\frac{d\bar{\eta}}{d\bar{\xi}} = -\frac{\lambda\bar{\xi}}{\bar{\eta}} + \lambda(\bar{\xi}^2 - 1). \tag{5.6}$$

Applying the transformation

$$\sqrt{\varepsilon}\bar{\eta} = \frac{\bar{\eta}}{\lambda} = \bar{y}, \quad \bar{\xi} = \bar{x}$$

we get from system (5.5) the van der Pol system (2.2). Thus we have the result

Theorem 5.3. *The algebraic ovales*

$$\bar{\xi}^2 + \varepsilon\bar{\eta}^2 = 1$$

and

$$\varepsilon^2\bar{\eta}^2 + \varepsilon\bar{\eta}\bar{\xi}\Big(2 - \frac{\bar{\xi}^2}{3}\Big) + (1+\varepsilon)\bar{\xi}^2 - \frac{7}{12}\bar{\xi}^4 + \frac{\bar{\xi}^6}{18} - 8\varepsilon - 3\sqrt{\varepsilon} - 18 = 0$$

form a global algebraic Poincaré-Bendixson annulus for system (5.5).

Annulus $\mathcal{A}_\varepsilon$ bounded by the ovals $\mathcal{I}_\varepsilon$ and Ω_ε together with limit cycle Γ_ε of system (5.5) for cases $\varepsilon = 0.15$ and $\varepsilon = 10$ is presented on the Figure 2.

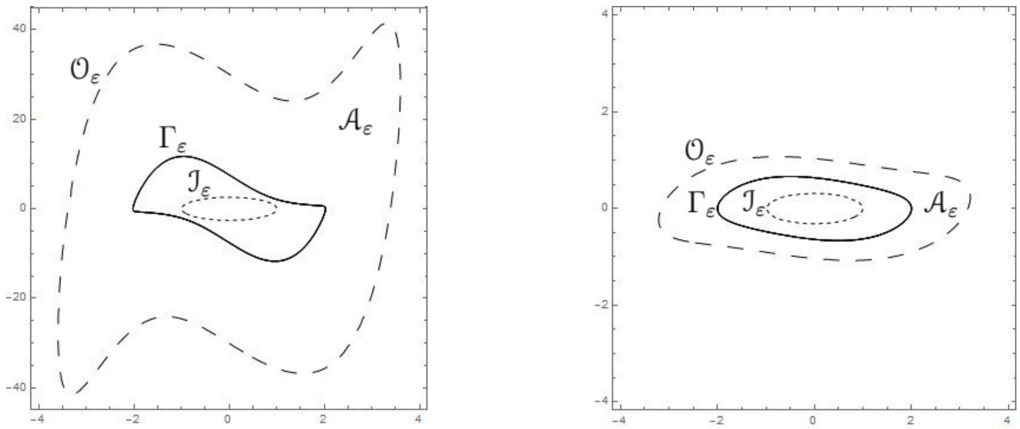

Fig. 2. Annulus $\mathcal{A}_\varepsilon$ with limit cycle Γ_ε of system (5.5) for $\varepsilon = 0.15$ (left) and $\varepsilon = 10$ (right).

6 Appendix

We consider planar autonomous systems

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) \tag{6.1}$$

in some open region $\mathcal{G} \subset \mathbb{R}^2$ under the assumption

(A_1). $P, Q \in C^1(\mathcal{G}, \mathbb{R})$.

We denote by X the vector field defined by (6.1). An important tool for the qualitative investigation of system (6.1) is the Dulac function (see e.g. [9]).

Definition 6.1. *Suppose the assumption (A_1) to be valid. A function $B \in C^1(\mathcal{G}, \mathbb{R})$ is called a Dulac function of system (6.1) in $\mathcal{G}$ if the expression*

$$div(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (gradB, X) + B\, divX$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}$ of measure zero.

The existence of a Dulac function implies the following estimate of the number of limit cycles of system (6.1) in $\mathcal{G}$ [2].

Proposition 6.2. *Let $\mathcal{G}$ be a p-connected ($p \geq 1$) region in $\mathbb{R}^2$, let the assumption (A_1) be satisfied. If there is a Dulac function B of (6.1) in $\mathcal{G}$, then (6.1) has not more than $p - 1$ limit cycles located entirely in $\mathcal{G}$.*

The method of Dulac function has been generalized by L. A. Cherkas in 1997 (see [1]). The corresponding generalized Dulac function, which is also called Dulac-Cherkas function (see [8]), is defined as follows.

Definition 6.3. *Suppose the assumption (A_1) is valid. A function $\Psi \in C^1(\mathcal{G}, \mathbb{R})$ is called a Dulac-Cherkas function of system (6.1) in $\mathcal{G}$ if there exists a real number $\kappa \neq 0$ such that*

$$\Phi := (grad\ \Psi, X) + \kappa\Psi\ div\ X > 0 \quad (< 0) \quad \text{in} \quad \mathcal{G}. \tag{6.2}$$

Remark 6.4. *In case $\kappa = 1$, Ψ is a Dulac function.*

Remark 6.5. *Condition (6.2) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no oval of this set is a limit cycle of (6.1).*

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ defined by

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}. \tag{6.3}$$

The following theorem can be found in [1].

Theorem 6.6. *Assume the assumption (A_1) to be valid. Let Ψ be a Dulac-Cherkas function of* (6.1) *in $\mathcal{G}$. Then any limit cycle Γ of* (6.1) *located entirely in $\mathcal{G}$ has the following properties:*

(i). Γ *does not intersect* $\mathcal{W}$.

(ii). Γ *is hyperbolic.*

(iii). *The stability of Γ is determined by the sign of the expression $k\Phi\Psi$ on Γ.*

Corollary 6.7. *Property (ii) has the strong consequence that the existence of a Dulac-Cherkas function implies that system (6.1) has no multiple limit cycle.*

The following result about the upper bound of the number of limit cycles has been proved in [8].

Theorem 6.8. *Let $\mathcal{G}$ be a p-connected region and suppose the assumption (A_1) to be valid. Let Ψ be a Dulac-Cherkas function of* (6.1) *in $\mathcal{G}$ such that $\mathcal{W}$ consists of s ovals in $\mathcal{G}$. Then system* (6.1) *has at most $p - 1 + s$ limit cycles in $\mathcal{G}$, and all limit cycles are hyperbolic.*

In the special case $p = s = 1$, Theorem 6.8 reads as follows.

Theorem 6.9. *Let $\mathcal{G}$ be a simply connected region and suppose the assumption (A_1) holds. Let Ψ be a Dulac-Cherkas function of* (6.1) *in $\mathcal{G}$ such that $\mathcal{W}$ consists of one oval in $\mathcal{G}$. Then system* (6.1) *has at most one limit cycle in $\mathcal{G}$.*

We note that the method of Dulac-Cherkas functions was also used by A. Gasull and H. Giacomini [5, 6].

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