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Precompact probability spaces in applied stochastic homogenization

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Abstract

We provide precompactness and metrizability of the probability space Ω for random measures and random coefficients such as they widely appear in stochastic homogenization and are typically given from data. We show that these properties are enough to implement the convenient two-scale formalism by Zhikov and Piatnitsky (2006). To further demonstrate the benefits of our approach we provide some useful trace and extension operators for Sobolev functions on Ω, which seem not known in literature. On the way we close some minor gaps in the Sobolev theory on Ω which seemingly have not been proven up to date.

1 Introduction

Stochastic homogenization has gained more and more interest during the last years with stochastic two-scale convergence methods being developed for applied homogenization problems in various versions [3, 14, 15, 10, 9, 8, 19, 21, 16]. A crucial point in these generalizations is the choice of the probability space $(\Omega, \sigma, \mathbb{P})$. More regularity of Ω implies less technical difficulties, while it also may restrict the range of applicability to very particular models of coefficient fields. Let us note for example that Papanicolaou and Varadhan [23] where able to show that their probability space is separable, but not more. In case of uniformly bounded coefficients, this was later picked up by Dal Maso and Modica [5] who showed the probability space to be indeed (pre)compact and separable.

Seemingly the assumptions on the probability space in the literature can be categorized in improving order of regularity as either

1 $(\Omega, \sigma, \mathbb{P})$ is a separable measure space, an assumption prevailing in [24, 30, 17, 18] and related work building upon these.

2 Ω is a separable metric space, σ the Borel algebra and $(\Omega, \sigma, \mathbb{P})$ a measure space, an assumption set up in [14] based on a first approach to derive probability spaces from random geometries.

3 Ω is a compact metric space, σ the Borel algebra and $(\Omega, \sigma, \mathbb{P})$ a measure space, an assumption set up in [29] and an approach which comes up with very convenient analysis.

However, let us note that the major advantage of 3. over 2. is the fact that $C(\Omega)$ is separable and dense in $L^p(\Omega)$, $1 \leq p < \infty$. The separability of $C(\Omega)$ further allows to characterize "typical" realizations in a very convenient way, see Lemma 3.3. In this work, we show that Ω can in many cases be constructed as precompact metric space (see Theorem 2.14 and Remark 2.15) and that we have enough regularity to fully apply any technical tools constructed in [29] and descendant work (Section 3).

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Our first main result in Theorem 2.14 treats non-negative stationary random measures μ_ω with a stationary function $a_\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mu_\omega)$ such that $\mathbb{E}\Big(\int_{[0,1]^d} a_\omega \mathrm{d}\mu_\omega\Big) + \mathbb{E}\big(\mu_\omega[0,1]^d\big) < \infty$ but also the original setting by Papanicolaou and Varadhan [23], where a_ω is positive measurable (see Example 2.16). Models of random geometries will also automatically fall into this class via the measurable embedding by ZÃd'hle [28]. In Theorem 2.19 we consider random measurable functions $f_\omega(x,u)$ that are stationary in x and continuous in u.

The benefits of the precompactness are the resulting structural properties of the Lebesgue- and Sobolev-spaces on Ω that we discuss in Section 3. We collect in Sections 3.1–3.3 some results that seem to be present in literature only in parts. In particular Lemma 3.6 seems completely new and for Theorem 3.11 there seems not to be a complete proof in the literature.

In particular, we will see that the (random) support of a random measure is the realization of the support of the Palm measure and reproduce as a special case a result from [14] where lower dimensional random Lipschitz structures $\Gamma(\omega)$ are the realization of a related set $\Gamma\subset\Omega$, i.e. $\chi_{\Gamma(\omega)}(x) = \chi_\Gamma(\tau_x\omega)$. Using the theory of Palm measures we will construct a trace operator on Γ and establish a Gauss-like theorem on Ω (Theorem 3.16).

In Section 4 we recall stochastic two-scale convergence in the sense of [29] but with the modification of a precompact Ω. The insights of Section 3.4 will help to get some additional insight in the limit behavior of traces. In Section 4.2 we use the developed methods to homogenize a Stokes system with Navier-slip condition. While neither the calculations nor the result are surprising since they simply reproduce the periodic calculations, we emphasize that the calculations can only be performed due to the functional analytic insights from Section 3.

2 Precompact probability spaces from data

2.1 Stationary random measures and the Vague topology

In what follows, we work with $\mathcal{M}\big(\mathbb{R}^d\big)$, the space of Radon measures on $\mathbb{R}^d$. Since some of the literature cited below works with boundedly regular Borel measures, let us note that on $\mathbb{R}^d$ the Borel, Baire and Radon measures coincide. In particular, all these measures have the inner regularity and are locally bounded, see e.g. [1] Corollary 40.4, p. 198, resp. Proposition 42.5, p. 207 or [2] Theorem 7.1.7.

Remark 2.1. For simplicity of notation, we cite the results in Sections 2.1–2.2 with signed $\mathbb{R}$-valued measures. However, all results below hold for $\mathbb{R}^N$-valued Radon measures, too.

The Vague topology on $\mathcal{M}\big(\mathbb{R}^d\big)$ is such that for every $f\in C_c(\mathbb{R}^d)$ the map

$$m_f: \quad \mu \mapsto \int_{\mathbb{R}^d} f\mathrm{d}\mu$$

is continuous. This topology is separable and generated by the metric

$$d(\mu_1,\mu_2) := \sum_{k\in\mathbb{N}} \frac{|m_{f_k}(\mu_1) - m_{f_k}(\mu_2)|}{1+|m_{f_k}(\mu_1) - m_{f_k}(\mu_2)|}\,, \tag{1}$$

where f_k are a countable dense subset of the Banach space $C_0(\mathbb{R}^d)$. This is a variant of the metrization of the weak* topology on $\mathcal{M}\big(\mathbb{R}^d\big)$, see [4] Theorem 3.28. An alternative metric is given in [6], (A2.6.1). A nice overview over the Vague topology is also given in [7], Chapter 13.4. The Vague sigma-algebra $\sigma_{\mathcal{M}}$ is generated by the open sets of the Vague topology. Of importance for us is the following result:

Lemma 2.2 (Compact sets in $\mathcal{M}(\mathbb{R}^d)$ [1, 2])**.** *A set $M \subset \mathcal{M}(\mathbb{R}^d)$ is precompact if and only if for every bounded set $A \subset \mathbb{R}^d$ it holds*

$$\sup_{\mu\in A} |\mu(A)| < \infty\,.$$

For $x \in \mathbb{R}^d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ we define

$$(\tau_x\mu)\,(\,\cdot\,) := \mu(\,\cdot\, + x)$$

the translation of μ by x. Note that

$$\forall f \in C_c(\mathbb{R}^d),\; x \in \mathbb{R}^d : \qquad \int_{\mathbb{R}^d} f(\,\cdot\, + x)\,\mathrm{d}(\tau_x\mu) = \int_{\mathbb{R}^d} f\,\mathrm{d}\mu\,.$$

Furthermore by definition of the topology we have continuity of the map

$$\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d)\,, \qquad (\mu, x) \mapsto \tau_x\mu\,.$$

A **warning** is in place: Often in the literature of random measures, the definition of τ reads $(\tau_x\mu)\,(\,\cdot\,) := \mu(\,\cdot\, - x)$ and hence the formulations of the theorems there differ in some cases by $\pm$.

Definition 2.3. Let $\Omega \subset \mathcal{M}(\mathbb{R}^d)$ be an arbitrary (in particular also non-measurable) set and let $\sigma_\Omega := \{A \cap \Omega :\; A \in \sigma_{\mathcal{M}}\}$ be the Vague sigma-algebra restricted to Ω.

1 A random measure is a measurable surjective mapping $\mu : \tilde\Omega \to \Omega$, $\tilde\omega \mapsto \mu_{\tilde\omega}$, where $\left(\tilde\Omega, \tilde\sigma, \tilde{\mathbb{P}}\right)$ is a probability space. Since $\mathbb{P} := \tilde{\mathbb{P}} \circ \mu^{-1}$ is a probability measure on Ω, we also find the following:

2 A probability measure $\mathbb{P}$ on (Ω, σ_Ω) is called a random measure. In order to highlight the measure aspect, we write the identity on Ω as

$$\Omega \to \mathcal{M}(\mathbb{R}^d)\,, \qquad \omega \mapsto \mu_\omega := \omega\,.$$

We also say that $(\Omega, \sigma_\Omega, \mathbb{P})$ induces the random measure $\omega \mapsto \mu_\omega$.

3 For for every bounded Borel sets $A_1, A_2, \dots, A_k \subset \mathbb{R}^d$, $k \in \mathbb{N}$ we denote by

$$F_k(A_1, \dots, A_k;\, x_1, \dots, x_k) = \mathbb{P}(\mu_\omega(A_i) \le x_i;\; i = 1, \dots, k)$$

the finite dimensional distributions (fidi distributions) of μ_ω.

4 A random measure $\mu_\bullet$ is called stationary if for every $x \in \mathbb{R}^d$ it holds $\tau_x\Omega \subset \Omega$ and the fidi distributions of $\mu_\bullet$ and $\tau_x\mu_\bullet$ are the same, i.e.

$$F_k(A_1, \dots, A_k;\, x_1, \dots, x_k) = F_k(A_1 + x, \dots, A_k + x;\, x_1, \dots, x_k)\,.$$

This is equivalent with $\mathbb{P} = \mathbb{P} \circ \tau_x$ for all $x \in \mathbb{R}^d$.

5 If $\Omega \subset \mathcal{M}(\mathbb{R}^d)^k$ and 1.–4. holds componentwise, we speak of a k-dimensional random measure.

The following result is obtained directly from the information collected above. Here, even if $\Omega \subset \mathcal{M}(\mathbb{R}^d)$ is not measurable, we observe that $A \subset \Omega$ is measurable with respect to σ_Ω iff there exists $A_{\mathcal{M}} \in \sigma_{\mathcal{M}}$ such that $A = A_{\mathcal{M}} \cap \Omega$.

Theorem 2.4. *Let $\Omega \subset \mathcal{M}\big(\mathbb{R}^d\big)^k$ be an arbitrary (also non-measurable) set and let σ_Ω be the Vague sigma-algebra projected on Ω. Let $(\Omega, \sigma_\Omega, \mathbb{P})$ be a stationary random measure which induces μ_ω. Then the family $(\tau_x)_{x\in\mathbb{R}^d}$ is a continuous dynamical system on Ω, i.e.*

1. $\tau_x \circ \tau_y = \tau_{x+y}$ *and* $\tau_0 = \mathrm{id}$,
2. *for every measurable $A \subset \Omega$ it holds $\mathbb{P}(A) = \mathbb{P}(\tau_{-x}A)$,*
3. *the evaluation $\Omega \times \mathbb{R}^d \to \Omega$, $(\omega, x) \mapsto \tau_x\omega$ is continuous.*

Furthermore, Ω is precompact if and only if for every bounded Borel set $U \subset \mathbb{R}^d$ it holds $f_U \in L^\infty(\Omega)$, where $f_U(\omega) := |\mu_\omega(U)|$.

The importance of stationary random measures for homogenization stems from the following result, which is due to Mecke [20]. In case $\Omega \subset \mathcal{M}\big(\mathbb{R}^d\big)$ is a measurable subset, it can be found also in Chapter 12 of [6].

Theorem 2.5 (Mecke [20, 6]: Existence of Palm measure)**.** *Let $(\Omega, \sigma, \mathbb{P})$ be a probability space with dynamic system τ satisfying Assumption 3.1 and let $\omega \mapsto \mu_\omega$ be a stationary random measure. Then there exists a unique measure $\mu_{\mathcal{P}}$ on Ω such that*

$$\int_\Omega \int_{\mathbb{R}^d} f(x, \tau_x\omega)\, \mathrm{d}\mu_\omega(x)\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d} \int_\Omega f(x, \omega)\, \mathrm{d}\mu_{\mathcal{P}}(\omega)\mathrm{d}x \tag{2}$$

for all $\mathcal{L} \otimes \mu_{\mathcal{P}}$-measurable non negative functions and all $\mathcal{L} \otimes \mu_{\mathcal{P}}$- integrable functions f. Furthermore for all $A \subset \Omega$, $u \in L^1(\Omega, \mu_{\mathcal{P}})$ there holds

$$\mu_{\mathcal{P}}(A) = \int_\Omega \int_{\mathbb{R}^d} g(x)\chi_A(\tau_x\omega)\mathrm{d}\mu_\omega(x)\mathrm{d}\mathbb{P} \tag{3}$$

$$\int_\Omega u(\omega)\mathrm{d}\mu_{\mathcal{P}} = \int_\Omega \int_{\mathbb{R}^d} g(x)u(\tau_x\omega)\mathrm{d}\mu_\omega(x)\mathrm{d}\mathbb{P} \tag{4}$$

for an arbitrary $g \in L^1(\mathbb{R}^n, \mathcal{L})$ with $\int_{\mathbb{R}^d} g(x)\mathrm{d}x = 1$ and $\mu_{\mathcal{P}}$ is σ-finite.

Remark 2.6. a) Setting $g(s) := \chi_{[0,1]^n}(s)$, the Palm measure is defined by (3).

b) For $\omega \mapsto \mathcal{L}$, we find $\mu_{\mathcal{P}} = \mathbb{P}$, the original probability measure.

Example 2.7. Typical examples for random measures are the Hausdorff measures on random piecewise $d - k$ -dimensional manifolds [28].

2.2 Ergodicity

Let $(\Omega, \sigma, \mathbb{P})$ be a probability space with mappings $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$ being a dynamical system:

- $\tau_x \circ \tau_y = \tau_{x+y}$ and $\tau_0 = \mathrm{id}$ (Group property)
- for every measurable $A \subset \Omega$ and every $x \in \mathbb{R}^d$ it holds $\mathbb{P}(A) = \mathbb{P}(\tau_{-x}A)$ (measure preserving)

- the evaluation $\Omega \times \mathbb{R}^d \to \Omega$, $(\omega, x) \mapsto \tau_x\omega$ is measurable (measurability of stationary evaluation)

A set $A \subset \Omega$ is *almost invariant* if $\mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0$. The family

$$\mathscr{I} = \left\{A \in \mathscr{F} \,:\, \forall x \in \mathbb{R}^d\, \mathbb{P}\left((A \cup \tau_x A) \setminus (A \cap \tau_x A)\right) = 0\right\} \tag{5}$$

of almost invariant sets is a σ-algebra and

$$\mathbb{E}\left(f|\mathscr{I}\right) \text{ denotes the expectation of } f \,:\, \Omega \to \mathbb{R} \text{ w.r.t. } \mathscr{I}\,. \tag{6}$$

$\mathbb{E}\left(f|\mathscr{I}\right)$ is a function which is constant on invariant sets and with $\mathbb{E}(\mathbb{E}\left(f|\mathscr{I}\right)) = \mathbb{E}(f)$.

Definition 2.8. A family $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is called convex averaging sequence if

(i) each A_n is convex

(ii) for every $n \in \mathbb{N}$ holds $A_n \subset A_{n+1}$

(iii) there exists a sequence r_n with $r_n \to \infty$ as $n \to \infty$ such that $B_{r_n}(0) \subseteq A_n$.

Theorem 2.9 (Ergodic Theorem [6] Theorems 10.2.II and also [27])**.** *Let $(A_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f \,:\, \Omega \to \mathbb{R}$ be measurable with $|\mathbb{E}(f)| < \infty$. Then for almost all $\omega \in \Omega$*

$$|A_n|^{-1} \int_{A_n} f(\tau_x\omega)\,\mathrm{d}x \to \mathbb{E}(f|\mathscr{I})\,. \tag{7}$$

Theorem 2.9 underlines the importance of $\mathbb{E}\left(f|\mathscr{I}\right)$. For convenience, one typically focuses on trivial $\mathscr{I}$, which is called ergodicity: $\mathbb{E}(f|\mathscr{I}) = \mathbb{E}(f)$ and

$$|A_n|^{-1} \int_{A_n} f(\tau_x\omega)\,\mathrm{d}x \to \mathbb{E}(\mathscr{I}) \tag{8}$$

Remark 2.10. According to the results in [27] the ergodic limits also hold for $A_n = n\mathbf{Q}$, where $\mathbf{Q}$ is a bounded domain with $0 \in \mathbf{Q}$.

A dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *mixing* if for every measurable $A, B \subset \Omega$ it holds

$$\lim_{\|x\|\to\infty} \mathbb{P}(A \cap \tau_x B) = \mathbb{P}(A)\,\mathbb{P}(B)\,. \tag{9}$$

One infers immediately that for $\hat\tau$ and $\tilde\tau$ mixing, also $\tau_x(\tilde\omega, \hat\omega) := (\tilde\tau_x\tilde\omega, \hat\tau_x\hat\omega)$ is mixing. On the other hand, the product of two ergodic dynamical systems is in general not ergodic, but in the intermediate regime, we can still be successful.

Lemma 2.11 (Ergodic times mixing is ergodic, see e.g. [13])**.** *Let $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and $(\hat\Omega, \hat{\mathscr{F}}, \hat{\mathbb{P}})$ be probability spaces with dynamical systems $(\tilde\tau_x)_{x\in\mathbb{R}^d}$ and $(\hat\tau_x)_{x\in\mathbb{R}^d}$ respectively. Let $\Omega := \tilde\Omega \times \hat\Omega$ be the usual product measure space with the notation $\omega = (\tilde\omega, \hat\omega) \in \Omega$ for $\tilde\omega \in \tilde\Omega$ and $\hat\omega \in \hat\Omega$. If $\tilde\tau$ is ergodic and $\hat\tau$ is mixing, then $\tau_x(\tilde\omega, \hat\omega) := (\tilde\tau_x\tilde\omega, \hat\tau_x\hat\omega)$ is ergodic.*

We denote by

$$\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I}) := \int_\Omega f\,\mathrm{d}\mu_{\mathcal{P}} \text{ the expectation of } f \text{ w.r.t. the } \sigma\text{-algebra } \mathscr{I} \text{ and } \mu_{\mathcal{P}}\,. \tag{10}$$

For random measures we find a more general version of Theorem 2.9.

Theorem 2.12 (Ergodic Theorem [6] 12.2.VIII). *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $(A_n)_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a convex averaging sequence, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega |f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$*

$$|A_n|^{-1}\int_{A_n} f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\,. \tag{11}$$

Given a bounded open (and convex) set $\mathbf{Q}\subset\Omega$, it is not hard to see that the following generalization holds:

Theorem 2.13 (General Ergodic Theorem [13]). *Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space, $\mathbf{Q}\subset\mathbb{R}^d$ be a convex bounded open set with $0\in\mathbf{Q}$, let $(\tau_x)_{x\in\mathbb{R}^d}$ be a dynamical system on Ω with invariant σ-algebra $\mathscr{I}$ and let $f:\,\Omega\to\mathbb{R}$ be measurable with $\int_\Omega |f|\,\mathrm{d}\mu_{\mathcal{P}}<\infty$. Then for $\mathbb{P}$-almost all $\omega\in\Omega$ it holds*

$$\forall\varphi\in C(\overline{\mathbf{Q}}):\quad n^{-d}\int_{n\mathbf{Q}}\varphi(\frac{x}{n})f(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\to\mathbb{E}_{\mu_{\mathcal{P}}}(f|\mathscr{I})\int_{\mathbf{Q}}\varphi\,. \tag{12}$$

2.3 Stationary random coefficients and precompact probability spaces

Theorem 2.14. *Let $(\Omega,\sigma,\mathbb{P})$ be a probability space, μ_ω be a k-dimensional stationary random measure with finite intensity and $a_\omega\in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mu_\omega)$ be a family of random functions. Then the following are equivalent.*

1 *The family $\mathrm{d}\mu_{a,\omega}:=a_\omega\mathrm{d}\mu_\omega$ is a stationary random measure with either $\mathbb{E}\big(\big|\mu_{a,\cdot}([0,1]^d)\big|\big)<\infty$ or a_ω is non-negative and measurable.*

2 *There exists a precompact metric space $\tilde\Omega\subset\mathcal{M}\big(\mathbb{R}^d\big)^k$ with Borel sigma-algebra $\tilde\sigma$ and a probability measure $\tilde{\mathbb{P}}$, a stationary random measure $\tilde\mu_\omega$ with Palm measure $\tilde\mu_{\mathcal{P}}$ and a function $\tilde a:\,\tilde\Omega\to\mathbb{R}$ with either $\tilde a\in L^1(\tilde\Omega;\tilde\mu_{\mathcal{P}})$ or $\tilde a$ non-negative and measurable such that $\tilde\mu_\omega$ and μ_ω as well as $\mathrm{d}\tilde\mu_{a,\omega}(x):=\tilde a(\tau_x\omega)\mathrm{d}\tilde\mu_\omega(x)$ and $\mu_{a,\omega}$ have the same fidi distributions respectively.*

Remark 2.15. Through monotone convergence, Theorem 2.14 can be extended to measurable a_ω.

Proof. Proof that 2 implies 1: By Theorem 2.5 we have

$$\mathbb{E}(|\mu_{a,\omega}(A)|)=\int_\Omega\left|\int_A\mathrm{d}\mu_{a,\omega}\right|\mathrm{d}\mathbb{P}(\omega)\le\int_\Omega\int_A|a(\tau_x\omega)|\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)<\infty$$

for every bounded open set $A\subset\mathbb{R}^d$ and in particular $\mathrm{d}\mu_{a,\omega}(x)$ is almost surely locally finite. Stationarity is obvious by definition.

Proof that 1 implies 2:

Step 1: For simplicity assume $k=1$, otherwise consider the application of $\arctan(\cdot)$ and $\tan(\cdot)$ in following calculations componentwise. We first assume that for every bounded open $A\subset\mathbb{R}^d$ it holds $\mu_\omega(A)\in L^\infty(\Omega;\mathbb{P})$ and $a_\omega\in L^\infty(\Omega;\mu_\omega)$ with $\|a_\omega\|_\infty\le 1$, $a_\omega\ge 0$. Then also $\mu_{a,\omega}(A)\in L^\infty(\Omega;\mathbb{P})$ and because both μ_ω and $\mu_{a,\omega}$ are stationary random measures we identify the space $\tilde\Omega:=(\mu_\bullet,\mu_{a,\bullet})(\Omega)\subset\mathcal{M}(\mathbb{R}^d)^2$. Moreover, there exist Palm measures $\tilde\mu_{\mathcal{P}}$ and $\tilde\mu_{a,\mathcal{P}}$ respectively on the space $\tilde\Omega$. For simplicity of notation, we drop the $\tilde\cdot$ in $\mu_{\mathcal{P}}$, $\mu_{a,\mathcal{P}}$ and Ω in the following.

Then

$$0\le\int_\Omega\int_A\mathrm{d}\mu_{a,\omega}\mathrm{d}\mathbb{P}<|A|\,\mu_{\mathcal{P}}(\Omega)$$

for every Borel set $A \subset \mathbb{R}^d$ of finite Lebesgue measure. Since μ_ω is a random measure, the Campbell-formula (3) yields for every $f \in C_c\big(\mathbb{R}^d\big)$ and every $g \in L^1(\Omega;\mu_\mathcal{P})$

$$\int_\Omega\int_{\mathbb{R}^d} f(x)g(\tau_x\omega)\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d}\int_\Omega f(x)g(\omega)\mathrm{d}\mu_\mathcal{P}(\omega)\,\mathrm{d}x\,. \tag{13}$$

For Borel measurable $A \subset \mathbb{R}^d$ with finite measure and measurable $A' \subset \Omega$ we calculate

$$\begin{aligned}\mu_{a,\mathcal{P}}(A') &= \int_{A'} \mathrm{d}\mu_{a,\mathcal{P}}(\omega) = \frac{1}{|A|}\int_A\int_{A'} \mathrm{d}\mu_{a,\mathcal{P}}(\omega)\,\mathrm{d}x\\ &= \frac{1}{|A|}\int_{A'}\int_A \mathrm{d}\mu_{a,\omega}(x)\,\mathrm{d}\mathbb{P}(\omega) = \frac{1}{|A|}\int_{A'}\int_A a_\omega(x)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \frac{1}{|A|}\int_{A'}\int_A \mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega) = \mu_\mathcal{P}(A')\end{aligned}$$

Hence $\mu_{a,\mathcal{P}} \ll \mu_\mathcal{P}$ and the Radon-NikodÃ¡m Theorem yields existence of $\tilde{a} \in L^1(\Omega;\mu_\mathcal{P})$ such that $\mathrm{d}\mu_{a,\mathcal{P}} = \tilde{a}\,\mathrm{d}\mu_\mathcal{P}$. We infer from (13)

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^d} f(x)g(\tau_x\omega)a_\omega(x)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega) &= \int_{\mathbb{R}^d}\int_\Omega f(x)g(\omega)\mathrm{d}\mu_{a,\mathcal{P}}(\omega)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d}\int_\Omega f(x)g(\omega)\tilde{a}(\omega)\mathrm{d}\mu_\mathcal{P}(\omega)\,\mathrm{d}x\\ &= \int_\Omega\int_{\mathbb{R}^d} f(x)g(\tau_x\omega)\,\tilde{a}(\tau_x\omega)\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)\,,\end{aligned}$$

which implies the claim also for $a_\omega \not\geq 0$.

Step 2: Suppose $\mu_\omega(A) \in L^1(\Omega;\mathbb{P})$ for some nonempty bounded open $A \subset \mathbb{R}^d$. We define the function $b_{r,\omega}(x) := \mu_\omega(\mathbb{B}_r(x)) + 1$ and $\hat{\mu}_{r,\omega}$ through $\mathrm{d}\hat{\mu}_{r,\omega} := b_{r,\omega}^{-1}\mathrm{d}\mu_\omega$. Then we find for every $x \in \mathbb{B}_{\frac{1}{2}}(0)$ that $b_{1,\omega}(x) \geq b_{\frac{1}{2},\omega}(0)$ and hence

$$\hat{\mu}_{1,\omega}\Big(\mathbb{B}_{\frac{1}{2}}(0)\Big) \le \int_{\mathbb{B}_{\frac{1}{2}}(0)} b_{1,\omega}^{-1}\mathrm{d}\mu_\omega \le \int_{\mathbb{B}_{\frac{1}{2}}(0)} b_{\frac{1}{2},\omega}^{-1}(0)\,\mathrm{d}\mu_\omega \le 1\,.$$

In particular, $\hat{\mu}_{1,\omega}$ with $\hat{a} := (a, b_{1,\omega}^{-1})$ satisfy the conditions of Step 1 and we afterwards use $\mathrm{d}\mu_\omega = b_{r,\omega}\mathrm{d}\hat{\mu}_{r,\omega}$.

Step 3: For unbounded a, consider $\hat{a} := 2\pi^{-1}\arctan a$ and use $\tilde{a} = \tan\frac{\pi}{2}\tilde{\hat{a}}$. In case a is only measurable we additionally use the integrability of $\hat{a}_\omega$ and $\tilde{\hat{a}}$ to obtain almost surely $\hat{a}_\omega(x) = \tilde{\hat{a}}(\tau_x\omega)$ and from there the general equivalence using non-negativity and the Campbell formula for all $B \subset \Omega$ and bounded open $A \subset \mathbb{R}^d$:

$$\int_B\int_A a_\omega - \tilde{a}(\tau_x\omega)\,\mathrm{d}\mu_\omega\,\mathrm{d}\mathbb{P} = 0\,.$$

□

Example 2.16. a) Let a_ω be a stationary random measurable function. We can use the idea of the proof to define $\hat{a}_\omega := \arctan a_\omega$ and find $\hat{a} \in L^1(\Omega)$ such that $\hat{a}_\omega(x) = \hat{a}(\tau_x\omega)$. Applying $a(\omega) := \tan(a(\omega))$ we have shown that the original probability space in [23] can be replaced by a precompact probability space.

b) Let us look on random families of functions

$$a:\,\Omega\to\mathscr{X}\,,\qquad\omega\mapsto a_\omega\,.$$

Here, $\mathscr{X}$ is a topological vector space of functions, such as the space of bounded continuous functions $\mathscr{X}=C_b(\mathbb{R}^d)$, the bounded α-Hölder-continuous functions $\mathscr{X}=C_b^{0,\alpha}(\mathbb{R}^d)$, the spaces of p-integrable functions $\mathscr{X}=L^p_{\rm loc}(\mathbb{R}^d)$, $1\leq p\leq\infty$ or many others, also $\mathbb{R}^D$-valued functions. For simplicity, we consider here $D=1$.

We furthermore assume that

$$\mathrm{id}_1:\;\mathscr{X}\hookrightarrow L^1_{\rm loc}(\mathbb{R}^d)\quad f\mapsto f\qquad\text{is continuous,}\tag{14}$$

a property which is satisfied by the above examples for $\mathscr{X}$. If now the values of a_ω are stationary, there exists $\tilde a\in L^1(\Omega)$ with $a_\omega(x)=\tilde a(\tau_x\omega)$ and we can assume w.l.o.g that Ω is precompact.

2.4 Further compact probability spaces

In [5] the authors showed that given $0<c_1<c_2<\infty$ and $p>1$ the functionals

$$F(u,A):=\int_A f(x,\nabla u(x))\,\mathrm{d}x$$

with the condition

$$\forall\,(x,u)\in\mathbb{R}^{2d}:\qquad c_1\,|u|^p\leq f(x,u)\leq c_2\,(|u|^p+1)$$

form a compact metric space.

There are further interesting examples:

Lemma. *Given $M,b>0$ the set*

$$C^{0;0,1}_{M,b}\left(R^{d+1}\right):=\left\{f\in C(\mathbb{R}^{d+1}):\;\forall x\in\mathbb{R}^d,u_1,u_2\in\mathbb{R}\;\begin{matrix}|f(x,u_1)-f(x,u_2)|\leq M\,|u_1-u_2|\\|f(x,0)|<b\end{matrix}\right\}$$

is a compact metric space with the metric

$$d(f_1,f_2):=\sum_{i\in\mathbb{N}}2^{-i}\frac{\|f_1-f_2\|_{\infty,i}}{1+\|f_1-f_2\|_{\infty,i}}\,,$$

$$\|f\|_{\infty,B_i}:=\sup\left\{|f(x,0)|+\frac{|f(x,u_1)-f(x,u_1)|}{|u_1-u_2|}\;:\;|x|\leq i,\;|u_1|\,,|u_2|\leq i\right\}\,.$$

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C^{0;0,1}_{M,b}\left(R^{d+1}\right)$. Then for every i it holds

$$\sup_k\left\{|f(x,u)|:\;|x|\leq i,\;|u|\leq i\right\}\leq b+Mi$$

There exists a subsequence, still indexed by k, and a function $f\in C(\mathbb{R}^{d+1})$ with $\|f\|_\infty<b$ such that

$$\forall i:\qquad\lim_{k\to\infty}\sup\left\{|f_k(x,u)-f(x,u)|:\;\max\left\{|x|\,,|u|\right\}\leq i\right\}\;=0\,.$$

It is straight forward that f lies in $C^{0;0,1}_{M,b}\left(R^{d+1}\right)$. □

With a little effort, this lemma can be further generalized.

Definition 2.17. A metric space U is hierarchically divisible if for every $k \in \mathbb{N}$ there exists a countable family $(A_{k,i})_{i\in\mathbb{Z}}$ of Borel-measurable sets $A_{k,i} \subset U$ such that:

- $A_{k,i} \cap A_{k,j} = \emptyset$ for $i \neq j$
- For every $A_{k,i}$ there exists $A_{k-1,j}$ such that $A_{k,i} \subset A_{k-1,j}$
- For every $u \in U$ there exists a sequence i_k, $k \in \mathbb{N}$, such that $\bigcap_k A_{k,i_k} = \{u\}$.

Example 2.18. In case $U = \mathbb{R}$ consider $A_{k,i} = \left[i2^{-k}, (i+1)2^{-k}\right)$.

Theorem 2.19. *Let U be a hierarchically divisible metric space with metric d and μ_ω a stationary random measure. Given a random measurable function $f_\omega : \mathbb{R}^d \times U \to \mathbb{R}$ stationary in $\mathbb{R}^d$ and $f_\omega(x, \cdot\,)$ almost surely continuous in U for every x and $\mathbb{E}\,|f_\omega(\cdot\,,u)| < \infty$ for every $u \in U$ then Ω can be chosen to be precompact and there exists a measurable $f : \Omega \times U \to \mathbb{R}$ such that for every $u \in U$ it holds $\int_\Omega |f(\cdot\,,u)|\,\mathrm{d}\mu_{\mathcal{P}} < \infty$ and almost surely $f_\omega(x,u) = f(\tau_x\omega, u)$ in the μ_ω sense. Furthermore, if $m_\omega : \mathbb{R}^d \times \mathbb{R} \to [0,+\infty)$ is stationary and almost surely continuous in $\mathbb{R}$ for every $x \in \mathbb{R}^d$ and if almost surely $|f_\omega(x,u_1) - f_\omega(x,u_2)| < m_\omega(x, d(u_1,u_2))$ then*

$$|f(\omega,u_1) - f(\omega,u_2)| \le m(\omega, d(u_1,u_2))\,. \tag{15}$$

Proof. We first consider $|f_\omega(x,u)| \le 1$ and assume $f \ge 0$ for simplicity. Due to our assumptions on f, for every $u \in U$ it holds $f_\omega(\cdot\,,u)\mathrm{d}\mu_\omega$ is precompact in $\mathcal{M}(\mathbb{R}^d)$. Hence it holds that $\omega \mapsto (\mathrm{d}\mu_\omega, f_\omega \mathrm{d}\mu_\omega)$ is a map $\Omega \to \tilde{\Omega} \subset \mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)^U$, where $\tilde{\Omega}$ is precompact in the product topology. According to Theorem 2.14 there exist measurable $f_u \in L^\infty(\Omega;\mu_{\mathcal{P}})$ such that $f_\omega(x,u) = f_u(\tau_x\omega)$. The same argument holds for m_ω and thus (15) holds.

It remains to show that $(\omega,u) \mapsto f_u(\omega)$ is measurable. We approximate f_ω by

$$\forall k \in \mathbb{N}: \qquad \tilde{f}_{\omega,k}(x,u) = m2^{-k} \quad \text{if } f_\omega(x,u) \in \left[m2^{-k},\,(m+1)2^{-k}\right),\ m \in \mathbb{Z}\,,$$
$$f_{\omega,k}(x,u) = \inf\left\{\tilde{f}_{\omega,k}(y,u) : y \in A_{k,i}\right\} \quad \text{if } x \in A_{k,i}\,.$$

Then the corresponding function $f_{u,k} \in L^\infty(\Omega;\mu_{\mathcal{P}})$ is constant on every $A_{k,i}$ and hence measurable. Furthermore, by monotone convergence of $f_{u,k} \nearrow f_u$ we find that f_u is measurable. □

Example 2.20. a) Let f_1, f_2 be two stationary random functions in $C^{0;0,1}_{1,1}(\mathbb{R}^{d+1})$. Let $\mathbb{X}(\omega) := (x_i)_{i\in\mathbb{N}}$ be a Poisson point process with intensity $\lambda < \infty$. This means that for every bounded domain $A \subset \mathbb{R}^d$

$$\mathbb{P}(\#(A \cap \mathbb{X}) = k) = e^{-\lambda|A|}\frac{\lambda^k\,|A|^k}{k!}\,.$$

We construct from $\mathbb{X}(\omega)$ the Voronoi tessellation with the open cells G_i corresponding to x_i respectively. We distribute the values $w_i \in \{1,2\}$ i.i.d to each cell G_i and introduce the random function

$$f_\omega(x,u) = \begin{cases} f_{w_i}(x,u) & \text{on } G_i \\ \max\{f_1(x,u), f_2(x,u)\} & \text{else} \end{cases}.$$

This is a random function which is piecewise continuous in x and Lipschitz in u.

b) If $\mathbf{P}(\omega)$ is a random perforated domain with boundary $\Gamma(\omega) = \partial\mathbf{P}(\omega)$ with the random Hausdorff measure $\mu_\omega = \mathcal{H}^{d-1}(\cdot \cap \Gamma(\omega))$, we consider f_ω on $\Gamma(\omega)$. If $\mathbf{P}$ is independent from $\mathbb{X}$, the probability for any $x \in \Gamma(\omega)$ to also lie in one of the boundaries ∂G_i is zero. Then $f_\omega|_{\Gamma(\omega)} = f_{w_i}$ for some G_i. Denoting $F_\omega = f_\omega|_{\Gamma(\omega)}$, Theorem 2.19 yields two functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $F : \Omega \times \mathbb{R} \to \mathbb{R}$. In combination with Lemma 3.6 below we will see that $F : \Gamma \times \mathbb{R} \to \mathbb{R}$, where Γ is the support of $\mu_{\mathcal{P}}$.

3 Lebesgue and Sobolev spaces on Ω

Based on Theorems 2.4 and 2.14 we make the following Assumption.

Assumption 3.1 (Central Assumption)**.** *Let Ω be a precompact metric space with Borel sigma-algebra σ and probability measure $\mathbb{P}$ such that $(\Omega,\sigma,\mathbb{P})$ is a probability space. There exists a dynamical system $(\tau_x)_{x\in\mathbb{R}^d}$ of measurable maps $\tau_x:\,\Omega\to\Omega$ satisfying the conditions:*

- *$\tau_x\circ\tau_y=\tau_{x+y}$ and $\tau_0=\mathrm{id}$ (Group property)*
- *for every measurable $A\subset\Omega$ and every $x\in\mathbb{R}^d$ it holds $\mathbb{P}(A)=\mathbb{P}(\tau_{-x}A)$ (measure preserving)*
- *the evaluation $\Omega\times\mathbb{R}^d\to\Omega$, $(\omega,x)\mapsto\tau_x\omega$ is continuous (continuity of stationary evaluation)*

3.1 Lebesgues spaces and continuous functions on Ω

Lemma 3.2. *Let Ω be a precompact metric space with Borel sigma-algebra σ. Then $C(\overline{\Omega})$ is separable and for every finite Borel measure μ on Ω the space $C(\overline{\Omega})$ is dense in $L^p(\Omega;\mu)$.*

Remark. Note that $C(\overline{\Omega})\subset C_b(\Omega)\subset L^p(\Omega)$ with $C_b(\Omega)$ the bounded continuous functions on Ω in general being much larger and not necessarily separable.

Proof of Lemma 3.2. Since $\overline{\Omega}$ is compact and metric, $C(\overline{\Omega})$ is separable. Furthermore, for every relatively open sets $A\subset\Omega$ there exists precisely one relatively open $\bar{A}\subset\overline{\Omega}$ such that $A=\bar{A}\cap\Omega$. Defining $\bar{\mu}(\bar{A}):=\mu(A)$ it holds that $\bar{\mu}$ is a Borel measure on $\overline{\Omega}$. Since Ω and $\overline{\Omega}$ are metric spaces, the simple functions

$$\sum_{i=1}^{K}\alpha_i\chi_{A_i}\,,\qquad\sum_{i=1}^{K}\alpha_i\chi_{\bar{A}_i}\,,\qquad K\in\mathbb{N},\ \alpha_i\in\mathbb{R}\tag{16}$$

lie dense in $L^p(\Omega;\mu)$ and $L^p(\overline{\Omega};\bar{\mu})$ respectively. In particular, $L^p(\Omega;\mu)$ and $L^p(\overline{\Omega};\bar{\mu})$ are isomorph with the isomorphism $L^p(\overline{\Omega};\bar{\mu})\to L^p(\Omega;\mu)$, $f\mapsto f|_\Omega$ for functions of type (16), but also (through approximation) for continuous functions $f\in C(\overline{\Omega})$. Since $C(\overline{\Omega})$ is dense in $L^p(\overline{\Omega};\bar{\mu})$ we conclude. □

Lemma 3.3. *Let Ω be a precompact metric space with Borel sigma-algebra σ with probability measure $\mathbb{P}$. There exists $\tilde{\Omega}\subset\Omega$ of full measure such that for every $\omega\in\tilde{\Omega}$ and every $f\in C(\overline{\Omega})$ the limit (8) holds. We call $\omega\in\tilde{\Omega}$* typical*.*

Proof. Take a countable dense family $F:=(f_k)_{k\in\mathbb{N}}\subset C(\overline{\Omega})$ and observe there exists $\tilde{\Omega}\subset\Omega$ of full measure such that for every $\omega\in\tilde{\Omega}$ and every $f_k\in F$ the limit (8) holds. Now approximate f by a sequence in F. The convergence is uniform by compactness of Ω and from here one concludes (8). □

We furthermore find due to (2) the following behavior.

Lemma 3.4. *Let Assumption 3.1 hold and let $f\in L^p(\Omega)$ for $1\leq p\leq\infty$. Then for almost every $\omega\in\Omega$ and for every bounded domain $\mathbf{Q}$ it holds $f_\omega\in L^p(\mathbf{Q})$ where*

$$f_\omega:\,\mathbb{R}^d\to\mathbb{R}\,,\qquad x\mapsto f(\tau_x\omega)\,.$$

We now come to a very important point that helps to interpret stochastic homogenization results when it comes to lower dimensional structures.

Definition 3.5 (See [2] Proposition 7.2.9)**.** The support of a Borel measure on a separable metric space is the intersection of all closed sets of full measure.

The following result was proved in [14] for $d-1$ -dimensional structures. The following Lemma generalizes the result to general random measures.

Lemma 3.6. *Let μ_ω be a stationary random measure on the precompact metric probability space Ω with Palm measure $\mu_{\mathcal{P}}$ and let $S(\omega)\subset\mathbb{R}^d$ and $S\subset\Omega$ be the support of μ_ω and $\mu_{\mathcal{P}}$ respectively. Then for almost every ω it holds $\chi_{S(\omega)}(x)=\chi_S(\tau_x\omega)$ in the sense of μ_ω and of the Lebesgue measure $\mathcal{L}$.*

Proof. Let $\phi\in C_c(\mathbb{R}^d)$ and $\psi\in C(\overline{\Omega})$. Then Theorem 2.5 implies

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^d}\phi(x)\psi(\tau_x\omega)\chi_{S(\omega)}(x)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)&=\int_\Omega\int_{\mathbb{R}^d}\phi(x)\psi(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)\\&=\int_\Omega\int_{\mathbb{R}^d}\phi(x)\psi(\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\omega)\,\mathrm{d}x\\&=\int_\Omega\int_{\mathbb{R}^d}\phi(x)\psi(\omega)\chi_S(\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\omega)\,\mathrm{d}x\\&=\int_\Omega\int_{\mathbb{R}^d}\phi(x)\psi(\tau_x\omega)\chi_S(\tau_x\omega)\,\mathrm{d}\mu_\omega(x)\,\mathrm{d}\mathbb{P}(\omega)\,.\end{aligned}$$

Since ϕ and ψ where arbitrary, we conclude. □

3.2 Sobolev spaces on Ω

The semigroup

$$\mathrm{T}(x):\ f\mapsto\mathrm{T}(x)f:=f(\tau_x\,\cdot\,)\,,$$

is strongly continuous [23, 30] and for the canonical basis $(\mathbf{e}_i)_{i=1,\dots d}$ the operators $\mathrm{T}_i(t)f:=\mathrm{T}(t\mathbf{e}_i)f$ define independent mutually commuting one-parameter strongly continuous semigroups on $L^p(\Omega)$ that jointly generate $(\mathrm{T}(x))_{x\in\mathbb{R}^d}$ on $L^p(\Omega)$. Each of these one-parameter semigroups has a generator

$$\mathrm{D}_if(\omega)=\lim_{t\to0}\frac{\mathrm{T}_i(t)f(\omega)-f(\omega)}{t}=\lim_{t\to0}\frac{f(\tau_{t\mathbf{e}_i}\omega)-f(\omega)}{t}$$

and D_if is called i-th derivative of f having the property

$$\int_\Omega g\mathrm{D}_if\mathrm{d}\mathbb{P}=-\int_\Omega f\mathrm{D}_ig\mathrm{d}\mathbb{P}\,.$$

The joint domain of all D_i equipped with the operator norm in $L^p(\Omega)$ is a Banach space

$$W^{1,p}(\Omega):=\left\{f\in L^p(\Omega)\,|\,\forall i=1,\dots,d:\ \mathrm{D}_if\in L^p(\Omega)\right\}\,,$$

$$\|f\|_{W^{1,p}(\Omega)}:=\|f\|_{L^p(\Omega)}+\sum_{i=1}^d\|\mathrm{D}_if\|_{L^p(\Omega)}\,.$$

Of course, in case $p = 2$ this defines a Hilbert space and otherwise a Banach space. We finally denote $\mathrm{D}_\omega f := (\mathrm{D}_1 f, \dots, \mathrm{D}_d f)^T$ the gradient with respect to ω and by $-\mathrm{div}_\omega$ the adjoint of D_ω. Sometimes we write $\nabla_\omega f := \mathrm{D}_\omega f$ to underline the gradient aspect. Similar to distributional derivatives in $\mathbb{R}^d$, we may define $\mathrm{D}_\omega^k f$ through iterated application of D_ω and

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) \mid \forall j = 1, \dots, k \; : \; \mathrm{D}_\omega^j f \in L^p(\Omega)^{d^j} \right\} .$$

Furthermore, we denote

$$C^1(\overline{\Omega}) := \left\{ f \in C(\overline{\Omega}) : \, \mathrm{D}_\omega f \in C(\overline{\Omega}; \mathbb{R}^d) \right\} .$$

The proofs of the following results can be found in parts in various other sources, the most cited being surely [30]. However, they seem to have never been collected in one source.

Lemma 3.7. *[23]Under Assumption 3.1 for every $f \in W^{1,p}(\Omega)$ for almost every $\omega \in \Omega$ it holds $f_\omega \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$. In particular, for every bounded domain $\mathbf{Q} \subset \mathbb{R}^d$ it holds*

$$\forall \psi \in C_c^1(\mathbf{Q}) \; : \qquad \int_{\mathbf{Q}} f_\omega \partial_i \psi = - \int_{\mathbf{Q}} \psi \left(\mathrm{D}_i f\right)_\omega . \tag{17}$$

Lemma 3.8. *Under Assumption 3.1 let $1 \le p < \infty$ and let $\eta \in C_c^\infty(\mathbb{R}^d)$. For every $f \in L^p(\Omega)$ let*

$$\left(\eta * f\right)(\omega) := \int_{\mathbb{R}^d} \eta(x) f(\tau_x \omega) \mathrm{d}x \, .$$

Then for every $k \in \mathbb{N}$ it holds $\eta * f \in W^{k,p}(\Omega)$ with $\mathrm{D}_i(\eta * f) = (\partial_i \eta) * f$ and almost every realization of $\mathscr{I}_\delta f$ is an element of $C^\infty(\mathbb{R}^d)$. Furthermore, the estimates

$$\|\eta * f\|^p_{L^p(\Omega)} \le \|\eta\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\Omega)} \, , \qquad \|\mathrm{D}_i(\eta * f)\|^p_{L^p(\Omega)} \le \|\partial_i \eta\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\Omega)} \tag{18}$$

hold and we have $\mathrm{D}_i \left(\eta * f\right) = \eta * \mathrm{D}_i f$. Continuity of f implies continuity of $\eta * f$ and $f \in C(\overline{\Omega})$ implies $\eta * f \in C^1(\overline{\Omega})$.

Proof. Let $k \in \mathbb{N}$ and observe

$$\begin{aligned}\|\eta * f\|^p_{L^p(\Omega)} &= \int_\Omega (2k)^{-d} \int_{(-k,k)^d} \left|\left(\eta * f\right)(\tau_y \omega)\right|^p \mathrm{d}y \, \mathrm{d}\mathbb{P}(\omega) \\ &\le \int_\Omega (2k)^{-d} \int_{(-k,k)^d} \int_{\mathbb{R}^d} \left|\eta(x) f(\tau_{y+x}\omega)\right|^p \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\mathbb{P}(\omega) \, .\end{aligned}$$

Due to the convolution inequality we have

$$\begin{aligned}\|\eta * f\|^p_{L^p(\Omega)} &\le \|\eta\|_{L^1(\mathbb{R}^d)} (2k)^{-d} \int_\Omega \|f_\omega\|^p_{L^p((-k-1,k+1)^d)} \mathrm{d}\mathbb{P}(\omega) \\ &\le \|\eta\|_{L^1(\mathbb{R}^d)} \left(\frac{k+1}{k}\right)^{-d} \int_\Omega |f(\omega)|^p \mathrm{d}\mathbb{P}(\omega)\end{aligned}$$

and since k is arbitrary, the we obtain $\|\eta * f\|^p_{L^p(\Omega)} \le \|\eta\|_{L^1(\mathbb{R}^d)} \|f\|_{L^p(\Omega)}$, the first part of (18).

In order to show $\mathscr{I}_\delta f \in W^{k,p}(\Omega)$ observe

$$\frac{1}{t} \left(\eta * f(\tau_{t\mathbf{e}_i}\omega) - \eta * f(\omega)\right) = \int_{\mathbb{R}^d} \frac{1}{t} \left(\eta(x + t\mathbf{e}_i) - \eta(x)\right) f(\tau_x \omega) \, .$$

Taking the limit $t \to 0$ in $L^p(\Omega)$ on both sides using Lebesgue's dominated convergence theorem implies

$$\mathrm{D}_i\left(\eta * f\right) = \int_{\mathbb{R}^d} \partial_i \eta(x) f(\tau_x \omega)\,, \tag{19}$$

and hence $\mathrm{D}_i\left(\mathscr{I}_\delta f\right) \in L^p(\Omega)$ with $\mathrm{D}_i(\eta * f) = (\partial_i \eta) * f$ and the second part of (18) follows. Equation (19) also shows that

$$\left(\eta * f\right)\left(\tau_y \omega\right) = \int_{\mathbb{R}^d} \eta(x) f\left(\tau_{x+y}\omega\right) \mathrm{d}x = \int_{\mathbb{R}^d} \eta(x-y) f\left(\tau_x \omega\right) \mathrm{d}x$$

and hence almost every realization of $\eta * f$ has C^∞-regularity. Furthermore, (19) implies

$$\begin{aligned}
\mathrm{D}_i\left(\mathscr{I}_\delta f\right) &= \lim_{t\to 0} \frac{1}{t}\left(\left(\eta * f\right)\left(\tau_{t\mathbf{e}_i}\omega\right) - \left(\eta * f\right)(\omega)\right) \\
&= \eta * \lim_{t\to 0} \frac{1}{t}\left(f(\tau_{t\mathbf{e}_i}\omega) - f(\omega)\right) \\
&= \eta * \mathrm{D}_i f\,,
\end{aligned}$$

where we used continuity of $f \mapsto \eta * f$ and strong convergence of $\frac{1}{t}\left(f(\tau_{t\mathbf{e}_i}\omega) - f(\omega)\right) \to \mathrm{D}_i f$.

The statement for continuous functions is obvious from the definition. □

Similar to $L^p(\mathbb{R}^d)$- and Sobolev spaces on $\mathbb{R}^d$, we can introduce a family of smoothing operators. Let $(\eta_\delta)_{\delta>0}$ be a standard sequence of mollifiers which are symmetric w.r.t. 0 and define

$$\mathscr{I}_\delta :\ L^p(\Omega) \to L^p(\Omega)\,, \qquad \mathscr{I}_\delta f(\omega) := \int_{\mathbb{R}^d} \eta_\delta(x) f\left(\tau_x \omega\right) \mathrm{d}x\,. \tag{20}$$

Lemma 3.9. *Under Assumption 3.1 for every $\delta > 0$, $1 \le p < \infty$, the operator $\mathscr{I}_\delta$ is unitary and selfadjoint. For every $f \in L^p(\Omega)$, $k \in \mathbb{N}$ it holds $\mathscr{I}_\delta f \in W^{k,p}(\Omega)$, $\mathscr{I}_\delta f \to f$ strongly in $L^p(\Omega)$, and almost every realization of $\mathscr{I}_\delta f$ is an element of $C^\infty(\mathbb{R}^d)$. In particular, $C^1(\overline{\Omega})$ is dense in is dense in $W^{1,p}(\Omega)$. Finally, for $f \in W^{1,p}(\Omega)$ it holds*

$$\lim_{\delta\to 0} \left\|\mathscr{I}_\delta f - f\right\|_{W^{1,p}(\Omega)} = 0 \tag{21}$$

and $\mathrm{D}_i \mathscr{I}_\delta f = \mathscr{I}_\delta \mathrm{D}_i f$.

Remark. The operator $\mathscr{I}_\delta$ can also be used to smooth Palm measures on Ω simultaneously with the random measures, see Section 1 of [29] for details.

Proof. The selfadjointness follows from the definition of $\mathscr{I}_\delta$, symmetry of η_δ and invariance of $\mathbb{P}$ w.r.t. τ_x. All other parts except for (21) follow from Lemma 3.8.

Finally, observe that the the convolution inequality and the strong continuity of $\mathrm{T}(x)$ yield

$$\begin{aligned}
\int_\Omega \left|\mathscr{I}_\delta f - f\right|^p &= \int_\Omega \left|\int_{\mathbb{R}^d} \eta_\delta(x)\left(f(\tau_x\omega) - f(\omega)\right)\mathrm{d}x\right|^p \mathrm{d}\mathbb{P}(\omega) \\
&\le \int_\Omega \|\eta_\delta\|_{L^1(\mathbb{R}^d)}^p \int_{[-\delta,\delta]^d} \left|\left(f(\tau_x\omega) - f(\omega)\right)\right|^p \mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega) \\
&\le \int_{[-\delta,\delta]^d} \int_\Omega \left|\left(f(\tau_x\omega) - f(\omega)\right)\right|^p \mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x \quad \to 0\,.
\end{aligned}$$

Since $\mathrm{D}_i \mathscr{I}_\delta f = \mathscr{I}_\delta \mathrm{D}_i f$, it also holds $\mathrm{D}_i \mathscr{I}_\delta f \to \mathrm{D}_i f$ strongly in $L^p(\Omega)$. □

3.3 Potentials and solenoidals

We denote by $L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ the set of measurable functions $f:\mathbb{R}^d\to\mathbb{R}^d$ such that $f|_{\mathbf{U}}\in L^p(\mathbf{U};\mathbb{R}^d)$ for every bounded domain $\mathbf{U}$ and we define

$$\begin{aligned} L^p_{\mathrm{pot,loc}}(\mathbb{R}^d) &:= \left\{u\in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\mid \forall\mathbf{U}\text{ bounded domain}, \exists\varphi\in W^{1,p}(\mathbf{U})\,:\, u=\nabla\varphi\right\}\,,\\ L^p_{\mathrm{sol,loc}}(\mathbb{R}^d) &:= \left\{u\in L^p_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\mid \int_{\mathbb{R}^d}u\cdot\nabla\varphi=0\ \forall\varphi\in C^1_c(\mathbb{R}^d)\right\}\,. \end{aligned}$$

Remark 3.10. The space $L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$ is invariant under convolution. This follows immediately from the fact that if $u=\nabla\varphi$ locally, then $\eta_\delta * u=\nabla\left(\eta_\delta*\varphi\right)$.

Recalling the notation for a realization $u_\omega(x):=u(\tau_x\omega)$ for $u\in L^p(\Omega)$, we can then define corresponding spaces on Ω through

$$\begin{aligned} L^p_{\mathrm{pot}}(\Omega) &:= \left\{u\in L^p(\Omega;\mathbb{R}^d)\,:\,u_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)\text{ for }\mathbb{P}-\text{a.e. }\omega\in\Omega\right\}\,,\\ L^p_{\mathrm{sol}}(\Omega) &:= \left\{u\in L^p(\Omega;\mathbb{R}^d)\,:\,u_\omega\in L^p_{\mathrm{sol,loc}}(\mathbb{R}^d)\text{ for }\mathbb{P}-\text{a.e. }\omega\in\Omega\right\}\,,\\ \mathcal{V}^p_{\mathrm{pot}}(\Omega) &:= \left\{u\in L^p_{\mathrm{pot}}(\Omega)\,:\,\int_\Omega u\,\mathrm{d}\mathbb{P}=0\right\}\,. \end{aligned} \tag{22}$$

The spaces $L^p_{\mathrm{pot}}(\Omega)$ and $W^{1,p}(\Omega)$ are connected as the following theorem shows.

Theorem 3.11. *For $1<p,q<\infty$ with $\frac1p+\frac1q=1$ the spaces $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $L^p_{\mathrm{sol}}(\Omega)$ are closed and it holds*

$$\left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp=L^q_{\mathrm{sol}}(\Omega)\,,\qquad \left(L^p_{\mathrm{sol}}(\Omega)\right)^\perp=\mathcal{V}^q_{\mathrm{pot}}(\Omega) \tag{23}$$

in the sense of duality. Furthermore, $W^{1,p}(\Omega)$ lies densely in $L^p(\Omega)$ and $\mathcal{V}^p_{\mathrm{pot}}(\Omega)=\mathscr{I}_\delta*\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ with

$$\mathcal{V}^p_{\mathrm{pot}}(\Omega)=\mathrm{closure}_{L^p}\left\{\mathrm{D}u\mid u\in W^{1,p}(\Omega)\right\}\,. \tag{24}$$

Remark 3.12. With (22) and (24) we have two different definitions of $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ at hand. (24) relates to [23], while (22) seems to be favored in [30] and many related works. However, the author is not aware of any proof that both definitions coincide, which is why we provide it here for the sake of completeness.

Proof. The density of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ follows from Lemma 3.9. We furthermore observe that $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ is invariant with respect to $\mathscr{I}_\delta$. In fact, let $u\in\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and consider $\omega\in\Omega$ such that $u_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$. Then

$$\left(\mathscr{I}_\delta u\right)_\omega(x)=\int_{\mathbb{R}^d}\eta_\delta(y)u\left(\tau_{x+y}\omega\right)\mathrm{d}y$$

and hence $\left(\mathscr{I}_\delta u\right)_\omega\in L^p_{\mathrm{pot,loc}}(\mathbb{R}^d)$ due to Remark 3.10. Furthermore, the space $L^p_{\mathrm{sol}}(\Omega)$ is closed as can be seen from the continuity of the expression

$$L^p(\Omega;\mathbb{R}^d)\to\mathbb{R}\,,\qquad u\mapsto\int_\Omega g(\omega)\int_{\mathbb{R}^d}u(\tau_x\omega)\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\,,$$

where $\varphi\in C^1_c(\mathbb{R}^d)$ and $g\in L^q(\Omega)$ are arbitrary.

It remains to show (23), (24) and closedness of $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$.

Step 1: We first show that $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $L^q_{\mathrm{sol}}(\Omega)$ are mutually orthogonal in the sense of duality. Let $\upsilon\in\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ and $p\in L^q_{\mathrm{sol}}(\Omega)$ and chose $\omega\in\Omega$ such that for $\upsilon^\varepsilon(x)=\upsilon(\tau_{\frac{x}{\varepsilon}}\omega)$, $p^\varepsilon(x)=p(\tau_{\frac{x}{\varepsilon}}\omega)$

and $\upsilon^\varepsilon \cdot p^\varepsilon$ the ergodic theorem holds. Thus, we get $\upsilon^\varepsilon \cdot p^\varepsilon \rightharpoonup \mathbb{E}(\upsilon \cdot p|\mathscr{I})$ weakly in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. It remains to show that $\upsilon^\varepsilon \cdot p^\varepsilon \rightharpoonup^* 0$. Since $\upsilon \in L^p_{\mathrm{pot}}(\Omega)$, we find for every $\varepsilon > 0$ some $u^\varepsilon \in W^{1,p}(\mathbf{Q})$ such that $\nabla u^\varepsilon = \upsilon^\varepsilon$ and $\int_{\mathbf{Q}} u^\varepsilon = 0$. By the PoincarÃľ inequality $u^\varepsilon \rightharpoonup u$ for some $u \in W^{1,p}(\mathbf{Q})$ and by the ergodic theorem $\nabla u^\varepsilon = \upsilon^\varepsilon \rightharpoonup^* \mathbb{E}(\upsilon|\mathscr{I}) = 0$. Because $u^\varepsilon \rightharpoonup u$ it holds $⨍_{\mathbf{Q}} u = 0$ and by the compactness $W^{1,p}(\mathbf{Q}) \hookrightarrow L^p(\mathbf{Q})$, we find $u^\varepsilon \to 0$ strongly in $L^p(\mathbf{Q})$. Therefore, for all $\psi \in C_c^\infty(\mathbf{Q})$, we find

$$\int_{\mathbf{Q}} \psi \upsilon^\varepsilon \cdot p^\varepsilon \, dx = \int_{\mathbf{Q}} \psi p^\varepsilon \cdot \nabla u^\varepsilon \, dx = -\int_{\mathbf{Q}} u^\varepsilon p^\varepsilon \cdot \nabla \psi \, dx \to 0 \quad \text{for } \varepsilon \to 0\,.$$

Therefore, we obtain

$$L^q_{\mathrm{sol}}(\Omega) \subset \left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \quad \text{and} \quad \mathcal{V}^p_{\mathrm{pot}}(\Omega) \subset (L^q_{\mathrm{sol}}(\Omega))^\perp\,. \tag{25}$$

Step 2: We prove (23) and closedness of $\mathcal{V}^p_{\mathrm{pot}}(\Omega)$ in case $p = 2$. From Step 1 we know that $L^2_{\mathrm{sol}}(\Omega) \subset \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$ and it remains to show that $\left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq L^2_{\mathrm{sol}}(\Omega)$. Let $u \in L^2(\Omega;\mathbb{R}^d)$ and use the decomposition $u = u_{\mathrm{pot}} + \tilde{u}$ where $u_{\mathrm{pot}} \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ and $\tilde{u} \in \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$. Since $\mathscr{I}_\delta$ is symmetric and $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ is invariant with respect to $\mathscr{I}_\delta$, we observe that

$$\forall \upsilon \in \mathcal{V}^2_{\mathrm{pot}}(\Omega) : \quad \langle \mathscr{I}_\delta \tilde{u},\, \upsilon \rangle = \langle \tilde{u},\, \mathscr{I}_\delta \upsilon \rangle = 0$$

and hence $\mathscr{I}_\delta \tilde{u} \in \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp$. In particular, for every $\varepsilon > 0$ and every $\phi \in L^2(\Omega)$ it holds

$$0 = \langle \mathscr{I}_\delta \tilde{u},\, \mathrm{D}_\omega \mathscr{I}_\varepsilon \phi \rangle = -\langle \mathrm{div}_\omega \mathscr{I}_\delta \tilde{u},\, \mathscr{I}_\varepsilon \phi \rangle$$

and as $\varepsilon \to 0$ it holds

$$0 = -\langle \mathrm{div}_\omega \mathscr{I}_\delta \tilde{u},\, \phi \rangle\,.$$

Since $\phi \in L^2(\Omega)$ was arbitrary, this implies $\sum_i \mathrm{D}_i \mathscr{I}_\delta \tilde{u} = 0$ almost everywhere, i.e. $\mathscr{I}_\delta \tilde{u} \in L^2_{\mathrm{sol}}(\Omega)$. Since $\mathscr{I}_\delta \tilde{u} \to \tilde{u}$ as $\delta \to 0$, the closedness of $L^2_{\mathrm{sol}}(\Omega)$ implies $\tilde{u} \in L^2_{\mathrm{sol}}(\Omega)$. Hence $L^2_{\mathrm{sol}}(\Omega) \supset \mathcal{V}^2_{\mathrm{pot}}(\Omega)^\perp$ and Step 1 implies $L^2_{\mathrm{sol}}(\Omega) = \mathcal{V}^2_{\mathrm{pot}}(\Omega)^\perp$ and closedness of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$.

Step 3: For $p \in [1,2]$ we deduce from Step 2

$$\left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq L^q(\Omega;\mathbb{R}^d) \cap \left(\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right)^\perp = L^q(\Omega;\mathbb{R}^d) \cap L^2_{\mathrm{sol}}(\Omega) \subseteq L^q_{\mathrm{sol}}(\Omega)\,. \tag{26}$$

Interchanging the role of $\mathcal{V}_{\mathrm{pot}}$ and L_{sol} yields

$$(L^p_{\mathrm{sol}}(\Omega))^\perp \subseteq \mathcal{V}^q_{\mathrm{pot}}(\Omega)\,. \tag{27}$$

Inclusions (25), (26) and (27) imply (23) for every $1 < p < \infty$.

Step 4: For $1 < p < \infty$ we denote

$$V := \left\{ \mathrm{D}\phi \mid \phi \in W^{1,p}(\Omega) \right\} \subset L^p_{\mathrm{pot}}(\Omega)\,.$$

Let $u \in L^q(\Omega;\mathbb{R}^d)$ satisfy

$$\forall \phi \in W^{1,p}(\Omega) : \qquad \langle u,\, \mathrm{D}_\omega \phi \rangle = 0\,.$$

According to Lemma 3.9, D_i and $\mathscr{I}_\delta$ commute for $\phi \in W^{1,p}(\Omega)$. Furthermore, $\mathscr{I}_\delta \phi \in W^{1,p}(\Omega)$ and hence

$$0 = \langle u,\, \mathrm{D}_\omega \mathscr{I}_\delta \phi \rangle = \langle u,\, \mathscr{I}_\delta \mathrm{D}_\omega \phi \rangle = -\langle \mathrm{div}_\omega \mathscr{I}_\delta u,\, \phi \rangle\,.$$

Since $\phi \in W^{1,p}(\Omega)$ was arbitrary and $W^{1,p}(\Omega)$ is dense in $L^p(\Omega)$, it follows $\mathrm{div}_\omega \mathscr{I}_\delta u = 0$, which implies $u \in L^q_{\mathrm{sol}}(\Omega)$ by closedness of $L^q_{\mathrm{sol}}(\Omega)$. To conclude, we have shown $L^q_{\mathrm{sol}}(\Omega) = \left(\mathcal{V}^p_{\mathrm{pot}}(\Omega)\right)^\perp \subseteq V^\perp \subseteq L^q_{\mathrm{sol}}(\Omega)$, and hence (24).

□

3.4 Spaces on $\mathbf{P} \subset \Omega$

Definition 3.13. [26]A domain $\mathbf{P} \subset \mathbb{R}^d$ is minimally smooth if there exist $\delta > 0$ and $M > 0$ such that for every $p \in \partial\mathbf{P}$ the set $\partial\mathbf{P} \cap \mathbb{B}_\delta(p)$ is the graph of a Lipschitz function with Lipschitz constant smaller or equal to M.

Let $\mathbf{P}(\omega)$ be a random closed set which is almost surely minimally smooth and let $\Gamma(\omega) := \partial\mathbf{P}(\omega)$. We summarize some results from [12, 25, 11] in the following Lemma. Some of the results have been proven explicitly, but particularly (28) is a consequence of the proofs in the literature.

Lemma 3.14. *If the random geometry $\mathbf{P}(\omega)$ is minimal smooth then there almost surely exists $C > 0$ depending only on (δ, M) and an extension operator $\mathcal{U}_\omega : W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$ such that for every bounded open $\mathbf{Q} \supset \mathbb{B}_1(0)$ it holds*

$$\forall u \in W^{1,p}(\mathbb{B}_\delta(\mathbf{Q}) \cap \mathbf{P}(\omega)) : \qquad \|\mathcal{U}_\omega u\|_{W^{1,p}(\mathbf{Q})} \leq C \|u\|_{W^{1,p}(\mathbb{B}_\delta(\mathbf{Q})\cap\mathbf{P}(\omega))} , \tag{28}$$

and a trace operator $\mathcal{T}_\omega : W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to L^p_{\mathrm{loc}}(\Gamma(\omega))$ such that

$$\forall u \in W^{1,p}(\mathbb{B}_\delta(\mathbf{Q}) \cap \mathbf{P}(\omega)) : \qquad \|\mathcal{T}_\omega u\|_{L^p(\mathbf{Q}\cap\Gamma(\omega))} \leq C \|u\|_{W^{1,p}(\mathbb{B}_\delta(\mathbf{Q})\cap\mathbf{P}(\omega))} . \tag{29}$$

If it is additionally assumed that the random inclusions $\mathbb{R}^d \backslash \mathbf{P}(\omega)$ have a finite maximal diameter independent from the realization then

$$\forall u \in W^{1,p}(\mathbb{B}_\delta(\mathbf{Q}) \cap \mathbf{P}(\omega)) : \qquad \|\nabla\mathcal{U}_\omega u\|_{L^p(\mathbf{Q})} \leq C \|\nabla u\|_{L^p(\mathbb{B}_\delta(\mathbf{Q})\cap\mathbf{P}(\omega))} .$$

We observe that $\chi_{\mathbf{P}(\omega)}(x)\,\mathrm{d}x$ and $\chi_{\Gamma(\omega)}(x)\,\mathrm{d}\mathcal{H}^{d-1}(x)$ are random measures (see e.g. [28, 14]), where $\mathcal{H}^{d-1}$ is the $d-1$ -dimensional Hausdorff measure and by Lemma 3.6 there exist $\mathbf{P} \subset \Omega$ and $\Gamma \subset \Omega$ such that for almost every ω it holds $\chi_{\mathbf{P}(\omega)} = \chi_{\mathbf{P}}(\tau_\bullet\omega)$ in the Lebesgue sense and $\chi_{\Gamma(\omega)} = \chi_\Gamma(\tau_\bullet\omega)$ in the sense of Hausdorff and Lebesgue.

We then introduce the function space

$$W^{1,p}(\mathbf{P}) := \left\{\chi_{\mathbf{P}} u : \quad u \in W^{1,p}(\Omega)\right\} ,$$
$$\|u\|_{W^{1,p}(\mathbf{P})} := \|u\|_{L^p(\mathbf{P})} + \|\mathrm{D}u\|_{L^p(\mathbf{P})} .$$

We will see in Section 4.1 that (28) can be used to prove the following:

Theorem 3.15. *There exists a continuous extension operator $\mathcal{U}_\Omega : W^{1,p}(\mathbf{P}) \to W^{1,p}(\Omega)$ such that $(\mathcal{U}_\Omega u)\,|_{\mathbf{P}} = u$. In particular, $W^{1,p}(\mathbf{P})$ is closed.*

For $u \in C^1(\overline{\Omega})$ we can define $\mathcal{T}_\Omega[u] := u|_\Gamma$ and observe that (29) implies for every $R > 1$

$$\|(\mathcal{T}_\Omega[u])_\omega\|_{L^p(\Gamma(\omega)\cap\mathbb{B}_R(0))} \leq C\left(\|u_\omega\|_{L^p(\mathbf{P}(\omega)\cap\mathbb{B}_{R+\delta}(0))} + \|\nabla u_\omega\|_{L^p(\mathbf{P}(\omega)\cap\mathbb{B}_{R+\delta}(0))}\right) ,$$

which yields by the ergodic theorem

$$\|\mathcal{T}_\Omega[u]\|_{L^p(\Gamma)} \leq C \|u\|_{W^{1,p}(\mathbf{P})}$$

and the operator $\mathcal{T}_\Omega$ can be extended to $W^{1,p}(\Omega)$.

Theorem 3.16. *Let Assumption 3.1 hold and $\mathbf{P}(\omega)$ be almost surely minimally smooth and let $\nu_{\Gamma(\omega)}$ be the outer normal of $\mathbf{P}(\omega)$ on $\Gamma(\omega) := \partial\mathbf{P}(\omega)$. Let $\mu_{\Gamma(\omega)}$ be the Hausdorff measure on $\Gamma(\omega)$ with Palm measure $\mu_{\Gamma,\mathcal{P}}$ and let Γ be the support of $\mu_{\Gamma,\mathcal{P}}$. Then there exists a measurable function $\nu_\Gamma : \Gamma \to \mathbb{S}^{d-1}$ such that almost surely $\nu_{\Gamma(\omega)}(x) = \nu_\Gamma(\tau_x\omega)$. Furthermore, for $\phi \in W^{1,p}(\mathbf{P})$ the function $\tilde{\phi} := \mathcal{T}_\Omega\phi$ is the only function in $L^p(\Gamma)$ such that for every $f \in C^1(\overline{\Omega};\mathbb{R}^d)$ it holds*

$$\int_{\mathbf{P}} \operatorname{div}_\omega(f\phi)\,\mathrm{d}\mathbb{P} = \int_\Gamma \tilde{\phi} f \cdot \nu_\Gamma \,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,. \tag{30}$$

and for $f \in W^{1,p}(\mathbf{P};\mathbb{R}^d)$ the function $\tilde{f} := \mathcal{T}_\Omega f\cdot\nu_\Gamma$ is the only function in $L^p(\Gamma)$ such that for every $\phi \in C^1(\overline{\Omega})$ it holds

$$\int_{\mathbf{P}} \operatorname{div}_\omega(f\phi)\,\mathrm{d}\mathbb{P} = \int_\Gamma \phi\tilde{f}\,\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,.$$

Proof. For $\delta > 0$ define $\chi_\delta(\omega) := (\eta_\delta * \chi_{\mathbf{P}})(\omega)$. We observe for fixed ω that

$$|\mathrm{D}_\omega\chi_\delta|(\tau_x\omega) = |\mathrm{D}_\omega(\eta_\delta * \chi_{\mathbf{P}})|(\tau_x\omega) = |\eta_\delta * (\mathrm{D}_\omega\chi_{\mathbf{P}})(\tau_\cdot\omega)|(x) = \left|\eta_\delta * \nabla\chi_{\mathbf{P}(\omega)}\right|(x)\,, \tag{31}$$

and hence for almost every ω we have $|\mathrm{D}_\omega\chi_\delta| \to \left|\nabla\chi_{\mathbf{P}(\omega)}\right| = \mathcal{H}^{d-1}(\Gamma(\omega)\cap\,\cdot\,)$ weakly. Then for $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $f \in C(\overline{\Omega})$ it holds by the Palm formula and (31)

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| &= \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\,|\mathrm{D}_\omega\chi_\delta|(\tau_x\omega)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\left|\eta_\delta * \nabla\chi_{\mathbf{P}(\omega)}\right|(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\left(\left|\nabla\chi_{\mathbf{P}(\omega)}\right|(\mathbb{B}_\delta(\mathrm{supp}\varphi))\right)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\,,\end{aligned}$$

where $\left|\nabla\chi_{\mathbf{P}(\omega)}\right| = \mathcal{H}^{d-1}(\,\cdot\cap\Gamma(\omega)) = \mu_{\Gamma(\omega)}$. From the ergodic theorem, the $\mathbb{P}$-almost sure pointwise weak convergence and the Lebesgue dominated convergence theorem, we conclude

$$\int_{\mathbb{R}^d}\varphi\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| \to \int_\Omega\int_{\mathbb{R}^d} f(\tau_x\omega)\varphi(x)\,\mathrm{d}\mu_{\Gamma(\omega)}(x)\,\mathrm{d}\mathbb{P}(\omega) = \int_{\mathbb{R}^d}\varphi\int_\Omega f\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,,$$

which implies $\int_\Omega f\,|\mathrm{D}_\omega\chi_\delta| \to \int_\Omega f\mathrm{d}\mu_{\Gamma,\mathcal{P}}$. In a similar way, we show $\int_\Omega f\mathrm{D}_\omega\chi_\delta \to \int_\Omega f\mathrm{d}\widetilde{\mu_{\Gamma,\mathcal{P}}}$, where $\widetilde{\mu_{\Gamma,\mathcal{P}}}$ is a $\mathbb{R}^d$-valued measure on Γ. Furthermore, for every e_i in the canonical basis of $\mathbb{R}^d$, $\mathrm{e}_i\cdot\widetilde{\mu_{\Gamma,\mathcal{P}}} \ll \mu_{\Gamma,\mathcal{P}}$, which implies by the Radon-Nikodym theorem the existence of a measurable ν_Γ with values in $\mathbb{S}^{d-1}$ such that $\widetilde{\mu_{\Gamma,\mathcal{P}}} = \nu_\Gamma\mu_{\Gamma,\mathcal{P}}$. The property $\nu_{\Gamma(\omega)}(x) = \nu_\Gamma(\tau_x\omega)$ follows from the fact that $\widetilde{\mu_{\Gamma,\mathcal{P}}}$ is the Palm measure of $\nabla\chi_{\mathbf{P}(\omega)}$.

For $f \in C^1(\overline{\Omega};\mathbb{R}^d)$ and $\phi \in C^1(\overline{\Omega})$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ it holds

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi\int_{\mathbf{P}}\operatorname{div}_\omega(f\phi) &= \int_\Omega\int_{\mathbf{P}(\omega)}\varphi\operatorname{div}(f\phi)_\omega\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega\int_{\Gamma(\omega)}\varphi\,\phi_\omega f_\omega\cdot\nu_{\Gamma(\omega)} + \int_\Omega\int_{\mathbf{P}(\omega)}\nabla\varphi\,\cdot\phi_\omega f_\omega\\ &= \int_{\mathbb{R}^d}\int_\Gamma\varphi\,\phi f\cdot\nu_\Gamma\mathrm{d}\mu_{\Gamma,\mathcal{P}}\,\mathrm{d}x + \int_{\mathbb{R}^d}\nabla\varphi\cdot\int_{\mathbf{P}}\phi_\omega f_\omega\,,\end{aligned}$$

which implies (30) by integration by parts in the last integral and a density argument. Now the statement follows from density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$. □

4 Stochastic two-scale convergence

We consider a stationary random measure $\omega \to \mu_\omega$ with Palm measure $\mu_{\mathcal{P}}$ and we define

$$\mu_\omega^\varepsilon(A) := \varepsilon^d \mu_\omega\big(\varepsilon^{-1}A\big)\,. \tag{32}$$

For the corresponding Lebesgue spaces we write $L^p(\Omega;\mu_{\mathcal{P}})$ or $L^p(\mathbf{Q};\mu_\omega^\varepsilon)$, where $\mathbf{Q}\subset\mathbb{R}^d$ is a convex domain with C^1-boundary.

Remark. The convexity of $\mathbf{Q}$ is sufficient but not necessary [27]. However, it is often assumed [6] in order to simplify the presentation of the ergodic theorems.

We recall (Lemma 3.3) that $\omega\in\Omega$ is typical if (8) holds for all $f\in C(\overline{\Omega})$ and that the typical ω together have full measure.

Definition 4.1. Let Assumption 3.1 hold, let $\omega\in\Omega$ be typical and let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence $u^\varepsilon\in L^2(\mathbf{Q};\mu_\omega^\varepsilon)$ and let $u\in L^2(\mathbf{Q};L^2(\Omega;\mu_{\mathcal{P}}))$ such that

$$\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^2(\mathbf{Q};\mu_\omega^\varepsilon)}<\infty\,,$$

and such that for every $\varphi\in C_c^\infty(\mathbf{Q})$, $\psi\in C(\overline{\Omega})$

$$\lim_{\varepsilon\to0}\int_{\mathbf{Q}}u^\varepsilon(x)\varphi(x)\psi\big(\tau_{\frac{x}{\varepsilon}}\omega\big)\,\mathrm{d}\mu_\omega^\varepsilon(x)=\int_{\mathbf{Q}}\int_\Omega u(x,\tilde\omega)\varphi(x)\psi(\tilde\omega)\,\mathrm{d}\mu_{\mathcal{P}}(\tilde\omega)\,\mathrm{d}x\,. \tag{33}$$

Then u^ε is said to be two-scale convergent to u, written $u^\varepsilon\stackrel{2s}{\rightharpoonup}u$.

Remark 4.2. a) If $\chi\in L^1(\Omega;\mu_{\mathcal{P}})$ we can extend the class of testfunctions from $\psi\in C(\overline{\Omega})$ to $\chi\psi$ since $\chi(\tau_{\frac{\cdot}{\varepsilon}}\omega)\,\mathrm{d}\mu_\omega^\varepsilon$ is again a random measure with Palm measure $\chi\mathrm{d}\mu_{\mathcal{P}}$.

b) The class of test functions can always be enriched by a countable number of functions $(\psi_i)_{i\in\mathbb{N}}\subset L^1(\Omega;\mu_{\mathcal{P}})$. This holds because (8) will still hold for all testfunctions on a set $\tilde\Omega\subset\Omega$ of full measure. This can then easily be accounted for in the classical proof of the following Lemma 4.3.

Lemma 4.3 ([29] Lemma 5.1)**.** *Let Assumption 3.1 hold. Let $1<p\leq\infty$ and let $\omega\in\Omega$ be typical and $u^\varepsilon\in L^p(\mathbf{Q};\mu_\omega^\varepsilon)$ be a sequence of functions such that $\|u^\varepsilon\|_{L^p(\mathbf{Q};\mu_\omega^\varepsilon)}\leq C$ for some $C>0$ independent of ε. Then there exists a subsequence of $(u^{\varepsilon'})_{\varepsilon'\to0}$ and $u\in L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))$ such that $u^{\varepsilon'}\stackrel{2s}{\rightharpoonup}u$ and*

$$\|u\|_{L^p(\mathbf{Q};L^p(\Omega;\mu_{\mathcal{P}}))}\leq\liminf_{\varepsilon'\to0}\left\|u^{\varepsilon'}\right\|_{L^p(\mathbf{Q};\mu_\omega^\varepsilon)}\,. \tag{34}$$

Proof. The proof follows exactly the lines of [29]. □

The following result is standard and has been proven often in literature. Of course, the proofs provided there remain valid in our setting, as they require only separability of $L^p(\Omega)$. Some references for the proof are [16, 22, 13, 29].

Theorem 4.4. *Under Assumption 3.1 for every typical $\omega\in\Omega$ and $1<p\leq\infty$ the following holds: If $u^\varepsilon\in W^{1,p}(\mathbf{Q};\mathbb{R}^d)$ for all ε and if there exists $0<C_u<\infty$ with*

$$\sup_{\varepsilon>0}\|u^\varepsilon\|_{L^p(\mathbf{Q})}+\varepsilon^\gamma\|\nabla u^\varepsilon\|_{L^p(\mathbf{Q})}<C_u$$

Then there exists $u\in L^p(\mathbf{Q}L^p(\Omega;\mathbb{P}))$ such that $u^\varepsilon\stackrel{2s}{\rightharpoonup}u$. Depending on the choice of γ, the following holds:

1 *If $\gamma = 0$, then $u \in W^{1,p}(\mathbf{Q})$ with $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\mathbf{Q})$ and there exists $\boldsymbol{v}_1 \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$ such that $\nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \nabla_x u + \boldsymbol{v}_1$ weakly in two scales.*

2 *If $\gamma \in (0,1)$ then $\varepsilon^\gamma \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \boldsymbol{v}_1$ for some $\boldsymbol{v}_1 \in L^p(\mathbf{Q};\mathcal{V}^p_{\mathrm{pot}}(\Omega))$.*

3 *If $\gamma = 1$ then $u \in L^p(\mathbf{Q}; W^{1,p}(\Omega))$ and $\varepsilon \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} \mathrm{D}_\omega u$.*

4 *If $\gamma > 1$ then $\varepsilon^\gamma \nabla u^\varepsilon \stackrel{2s}{\rightharpoonup} 0$.*

4.1 Two-scale convergence on perforated domains

We consider a random open connected set $\mathbf{P}(\omega)$ with random measure $\chi_{\mathbf{P}(\omega)}(x)\mathrm{d}x$ and Palm measure $\chi_{\mathbf{P}}\mathbb{P}$ or and $\Gamma(\omega) := \partial\mathbf{P}(\omega)$ with random measure $\mu_{\Gamma(\omega)}(A) := \mathcal{H}^{d-1}(A)$ and with Palm measure $\mu_{\Gamma,\mathcal{P}}$ located on $\Gamma \subset \Omega$ with $\chi_{\Gamma(\omega)} = \chi_\Gamma(\tau.\omega)$.

Moreover, in view of (32), we write $\mu^\varepsilon_{\Gamma(\omega)}(A) := \varepsilon^d \mu_{\Gamma(\omega)}(\varepsilon^{-1}A) = \varepsilon\mathcal{H}^{d-1}(A \cap \varepsilon\Gamma(\omega))$. In case of $\mu_\omega = \chi_{\mathbf{P}(\omega)}\mathcal{L}$, we sometimes drop the notation μ^ε_ω in the spaces and norms.

Definition 4.5 (Uniform and weak extension property)**.** We say that $\mathbf{P}^\varepsilon(\omega) := \varepsilon\mathbf{P}(\omega)$ has the uniform extension property on $\mathbf{Q}$ if for almost every ω there exists $C_\omega > 0$ and a linear extension operator

$$\mathcal{U}:\, W^{1,p}_{\mathrm{loc}}(\mathbf{P}(\omega)) \to W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$$

such that

$$\mathcal{U}_\varepsilon[u](x) := \mathcal{U}[u(\varepsilon\cdot)]\left(\frac{x}{\varepsilon}\right)$$

satisfies the following: For every bounded open $A \subset \mathbb{R}^d$ it holds $(\mathcal{U}_\varepsilon u)\,|_A$ depends only on $u|_{\mathbf{P}^\varepsilon(\omega)\cap\mathbb{B}_\varepsilon(A)}$ and for every $\varepsilon > 0$ and $u \in W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \mathbf{P}^\varepsilon(\omega))$

$$\left\|\nabla\mathcal{U}_\varepsilon u\right\|_{L^p(\mathbf{Q})} \le C_\omega \left\|\nabla u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\,, \quad \left\|\mathcal{U}_\varepsilon u\right\|_{L^p(\mathbf{Q})} \le C_\omega \left\|u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\,. \tag{35}$$

Furthermore, $\mathbf{P}^\varepsilon(\omega)$ has the weak extension principle, if instead of (35) the following holds:

$$\left\|\mathcal{U}_\varepsilon u\right\|_{L^p(\mathbf{Q})} + \varepsilon\left\|\nabla\mathcal{U}_\varepsilon u\right\|_{L^p(\mathbf{Q})} \le C_\omega\left(\left\|u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon\left\|\nabla u\right\|_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\right)\,. \tag{36}$$

$\mathbf{P}^\varepsilon(\omega)$ has the uniform trace property on $\mathbf{Q}$ if for almost every ω there exists $C_\omega > 0$ such that the trace operator satisfies

$$\mathcal{T}:\, W^{1,p}(\mathbb{B}_\varepsilon(\mathbf{Q}) \cap \varepsilon\mathbf{P}(\omega)) \to L^p(\mathbf{Q} \cap \varepsilon\Gamma(\omega))$$

with

$$\varepsilon\left\|\mathcal{T}u\right\|^p_{L^p(\mathbf{Q}\cap\varepsilon\Gamma(\omega))} \le C_\omega\left(\left\|u\right\|^p_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))} + \varepsilon^p\left\|\nabla u\right\|^p_{L^p(\mathbb{B}_\varepsilon(\mathbf{Q})\cap\mathbf{P}^\varepsilon(\omega))}\right)\,.$$

Defining

$$W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q}\cap\mathbf{P}^\varepsilon(\omega)) := \left\{u \in W^{1,p}(\mathbf{P}^\varepsilon(\omega)):\; u|_{\mathbb{R}^d\setminus\mathbf{Q}} = 0\right\}$$

we obtain the following standard result (e.g. [25]) for which we provide the very short proof for convenience.

Lemma 4.6. *Let $1 < p \leq \infty$ and let $\mathbf{P}(\omega)$ be a random open domain such that almost surely $\mathbf{P}^{\varepsilon}(\omega)$ has the uniform extension property on $\mathbf{Q}$. Then for almost every $\omega \in \Omega$ the following holds:*

If $u^{\varepsilon} \in W^{1,p}_{0,\partial\mathbf{Q}}(\mathbf{Q} \cap \mathbf{P}^{\varepsilon}(\omega))$ for all ε with $\sup_{\varepsilon} \|u^{\varepsilon}\|_{L^p(\mathbf{Q}_1^{\varepsilon}(\omega))} + \|\nabla u^{\varepsilon}\|_{L^p(\mathbf{Q}_1^{\varepsilon}(\omega))} < C$ for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and functions $u \in W^{1,p}_0(\mathbf{Q})$ and $v \in L^p(\mathbf{Q}; \mathcal{V}^p_{\mathrm{pot}}(\Omega))$ such that

$$u^{\varepsilon'} \xrightarrow{2s} \chi_{\mathbf{P}} u \quad \textit{and} \quad \nabla u^{\varepsilon'} \xrightarrow{2s} \chi_{\mathbf{P}} \nabla u + \chi_{\mathbf{P}} v \qquad \textit{as } \varepsilon' \to 0\,, \tag{37}$$

$$\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} u \quad \textit{and} \quad \nabla \mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} \nabla u + v \qquad \textit{as } \varepsilon' \to 0\,. \tag{38}$$

Furthermore, $\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^d)$).

Proof. (38) is a consequence of (35) and Theorem 4.4. (37) is a consequence of (38) with Remark 4.2 for $\chi = \chi_{\mathbf{P}}$. Finally, $u \in W^{1,p}_0(\mathbf{Q})$ follows from the claim that $(\mathcal{U}_{\varepsilon} u)\,|_A$ depends only on $u|_{\mathbf{P}^{\varepsilon}(\omega) \cap \mathbb{B}_{\varepsilon}(A)}$. □

Lemma 4.7. *Let $1 < p \leq \infty$ and let $\mathbf{P}(\omega)$ be a random open domain such that almost surely $\mathbf{P}^{\varepsilon}(\omega)$ has the weak extension property on $\mathbf{Q}$. Then for almost every $\omega \in \Omega$ the following holds:*

If $u^{\varepsilon} \in W^{1,p}(\mathbb{B}_{\varepsilon}(\mathbf{Q}) \cap \mathbf{P}^{\varepsilon}(\omega))$ for all ε with

$$\sup_{\varepsilon} \|u^{\varepsilon}\|_{L^p(\mathbb{B}_{\varepsilon}(\mathbf{Q}) \cap \mathbf{P}^{\varepsilon}(\omega))} + \varepsilon \|\nabla u^{\varepsilon}\|_{L^p(\mathbb{B}_{\varepsilon}(\mathbf{Q}) \cap \mathbf{P}^{\varepsilon}(\omega))} < C$$

for C independent from $\varepsilon > 0$ then there exists a subsequence denoted by $u^{\varepsilon'}$ and a function $u \in L^p(\mathbf{Q}; W^{1,p}(\mathbf{P}))$ such that

$$u^{\varepsilon'} \xrightarrow{2s} u \quad \textit{and} \quad \varepsilon' \nabla u^{\varepsilon'} \xrightarrow{2s} \mathrm{D}_{\omega} u \qquad \textit{as } \varepsilon' \to 0\,.$$

Furthermore, if $\mathbf{P}^{\varepsilon}(\omega)$ has the uniform trace property and

$$\sup_{\varepsilon} \varepsilon \, \|\mathcal{T} u^{\varepsilon}\|^p_{L^p(\mathbf{Q} \cap \varepsilon\Gamma(\omega))} < \infty$$

then $\mathcal{T} u^{\varepsilon'} \xrightarrow{2s} \mathcal{T}_{\Omega} u$.

Proof of Lemma 4.7 and Theorem 3.15. The bound on $u^{\varepsilon'}$ and (36) together with Theorem 4.4 imply

$$\mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} \tilde{u} \quad \text{and} \quad \nabla \mathcal{U}_{\varepsilon'} u^{\varepsilon'} \xrightarrow{2s} \mathrm{D}\tilde{u} \qquad \text{as } \varepsilon' \to 0$$

for some $\tilde{u} \in W^{1,p}(\Omega)$. Hence $u^{\varepsilon'} \xrightarrow{2s} u = \chi_{\mathbf{P}} \tilde{u}$.

If $u \in C^1(\overline{\Omega})$ then the uniform trace property implies for $u^{\varepsilon}(x) := \chi_{\mathbf{P}(\omega)}(x) u\big(\tau_{\frac{x}{\varepsilon}} \omega\big)$ that $\mathcal{T} u^{\varepsilon} \xrightarrow{2s} v$ for some $v \in L^p(\Omega; \mu_{\Gamma,\mathcal{P}})$. On the other hand, a direct calculation shows that $\mathcal{T} u^{\varepsilon} = (\mathcal{T}_{\Omega} u)\,\big(\tau_{\frac{x}{\varepsilon}} \omega\big)$ and hence $v = \mathcal{T}_{\Omega} u$. Using $\phi \in C^1_c(\mathbf{Q})$ and $\psi \in C^1(\overline{\Omega})^d$ we find with $\tilde{\phi}_{\varepsilon}(x) := \phi(x)\psi(\tau_{\frac{x}{\varepsilon}} \omega)$

$$\begin{aligned}
\int_{\mathbf{Q}} \int_{\Gamma} \phi\psi \cdot \nu_{\Gamma} v \, \mathrm{d}\mu_{\Gamma,\mathcal{P}} \, \mathrm{d}x &= \lim_{\varepsilon \to 0} \varepsilon \int_{\mathbf{Q} \cap \Gamma^{\varepsilon}(\omega)} \phi(x)\psi(\tau_{\frac{x}{\varepsilon}} \omega) \cdot \nu_{\Gamma(\omega)} u^{\varepsilon}(x) \mathrm{d}\mathcal{H}^{d-1}(x) \\
&= \lim_{\varepsilon \to 0} \int_{\mathbf{Q} \cap \mathbf{P}^{\varepsilon}(\omega)} \left(\varepsilon \nabla u^{\varepsilon} \tilde{\phi}_{\varepsilon} + u^{\varepsilon} \left(\psi_{\omega}(\frac{\cdot}{\varepsilon}) \, \varepsilon \nabla \phi + \phi \, [\mathrm{D}_{\omega} \psi] \, (\frac{\cdot}{\varepsilon}) \right) \right) \\
&= \int_{\mathbf{Q}} \int_{\mathbf{P}} \phi \mathrm{div}_{\omega} \, (\psi u) \, \mathrm{d}\mathbb{P} \, \mathrm{d}x\,.
\end{aligned}$$

From Theorem 3.16 we conclude that $v = \mathcal{T}_{\Omega} u$.

To prove Theorem 3.15, if $u \in C^1(\overline{\Omega})$ we apply Lemma 4.7 to $u^{\varepsilon}(x) := \chi_{\mathbf{P}(\omega)}(x) u\big(\tau_{\frac{x}{\varepsilon}} \omega\big)$ and denote the two-scale limit by $\mathcal{U}_{\Omega} u$. Since this operator is linear and continuous, we can extend it to $W^{1,p}(\mathbf{P})$. □

4.2 Homogenization of a Stokes system with Navier-slip

In the setting of Section 4.1, we assume that $\mathbf{P}(\omega)$ is almost surely minimally smooth with (δ, M) independent from ω and that $\mathbb{R}^d \backslash \mathbf{P}(\omega)$ consists of inclusions of uniformly bounded diameter. Then $\mathbf{P}(\omega)$ has the strong extension property [12] and the trace property [25], see Lemma 3.14.

Given a bounded domain $\mathbf{Q} \subset \mathbb{R}^d$ and $\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega) := \mathbf{Q} \cap \varepsilon\mathbf{P}(\omega)$, $\Gamma^\varepsilon(\omega) := \mathbf{Q} \cap \varepsilon\Gamma(\omega)$ we write $\upsilon^\varepsilon_\tau$ for the tangential part of the vector valued function υ^ε on $\Gamma^\varepsilon(\omega)$ and consider the following problem:

$$
\begin{aligned}
-\varepsilon^2 \Delta \upsilon^\varepsilon + \nabla p^\varepsilon &= g && \text{on } \mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)\,, && (39)\\
\nabla \cdot \upsilon^\varepsilon &= 0 && \text{on } \mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)\,, && (40)\\
\left(-\varepsilon \nabla \upsilon^\varepsilon + \varepsilon^{-1} p^\varepsilon Id\right) \cdot \nu_{\Gamma^\varepsilon} &= \upsilon^\varepsilon_\tau && \text{on } \Gamma^\varepsilon(\omega)\,, && (41)\\
\upsilon^\varepsilon \cdot \nu_{\Gamma^\varepsilon} &= 0 && \text{on } \Gamma^\varepsilon(\omega)\,, && (42)\\
\upsilon^\varepsilon &= 0 && \text{on } \partial\mathbf{Q} \cap \varepsilon\mathbf{P}(\omega)\,, && (43)
\end{aligned}
$$

which is a Stokes system with Navier-slip condition (41). By a standard calculation, we observe that for some C independent from ε:

$$
\|\upsilon^\varepsilon\|_{L^2(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))} + \varepsilon \|\nabla \upsilon^\varepsilon\|_{L^2(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))} + \varepsilon^{\frac{1}{2}} \|\upsilon^\varepsilon\|_{L^2(\Gamma^\varepsilon(\omega))} + \|p^\varepsilon\|_{H^1(\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega))} \leq C \|g\|_{L^2(\mathbf{Q})}\,.
$$

We find $\upsilon^\varepsilon \stackrel{2s}{\rightharpoonup} \upsilon$, where $\upsilon \in L^2(\mathbf{Q}; W^{1,2}(\mathbf{P}))$ and $\mathcal{T}\upsilon^\varepsilon \stackrel{2s}{\rightharpoonup} \mathcal{T}_\Omega \upsilon$ as well as $\mathcal{U}_{\varepsilon,\omega} p^\varepsilon \rightharpoonup p$ weakly in $H^1(\mathbf{Q})$. In order to pass to the homogenization limit in the Stokes system, we introduce the space

$$
W^{1,p}_{\mathrm{sol}}(\mathbf{P}) := \left\{ u \in W^{1,p}(\mathbf{P})^d :\ \mathrm{div}_\omega u = 0 \right\}\,,
$$

which is a separable space and choose a countable dense family $\Psi := (\psi_i)_{i\in\mathbb{N}} \subset W^{1,2}_{\mathrm{sol}}(\mathbf{P})$.

Choosing some $\psi \in W^{1,p}(\mathbf{P})$ we find with $\psi^\varepsilon_\omega(x) := \psi\big(\tau_{\frac{x}{\varepsilon}}\omega\big)$ and with help of Theorem 3.16 and Lemma 4.7

$$
\begin{aligned}
0 = \varepsilon \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} \psi^\varepsilon_\omega \,\mathrm{div}\, \upsilon^\varepsilon &= -\int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} (\mathrm{D}_\omega \psi) \left(\tau_{\frac{\cdot}{\varepsilon}}\omega\right) \cdot \upsilon^\varepsilon + \varepsilon \int_{\mathbf{Q}\cap\Gamma^\varepsilon(\omega)} \psi^\varepsilon_\omega\, \upsilon^\varepsilon \cdot \nu_{\Gamma^\varepsilon(\omega)}\\
\to -\int_{\mathbf{Q}}\int_{\mathbf{P}} (\mathrm{D}_\omega\psi)\cdot\upsilon + \int_{\mathbf{Q}}\int_\Omega \psi\,\upsilon\cdot\nu_\Gamma \mathrm{d}\mu_{\Gamma,\mathcal{P}} &= \int_{\mathbf{Q}}\int_{\mathbf{P}} \psi \mathrm{div}_\omega \upsilon\,,
\end{aligned}
$$

which implies $\mathrm{div}_\omega \upsilon = 0$ and hence $\upsilon \in W^{1,2}_{\mathrm{sol}}(\mathbf{P})$. On the other hand, since $\upsilon^\varepsilon \cdot \nu_{\Gamma^\varepsilon(\omega)} = 0$, we find $\upsilon \cdot \nu_\Gamma = 0$. This implies

$$
\forall \psi \in W^{1,p}(\mathbf{P}): \qquad \int_{\mathbf{Q}}\int_{\mathbf{P}} (\mathrm{D}_\omega \psi)\cdot \upsilon = 0 \tag{44}
$$

In what follows we will use $\tilde{\varphi}, \phi \in C^1_c(\mathbf{Q})$ and $\psi \in \Psi$ and write $\varphi(x) = \tilde{\varphi}(x)$ and $\phi^\varepsilon_\omega(t,x) = \phi(x)\psi(\tau_{\frac{x}{\varepsilon}}\omega)$. Then testing (39) with ϕ^ε_ω and exploiting (40), (41)–(42) we find

$$
\varepsilon^2 \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} \nabla \upsilon^\varepsilon : \nabla \phi^\varepsilon_\omega - \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} p^\varepsilon \mathrm{div}\phi^\varepsilon_\omega + \int_{\Gamma^\varepsilon(\omega)} \upsilon^\varepsilon_\tau \cdot \phi^\varepsilon_\omega = \int_{\mathbf{Q}^\varepsilon_{\mathbf{P}}(\omega)} g \cdot \phi^\varepsilon_\omega\,.
$$

We exploit $\nabla\phi^\varepsilon_\omega = \left(\psi \otimes \nabla\phi + \frac{1}{\varepsilon}\phi\nabla_\omega\psi\right)$ and $\mathrm{div}\phi^\varepsilon_\omega = \psi\cdot\nabla\phi$ and pass to the two-scale limit in the above weak formulation to obtain

$$
\int_{\mathbf{Q}}\int_{\mathbf{P}} \mathrm{D}_\omega \upsilon \phi \mathrm{D}_\omega \psi - \int_{\mathbf{Q}}\int_{\mathbf{P}} p\,\psi\cdot\nabla\phi + \int_{\mathbf{Q}}\int_\Omega \upsilon_\tau \cdot \phi\psi\,\mathrm{d}\mu_{\Gamma,\mathcal{P}} = \int_{\mathbf{Q}}\int_{\mathbf{P}} g\cdot\phi\psi\,. \tag{45}
$$

Because of (44) we have to account for $\mathrm{div}_\omega \psi = 0$ for $\psi \in \Psi$ by adding an integral $\int_{\mathbf{Q}} \int_{\mathbf{P}} q \,\mathrm{div}_\omega \psi \phi$ for some $q \in L^2(\mathbf{Q}; W^{1,2}(\mathbf{P}))$. Then we find that (45) is a weak formulation of

$$\begin{aligned} -\mathrm{div}_\omega \mathrm{D}_\omega \upsilon + \nabla p + \mathrm{D}_\omega q &= g\,, && \text{on } \mathbf{Q}\times\mathbf{P} \\ \mathrm{div}_\omega \upsilon &= 0 && \text{on } \mathbf{Q}\times\mathbf{P} \\ -\mathrm{D}_\omega \upsilon + q\mathrm{Id} &= \upsilon_\tau && \text{on } \mathbf{Q}\times\Gamma \\ \upsilon\cdot\nu_\Gamma &= 0 && \text{on } \mathbf{Q}\times\Gamma \end{aligned}$$

If e_i is the i-th coordinate vector, we solve the systems

$$\begin{aligned} -\mathrm{div}_\omega(\nu \mathrm{D}_\omega u_i) + \mathrm{D}_\omega \Pi_i &= e_i \quad \text{on } \mathbf{P}\,, \\ \mathrm{div}_\omega u_i &= 0 \quad \text{on } \mathbf{P}\,, \qquad (46) \\ -\mathrm{D}_\omega u_i + q\mathrm{Id} &= u_{i,\tau} \quad \text{on } \Gamma \\ u_i\cdot\nu_\Gamma &= 0 \quad \text{on } \Gamma \qquad (47) \end{aligned}$$

by passing to the limit in (39)–(43) for $g = e_i$ and noting that there is no x-dependence of the solution in the limit, i.e. $\nabla p = 0$. We note at this point that we have not proved a PoincarÃľ inequality on $\mathbf{P}$ and Γ of the form $\|\upsilon\|_{L^2(\mathbf{P})} \leq C\left(\|\mathrm{D}_\omega \upsilon\|_{L^2(\mathbf{P})} + \|\upsilon_\tau\|_{L^2(\Gamma)}\right)$, which would make a direct proof more involved.

Finally, it is easy to see that $q = \sum_i (g - \nabla_x p)_i \Pi_i$ and $\upsilon := \sum_i (g - \nabla_x p)_i u_i$. Defining the matrix $K_{i,j} := \int_{\mathbf{P}} u_i \cdot e_j$ we find the homogenized system

$$\int_{\mathbf{P}} \upsilon = \sum_j (e_j\cdot\upsilon)\, e_j = K\,(f - \nabla_x p)\ . \qquad (48)$$

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