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Gibbs point processes on path space: Existence, cluster expansion and uniqueness

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Abstract

We study a class of infinite-dimensional diffusions under Gibbsian interactions, in the context of marked point configurations: the starting points belong to $\mathbb{R}^d$, and the marks are the paths of Langevin diffusions. We use the entropy method to prove existence of an infinite-volume Gibbs point process and use cluster expansion tools to provide an explicit activity domain in which uniqueness holds.

Introduction

In this work we consider infinitely many Langevin diffusions in interaction: through the lens of Gibbs point process theory, we see a diffusion – starting in $x \in \mathbb{R}^d$ and with displacement $\big(m(s),\ s \in [0,1]\big)$ – as a marked point $\boldsymbol{x} = (x, m) \in \mathscr{E} := \mathbb{R}^d \times C_0$, where C_0 is the space of continuous paths $\big(m(s), s \in [0,1]\big)$ starting at $m(0) = 0$. On this state space we then consider a pair potential Φ that acts on both the starting points and the trajectories of the marked points. This leads to a Gibbsian energy functional H, with (finite but) not uniformly bounded interaction range, for which the questions of existence and uniqueness of Gibbs point processes are far from trivial. In particular, we note how the random marks are a priori unbounded.

We wish to remark that, while in this work we investigate Gibbs point processes on path space, the existence and uniqueness methods we describe appear to be more general, and could be applied to general marked models with pair interactions.

In this setting we can start from interactions which are common for classical systems in $\mathbb{R}^d$, like the Lennard–Jones pair potential, and use them to describe interactions between paths instead. We remark, however, that the typical potentials that we consider (see Example 1) need a hard-core repulsion near the origin in order to satisfy the stability conditions that are required in the method.

In Section 2 we tackle the existence question, via the Dobrushin–Lanford–Ruelle description of Gibbs point processes. Under some stability assumptions for H, we are able to prove (in Theorem 1) the existence of at least one infinite-volume Gibbs point process on path space P^z with energy functional H, for any activity z and inverse temperature β, by applying the entropy method presented for the general marked setting in [16].

Moreover, we also show that, for any $N \geq 1$, the N-point correlation function ρ_N of these Gibbs point processes satisfy a (point-dependent) *Ruelle bound* of the following form: there exists a function $\boldsymbol{c}\colon \mathscr{E} \to \mathbb{R}_+$ such that, for almost any finite path configuration $\big(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\big) \in \mathscr{E}^N$,

$$\rho_N(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) \leq \prod_{i=1}^{N} \boldsymbol{c}(\boldsymbol{x}_i).$$

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In Sections 3 and 4 we present, as a novel result, an explicit activity domain where uniqueness of the Gibbs point process holds. This is obtained with the approach of cluster expansion and the Kirkwood–Salsburg equations – a method which was first developed for lattice systems in the 1980s (see e.g. [8]) and then extended to the continuous case (see e.g. [9, 11]). We are hopeful these techniques and assumptions – presented here making use of the specificity of the path space properties – could be adapted to different marked settings.

In the case of unmarked continuous point processes, the technique relies on considering a series expansion of the correlation functions. As presented by D. Ruelle in [19], one first shows that the correlation functions of a Gibbs point process can be expressed as an absolutely converging series of cluster terms, and then proves uniqueness by considering a system of integral equations – the so-called Kirkwood–Salsburg equations – that the correlation functions satisfy. In fact, these equations can be reformulated as a fixed-point problem for an operator $\boldsymbol{K}_z$ in an appropriately chosen Banach space, having therefore a unique solution.

The cluster expansion approach is actually well adapted to the marked setting. Indeed, S. Poghosyan and D. Ueltschi develop, in [14], abstract techniques that can be used both in the classical and in the marked setting, under assumptions of so-called *modified-regularity* of the interaction. These assumptions and techniques are further developed in [15] by S. Poghosyan and H. Zessin, proving uniqueness of infinite-volume Gibbs point processes for potentials satisfying a certain stability condition (which they refer to as *Penrose stability*). Some similar result is presented by S. Jansen in [3], but making strong use of the repulsive nature of the interaction she considers. These techniques can be restrictive in our setting of unbounded marks (see Example 4), so we use here a different approach: inspired by the work [5] of T. Kuna, our approach relies on some tree-graph estimates, that allow to prove a Ruelle bound for the correlation functions of infinite-volume Gibbs point processes.

A key point, presented in Section 3 under a different set of assumptions than that of Section 2, consists in using cluster expansion to obtain a Ruelle bound for the correlation functionals of a Gibbs point process of activity z. In particular, we show that, under an additional regularity assumption for the interaction potential Φ, there exists an activity threshold $\mathfrak{z}_{Ru}(\beta) > 0$ such that, for any $z \in (0, \mathfrak{z}_{Ru}(\beta))$, the correlation functions $\rho_N^{(P)}$ of any Gibbs point process P with activity z and inverse temperature β satisfy a Ruelle bound as above, but where $\boldsymbol{c}$ is uniformly bounded: for almost any $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^N$, $\rho_N^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq \boldsymbol{c}^N$. This shows that the correlation functions belong to a certain Banach space $\mathbb{X}_{\boldsymbol{c}}$.

In Section 4, after showing that there exists an activity threshold $\mathfrak{z}_{\text{crit}}(\beta) > 0$ such that, for any $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, the norm of the Kirkwood–Salsburg operator $\boldsymbol{K}_z$ in $\mathbb{X}_{\boldsymbol{c}}$ is bounded by 1, we show that the associated equations have a unique solution and obtain the following uniqueness domain (in Theorem 2): for any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, there exists a unique infinite-volume Gibbs point process P with activity z and inverse temperature β associated to the energy functional H.

1 The setting

We consider infinitely-many independent gradient diffusions and add a dependence between them by introducing an *interaction energy* in the context of marked Gibbs point processes. In this setting, we adopt the DLR description and set up the existence and uniqueness questions that are explored in the later sections.

1.1 Infinite-dimensional free system of Langevin diffusions

The basic mathematical object of this work is the following Langevin dynamics on $\mathbb{R}^d$:

$$dX(s) = dB(s) - \frac{1}{2}\nabla V\big(X(s)\big)ds, \quad s \in [0,1], \tag{1}$$

where B is a standard $\mathbb{R}^d$-valued Brownian motion, and $V\colon \mathbb{R}^d \to \mathbb{R}$ is a smooth potential satisfying, outside of some compact subset of $\mathbb{R}^d$,

$$\exists \delta', b_1, b_2 > 0, \quad V(x) \geq b_1|x|^{d+\delta'} \text{ and } \Delta V(x) - \frac{1}{2}|\nabla V(x)|^2 \leq -b_2|x|^{2+2\delta'}. \tag{2}$$

It is a known result (see e.g. [17]) that, under these conditions, there exists a unique solution to the SDE (1), which generates an *ultracontractive* semigroup (see [4, 2]). Moreover, for any $\delta < \delta'/2$,

$$\mathbb{E}\left[e^{\sup_{s\in[0,1]}|X(s)-X(0)|^{d+2\delta}}\right] < +\infty. \tag{3}$$

For the rest of this work, let $\delta > 0$ be fixed.

1.2 The system with Gibbsian interaction

Consider now that any (continuous) path $\boldsymbol{x}$ on $[0,1]$ can be decomposed into its initial location x and a (shifted) path m starting from 0. In other words, we identify $\boldsymbol{x}$ with the pair $(x,m) \in \mathscr{E} := \mathbb{R}^d \times C_0$, where C_0 is the space of continuous paths on $[0,1]$ starting at 0. The space C_0, endowed with the norm $\|m\|$ given by the maximum displacement of the trajectory m, that is $\|m\| := \sup_{s\in[0,1]}|m(s)|$, is a normed space.

On C_0, we consider the measure $\mathbf{R}$, given by the law of the process X solution of (1) starting at $X(0) = 0$. Notice that, thanks to (3), for any $\delta < \delta'/2$,

$$\int_{C_0} e^{\|m\|^{d+2\delta}}\mathbf{R}(dm) < +\infty. \tag{4}$$

We consider point measures on the product state space $\mathscr{E}$. More precisely, we take the following product measure on $\mathscr{E}$:

$$\lambda(dx, dm) = dx \otimes \mathbf{R}(dm).$$

We denote by $\mathscr{M}$ the space of simple point measures (*configurations*) on $\mathscr{E}$, i.e. of all σ-finite measures of the form

$$\gamma = \sum_i \delta_{\boldsymbol{x}_i},\ \boldsymbol{x}_i = (x_i, m_i) \in \mathscr{E},\ \text{with } \boldsymbol{x}_i \neq \boldsymbol{x}_j \text{ if } i \neq j.$$

Since the configurations are simple, we identify them with the subset of their atoms:

$$\gamma \equiv \big\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \ldots\big\} \subset \mathscr{E}.$$

Moreover, for two disjoint configurations $\gamma, \xi \in \mathscr{M}$, we denote by $\gamma\xi$ their concatenation: $\gamma\xi := \gamma \cup \xi$. For $\gamma \in \mathscr{M}$, $|\gamma|$ denotes the number of its points; $\mathscr{M}_f \subset \mathscr{M}$ is the subset of *finite* configurations, i.e. with $|\gamma| < +\infty$. We denote by $\underline{o}$ the configuration supported on the empty set.

For any $\Lambda \subset \mathbb{R}^d$, $\mathscr{M}_\Lambda \subset \mathscr{M}$ denotes the subset of point measures with support in $\Lambda \times C_0$, and $\gamma_\Lambda \coloneqq \gamma \cap (\Lambda \times C_0)$. Let $\mathscr{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$, and $\mathscr{B}_b(\mathbb{R}^d)$ the set of bounded Borel subsets of $\mathbb{R}^d$, which we often call *finite volumes*. For $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, $|\Lambda|$ denotes its volume.

We denote by $\mathscr{P}(\mathscr{M})$ (resp. $\mathscr{P}(\mathscr{M}_\Lambda)$) the set of probability measures (or *point processes*) on $\mathscr{M}$ (resp. $\mathscr{M}_\Lambda$). Finally, let $\mathbb{N}^* \coloneqq \mathbb{N} \setminus \{0\}$.

We consider the following measure (of infinite mass):

Definition 1.1. Fix $z > 0$. We define the measure $\tilde{\pi}^z = 1 + \sum_{N=1}^{+\infty} \dfrac{z^N}{N!} \lambda^{\otimes N}$ on $\mathscr{M}_f$.

For any finite volume Λ, we consider as reference probability measure the *marked Poisson point process* $\pi_\Lambda^z \in \mathscr{P}(\mathscr{M}_\Lambda)$ with intensity parameter z, defined by renormalising the restriction $\tilde{\pi}_\Lambda^z$ of $\tilde{\pi}^z$ to $\mathscr{M}_\Lambda$ as follows:

$$\pi_\Lambda^z(d\gamma) = e^{-z|\Lambda|} \tilde{\pi}_\Lambda^z(d\gamma).$$

As a modification of the Poisson point process, we introduce an interaction between the paths by considering the finite-volume Gibbs point process associated to an energy functional H. More precisely:

Definition 1.2. An *energy functional* $H\colon \mathscr{M}_f \to \mathbb{R} \cup \{+\infty\}$ is a measurable functional on the set of finite configurations, with $H(\underline{o}) = 0$ by convention. In this work we consider the energy of a finite number $N \geq 1$ of paths to be defined, for any $\gamma = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \in \mathscr{M}_f$, by the sum of a self-interaction term and a pair-potential term:

$$H(\gamma) \coloneqq \sum_{i=1}^{N} \Psi(\boldsymbol{x}_i) + \beta \sum_{1 \leq i < j \leq N} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathbb{R} \cup \{+\infty\}, \tag{5}$$

where $\beta > 0$ is the *inverse temperature*.

We denote the *pair-interaction* component of the energy as

$$E_\Phi(\gamma) \coloneqq \sum_{1 \leq i < j \leq N} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j),$$

and the *conditional energy* of any path $\boldsymbol{x} \in \mathscr{E}$ given any $\xi \in \mathscr{M}$ as

$$E_\Phi\left(\boldsymbol{x} \,|\, \xi\right) \coloneqq \sum_{\boldsymbol{y} \in \xi} \Phi(\boldsymbol{x}, \boldsymbol{y}).$$

Note that this infinite sum is not always well defined (see Assumption 2).

Finally, for any $\gamma \in \mathscr{M}$, let

$$E_\Phi\left(\gamma \,|\, \xi\right) \coloneqq \sum_{\boldsymbol{x} \in \gamma} E_\Phi\left(\boldsymbol{x} \,|\, \xi\right).$$

be the conditional energy of the configuration γ given the configuration ξ.

We specify later a growth condition on the self potential Ψ, and consider different sets of assumptions on the pair potential $\Phi\colon \mathscr{E} \times \mathscr{E} \to \mathbb{R} \cup \{+\infty\}$.

Definition 1.3. Let H be an energy functional as in (5). For any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, the free-boundary-condition *finite-volume Gibbs point process* on Λ with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$ is the probability measure $P_\Lambda^{z,\beta}$ on $\mathscr{M}_\Lambda$ defined by

$$P_\Lambda^{z,\beta}(d\gamma) \coloneqq \frac{1}{Z_\Lambda^z} e^{-\beta H(\gamma_\Lambda)} \, \pi_\Lambda^z(d\gamma), \tag{6}$$

where the *partition function* Z_Λ^z is the renormalisation constant.

In this work we investigate the existence and uniqueness of an infinite-volume Gibbs point process, in the following sense:

Definition 1.4. Let H be an energy functional as in (5). A probability measure P on $\mathscr{M}$ is said to be an *infinite-volume Gibbs point process* with energy functional H, activity $z > 0$ and inverse temperature $\beta > 0$, denoted $P \in \mathscr{G}_{z,\beta}(H)$, if it satisfies, for any $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$ and any positive, bounded, and measurable functional $F : \mathscr{M} \to \mathbb{R}$, the following *DLR equation* (for Dobrushin–Landford–Ruelle)

$$\int_{\mathscr{M}} F(\gamma)\, P(d\gamma) = \int_{\mathscr{M}} \frac{1}{Z^z_\Lambda(\xi)} \int_{\mathscr{M}_\Lambda} F(\gamma_\Lambda \xi_{\Lambda^c}) e^{-\beta\left(H(\gamma_\Lambda)+E_\Phi(\gamma_\Lambda|\xi_{\Lambda^c})\right)} \pi^z_\Lambda(d\gamma)\, P(d\xi), \qquad \text{(DLR)}$$

where the partition function $Z^z_\Lambda(\xi)$ depends on the boundary condition ξ.

A concept that will help in showing that such an infinite-volume measure exists is that of tempered configuration. For such a configuration γ, the number $|\gamma_\Lambda|$ of its points in any finite volume Λ, should grow sublinearly w.r.t. the volume, while the norm $\|m\|$ of its marks should grow as a fractional power of it. More precisely,

Definition 1.5. The set of *tempered path configurations* is given by the increasing union $\mathscr{M}^{\text{temp}} := \bigcup_{\mathbf{t}\in\mathbb{N}^*} \mathscr{M}^{\mathbf{t}}$, where

$$\mathscr{M}^{\mathbf{t}} := \Big\{\gamma \in \mathscr{M} : \forall l \in \mathbb{N}^*, \sum_{\substack{(x,m)\in\gamma \\ |x|\le l}} (1 + \|m\|^{d+2\delta}) \le \mathbf{t} l^d\Big\}. \tag{7}$$

We denote by $\mathscr{G}^{\text{temp}}_{z,\beta}(H) := \mathscr{G}_{z,\beta}(H) \cap \mathscr{P}(\mathscr{M}^{\text{temp}})$ the set of *tempered Gibbs point processes*, i.e. those whose support is included in the tempered configurations.

In what follows we show that a tempered Gibbs point process associated to infinitely-many interacting Langevin dynamics exists as soon as the interaction energy satisfies some quite natural assumptions.

Assumption 1 (Self interaction growth and stability)**.**

$(\mathscr{H}_{\text{self}})$ The self potential $\Psi : \mathscr{E} \to \mathbb{R} \cup \{+\infty\}$ acting on each path is bounded from below by the opposite of a power of its maximum displacement, i.e.

$$\exists A_\Psi > 0 : \inf_{x\in\mathbb{R}^d} \Psi(x,m) \ge -A_\Psi \|m\|^{d+\delta}. \tag{8}$$

$(\mathscr{H}_{\text{st.}})$ The pair potential $\Phi : \mathscr{E} \times \mathscr{E} \to \mathbb{R} \cup \{+\infty\}$ between two paths is a symmetric functional that satisfies the following *stability* condition: there exists a constant $B_\Phi \ge 0$ such that for any finite configuration $\gamma = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\} \in \mathscr{M}_f$,

$$E_\Phi(\gamma) = \sum_{1\le i<j\le N} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) \ge -B_\Phi N. \tag{9}$$

2 Existence of a Gibbs point process of diffusions

The proof of the existence of an infinite-volume Gibbs point process that we describe here makes use of the specific entropy functional as a tightness tool, as in the general approach presented in [16]. In order for our path model of interacting Langevin diffusions to fit the setting of the aforementioned paper, in this section we consider energy functionals H that satisfy, in addition to Assumption 1, the following:

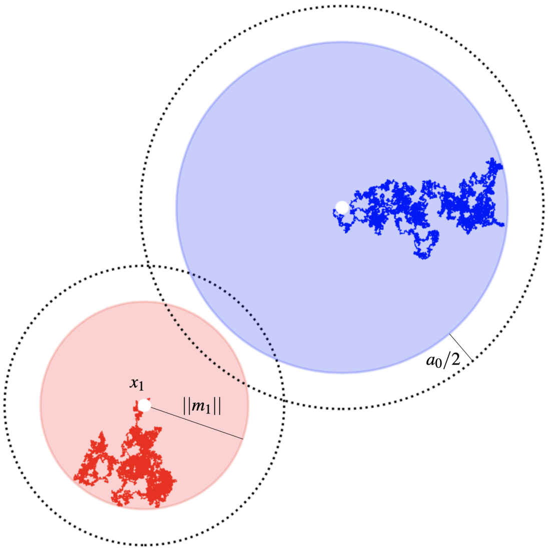

Figure 1: *Two interacting paths of a Langevin diffusion in* $\mathbb{R}^2$*. Each circle is centred in the starting point, while the radii of the coloured circles correspond to their maximum displacement in the time interval* $[0,1]$*; the dotted circles represent the security distance* $a_0/2$ *introduced in* (10).

Assumption 2 (Range and local stability)**.** The pair potential $\Phi : \mathscr{E} \times \mathscr{E} \to \mathbb{R} \cup \{+\infty\}$ is such that

$(\mathscr{H}_{\mathsf{r}})$ Two diffusions $\boldsymbol{x}_i, \boldsymbol{x}_j$ do not interact whenever they start *too far away*: there exists a constant $a_0 \geq 0$ such that

$$\Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) = 0 \text{ whenever } |x_i - x_j| > a_0 + \|m_i\| + \|m_j\|. \tag{10}$$

$(\mathscr{H}_{\mathsf{loc.st}})$ There exists a constant $\bar{B}_\Phi \geq 0$ such that, for any path $\boldsymbol{x} \in \mathscr{E}$, for any configuration $\xi \in \mathscr{M}^{\mathsf{temp}}$,

$$E_\Phi(\boldsymbol{x} \,|\, \xi) = \sum_{\boldsymbol{y} \in \xi} \Phi(\boldsymbol{x}, \boldsymbol{y}) \geq -\bar{B}_\Phi(1 + \|m\|^{d+\delta}). \tag{11}$$

Remark. We briefly comment on these two assumptions: the expression in (11) is well defined since, as we will see in the proof of Theorem 1, the range assumption $(\mathscr{H}_{\mathsf{r}})$ implies that the infinite sum of the conditional energy of $\boldsymbol{x}$ given ξ is actually given by a finite (random) number of terms.

It is easy to show that the following Lemma holds for the support of any Gibbs point process:

Lemma 2.1. For any activity $z > 0$ and inverse temperature $\beta > 0$, any infinite-volume Gibbs point process $P \in \mathscr{G}^{\mathsf{temp}}_{z,\beta}(H)$ is supported on configurations with locally finite energy, that is configurations $\gamma \in \mathscr{M}^{\mathsf{temp}}$ such that, for any $\Delta \in \mathscr{B}_b(\mathbb{R}^d)$, $E_\Phi(\gamma_\Delta) < +\infty$. Note that this is true also whenever Φ takes infinite values.

Example 1. Consider the following class of interactions, described by a path pair potential of the form

$$\Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) = \Big(\int_0^1 \phi(|x_i - x_j + m_i(s) - m_j(s)|) ds \Big) 1_{[0, a_0 + \|m_i\| + \|m_j\|]}(|x_i - x_j|), \tag{12}$$

with ϕ given by the sum of two potentials on $\mathbb{R}_+$: $\phi = \phi_{hc} + \phi_l$, where

- The potential ϕ_{hc} is pure *hard core* at some diameter $R > 0$, that is

$$\phi_{hc}(u) = (+\infty) 1_{[0,R)}(u).$$

- The potential ϕ_l satisfies a stability property, i.e. there exists a constant $B_\phi \geq 0$ such that, for any admissible configuration $\{y_1, \dots, y_N\}$, $N \geq 1$, the following holds (see [19], paragraph

3.2.5):

$$\sum_{i=1}^{N} \phi_l(|y_i|) \geq -2B_\phi, \tag{13}$$

where a finite configuration $\{y_1, \dots, y_N\} \subset \mathbb{R}^d$, $N \geq 1$, is called *admissible* if, for any pair $y_i \neq y_j$, $\phi(|y_i - y_j|) < +\infty$.

Note how the coefficient a_0 here plays the role of a *sensitivity parameter* (see Figure 1): if the pair potential ϕ is repulsive (i.e. positive), then a_0 can take any finite positive value. If instead ϕ is attractive (i.e. negative) on some region, a_0 should be chosen in such a way that ϕ remains attractive on $[a_0, +\infty)$: $\phi(u) \leq 0$ if $u \geq a_0$ (see Figure 2). We now show that this class of potentials satisfy Assumption 2.

Proof. Firstly, thank to the previous Lemma, we can actually restrict our study to the admissible configurations. It is easy to see that the stability (9) of the potential Φ holds with $B_\Phi = B_\phi$. Moreover, setting $\mathfrak{l}(\mathbf{t}) := 2^{\frac{d+\delta}{\delta}-1}\mathbf{t}^{\frac{1}{\delta}}$, one can see that the range of the interaction is bounded by

$$\mathfrak{r}(\gamma, \Lambda) = 2\mathfrak{l}(\mathbf{t}) + 2 \sup_{\boldsymbol{x} \in \gamma_\Lambda} \|m\| + 1 + a_0,$$

i.e. for any $\boldsymbol{x} = (x, m) \in \mathscr{E}$ and $\xi \in \mathscr{M}^{\mathbf{t}}$, $\mathbf{t} \geq 1$, setting $\Delta := B(x, \mathfrak{r}(\gamma, \Lambda))$, the conditional energy $E_\Phi(\boldsymbol{x} \,|\, \xi \setminus \{x\})$ of $\boldsymbol{x}$ given ξ is actually given by $E_\Phi\left(\boldsymbol{x} \,|\, \xi_{\Delta \setminus \{x\}}\right)$: it is a finite sum, and is bounded from below by $-2B_\phi$.

$$\begin{aligned} E_\Phi\left(\boldsymbol{x} \,|\, \xi_{\Delta \setminus \{x\}}\right) &= \Big(\int_0^1 \sum_{\boldsymbol{x}_i \in \xi_{\Delta \setminus \{x\}}} \phi(|x - x_i + m(s) - m_i(s)|) ds \Big) 1_{\{|x - x_i| \leq a_0 + \|m\| + \|m_i\|\}} \\ &\geq -2B_\phi. \end{aligned}$$

Notice how, under these conditions, the trajectories of two interacting paths $\boldsymbol{x}_1 = (x_1, m_1)$ and $\boldsymbol{x}_2 = (x_2, m_2)$ are allowed to intersect, but at each time s the paths keep at a distance of at least R; the hard-core component, indeed, imposes $|x_1 + m_1(s) - x_2 + m_2(s)| \geq R$ for any $s \in [0, 1]$. □

A particular case: Let ϕ be given by the sum of a hard-core component and a shifted Lennard–Jones potential, i.e.

$$\phi(u) = \phi_{hc}(u) + \phi_{LJ}(u - R) 1_{[R, +\infty)}(u), \; u \in \mathbb{R}_+,$$

where $\phi_{LJ}(u) = \frac{a}{u^{12}} - \frac{b}{u^6}$, $a, b > 0$. Pictured in Figure 2 is an example with $R = 1$. We remark that this potential has a non-integrable growth in a neighbourhood of its hard core component; in particular, it does not satisfy Assumption 3 below, which is used for the uniqueness proof.

Example 2. One can consider a class of *translation-invariant* pair potentials. More precisely, let Φ be invariant by translation: $\Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) = \Phi(\boldsymbol{x}_i - \boldsymbol{x}_j)$, with

$$\Phi(\boldsymbol{x}) = \Big(\int_0^1 \phi(|x + m(s)|) ds \Big) 1_{\{|x| \leq a_0 + \|m\|\}},$$

where ϕ is given by the above sum of a hard-core component and a shifted Lennard–Jones potential.

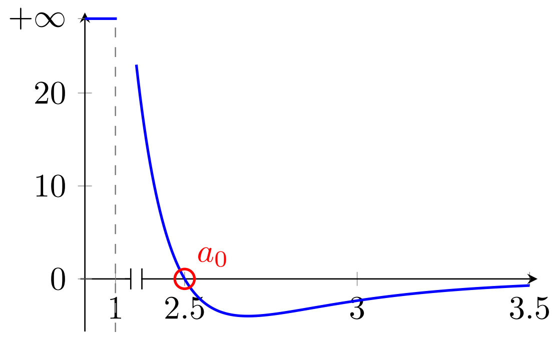

Figure 2: *A shifted Lennard–Jones potential* $\phi_{LJ}(u-1) = 16\big(\big(\frac{3/2}{u-1}\big)^{12} - \big(\frac{3/2}{u-1}\big)^{6}\big)$ *with hard core diameter* $R = 1$*; it is always negative after* $a_0 = 2.5$*, and explodes as* $x \to 1^+$.

Definition 2.2. Consider a configuration $\gamma \in \mathscr{M}$. For any $N \geq 1$, its *factorial measure of order* N is given by

$$\gamma^{(N)}(d\boldsymbol{x}_1, \ldots, d\boldsymbol{x}_N) := \gamma(d\boldsymbol{x}_1)(\gamma \setminus \{\boldsymbol{x}_1\})(d\boldsymbol{x}_2) \ldots (\gamma \setminus \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N-1}\})(d\boldsymbol{x}_N).$$

By taking the expectation under a point process P we obtain its N*-th factorial moment measure*: a measure $\alpha_N^{(P)}$ on $\mathscr{E}^N$ defined by

$$\alpha_N^{(P)}(\cdot) := \mathbb{E}_P[\gamma^{(N)}(\cdot)].$$

For any point process P, one can consider, for any $N \geq 1$, its N-point correlation function, defined as the Radon–Nikodym derivative of its N-th factorial moment measure $\alpha_N^{(P)}$ with respect to the product measure $(z\sigma)^{\otimes N}$, where

$$\sigma(d\boldsymbol{x}) := e^{-\Psi(\boldsymbol{x})}\lambda(d\boldsymbol{x}).$$

Proposition 2.3 ([12])**.** Let $P \in \mathscr{G}_{z,\beta}(H)$, $z > 0$, $\beta > 0$. Its N*-point correlation function* admits, for $\sigma^{\otimes N}$-almost all $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^N$, the following representation:

$$\rho_N^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = e^{-\beta E_\Phi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \,|\, \xi)} P(d\xi), \tag{14}$$

as soon as this expression is well defined.

Remark. Note that $\rho_N^{(P)}(\cdot)$ is a symmetric function, as for any $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^k$ and any permutation $\{i_1, \ldots, i_N\}$, $\rho_N^{(P)}(\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_N}) = \rho_N^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$.

We can now state our existence result.

Theorem 1. Let H be an energy functional as in (5), satisfying Assumptions 1+2. For any $z > 0$ and $\beta > 0$, there exists at least one infinite-volume tempered Gibbs point process $P^{z,\beta} \in \mathscr{G}_{z,\beta}^{\text{temp}}(H)$. Moreover, for any $N \geq 1$, the N-point correlation function of $P^{z,\beta}$ exists and can be written as in (14).

Proof. Let $z > 0$, $\beta > 0$. In order to apply the existence result of [16] to this path space context, we show that a stability condition holds both for the energy of a finite configuration and for the conditional energy, and that the random interaction range is finite (possibly unbounded). These conditions are called in [16], $(\mathscr{H}_{\text{st.}})$, $(\mathscr{H}_{\text{r}})$, and $(\mathscr{H}_{\text{loc.st}})$.

Step 1. We start by noting that (13) implies that the potential ϕ – defined on the location space $\mathbb{R}^d$ – is *stable* in the sense of Ruelle (see [20]), with stability constant B_ϕ, i.e.

$$\forall N \geq 1,\, \forall \{y_1, \ldots, y_N\} \subset \mathbb{R}^d, \quad \sum_{1 \leq i < j \leq N} \phi(|y_i - y_j|) \geq -B_\phi N.$$

The conditions (8) and (9) – on the self interaction and pair potential, respectively, yield the following *stability* for the energy of a finite number of paths:

$$\forall \gamma \in \mathscr{M}_f, \quad H(\gamma) \geq -(B_\phi \vee A_\Psi)\big(|\gamma| + \sum_{(x,m)\in\gamma} \|m\|^{d+\delta}\big).$$

Step 2. We now focus on analysing the *range of the interaction*: we show that for any tempered configuration $\gamma \in \mathscr{M}^{\mathbf{t}}$, $\mathbf{t} \geq 1$, and for any finite volume Λ, there exists a positive number $\mathfrak{r} = \mathfrak{r}(\gamma, \Lambda)$ such that

$$E_\Phi\left(\boldsymbol{x} \,|\, \xi\right) = \sum_{\substack{\boldsymbol{y}\in\xi \\ 0<|y-x|\leq\mathfrak{r}}} \Phi(\boldsymbol{x}, \boldsymbol{y}). \tag{15}$$

Set $\mathfrak{l}(\mathbf{t}) := 2^{\frac{d+\delta}{\delta}-1}\mathbf{t}^{\frac{1}{\delta}}$. Using the definition of tempered configurations, one has that, for all $l \geq \mathfrak{l}(\mathbf{t})$ and for any $\boldsymbol{x} \in \gamma \in \mathscr{M}^{\mathbf{t}}$ such that $|x| > 2l + 1 + a_0$,

$$|x| - \|m\| \overset{(7)}{\geq} |x| - \frac{1}{2}\lceil |x| \rceil \geq l + a_0.$$

Thanks to condition $(\mathscr{H}_{\mathrm{r}})$, this means that the range of the interaction is bounded by

$$\mathfrak{r}(\gamma, \Lambda) = 2\mathfrak{l}(\mathbf{t}) + 2\sup_{\boldsymbol{x}\in\gamma_\Lambda} \|m\| + 1 + a_0.$$

Step 3. Fix $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$, and consider, for $\gamma \in \mathscr{M}$ and $\xi \in \mathscr{M}^{\mathrm{temp}}$, the conditional energy of γ_Λ given ξ_{Λ^c}, that is:

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) := H(\gamma_\Lambda) + E_\Phi\left(\gamma_\Lambda \,|\, \xi_{\Lambda^c}\right).$$

Thanks to (15), denoting $\Delta := \Lambda \oplus B(0, \mathfrak{r}(\gamma, \Lambda))$, we have

$$E_\Phi\left(\gamma_\Lambda \,|\, \xi_{\Lambda^c}\right) = E_\Phi\left(\gamma_\Lambda \,|\, \xi_{\Delta\setminus\Lambda}\right) = \sum_{\boldsymbol{x}_i\in\gamma_\Lambda} \sum_{\boldsymbol{x}_j\in\xi_{\Delta\setminus\Lambda}} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j). \tag{16}$$

It is unfortunately not true – as used instead in Section 4 of [16] – that we can control the cardinality of the second sum, i.e. the number of points of ξ_Δ, *uniformly in* γ. On the other hand, thanks to Lemma 2.1, we can assume that ξ_Δ is of finite energy, and therefore use (13) to estimate

$$\begin{aligned}
&\sum_{\boldsymbol{x}_i\in\gamma_\Lambda} \sum_{\boldsymbol{x}_j\in\xi_{\Delta\setminus\Lambda}} \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j) \\
&\quad = \int_0^1 \sum_{\boldsymbol{x}_i\in\gamma_\Lambda} \sum_{\boldsymbol{x}_j\in\xi_{\Delta\setminus\Lambda}} \phi(|x_i - x_j + m_i(s) - m_j(s)|)ds\, 1_{\{|x_i-x_j|\leq a_0+\|m_i\|+\|m_j\|\}} \\
&\quad \overset{(13)}{\geq} \int_0^1 \sum_{\boldsymbol{x}_i\in\gamma_\Lambda} -2B_\phi \geq -2B_\phi|\gamma_\Lambda|.
\end{aligned}$$

Together with the stability of $\gamma_\Lambda \mapsto H(\gamma_\Lambda)$, this yields the following lower bound for the conditional energy:

$$H_\Lambda(\gamma_\Lambda \xi_{\Lambda^c}) \geq -(A_\Psi \vee 2B_\phi) \sum_{\boldsymbol{x}\in\gamma_\Lambda} (1 + \|m\|^{d+\delta}),$$

We can now apply, having checked its three conditions $(\mathscr{H}_{\mathrm{st.}})$, $(\mathscr{H}_{\mathrm{r}})$, and $(\mathscr{H}_{\mathrm{loc.st}})$, Theorem 1 of [16]: there exists an infinite-volume Gibbs measure $P^{z,\beta} \in \mathscr{G}^{\mathrm{temp}}_{z,\beta}(H)$.

The correlation functions of a Gibbs point process can be written as in (14) whenever the term $e^{-\sum_i \sum_{\boldsymbol{y}\in\xi} \Phi(\boldsymbol{x}_i,\boldsymbol{y})}$ is well defined. Thanks to (11), this is indeed the case, as we have $\sum_{\boldsymbol{y}\in\xi} \Phi(\boldsymbol{x}, \boldsymbol{y}) \geq -\bar{B}_\Phi(1 + \|m\|^{d+\delta})$. □

Proposition 2.4. For any $N \geq 1$, the N-point correlation function $\rho_N^{(P^{z,\beta})}$ of any Gibbs point process $P^{z,\beta}$ constructed above satisfy a *Ruelle bound*: for $\sigma^{\otimes N}$-almost all $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^N$,

$$\rho_N^{(P^{z,\beta})}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i), \tag{17}$$

where $\boldsymbol{c}(x,m) := \exp\left(\beta B_\Phi + \beta \bar{B}_\Phi(1 + \|m\|^{d+\delta})\right)$.

Proof. Putting together (9) and (11), we estimate

$$\rho_N^{(P^{z,\beta})}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq e^{\beta B_\Phi N} \int_{\mathscr{M}^{\text{temp}}} e^{\beta \sum_{i=1}^N \bar{B}_\Phi (1+\|m_i\|^{d+\delta})} P^{z,\beta}(d\xi),$$

yielding the desired bound. □

Example 1 (continued)**.** For the class of potentials described in Example 1, the Ruelle bound holds uniformly in $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^N$, and is of the form

$$\rho_N^{(P^{z,\beta})}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq e^{3\beta B_\phi N}. \tag{18}$$

3 Ruelle bounds for correlation functions

Suppose you have a pair potential Φ – not necessarily satisfying the assumptions of the previous section – and that you already have an infinite-volume Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$, not necessarily constructed as above. In this section – under an additional regularity condition (Assumption 3) – we use tools from cluster expansion (see, for example, [18, 20]) to find a domain of activity $(0, \mathfrak{z}_{Ru}(\beta))$ such that, for any $z \in (0, \mathfrak{z}_{Ru}(\beta))$, the correlation functions of P exist and satisfy a Ruelle bound.

An important tool is given by the Ursell kernel (see the work by R.A. Minlos and S. Poghosyan in [10]), introduced in Section 3.2. As this method requires the correlation functions have a specific representation, we assume a priori here that, for any $N \geq 1$, the expression (14) for the N-point correlation function $\rho_N^{(P)}$ of any $P \in \mathscr{G}_{z,\beta}(H)$ is well defined.

3.1 Correlation functions

While we have so far decomposed the energy functional in (5) into self- and pair-interaction terms, in order to set ourselves in the framework of cluster expansion – that typically deals exclusively with pair interactions – in what follows we include the self-interaction term in the reference measure, and define, for $z > 0$, the measure

$$\tilde{\pi}^{z\sigma} = \sum_{N=0}^{+\infty} \frac{z^N}{N!} \sigma^{\otimes N},$$

and the corresponding Poisson point process $\pi^{z\sigma}$. The finite-volume Gibbs point process $P_\Lambda^{z,\beta}$ defined in (6) on $\mathscr{M}_\Lambda$ can then be equivalently defined using $\pi^{z\sigma}$ and just the pair interaction $E_\Phi(\gamma) = \sum_{\{\boldsymbol{x},\boldsymbol{y}\}\subset\gamma} \Phi(\boldsymbol{x},\boldsymbol{y})$ (in place of the full energy functional H):

$$P_\Lambda^{z,\beta}(d\gamma) = \frac{1}{Z_\Lambda^{z\sigma}} e^{-\beta E_\Phi(\gamma_\Lambda)} \, \pi_\Lambda^{z\sigma}(d\gamma),$$

where $Z_\Lambda^{z\sigma}$ is the normalisation constant.

As we already mentioned, the proof of the uniqueness of the Gibbs point process revolves around the study of its correlation functions, which we now introduce. We start by introducing a finite-volume correlation function induced by the interaction Φ:

Definition 3.1. Let $z > 0, \beta > 0$. For any finite volume $\Lambda \subset \mathbb{R}^d$, the *finite-volume correlation function* $\rho_\Lambda^{(z,\beta)}$ in Λ (with free boundary condition) is given, for any $\gamma \in \mathscr{M}_\Lambda$, by

$$\rho_\Lambda^{(z,\beta)}(\gamma) = \frac{1}{\tilde{Z}_\Lambda^{z\sigma}} \int_{\mathscr{M}_\Lambda} e^{-\beta E_\Phi(\xi\gamma)} \tilde{\pi}_\Lambda^{z\sigma}(d\xi),$$

where $\tilde{Z}_\Lambda^{z\sigma}$ is the normalisation constant.

Remark. Note that, from the stability (9) of the pair potential Φ, there exists a functional

$$\boldsymbol{i} : \mathscr{M}_f \setminus \{\underline{o}\} \to \mathscr{E}$$

such that for any non-empty path configuration γ there exists a path $\boldsymbol{i}(\gamma) \in \gamma$ where the sum of its interactions with the other paths in γ is bounded from below:

$$\forall \gamma \in \mathscr{M}_f \setminus \{\underline{o}\}, \quad E_\Phi\left(\boldsymbol{i}(\gamma) \,|\, \gamma \setminus \{\boldsymbol{i}(\gamma)\}\right) \geq -2B_\Phi. \tag{19}$$

As a consequence, Φ is bounded from below by $-2B_\Phi$:

$$\inf \Phi(\boldsymbol{x}, \boldsymbol{y}) \geq -2B_\Phi. \tag{20}$$

In Example 1 below we make use of (20), while (19) is used in Proposition 3.8.

In the following we fix the inverse temperature parameter $\beta > 0$, and consider energy functionals H such that, additionally to Assumption 1, the following holds:

Assumption 3 (regularity)**.** The pair potential Φ satisfies the following uniform *regularity condition* (for some, and therefore any, $\beta > 0$):

$$\boldsymbol{C}(\beta) \coloneqq \sup_{\boldsymbol{x} \in \mathscr{E}} \int_{\mathscr{E}} |e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y})} - 1| \sigma(d\boldsymbol{y}) < +\infty.$$

Example 1 (continued)**.** Suppose the potential $\phi = \phi_{hc} + \phi_l$ is *integrable* outside of the hard core, that is

$$\|\phi\|_{R_+} \coloneqq \int_R^{+\infty} |\phi_l(u)| u^{d-1} du < +\infty,$$

then Assumption 3 holds. Indeed, since for any $x \in (-\infty, +\infty]$,

$$|e^{-x} - 1| \leq x^- e^{x^-} + (1 - e^{-x^+}) \leq |x| e^{x^-}, \tag{21}$$

where $x^- \coloneqq \max(0, -x)$ and $x^+ \coloneqq \max(0, x)$ are the negative and positive part of x, respectively, we have

$$|e^{-\beta\Phi} - 1| \leq \beta |\bar{\Phi}| e^{2\beta B_\phi},$$

where we denote by $\bar{\Phi}$ the *truncated pair potential*, defined, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{E}$, by

$$\bar{\Phi}(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 1 & \text{if } \Phi(\boldsymbol{x}, \boldsymbol{y}) = +\infty \\ \Phi(\boldsymbol{x}, \boldsymbol{y}) & \text{otherwise.} \end{cases}$$

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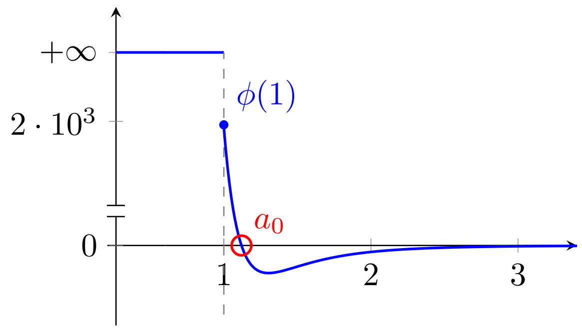

Figure 3: *The sum of a hard-core potential ϕ_{hc} and the Lennard–Jones potential ϕ_{LG}. The integrable component ϕ_{LG} of the potential has a maximum in $\phi_{LG}(1)$.*

Let $\bar{\phi}(u) := 1_{\{u<R\}} + \phi(u)1_{\{u\geq R\}}$. Using the above bound, we can estimate, for any $\boldsymbol{x}_1 \in \mathscr{E}$,

$$\begin{aligned}
&\int_{\mathscr{E}} |e^{-\beta\Phi(\boldsymbol{x}_1,\boldsymbol{x}_2)} - 1|\sigma(d\boldsymbol{x}_2) \leq e^{2\beta B_\phi} \int_{\mathscr{E}} \beta|\bar{\Phi}(\boldsymbol{x}_1,\boldsymbol{x}_2)|\,\sigma(d\boldsymbol{x}_2) \\
&\leq e^{2\beta B_\phi} \int_{\mathscr{E}} \int_0^1 \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))|ds\, 1_{\{|x_2-x_1|\leq a_0+\|m_2\|+\|m_1\|\}}\, \sigma(d\boldsymbol{x}_2) \\
&\overset{(8)}{\leq} e^{2\beta B_\phi} \int_{C_0} \int_0^1 \int_{\mathbb{R}^d} \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))|\, dx_2\, ds\, e^{A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2) \\
&\leq e^{2\beta B_\phi}\beta\left(b_d R^d + \|\phi\|_{R_+}\right) \int_{C_0} e^{A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2),
\end{aligned}$$

which is finite thanks to the ultra-contractivity assumption, see (4).

A particular case: Suppose the potential ϕ is given by the sum of a hard-core potential ϕ_{hc} in $[0, R)$ and the Lennard–Jones potential $\phi_l \equiv \phi_{LJ}$ in $[R, +\infty)$. In particular, it is finite in $[R, +\infty)$, with maximum $\phi_l(R)$. Pictured in Figure 3 is an example with $R = 1$.

3.2 Cluster expansion: Ursell kernel and tree-graph estimates

In this subsection, after introducing the *Ursell kernel*, we use it to rewrite the correlation functions of a Gibbs point process and – following an approach inspired by [1, 5] – use *tree-graph estimates* to obtain a Ruelle bound for them. Our innovation comes from being able to obtain that the correlation functions of any Gibbs point process satisfy a Ruelle bound with the same constant $\boldsymbol{c}_z$, uniformly in the finite volume, therefore yielding uniqueness in the set of tempered Gibbs point processes.

We consider here *undirected connected graphs*. For any non-empty set $A \subset \mathbb{R}^d$, a *graph* G on A is given by a pair (V, E): the vertex set V is a subset of A, and the set of edges is a subset of $\{\{x, y\} \subset A : x \neq y\}$. Indeed, for a graph $G = (V, E)$ on A, we write $\{x, y\} \in G$ to denote the edge $xy \in E$ between two vertices $x, y \in V$. A *tree* T is a connected graph without loops. We also introduce the following notations:

- $\mathcal{C}_n(A)$ denotes the set of all undirected connected graphs with n vertices belonging to A.
- $\mathcal{T}(A)$ denotes the set of all *trees* on A.

Note that the notion of graph $G = (V, E) \in \mathcal{C}_n(A)$ does not depend on the possible orderings of the points of the vertex set $V = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$. Moreover, when there is no risk of confusion, we identify a graph G on $\{x_1, \ldots, x_n\}$ with the corresponding one on the index set $\{1, \ldots, n\} \in \mathbb{N}$ (i.e. where the edge $\{x_i, x_j\}$ corresponds with the edge $\{i, j\}$, see Figure 4).

When using these notations on a finite configuration $\gamma \subset \mathbb{R}^d \times C_0$, with an abuse of notations, we write $C_n(\gamma)$ as shorthand for $C_n(\text{proj}_{\mathbb{R}^d}(\gamma))$ (analogously for $\mathcal{T}$).

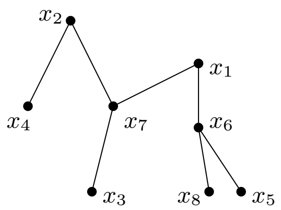

Figure 4: *Example of a treee $T \in \mathcal{T}(A)$, where $A = \{x_1, \dots, x_8\} \subset \mathbb{R}^2$. It can be equivalently described by placing the points of A on the vertices of a tree $\tilde{T}$ on $\{1, \dots, 8\} \in \mathbb{N}$. More precisely, $\tilde{T}$ on $\{1, \dots, 8\}$ is constructed by placing an edge $\{i, j\} \in \tilde{T}$ if and only if there is an edge $\{x_i, x_j\} \in T$.*

Definition 3.2. For any two measurable functionals $F, G\colon \mathscr{M}_f \to \mathbb{R}$, define their $*$*-product* by

$$(F * G)(\gamma) := \sum_{\xi \subset \gamma} F(\gamma \setminus \xi) G(\xi), \quad \gamma \in \mathscr{M}_f.$$

with identity $1^*(\gamma) := 1_{\{\gamma = \underline{o}\}}$. The space of measurable functionals with this operation is an algebra $\mathscr{A}$. Moreover, the set

$$\mathscr{A}_0 := \{F \in \mathscr{A} \colon F(\underline{o}) = 0\}$$

is an ideal of $\mathscr{A}$. The *exponential* and *logarithm operators* are defined by

$$\exp^* F := \sum_{n \geq 0} \tfrac{1}{n!} F^{*n}, \quad \log^*(1^* + F) := \sum_{n \geq 1} \tfrac{(-1)^{n-1}}{n} F^{*n}.$$

Definition 3.3 (Ursell function and kernel)**.** We introduce the two following notions:

- The *Ursell function* $k\colon \mathscr{M}_f \to \mathbb{R}$ is a functional on finite configurations, defined by setting

$$k(\gamma) := \log^*(e^{-\beta E_\Phi})(\gamma), \quad \gamma \in \mathscr{M}_f.$$

  Equivalently ([1], Proposition 4.3), $k(\underline{o}) = 0$ and, for any γ with $|\gamma| = n \geq 1$,

$$k(\gamma) = \sum_{G \in C_n(\gamma)} \prod_{\{\boldsymbol{x}, \boldsymbol{y}\} \in G} \left(e^{-\beta \Phi(\boldsymbol{x}, \boldsymbol{y})} - 1\right).$$

- The *Ursell kernel* $\bar{k}\colon \mathscr{M}_f \times \mathscr{M}_f \to \mathbb{R}$ is defined on disjoint configurations by

$$\bar{k}(\gamma, \xi) := \left[\exp^*(-k) * (e^{-\beta E_\Phi})(\gamma, \cdot)\right](\xi), \quad \gamma, \xi \in \mathscr{M}_f, \ \gamma \cap \xi = \underline{o}.$$

The Ursell kernel relates to the Ursell function as follows:

Lemma 3.4 ([1], Lemma 4.6)**.** For any finite configuration $\gamma \neq \underline{o}$,

$$\forall \boldsymbol{x} \in \gamma, \quad \bar{k}(\{\boldsymbol{x}\}, \gamma \setminus \{\boldsymbol{x}\}) = k(\gamma).$$

Moreover, it provides a new expression for the correlation functions:

Lemma 3.5 ([1], Proposition 4.5)**.** Let $\gamma \in \mathscr{M}_\Lambda$, $\Lambda \in \mathscr{B}_b(\mathbb{R}^d)$. If $\int_{\mathscr{M}_\Lambda} |k(\xi)| \tilde{\pi}^{z\sigma}(d\xi) < +\infty$, then

$$\rho_\Lambda^{(z,\beta)}(\gamma) = \int_{\mathscr{M}_\Lambda} \bar{k}(\gamma, \xi) \tilde{\pi}^{z\sigma}(d\xi). \tag{22}$$

Lemma 3.6 ([1], Remark 4.8)**.** The Ursell kernel $\bar{k}$ is the unique solution of the so-called non-integrated Kirkwood–Salsburg equation

$$\begin{cases} \bar{k}(\gamma,\xi) = e^{-\beta \sum_{\boldsymbol{y}\in\gamma\setminus\{\boldsymbol{x}\}} \Phi(\boldsymbol{x},\boldsymbol{y})} \sum_{\eta\subset\xi} k_\eta(\boldsymbol{x}) \bar{k}\big((\gamma\setminus\{\boldsymbol{x}\})\eta, \xi\setminus\eta\big) \\ \bar{k}(\underline{o},\xi) = 1_{\{\xi=\underline{o}\}}, \end{cases} \tag{23}$$

where $k_\eta(\boldsymbol{x}) := \prod_{\boldsymbol{y}\in\eta} \big(e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y})} - 1\big)$, and $\boldsymbol{x} \in \gamma$ is chosen arbitrarily.

We now introduce a second functional Q, which satisfies a similar equation to (23), dominates the Ursell kernel, and its simpler expression allows for more convenient computations.

Definition 3.7. Consider a functional Q on $\mathscr{M}_f \times \mathscr{M}_f$ defined as follows: for any $\xi \in \mathscr{M}_f$, $Q(\underline{o},\xi) = 1_{\{\xi=\underline{o}\}}$, and for any $\gamma = \{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\}$, $N \geq 1$,

$$Q(\gamma,\xi) := \sum_{\substack{\xi_1,\dots,\xi_N\subset\xi \\ \xi_i\cap\xi_j=\underline{o}\,\forall i\neq j}} Q(\{\boldsymbol{x}_1\},\xi_1)\cdots Q(\{\boldsymbol{x}_N\},\xi_N),$$

where

$$\begin{cases} Q(\{\boldsymbol{x}\},\xi) := e^{2\beta B_\Phi(|\xi|+1)} \sum_{T\in\mathcal{T}(\{x\}\cup\xi)} \prod_{\{\boldsymbol{y}_1,\boldsymbol{y}_2\}\in T} |e^{-\beta\Phi(\boldsymbol{y}_1,\boldsymbol{y}_2)} - 1| & \text{if } \xi \neq \underline{o} \\ Q(\{\boldsymbol{x}\},\underline{o}) = e^{2\beta B_\Phi}. \end{cases} \tag{24}$$

Proposition 3.8 ([1], Proposition 4.10)**.** The functional Q defined above is the unique solution of

$$\begin{cases} Q(\gamma,\xi) = e^{2\beta B_\Phi} \sum_{\eta\subset\xi} |k_\eta(\boldsymbol{i}(\gamma))| Q(\gamma\setminus\boldsymbol{i}(\gamma)\cup\eta, \xi\setminus\eta) \\ Q(\underline{o},\xi) = 1_{\{\xi=\underline{o}\}}, \end{cases}$$

where the functional $\boldsymbol{i}$ was defined in (19).

Corollary 3.9 ([1], Proposition 4.11)**.** For any $\gamma = \{\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\}$, $N \geq 1$, and $\xi \in \mathscr{M}_f$ such that $\gamma\cap\xi = \underline{o}$, we have

$$\begin{aligned} |\bar{k}(\gamma,\xi)| &\leq Q(\gamma,\xi) \\ &= \sum_{\substack{\xi_1,\dots,\xi_N\subset\xi \\ \xi_i\cap\xi_j=\underline{o}\,\forall i\neq j}} Q(\{\boldsymbol{x}_1\},\xi_1)\dots Q(\{\boldsymbol{x}_N\},\xi_N), \end{aligned}$$

and

$$|k(\gamma)| \leq e^{2\beta B_\Phi|\gamma|} \sum_{T\in\mathcal{T}(\gamma)} \prod_{\{\boldsymbol{x}_i,\boldsymbol{x}_j\}\in T} |e^{-\Phi(\boldsymbol{x}_i,\boldsymbol{x}_j)} - 1|.$$

Lemma 3.10. For any finite volume $\Lambda \subset \mathbb{R}^d$ and $N \geq 1$, for λ-a.a. $\boldsymbol{x} \in \mathscr{E}$,

$$\begin{aligned} &\int_{(\Lambda\times C_0)^N} Q(\{\boldsymbol{x}\},\{\boldsymbol{y}_1,\dots,\boldsymbol{y}_N\})\sigma(d\boldsymbol{y}_1)\cdots\sigma(d\boldsymbol{y}_N) \\ &\qquad \leq e^{2\beta B_\Phi(N+1)} C(\beta)^{N-1}(N+1)^{N-1} \int_{\Lambda\times C_0} |e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y})} - 1|\sigma(d\boldsymbol{y}). \end{aligned}$$

Proof. Using (24), we rewrite the l.h.s. as

$$e^{2\beta B_\Phi(N+1)} \sum_{T\in\mathcal{T}([N+1])} \underbrace{\int_{(\Lambda\times C_0)^N} \prod_{\{i,j\}\in T} |e^{-\beta\Phi(\boldsymbol{y}_i,\boldsymbol{y}_j)}-1|\,\sigma(d\boldsymbol{y}_1)\cdots\sigma(d\boldsymbol{y}_N)}_{=:I_N},$$

where we set $\boldsymbol{y}_{N+1} := \boldsymbol{x}$, and $[N+1] := \{1,\dots,N+1\}$. We estimate I_N by induction on $N\geq 1$:

- For $N=1$,
$$I_1 = \int_{\Lambda\times C_0} |e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y}_1)}-1|\sigma(d\boldsymbol{y}_1).$$
- For the inductive step, assume that, for all $T\in\mathcal{T}([N])$,
$$\int_{(\Lambda\times C_0)^{N-1}} \prod_{\{i,j\}\in T} |e^{-\beta\Phi(\boldsymbol{y}_i,\boldsymbol{y}_j)}-1| \bigotimes_{i=1}^{N-1}\sigma(d\boldsymbol{y}_i) \leq \boldsymbol{C}(\beta)^{N-2}\int_{\Lambda\times C_0}|e^{-\beta\Phi(\boldsymbol{y}_N,\boldsymbol{y})}-1|\sigma(d\boldsymbol{y}).$$
- Let $T\in\mathcal{T}([N+1])$ be given, and root it in $\boldsymbol{y}_{N+1}$. There exists then an edge $\{j_1,j_2\}\in T$, where $\boldsymbol{y}_{j_1}$ is a leaf, and $\boldsymbol{y}_{j_1}\neq \boldsymbol{y}_{j_{N+1}}$. We obtain
$$\begin{aligned}&\int_{(\Lambda\times C_0)^N} \prod_{\{i,j\}\in T} |e^{-\beta\Phi(\boldsymbol{y}_i,\boldsymbol{y}_j)}-1| \bigotimes_{i=1}^{N}\sigma(d\boldsymbol{y}_i)\\ &= \int_{(\Lambda\times C_0)^{N-1}} \underbrace{\int_{\Lambda\times C_0} |e^{-\beta\Phi(\boldsymbol{y}_{j_1},\boldsymbol{y}_{j_2})}-1|\,\sigma(d\boldsymbol{y}_{j_1})}_{\leq \boldsymbol{C}(\beta)} \prod_{\{i,j\}\in T\setminus\{\{j_1,j_2\}\}} |e^{-\beta\Phi(\boldsymbol{y}_i,\boldsymbol{y}_j)}-1| \bigotimes_{\substack{i=1\\ i\neq j_1}}^{N}\sigma(d\boldsymbol{y}_i)\\ &\leq \boldsymbol{C}(\beta)\int_{(\Lambda\times C_0)^{N-1}} \prod_{\{i,j\}\in T\setminus\{\{j_1,j_2\}\}} |e^{-\beta\Phi(\boldsymbol{y}_i,\boldsymbol{y}_j)}-1| \bigotimes_{\substack{i=1\\ i\neq j_1}}^{N}\sigma(d\boldsymbol{y}_i).\end{aligned}$$
  We can then use the inductive step to prove the assertion.

Moreover,

$$e^{2\beta B_\Phi(N+1)} \sum_{T\in\mathcal{T}([N+1])} I_N \leq e^{2\beta B_\Phi(N+1)} \sum_{T\in\mathcal{T}([N+1])} \boldsymbol{C}(\beta)^{N-1}\int_{\Lambda\times C_0}|e^{-\beta\Phi(\boldsymbol{y}_{N+1},\boldsymbol{y})}-1|\sigma(d\boldsymbol{y}),$$

and the claim follows, since the number of elements of $\mathcal{T}([N+1])$ is $(N+1)^{N-1}$ (see Theorem 4.1.3 of [13]). □

Lemma 3.11. Define the threshold activity

$$\mathfrak{z}_{Ru}(\beta) := (\boldsymbol{C}(\beta)e^{2\beta B_\Phi+1})^{-1}, \tag{25}$$

and let $z<\mathfrak{z}_{Ru}(\beta)$. For any finite volume $\Lambda\subset\mathbb{R}^d$, for $\tilde{\pi}^{z\sigma}$-a.a. $\gamma\in\mathscr{M}_f$, if $|\gamma| = N\geq 1$,

$$\int_{\mathscr{M}_\Lambda} |\bar{k}(\gamma,\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq \boldsymbol{c}_z^N,$$

where

$$\boldsymbol{c}_z := e^{2\beta B_\Phi}\left(1+\frac{e}{\sqrt{2\pi}}\log\left(\frac{1}{1-z/\mathfrak{z}_{Ru}(\beta)}\right)\right) < +\infty. \tag{26}$$

Moreover, for any $z<\mathfrak{z}_{Ru}(\beta)$,

$$\int_{\mathscr{M}_\Lambda} |k(\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) < +\infty. \tag{27}$$

Proof. Let $\gamma = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$. From Corollary 3.9,

$$\int_{\mathscr{M}_\Lambda} |\bar{k}(\gamma,\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq \prod_{i=1}^N \int_{\mathscr{M}_\Lambda} Q(\{\boldsymbol{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi).$$

Thanks to Lemma 3.10,

$$\int_{\mathscr{M}_\Lambda} Q(\{\boldsymbol{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi) = \sum_{N=0}^{+\infty} \frac{z^N}{N!} \int_{(\Lambda\times C_0)^N} Q(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_1,\ldots,\boldsymbol{y}_N\})\,\sigma(d\boldsymbol{y}_1)\cdots\sigma(d\boldsymbol{y}_N)$$

$$= e^{2\beta B_\Phi} + \sum_{N=1}^{+\infty} \frac{z^N}{N!} e^{2B_\phi(N+1)} C(\beta)^{N-1}(N+1)^{N-1} \underbrace{\int_{\Lambda\times C_0} |e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y})} - 1|\sigma(d\boldsymbol{y})}_{\leq C(\beta)}$$

$$\leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}} \sum_{N=1}^{+\infty} \frac{(z\,C(\beta)e^{2\beta B_\Phi+1})^N}{N^{3/2}}\right) \leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}} \sum_{N=1}^{+\infty} \frac{(z\,C(\beta)e^{2\beta B_\Phi+1})^N}{N}\right),$$

where, in the third step, we used the inequality $(N+1)^{N-1} \leq \frac{1}{\sqrt{2\pi}} e^{N+1} \frac{N!}{(N+1)^{3/2}}$, which is a consequence of Stirling's formula: for any $n \geq 0$,

$$\sqrt{2\pi} n^{n+1/2} e^{-n} e^{1/(12n+1)} \leq n! \Rightarrow n^{n-2} \leq \frac{1}{\sqrt{2\pi}} e^n \frac{(n-1)!}{n^{3/2}}.$$

For $z < (C(\beta)e^{2\beta B_\Phi+1})^{-1} =: \mathfrak{z}_{Ru}(\beta)$, the above series converges, and we obtain

$$\int_{\mathscr{M}_\Lambda} Q(\{\boldsymbol{x}_i\},\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi) \leq e^{2\beta B_\Phi}\left(1 + \frac{e}{\sqrt{2\pi}} \log\left(\frac{1}{1 - z/\mathfrak{z}_{Ru}(\beta)}\right)\right) =: \boldsymbol{c}_z.$$

By using Corollary 3.9, and proceeding similarly to the proof of Lemma 3.10, we obtain that, for $z < \mathfrak{z}_{Ru}(\beta)$,

$$\int_{\mathscr{M}_\Lambda} |k(\xi)|\tilde{\pi}^{z\sigma}_\Lambda(d\xi) < +\infty.$$

□

Remark. Note that $\boldsymbol{c}_z$ depends on z but is uniform in Λ; moreover, $\boldsymbol{c}_0 = e^{2\beta B_\Phi}$.

3.3 A Ruelle bound for correlation functions

As a consequence of (27), we can use the representation (22) of the correlation function

$$\rho^{(z,\beta)}_\Lambda(\gamma) = \int_{\mathscr{M}_\Lambda} \bar{k}(\gamma,\xi)\tilde{\pi}^{z\sigma}_\Lambda(d\xi),$$

and use the above tree-graph estimates to obtain the following Ruelle bound:

Proposition 3.12. Let $\beta > 0$ and $\mathfrak{z}_{Ru}(\beta)$ as defined in (25). For a pair potential Φ satisfying Assumptions 1+3, for any activity $z \in (0, \mathfrak{z}_{Ru}(\beta))$ and any finite volume $\Lambda \subset \mathbb{R}^d$, the finite-volume correlation function $\rho^{(z,\beta)}_\Lambda$ satisfies, for $\tilde{\pi}^{z\sigma}$-a.a. $\gamma \in \mathscr{M}_\Lambda$,

$$\rho^{(z,\beta)}_\Lambda(\gamma) \leq \boldsymbol{c}_z^{|\gamma|}, \tag{28}$$

where the constant $\boldsymbol{c}_z$ is defined in (26). Moreover, a similar bound holds for the N-point correlation functions of any $P \in \mathscr{G}_{z,\beta}(H)$: for any $z \in (0, \mathfrak{z}_{Ru}(\beta))$, for any $N \geq 1$, for $\sigma^{\otimes N}$-almost all $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathscr{E}^N$

$$\rho^{(P)}_N(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq \boldsymbol{c}_z^N. \tag{29}$$

Proof. Fix $z < \mathfrak{z}_{Ru}(\beta)$. The first statement is an immediate consequence of Lemma 3.11. Moreover, as the right hand side of (28) does not depend on Λ, this bound also holds in the limit as $\Lambda \uparrow \mathbb{R}^d$, so for the limiting correlation function $\rho_f^{(z)}(\gamma) := \int_{\mathscr{M}_f} \bar{k}(\gamma, \xi)\tilde{\pi}^{z\sigma}(d\xi)$, $\gamma \in \mathscr{M}_f$.

For the second statement, consider $\gamma = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$. It is known (see [15], Lemmas 12 and 15), that the limiting correlation functional $\rho_f^{(z)}(\gamma)$ coincides with the correlation function $\rho_N^{(P)}(\gamma)$ whenever the expression in (14) is well defined. As this is true thanks to (14), the Ruelle bound (29) holds for any $P \in \mathscr{G}_{z,\beta}(H)$. □

4 Uniqueness via the Kirkwood–Salsburg equations

We are in the following situation: we have an infinite-volume Gibbs point process $P \in \mathscr{G}_{z,\beta}(H)$ associated to a potential Φ (not necessarily constructed as in Section 2) and whose correlation functions satisfy a Ruelle bound, and wish to understand whether it is indeed the unique such process associated to Φ and with activity z.

In this section we assume that, additionally to Assumption 1, the correlation functions of any $P \in \mathscr{G}_{z,\beta}(H)$ can be represented as in (14):

$$\rho_N^{(P)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \,|\, \xi)} P(d\xi).$$

The uniqueness proof is structured as follows: we prove that the correlation functions of a Gibbs point process satisfy the Kirkwood–Salsburg equations. Moreover, thanks to the Ruelle bounds, these correlation functions belong to an appropriate Banach space, where these equations have at most one solution. From this, we obtain the uniqueness of the Gibbs point process P.

4.1 The Kirkwood–Salsburg equations

The key of this part is to show that the correlation functions $(\rho_N^{(P)})_N$ of any $P \in \mathscr{G}_{z,\beta}(H)$ solve, for all $N \geq 1$, for $\sigma^{\otimes(N+1)}$-almost all $(\boldsymbol{x}_0, \dots, \boldsymbol{x}_N) \in \mathscr{E}^{N+1}$, the sequence of *Kirkwood–Salsburg equations*

$$\begin{aligned} \rho_{N+1}^{(P)}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_N) &= e^{-\beta E_\Phi(\boldsymbol{x}_0 \,|\, \boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} \Big(\rho_N^{(P)}(\boldsymbol{x}_1, \dots \boldsymbol{x}_N) \\ &+ \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1)\rho_{N+k}^{(P)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N, \boldsymbol{y}_1, \dots, \boldsymbol{y}_k)\sigma^{\otimes k}(d\boldsymbol{y}_1, \dots, d\boldsymbol{y}_k)\Big), \end{aligned} \qquad (\mathrm{KS})_z$$

where, by convention, $\rho_0^{(P)} = 1$.

Note that the different nature of the Ruelle bounds of Section 2 and 3 – the former allows for a dependence on the marks of the N points that the latter does not – requires two different approaches. We first treat, in Section 4.2, the simpler case where the Ruelle bound holds for a constant $\boldsymbol{c} > 0$; in Section 4.3 we consider the situation in which the Ruelle bound holds for a positive function $\boldsymbol{c} \colon \mathscr{E} \to \mathbb{R}_+$. Accordingly, the Banach space in which we prove uniqueness is defined as follows:

Definition 4.1. The Banach space $\mathbb{X}_{\boldsymbol{c}}$ is the set of all sequences $r = (r_N)_N$ such that

$$\exists\, \mathfrak{b}_r \geq 0 : \forall N \geq 1,\ |r_N(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)| \leq \mathfrak{b}_r \prod_{i=1}^{N} \boldsymbol{c}(\boldsymbol{x}_i),$$

endowed with the norm $\|r\|_{\boldsymbol{c}}$ equal to the smallest such $\mathfrak{b}_r$.

Note that, in the case of $\boldsymbol{c} > 0$ constant, the right hand side reads $\mathfrak{b}_r \boldsymbol{c}^N$.

We can then interpret the Kirkwood–Salsburg equations as an operator acting on the Banach space $\mathbb{X}_{\boldsymbol{c}}$.

Definition 4.2. Consider the *Kirkwood–Salsburg operator* $\boldsymbol{K}_z$, $z > 0$, acting on $\mathbb{X}_{\boldsymbol{c}}$, given by

$$(\boldsymbol{K}_z r)_1(\boldsymbol{x}_0) = \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1) r_{N+k}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \ldots, d\boldsymbol{y}_k)\big);$$

$$(\boldsymbol{K}_z r)_{N+1}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N) = e^{-\beta \sum_{i=1}^{N} \Phi(\boldsymbol{x}_0,\boldsymbol{x}_i)} \big(r_N(\boldsymbol{x}_1, \ldots \boldsymbol{x}_N) \tag{30}$$
$$+ \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1) r_{N+k}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \ldots, d\boldsymbol{y}_k)\big),\ N \geq 1.$$

The Kirkwood–Salsburg equations(KS)$_z$ can now be rewritten as the following fixed-point problem in the Banach space $\mathbb{X}_{\boldsymbol{c}}$:

$$r = \boldsymbol{K}_z r + \underline{1}_z,$$

where $\underline{1}_z = (\underline{1}_{z,N})_N$ is given by $\underline{1}_{z,1}(\boldsymbol{x}_1) = 1$, $\underline{1}_{z,N} = 0$ for $N \geq 2$.

4.2 The case of uniform Ruelle bounds

In this subsection, we work with energy functionals E_Φ and activities $z > 0$ such that Assumptions 1+3+4 hold. We consider the case of a Ruelle bound that holds for a constant $\boldsymbol{c}$, *uniformly* in the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$, that is:

Assumption 4 (Uniform Ruelle bound)**.** Assume there exists a constant $\boldsymbol{c} > 0$ such that, for any $P \in \mathscr{G}_{z,\beta}(H)$, for any $N \geq 1$, for $\sigma^{\otimes N}$-almost all $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathscr{E}^N$, its correlation function $\rho_N^{(P)}$ satisfy, uniformly in $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$, the following Ruelle bound:

$$\rho_N^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq \boldsymbol{c}^N. \tag{31}$$

Example 3. In what we have seen above, this holds

- For any $z > 0$, for Φ as in Example 1, $P = P^{z,\beta}$ (see (18)), with $\boldsymbol{c} = e^{3\beta B_\phi}$.
- For $z \in (0, \mathfrak{z}_{Ru}(\beta))$, under Assumption 3, with $\boldsymbol{c} = \boldsymbol{c}_z$ as defined in (26).

Proposition 4.3. Let $z > 0$, $\beta > 0$. Under Assumptions 1+3+4, the correlation functions $(\rho_N^{(P)})_N$ of any $P \in \mathscr{G}_{z,\beta}(H)$ solve, for all $N \geq 1$, for $\sigma^{\otimes(N+1)}$-almost all $(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N) \in \mathscr{E}^{N+1}$, the *Kirkwood–Salsburg equation* (KS)$_z$ defined above.

Proof. Thanks to the stability of Φ, we can define $\boldsymbol{i}$ as in (19), and assume, without loss of generality, that $\boldsymbol{x}_0 = \boldsymbol{i}(\gamma)$.

We note first that the absolute convergence of the right hand side of (30) is guaranteed by the Ruelle bound and the Ruelle regularity condition. Indeed,

$$\sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^{k} |e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1| \rho_{N+k}^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \ldots, d\boldsymbol{y}_k)\big)$$
$$\leq \sum_{k=1}^{+\infty} \frac{z^k}{k!} C(\beta) \boldsymbol{c}_z^{N+k} = \boldsymbol{c}_z^N \sum_{k=1}^{+\infty} \frac{(z\boldsymbol{c}_z C(\beta))^k}{k!} \leq \boldsymbol{c}_z^N e^{z\boldsymbol{c}_z C(\beta)}.$$

Consider the $(N+1)$-point correlation function of a Gibbs point process P:

$$\rho_{N+1}^{(P)}(\boldsymbol{x}_0,\dots,\boldsymbol{x}_N) = e^{-\beta E_\Phi(\boldsymbol{x}_0,\dots,\boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_0,\dots,\boldsymbol{x}_N \,|\, \xi)} P(d\xi)$$
$$= e^{-\beta E_\Phi(\boldsymbol{x}_0 \,|\, \boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_0 \,|\, \xi)} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \,|\, \xi)} P(d\xi).$$

Using the factorial measure $\xi^{(k)}$, we have the following expansion:

$$e^{-\beta E_\Phi(\boldsymbol{x}_0 \,|\, \xi)} = 1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1)\, \xi^{(k)}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k),$$

which is indeed absolutely convergent, since using the GNZ equations (see [12]) one has:

$$\int_{\mathscr{M}} \left(1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1|\, \xi^{(k)}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k) \right) P(d\xi)$$
$$\overset{\text{(GNZ)}}{=} 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1|\, e^{-\beta E_\Phi(\boldsymbol{y}_1,\dots,\boldsymbol{y}_k)}$$
$$\int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{y}_1,\dots,\boldsymbol{y}_k \,|\, \xi)} P(d\xi)\, \sigma^{\otimes k}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k) \tag{32}$$
$$= 1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1|\, \rho_k^{(P)}(\boldsymbol{y}_1,\dots,\boldsymbol{y}_k) \sigma^{\otimes k}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k)$$
$$\overset{(31)}{\leq} 1 + \sum_{k=1}^{+\infty} \frac{(z\boldsymbol{c})^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} |e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1|\, \sigma^{\otimes k}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k) \leq e^{z\boldsymbol{c}C(\beta)} < +\infty,$$

where in the last line we used the Ruelle bound and the regularity assumption 3. We can then exchange summation over k and integration over $\mathscr{M}$, yielding

$$e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_0 \,|\, \xi)} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \,|\, \xi)} P(d\xi)$$
$$= e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)} \int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \,|\, \xi)} P(d\xi)$$
$$+ \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{\mathscr{M}} \int_{\mathscr{E}^k} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) - \beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N \,|\, \xi)} \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1)\, \xi^{(k)}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k)\, P(d\xi)$$
$$= \rho_N^{(P)}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)$$
$$+ \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N,\boldsymbol{y}_1,\dots,\boldsymbol{y}_k)} \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1)$$
$$\int_{\mathscr{M}} e^{-\beta E_\Phi(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N,\boldsymbol{y}_1,\dots,\boldsymbol{y}_k \,|\, \xi)} P(d\xi)\, \sigma^{\otimes k}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k)$$
$$= \rho_N^{(P)}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)$$
$$+ \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int_{\mathscr{E}^k} \prod_{j=1}^{k} (e^{-\beta\Phi(\boldsymbol{x}_0,\boldsymbol{y}_j)} - 1) \rho_{N+k}^{(P)}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N,\boldsymbol{y}_1,\dots,\boldsymbol{y}_k)\, \sigma^{\otimes k}(d\boldsymbol{y}_1,\dots,d\boldsymbol{y}_k),$$

and concluding the proof. □

Proposition 4.4. Under Assumptions 1+3+4, for any $z > 0$, $\beta > 0$, and any $\boldsymbol{c} > 0$, the Kirkwood–Salsburg operator $\boldsymbol{K}_z$ is a bounded operator in $\mathbb{X}_{\boldsymbol{c}}$.

Moreover, there exists a critical threshold

$$0 < \mathfrak{z}_{\text{crit}}(\beta) \coloneqq \inf\{z > 0 : \boldsymbol{c}_z^{-1} e^{2\beta B_\Phi + z\boldsymbol{c}_z C(\beta)} > 1\} \leq \mathfrak{z}_{Ru}(\beta) \tag{33}$$

such that, for any $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, $\boldsymbol{K}_z$ is a contraction in $\mathbb{X}_{\boldsymbol{c}_z}$, where $\boldsymbol{c}_z$ is defined in (26), and $\mathfrak{z}_{Ru}(\beta)$ is defined in (25). For such activities there exists then at most one solution of (KS)$_z$ in $\mathbb{X}_{\boldsymbol{c}_z}$.

Proof. For any $r \in \mathbb{X}_{\boldsymbol{c}}$, with $\|r\|_{\boldsymbol{c}} \leq 1$, we estimate

$$\begin{aligned}
|(\boldsymbol{K}_z r)_{N+1}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_N)| &\leq e^{-\sum_{i=1}^N \Phi(\boldsymbol{x}_0 - \boldsymbol{x}_i)} \Big(\boldsymbol{c}^N \\
&\quad + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k |e^{-\Phi(\boldsymbol{x}_0, \boldsymbol{y}_j)} - 1| \boldsymbol{c}^{N+k} \, \sigma^{\otimes k}(d\boldsymbol{y}_1, \ldots, d\boldsymbol{y}_k) \Big) \\
&\leq e^{2\beta B_\Phi} \boldsymbol{c}^N \Big(1 + \sum_{k=1}^{+\infty} \frac{(z\boldsymbol{c})^k}{k!} \int \prod_{j=1}^k |e^{-\Phi(\boldsymbol{x}_0, \boldsymbol{y}_j)} - 1| \, \sigma^{\otimes k}(d\boldsymbol{y}_1, \ldots, d\boldsymbol{y}_k) \Big) \\
&\leq e^{2\beta B_\Phi} \boldsymbol{c}^N \Big(1 + \sum_{k=1}^{+\infty} \frac{(z\boldsymbol{c})^k}{k!} C(\beta)^k \Big) \\
&= \boldsymbol{c}^{N+1} \boldsymbol{c}^{-1} e^{2\beta B_\Phi + zC(\beta)\boldsymbol{c}}.
\end{aligned}$$

The Kirkwood–Salsburg operator is then bounded in $\mathbb{X}_{\boldsymbol{c}}$: $|||\boldsymbol{K}_z|||_{\boldsymbol{c}} \leq \boldsymbol{c}^{-1} e^{2\beta B_\phi + zC(\beta)\boldsymbol{c}}$.

Consider now $\boldsymbol{c}_z$ as defined in (26), and set $f(z) \coloneqq \frac{e^{2\beta B_\phi + z\boldsymbol{c}_z C(\beta)}}{\boldsymbol{c}_z}$. We have $f(0) = 1$ and

$$f'(z) = e^{2\beta B_\Phi} \frac{e^{z\boldsymbol{c}_z C(\beta)}}{\boldsymbol{c}_z^2} \big(C(\beta)(\boldsymbol{c}_z^2 + z\boldsymbol{c}_z' \boldsymbol{c}_z - \boldsymbol{c}_z' \big).$$

so that $f'(0) < 0$. Indeed,

$$\text{sign } f'(0) = \text{sign } \big(C(\beta)\boldsymbol{c}_0^2 - \boldsymbol{c}_0' \big) = \text{sign } \big(C(\beta) e^{4\beta B_\Phi} (1 - e^2/\sqrt{2\pi}) \big) = -1.$$

(see Figures 5 and 6), The set $\{z > 0 : \boldsymbol{c}_z^{-1} e^{2\beta B_\Phi + z\boldsymbol{c}_z C(\beta)} < 1\}$ is then non-empty, and defining

$$\mathfrak{z}_{\text{crit}}(\beta) \coloneqq \inf\{z > 0 : \boldsymbol{c}_z^{-1} e^{2\beta B_\Phi + z\boldsymbol{c}_z C(\beta)} > 1\},$$

we have that, for any $z < \mathfrak{z}_{\text{crit}}(\beta)$, the norm of $\boldsymbol{K}_z$ in $\mathbb{X}_{\boldsymbol{c}_z}$ is smaller than 1, so that it is a contraction in $\mathbb{X}_{\boldsymbol{c}_z}$.

Finally, note that, since $\lim_{z \to \mathfrak{z}_{Ru}(\beta)^-} \boldsymbol{c}_z = +\infty$ and $\boldsymbol{c}_z^{-1} e^{2\beta B_\Phi + z\boldsymbol{c}_z C(\beta)} = +\infty$ for $z \geq \mathfrak{z}_{Ru}(\beta)$, we have that $\mathfrak{z}_{\text{crit}}(\beta) \leq \mathfrak{z}_{Ru}(\beta)$. ☐

Example 1 (continued)**.** Consider a potential Φ in the class of Example 1. The Ruelle bound is satisfied for $\boldsymbol{c} = e^{3\beta B_\phi}$ (see (18)). For such a value of $\boldsymbol{c}$, the Kirkwood–Salsburg operator $\boldsymbol{K}_z$ on $\mathbb{X}_{\boldsymbol{c}}$ is a contraction as soon as $\boldsymbol{c}^{-1} e^{2\beta B_\Phi + z\boldsymbol{c} C(\beta)} < 1$, that is for $z < \beta B_\phi (C(\beta) e^{3\beta B_\phi})^{-1}$.

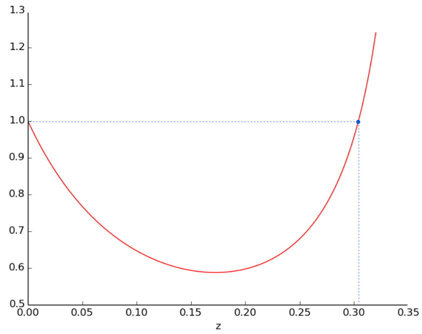

Figure 5: *Let $B_\phi = 0$ (i.e. a repulsive potential), $\beta = 1$, $C(1) = 1$. Plot of $z \mapsto \mathbf{c}_z^{-1} e^{z\mathbf{c}_z}$. The curve explodes as z approaches $\mathfrak{z}_{RB}(1) = 1/e \simeq 0.37$, and the uniqueness domain is $(0, \mathfrak{z}_{crit}(1))$, where $\mathfrak{z}_{crit}(1) \simeq 0.304$.*

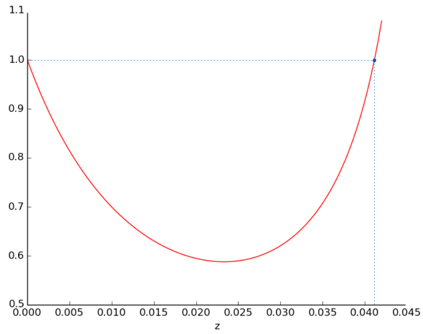

Figure 6: *Let $B_\phi = 1$, $\beta = 1$, $C(1) = 1$. Plot of $z \mapsto \mathbf{c}_z^{-1} e^{2+z\mathbf{c}_z}$. The curve explodes as z approaches $\mathfrak{z}_{RB}(1) \simeq 0.05$, and the uniqueness domain is $(0, \mathfrak{z}_{crit}(1))$, where $\mathfrak{z}_{crit}(1) \simeq 0.041$.*

4.3 The case of non-uniform Ruelle bounds

In this subsection we allow for a weaker notion of stability, in particular, we work under the following weakening of Assumption 1:

Assumption 1′ (Weak stability)**.** Consider an energy functional H as in (5), where the self-potential Ψ satisfies (8), but for which the stability condition (9) of the pair potential Φ is replaced by a weaker one:

$(\mathscr{H}_{\text{w.st.}})$ The pair potential between two paths is given by a symmetric functional $\Phi : \mathscr{E} \times \mathscr{E} \to \mathbb{R} \cup \{+\infty\}$ such that, for any $\{\boldsymbol{x}_0, \dots, \boldsymbol{x}_N\} \subset \mathscr{E}$, there exist a function $\boldsymbol{b}\colon \mathscr{E} \to \mathbb{R}_+$ and some $\boldsymbol{x} \in \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_N\}$ (w.l.o.g. $\boldsymbol{x}_0$) with

$$\sum_{i=1}^{N} \Phi(\boldsymbol{x}_0, \boldsymbol{x}_i) \geq -\boldsymbol{b}(\boldsymbol{x}_0). \tag{34}$$

Remark. We know from (19) that $(\mathscr{H}_{\text{w.st.}})$ holds whenever Φ is a stable potential. Conversely, if Φ satisfies $(\mathscr{H}_{\text{w. st.}})$ for a constant $\boldsymbol{b} \equiv B_\Phi$, then it is also stable for that same constant.

While in the previous section we assumed that the pair potential Φ satisfied a *uniform* regularity condition (Assumption 3), here we work with potentials Φ that satisfy the following *weighted* regularity condition (cf. [14]):

Assumption 3′ (Weighted regularity)**.** There exist a function $\boldsymbol{a}\colon \mathscr{E} \to \mathbb{R}_+$ and a critical activity $\mathfrak{z}_{\text{crit}}(\beta) > 0$ such that, for any $\boldsymbol{x} \in \mathscr{E}$,

$$\mathfrak{z}_{\text{crit}}(\beta) \int e^{\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{b}(\boldsymbol{y})} |e^{-\beta\Phi(\boldsymbol{x},\boldsymbol{y})} - 1| \, \sigma(d\boldsymbol{y}) \leq \boldsymbol{a}(\boldsymbol{x}), \tag{35}$$

with

$$\int_{\mathscr{E}} e^{\boldsymbol{a}(\boldsymbol{x})+\boldsymbol{b}(\boldsymbol{x})} \sigma(d\boldsymbol{x}) < +\infty. \tag{36}$$

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ as above. We also assume that the correlation functions satisfy a Ruelle bound of the following form:

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Assumption 4′ (Non-uniform Ruelle bound)**.** For any $P \in \mathscr{G}_{z,\beta}(H)$, for any $N \geq 1$, for $\sigma^{\otimes N}$-almost all $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathscr{E}^N$, the following holds:

$$\rho_N^{(P)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \leq \prod_{i=1}^{N} e^{\boldsymbol{a}(\boldsymbol{x}_i)+\boldsymbol{b}(\boldsymbol{x}_i)}.$$

Example 4. Consider a potential $\phi = \phi_{hc} + \phi_l$, satisfying Assumptions 1 and 2, given by the sum of a hard core potential (with hard core diameter $R > 0$) and a bounded potential ϕ_l, on $[R, +\infty)$:

$$\exists M_\phi > 0 : \phi_l(u) \leq M_\phi \; \forall u \geq R.$$

In particular, we recall from (11) that there exists a constant $\bar{B}_\Phi \geq 0$ such that, for any $\boldsymbol{x} = (x, m) \in \mathscr{E}$, for any $\xi \in \mathscr{M}^{\text{temp}}$,

$$E_\Phi(\boldsymbol{x} \,|\, \xi) \geq -\bar{B}_\Phi(1 + \|m\|^{d+\delta}). \tag{37}$$

We show here that there exist functions $\boldsymbol{a}$ and $\boldsymbol{b}$, and a threshold activity $\mathfrak{z}_{\text{crit}}(\beta) > 0$ such that Assumptions 3′ and 4′ hold for any $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$.

Proof. Using (21), we have $|e^{-\beta\Phi} - 1| \leq |\beta\bar{\Phi}|e^{2\beta B_\Phi}$, and the weighted regularity condition follows as soon as

$$ze^{2\beta B_\Phi} \int_{\mathscr{E}} e^{\boldsymbol{a}(\boldsymbol{x}_2)} \int_0^1 \beta|\bar{\phi}(x_2 + m_2(s) - x_1 - m_1(s))| ds \; 1_{\{|x_2 - x_1| \leq a_0 + \|m_2\| + \|m_1\|\}} \, \sigma(d\boldsymbol{x}_2) \leq \boldsymbol{a}(\boldsymbol{x}_1).$$

Considering a function $\boldsymbol{a}$ of the form $\boldsymbol{a}(x, m) = \boldsymbol{a}(m) = A(1 + \|m\|^{d+\delta})$, for some constant $A > 0$ to be determined, and recalling that $\Psi(x, m) \geq -A_\Psi \|m\|^{d+\delta}$, this reduces to

$$z\beta e^{2\beta B_\Phi} \int_{C_0} e^{A(1+\|m_2\|^{d+\delta})} \int_{\mathbb{R}^d} \int_0^1 \big|\phi\big(|x_1 + m_1(s) - x_2 - m_2(s)|\big)\big| ds 1_{\{|x_1 - x_2| \leq a_0 + \|m_1\| + \|m_2\|\}} \\ dx_2 \; e^{A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2) \leq A(1 + \|m_1\|^{d+\delta}).$$

Estimating the left hand side leads to:

$$z\beta e^{2\beta B_\Phi} \int_{C_0} \left(b_d R^d + M_\phi k_d b_d (a_0^d + \|m_1\|^d + \|m_2\|^d)\right) e^{A(1+\|m_2\|^d) + A_\Psi \|m_2\|^{d+\delta}} \mathbf{R}(dm_2),$$

where k_d is such that $(x + y + z)^d \leq k_d(x^d + y^d + z^d)$, and b_d the volume of the unit ball in $\mathbb{R}^d$. Setting

$$\upsilon_A := \int e^{A(1+\|m\|^{d+\delta}) + A_\Psi \|m\|^{d+2\delta}} \mathbf{R}(dm),$$

which is finite thanks to the definition of the measure R, we can fix A by the following (note that $A \geq \bar{B}_\Phi$; the reason for this choice will be apparent shortly):

$$A := \sup_{u \geq 0} \frac{\bar{B}_\Phi(1 + u^{d+\delta}) \;\vee\; b_d\left(R^d + M_\phi k_d (a_0^d + u^d + 1)\right)}{1 + u^{d+\delta}} < +\infty,$$

so that the regularity assumption is satisfied for $\boldsymbol{a}(x, m) = A(1 + \|m_1\|^{d+\delta})$, $\boldsymbol{b} \equiv 2B_\Phi$, and

$$\mathfrak{z}_{\text{crit}}(\beta) := (\upsilon_A \beta e^{2\beta B_\Phi})^{-1}.$$

Note that, as the value of the integral υ_A is not easily controlled, the threshold activity $\mathfrak{z}_{\text{crit}}(\beta) > 0$ may be very small.

From the representation (14) of the correlation functions and (37), the Ruelle bound of Assumption 4′ follows as well, as $A \geq \bar{B}_\Phi$ by construction. ☐

It is easy to see that an analogous to Proposition 4.3 holds also for the case of non-uniform Ruelle bound. Indeed, the computations are the same, except for using Assumption 3′ to prove the absolute convergence of the series in (32).

Proposition 4.5. Let Φ such that Assumptions 1′+3′+4′ hold, and set $\boldsymbol{c}(\boldsymbol{x}) := e^{\boldsymbol{a}(\boldsymbol{x})+\boldsymbol{b}(\boldsymbol{x})}$. For any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, the operator $\boldsymbol{K}_z$ is a contraction in $\mathbb{X}_{\boldsymbol{c}}$. For such activities there exists then at most one solution of $(\text{KS})_z$ in $\mathbb{X}_{\boldsymbol{c}}$.

Proof. For any $r \in \mathbb{X}_{\boldsymbol{c}}$, with $\|r\|_{\boldsymbol{c}} \le 1$, we estimate

$$
\begin{aligned}
|(\boldsymbol{K}_z r)_{N+1}(\boldsymbol{x}_0, \dots, \boldsymbol{x}_N)| &\le e^{-\sum_{i=1}^N \Phi(\boldsymbol{x}_0 - \boldsymbol{x}_i)} \Bigg(\prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) \\
&\quad + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k |e^{-\Phi(\boldsymbol{x}_0, \boldsymbol{y}_j)} - 1| \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) \prod_{j=1}^k \boldsymbol{c}(\boldsymbol{y}_j)\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \dots, d\boldsymbol{y}_k) \Bigg) \\
&\overset{(34)}{\le} e^{\boldsymbol{b}(\boldsymbol{x}_0)} \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) \Big(1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k \boldsymbol{c}(\boldsymbol{y}_j) |e^{-\Phi(\boldsymbol{x}_0, \boldsymbol{y}_j)} - 1|\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \dots, d\boldsymbol{y}_k) \Big) \\
&= e^{\boldsymbol{b}(\boldsymbol{x}_0)} \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) \Big(1 + \sum_{k=1}^{+\infty} \frac{z^k}{k!} \int \prod_{j=1}^k e^{\boldsymbol{a}(\boldsymbol{y})+\boldsymbol{b}(\boldsymbol{y})} |e^{-\Phi(\boldsymbol{x}_0, \boldsymbol{y}_j)} - 1|\, \sigma^{\otimes k}(d\boldsymbol{y}_1, \dots, d\boldsymbol{y}_k) \Big) \\
&\overset{(35)}{\le} e^{\boldsymbol{b}(\boldsymbol{x}_0)} \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) \sum_{k=0}^{+\infty} \frac{(z/\mathfrak{z}_{\text{crit}}(\beta))^k \boldsymbol{a}^k(\boldsymbol{x}_0)}{k!} \\
&= e^{\boldsymbol{b}(\boldsymbol{x}_0)} \prod_{i=1}^N \boldsymbol{c}(\boldsymbol{x}_i) e^{\boldsymbol{a}(\boldsymbol{x}_0) z/\mathfrak{z}_{\text{crit}}(\beta)} < \prod_{i=0}^N \boldsymbol{c}(\boldsymbol{x}_i).
\end{aligned}
$$

The Kirkwood–Salsburg operator is then a contraction: $|||\boldsymbol{K}_z|||_{\boldsymbol{c}} < 1$. □

4.4 Uniqueness domain

We can now state the main result of this section. Recall that, both for the uniform and the non-uniform Ruelle bound setting we have a critical threshold $\mathfrak{z}_{\text{crit}}(\beta) > 0$ such that for any $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, the Kirkwood–Salsburg operator $\boldsymbol{K}_z$ is a contraction in $\mathbb{X}_{\boldsymbol{c}}$. In the former case $\boldsymbol{c} > 0$ is a constant, while in the latter it is a non-negative function $\boldsymbol{c} : \mathscr{E} \to \mathbb{R}_+$.

Theorem 2. Let H be an energy functional as in (5), satisfying either Assumptions 1+3+4 or Assumptions 1′+3′+4′. For any $\beta > 0$ and $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, there exists at most one infinite-volume Gibbs point process P in $\mathscr{G}_{z,\beta}(H)$.

Proof. Let $\beta > 0$, $z \in (0, \mathfrak{z}_{\text{crit}}(\beta))$, and consider two Gibbs point processes $P, \hat{P} \in \mathscr{G}_{z,\beta}(H)$.

(i) We know from Proposition 4.3, that the correlation functions $\rho^{(P)}$ and $\rho^{(\hat{P})}$ both satisfy the Kirkwood–Salsburg equations $(\text{KS})_z$.

(ii) By assumption, $\rho^{(P)}$ and $\rho^{(\hat{P})}$ satisfy a Ruelle bound for the same $\boldsymbol{c}$, and are therefore both elements of $\mathbb{X}_{\boldsymbol{c}}$.

(iii) For $z < \mathfrak{z}_{\text{crit}}(\beta)$, $(\text{KS})_z$ has a unique solution, so that the correlation functions of $\hat{P}$ – and therefore its factorial moment measures $(\alpha_N^{(\hat{P})})_N$ – must coincide with those of P.

(iv) For any $N \geq 1$ and any bounded $\Gamma \subset \mathscr{E}$, we compute

$$\begin{aligned}\alpha_N^{(P)}(\Gamma^N) &= \mathbb{E}\left[|\gamma_\Gamma|\big(|\gamma_\Gamma|-1\big)\dots\big(|\gamma_\Gamma|-N+1\big)\right] \\ &= \int_{\Gamma^N} \rho_N(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) z^N \sigma(d\boldsymbol{x}_1)\dots\sigma(d\boldsymbol{x}_N) \\ &\leq \int_{\Gamma^N} \prod_{i=1}^N (z\boldsymbol{c}(\boldsymbol{x}_i))\sigma(d\boldsymbol{x}_1)\dots\sigma(d\boldsymbol{x}_N) = (zc_\Gamma)^N,\end{aligned}$$

with $c_\Gamma := \int_{\Gamma^N} \boldsymbol{c}(\boldsymbol{x})\sigma(d\boldsymbol{x})$. We have used here the fact that the Ruelle bound holds either for $\boldsymbol{c}$ constant (under Assumption 4) or integrable (under Assumptions 3′+4′). We can then conclude that $P = \hat{P}$ (see [7, 6]); in other words, $\mathscr{G}_{z,\beta}(H) = \{P\}$. □

Example 1 (continued)**.** Consider here a potential $\phi = \phi_{hc}$ with a pure hard core at some diameter $R > 0$, i.e. $\phi_l \equiv 0$. Taking $a_0 = R$ in the range Assumption 2 yields a path potential Φ (stable, with stability constant $B_\phi = 0$) of the form

$$\Phi(\boldsymbol{x}_1, \boldsymbol{x}_2) = (+\infty)\, 1_{[0,R)}\big(\inf_{s\in[0,1]} |x_1 + m_1(s) - x_2 - m_2(s)|\big).$$

Under this interaction, two Langevin diffusions are forbidden from coming closer than R to each other, at any given time $s \in [0, 1]$.

For such a choice of Φ – which satisfies Assumptions 1+2+3+4 – the Gibbs point process $P^{z,\beta}$ constructed in Theorem 1 is the unique element of $\mathscr{G}^{\text{temp}}_{z,\beta}(H)$.

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