Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

An existence result for a class of nonlinear magnetorheological composites

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submitted: January 12, 2021

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No. 2804
Berlin 2021

2010 *Mathematics Subject Classification.* 35A15, 35J60, 74F10.

Key words and phrases. Magnetorheological fluids, augmented variational formulation, fixed point methods, weak solutions.

G. Nika gratefully acknowledges the funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – The Berlin Mathematics Research Center MATH+ (EXC-2046/1, project ID: 390685689) in project AA2-10.

Edited by

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Abstract

We prove existence of a weak solution for a nonlinear, multi-physics, multi-scale problem of magnetorheological suspensions introduced in Nika $\&$ Vernescu (Z. Angew. Math. Phys., 71(1):1–19, '20). The hybrid model couples the Stokes' equation with the quasi-static Maxwell's equations through the Lorentz force and the Maxwell stress tensor. The proof of existence is based on: i) the augmented variational formulation of Maxwell's equations, ii) the definition of a new function space for the magnetic induction and the proof of a Poincaré type inequality, iii) the Altman-Shinbrot fixed point theorem when the magnetic Reynold's number, $\mathrm{R_m}$, is small.

Introduction

The use of suspensions of rigid particles as *smart materials* is of great interest, as their rheological properties can be reversibly changed by the interaction with a magnetic or electric field. The ability of magnetorheological fluids [PV00], [Ver02], [Rab48] to modify their rheology from liquid to a semi-solid state under the presence of an external magnetic field in a matter of milliseconds make them desirable in many industries [LFS01, dVKHA11].

The modelling of magnetorheological and electrorheological fluids has been mostly explored from thermodynamically consistent, phenomenological point of view ([BD05, Ruz00, RR01]). While this approach is well founded, it does not allow for explicit control of the material properties. The theory of periodic homogenization, specifically designed to treat problems for multiscale heterogeneous materials, allows to derive the effective properties of the aforementioned heterogeneous materials based on the properties of the constituents at the microscale, allowing thus for the design of materials with specified properties [NC19], [ACN20].

The derivation of effective models of magnetorheological and electrorheological fluids using homogenization has been carried out in [L85́, LH88, PV00, Ver02]. The microscale problems used to derive the effective models in these works were one-way coupled systems of Stokes or Navier-Stokes equations with quasi-static Maxwell's equations. In [NV20] a fully coupled model between Stokes' equations and the quasi-static Maxwell's equations through the Lorentz force was used to derive a class of nonlinear magnetorheological composites. Numerical results, for this model, showed that particle-chain microstructures have a non-linear contribution to the magnetorheological effect. Furthermore, in [NVar] it was shown numerically that for particles of fixed volume fraction there is a decrease in the strength of the magnetorheological effect as the surface-to-volume ratio increases.

In this work we prove existence of a weak solution to the model introduced in [NV20], describing the stationary flow of rigid, magnetizable particles in a non-conducting fluid, distributed periodically with period ϵ under the influence of a magnetic field. We use an augmented variational formulation that encapsulates the fact that the magnetic induction does not possess a full weak derivative in L^2, rather, due to the material properties, the derivatives are split into a divergence part and a rotation part that respectively belong in L^2. We also introduce a new function space for the magnetic induction and prove a Poincare type inequality for this new space.

The proof of existence is based on the Altman-Shinbrot fixed point theorem [Alt57], [Shi64] and relies in the augmented variational formulation of Maxwell's equations when the magnetic Reynold's number, $\mathrm{R_m}$, is small. In more traditional fixed point arguments like Leray-Schauder that was employed by Ladyzhenskaya to show existence for the nonlinear stationary Navier-Stokes, require that the defined operator be completely continuous which is a consequence of the Sobolev embedding of H^1 into L^2 for three dimensions. However, in our case this is not possible since the magnetic induction does not possess full weak derivatives. In contrast the Altman-Shinbrot fixed point argument requires that the defined operator be continuous only in the weak topology of the underlying space.

The article is organized as follows: In Section 1 we introduce the model describing the suspension in the two component domain. Section 2 introduces the function spaces for the variational framework of the problem and certain auxiliary results regarding embeddings and Poincaré's inequality while in Section 3 we write down the augmentned variational formulation and prove its equivalence to the strong form a.e. in the domain Ω. This is done in Theorem 3.1. Moreover, we define the function space $\mathcal{W}^\epsilon$, where the magnetic induction belongs and prove that it is a Hilbert space. Furthermore, we prove a Poincaré type inequality for $\mathcal{W}^\epsilon$ in Theorem 2.1 using the global div-curl lemma ([Tar79], [Mur78]). Section 4 is dedicated to the existence proof. The proof relies on the Altman-Shinbrot fixed point theorem [Alt57], [Shi64] when $\mathrm{R_m}$ is small and the main result of this section is stated in Theorem 4.2. Finally, Section 5 is devoted to concluding remarks and comments.

Notation

Throughout the paper we will make use of the following notation:

- In addition to the standard Sobolev space $H^1(\Omega)$ we define the following spaces:

$$H^1_{\Gamma_0}(\Omega) = \left\{ w \in H^1(\Omega) \mid \left. w \right|_{\Gamma_0} = 0 \text{ on } \Gamma_0 \right\},$$

$$H(\mathrm{div};\Omega) = \left\{ \boldsymbol{w} \in L^2(\Omega) \mid \mathrm{div}\, \boldsymbol{w} \in L^2(\Omega) \right\},$$

$$H(\mathrm{curl};\Omega) = \left\{ \boldsymbol{w} \in L^2(\Omega) \mid \mathrm{curl}\, \boldsymbol{w} \in L^2(\Omega;\mathbb{R}^d) \right\},$$

  where the div and curl operators are understood in the sense of distributions and $\left. w \right|_{\Gamma_0}$ is the usual trace operator. Naturally, the above spaces are Hilbert spaces when they are equipped with their corresponding graph norms. Moreover, we will make use of fractional Sobolev spaces defined e.g. in [LM72].

- $\chi_\Omega(\boldsymbol{x})$ is the indicator function over some set Ω such that,

$$\chi_\Omega(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \boldsymbol{x} \in \Omega, \\ 0 \text{ otherwise }. \end{cases} \tag{0.1}$$

- Throughout the article we employ the Einstein summation notation of repeated indices while the expressions "mesänd "$\mathrm{mes_{d-1}}$ßtand for the Lebesgue measure and for the $d-1$ surface measure.

1 The model

Assume Ω is an open, bounded, multiply connected subset of $\mathbb{R}^d$, $d = 2$ or 3 lying in vacuum. Let $Y = [-1/2, 1/2)^d$ be the unit cube in $\mathbb{R}^d$, and $\mathbb{Z}^d$ be the set of all d–dimensional vectors with integer components. For every positive ϵ, let $N(\epsilon)$ be the set of all points $\ell \in \mathbb{Z}^d$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N(\epsilon)|$ their total number. Let Y_1 be the closure of an open, connected set with sufficiently smooth boundary S, compactly included in Y and $Y_2 := Y \backslash \overline{Y}_1$. For every $\epsilon > 0$ and $\ell \in N(\epsilon)$ we consider the set $Y^\ell_{i\,\epsilon} \subset\subset \epsilon(\ell + Y)$, where $Y^\ell_{i\,\epsilon} = \epsilon(\ell + Y_i)$ for $i = 1, 2$. The set $Y^\ell_{1\,\epsilon}$ represents one of the rigid particles suspended in the fluid, and $S^\epsilon_\ell = \epsilon(\ell + S)$ denotes its surface (see Fig. 1).

We now define the following subsets of Ω : $\Omega_{1\epsilon} = \bigcup\limits_{\ell \in N(\epsilon)} Y^\ell_{1\,\epsilon}$, $\Omega_{2\epsilon} = \Omega \backslash \overline{\Omega}_{1\epsilon}$. Here, $\Omega_{1\epsilon}$ is the domain occupied by the rigid particles and $\Omega_{2\epsilon}$ the domain occupied by the ambient surrounding fluid of viscosity $\nu \equiv$

1. We denote by $\Gamma := \partial\Omega$ the boundary of Ω. By Γ_0 we denote the exterior component of Γ and by S_ℓ^ϵ, $\ell = 1, \cdots, N(\epsilon)$ the remaining finite number of components. The vectors $\boldsymbol{n}$ and $\boldsymbol{n}^\epsilon$ indicate the unit normal on Γ_0 and the unit normal to S_ϵ^ℓ respectively with both unit normals pointing outwards. Moreover, by $[\![\cdot]\!]$ we indicate the jump discontinuity between the fluid and the rigid part.

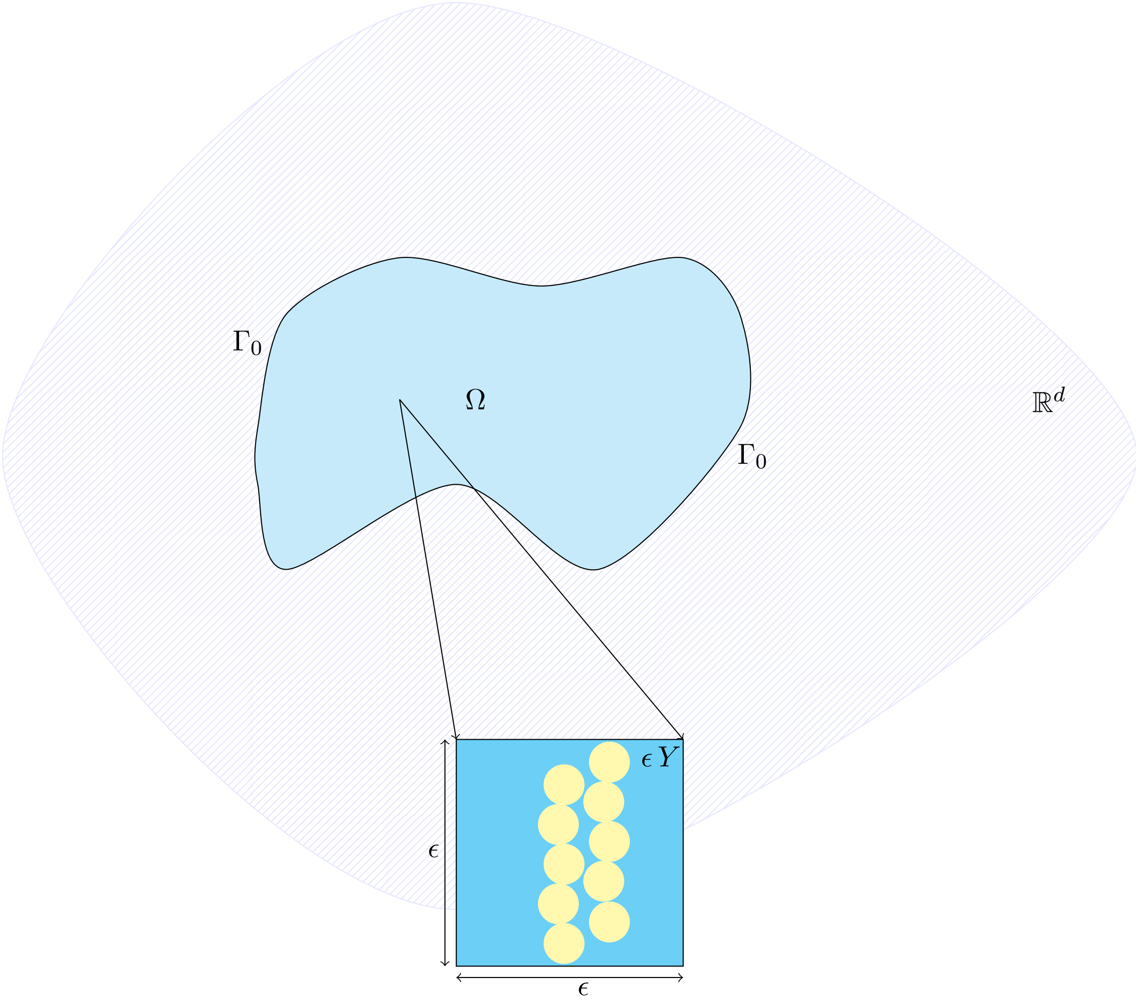

Figure 1: Schematic of the periodic suspension of rigid magnetizable particles in a non-conducting, non-magnetizable fluid. The periodic cell $\epsilon\, Y$ contains a potential geometric realization of a magnetizible, spherical, rigid particles in a chain structure.

The magnetorheological problem considered in [NV20] after non-dimensionalizing and assuming that the flow is at low Reynolds numbers was the following,

$$\begin{aligned} -\mathrm{div}\,(\sigma^\epsilon) &= \mathbf{0} && \text{in } \Omega_{2\epsilon}, \\ \sigma^\epsilon &= 2\, e(\boldsymbol{v}^\epsilon) - p^\epsilon I && \text{in } \Omega_{2\epsilon}, \\ \mathrm{div}\,(\boldsymbol{v}^\epsilon) &= 0 && \text{in } \Omega_{2\epsilon}, \\ e(\boldsymbol{v}^\epsilon) &= 0 && \text{in } \Omega_{1\epsilon}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \mathrm{div}\,(\boldsymbol{B}^\epsilon) &= 0 && \text{in } \Omega, \\ \mathrm{curl}\,(\widehat{\mu}^\epsilon \boldsymbol{B}^\epsilon) &= \mathrm{R_m}\, \boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}} && \text{in } \Omega. \end{aligned} \tag{1.2}$$

with compatibility conditions,

$$\operatorname{div}\left(\mathrm{R_m}\,\boldsymbol{v}^\epsilon \times\, \boldsymbol{B}^\epsilon\,\chi_{\Omega_{1\epsilon}}\right) = 0 \text{ in } \Omega, \quad \langle \mathrm{R_m}\,\boldsymbol{v}^\epsilon \times\, \boldsymbol{B}^\epsilon \cdot \boldsymbol{n}^\epsilon \mid 1\rangle_{H^{1/2}(S_\ell^\epsilon),H^{1/2}(S_\ell^\epsilon)} = 0, \tag{1.3}$$

and interface and exterior boundary conditions,

$$[\![\boldsymbol{v}^\epsilon]\!] = \boldsymbol{0} \text{ on } S_\ell^\epsilon, \;\; \boldsymbol{v}^\epsilon = \boldsymbol{0}, \;\; \boldsymbol{B}^\epsilon \cdot \boldsymbol{n} = \boldsymbol{c} \cdot \boldsymbol{n} \text{ on } \Gamma_0. \tag{1.4}$$

When the MR fluid is submitted to a magnetic field, the rigid particles are subjected to a force that makes them behave like a dipole aligned in the direction of the magnetic field. This force can be written in the form,

$$\boldsymbol{F}^\epsilon := -\frac{1}{2}\,|\boldsymbol{H}^\epsilon|^2\,\nabla\,\mu^\epsilon,$$

where $|\cdot|$ represents the standard Euclidean norm. The force can be written in terms of the Maxwell stress,

$$\tau_{ij}^\epsilon = \widehat{\mu}^\epsilon\, B_i^\epsilon\, B_j^\epsilon - \frac{1}{2}\,\widehat{\mu}^\epsilon\, B_k^\epsilon\, B_k^\epsilon\,\delta_{ij}, \tag{1.5}$$

as $\boldsymbol{F}^\epsilon = \operatorname{div}(\tau^\epsilon) - \boldsymbol{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon\,\boldsymbol{B}^\epsilon)$. Since the magnetic permeability is considered constant in each phase, it follows that the force is zero in each phase. Therefore, we deduce that

$$\operatorname{div}(\tau^\epsilon) = \begin{cases} 0 & \text{if } \boldsymbol{x} \in \Omega_{2\epsilon}, \\ \boldsymbol{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon\,\boldsymbol{B}^\epsilon) & \text{if } \boldsymbol{x} \in \Omega_{1\epsilon}. \end{cases} \tag{1.6}$$

Lastly, we remark that unlike the viscous stress σ^ϵ, the Maxwell stress is present in the entire domain Ω. Hence, we can write the balance of forces and torques in each particle as,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \boldsymbol{n}^\epsilon\, ds + \alpha \int_{S_\ell^\epsilon} [\![\tau^\epsilon \boldsymbol{n}^\epsilon]\!]\; ds - \alpha \int_{T_\ell^\epsilon} \boldsymbol{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon\,\boldsymbol{B}^\epsilon)\; d\boldsymbol{x}, \\ 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \boldsymbol{n}^\epsilon \times (\boldsymbol{x} - \boldsymbol{x}_c^\ell)\, ds + \alpha \int_{S_\ell^\epsilon} [\![\tau^\epsilon \boldsymbol{n}^\epsilon]\!] \times (\boldsymbol{x} - \boldsymbol{x}_c^\ell)\, ds \\ &\quad - \alpha \int_{T_\ell^\epsilon} (\boldsymbol{B}^\epsilon \times \operatorname{curl}(\widehat{\mu}^\epsilon\,\boldsymbol{B}^\epsilon)) \times (\boldsymbol{x} - \boldsymbol{x}_c^\ell)\, d\boldsymbol{x}. \end{aligned} \tag{1.7}$$

Here $\boldsymbol{v}^\epsilon$ represents the fluid velocity field, p^ϵ the pressure, $e(\boldsymbol{v}^\epsilon)$ the strain rate, $\boldsymbol{n}^\epsilon$ the unit normal to S_ℓ^ϵ, $\boldsymbol{n}$ is the unit normal to Γ_0, $\boldsymbol{B}^\epsilon$ is the magnetic induction and it is related to the magnetic field $\boldsymbol{H}^\epsilon$ by $\boldsymbol{B}^\epsilon = \mu^\epsilon \boldsymbol{H}^\epsilon$, where $0 < \mu^\epsilon$ is the magnetic permeability of the material and $\widehat{\mu}^\epsilon = (\mu^\epsilon)^{-1}$, $\boldsymbol{x}_c^\ell$ is the center of mass of the rigid particle T_ℓ^ϵ, α is the Alfven number, and $\mathrm{R_m}$ is the magnetic Reynolds number. Moreover, $\boldsymbol{c}\cdot\boldsymbol{n}$ is a transmission condition on the outer boundary indicating that a magnetic field $\boldsymbol{c}$ exterior to the domain Ω is present. Finally, we remark that condition $e(\boldsymbol{v}^\epsilon) = 0$ in $\Omega_{1\epsilon}$ means that $\boldsymbol{v}^\epsilon = \boldsymbol{V}^{\ell,\epsilon} + \boldsymbol{\omega}^{\ell,\epsilon} \times (\boldsymbol{x} - \boldsymbol{x}_c^\ell)$ in $\Omega_{1\epsilon}$ where $\boldsymbol{V}^{\ell,\epsilon}$ is a constant translational velocity and $\boldsymbol{\omega}^{\epsilon,\ell}$ is a constant rotational velocity for each particle.

2 Function spaces and auxiliary results

We begin with a collection of results proved in [GLN] regarding a non-homogeneous domain containing subdomains of i.e. different piece-wise constant magnetic permeability, say, μ_i, $i = 1, \ldots, \kappa$, where κ is the number of subdomains. These domains occur naturally in problems of electromagnetism (see Fig. 2).

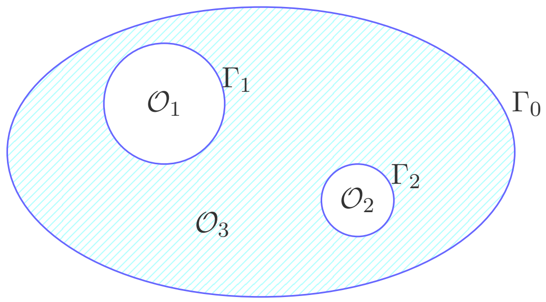

Figure 2: A schematic of a non-homogeneous domain. Namely, a finite multiply connect region $\mathcal{O}$ containing two sub-regions. The open set $\mathcal{O}$ is defined as $\mathcal{O} := \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \Gamma_1 \cup \Gamma_2$

.

Proposition 2.1. *Let $\mathcal{O} \subset \mathbb{R}^d$ be any open, bounded, multiply connected set with boundary $\Gamma := \partial\mathcal{O}$ of class C^2. The exterior boundary will be denoted by Γ_0 and by $\Gamma_j,\ j = 1, \dots, \kappa - 1$, the other components of Γ.*

Define $\mathcal{Y}$ to be the Hilbert space of vector fields,

$$\mathcal{Y} := \Big\{ \boldsymbol{v} \in L^2(\mathcal{O}; \mathbb{R}^d) \mid \operatorname{div} \boldsymbol{v} \in L^2(\mathcal{O}), \operatorname{curl}(\widehat{\mu}\boldsymbol{v}) \in L^2(\mathcal{O}; \mathbb{R}^d), \boldsymbol{v} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma_0) \Big\}, \tag{2.1}$$

for the norm,

$$\|\boldsymbol{v}\|_{\mathcal{Y}} := \|\boldsymbol{v}\|_{L^2(\mathcal{O};\mathbb{R}^d)} + \|\operatorname{div} \boldsymbol{v}\|_{L^2(\mathcal{O})} + \|\operatorname{curl}(\widehat{\mu}\boldsymbol{v})\|_{L^2(\mathcal{O};\mathbb{R}^d)} + \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{H^{1/2}(\Gamma_0)}, \tag{2.2}$$

then for all $\boldsymbol{v} \in \mathcal{Y}$ we have, $\boldsymbol{v}_{|\mathcal{O}_i} \in H^1(\mathcal{O}_i; \mathbb{R}^d)$ for $i = 1, \dots, \kappa$ where $\boldsymbol{v}_{|\mathcal{O}_i}$ is the restriction of $\boldsymbol{v}$ to $\mathcal{O}_i$, $0 < \mu = \widehat{\mu}^{-1}$ is constant in $\mathcal{O}_i$ and

$$\left\| \boldsymbol{v}_{|\mathcal{O}_i} \right\|_{H^1(\mathcal{O}_i;\mathbb{R}^d)} \le C_{\mathcal{O}_i} \|\boldsymbol{v}\|_{\mathcal{Y}}. \tag{2.3}$$

Proof. Let $\boldsymbol{v} \in \mathcal{Y}$ and define $\pi \in H^1(\mathcal{O})$ as the solution to the following Neumann problem,

$$\begin{aligned} \operatorname{div}(\mu \nabla \pi) &= \operatorname{div} \boldsymbol{v} \text{ in } \mathcal{O}, \\ [\![\mu\, \partial_n \pi]\!] &= 0 \text{ on } \Gamma_j,\ j = 1, \dots, \kappa - 1, \\ \mu\, \partial_n \pi &= \boldsymbol{v} \cdot \boldsymbol{n} \text{ on } \Gamma_0. \end{aligned} \tag{2.4}$$

Take $\boldsymbol{u} = \boldsymbol{v} - \mu \nabla \pi$ and note that $\operatorname{div} \boldsymbol{u} = 0$ in $\mathcal{O}$, $\operatorname{curl}(\widehat{\mu}\boldsymbol{u}) = \operatorname{curl}(\widehat{\mu}\boldsymbol{v}) \in L^2(\mathcal{O}; \mathbb{R}^d)$, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_0. Then $\boldsymbol{u}_{|\mathcal{O}_i} \in H^1(\mathcal{O}_i; \mathbb{R}^d)$ by [DL72, Theorem 6.2, page 355].

It remains to prove that $\pi_{|\mathcal{O}_i} \in H^2(\mathcal{O}_i)$ and inequality (2.3). This is the result of [LU68, Chap. 3, Sec. 16, Eq. 16.12, pg. 212]. For a sketch of the proof in this particular case one can also consult [GLN]. □

Lemma 2.1. *Let $\boldsymbol{v} \in \{\boldsymbol{w} \in H(\operatorname{div}, \mathcal{O}) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \mathcal{O}\}$ and define $\mathcal{Y}_{\Gamma_0} = \Big\{ \boldsymbol{w} \in \mathcal{Y} \mid \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_0 \Big\}$, $\mathcal{Y}^0_{\Gamma_0} = \Big\{ \boldsymbol{w} \in \mathcal{Y}_{\Gamma_0} \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \mathcal{O} \Big\}$. There exists a vector potential $\boldsymbol{w} \in \mathcal{Y}^0_{\Gamma_0}$ such that*

$$\operatorname{curl}(\widehat{\mu}\boldsymbol{w}) = \boldsymbol{v}, \quad \|\widehat{\mu}\boldsymbol{w}\|_{\mathcal{Y}^0_{\Gamma_0}} \le c\, \|\boldsymbol{v}\|_{L^2(\mathcal{O};\mathbb{R}^d)}, \tag{2.5}$$

where $c := c(\mathcal{O})$. Moreover, there exists $\xi \in (3, 6]$ such that

$$\widehat{\mu}\,\boldsymbol{w} \in L^{\xi}(\mathcal{O}; \mathbb{R}^d) \text{ and } \|\widehat{\mu}\,\boldsymbol{w}\|_{L^{\xi}(\mathcal{O};\mathbb{R}^d)} \leq c\, \|\boldsymbol{v}\|_{L^2(\mathcal{O};\mathbb{R}^d)}\,. \tag{2.6}$$

Proof. This is [Dru, Prop. 2.2]. □

Proposition 2.2. *The space $\mathcal{Y}$ is embedded into $L^q(\mathcal{O}, \mathbb{R}^d)$ for $q \in [1, 2d/(d-2)]$ with the embedding being continuous.*

Proof. This is [GLN, Prop. 3.4] which is an extension result of [Dru, Prop. 2.6 (2)] when the normal trace on Γ_0 belongs in $H^{1/2}(\Gamma_0)$. □

Proposition 2.3. *Define a new norm on $\mathcal{Y}$ by*

$$[\boldsymbol{v}]_{\mathcal{Y}} := \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\mathcal{O})} + \|\mathrm{curl}\,(\widehat{\mu}\,\boldsymbol{v})\|_{L^2(\mathcal{O};\mathbb{R}^d)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{1/2}(\Gamma_0)}\,, \tag{2.7}$$

then $\mathcal{Y}$ is also a Hilbert space with norm $[\cdot]_{\mathcal{Y}}$.

Proof. It is evident that if $\boldsymbol{v} = \boldsymbol{0}$ then $[\boldsymbol{v}]_{\mathcal{Y}} = 0$. For the other direction we have, $[\boldsymbol{v}]_{\mathcal{Y}} = 0$ implies that $\mathrm{curl}\,(\widehat{\mu}\,\boldsymbol{v}) = \boldsymbol{0}$ in Ω. Hence, $\boldsymbol{v}$ can be written as, $\boldsymbol{v} = -\mu\,\nabla\theta$. Thus, we get that θ satisfies the following elliptic problem,

$$\begin{aligned} -\mathrm{div}\,(\mu\,\nabla\theta) &= 0 \text{ in } \mathcal{O}, \\ \mu\,\nabla\theta\cdot\boldsymbol{n} &= 0 \text{ on } \Gamma_0. \end{aligned} \tag{2.8}$$

The above problem has a unique solution, $\theta = 0$ (if we fix constants) and the result follows. We remark that $\mathcal{Y}$ is complete which follows by similar arguments used to show the completeness of the classical $H(\mathrm{div}; \mathcal{O})$ or $H(\mathrm{curl}; \mathcal{O})$ spaces (see [Tem84]). □

Theorem 2.1 (Poincaré type inequality for $(\mathcal{Y}, [\cdot]_{\mathcal{Y}})$)**.** *There exists a constant, $c := c(\mathcal{O})$, such that*

$$\|\boldsymbol{w}\|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq c\,[\boldsymbol{w}]_{\mathcal{Y}}, \tag{2.9}$$

for all $\boldsymbol{w} \in \mathcal{Y}$.

Proof. We proceed by contradiction. If (2.9) is false then there exists a sequence $\boldsymbol{w}_n \in \mathcal{Y}$, such that

$$\|\boldsymbol{w}_n\|_{L^2(\mathcal{O};\mathbb{R}^d)} > n\,[\boldsymbol{w}_n]_{\mathcal{Y}} \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

We can suppose that $\|\boldsymbol{w}_n\|_{L^2(\mathcal{O};\mathbb{R}^d)} = 1$. Then as $n \to \infty$ and up to a, non-relabeled, subsequence we have:

$$\boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ in } H(\mathrm{div}; \mathcal{O}), \quad \widehat{\mu}\,\boldsymbol{w}_n \rightharpoonup \widehat{\mu}\,\boldsymbol{w} \text{ in } H(\mathrm{curl}; \mathcal{O}), \tag{2.11}$$

$$\mathrm{div}\,\boldsymbol{w}_n \to 0 \text{ in } L^2(\mathcal{O}), \quad \mathrm{curl}\,(\widehat{\mu}\,\boldsymbol{w}_n) \to 0 \text{ in } L^2(\mathcal{O}; \mathbb{R}^d), \quad \boldsymbol{w}_n\cdot\boldsymbol{n} \to 0 \text{ in } H^{1/2}(\Gamma_0). \tag{2.12}$$

Decompose $\widehat{\mu}\,\boldsymbol{w}_n$ using Helmholtz decomposition as $\widehat{\mu}\,\boldsymbol{w}_n = \nabla\,p_n + \mathrm{curl}\,\boldsymbol{f}_n$. Denote by $\boldsymbol{g}_n := \mathrm{curl}\,\boldsymbol{f}_n$, then $\boldsymbol{g}_n \in L^2(\mathcal{O}; \mathbb{R}^d)$, $\mathrm{div}\,\boldsymbol{g}_n = 0$, $\mathrm{curl}\,\boldsymbol{g}_n \in L^2(\mathcal{O}; \mathbb{R}^d)$, and $\boldsymbol{g}_n\cdot\boldsymbol{n} = 0$ on Γ_0 in the sense of distributions. By theorem, [FT78, Prop. 1.4, pg. 41] or [DL72, Thm 6.1, pg. 354] $\boldsymbol{g}_n \in H^1(\mathcal{O}; \mathbb{R}^d)$ and is bounded uniformly. By the compact embedding of H^1 into L^2, $\boldsymbol{g}_n \to \boldsymbol{g}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$ or $\mathrm{curl}\,\boldsymbol{f}_n \to \mathrm{curl}\,\boldsymbol{f}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$. Hence, if $\widehat{\mu}_0 := \min_i \widehat{\mu}_i$ denotes the the smallest of the $\widehat{\mu}_i$, we have

$$
\begin{aligned}
0 < \widehat{\mu}_0 &\le \int_{\mathcal{O}} \widehat{\mu}\, \boldsymbol{w}_n \cdot \boldsymbol{w}_n \, d\boldsymbol{x} \\
&= \int_{\mathcal{O}} \nabla p_n \cdot \boldsymbol{w}_n \, d\boldsymbol{x} + \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{f}_n \cdot \boldsymbol{w}_n \, d\boldsymbol{x} \\
&= \int_{\Gamma_0} p_n \boldsymbol{w}_n \cdot \boldsymbol{n} \, ds - \int_{\mathcal{O}} q_n \cdot \operatorname{div} \boldsymbol{w}_n \, d\boldsymbol{x} + \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{f}_n \cdot \boldsymbol{w}_n \, d\boldsymbol{x}.
\end{aligned} \tag{2.13}
$$

Since p_n remains bounded in $H^1(\mathcal{O})$, $\operatorname{curl} \boldsymbol{f}_n \to \operatorname{curl} \boldsymbol{f}$ in $L^2(\mathcal{O}; \mathbb{R}^d)$, and using (2.11) and (2.12) we can pass to the limit as $n \to \infty$. Noting further from (2.12) that the $\operatorname{curl} \boldsymbol{f} = \boldsymbol{0}$ in $\mathcal{O}$ we obtain that $0 < \widehat{\mu}_0 = 0$, which is a contradiction.

Sometimes this is referred to as the global div-curl lemma (see [Sch18]). □

Corollary 2.1. *The norms $\|\cdot\|_{\mathcal{Y}}$ and $[\cdot]_{\mathcal{Y}}$ are equivalent norms on $\mathcal{Y}$*

Proof. It is a consequence of Theorem 2.1. □

3 Augmented variational formulation

3.1 Assumptions

We frame the magnetorheological model (1.1)–(1.2) under the following general **Assumptions (A)**.

- We assume Ω is a bounded, multiply connected domain such that $\operatorname{mes}_{d-1}(\Gamma) > 0$ and $\operatorname{mes}_{d-1}(S_\ell^\epsilon) > 0$ for $\ell = 1, \dots, N(\epsilon)$.
- Γ_0 and S_ℓ^ϵ are surfaces of class C^2, $S_p^\epsilon \cap S_q^\epsilon = \emptyset$ for $p, q \in N_\epsilon$ with $p \neq q$, and $\Gamma_0 \cap S_\ell^\epsilon = \emptyset$ for every $\ell \in N_\epsilon$.
- The magnetic permeability of the magnetorheological fluid, μ^ϵ, is assumed be a piece-wise constant function with values $\mu^\epsilon(\boldsymbol{x}) = \mu_1$ if $\boldsymbol{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\boldsymbol{x}) = \mu_2$ if $\boldsymbol{x} \in \Omega_{2\epsilon}$ with $0 < \mu_2 < \mu_1 < +\infty$.

3.2 Variational formulation

To properly establish a weak solution to the system of equations (1.1), (1.2), (1.3), (1.4), and (1.7) we need appropriate variational formulations and function spaces. We begin by defining the following function spaces,

$$
\mathcal{V}^\epsilon = \left\{ \boldsymbol{v} \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^d) \mid \operatorname{div}(\boldsymbol{v}) = 0 \text{ in } \Omega, e(\boldsymbol{v}) = 0 \in \Omega_{1\epsilon} \right\}. \tag{3.1}
$$

It is clear that $\mathcal{V}^\epsilon$ is a closed subspace of $H^1_{\Gamma_0}(\Omega; \mathbb{R}^d)$ and thus a Hilbert space with the induced $H^1_{\Gamma_0}(\Omega; \mathbb{R}^d)$ inner product which by Korn's inequality is equivalent to

$$
(\boldsymbol{v} \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} = \int_{\Omega_{2\epsilon}} 2\, e(\boldsymbol{v}) : e(\boldsymbol{\phi}) \, d\boldsymbol{x}. \tag{3.2}
$$

The corresponding norm will be denoted by $\|\cdot\|_{\mathcal{V}^\epsilon}$. Furthermore, we define the function space,

$$
\mathcal{W}^\epsilon = \left\{ \boldsymbol{w} \in L^2(\Omega; \mathbb{R}^d) \mid \operatorname{div} \boldsymbol{w} \in L^2(\Omega),\ \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{w}) \in L^2(\Omega; \mathbb{R}^d), \boldsymbol{w} \cdot \boldsymbol{n} \in H^{1/2}(\Gamma_0) \right\}, \tag{3.3}
$$

equipped with the inner product,

$$(\boldsymbol{h} \mid \boldsymbol{\psi})_{\mathcal{W}^\epsilon} = \int_\Omega \operatorname{div}(\boldsymbol{h}) \operatorname{div}(\boldsymbol{\psi})\, d\boldsymbol{x} + \int_\Omega \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{h}) \cdot \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{\psi})\, d\boldsymbol{x} + \int_{\Gamma_0} (\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{\psi}\cdot\boldsymbol{n}) ds, \tag{3.4}$$

while the corresponding norm will be denoted by $[\cdot]_{\mathcal{W}_\epsilon}$. It is evident that $(\mathcal{W}^\epsilon, [\cdot]_{\mathcal{W}^\epsilon})$ is a Hilbert space from Proposition 2.3, since $\mathcal{W}^\epsilon$ is the Hilbert space $\mathcal{Y}$ with $\widehat{\mu} := \widehat{\mu}^\epsilon$ and $\mathcal{O}$ is now the domain Ω.

The variational formulation of (1.1), (1.2), (1.3), (1.4) and (1.7) is written in two steps. First, we write down the variational formulation of the Stokes' equations and the Maxwell equations separately and then add the resulting variational problems. The variational formulation of the Stokes' equation reads: Find $\boldsymbol{u}^\epsilon \in \mathcal{V}^\epsilon$ such that,

$$(\boldsymbol{u}^\epsilon \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} + \alpha \int_{\Omega_{2\epsilon}} \tau^\epsilon : e(\phi)\, d\boldsymbol{x} = \boldsymbol{0} \ \text{ for all } \boldsymbol{\phi} \in \mathcal{V}^\epsilon. \tag{3.5}$$

For the quasi-static Maxwell's equations, we consider an augmented variational formulation in $\mathcal{W}^\epsilon$ [Jr05]. Find $\boldsymbol{B}^\epsilon \in \mathcal{W}^\epsilon$ such that

$$\frac{\alpha}{\mathrm{R_m}} (\boldsymbol{B}^\epsilon \mid \boldsymbol{\psi})_{\mathcal{W}^\epsilon} = \alpha \int_{\Omega_{1\epsilon}} \boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon \cdot \operatorname{curl}(\boldsymbol{\psi})\, d\boldsymbol{x} + \frac{\alpha}{\mathrm{R_m}} \int_{\Gamma_0} (\boldsymbol{c}\cdot\boldsymbol{n})(\boldsymbol{\psi}\cdot\boldsymbol{n})\, ds, \tag{3.6}$$

for all $\boldsymbol{\psi} \in \mathcal{W}^\epsilon$.

Hence, the variational formulation of (1.1), (1.2), (1.3), (1.4) and (1.7) reads: Find $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ such that

$$\begin{aligned}(\boldsymbol{u}^\epsilon \mid \boldsymbol{\phi})_{\mathcal{V}^\epsilon} + \frac{\alpha}{\mathrm{R_m}} (\boldsymbol{B}^\epsilon \mid \boldsymbol{\psi})_{\mathcal{W}^\epsilon} = &- \alpha \int_{\Omega_{2\epsilon}} \tau^\epsilon : e(\boldsymbol{\phi})\, d\boldsymbol{x} + \alpha \int_{\Omega_{1\epsilon}} \boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon \cdot \operatorname{curl}(\boldsymbol{\psi})\, d\boldsymbol{x} \\ &+ \frac{\alpha}{\mathrm{R_m}} \int_{\Gamma_0} (\boldsymbol{c}\cdot\boldsymbol{n})(\boldsymbol{\psi}\cdot\boldsymbol{n})\, ds \ \text{ for all } (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon.\end{aligned} \tag{3.7}$$

Theorem 3.1. *The pair* $(\boldsymbol{v}^\epsilon, \boldsymbol{B}^\epsilon)$ *satisfies* (1.1), (1.2), (1.3), (1.4) *and* (1.7) *if and only if it is a weak solution to* (3.7).

Proof. It is clear that if $(\boldsymbol{v}^\epsilon, \boldsymbol{B}^\epsilon)$ satisfies (1.1), (1.2), (1.3), (1.4) and (1.7) then it is a solution to (3.7). To see this, multiply the Stokes' equations by a test function in $\boldsymbol{\phi} \in \mathcal{V}^\epsilon$ and carry out the varational formulation as in [Tem84], [GMV14], [NV16, Appendix]. For Maxwell's equations multiply the divergence part by $\frac{\alpha}{\mathrm{R_m}} \operatorname{div} \boldsymbol{\psi}$, the rotational part by $\frac{\alpha}{\mathrm{R_m}} \operatorname{curl} \boldsymbol{\psi}$, and the exterior boundary condition by $\frac{\alpha}{\mathrm{R_m}}\, \boldsymbol{\psi}\cdot\boldsymbol{n}$, respectively.

For the other direction we have: Take $(\boldsymbol{\phi}, \boldsymbol{0})$ as a test function in (3.7) and obtain the variational formulation of Stokes' equation (3.5) from which we can recover Stokes' equation, boundary conditions, and balance of forces and torques in the distributional sense as usual (see [Tem84], [GMV14]). On the other hand if we take $(\boldsymbol{0}, \boldsymbol{\psi})$ as a test function in (3.7) we obtain (3.6). In order to recover Maxwell's equations we need to introduce appropriate test functions on each domain $\Omega_{1\epsilon}$ and $\Omega_{2\epsilon}$. To this end if we let $\zeta^\delta : \mathbb{R}^d \to [0,1]$ be a smooth cut-off function defined by

$$\zeta^\delta(\boldsymbol{x}) = \begin{cases} 1 & \text{if } d(\boldsymbol{x}, \Gamma_0) < \delta, \\ 0 & \text{if } d(\boldsymbol{x}, \Gamma_0) > 2\,\delta, \end{cases} \tag{3.8}$$

where δ is chosen in a way that the inner most neighbourhood does not intersect the rigid particles. Following [Lad14], define $\boldsymbol{d}(\boldsymbol{x}) := (c_2 x_3, c_3 x_1, c_1 x_2)$ where the vector field $\boldsymbol{c} = (c_1, c_2, c_3)$ is the constant vector field from the outer transmission condition on the boundary Γ_0. Set $\boldsymbol{a}^\delta(\boldsymbol{x}) := \operatorname{curl}\left(\zeta^\delta(\boldsymbol{x})\,\boldsymbol{d}(\boldsymbol{x})\right)$ then $\boldsymbol{a}^\delta(\boldsymbol{x})$ is a divergence free vector field that is zero in the domain $\Omega^\delta := \{\boldsymbol{x} \in \Omega \mid d(\boldsymbol{x}, \Gamma_0) < 2\delta\}$ and equals $\boldsymbol{c}$ in the δ neighbourhood of Γ_0.

Moreover, by Proposition 2.1 we have that $\boldsymbol{B}^\epsilon \in H^1(\Omega_{i\epsilon}),\ i = 1, 2$ and by the classical Sobolev embedding of H^1 into L^q for $1 \le q < 2d/(d-2)$ we have that $\boldsymbol{B}^\epsilon \in L^4(\Omega; \mathbb{R}^d)$. Likewise, for $\boldsymbol{v}^\epsilon$, namely, $\boldsymbol{v}^\epsilon \in L^4(\Omega; \mathbb{R}^d)$. Thus, $\mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}} \in L^2(\Omega; \mathbb{R}^d)$ by the Cauchy–Schwartz inequality. Using (1.3) we also have that $\operatorname{div}(\mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}}) = 0$ in Ω. Therefore, $\mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}} \in \{\boldsymbol{v} \in H(\operatorname{div}, \Omega) \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$. By Lemma 2.1 there exists a $\boldsymbol{w} \in \mathcal{Y}^0_{\Gamma_0}$ such that,

$$\operatorname{curl}(\widehat{\mu}_\epsilon \boldsymbol{w}) = \mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}}. \tag{3.9}$$

Setting $\boldsymbol{\psi} := \boldsymbol{B}^\epsilon - \boldsymbol{w} - \boldsymbol{a}^\delta \in \mathcal{W}^\epsilon$ we reduce (3.6) to the following,

$$\begin{aligned} &\int_\Omega \operatorname{div} \boldsymbol{B}^\epsilon \operatorname{div}(\boldsymbol{B}^\epsilon - \boldsymbol{w} - \boldsymbol{a}^\delta)\, d\boldsymbol{x} \\ &+ \int_\Omega (\operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{B}^\epsilon) - \mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}}) \cdot \operatorname{curl}(\widehat{\mu}^\epsilon (\boldsymbol{B}^\epsilon - \boldsymbol{w} - \boldsymbol{a}^\delta))\, d\boldsymbol{x} \\ &+ \int_{\Gamma_0} ((\boldsymbol{B}^\epsilon - \boldsymbol{c}) \cdot \boldsymbol{n})((\boldsymbol{B}^\epsilon - \boldsymbol{w} - \boldsymbol{a}^\delta) \cdot \boldsymbol{n}) ds = 0 \end{aligned} \tag{3.10}$$

Using the properties of the vector fields $\boldsymbol{w}$ and $\boldsymbol{a}^\delta$ we obtain:

$$\int_\Omega |\operatorname{div}\boldsymbol{B}^\epsilon|^2\, d\boldsymbol{x} + \int_\Omega |\operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{B}^\epsilon) - \mathrm{R_m}\boldsymbol{v}^\epsilon \times \boldsymbol{B}^\epsilon \chi_{\Omega_{1\epsilon}}|^2\, d\boldsymbol{x} + \int_{\Gamma_0} |(\boldsymbol{B}^\epsilon - \boldsymbol{c}) \cdot \boldsymbol{n}|^2\, ds = 0 \tag{3.11}$$

Since the expression above is a sum of squares that is equal to zero, each integral must be equal to zero and the claim follows. □

4 Existence of a weak solution via the Altman-Shinbrot fixed point theorem

4.1 M. Shinbrot's fixed point argument

To prove existence we employ the fixed point argument of Altman-Shinbrot [Shi64], [Alt57]. For the readers convenience, we recall the main theorem and corollaries of M. Shinbrot's fixed point argument whose proofs can be found in [Shi64].

In what follows, $\mathcal{H}$ denotes a real or complex Hilbert space, and $\mathcal{S}_r$ and $\mathcal{B}_r$ will denote the sphere and the closed unit ball of radius r centered at zero:

$$\mathcal{S}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} = r\}, \quad \mathcal{B}_r = \{x \in \mathcal{H} \mid \|x\|_{\mathcal{H}} \le r\}.$$

Theorem 4.1. *Let H be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. If there is a positive constant r such that*

$$\Re(Hx, x) \le \|x\|^2_{\mathcal{H}} \text{ for all } x \in \mathcal{B}_r, \tag{4.1}$$

then H has a fixed point in $\mathcal{B}_r$.

Corollary 4.1. *Let G be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. Let y be an element of $\mathcal{H}$. Let y be an element of $\mathcal{H}$. If there exists a positive r such that either,*

$$\Re(Gx - y, x) \geq 0 \text{ for all } x \in \mathcal{S}_r,$$

or

$$\Re(Gx - y, x) \leq 0 \text{ for all } x \in \mathcal{S}_r,$$

then y is in the range of G.

Corollary 4.2. *Let G be an operator on the separable Hilbert space $\mathcal{H}$, continuous in the weak topology on $\mathcal{H}$. Let y be an element of $\mathcal{H}$. Then, zero is in the range of G if (Gx, x) is of one sign on some sphere $\mathcal{S}_r$.*

4.2 Existence

For all $\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon, \boldsymbol{\phi}, \boldsymbol{\psi}$ we define the following expression $\mathcal{Q}$ by,

$$\mathcal{Q}[(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})] := -\alpha \int_{\Omega_{2\epsilon}} \widehat{\mu}_2\, \boldsymbol{B}^\epsilon \otimes \boldsymbol{B}^\epsilon : e(\boldsymbol{\phi})\, d\boldsymbol{x} + \alpha \int_{\Omega_{1\epsilon}} \boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon \cdot \operatorname{curl}\left(\widehat{\mu}_1\, \boldsymbol{\psi}\right)\, d\boldsymbol{x}. \tag{4.2}$$

We can immediately see that by combining the results of Proposition 2.1 and Theorem 2.1 with classical Sobolev embedding theorems of $L^q,\ q \in [1, 2d/(d-2))$ into H^1 we obtain,

$$|\mathcal{Q}[(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})]| \leq c\, |||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)|||^2\ |||(\boldsymbol{\phi}, \boldsymbol{\psi})|||\,, \tag{4.3}$$

where $|||(-,\cdot)||| := \|-\|_{\mathcal{V}^\epsilon} + \frac{\alpha}{\mathrm{R_m}}[\cdot]_{\mathcal{W}^\epsilon}$ and c is a generic constant depending on $\Omega_{i\epsilon}, \alpha, \widehat{\mu}_i$ for $i = 1, 2$.

Thus, we can write (3.7) as: Find $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ such that,

$$(\boldsymbol{u}^\epsilon \mid \boldsymbol{\phi}) + \frac{\alpha}{\mathrm{R_m}}\,(\boldsymbol{B}^\epsilon \mid \boldsymbol{\psi}) - \mathcal{Q}[(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})] = \int_{\Omega} (\boldsymbol{c} \cdot \boldsymbol{n})(\boldsymbol{\phi} \cdot \boldsymbol{n})\, ds, \tag{4.4}$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$.

The Cauchy-Schwartz inequality and the definition of the norm $\frac{\alpha}{\mathrm{R_m}}[\cdot]_{\mathcal{W}^\epsilon}$ make the right hand side of equation (4.4) a bounded linear functional of $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$. Using Riesz's theorem, we can express the right hand side of (4.4) as the scalar product of a well determined element $(\boldsymbol{f}, \boldsymbol{g}) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ by $(\boldsymbol{\phi}, \boldsymbol{\psi})$.

Likewise, if we fix $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ and take into account the estimate (4.3), we can write the left hand side as a product of an element in $\mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$, denoted by $\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)$ that depends nonlinearly on $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)$, by $(\boldsymbol{\phi}, \boldsymbol{\psi})$.

Therefore, we can re-write (4.4) using the operator $\mathcal{F}$ as,

$$(\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})) = ((\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{\phi}, \boldsymbol{\psi})), \tag{4.5}$$

for all $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ where,

$$(\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})) := (\boldsymbol{u}^\epsilon \mid \boldsymbol{\phi}) + \frac{\alpha}{\mathrm{R_m}}\,(\boldsymbol{B}^\epsilon \mid \boldsymbol{\psi}) - \mathcal{Q}[(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{\phi}, \boldsymbol{\psi})], \tag{4.6}$$

and

$$((\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{\phi}, \boldsymbol{\psi})) := \int_{\Omega} (\boldsymbol{c} \cdot \boldsymbol{n})(\boldsymbol{\phi} \cdot \boldsymbol{n})\, ds. \tag{4.7}$$

Hence, searching for a solution to (3.7) reduces to showing that at least one solution exists to the above nonlinear operator equation.

Lemma 4.1. *The nonlinear operator $\mathcal{F} : (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \mapsto \mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)$ is continuous in the weak topology of the product space $\mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$.*

Proof. Assume that $(\boldsymbol{u}^\epsilon_\kappa, \boldsymbol{B}^\epsilon_\kappa)$ is a weakly convergent sequence in $\mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$ to $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)$ as $\kappa \to +\infty$ then,

$$
\begin{aligned}
&|(\mathcal{F}(\boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}^\epsilon); (\boldsymbol{\psi}, \boldsymbol{\psi}))| \\
&= \left| (\boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}^\epsilon \mid \boldsymbol{\phi}) + \frac{\alpha}{R_m} (\boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}^\epsilon \mid \boldsymbol{\psi}) - \mathcal{Q}[(\boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}, \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}); (\boldsymbol{\phi}, \boldsymbol{\psi})] \right| .
\end{aligned}
\tag{4.8}
$$

By Hölder's inequality and the embedding of $H^1(\Omega_{i\epsilon}; \mathbb{R}^d)$ into $L^q(\Omega_{i\epsilon}; \mathbb{R}^d)$, $i = 1, 2$ with $1 \leq q < 2d/(d-2)$ we have,

$$
\begin{aligned}
&|\mathcal{Q}[(\boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}, \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}); (\boldsymbol{\phi}, \boldsymbol{\psi})]| \\
&\leq c \left\| \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}^\epsilon \right\|^2_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)} \left\| e(\boldsymbol{\phi}) \right\|_{L^2(\Omega_{2\epsilon}; \mathbb{R}^{d\times d})} \\
&\leq c \left\| \boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}^\epsilon \right\|_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)} \left\| \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}^\epsilon \right\|_{L^4(\Omega_{1\epsilon}; \mathbb{R}^d)} \left\| \operatorname{curl}(\widehat{\mu}_1 \boldsymbol{\psi}) \right\|_{L^2(\Omega_{2\epsilon}; \mathbb{R}^d)} ,
\end{aligned}
\tag{4.9}
$$

for generic constant $c := c(\Omega_{i\epsilon}, a, \mathrm{R_m}, \widehat{\mu}_i)$, $i = 1, 2$. Moreover, since the above embedding of $H^1(\Omega_{i\epsilon}; \mathbb{R}^d)$ into $L^q(\Omega_{i\epsilon}; \mathbb{R}^d)$ is compact we can extract strongly κ convergent subsequences (not relabelled) in $L^4(\Omega_{i\epsilon}; \mathbb{R}^d)$ of $\boldsymbol{u}^\epsilon_\kappa$ and $\boldsymbol{B}^\epsilon_\kappa$ to $\boldsymbol{u}^\epsilon$ and $\boldsymbol{B}^\epsilon$, respectively.

Passing to the limit as $\kappa \to +\infty$ in (4.8) we have,

$$
\lim_{\kappa \to +\infty} (\mathcal{F}(\boldsymbol{u}^\epsilon_\kappa - \boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon_\kappa - \boldsymbol{B}^\epsilon); (\boldsymbol{\psi}, \boldsymbol{\psi})) = 0. \tag{4.10}
$$

□

Lemma 4.2. *If the magnetic Reynolds number, $\mathrm{R_m}$, is small then*

$$
(\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) \geq \frac{1}{2} \left|\left|\left| (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \right|\right|\right|^2 , \tag{4.11}
$$

for all $(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \in \mathcal{V}^\epsilon \times \mathcal{W}^\epsilon$.

Proof. We prove the lemma in two steps. In step 1 we obtain an estimate of the magnetic induction in terms of $\mathrm{R_m}$ using Proposition 2.2. In step 2. we obtain an estimate for $\mathcal{Q}$. Combining both steps gives bounds on $\mathrm{R_m}$ for the existence of solutions.

Step 1: We begin with a bound on $\boldsymbol{B}^\epsilon$ in $L^q(\Omega; \mathbb{R}^d)$ for $q \in (1, 2d/(d-2)]$. By Proposition 2.2 we have that

$$
\begin{aligned}
\left\| \boldsymbol{B}^\epsilon \right\|_{L^q(\Omega; \mathbb{R}^d)} &\leq c[\boldsymbol{B}^\epsilon]_{\mathcal{W}^\epsilon} \\
&= c(\left\| \operatorname{curl}(\widehat{\mu}^\epsilon \boldsymbol{B}^\epsilon) \right\|_{L^2(\Omega; \mathbb{R}^d)} + \left\| \boldsymbol{B}^\epsilon \cdot \boldsymbol{n} \right\|_{H^{1/2}(\Gamma_0)}) \\
&\leq c(\left\| \mathrm{R_m} \boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon \right\|_{L^2(\Omega_{1\epsilon}; \mathbb{R}^d)} + |\boldsymbol{c}| \mathrm{mes}_{d-1}(\Gamma_0)).
\end{aligned}
\tag{4.12}
$$

On the rigid particles the velocity takes the form $\boldsymbol{u}^\epsilon = \boldsymbol{V}^{\ell,\epsilon} + \boldsymbol{\omega}^{\ell,\epsilon} \times (\boldsymbol{x} - \boldsymbol{x}^\ell_c)$, $\ell = 1, \ldots, N(\epsilon)$ with the translational and rotational velocity $\boldsymbol{V}^{\ell,\epsilon}$ and $\boldsymbol{\omega}^{\ell,\epsilon}$, respectively, being constant. Additionally, the term $|\boldsymbol{x} - \boldsymbol{x}^\ell_c|$ is such that $|\boldsymbol{x} - \boldsymbol{x}^\ell_c| < \mathrm{diam}(T^\epsilon_\ell) < \epsilon \ll 1$. Hence,

$$
c \left\| \mathrm{R_m} \boldsymbol{u}^\epsilon \times \boldsymbol{B}^\epsilon \right\|_{L^2(\Omega_{1\epsilon}; \mathbb{R}^d)} \leq c_\epsilon \mathrm{R_m} \left\| \boldsymbol{B}^\epsilon \right\|_{L^2(\Omega_{1\epsilon}; \mathbb{R}^d)} \leq c_\epsilon \mathrm{R_m} \left\| \boldsymbol{B}^\epsilon \right\|_{L^q(\Omega; \mathbb{R}^d)} , \tag{4.13}
$$

for $q \in [2, 2d/(d-2)]$.

Combining (4.12) and (4.13) we obtain the following L^q, $q \in [2, 2d/(d-2)]$, bound for $\boldsymbol{B}^\epsilon$,

$$\|\boldsymbol{B}^\epsilon\|_{L^q(\Omega;\mathbb{R}^d)} \leq \frac{c|\boldsymbol{c}|\mathrm{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathrm{R_m}}, \tag{4.14}$$

if $\mathrm{R_m} < 1/c_\epsilon$.

Step 2: By Korn's inequality, Hölder's inequality, and (4.14) we can bound $\mathcal{Q}$ by,

$$\begin{aligned} |\mathcal{Q}\,[(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)]| &\leq c\,\|\boldsymbol{B}^\epsilon\|_{L^4(\Omega_{2\epsilon};\mathbb{R}^d)}\,\|\boldsymbol{B}^\epsilon\|_{L^4(\Omega_{2\epsilon};\mathbb{R}^d)}\,\|e(\boldsymbol{u}^\epsilon)\|_{L^2(\Omega_{2\epsilon};\mathbb{R}^{d\times d})} \\ &\leq c\,\|\boldsymbol{v}^\epsilon\|_{L^4(\Omega_{1\epsilon};\mathbb{R}^d)}\,\|\boldsymbol{B}^\epsilon\|_{L^4(\Omega_{1\epsilon};\mathbb{R}^d)}\,\|\mathrm{curl}(\widehat{\mu}^\epsilon \boldsymbol{u}^\epsilon)\|_{L^2(\Omega_{1\epsilon};\mathbb{R}^d)} \\ &\leq \frac{c|\boldsymbol{c}|\mathrm{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathrm{R_m}}\,|||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)|||^2 . \end{aligned} \tag{4.15}$$

Hence we obtain,

$$(\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) = |||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)|||^2 - \frac{c|\boldsymbol{c}|\mathrm{mes}_{d-1}(\Gamma_0)}{1 - c_\epsilon \mathrm{R_m}}\,|||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)|||^2 , \tag{4.16}$$

if $\mathrm{R_m} > (1 - 2\,c\,|\boldsymbol{c}|\,\mathrm{mes}_{d-1}(\Gamma_0))/c_\epsilon$ the result follows. □

Theorem 4.2. *Given the assumptions in Subsection 3.1, if the magnetic Reynolds number,* $\mathrm{R_m}$*, is small then problem* (4.5) *admits at least one weak solution.*

Proof. According to [Shi64, Corollary 2] (see also [Fin65], [SP68]) if we can show that there exists a number r such that

$$(\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) - (\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) \geq 0 \text{ for all } (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) \text{ with } |||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)||| = r \tag{4.17}$$

then equation (4.5) has at least one solution. Hence, by Lemma 4.2 we have,

$$\begin{aligned} (\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon) - (\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) &= (\mathcal{F}(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) - ((\boldsymbol{f}, \boldsymbol{g}); (\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)) \\ &\geq \frac{1}{2}\,|||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)|||^2 - |||(\boldsymbol{f}, \boldsymbol{g})|||\;|||(\boldsymbol{u}^\epsilon, \boldsymbol{B}^\epsilon)||| \\ &\geq 0, \end{aligned} \tag{4.18}$$

if we select $r = 2\,|||(\boldsymbol{f}, \boldsymbol{g})|||$. □

5 Conclusions

We proved existence of a weak solution to coupled system of Stokes' equations and quasi-static Maxwell's equations under moderate magnetic field strength using the Altman-Shinbrot fixed point theorem. The novelty of the Altman-Shinbrot fixed point argument was that the operator constructed need only by continuous in the weak topology of the underlying function space (and not completely continuous as is required by the fixed point theorem of Leray-Schauder). This is useful due to the fact that the magnetic induction (or magnetic field) do not possess full derivatives in L^2 due to material inhomogeneities.

By and large, the existence result holds true when the magnetic Reynold's number, $\mathrm{R_m}$ is small. The case of $\mathrm{R_m} \equiv 0$ can be thought off as a limit case of the above model. When $\mathrm{R_m} \equiv 0$ the system becomes weakly coupled and, existence and uniqueness follow by invoking the Lax-Milgram lemma once higher integrability of the magnetic induction is established (for details one can consult [GLN]).

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