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Rough invariance principle for delayed regenerative processes

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Abstract

We derive an invariance principle for the lift to the rough path topology of stochastic processes with delayed regenerative increments under an optimal moment condition. An interesting feature of the result is the emergence of area anomaly, a correction term in the second level of the limiting rough path which is identified as the average stochastic area on a regeneration interval. A few applications include random walks in random environment and additive functionals of recurrent Markov chains. The result is formulated in the p-variation settings, where a rough Donsker Theorem is available under the second moment condition. The key renewal theorem is applied to obtain an optimal moment condition.

1 Introduction

Donsker's invariance principle states that a diffusively rescaled centered random walk on $\mathbb{R}^d$ with jumps of finite variance converges in distribution to a Brownian motion in the Skorohod topology. Regenerative processes are a more general class: they are assumed to contain an infinite subsequence of times on which the induced process is a random walk. The natural strategy to prove an invariance principle here is to first prove it for that subsequence, and then to show that the fluctuations of the original process coincide with the ones of the approximating sequence in the limit. However, when lifting regenerative processes to the rough path space a surprising feature appears in the limiting rough path. The first level is naturally, the Brownian motion defined by the covariance matrix achieved in the classical case, whereas the second level (see Section 1.1 for more details) does not coincide with the iterated integral of the Brownian motion, but should be corrected by a deterministic process. The latter is called 'area anomaly' and is linear in time and identified in terms of the stochastic area on a regeneration time interval, see for example (5) below. This provides non-trivial and new information on the limiting path which is not captured in the classical invariance principle. This information is crucial in order to describe the limit of stochastic differential equations (SDE) of the form

$$Y_t^{(n)} = Y_0 + \int_0^t b(Y_s^{(n)})\mathrm{d}s + \int_0^t \sigma(Y_s^{(n)})\mathrm{d}X_s^{(n)}, \quad t \in [0, T],$$

where the driver $X^{(n)}$ is a linearly interpolated rescaled regenerative process, b and σ are smooth functions and the integral is in the sense of Riemann-Stieltjes. Even though $X^{(n)}$ converges weakly to a Brownian motion B, the limit of $Y^{(n)}$ does not satisfy the SDE with b and σ driven by B, but the is an additional drift which is explicit in terms of the area correction of the second level of the limiting rough path (denoted by Γ in (5) below), see e.g [6] line (1.1) and the discussion below it.

In this work we optimize the moment condition on regeneration intervals that was assumed in [21]. The main obstacle of [21] is that it relies on the rough path extension of Donsker's Theorem [2] which is based on Kolmogorov's tightness criterion on Hölder rough paths. As mentioned already in [2], this was

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costly and therefore the assumed moment condition was not optimal. Instead of considering the somewhat heavy algebraic framework in Hölder rough path formalism, we consider the parametrization-free p-variation settings which fits better to jump processes and discrete-time processes.

Recently, a new machinery was introduced to deal with regularity for jump processes in the rough path topology. The main tool is the Lépingle Burkholder-Davis-Gundy (BDG) inequality lifted to the p-variation rough path settings. The latter provides an equivalence between the $(2q)$-th moment of the p-variation norm of a local martingale and the q-th moment of its quadratic variation, and allows to obtain the rough version of Donsker's Theorem under the second moment condition, as in the classical case. The proof is by now standard, however for completeness we sketch it in the proof of our main result.

In order to optimize the moment condition on a regeneration interval for processes with regenerative increments so that the rough path version of Donsker's Theorem is used under not more than the second moment, we apply the Key Renewal Theorem. This theorem roughly says that the mass function of the process in a regeneration interval around a fixed time (also called 'age' in the renewal theory jargon) is approaching a density which is proportional to the uniform measure of an independent copy of the interval, namely, its size-biased version. This result is sharp and in particular guarantees that in the limit as time goes to infinity the m-th moment of a regeneration interval around a deterministic time, and the $(m+1)$-st moment of a fixed regeneration interval are equal up to a constant. The result extends the classical Donsker Theorem for these processes with no extra regularity assumption.

There has been some progress related to random walks in the rough topology in the past two decades. The closest works, generalized in this paper are [21, 23, 22]. In the context of semimartingales and rough paths with jumps [3, 8, 4], CLT on nilpotent covering graphs and crystal lattices [15, 24, 16], additive functional of Markov process and random walks in random environment [6]. For homogenization in the continuous settings [5, 18, 19], and for additive functionals of fractional random fields [11, 12, 13].

In the remaining part of this section we shall present the model and the main result, Theorem 1.5, after introducing the necessary rough path theory elements in Section 1.1. In Chapter 2 we shall mention some examples to which our main result applies, whereas its proof is given in Chapter 3. We also included two short appendices which might be useful in other context.

1.1 Preliminaries on rough paths

For two families $(a_i)_{i\in I}, (b_i)_{i\in I}$ of real numbers indexed by I, we write $a_i \lesssim b_i$ if there is a positive constant c so that $a_i \leqslant cb_i$ for all $i \in I$. We write $a_n \approx b_n$ whenever $a_n - b_n \to 0$ as $n \to \infty$. Set $\mathbb{N} = \{1, 2, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\Delta_T := \{(s,t) : 0 \leqslant s < t \leqslant T\}$ for $T > 0$. For a function $X : [0,T] \to \mathbb{R}^d$ we set $X_{s,t} := X_t - X_s$. We interpret X also as a function on Δ_T given by $(s,t) \mapsto X_{s,t}$. For a metric space E we write $C([0,T], E)$ resp. $D([0,T], E))$ for the E-valued continuous resp. càdlàg functions on $[0,T]$. We write $C(\Delta_T, E)$ resp. $D(\Delta_T, E))$ for the space of E-valued functions $X : \Delta_T \to E$ so that $t \mapsto X_{s,t}$ is continuous resp. càdlàg on $[s,T]$, for every $s \in [0,T]$. By convention, whenever $E = \mathbb{R}^d$, we write $|\cdot|$ for the d-dimensional Euclidean norm. Also, the expectation of a vector (or matrix) valued random variable is understood coordinate- (or entry-) wise. For $x, y \in \mathbb{R}^d$ we write $x \otimes y \in \mathbb{R}^{d\times d}$ for the tensor product $(x \otimes y)_{i,j} = x_i y_j, i, j = 1, \ldots, d$, and $x^{\otimes 2}$ for $x \otimes x$. Whenever $(x_n)_n$ is a sequence of elements in $\mathbb{R}^d$ we write $x_n^i, i = 1, ..., d$, for their components.

For brevity, we shall focus on the necessary objects needed for introducing our results. We follow closely Chapter 2 of [6]. The reader is referred to Section 5 of [8] for details on Itô p-variation rough paths with jumps and to [9] and [10] for an extensive account of the theory of rough paths.

For a normed space $(E, \|\cdot\|_E)$, and a function $X \in C([0,T],E)$ or $X \in C(\Delta_T, E)$ we write $\|X\|_{\infty,[0,T]} := \sup_{(s,t)\in\Delta_T} \|X_{s,t}\|_E$ to denote the *uniform norm* of X, and for any $p \in (0,\infty)$, we write $\|X\|_{p,[0,T]} := \left(\sup \sum_{[s,t]\in\mathcal{P}} \|X_{s,t}\|_E^p\right)^{1/p}$, where the supremum is over all finite partitions $\mathcal{P}$ of $[0,T]$, to denote its p-*variation norm*. A continuous rough path is a pair of functions $(X, \mathbb{X}) \in C([0,T],\mathbb{R}^d) \times C(\Delta_T, \mathbb{R}^{d\times d})$ satisfying Chen's relation, that is

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,r} - \mathbb{X}_{r,t} = X_{s,r} \otimes X_{r,t} \quad \text{for all } \ 0 \leqslant s < r < t \leqslant T. \tag{1}$$

Definition 1.1 (p-variation rough path space)**.** *For $p \in [2,3)$, $C_p([0,T], \mathbb{R}^d \times \mathbb{R}^{d\times d})$ is the space of all continuous rough paths $(X,\mathbb{X})$ such that*

$$|||(X,\mathbb{X})|||_{p,[0,T]} := |X_0| + \|X\|_{p,[0,T]} + \|\mathbb{X}\|_{p/2,[0,T]} < \infty. \tag{2}$$

The (uniform) p-variation distance $\sigma_{p,[0,T]}$ *is defined by taking the norm of the path defined by the differences increment-wise:*

$$\sigma_{p,[0,T]}((X,\mathbb{X}),(Y,\mathbb{Y})) := |||(X-Y, \mathbb{X}-\mathbb{Y})|||_{p,[0,T]} \, .$$

We refer to X as the first level of the rough path $(X,\mathbb{X})$ and to $\mathbb{X}$ as its second level. We shall now state a simple and useful sufficient condition for convergence in p-variation based on the convergence in the uniform topology together with tightness of the p-variation norms, see Lemma 2.3 in [6] and which is based on Theorem 6.1 of [8].

Lemma 1.2 (Sufficient condition for convergence in p-variation)**.** *Assume that $(Z_n, \mathbb{Z}_n)_{n\in\mathbb{N}}$ is a sequence of continuous rough paths and let $p_0 \in (2,3)$. Assume also that there exists a continuous rough path $(Z,\mathbb{Z})$ such that $(Z_n,\mathbb{Z}_n) \to (Z,\mathbb{Z})$ in distribution in the uniform topology and that the family of real valued random variables $(\|(Z_n,\mathbb{Z}_n)\|_{p_0,[0,T]})_{n\in\mathbb{N}}$ is tight. Then $(Z_n,\mathbb{Z}_n) \to (Z,\mathbb{Z})$ in distribution in the p_0-variation uniform topology $C_p([0,T],\mathbb{R}^d \times \mathbb{R}^{d\times d})$ for all $p \in (p_0, 3)$.*

1.2 Main result

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a discrete time stochastic process on $\mathbb{R}^d$ defined on a probability space $(S,\mathcal{F},\mathbb{P})$ and let $\mathbb{E}$ be the corresponding expectation. Assume that X has a delayed regenerative increments, that is there exists a sequence $0 =: \tau_0 < \tau_1 < \tau_2 <$... of $\mathcal{F}$-measurable $\mathbb{N}_0$-valued random variables so that $(T_k, \{X_{\tau_k,\tau_k+m}, 0 \leqslant m \leqslant T_k\})_{k\in\mathbb{N}}$ is an i.i.d. family independent of $(T_0, \{X_{0,m}, 0 \leqslant m \leqslant \tau_1\})$ under $\mathbb{P}$, where $T_k = \tau_{k+1} - \tau_k$ are the *regeneration intervals* and $X_{\ell,k} := X_k - X_\ell$ are the increments. Assume that $\mathbb{E}[X_{\tau_1,\tau_2}] = 0$ and that $\gcd\{j : p_j > 0\} = d$ for some $d \in \mathbb{N}$, where $p_j = \mathbb{P}(T_1 = j)$, that is, T_1 is d-arithmetic. For any sequence $(X_k)_{k\in\mathbb{N}_0}$ of elements in $\mathbb{R}^d$ we define

$$X_t^{(n)} := \frac{1}{\sqrt{n}} X_{\lfloor nt\rfloor} + \frac{nt - \lfloor nt\rfloor}{\sqrt{n}}(X_{\lfloor nt\rfloor+1} - X_{\lfloor nt\rfloor}) \quad \text{and}$$

$$\mathbb{X}_{s,t}^{(n)} := \frac{1}{n} I^{\mathrm{Str}}_{\lfloor ns\rfloor,\lfloor nt\rfloor}(X) + \frac{n(t-s) - \lfloor nt\rfloor + \lfloor ns\rfloor}{n}\big(I^{\mathrm{Str}}_{\lfloor ns\rfloor,\lfloor nt\rfloor+1}(X) - I^{\mathrm{Str}}_{\lfloor ns\rfloor,\lfloor nt\rfloor}(X)\big), \tag{3}$$

where for positive integers $M < N$

$$I^{\mathrm{Str}}_{M,N}(X) := \sum_{M+1\leqslant k\leqslant N} \left(X_{M,k-1} \otimes X_{k-1,k} - \frac{1}{2} X_{k-1,k} \otimes X_{k-1,k} \right).$$

Remark 1.3. *Note that*

$$\mathbb{X}^{(n)}_{s,t} := \int_{(s,t]} \int_{(s,u)} \mathrm{d}X^{(n)}_v \otimes \mathrm{d}X^{(n)}_u,$$

where the integration with respect to $\mathrm{d}X^{(n)}_v$ is in the sense of Riemann-Stieltjes.

We shall now formulate the main regularity assumption. We remind the reader that for $Y \in \mathbb{R}^d$ we write Y^i_k for the i-th component of Y, $i \in \{1, ..., d\}$.

Assumption 1.4. *For all $i \in \{1, ..., d\}$, $m \in \{0, 1\}$ and $p \in \{0, 2\}$*

$$0 < \mathbb{E}\left[\left(\Xi^i_m\right)^p T_k \right] < \infty,$$

where $\Xi^i_m := \sup\{|X^i_{\tau_m,\tau_m+k}| : 0 \leqslant k \leqslant T_m\}$.

Theorem 1.5. *Let X be a discrete time stochastic process satisfying Assumption 1.4. Assume that $\mathbb{E}[X_{\tau_k,\tau_{k+1}}] = 0$ for every $k \in \mathbb{N}_0$. Then, $(X^{(n)}, \mathbb{X}^{(n)})_{n\in\mathbb{N}}$ converges in distribution to $(B, \mathbb{B} + \cdot\Gamma)$ in $C_p([0,T], \mathbb{R}^d \times \mathbb{R}^{d\times d})$ for every $p > 2$, where B is a centered Brownian motion with covariance*

$$[B, B]_t = \frac{\mathbb{E}[X^{\otimes 2}_{\tau_1,\tau_2}]}{\mathbb{E}[T_1]} t, \tag{4}$$

$\mathbb{B}$ is the Stratonovich iterated integral of B, that is $\mathbb{B}_{s,t} = \int_{(s,t]} B_{s,u} \otimes \circ\mathrm{d}B_u$ and $\Gamma \in \mathbb{R}^{d\times d}$ is given by

$$\Gamma = \frac{\mathbb{E}[A_{\tau_1,\tau_2}(X)]}{\mathbb{E}[T_1]}, \tag{5}$$

where $A_{M,N}(X) = \mathrm{Antisym}(\mathbb{X}^{(1)}_{M,N})$ is the antisymmetric part of the matrix $\mathbb{X}^{(1)}_{M,N}$. The notation $\mathbb{B} + \cdot\Gamma$ above is for $\left(\mathbb{B}_{s,t} + (t-s)\Gamma\right)_{(s,t)\in\Delta_T}$.

Remark 1.6. *Note that the positivity condition for the moment in Assumption 1.4 is assumed in order to avoid degeneracies. Indeed, it can be omitted if we accept a degenerate formulation of the invariance principle. More accurately, if this is violated, then $X^i_n = 0$ for all times for some coordinate i, for which one can say that the invariance principle holds with a singular covariance matrix. Note also that Assumption 1.4 holds whenever τ_1 has a third moment and $\frac{|X_{\ell,k}|}{k-\ell}$ are uniformly bounded from above by a constant a.s., for example, whenever the process X has values in $\mathbb{Z}^d$, with nearest-neighbor jumps.*

Remark 1.7 (Optimality of the result)**.** *Let $X_n = \sum_{k=1}^n \xi_k$ be a centered random walk on $\mathbb{R}^d$, that is $\mathbb{E}[\xi^i_k] = 0, i = 1, \dots, d$, where $\xi_k = (\xi^1_k, \dots, \xi^d_k)$ are $\mathbb{R}^d$-valued i.i.d random variables. Then $X = (X_n)_{n\in\mathbb{N}_0}$ has, trivially, delayed regenerative increments: here $T_k = 1$ and $\Xi^i_k = |\xi^i_k|$, $i = 1, \dots, d$, $k = 0, 1, \dots$ Therefore Assumption 1.4 is equivalent in this case to $\mathbb{E}\left[\sum_{i=1}^d |\xi^i_1|^2\right] < \infty$, which is a necessary and sufficient condition for the classical central limit theorem, cf. Theorem 4 in Chapter 7 of [14].*

2 Examples

Positive recurrent countable Markov chains. Assume that $(Y_k)_{k\in\mathbb{N}}$ is a positive recurrent irreducible Markov chain taking values in some measurable space $\boldsymbol{S}$, that is

$$\mathbb{P}(T_x^+ < \infty | X_0 = x) = 1 \,\text{for some}\, x \in \boldsymbol{S}$$

where $T_x^+ = \inf\{k \in \mathbb{N} : Y_k = x\}$. Assume moreover that $\mathbb{E}[(T_x^+)^3 | X_0 = x] < \infty$. Then the conditions of Theorem 1.5 hold under $\mathbb{P}(\cdot | X_0 = x)$ for the sequence

$$X_n := \sum_{k=0}^{n} f(Y_k) - n\frac{\mathbb{E}[D|Y_0 = x]}{\mathbb{E}[T_x^+|Y_0 = x]}, n \in \mathbb{N}_0,$$

where $D = \sum_{k=0}^{T_x^+ - 1} f(Y_k)$, and $f : \boldsymbol{S} \to \mathbb{R}^d$ is any bounded measurable function.

All the examples considered in Section 5 of [21] are applicable here. In particular, the result applies to **Random walks in periodic environment** ([21], Section 5.2), where the periodicity assumption can be easily relaxed. For example, it can include i.i.d. impurities, as long as the regenerative structure is kept. It applies also to **Random walks on covering graphs and hidden Markov chains**, (Chapter 5.2 of [21]). The examples there immediately satisfy Assumption 1.4. Moreover, these can be extended, allowing infinite modulating systems and covering graphs with infinite structure. Another application is to the so-called **Ballistic random walks in random environment** (Section 5.1 of [21]). Lastly, for the example of **Random walks in Dirichlet environments**, the rough invariance principle, Theorem 5.5 of [21] is shown in the ballistic regime (more accurately, whenever a condition which is denoted by $(T)_\gamma$ holds for some $\gamma \in (0, 1)$, see Chapter 6.1 of [26] for teh definition and more details) in the sense of Hölder rough paths only whenever the trap parameter κ satisfies $\kappa > 8$ in the α-Hölder rough path topology for all $\alpha < \frac{1}{2} - \frac{1}{(\kappa/2)^*}$, where $(\kappa/2)^* = \min\{\lfloor \kappa/2 \rfloor, 2\lfloor \kappa/4 \rfloor\}$. The improvement in the present work is that the rough invariance principle - Theorem 1.5 below - applies already whenever the trap parameter satisfies $\kappa > 3$, and moreover it holds for all $p > 2$, which corresponds to all $\alpha < 1/2$ in the α-Hölder settings.

3 Proof of Theorem 1.5

Set $Z_k := X_{\tau_k} = \sum_{\ell=0}^{k-1} Y_\ell$ for $k \geq 0$, where $Y_\ell := X_{\tau_\ell, \tau_{\ell+1}}$. Then $Z = (Z_k)_{k\in\mathbb{N}_0}$ is a random walk with square integrable jumps. Therefore,

$$(Z^{(n)}, \mathbb{Z}^{(n)}) \text{ converges in distribution to } (B^Z, \mathbb{B}^Z) \text{ in } C_p([0,T], \mathbb{R}^d) \tag{6}$$

for all $p > 2$, where B^Z is a centered Brownian motion with covariance

$$[B^Z, B^Z]_t = \mathbb{E}[Y_2 \otimes Y_2]t$$

and $\mathbb{B}^Z$ is the Stratonovich iterated integral of B^Z. Indeed, this is a rough path version of Donsker's Theorem and for completeness, we now sketch the proof. By Lemma 1.2 it is enough to prove first convergence in distribution in the uniform topology and then to show tightness for the sequence of p-variation norms. The convergence in the uniform topology for the path is Donsker's Theorem, cf. [7, 1], while for the iterated integral it follows by Theorem 2.2 of Kurtz-Protter [17] (with the slight modification to Stratonovich's integral rather then Itô's). Tightness of the p-variation norm of the path follows from Lépingle's p-variation inequality [20] combined with the BDG inequality, whereas tightness for the $p/2$-variation norm of the iterated integrals follows from Theorem 1.1 of [27], which is an off-diagonal version to the Lépingle p-variation BDG inequality (one can also use [25] or Proposition 3.8 of [6]).

Next, we treat the rescaled lift $(X^{(n)}, \mathbb{X}^{(n)})_{n\in\mathbb{N}}$. We shall work on each level separately. Let us first identify the limit by proving the convergence of the finite-dimensional distributions. For ease of notation we shall only show the one-dimensional distributions, proving the convergence for higher dimensions is done similarly. For any $u \geqslant 0$ let $\kappa(u)$ be the unique random integer k so that $\tau_k \leqslant u < \tau_{k+1}$. Note that $\kappa(u)$ is measurable with respect to $\sigma(T_k : k < u)$, since $\tau_k = \sum_{\ell=0}^{k-1} T_\ell$. Observe that

$$|X_t^{(n),i} - Z_{\kappa(nt)/n}^{(n),i}| = \left| \frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor}^i + \frac{nt - \lfloor nt \rfloor}{\sqrt{n}} (X_{\lfloor nt \rfloor + 1}^i - X_{\lfloor nt \rfloor}^i) - \frac{1}{\sqrt{n}} X_{\tau_{\kappa(nt)}}^i \right| \leqslant \frac{1}{\sqrt{n}} |\Xi_{\kappa(nt)}^i|.$$

Next, note that as $\tau_{\kappa(nt)} \leqslant nt < \tau_{\kappa(nt)+1}$

$$\frac{\kappa(nt)}{\kappa(nt)+1} \frac{\kappa(nt)+1}{\tau_{\kappa(nt)+1}} \leqslant \frac{\kappa(nt)}{nt} \leqslant \frac{\kappa(nt)}{\tau_{\kappa(nt)}}.$$

The weak law of large numbers for $(\tau_k)_{k\in\mathbb{N}_0}$ implies that

$$\frac{\kappa(nt)}{nt} \xrightarrow[n\to\infty]{} \mathbb{E}[T_1]^{-1} =: \beta \tag{7}$$

in probability with respect to $\mathbb{P}$, where we used the fact that $1 \leqslant \mathbb{E}[T_1] < \infty$ by assumption. In particular, $\mathbb{P}(\kappa(n) > 2\beta n) \xrightarrow[n\to\infty]{} 0$ and therefore for every $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}(\|X^{(n),i} - Z_{\kappa(n\cdot)/n}^{(n),i}\|_{\infty,[0,T]} > \varepsilon) &\leqslant \mathbb{P}\left(\|\Xi_{\kappa(n\cdot)}^i\|_{\infty,[0,T]} > \varepsilon\sqrt{n}\right) \\ &\leqslant \mathbb{P}\left(\max_{0\leqslant m\leqslant nT2\beta} \Xi_m^i > \varepsilon\sqrt{n}\right) + o(1). \end{aligned}$$

But since the maximum of order n i.i.d random variables with a finite second moment is sub-diffusive in probability the first term also vanishes in probability. Indeed,

$$\begin{aligned} \mathbb{P}\left(\max_{0\leqslant m\leqslant cn} \Xi_m^i > \varepsilon\sqrt{n}\right) &= 1 - \left(1 - \mathbb{P}\left(\Xi_0^i > \varepsilon\sqrt{n}\right)\right)\left(1 - \mathbb{P}\left(\Xi_1^i > \varepsilon\sqrt{n}\right)\right)^{\lfloor cn \rfloor} \\ &\approx 1 - \left(1 - \mathbb{P}\left(\Xi_1^i > \varepsilon\sqrt{n}\right)\right)^{\lfloor cn \rfloor} \\ &\approx 1 - \exp\left(-cn\mathbb{P}\left(\Xi_1^i > \varepsilon\sqrt{n}\right)\right), \end{aligned}$$

which tends to 0 as $n \to \infty$. This holds since $\mathbb{P}(|\Xi_1^i|^2 > \varepsilon^2 n) \leqslant \mathbb{P}(|\Xi_1^i|^2 > \varepsilon^2 j)$ for every $j \leqslant n$ and so

$$(n-k)\mathbb{P}(|\Xi_1^i|^2 > \varepsilon^2 n) \leqslant \sum_{j=k+1}^{n} \mathbb{P}(|\Xi_1^i|^2 > \varepsilon^2 j), \quad k \leqslant n.$$

Therefore,

$$\limsup_{n\to\infty} n\mathbb{P}\left(\Xi_1^i > \varepsilon\sqrt{n}\right) = \limsup_{n\to\infty}(n-k)\mathbb{P}(|\Xi_1^i|^2 > \varepsilon^2 n) \leqslant \sum_{j=k+1}^{\infty} \mathbb{P}(\varepsilon^{-2}|\Xi_1^i|^2 > j) \xrightarrow[k\to\infty]{} 0,$$

since the right hand side is summable as $\mathbb{E}[|\Xi_1^i|^2] < \infty$.

Next, since the maximum of linear interpolations of any finite sequence on any bounded interval is obttained on the end points, we have

$$\begin{aligned}\|Z^{(n),i}_{\kappa(n\cdot)/n} - Z^{(n),i}_{\cdot\beta}\|_{\infty,[0,T]} &\leqslant \frac{1}{\sqrt{n}} \max_{0\leqslant m\leqslant T\beta n} |Z^i_{\kappa(m/\beta)} - Z^i_m| \\ &\leqslant \frac{1}{\sqrt{n}} \max_{0\leqslant m\leqslant Tn\beta} |\Xi^i_m| \max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m|.\end{aligned}$$

Thus for $R > 0$

$$\begin{aligned}&\mathbb{P}(\|Z^{(n),i}_{\kappa(n\cdot)/n} - Z^{(n),i}_{\cdot\beta}\|_{\infty,[0,T]} > \varepsilon) \\ &\leqslant \mathbb{P}\left(\max_{0\leqslant m\leqslant T\beta n} |Z^i_{\kappa(m/\beta)} - Z^i_m| > \varepsilon\sqrt{n}, \max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m| \leqslant R\sqrt{n}\right) \\ &\quad + \mathbb{P}\left(\max_{0\leqslant m\leqslant T\beta n} |Z^i_{\kappa(m/\beta)} - Z^i_m| > \varepsilon\sqrt{n}, \max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m| > R\sqrt{n}\right) \\ &\leqslant \mathbb{P}\left(\max_{0\leqslant k,m\leqslant T\beta n, |k-m|\leqslant R\sqrt{n}} |Z^i_k - Z^i_m| > \varepsilon\sqrt{n}\right) + \mathbb{P}\left(\max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m| > R\sqrt{n}\right) \\ &\leqslant 3T\beta\sqrt{n}\mathbb{P}\left(\max_{k\leqslant R\sqrt{n}} |Z^i_k| > \varepsilon\sqrt{n}\right) + \mathbb{P}\left(\max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m| > R\sqrt{n}\right).\end{aligned}$$

To deal with the first term we use a standard two pairs estimate (see Example 10.1 of Billingsley [1]), to find some $K > 0$ and $\Psi(\lambda) \xrightarrow[\lambda\to\infty]{} 0$ so that for any fixed $R > 0$

$$\begin{aligned}\sqrt{n}\mathbb{P}\left(\max_{k\leqslant R\sqrt{n}} |Z^i_k| > \varepsilon\sqrt{n}\right) &\leqslant \sqrt{n}\frac{K}{\left(\varepsilon n^{1/4}/\sqrt{R}\right)^4} + \sqrt{n}\frac{\Psi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{R}}\right)}{\left(\varepsilon n^{1/4}/\left(2\sqrt{R}\right)\right)^2} \\ &= \frac{KR^2}{\varepsilon^4\sqrt{n}} + \frac{4R}{\varepsilon^2}\Psi\left(\frac{\varepsilon\sqrt{n}}{\sqrt{R}}\right) \xrightarrow[n\to\infty]{} 0.\end{aligned} \tag{8}$$

By the central limit theorem for renewal processes with finite variance

$$\mathbb{P}\left(\max_{0\leqslant m\leqslant Tn\beta} |\kappa(m/\beta) - m| > R\sqrt{n}\right) \xrightarrow[n\to\infty]{} \mathbb{P}(\max_{0\leqslant t\leqslant T} c|W_t| > R),$$

where W is a Brownian motion and $c > 0$ is some constant. As R can be chosen arbitrarily large, $\limsup_{n\to\infty} \mathbb{P}(\|Z^{(n),i}_{\kappa(n\cdot)/n} - Z^{(n),i}_{\cdot\beta}\|_{\infty,[0,T]} > \varepsilon) = 0$. Therefore,

$$\|X^{(n),i} - Z^{(n),i}_{\cdot\beta}\|_{\infty,[0,T]} \xrightarrow[n\to\infty]{} 0 \qquad \text{in probability}.$$

Applying Slutsky's Theorem in the Skorohod topology, Theorem B.1, to

$$X^{(n)} = X^{(n)} - Z^{(n)}_{\beta\cdot} + Z^{(n)}_{\beta\cdot} \tag{9}$$

we deduce that $X^{(n)} \xrightarrow[n\to\infty]{} B^Z_{\beta\cdot} =: B$ in distribution with respect to $\mathbb{P}$, so that B is a d-dimensional Brownian motion with a covariance matrix given in (4), as desired.

Next, we shall show tightness of the p-variation norms for $p > 2$. Note that $\kappa(u) \leqslant u$ for any $u \geqslant 0$. Indeed, $T_i \in \mathbb{N}$ for all $i \in \mathbb{N}$ by assumption and therefore $\kappa(u) \leqslant \tau_{\kappa(u)} \leqslant u$. Now, since

$$|X^{(n),i}_{s,t} - Z^{(n),i}_{\kappa(ns)/n,\kappa(nt)/n}| \leqslant \frac{1}{\sqrt{n}}(|\Xi^i_{\kappa(ns)}| + |\Xi^i_{\kappa(nt)+1}|),$$

for any partition $0 = t_0 < t_1 < \cdots < t_m = T$, we have

$$\sum_{r=1}^{m} |X^i_{nt_{r-1},nt_r} - Z^i_{\kappa(nt_{r-1}),\kappa(nt_r)}|^2 \lesssim \sum_{0 \leqslant k < nT} \#\{1 \leqslant r \leqslant m : \kappa(nt_r) = k\}|\Xi^i_k|^2 \leqslant \sum_{0 \leqslant k < nT} T_k |\Xi^i_k|^2.$$

Hence,

$$\mathbb{E}[\|X^{(n)} - Z^{(n)}_{\kappa(n\cdot)/n}\|^2_{2,[0,T]}] \lesssim \frac{1}{n}((n-1)\mathbb{E}[T_1|\Xi_1|^2] + \mathbb{E}[T_0|\Xi_0|^2]) \lesssim 1.$$

Next, since $\kappa(nT) \leqslant nT$ we have that $\left(\frac{1}{\sqrt{n}} Z^i_{\kappa(\ell)}\right)_{\ell \leqslant nT}$ is a subsequence of $\left(\frac{1}{\sqrt{n}} Z^i_k\right)_{k \leqslant nT}$ and therefore $\mathbb{E}[\|Z^{(n),i}_{\kappa(n\cdot)/n}\|^2_{p,[0,T]}] \leqslant \mathbb{E}[\|Z^{(n),i}\|^2_{p,[0,T]}] \lesssim 1$, see (**??**).

Using the triangle inequality together with the fact that the p-variation norms are monotonically decreasing in p,

$$\mathbb{E}[\|X^{(n)}\|^2_{p,[0,T]}] \lesssim \mathbb{E}[\|X^{(n)} - Z^{(n),i}_{\kappa(n\cdot)/n}\|^2_{2,[0,T]}] + \mathbb{E}[\|Z^{(n),i}_{\kappa(n\cdot)/n}\|^2_{p,[0,T]}] \lesssim 1,$$

as desired.

We shall now treat the second level. Let us show first convergence in the Skorohod topology. Note that for $\ell < k$ we have the following decomposition which is a consequence of Chen's relation (1)

$$I^{\mathrm{Str}}_{\tau_\ell,\tau_k}(X) = I^{\mathrm{Str}}_{\ell,k}(Z) + \sum_{\ell+1 \leqslant u \leqslant k} A_{\tau_{u-1},\tau_u}(X). \tag{10}$$

Indeed, $I^{\mathrm{Str}}_{\tau_{u-1},\tau_u}(X) = \mathrm{sym}(I^{\mathrm{Str}}_{\tau_{u-1},\tau_u}(X)) + A_{\tau_{u-1},\tau_u}(X)$, and by a direct computation $\mathrm{sym}(I^{\mathrm{Str}}_{\tau_{u-1},\tau_u}(X)) = \frac{1}{2} Z_{u-1,u} \otimes Z_{u-1,u}$. Therefore,

$$\begin{aligned} I^{\mathrm{Str}}_{\tau_\ell,\tau_k}(X) &= I^{\mathrm{Str}}_{\ell,k}(Z) + \sum_{\ell+1 \leqslant u \leqslant k} I^{\mathrm{Str}}_{\tau_{u-1},\tau_u}(X) - \frac{1}{2} \sum_{\ell+1 \leqslant u \leqslant k} Z_{u-1,u} \otimes Z_{u-1,u} \\ &= I^{\mathrm{Str}}_{\ell,k}(Z) + \sum_{\ell+1 \leqslant u \leqslant k} A_{\tau_{u-1},\tau_u}(X). \end{aligned}$$

By Remark 2.2 of [6], it is enough to prove the convergence of $\mathbb{X}^{(n)}_t := \mathbb{X}^{(n)}_{0,t}$. By (10), we have for any $t \geqslant 0$

$$\begin{aligned} \frac{1}{n}\left|\mathbb{X}^{(1)}_{nt} - \mathbb{Z}_{\kappa(nt)} - \sum_{1 \leqslant u \leqslant \kappa(nt)} A_{\tau_{u-1},\tau_u}(X)\right| &= \frac{1}{n}|\mathbb{X}^{(1)}_{\tau_{\kappa(nt)},nt} + X_{0,\tau_{\kappa(nt)}} \otimes X_{\tau_{\kappa(nt)},nt}| \\ &\leqslant \frac{1}{n}|\Xi^{\otimes 2}_{\kappa(nt)}| + \frac{1}{n}|Z_{\kappa(nt)}| \otimes |\Xi_{\kappa(nt)}|. \end{aligned}$$

Therefore

$$\begin{aligned} &\sup_{0 \leqslant s,t \leqslant T} \frac{1}{n}\left|\mathbb{X}^{(1)}_{ns,nt} - \mathbb{Z}_{\kappa(ns),\kappa(nt)} - \sum_{\kappa(ns) < u \leqslant \kappa(nt)} A_{\tau_{u-1},\tau_u}(X)\right| \\ &\qquad \leqslant 2 \sup_{0 \leqslant t \leqslant T} \frac{1}{n}(|\Xi^{\otimes 2}_{\kappa(nt)}| + |Z_{\kappa(nt)}| \otimes |\Xi_{\kappa(nt)}|). \end{aligned}$$

But $\sup_{0\leqslant t\leqslant T}\frac{1}{n}(|\Xi^{\otimes 2}_{\kappa(nt)}|+|Z_{\kappa(nt)}|\otimes|\Xi_{\kappa(nt)}|)\xrightarrow[n\to 0]{}0$ in probability. Indeed, we have already seen that $\mathbb{P}\left(\|\Xi^i_{\kappa(n\cdot)}\|_{\infty,[0,T]}>\varepsilon\sqrt{n}\right)\to 0$ for any $\varepsilon>0$ and for the second term

$$\mathbb{P}(\sup_{0\leqslant t\leqslant T}|Z_{\kappa(nt)}\otimes\Xi_{\kappa(nt)}|>\varepsilon n)\quad\leqslant\quad\mathbb{P}\left(\|Z_{n\cdot}\|_{\infty,[0,T]}>R\sqrt{n}\right)+\mathbb{P}\left(\max_{0\leqslant k\leqslant nT}|\Xi_k|>\varepsilon\sqrt{n}/R\right).$$

But for any $R>0$ the right term vanishes as $n\to\infty$, while the left term converges to $\mathbb{P}(\max_{0\leqslant t\leqslant T}|cW_t|>R)$, where W is a Brownian motion and $c>0$ is a constant, which vanishes as $R\to\infty$. By Slutsky's Theorem the convergence of $\mathbb{X}^{(n)}$ in distribution holds if

$$\frac{1}{n}\mathbb{Z}_{\kappa(n\cdot)}+\frac{1}{n}\sum_{1\leqslant u\leqslant\kappa(n\cdot)}A_{\tau_{u-1},\tau_u}(X)\xrightarrow[n\to\infty]{}\mathbb{B}_{0,\cdot}+\Gamma\cdot$$

in distribution in the uniform topology. To achieve the last convergence, first note that $(A_{\tau_{u-1},\tau_u}(X))_{u\in\mathbb{N}}$ are independent random matrices with the same law for $u\geqslant 2$. Note also that $|A_{\tau_{u-1},\tau_u}(X)|\leqslant 4|\Xi_{u-1}|^{\otimes 2}T_{u-1}$ which implies that $\mathbb{E}[|A_{\tau_{u-1},\tau_u}(X)|]<C$ for $u=1,2$ by Assumption 1.4. Hence the weak law of large numbers for the sum yields

$$\frac{1}{n}\sum_{1\leqslant k\leqslant n}A_{\tau_{k-1},\tau_k}(X)\xrightarrow[n\to\infty]{}\mathbb{E}[A_{\tau_1,\tau_2}(X)]$$

in probability. Together with the convergence in probability $\frac{\kappa(nt)}{nt}\xrightarrow[n\to\infty]{}\mathbb{E}[T_1]^{-1}$ which is the consequence of Assumption 1.4. with $\alpha=0$, we deduce that

$$\frac{1}{n}\sum_{1\leqslant k\leqslant\kappa(nt)}A_{\tau_{k-1},\tau_k}(X)\xrightarrow[n\to\infty]{}\frac{\mathbb{E}[A_{\tau_1,\tau_2}(X)]}{\mathbb{E}[T_1]}=:\Gamma\quad\text{almost surely}.$$

Moreover, since $\frac{\mathbb{E}[|A_{\tau_1,\tau_2}(X)|]}{\mathbb{E}[T_1]}<\infty$ we have moreover

$$\left\|\left|\frac{1}{n}\sum_{1\leqslant k\leqslant\kappa(n\cdot)}A_{\tau_{k-1},\tau_k}(X)-\Gamma\cdot\right\|\right|_{\infty,[0,T]}\xrightarrow[n\to 0]{}0\quad\text{in probability}.$$

Using Slutsky's Theorem again together with (6) it is left to show that

$$\|\mathbb{Z}^{(n)}_{\kappa(n\cdot)/n}-\mathbb{Z}^{(n)}_{\cdot\beta}\|_{\infty,[0,T]}\xrightarrow[n\to\infty]{}0\text{ in probability.}$$

As for the case of the first level we use (3) to reduce the last convergence to showing that $\|\mathbb{Z}^{(n)}_{\kappa(n\cdot)/n}-\mathbb{Z}^{(n)}_{\cdot\beta}\|_{\infty,[0,T]}1_{\left\{\max_{0\leqslant m\leqslant Tn}|\kappa(m)-m\beta|\leqslant R\sqrt{n}\right\}}\xrightarrow[n\to\infty]{}0$ in probability. Fix $\varepsilon>0$.

$$\begin{aligned}
&\mathbb{P}\left(\max_{0\leqslant k,m\leqslant T\beta n,|k-m|\leqslant R\sqrt{n}}|\mathbb{Z}^{i,j}_{0,m}-\mathbb{Z}^{i,j}_{0,k}|>\varepsilon n\right)\\
&\quad\lesssim\quad\mathbb{P}\left(\max_{0\leqslant k,m\leqslant T\beta n,|k-m|\leqslant R\sqrt{n}}|Z^i_{k,m}|\qquad\qquad|Z^j_{0,k}|>\varepsilon n\right)\\
&\quad\leqslant\quad\mathbb{P}\left(\max_{0\leqslant k,m\leqslant T\beta n,|k-m|\leqslant R\sqrt{n}}|Z^i_{k,m}|\quad>\sqrt{\varepsilon n}/R\right)+\mathbb{P}\left(\max_{0\leqslant k\leqslant nT}|Z^j_{0,k}|>\sqrt{\varepsilon n}R\right)\\
&\quad\leqslant\quad\frac{\sqrt{n}}{R}\mathbb{P}\left(\max_{0\leqslant k\leqslant R\sqrt{n}}|Z^i_{0,k}|>\sqrt{\varepsilon n}/R\right)\qquad+\mathbb{P}\left(\max_{0\leqslant k\leqslant nT}|Z^j_{0,k}|>\sqrt{\varepsilon n}R\right).
\end{aligned}$$

Now, for any fixed R the first term converges to zero by (8) while the limsup of the right term is bounded by $\mathbb{P}\left(\max_{0\leqslant t\leqslant T}|cW_t| > \sqrt{\varepsilon}R\right)$, which tends to zero as $R\to\infty$. Therefore we have proved the convergence is the uniform topology.

To end, we shall now prove tightness of the $p/2$-variation norms, $p>2$. As in the estimate for the first level, we observe that for $0=t_0<t_1<\cdots<t_m=T$ and coordinates $1\leqslant i,j\leqslant d$ we have

$$\begin{aligned}\sum_{r=1}^{m}|A^{i,j}_{nt_{r-1},nt_r}| &= \sum_{0\leqslant u<nT}\ \sum_{0\leqslant r\leqslant m:\kappa(nt_r)=u}|A^{i,j}_{nt_{r-1},nt_r}| \\ &\lesssim \sum_{0\leqslant u<nT}\#\{1\leqslant r\leqslant m:\kappa(nt_r)=u\}|\Xi^i_u\Xi^j_u| \\ &\leqslant \sum_{0\leqslant u<nT}T_u|\Xi^i_u\Xi^j_u|,\end{aligned}$$

which implies that $\mathbb{E}\left[\frac{1}{n}\left\||\left(\sum_{\kappa(ns)\leqslant u\leqslant\kappa(nt)}A_{\tau_{u-1},\tau_u}(X)\right)_{0\leqslant s<t\leqslant T}\right\||_{1,[0,T]}\right]\lesssim 1$. Also,

$$\left|\mathbb{X}^{(1)}_{ns,nt}-\mathbb{Z}_{\kappa(ns),\kappa(nt)}-\textstyle\sum_{\kappa(ns)\leqslant u\leqslant\kappa(nt)}A_{\tau_{u-1},\tau_u}(X)\right| \leqslant |\Xi^{\otimes 2}_{\kappa(ns),\kappa(nt)}|+|Z_{\kappa(ns),\kappa(nt)}|\otimes|\Xi_{\kappa(ns),\kappa(nt)}|,$$

and so

$$\mathbb{E}\left[\frac{1}{n}\left\||\left(\mathbb{X}^{(1)}_{ns,nt}-\mathbb{Z}_{\kappa(ns),\kappa(nt)}-\sum_{\kappa(ns)\leqslant u\leqslant\kappa(nt)}A_{\tau_{u-1},\tau_u}(X)\right)_{0\leqslant s<t\leqslant T}\right\||_{1,[0,T]}\right]\lesssim 1.$$

Observe that $\mathbb{E}[\|\mathbb{Z}^{(n)}\|_{p/2,[0,T]}]\lesssim 1$, see (**??**). Using the triangle inequality together with the fact that the p-variation norms are monotonically decreasing $\mathbb{E}[\|\mathbb{X}^{(n)}\|_{p/2,[0,T]}]\lesssim 1$, as desired. □

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A Key renewal theorem

In this section we show that the moment condition of Assumption 1.4 is asymptotically equivalent to a second moment condition on $\Xi_{\kappa(n)}$.

Theorem A.1 (Key renewal theorem). *Assume that* $p_j\geqslant 0, j\in\mathbb{N}_0, p_0=0, \sum_{j\in\mathbb{N}_0}p_j=1$ *so that* $\gcd\{j:p_j>0\}=d\in\mathbb{N}$. *If* $(b_n)_{n\in\mathbb{N}_0}$ *is a summable sequence of non-negative real numbers, then the equation*

$$a_n=\sum_{m=0}^{n}b_{n-m}u_m \tag{11}$$

has a unique solution satisfying

$$\lim_{n\to\infty}a_n=\frac{\sum_{m\in\mathbb{N}_0}b_m}{\sum_{j\in\mathbb{N}_0}jp_j}, \tag{12}$$

where $u_m = \sum_{k\in\mathbb{N}_0} p^{*k}(m)$ *is the* k*-fold convolution of p evaluated at* m*, that is* $p^{*k}(m) = \mathbb{P}\left(\sum_{j=1}^{k} T_j = m\right)$*, where* $(T_k)_{k\geqslant 2}$ *is a sequence of independent random variables so that* $T_k, k \geqslant 2$ *all have the same probability mass function* p*. Moreover, This to be understood even when* $\sum_{j\in\mathbb{N}_0} jp_j = \infty$*, in which case the limit on the right side of* (12) *is* 0*.*

Lemma A.2. *Let* Ξ_k *as defined in Assumption 1.4, then for* $r, \ell \in \mathbb{N}_0$

$$\mathbb{E}[|\Xi_{\kappa(n)}|^{\otimes r} T_{\kappa(n)}^{\ell}] \xrightarrow[n\to\infty]{} \frac{\mathbb{E}[|\Xi_2|^{\otimes r} T_2^{\ell+1}]}{\mathbb{E}[T_2]},$$

whenever the right hand side is finite.

Proof. Let $b_n = \mathbb{E}[|\Xi_2|^{\otimes r} T_2^{\ell} 1_{T_2>n}]$, then

$$\sum_{n\in\mathbb{N}_0} b_n = \sum_{n\in\mathbb{N}_0}\sum_{k>n} \mathbb{E}[|\Xi_2|^{\otimes r} T_2^{\ell} 1_{T_2=k}] = \sum_{k\in\mathbb{N}} k\mathbb{E}[|\Xi_2|^{\otimes r} k^{\ell} 1_{T_2=k}] = \mathbb{E}[|\Xi_2|^{\otimes r} T_2^{\ell+1}].$$

By The key renewal theorem there is a unique solution $(a_n)_{n\in\mathbb{N}_0}$ to the equation (A.1), and it satisfies the limit in (12). By the last computation the right hand side of (12) is exactly the one in the wanted assertion. It is therefore enough to show that $a_n = \mathbb{E}[|\Xi_{\kappa(n)}|^{\otimes r} T_{\kappa(n)}^{\ell}]$. Indeed,

$$\begin{aligned}
\mathbb{E}[|\Xi_{\kappa(n)}|^{\otimes r} T_{\kappa(n)}^{\ell}] &= \sum_{k\in\mathbb{N}_0} \mathbb{E}[|\Xi_k|^{\otimes r} T_k^{\ell} 1_{\kappa(n)=k}] = \sum_{k\in\mathbb{N}_0} \mathbb{E}[|\Xi_k|^{\otimes r} T_k^{\ell} 1_{\tau_k\leqslant n, \tau_k+T_k>n}] \\
&= \sum_{k\in\mathbb{N}_0}\sum_{m=0}^{n} \mathbb{E}[|\Xi_k|^{\otimes r} T_k^{\ell} 1_{T_k>n-m} 1_{\tau_k=m}] = \sum_{k\in\mathbb{N}_0}\sum_{m=0}^{n} \mathbb{E}[|\Xi_k|^{\otimes r} T_k^{\ell} 1_{T_k>n-m}]\mathbb{P}[\tau_k = m] \\
&= \sum_{m=0}^{n} \mathbb{E}[|\Xi_2|^{\otimes r} T_2^{\ell} 1_{T_2>n-m}] \sum_{k\in\mathbb{N}_0} \mathbb{P}[\tau_k = m] = \sum_{m=0}^{n} b_{n-m} u_m = a_n,
\end{aligned}$$

where in the fourth equality we used independence. □

Note that one could use the argument for b_n of the form $\mathbb{E}[f(\{X\tau_1, \tau_1 + k\}_{0\leqslant k\leqslant T_1}) g(T_1)]$, where f, g are real functions, as long as (b_n) is an absolutely summable sequence (of well-defined finite elements).

B Slutsky's Theorem in the Skorohod topology

Theorem B.1. *Let* $X^n, Y^n \in D([0,T],\mathbb{R})$*, the Skorohod space of càdlàg functions (or* $X^n, Y^n \in C([0,T],\mathbb{R})$*) so that* $X^n \to X$ *in distribution in* $D([0,T],\mathbb{R})$*,* $X \in C([0,T],\mathbb{R})$ *and* $\|Y^n - f\|_{\infty,[0,T]} \to 0$ *in probability, for a deterministic continuous function* f*. Then* $X^n + Y^n \to X + f$ *in distribution in* $D([0,T],\mathbb{R})$ *(or in* $C([0,T],\mathbb{R})$*, resp.).*

Sketch of proof. We shall show the case $D = D([0,T],\mathbb{R})$. Let $\Phi \in C_b(D,\mathbb{R})$. Since $g \mapsto \Phi(g + f)$ is bounded and continuous in D by the continuity of f, we have

$$|\mathbb{E}[\Phi(X^n + Y^n) - \Phi(X + f)]| = |\mathbb{E}[\Phi(X^n + Y^n) - \Phi(X^n + f)]| + o(1) \text{ as } n \to \infty.$$

However, since the Skorohod distance d on D is controlled by the uniform distance which is homogeneous $d(g+h,f+h)\leqslant \|f-g\|_{\infty,[0,T]}$ for any $h\in D$, we have

$$\begin{aligned}&|\mathbb{E}[\Phi(X^n+Y^n)-\Phi(X^n+f)]|\\&\leqslant 2\|\Phi\|_\infty\mathbb{P}[\|Y^n-f\|_{\infty,[0,T]}\geqslant\varepsilon]+|\mathbb{E}[(\Phi(X^n+Y^n)-\Phi(X^n+f))1_{d(X^n+Y^n,X^n+f)<\varepsilon}]|\\&\simeq|\mathbb{E}[(\Phi(X^n+Y^n)-\Phi(X^n+f))1_{d(X^n+Y^n,X^n+f)<\varepsilon}]|.\end{aligned}$$

We shall that $\limsup_{n\to\infty}|\mathbb{E}[\Phi(X^n+Y^n)-\Phi(X+f)]|=0$. Fix $\eta>0$. By tightness of X^n, $\mathbb{P}(X^n\notin K_\eta)<\frac{\eta}{4\|\Phi\|_\infty}$ for a compact K_η. Since Φ is continuous and $\tilde{K}_\eta:=K_\eta+f:=\{g+f: g\in K_\eta\}$ is compact, for all small enough $\varepsilon>0$ we have

$$\begin{aligned}&|\mathbb{E}[(\Phi(X^n+Y^n)-\Phi(X^n+f))1_{d(X^n+Y^n,X^n+f)<\varepsilon}]|\\&\leqslant 2\|\Phi\|_\infty|\mathbb{P}(X^n\notin K_\eta)|+|\mathbb{E}[(\Phi(X^n+Y^n)-\Phi(X^n+f))1_{d(X^n+Y^n,X^n+f)<\varepsilon,X^n+f\in\tilde{K}_\eta}]|\\&<\eta/2+\eta/2\mathbb{P}(d(X^n+Y^n,X^n+f)<\varepsilon,X^n+f\in\tilde{K}_\eta)<\eta.\end{aligned}$$

□

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