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Optimal control and directional differentiability for elliptic quasi-variational inequalities

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Abstract

We focus on elliptic quasi-variational inequalities (QVIs) of obstacle type and prove a number of results on the existence of solutions, directional differentiability and optimal control of such QVIs. We give three existence theorems based on an order approach, an iteration scheme and a sequential regularisation through partial differential equations. We show that the solution map taking the source term into the set of solutions of the QVI is directionally differentiable for general unsigned data, thereby extending the results of our previous work which provided a first differentiability result for QVIs in infinite dimensions. Optimal control problems with QVI constraints are also considered and we derive various forms of stationarity conditions for control problems, thus supplying among the first such results in this area.

1 Introduction

Quasi-variational inequalities (QVIs) are generalisations of variational inequalities (VIs) where the constraint set in which the solution is sought depends on the unknown solution itself. The very nature of the dependency of the constraint set on the solution intrinsically leads to a complicated and challenging mathematical structure since it significantly amplifies the nonlinear and nonsmooth nature of VIs. Another attribute that fundamentally distinguishes QVIs from VIs is the lack of uniqueness of solutions (in general) which then necessitates the consideration of multi-valued or set-valued solution mappings. QVIs arise in a multitude of models describing phenomena in fields such as biology, physics, economics and social sciences amongst others. First introduced by Bensoussan and Lions [14, 40] in the study of stochastic impulse controls, specific applications involving QVIs are thermoforming processes [3], the formation and growth of lakes, rivers and sandpiles [51, 12, 50, 48, 50, 48, 13], games in the context of generalised Nash equilibrium problems [29, 23, 46], and magnetisation of superconductors [38, 11, 49, 54]. See [4, 9] for additional details and references.

In this paper, we focus on elliptic QVIs of obstacle type or *compliant obstacle problems*. These have the form

$$\text{find } y \in \mathbf{K}(y) : \langle Ay - f, y - v \rangle \leq 0 \quad \forall v \in \mathbf{K}(y) \text{ where } \mathbf{K}(y) := \{v \in V : v \leq \Phi(y)\}. \tag{1}$$

Here $f \in V^*$ is data, $\Phi \colon V \to V$ is a given *obstacle map*, and V is a Hilbert space possessing an ordering $\leq$ which is used in the definition of the constraint set (we shall be more precise below). Let us define $\mathbf{Q}$ to be the solution map associated to (1), so that it reads $y \in \mathbf{Q}(f)$. We develop in this paper theory addressing the matters of existence for (1), directional differentiability of $\mathbf{Q}$ and stationarity conditions for optimal control problems with QVI constraints of the form

$$\min_{\substack{u \in U_{ad} \\ y \in \mathbf{Q}(u)}} \frac{1}{2} \|y - y_d\|_H^2 + \frac{\nu}{2} \|u\|_U^2 . \tag{2}$$

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Different methodologies exist for the mathematical treatment of existence for QVIs. There is an approach based on order that was pioneered by Tartar [58] which relies on the existence of subsolutions and supersolutions to guarantee existence of solutions (typically, one takes 0 as a subsolution which would hold under sign conditions on the source term). In certain cases, the QVI can be expressed as a generalized equation and it therefore belongs to a more general problem class [35, 36, 24, 34, 25]. In problems involving constraints on derivatives (which is not the case under consideration in this paper), special forms of regularisation of the constraint that modify the partial differential operator may be suitable, see [54, 43, 7, 8]. For more details, we refer the reader to [4]. We discuss in §2 appropriate conditions on the function spaces and the obstacle map Φ for $\mathbf{Q}$ to be well defined. One approach relies on an iteration argument where a contraction-type property of Φ is used. Another existence result is given for source terms bounded from below by a non-negative function using the aforementioned Birkhoff–Tartar theory, and we also study a sequential regularisation approach of the QVI by PDEs where the QVI constraint is handled by a smoothing via Moreau–Yosida.

Literature on the differentiability and sensitivity analysis for solution maps associated to QVIs in infinite dimensions is almost non-existent: our contributions [3, 5] appear to be the first ones that address these issues. In [3], we give a first directional differentiability result for the solution map taking the source term into the set of solutions for non-negative sources and directions whilst in [5] we studied continuity properties related to minimal and maximal solution mappings of QVIs. In §3, we derive directional differentiablity results for $\mathbf{Q}$. We extend here our previous work [3] which provided differentiability results for source and direction terms that are non-negative; in this paper we shall remove this restriction in our Theorems 3.12 and 3.15 which utilise the new results from the preceding section. We give a characterisations of the QVI that is satisfied by the directional derivative of $\mathbf{Q}$ as a complementarity system and in §3.3 we also prove a continuity result that shows that the derivative depends continuously on the direction under some assumptions. This gives a comprehensive answer to the question of sensitivity analysis of QVIs.

The scarcity of work done on the optimal control of QVIs in infinite dimensions is unsurprisingly even more pronounced; see [2, 5, 20, 21, 45] for some of the very few contributions. In our work [5], in addition to stability properties we also provided results on the optimal control of minimal and maximal solutions of QVIs. While this article was under preparation, we note that [61] has appeared wherein the author considers elliptic QVIs and their differential sensitivity and optimal control but for Frèchet differentiable obstacle maps Φ; we assume only Hadamard or bounded differentiability of Φ for the differentiability results. For QVIs in the finite dimensional setting, see [44] and the references therein. In sharp contrast, control problems with VI constraints has attracted wide attention: see for example [10, 42, 16, 15, 33, 32, 31, 60] and the references therein. We shall consider in §4 the optimal control problem (2) where existence of the optimal control will be shown using a standard calculus of variations argument. Then we turn our attention to the derivation of stationarity conditions for the optimal control and state. There are a number of concepts of stationarity for these types of control problems, see [32] for a discussion. We work on obtaining first Bouligand stationarity in §5.1, then $\mathcal{E}$-almost C-stationarity conditions [31, 30] in §5.2 by approximating the QVI control-to-state map through PDEs (as done in §2.3) and then passing to the limit. In §5.3 we provide a strong stationarity result.

1.1 Contributions of the paper

We summarise the main results of this work.

- **Existence for** (1):

- Theorem 2.3: iteration by solutions of VIs using complete continuity of Φ,
- Theorem 2.6: Birkhoff–Tartar order approach under lower bounds on Φ and the source term,
- Theorem 2.11: sequential regularisation by PDEs under complete continuity of Φ.

- **Directional differentiability for QVIs**:
  - Theorem 3.12: for locally boundedly differentiable (see Definition 3.8) maps Φ with no restriction on the sign on the source and direction terms,
  - Theorem 3.15: for locally Hadamard differentiable maps Φ for source/direction terms bounded from below.

- **Properties**:
  - Proposition 2.1: complementarity characterisations of the QVI (1),
  - Propositions 3.13 and 3.16: complementarity characterisations of the QVI satisfied by the directional derivative of the solution map,
  - Proposition 3.18: uniqueness for the QVI satisfied by directional derivative and continuity properties.

- **Optimal control**:
  - Theorem 4.1: existence of optimal controls for (2).

- **Stationarity conditions for** (2):
  - Proposition 5.1: Bouligand stationarity,
  - Theorem 5.4: $\mathcal{E}$-almost C stationarity,
  - Theorem 5.6: strong stationarity.

1.2 Basic assumptions and notations

We make some standing assumptions that are necessary throughout the paper, except where mentioned otherwise.

Let $V \subset H$ be an embedding of separable Hilbert spaces and suppose that there exists an ordering to elements of H via a closed convex cone H_+ satisfying $H_+ = \{h \in H : (h, g) \geq 0 \quad \forall g \in H_+\}$; the ordering then is $h_1 \leq h_2$ if and only if $h_2 - h_1 \in H_+$. This also induces an ordering for V in the obvious way and we write $V_+ := \{v \in V : v \geq 0\}$. It also induces an ordering for V^* via

$$V^*_+ := \{f \in V^* : \langle f, v\rangle \geq 0 \quad \forall v \in V_+\},$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V^*,V}$ is the standard duality pairing. We write h^+ for the orthogonal projection of $h \in H$ onto the space H_+ and we have the decomposition $h = h^+ - h^-$. We suppose that $v \in V$ implies that $v^+ \in V$ and that there exists a constant $C > 0$ such that for all $v \in V$,

$$\left\|v^+\right\|_V \leq C \left\|v\right\|_V.$$

An example of such a space V is the Sobolev space $V = W^{1,p}(\Omega)$ over a domain Ω for $1 \leq p \leq \infty$ with $H = L^2(\Omega)$ (see [1] for a definition). The ordering relation $u \leq v$ in this case is equivalent to '$u \leq v$ a.e. in Ω' as expected.

We take the obstacle map $\Phi\colon V \to V$ to be increasing and $A\colon V \to V^*$ to be a linear operator that satisfies the following properties for all $u, v \in V$:

$$\langle Au, v\rangle \leq C_b \|u\|_V \|v\|_V, \qquad \text{(boundedness)}$$

$$\langle Au, u\rangle \geq C_a \|u\|_V^2, \qquad \text{(coercivity)}$$

$$\langle Au^+, u^-\rangle \leq 0. \qquad \text{(T-monotonicity)}$$

Since later V will be assumed to be part of a Gelfand triple with a pivot space H, we will only rarely need to use the inner product on V and when we do so this will always be denoted by $(\cdot,\cdot)_V$ (with the subscript). The identity operator will be denoted by I.

We denote continuous, dense, and compact embeddings of spaces by $\hookrightarrow$, $\overset{d}{\hookrightarrow}$, and $\overset{c}{\hookrightarrow}$ respectively. The notation $B_R(u)$ will be used to mean the closed ball in V of radius R centred at u.

2 Existence for QVIs

We begin by discussing three existence results for (1), reproduced here:

$$y \leq \Phi(y) : \langle Ay - f, y - v\rangle \leq 0 \quad \forall v \in V : v \leq \Phi(y),$$

involving different approaches. We begin by obtaining existence through iteration by solutions of VIs. Then we consider a translation of the theory by Birkhoff–Tartar for source terms that are bounded from below and we finish by considering a sequential regularisation approach through PDEs. These existence results entail different assumptions. The third approach, which is useful for purposes of numerical realisation, requires only complete continuity of Φ and furthermore the assumption of Φ being increasing can be dropped. The first approach also requires complete continuity in addition to having either a smallness condition on the boundedness of Φ or a non-empty intersection of the constraint sets associated to the QVI for varying obstacles. The second approach does not need any compactness but instead one needs the obstacle map and the data to be bounded from below in a certain sense.

Before we proceed, let us give the following characterisation of (1).

Proposition 2.1. *The problem* (1) *is equivalent to the complementarity system*

$$\begin{aligned} \xi &:= f - Ay,\\ \xi &\geq 0,\\ \langle \xi, \Phi(y) - y\rangle &= 0,\\ 0 &\leq \Phi(y) - y. \end{aligned}$$

Proof. The proof is standard. By definition, ξ satisfies $\langle \xi, y - v\rangle \geq 0$ for all feasible v. Setting $v = \Phi(y)$ and then $v = 2y - \Phi(y)$, we obtain the the orthogonality condition for ξ. Testing with $v = y - \varphi$ for $\varphi \geq 0$ a.e. gives the stated non-negativity. □

2.1 Iteration scheme

Let $S\colon V^* \times V \to V$ be the usual solution mapping associated to the class of VIs under consideration, i.e. $y = S(f, \psi)$ solves

$$y \leq \Phi(\psi) : \langle Ay - f, y - v\rangle \leq 0 \quad \forall v \in V : v \leq \Phi(\psi).$$

Take a source term $f \in V^*$ and set $y_0 := A^{-1}f = S(f, \infty)$. The function $y_1 := S(f, y_0)$ satisfies $y_1 \leq S(f, \infty) \equiv y_0$ by the comparison principle [53, §4:5], and defining

$$y_n := S(f, y_{n-1}),$$

we see that $y_n \leq y_{n-1}$ by repeated applications of the comparison principle. Hence $\{y_n\}$ is monotonically decreasing and each y_n satisfies

$$y_n \in V, y_n \leq \Phi(y_{n-1}) : \langle Ay_n - f, y_n - v\rangle \leq 0 \quad \forall v \in V : v \leq \Phi(y_{n-1}). \tag{3}$$

We look for a uniform bound on $\{y_n\}$. When the obstacle map is such that it always dominates some given function $v_0 \in V$, this is easy since we may test with $v = v_0$. Otherwise, we need the following.

Lemma 2.2. *If*

$$\|\Phi(v)\|_V \leq C_X \|v\|_V \quad \forall v \in V \text{ where } C_X < \frac{C_a}{C_b}, \tag{4}$$

then $\{y_n\}$ is bounded in V.

Proof. Since $y_n \leq y_{n-1}$ and Φ is increasing, $\Phi(y_n) \leq \Phi(y_{n-1})$ and so $\Phi(y_n)$ is a valid test function in (3) and we obtain

$$\begin{aligned} C_a \|y_n\|_V^2 &\leq \langle Ay_n, \Phi(y_n)\rangle + \langle f, y_n - \Phi(y_n)\rangle \\ &\leq C_b \|y_n\|_V \|\Phi(y_n)\|_V + \|f\|_{V^*} \|y_n - \Phi(y_n)\|_V \\ &\leq C_b C_X \|y_n\|_V^2 + (1 + C_X) \|f\|_{V^*} \|y_n\|_V . \end{aligned}$$

From this, we deduce that under the condition on C_X in (4), y_n is bounded in V. □

The assumption (4) places a limitation on the variation on the bound of the constraint map Φ which implies uniqueness of solutions for (1).

Theorem 2.3. *For any $f \in V^*$, under the assumptions*

$$\text{either there exists } v_0 \in V : \Phi(v) \geq v_0 \text{ for all } v \in V, \text{ or } (4), \tag{5}$$

$$\Phi\colon V \to V \text{ is completely continuous}, \tag{6}$$

there exists a solution $y \in \mathbf{Q}(f) \cap (-\infty, A^{-1}f]$ which is the weak limit of the sequence $\{y_n\}$ defined above.

Proof. We obtain, thanks to monotonicity and the above lemma that $y_n \rightharpoonup y$ in V (for the full sequence) for some y. Taking $v^* \in V$ with $v^* \leq \Phi(y)$ and taking as test function $v_n = v^* - \Phi(y) + \Phi(y_{n-1})$, which is feasible for the VI for y_n and strongly converges to v^*, we can easily pass to the limit in (3) and we find $y \in \mathbf{Q}(f)$ in the stated interval. □

We have shown that $\mathbf{Q}\colon V^* \rightrightarrows V$ is well defined under the above circumstances.

Remark 2.4. *If the source term f is non-negative (i.e. if $f \in V^*_+$), $\Phi\colon H \to V$, and $\Phi(0) \geq 0$, then the function 0 acts a subsolution for the map $S(f, \cdot)$ which, in combination with the supersolution y_0 defined as above, allows us to directly apply the theory of fixed points in vector lattices of Tartar–Birkhoff and obtain existence of solutions for (1) in the interval $[0, y_0]$. In this case one does not need the assumption (5). This was the approach taken in [3].*

2.2 Translation of Birkhoff–Tartar order approach

In this section, we translate the Birkhoff–Tartar-type existence results for QVIs with non-negative source terms (see Remark 2.4) to QVIs with source terms that are allowed to be negative. This leads to different assumptions than those made in §2.1. The bedrock of this technique, as detailed in the introduction, is the result of Tartar [58] that gives existence of fixed points for increasing maps that possess subsolutions and supersolutions, see also [6, Chapter 15, §15.2].

Let $G \in V^*_+$ be given such that $\Phi\colon H \to V$ satisfies the property

$$\Phi(-A^{-1}(rG)) \geq -A^{-1}(rG) \text{ for all } r \geq 1 \text{ arbitrarily close to } 1. \tag{7}$$

We think of G as a 'lower bound' function. The next example illustrates the existence of such a function G to a map Φ related to solution maps of elliptic PDEs.

Example 2.5. *Suppose $V \subset H \subset V^*$ is a Gelfand triple with $H = L^2(\Omega)$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Let $\Phi(u) = \varphi$ be defined as the solution of*

$$B\varphi = f_0 + u,$$

where $B\colon V \to V^*$ is a bounded linear and coercive operator which is T-monotone and $f_0 \geq 0$ is data. The interest in such obstacle mappings is not merely academic, see [3] for some applications. We claim that if G is such that

$$BA^{-1}G \geq A^{-1}G,$$

then (7) is satisfied. To see this, set $v := \Phi(-A^{-1}rG)$ so that $Bv = f_0 - A^{-1}rG$. Adding the same term to both sides, we obtain $B(v + A^{-1}rG) = f_0 + BA^{-1}rG - A^{-1}rG$. Test this with the function $(v + A^{-1}rG)^-$ to obtain

$$\langle B(v + A^{-1}rG)^-, (v + A^{-1}rG)^- \rangle = \int_\Omega -(f_0 + BA^{-1}rG - A^{-1}rG)(v + A^{-1}rG)^-,$$

and the right-hand side of this is less than zero since $f_0 \geq 0$.

While the assumption of the existence of such a G may appear to be restrictive, note that choosing $G \equiv 0$ recovers the results of [3] which has been successfully applied to an application in thermoforming.

Theorem 2.6. *Let $G \in V^*_+$ and $r \geq 1$ be according to (7). Given $f \in V^*$ with $f \geq -G$, there exist solutions $y_r \in \mathbf{Q}(f) \cap [-A^{-1}(rG), A^{-1}f]$.*

Furthermore, defining the family of obstacle maps $\Psi_\tau\colon H \to V$ by

$$\Psi_\tau(w) := \Phi(w - A^{-1}(\tau G)) + A^{-1}(\tau G), \tag{8}$$

the function $w_r := y_r + A^{-1}(rG)$ satisfies

$$w_r \leq \Psi_r(w_r) : \langle Aw_r - (f + rG), w_r - v \rangle \leq 0 \quad \forall v \in V : v \leq \Psi_r(w_r)$$

and lies in $[0, A^{-1}(f + rG)]$. We write $w_r \in \mathbf{H}_r(f + rG)$.

Proof. Firstly, observe that by the assumptions on Φ, we have for all $\tau \geq 1$ that the obstacle map Ψ_τ (defined in (8)) is increasing and satisfies $\Psi_\tau(0) \geq 0$. Hence, for $f \in V^*_+$, there exist [58] solutions $w_\tau \in [0, A^{-1}f]$ to the QVI

$$\text{find } w \in V, w \leq \Psi_\tau(w) : \langle Aw - f, w - v\rangle \leq 0 \quad \forall v \in V : v \leq \Psi_\tau(w),$$

and we write $w \in \mathbf{H}_\tau(f)$. We therefore have the existence of $w_r \in \mathbf{H}_r(f + rG)$ for any $r \geq 1$ satisfying the inequality in the statement of the theorem. Define now $y_r := w_r - A^{-1}(rG)$ which, since

$$w_r \leq \Psi_r(w_r) = \Phi(w_r - A^{-1}(rG)) + A^{-1}(rG),$$

is feasible in the sense that $y_r \leq \Phi(y_r)$. Furthermore, $Ay_r - f = Aw_r - (f + rG)$ and if $\varphi := v - A^{-1}(rG)$ then $y_r - \varphi = w_r - v$ and we have

$$y_r \leq \Phi(y_r) : \langle Ay_r - f, y_r - \varphi\rangle \leq 0 \quad \forall \varphi : \varphi \leq \Psi_r(w_r) - A^{-1}(rG) = \Phi(w_r - A^{-1}(rG)) = \Phi(y_r).$$

This shows existence for (1).

Remark 2.7. *If* (7) *holds at some particular* $r \geq 1$ *(and rather than for all* r *close to* 1*) then the results of this section clearly still hold for that value of* r*. The assumption* (7) *is phrased as it is due to necessity in later sections.*

□

2.3 Sequential regularisation by PDEs

In this section, we obtain existence results for (1) by regularising the QVI by PDEs. There has been considerable effort on various aspects and methods of regularisation of VIs by PDEs; see for example [27, §3.2] for an approach similar to what we consider here and [39] and [37, §IV] for a penalisation involving approximations to the Heaviside graph (see also [53, §5:3] on this). We make use of and adapt the work of Hintermüller and Kopacka [31] for VIs in this section. We take $H := L^2(\Omega)$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and work in the Gelfand triple setting (V, H, V^*) with $V \overset{c}{\hookrightarrow} H$.

For $\rho > 0$, let $m_\rho(\cdot) \equiv \max^g_{\epsilon(\rho)}(0, \cdot)$ be the following regularisation of the positive part function $(\cdot)^+ = \max(0, \cdot)$:

$$m_\rho(r) := \begin{cases} 0 & : r \leq 0 \\ \frac{r^2}{2\epsilon} & : 0 < r < \epsilon \\ r - \frac{\epsilon}{2} & : r \geq \epsilon; \end{cases}$$

here, $\epsilon = \epsilon(\rho) > 0$ is a smoothing parameter utilised for ensuring differentiability at 0 and ρ is a penalty parameter which we send to zero later (this is the so-called global penalisation used in [31]). We suppose that $\{\epsilon(\rho)\}$ is a bounded sequence; since m_ρ is an exact penalisation of the associated constraint set, it is not necessary to drive $\epsilon \to 0$. Since $m_\rho \colon \mathbb{R} \to \mathbb{R}$ is C^1 with $m'_\rho \in [0, 1]$, by [18, Lemma 2.83], $m_\rho \colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ for $p \in [1, \infty)$. We will assume that V is such that this property holds, i.e., that $m_\rho \colon V \to V$.

We consider the penalisation[1]

$$Ay_\rho + \frac{1}{\rho} m_\rho(y_\rho - \Phi(y_\rho)) = f \tag{9}$$

[1]For the results of this section, it would be sufficient to simply consider $\max(0, \cdot)$ instead of m_ρ, but in anticipation of the optimal control problem that we shall later study (in particular when we derive optimality conditions), it becomes useful to smooth out the $\max$ function like we have done here.

of (1) and study the convergence properties of its solution as $\rho \to 0$.

Remark 2.8. *In fact, we do not need Φ to be increasing for the results in this section.*

Lemma 2.9. *Let $\Phi\colon V \to H$ be completely continuous. Given $f \in V^*$, there exists a solution $y_\rho \in V$ of* (9)*. Furthermore, every solution satisfies*

$$\|y_\rho\|_V \le C_a^{-1}\left(\|f\|_{V^*} + \frac{C\epsilon}{\rho}\right), \tag{10}$$

where C is the constant of continuity for the embedding $V \hookrightarrow L^1(\Omega)$.

Proof. Since Φ is completely continuous, it is compact and therefore bounded. We see that, using $H \hookrightarrow V^*$ and the Lipschitz continuity of m_ρ [31, Lemma 2.5 (v)],

$$\left\|Ay + \frac{1}{\rho}m_\rho(y-\Phi(y))\right\|_{V^*} \le C_b\|y\|_V + \frac{C}{\rho}\|y-\Phi(y)\|_H,$$

so that $A + \frac{1}{\rho}m_\rho(\mathrm{I}-\Phi)$ is a bounded operator. Let us show that it is also coercive. Observe that

$$\begin{aligned}\frac{1}{\rho}\int_\Omega m_\rho(y-\Phi(y))y = \frac{1}{2\epsilon\rho}\int_{\{0<y-\Phi(y)<\epsilon\}}(y-\Phi(y))^2 y + \frac{1}{\rho}\int_{\{y-\Phi(y)>\epsilon\}\cap\{y\ge 0\}}\left(y-\Phi(y)-\frac{\epsilon}{2}\right)y\\ + \frac{1}{\rho}\int_{\{y-\Phi(y)>\epsilon\}\cap\{y<0\}}\left(y-\Phi(y)-\frac{\epsilon}{2}\right)y.\end{aligned}$$

The first term on the right-hand side can be bounded as

$$\frac{1}{2\epsilon\rho}\left|\int_{\{0<y-\Phi(y)<\epsilon\}}(y-\Phi(y))^2 y\right| \le \frac{\epsilon}{2\rho}\int_\Omega |y| \le \frac{C\epsilon}{2\rho}\|y\|_V$$

where we used that $V \hookrightarrow H \hookrightarrow L^1(\Omega)$ are continuous embeddings. The second term can be neglected:

$$\frac{1}{\rho}\int_{\{y-\Phi(y)>\epsilon\}\cap\{y\ge 0\}}\left(y-\Phi(y)-\frac{\epsilon}{2}\right)y \ge \frac{\epsilon}{2\rho}\int_{\{y-\Phi(y)>\epsilon\}\cap\{y\ge 0\}}|y| \ge 0,$$

whilst the third term is

$$\frac{1}{\rho}\int_{\{y-\Phi(y)>\epsilon\}\cap\{y<0\}}\left(y-\Phi(y)-\frac{\epsilon}{2}\right)y \le \frac{\epsilon}{2\rho}\int_\Omega |y| \le \frac{C\epsilon}{2\rho}\|y\|_V$$

since on the domain of integration, we have $(y-\Phi(y)-\epsilon/2)\,y \le \epsilon y/2$. Hence,

$$\langle Ay, y\rangle + \frac{1}{\rho}\int_\Omega m_\rho(y-\Phi(y))y \ge C_a\|y\|_V^2 - \frac{C\epsilon}{\rho}\|y\|_V,$$

and we see that if we divide both sides by $\|y\|_V$ and take the limit $\|y\|_V \to \infty$, the resulting right-hand side diverges and the operator is coercive.

Using the complete continuity and $V \overset{c}{\hookrightarrow} H$, the term $\rho^{-1}m_\rho(y-\Phi(y))$ is completely continuous, giving the pseudo-montonicity of the full elliptic operator. Then standard results (eg. [56, §2, Lemma 2.1, Example 2.B and Corollary 2.2]) yield existence. The estimate stated in the lemma is a simple consequence of the above coercivity estimate. □

Remark 2.10. *In case there exists an element $v_0 \in V$ such that $v_0 \leq \Phi(v)$ for all $v \in V$ (this is true in the VI case, for example), then the bound* (10) *on y can be replaced with a different bound which is independent of ϵ and ρ. Indeed, omitting the subscript ρ in y_ρ for ease of reading, testing* (9) *with $y - v_0$ and manipulating with*

$$\begin{aligned}\int_\Omega m_\rho(y - \Phi(y))(y - v_0) &= \int_\Omega (m_\rho(y - \Phi(y)) - m_\rho(v_0 - \Phi(y)))(y - v_0)\\ &\quad + \int_\Omega m_\rho(v_0 - \Phi(y)))(y - v_0)\\ &\geq 0\end{aligned}$$

(by monotonicity and because $m_\rho \equiv 0$ on $(-\infty, 0]$), we have

$$\begin{aligned}C_a \|y\|_V^2 &\leq C_b \|y\|_V \|v_0\|_V + \|f\|_{V^*} \|y\|_V + \|f\|_{V^*} \|v_0\|_V\\ &\leq \frac{C_a}{3} \|y\|_V^2 + \frac{3C_b^2}{4C_a} \|v_0\|_V^2 + \frac{3}{4C_a} \|f\|_{V^*}^2 + \frac{C_a}{3} \|y\|_V^2 + \frac{1}{2} \|f\|_{V^*}^2 + \frac{1}{2} \|v_0\|_V^2 .\end{aligned}$$

This gives the uniform bound

$$\frac{C_a}{3} \|y\|_V^2 \leq \left(\frac{3C_b^2}{4C_a} + \frac{1}{2}\right) \|v_0\|_V^2 + \left(\frac{3}{4C_a} + \frac{1}{2}\right) \|f\|_{V^*}^2 .$$

Thanks to this lemma, for every source term $f_\rho \in V^*$, the following equation has a solution y_ρ:

$$Ay_\rho + \frac{1}{\rho} m_\rho(y_\rho - \Phi(y_\rho)) = f_\rho. \tag{11}$$

We write the possibly multivalued solution mapping associated to this equation as $\mathbf{P}_\rho \colon V^* \rightrightarrows V$, so (11) reads $y_\rho \in \mathbf{P}_\rho(f_\rho)$. The next theorem shows that solutions of QVIs can be approximated by solutions of (11) if we choose the parameter ϵ such that $\{\epsilon(\rho)/\rho\}_\rho$ is bounded; note that is a requirement special to our QVI case and was not necessary in the setting of [31].

Theorem 2.11. *Let V be a Hilbert space, $\{\epsilon(\rho)/\rho\}$ be bounded*[2]*, and let* (6) *hold. Take a sequence $f_\rho \to f$ in V^*. Then there exists a subsequence $\{\rho_n\}_n$ and elements $y_{\rho_n} \in \mathbf{P}_{\rho_n}(f_{\rho_n})$ such that $y_{\rho_n} \to y$ in V where $y \in \mathbf{Q}(f)$.*

Proof. The proof is in four steps and is similar to the proof of Theorem 2.3 of [31].

1. Uniform estimates and feasibility of limit. For each ρ, let y_ρ be a solution of (11) (such a selection is possible due to the axiom of countable choice). By Lemma 2.9, it satisfies the bound

$$C_a \|y_\rho\|_V \leq \|f_\rho\|_{V^*} + \frac{C\epsilon}{\rho},$$

and this is bounded because we took $\epsilon(\rho)/\rho$ to be bounded, and hence for a subsequence (which we do not attempt to differentiate for ease of reading), $y_\rho \rightharpoonup y$ in V to some y. Rearranging the equality (11),

$$\|m_\rho(y_\rho - \Phi(y_\rho))\|_{V^*} = \rho \|f_\rho - Ay_\rho\|_{V^*} \leq C\rho$$

[2]See Remark 2.13 regarding this assumption.

and therefore $m_\rho(y_\rho - \Phi(y_\rho)) \to 0$ in V^* as $\rho \to 0$. Since $\epsilon(\rho)/\rho$ is bounded, we have $\epsilon(\rho) \to 0$; we use this in the following calculation:

$$\begin{aligned}\left\|\max(0, y - \Phi(y)) - \max_{\epsilon(\rho)}(0, y_\rho - \Phi(y_\rho))\right\|_{V^*} &\leq \|\max(0, y - \Phi(y)) - \max(0, y_\rho - \Phi(y_\rho))\|_H \\ &\quad + \left\|\max(0, y_\rho - \Phi(y_\rho)) - \max_{\epsilon(\rho)}(0, y_\rho - \Phi(y_\rho))\right\|_H \\ &\leq \|y - y_\rho\|_H + \|\Phi(y) - \Phi(y_\rho)\|_H + \frac{\epsilon(\rho)}{2}|\Omega|^{1/2} \\ &\to 0\end{aligned}$$

using [31, Lemma 2.1 (iv)] (in fact, up to here, complete continuity of $\Phi\colon V \to H$ would suffice rather than (6)). Hence we find $\max(0, y - \Phi(y)) = 0$, which tells us that $y \leq \Phi(y)$.

2. Monotonicity formula. For $v \in V$, we get by adding and subtracting the same term and using the monotonicity of m_ρ,

$$\begin{aligned}m_\rho(y_\rho - \Phi(y_\rho))(y_\rho - v) &= (m_\rho(y_\rho - \Phi(y_\rho)) - m_\rho(v - \Phi(y_\rho)))(y_\rho - \Phi(y_\rho) + \Phi(y_\rho) - v) \\ &\quad + m_\rho(v - \Phi(y_\rho))(y_\rho - v) \\ &\geq m_\rho(v - \Phi(y_\rho))(y_\rho - v). \end{aligned} \tag{12}$$

3. Passage to the limit. Test the equation (11) with $y_\rho - v$ for $v \in V$ and use (12) to find

$$\langle Ay_\rho, y_\rho\rangle + \frac{1}{\rho}\int_\Omega m_\rho(v - \Phi(y_\rho))(y_\rho - v) \leq \langle f_\rho, y_\rho - v\rangle + \langle Ay_\rho, v\rangle. \tag{13}$$

Now, choose an arbitrary $v^* \in V$ with $v^* \leq \Phi(y)$ and select the test function to be

$$v_\rho = v^* - \Phi(y) + \Phi(y_\rho).$$

This satisfies $v_\rho \to v^*$ in V and $v_\rho \leq \Phi(y_\rho)$. With this choice, the second term on the left-hand side of (13) of the above inequality is equal to zero by definition of m_ρ. Hence we find

$$\langle Ay_\rho, y_\rho\rangle \leq \langle f_\rho, y_\rho - v_\rho\rangle + \langle Ay_\rho, v_\rho\rangle.$$

Take the limit inferior as $\rho \to 0$ and use weak lower semicontinuity to get $y \in \mathbf{Q}(f)$.

4. Strong convergence. Define $v_\rho := y + \Phi(y_\rho) - \Phi(y)$ which has the properties

$$\begin{aligned}&v_\rho \to y \text{ in } V,\\ &v_\rho \leq \Phi(y_\rho),\\ &y_\rho - v_\rho = (y_\rho - y) + (\Phi(y) - \Phi(y_\rho)) \rightharpoonup 0 \text{ in } V,\end{aligned}$$

the first holding due to complete continuity since we already have $y_\rho \rightharpoonup y$ in V. By coercivity we obtain the estimate

$$\begin{aligned}\langle A(y_\rho - v_\rho), y_\rho - v_\rho\rangle &\geq C_a \|(y_\rho - y) + (\Phi(y) - \Phi(y_\rho))\|_V^2 \\ &= C_a \|y_\rho - y\|_V^2 + C_a \|\Phi(y) - \Phi(y_\rho)\|_V^2 + 2C_a(y_\rho - y, \Phi(y) - \Phi(y_\rho))_V.\end{aligned}$$

Testing (11) appropriately, we have

$$\langle A(y_\rho - v_\rho), y_\rho - v_\rho\rangle = \langle f_\rho, y_\rho - v_\rho\rangle - \frac{1}{\rho}\int_\Omega m_\rho(y_\rho - \Phi(y_\rho))(y_\rho - v_\rho) - \langle Av_\rho, y_\rho - v_\rho\rangle$$

and to this we apply the monotonicity formula and the above calculation to find

$$\begin{aligned} C_a \|y_\rho - y\|_V^2 &\le -C_a \|\Phi(y) - \Phi(y_\rho)\|_V^2 - 2C_a(y_\rho - y, \Phi(y) - \Phi(y_\rho))_V \\ &\quad + \langle f_\rho, y_\rho - v_\rho\rangle - \frac{1}{\rho}\int_\Omega m_\rho(v_\rho - \Phi(y_\rho))(y_\rho - v_\rho) - \langle Av_\rho, y_\rho - v_\rho\rangle \\ &\le -C_a \|\Phi(y) - \Phi(y_\rho)\|_V^2 - 2C_a(y_\rho - y, \Phi(y) - \Phi(y_\rho))_V + \langle f_\rho, y_\rho - v_\rho\rangle \\ &\quad - \langle Av_\rho, y_\rho - v_\rho\rangle. \qquad \text{(since } v_\rho \le \Phi(y_\rho)\text{)} \end{aligned}$$

By complete continuity of Φ, the first and second terms converge to zero (the second being the inner product of a weakly and a strongly convergent sequence). This reasoning also applies to the third term and fourth term. Hence $y_\rho \to y$ strongly in V. □

Remark 2.12. *If $\mathbf{Q}(f)$ is a singleton, then the convergence result of the previous theorem holds for the entire sequence and not just a subsequence because the limit $y = \mathbf{Q}(f)$ is unique.*

Remark 2.13. *In the situation of Remark 2.10, the requirement that $\{\epsilon(\rho)/\rho\}$ is bounded for Theorem 2.11 is unnecessary. Since $\{\epsilon(\rho)\}$ is bounded, we have (for a subsequence that we relabelled) $\epsilon(\rho) \to \bar\epsilon$ for some $\bar\epsilon \ge 0$ and we can replace the calculation in the first step of the proof of the previous theorem by*

$$\begin{aligned} \left\|\max_{\bar\epsilon}(0, y - \Phi(y)) - \max_{\epsilon(\rho)}(0, y_\rho - \Phi(y_\rho))\right\|_{V^*} &\le \left\|\max_{\bar\epsilon}(0, y - \Phi(y)) - \max_{\bar\epsilon}(0, y_\rho - \Phi(y_\rho))\right\|_H \\ &\quad + \left\|\max_{\bar\epsilon}(0, y_\rho - \Phi(y_\rho)) - \max_{\epsilon(\rho)}(0, y_\rho - \Phi(y_\rho))\right\|_H \\ &\le \|y - y_\rho\|_H + \|\Phi(y) - \Phi(y_\rho)\|_H + \frac{3}{2}|\bar\epsilon - \epsilon(\rho)| \\ &\to 0 \end{aligned}$$

with the convergence due to [31, Lemma 2.1 (iv) and (v)].

Example 2.14. *The prototypical example for Φ to have in mind is a map given by the inverse of a partial differential operator such as*

$$\Phi(w) := L^{-1}w + f_0,$$

for example with $L\colon V \to V^*$ a second-order linear elliptic operator on a bounded Lipschitz domain Ω and $f_0 \in V$. The validity of elliptic regularity and continuous dependence estimates for L would give compactness properties for Φ and weak maximum principles yield the increasing property. See [3, §1.2] for more details on this and on an application to fluid flow.

We shall need more assumptions on the obstacle map in the forthcoming sections and these will mainly be differentiability requirements on Φ. In case of the example above with a linear L, these can be checked without great difficulty. In [3, §6] we studied in substantial detail an application in thermoforming and the mathematical model given there of the thermoforming process involves a QVI with a nonlinear obstacle mapping Φ related to a solution of a PDE and we showed that all desired assumptions (including those on differentiability) were satisfied.

3 Directional differentiability

In this section, we extend the results of our previous work [3] which dealt with directional differentiability of the solution map $\mathbf{Q}$ associated to (1) for non-negative source terms and directions. Here, we shall see that similar results hold for

(a) unsigned source and direction terms and

(b) for source and direction terms bounded from below

by using two different approaches. Formally, the goal is to show that there exists a $\mathbf{Q}'(f)(d) \in V$ such that

$$\lim_{s\to 0^+} \frac{\mathbf{Q}(f+sd)-\mathbf{Q}(f)}{s} = \mathbf{Q}'(f)(d).$$

This is merely a formal limit since $\mathbf{Q}\colon V \rightrightarrows V$ is set valued and not single valued in general, however in case $\mathbf{Q}\colon V \to V$ is single valued, it is precise. It is important to obtain such a sensitivity result not only for applications but also for the procurement of strong stationarity conditions for optimal control problems with QVI constraints, a topic that we will address in §5.3.

In order to show differentiability, in the case (a), we will consider an iteration argument similar to that in §2.1 and for (b) we shall utilise the results of §2.2 and apply the results of our earlier work [3]. In both cases, we fundamentally require the differentiability result for VIs [41] for which more structure on the function space framework is required in the form of the next assumption.

Assumption 3.1. *Suppose that $H := L^2(X;\mu)$ where X is a locally compact topological space which is σ-compact and μ is a Radon measure on X and let $V \subset H \subset V^*$ be a Gelfand triple. Furthermore, we assume that*

$$V \cap C_c(X) \overset{d}{\hookrightarrow} C_c(X) \qquad \textit{and} \qquad V \cap C_c(X) \overset{d}{\hookrightarrow} V.$$

This allows us to define the notions of capacity, quasi-continuity and related concepts, consult [41, §3], [28, §3] and [17, §6.4.3] for more details. We will typically choose X to be Ω or its closure $\overline{\Omega}$ (where $\Omega \subset \mathbb{R}^n$ is a sufficiently regular domain) depending on the choice of V.

Remark 3.2. *Here are some concrete examples taken from [3, §1.2].*

1 *Let Ω be a bounded Lipschitz domain, $V = H^1_0(\Omega)$ or $V = H^1(\Omega)$ and let A be the linear second-order elliptic operator*

$$\langle Au, v\rangle = \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^n \int_\Omega b_i \frac{\partial u}{\partial x_i} v + \int_\Omega c_0 uv$$

with coefficients $a_{ij}, b_i, c_0 \in L^\infty(\Omega)$ such that for all $\xi \in \mathbb{R}^n$ and for some $C > 0$, $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq C|\xi|^2$ a.e., and $c_0 \geq \lambda > 0$ with λ a constant. The space X is

$$X := \begin{cases} \Omega : \text{ if } V = H^1_0(\Omega) \\ \overline{\Omega} : \text{ if } V = H^1(\Omega). \end{cases}$$

2 *Let Ω be the half space of $\mathbb{R}^d$ for $d \geq 2$, $A = -\Delta + \mathrm{Id}$ with $V = H^1(\Omega)$ and $X := \overline{\Omega}$. We could also have chosen $\Omega = X := \mathbb{R}^d$ for any $d \geq 1$.*

3 *Let $V = H^s(\Omega)$, $s \in (0,1)$, on a bounded Lipschitz domain Ω, where the classical fractional Sobolev space $H^s(\Omega)$ is defined as the subspace of $L^2(\Omega)$ with the following norm finite:*

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega u^2 + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\right)^{\frac{1}{2}}. \tag{14}$$

Set $\langle Au, v\rangle = (u,v)_{H^s(\Omega)}$. In this case, $X := \overline{\Omega}$. More details of fractional Sobolev spaces and fractional Laplace operators can be found in [57, 19].

4 *The singular integral definition of the fractional Laplacian for sufficiently smooth functions $u\colon \mathbb{R}^d \to \mathbb{R}$ is*

$$(-\Delta)^s u(x) := c\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y, \quad \text{where} \quad c = \frac{4^s\Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|},$$

again for $s \in (0,1)$. Pick $V = H^s(\mathbb{R}^d)$ (this space is defined through (14) *with the obvious modifications) and define the operator*

$$\langle Au, v\rangle := \int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \int_\Omega uv,$$

and here we choose $X := \mathbb{R}^d$.

When we talk about the *active set* or *coincidence set* of a solution y to a QVI related to an obstacle map Φ, we mean the set defined through

$$\mathcal{A}(y) := \{x \in X : y(x) = \Phi(y)(x)\} \quad \text{for } y \in V.$$

This set is quasi-closed and is defined up to sets of capacity zero. It is important to note that the set of points defining the active set is taken over X; in the context of the examples above, this can sometimes be $X = \overline{\Omega}$ and not merely Ω.

3.1 Differentiability for unsigned sources and directions

In this section, we shall fix an arbitrary $f \in V^*$ and take an arbitrary but fixed $y \in \mathbf{Q}(f)$[3]. Since we study differentiability of QVIs, we need some differentiability for the constraint set mapping. Recall the notation $B_R(y) \subset V$ to stand for the closed ball in V of radius R centred on u. We will henceforth assume that

$$\text{there exists } \epsilon > 0 \text{ s.t. } \Phi\colon V \to V \text{ is Hadamard directionally differentiable on } B_\epsilon(y). \tag{15}$$

Remark 3.3. *Note that this is an assumption on the differentiability of Φ on a small ball around the fixed element y, that is, it is a* local *assumption and we do not ask for it hold globally on the whole of V. We shall introduce more local assumptions in the course of the paper and one should bear in mind that these local assumptions are stated in terms of a fixed element y which, in later sections, needs to be modified appropriately (for example in §5 such assumptions should be evaluated at the function that we call y^*). This should become apparent from the context.*

[3]This is possible: for example, by Theorem 2.3 under certain assumptions.

Pick a direction $d \in V^*$ and construct, like in §2.1, the sequence

$$\begin{aligned} y_0^s &:= y, \\ y_n^s &:= S(f + sd, y_{n-1}^s). \end{aligned} \tag{16}$$

The idea here is to expand each y_n^s in terms of y, a directional derivative and a remainder term (both of these would depend on n) and then to pass to the limit in such an expansion. The natural way to proceed would be to obtain a uniform bound on $\{y_n^s\}$ which would result in the existence of a weakly convergent *subsequence* $\{y_{n_j}^s\}$. This is not enough to identify the limit of $\{y_{n_j}^s\}$ due to the $(n-1)$ term in the definition of y_n^s, so one would need convergence of the whole sequence which holds true when, for example, one has monotonicity. However, in contrast to the sequence considered in §2.1, we do not obtain any monotonicity of $\{y_n^s\}$ since we do not assume a sign on d. Therefore, for a convergence of the full sequence, we instead look for a contraction of the map associated to $\{y_n^s\}$ on some closed ball.

Lemma 3.4. *Assume that*

$$\exists \epsilon > 0 : \|\Phi'(z)(v)\|_V \leq C_\Phi \|v\|_V \quad \forall z \in B_\epsilon(y), \forall v \in V, \text{ where } C_\Phi < (1 + C_a^{-1}C_b)^{-1}. \tag{17}$$

Then for any $0 < R \leq \epsilon$, $S(f + sd, \cdot)\colon B_R(y) \to B_R(y)$ *is a contraction whenever*

$$s \leq C_a \|d\|_{V^*}^{-1} R(1 - (1 + C_b C_a^{-1})C_\Phi).$$

Proof. Let $v \in B_R(y)$; we want to show that $S(f + sd, v) \in B_R(y)$. Observe that, using $y = S(f, y)$, continuous dependence (eg. [3, Equation (21)]) and the mean value theorem [47, §2, Proposition 2.29],

$$\begin{aligned} \|S(f + sd, v) - y\|_V &\leq (1 + C_b C_a^{-1}) \sup_{\lambda \in (0,1)} \|\Phi'(\lambda v + (1-\lambda)y)(v - y)\|_V + C_a^{-1} s \|d\|_{V^*} \\ &\leq (1 + C_b C_a^{-1}) C_\Phi \|v - y\|_V + C_a^{-1} s \|d\|_{V^*} \\ &\qquad \text{(since } \lambda v + (1-\lambda)y \in B_R(y) \subset B_\epsilon(y)) \\ &\leq (1 + C_b C_a^{-1}) C_\Phi R + C_a^{-1} s \|d\|_{V^*}, \end{aligned}$$

and, using the fact that $(1 + C_b C_a^{-1})C_\Phi$ equals a constant strictly less than 1, the right-hand side is bounded above by R under the stated assumption. This shows that $S(f + sd, \cdot)$ maps $B_R(y)$ into itself.

To see that the map is a contraction, take $v, w \in B_R(y)$ and observe that

$$\begin{aligned} \|S(f + sd, v) - S(f + sd, w)\|_V &\leq (1 + C_a^{-1} C_b) \sup_{\lambda \in (0,1)} \|\Phi'(\lambda w + (1-\lambda)v)(w - v)\|_V \\ &\leq C_\Phi (1 + C_a^{-1} C_b) \|z_2 - z_1\|_V. \end{aligned}$$

□

Remark 3.5. *Observe that when* Φ *is linear, the boundedness condition* (4) *is implied by assumption* (17).

Under (17), we have that each $y_n^s \in B_R(y)$. By applying the Banach fixed point theorem, we obtain the following existence and convergence result.

Lemma 3.6. *Given* $f, d \in V^*$ *and* $y \in \mathbf{Q}(f)$*, under the assumptions of the previous lemma, there exists* $y^s \in \mathbf{Q}(f + sd) \cap B_R(y)$ *such that* $y_n^s \to y^s$ *in* V *(where* y_n^s *is defined in* (16)*).*

Now, making use of the differentiability result for VIs provided by Mignot [41, Theorem 3.3], we can expand $y_1^s = S(f + sd, y)$ as follows:

$$y_1^s = y + s\delta_1 + o_1(s),$$

where $s^{-1}o_1(s) \to 0$ as $s \to 0^+$ and $\delta_1 = \partial S(f, y)(d)$ is the directional derivative of $S(f, \cdot)$ in the direction d, and this satisfies the VI

$$\begin{aligned} \delta_1 \in \mathcal{K}^y : \quad & \langle A\delta_1 - d, \delta_1 - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y, \\ \mathcal{K}^y &:= \{w \in V : w \leq 0 \text{ q.e. on } \mathcal{A}(y) \text{ and } \langle Ay - f, w\rangle = 0\}. \end{aligned} \tag{18}$$

Here, 'q.e.' stands for quasi-everywhere and a statement holds quasi-everywhere if it holds everywhere except on a set of capacity zero. To acquire an expansion formula for a general y_n^s, define

$$\delta_n := \partial S(f, y)[d - A\Phi'(y)(\Phi'(y)[...\Phi'(y)[\Phi'(y)(\delta_0) + \delta_1] + \delta_2...] + \delta_{n-2}] + \delta_{n-1})] \quad \text{for } n > 1$$

and

$$\alpha_n := \begin{cases} \delta_1 & : \text{if } n = 1 \\ \Phi'(y)[\Phi'(y)[...\Phi'(y)[\Phi'(y)(\delta_1) + \delta_2] + \delta_3...] + \delta_{n-1}] + \delta_n & : \text{if } n \geq 2, \end{cases}$$

and observe the recursion formula

$$\alpha_n = \Phi'(y)[\alpha_{n-1}] + \delta_n \text{ for } n > 1. \tag{19}$$

In exactly the same way as in [3, Proposition 2], we obtain the following result (the proof is by induction and we omit it here).

Proposition 3.7. *Under the assumptions of the previous lemma, for $n \geq 1$,*

$$y_n^s = y + s\alpha_n + o_n(s) \tag{20}$$

where $\alpha_n = \alpha_n(d)$ is positively homogeneous in the direction d and satisfies the VI

$$\begin{aligned} &\alpha_n \in \mathcal{K}^y(\alpha_{n-1}) : \langle A\alpha_n - d, \alpha_n - \varphi\rangle \leq 0 \qquad \forall \varphi \in \mathcal{K}^y(\alpha_{n-1}), \\ &\quad \mathcal{K}^y(\alpha_{n-1}) := \{\varphi \in V : \varphi \leq \Phi'(y)(\alpha_{n-1}) \text{ q.e. on } \mathcal{A}(y) \text{ and } \langle Ay - f, \varphi - \Phi'(y)(\alpha_{n-1})\rangle = 0\}, \end{aligned}$$

with $s^{-1}o_n(s) \to 0$ as $s \to 0^+$.

It remains then to pass to the limit in (20) and to identify the corresponding limits. To this end, observe that $s\alpha_n + o_n(s) = y_n^s - y \to y^s - y$ in V. Assumption (17) provides the existence of a constant $c > 0$ such that

$$\|\Phi'(y)(v)\|_V \leq \frac{C_a - c}{C_b} \|v\|_V \,,$$

and thus the sequence $\{\alpha_n\}$ is bounded exactly as shown in the proof of [3, Theorem 6] and we have the existence of a subsequence $\{n_j\}$ with

$$\alpha_{n_j} \rightharpoonup \alpha \text{ in } V \qquad \text{and} \qquad o_{n_j}(s) \rightharpoonup o^*(s) \text{ in } V.$$

We can pass to the limit in (20) along this subsequence to obtain

$$y^s = y + s\alpha + o^*(s), \tag{21}$$

and it is left for us to show that o^* is a remainder term and to characterise α suitably. For this, we need some more notation. Let $S_0\colon V^* \to V$ be the map $f \mapsto u$ of the following VI with trivial lower obstacle:

$$u \in V_+ : \langle Au - f, u - v\rangle \leq 0 \quad \forall v \in V_+,$$

and denote the remainder term associated to the derivative formula of S_0 by $o(\cdot, \cdot\,; \cdot)$, that is,

$$o(s, h; f) := \frac{S_0(f + sh) - S_0(f) - sS_0'(f)(h)}{s}.$$

Similarly, we denote the remainder term associated to Φ by $l(\cdot, \cdot\,; \cdot)$.

The idea in [3] was to show that the convergence $s^{-1}o_n(s) \to 0$ as $s \to 0^+$ is *uniform* in n, which is sufficient to commute the limits $s \to 0^+$ and $n \to \infty$ for $s^{-1}o_n(s)$, giving the desired behaviour $s^{-1}o^*(s) \to 0$ as $s \to 0^+$. This unformity was shown by the derivation of the estimate (see [3, Lemma 14])

$$\begin{aligned}
\|o_n(s)\|_V \leq C^{n-1}\|o_1(s)\|_V &+ C^{n-2}\big((1 + C_a^{-1}C_b)\|l(s, \alpha_1; y)\|_V \\
&\qquad\qquad + \|o(s, A\Phi'(y)(\alpha_1) - d; A\Phi(y) - f)\|_V\big) \\
&+ C^{n-3}\big((1 + C_a^{-1}C_b)\|l(s, \alpha_2; y)\|_V + \|o(s, A\Phi'(y)(\alpha_2) - d; A\Phi(y) - f)\|_V\big) \\
&+ \ldots + (1 + C_a^{-1}C_b)\|l(s, \alpha_{n-1}; y)\|_V + \|o(s, A\Phi'(y)(\alpha_{n-1}) - d; A\Phi(y) - f)\|_V
\end{aligned}$$

for a constant $C < 1$, and then the following quantity was shown to vanish uniformly in the limit $s \to 0^+$:

$$(1 + C_a^{-1}C_b)\frac{\|l(s, \alpha_n; y)\|_V}{s} + \frac{\|o(s, A\Phi'(y)(\alpha_n) - d; A\Phi(y) - f)\|_V}{s}.$$

In the setting of [3] (where $f, d \in V_+^*$), this indeed converges to zero uniformly in n because by [3, Lemma 12], $\alpha_n \to \alpha$ in V for the whole sequence and thus the Hadamard differentiability (and hence compact differentiability, see [55, Proposition 3.3]) of Φ and S_0 directly gives the uniform convergence. This argument is not directly applicable in our setting because we do not have convergence of the whole sequence $\{\alpha_n\}$ nor $\{A\Phi'(y)(\alpha_n)\}$ (we merely know that a subsequence converges). This means that these sets are no longer guaranteed to be embedded into compact sets and so the compact differentiability of Φ is no longer of help. Thus, we need a strengthening of the Hadamard differentiability assumption and the right notion in this setting is that of **bounded directional differentiability**.

Definition 3.8. A map $T\colon X \to Y$ between Banach spaces is said to be *boundedly directionally differentiable* at $x \in X$ if there exists a positively homogeneous map $T'(x)\colon X \to Y$ such that

$$\lim_{s\to 0^+} \frac{T(x + sh) - T(x) - sT'(x)(h)}{s} = 0 \quad \text{uniformly in } h \text{ on bounded subsets of } Y.$$

Fréchet differentiable operators are boundedly directionally differentiable. See [55, §2] for more details and further references. Now we adapt the proof of [3, Lemma 14] under this context.

Proposition 3.9. *Assume* (15), (17), *and*

$$\Phi\colon V \to V \text{ is boundedly directionally differentiable at } y, \tag{22}$$

$$\Phi'(y)\colon V \to V \text{ is completely continuous.} \tag{23}$$

Then $s^{-1}o^*(s) \to 0$ *as* $s \to 0$.

Proof. Define

$a_n(s) := \|o_n(s)\|_V$ and $b_n(s) := (1+C_a^{-1}C_b)\,\|l(s,\alpha_n;y)\|_V + \|o(s,A\Phi'(y)(\alpha_n);A\Phi(y)-f)\|_V\,.$

From the proof of [3, Lemma 14], we see that a_n satisfies the following recurrence inequality:

$$a_n(s) \le Ca_{n-1}(s) + b_{n-1}(s),$$

where the constant $C < 1$ by the assumption on C_Φ in (17). This implies

$$a_n(s) \le C^{n-1}a_1(s) + C^{n-2}b_1(s) + C^{n-3}b_2(s) + ... + Cb_{n-2}(s) + b_{n-1}(s). \tag{24}$$

Consider

$$\frac{b_{n-1}(s)}{s} = \frac{(1+C_a^{-1}C_b)\,\|l(s,\alpha_{n-1};y)\|_V}{s} + \frac{\|o(s,A\Phi'(y)(\alpha_{n-1});A\Phi(y)-f)\|_V}{s}.$$

Since $\{\alpha_n\}$ is bounded, we know that the first term on the right-hand side converges to zero uniformly in n by definition of Φ being boundedly directionally differentiable at y. The compactness of $\Phi'(y)(\cdot)\colon V \to V$ implies that $A\Phi'(y)(\cdot)\colon V \to V^*$ is compact. By definition, the image of a bounded set under a compact map is relatively compact, meaning that $\overline{\{A\Phi'(y)(\alpha_{n-1})\}}$ is a compact set in V^*. Since the remainder term o above arises from the Hadamard (and hence compact) differentiability of the solution map associated to VIs, it follows that $o(s,h)/s \to 0$ uniformly for h belonging to the compact set $\overline{\{A\Phi'(y)(\alpha_{n-1})\}}$. Because $\{A\Phi'(y)(\alpha_{n-1})\} \subset \overline{\{A\Phi'(y)(\alpha_{n-1})\}}$, we have that

$$\frac{o(s,h;A\Phi(y)-f)}{s} \to 0 \quad \text{uniformly in } h \in \{A\Phi'(y)(\alpha_{n-1})\}$$

which then gives

$$\frac{b_{n-1}(s)}{s} \to 0 \quad \text{uniformly in } n.$$

These facts along with (24) imply that for every $\epsilon > 0$, there exists an s_0 independent of n such that

$$\frac{\|o_n(s)\|_V}{s} \le \epsilon \quad \text{when } s \le s_0$$

which means precisely that $s^{-1}o_n(s) \to 0$ as $s \to 0^+$ uniformly in n. Finally, using the weak convergence of the subsequence o_{n_j}, taking the liminf as $n_j \to \infty$ and using the weak lower semicontinuity of norms in the above inequality for $n = n_j$, we deduce that $s^{-1}o^*(s) \to 0$ as $s \to 0^+$. □

Remark 3.10. *Assumptions* (15)*,* (6)*,* (23)*,* (17) *are related to assumptions (A1), (A2), (A3), (A5) of the paper [3].*

As a byproduct of the above result, we find that the whole sequence $\{\alpha_n\}$ indeed converges.

Lemma 3.11. *Under the assumptions of the previous proposition, $\alpha_n \to \alpha$ in V (for the whole sequence).*

Proof. Consider the difference quotient

$$r_n(s) := \alpha_n + \frac{o_n(s)}{s} = \frac{y_n^s - y}{s}$$

which, thanks to the strong convergence of y_n^s and (21), is such that

$$\lim_{n\to\infty} r_n(s) = \frac{y^s - y}{s} = \alpha + \frac{o^*(s)}{s}$$

(this limit and the ones below are all taken in V). We claim that

$$\lim_{s\to 0^+} r_n(s) = \alpha_n \quad \text{uniformly in } n.$$

This follows because the quantity $r_n(s) - \alpha_n = o_n(s)/s$ converges to zero as $s \to 0^+$ uniformly in n as we have seen in the proof of Proposition 3.9, and the Moore–Osgood theorem [22, §I.7, Lemma 6] then applies, giving the existence of iterated limits as well as commutability and we get

$$\alpha = \lim_{s\to 0^+}\left(\alpha + \frac{o^*(s)}{s}\right) = \lim_{s\to 0^+}\lim_{n\to\infty} r_n(s) = \lim_{n\to\infty}\lim_{s\to 0^+} r_n(s) = \lim_{n\to\infty} \alpha_n$$

with the first equality thanks to Proposition 3.9. □

This strong convergence opens the door for the characterisation of the directional derivative as in [3] — namely, it allows us to pass to the limit in the recurrence formula (19) which involves the terms α_n and α_{n-1} (for which arguments using convergences of subsequences would not be viable). See §5.1 and §5.2 in [3] for more details. Finally, we obtain the following theorem.

Theorem 3.12. *Given $f \in V^*$ and $d \in V^*$, for every $y \in \mathbf{Q}(f)$, under assumption* (6)*, the local assumptions* (15)*,* (17)*,* (22)*,* (23)*, and Assumption 3.1, there exists $y^s \in \mathbf{Q}(f + sd) \cap B_R(y)$ (where $0 < R \leq \epsilon$) and $\alpha = \alpha(d) \in V$ such that*

$$y^s = y + s\alpha + o(s)$$

holds where $s^{-1}o(s) \to 0$ as $s \to 0^+$ in V and α satisfies the QVI

$$\begin{aligned} &\alpha \in \mathcal{K}^y(\alpha) : \langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y(\alpha),\\ &\qquad \mathcal{K}^y(w) := \{\varphi \in V : \varphi \leq \Phi'(y)(w) \text{ q.e. on } \mathcal{A}(y) \text{ and } \langle Ay - f, \varphi - \Phi'(y)(w)\rangle = 0\}. \end{aligned} \tag{25}$$

The directional derivative $\alpha = \alpha(d)$ is positively homogeneous in d.

Furthermore, if $d \in V_+^*$ or $-d \in V_+^*$, (22) *can be omitted.*

Proof. The proof has been sketched above and we detail here the final claim. Indeed, supposing $d \geq 0$, we easily obtain $y_{n+1}^s \geq y_n$ which directly implies that $\alpha_{n+1} \geq \alpha_n$, leading to $\alpha_n \rightharpoonup \alpha$ for the full sequence. This fact then implies (along the same lines as [3, Lemma 5.4]) that $\alpha_n \to \alpha$ in V (we get this long before Lemma 3.11, which becomes superfluous) so that $\{\alpha_n\}$ belongs to a compact set in V and the Hadamard differentiability for Φ is enough for Proposition 3.9. If instead $d \leq 0$, the inequalities above are merely flipped. □

In the theorem, the existence of a particular $y \in \mathbf{Q}(f)$ is *assumed*; conditions under $\mathbf{Q}(f)$ is non-empty were given in the existence results of §2. Observe that the theorem generalises the result of Theorem 1.6 in [3].

We now look for an analogue of the complementarity characterisation of Proposition 2.1 for the QVI (25) satisfied by the directional derivative. First, recall $\mathcal{K}^y$ from (18) and that the *polar cone* of a set $M \subset V$ is defined

$$M^\circ = \{g \in V^* : \langle g, v\rangle \leq 0 \quad \forall v \in M\}.$$

Proposition 3.13. *The QVI* (25) *is equivalent to the complementarity system*

$$\alpha - \Phi'(y)(\alpha) \in \mathcal{K}^y,$$
$$\xi_d = d - A\alpha,$$
$$\xi_d \in (\mathcal{K}^y)^\circ,$$
$$\langle \xi_d, \Phi'(y)(\alpha) - \alpha \rangle = 0.$$

Proof. Observe that $\alpha - \Phi'(y)(\alpha)$ belongs to the set $\mathcal{K}^y$. Define $\xi_d := d - A\alpha$ which by definition satisfies

$$\alpha - \Phi'(y)(\alpha) \in \mathcal{K}^y : \langle \xi_d, \alpha - v \rangle \geq 0 \quad \forall v \in V : v - \Phi'(y)(\alpha) \in \mathcal{K}^y.$$

Taking $v = \Phi'(y)(\alpha)$ here and then $v = 2\alpha - \Phi'(y)(\alpha)$ (which is feasible since $v - \Phi'(y)(\alpha)$ is twice a function that belongs to $\mathcal{K}^y$) shows the orthogonality condition.

Let $w \in \mathcal{K}^y$ and select $v = \alpha + w$ (this is feasible since $v - \Phi'(y)(\alpha) = \alpha - \Phi'(y)(\alpha) + w \in \mathcal{K}^y + \mathcal{K}^y$ and the critical cone is closed under addition). With this choice, we obtain

$$\langle \xi_d, w \rangle \leq 0 \quad \forall w \in \mathcal{K}^y,$$

meaning precisely that $\xi_d \in (\mathcal{K}^y)^\circ$. □

3.2 Differentiability for sources and directions bounded from below

We now prove differentiability using different assumptions. Namely we consider source and direction terms that are bounded from below by a negative functional and we drop the bounded differentiability assumption on Φ, like in §2.2. By making a transformation, we will rewrite the QVI as another QVI involving non-negative source and direction to which we directly apply [3]. The advantage of this approach in contrast to the previous section is that, as mentioned, Hadamard differentiability is sufficient.

Indeed, like in §2.2, given a lower bound functional $G \in V^*_+$, take a source term $f \geq -G$ and fix $w \in \mathbf{H}_1(f + G)$ where we recall that the notation $\mathbf{H}$ was defined in Theorem 2.6. In addition to assumption (6), we need the following. Assume that

$$\text{(15) holds with } y \text{ replaced with } w - A^{-1}G. \tag{26}$$

$$\text{(17) holds with } y \text{ replaced with } w - A^{-1}G. \tag{27}$$

$$\text{(23) holds with } y \text{ replaced with } w - A^{-1}G. \tag{28}$$

These hypotheses imply that Ψ_1 satisfies the assumptions made on the obstacle map in Theorem 1 of [3].

Remark 3.14. *Strictly speaking, in [3], instead of* (15) *or* (26) *we assumed the stronger condition that $\Phi\colon V \to V$ is (globally) Hadamard differentiable. That, however, is not necessary as the local condition* (15) *or* (26) *suffices upon inspection of the proofs in [3].*

Take now a direction $d \in V^*$ such that $d \geq -G$; by [3, Theorem 1], we know that there exists a $w_s \in \mathbf{H}_1(f + G + s(d + G)) \cap [w, A^{-1}(f + G + s(d + G))]$ such that

$$w_s = w + s\beta + o(s)$$

where o is a remainder term and $\beta = \beta(f+G; d+G)$ is a directional derivative satisfying

$$\beta \in \mathcal{K}^w_{\Psi_1}(\beta) : \langle A\beta - (d+G), \beta - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^w_{\Psi_1}(\beta),$$
$$\mathcal{K}^w_{\Psi_1}(v) := \{\varphi \in V : \varphi \leq \Psi_1'(w)(v) \text{ q.e. on } \mathcal{A}_{\Psi_1}(w) \text{ and } \langle Aw - (f+G), \varphi - \Psi_1'(w)(v)\rangle = 0\}.$$

Here the set $\mathcal{A}_{\Psi_1}(w) = \{w = \Psi_1(w)\}$ is the active set associated to the obstacle map Ψ_1 and we used the fact that $\Psi_r'(w)(d) = \Phi'(w - A^{-1}(rG))(d)$.

We know that since $w \in \mathbf{H}_1(f+G)$ and $w_s \in \mathbf{H}_1(f+sd+(1+s)G)$, by Theorem 2.6,

$$y := w - A^{-1}G \in \mathbf{Q}(f) \qquad \text{and} \qquad y_s := w_s - A^{-1}(1+s)G \in \mathbf{Q}(f+sd),$$

and we have, using the above expansion formula and the linearity of A^{-1},

$$y_s = y + s(\beta - A^{-1}G) + o(s).$$

This gives us a differentiability formula for source terms and directions satisfying $f, h \geq -G$.

Theorem 3.15. *Let $f, d \in V^*$ with $f \geq -G$ a source term and let $d \geq -G$ be a direction. For any $y \in \mathbf{Q}(f) \cap [-A^{-1}G, A^{-1}f]$, under assumptions* (7), (6)*, the local assumptions* (26), (27), (28)*, and Assumption 3.1, there exists $y_s \in \mathbf{Q}(f+sd) \cap [y - sA^{-1}G, A^{-1}(f+sd)]$ and $\alpha = \alpha(d) \in V$ such that*

$$y_s = y + s\alpha + o(s)$$

holds where $s^{-1}o(s) \to 0$ as $s \to 0^+$ in V and α satisfies the QVI

$$\alpha \in \mathcal{K}^y_G(\alpha) : \langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y_G(\alpha),$$
$$\mathcal{K}^y_G(v) := \{\varphi \in V : \varphi \leq \Phi'(y)(v + A^{-1}G) - A^{-1}G \text{ q.e. on } \mathcal{A}(y) \text{ and}$$
$$\langle Ay - f, \varphi + A^{-1}G - \Phi'(y)(v + A^{-1}G)\rangle = 0\}. \tag{29}$$

Naturally, when $G \equiv 0$ we recover the results of [3]. The constraint set above depends on the function G that was used to set up the transformation; this is natural since the 'base' function y and the perturbation y_s also depend on G. Note the identity

$$\mathcal{K}^y_G(\alpha) = \mathcal{K}^y(\alpha + A^{-1}G) - A^{-1}G.$$

We now state a complementarity characterisation for the QVI satisfied by α. The proof is similar to that of Proposition 3.13 and is omitted.

Proposition 3.16. *The QVI* (29) *is equivalent to the complementarity system*

$$\begin{aligned}
&\alpha + A^{-1}G - \Phi'(y)(\alpha + A^{-1}G) \in \mathcal{K}^y,\\
&\xi_d = d - A\alpha,\\
&\xi_d \in (\mathcal{K}^y)^\circ,\\
&\langle \xi_d, \Phi'(y)(\alpha + A^{-1}G) - \alpha - A^{-1}G\rangle = 0.
\end{aligned}$$

3.3 Continuity properties of the directional derivative

We now study the conditions under which continuity of the map taking the direction d into the directional derivative α in (25) and (29) is assured. In the next lemma, note that we do not require (27) since we only need boundedness of $\Phi'(y)$.

Lemma 3.17. *Let the local assumptions* (17) *and* (23) *(or* (27) *and* (28)*) hold. If $d_n \to d$ in V^*, then there exists a subsequence $\{n_j\}$ and solutions α_{n_j} of the QVI* (25) *(or* (29)*) with source term d_{n_j} such that*

$$\alpha_{n_j} \to \alpha \quad \textit{in}\, V$$

where α is a solution of (25) *(or* (29)*) with source term d.*

Proof. We again prove just the case for the QVI (25) obtained in §3. First observe that (17) implies

$$\|\Phi'(y)(v)\|_V \le C_\Phi \|v\|_V \quad \text{where } C_\Phi < (1 + C_a^{-1}C_b)^{-1}.$$

The derivative α_n associated to d_n satisfies

$$\alpha_n \in \mathcal{K}^y(\alpha_n) : \langle A\alpha_n - d_n, \alpha_n - v\rangle \le 0 \quad \forall v \in \mathcal{K}^y(\alpha_n).$$

We choose $v = \Phi'(y)(\alpha_n)$ as a test function. This leads to

$$\begin{aligned} C_a \|\alpha_n\|_V^2 &\le \|d_n\|_{V^*} \|\alpha_n\|_V + (C_b \|\alpha_n\|_V + \|d_n\|_{V^*}) \|\Phi'(y)(\alpha_n)\|_V \\ &\le C \|\alpha_n\|_V + (C_b \|\alpha_n\|_V + C)C_\Phi \|\alpha_n\|_V , \end{aligned}$$

and we see that since $C_a - C_b C_\Phi$ is strictly positive, $\{\alpha_n\}$ is bounded and we obtain, for a subsequence,

$$\alpha_n \rightharpoonup \alpha \quad \text{in } V$$

for some $\alpha \in V$ that we need to identify. We first prove that the above convergence is also strong. Indeed, take $n, m \in \mathbb{N}$ and in the inequality for α_n, take the test function $v_n = \alpha_m - \Phi'(y)(\alpha_m) + \Phi'(y)(\alpha_n)$ which is clearly feasible, whilst in the inequality for α_m, set $v = \alpha_n - \Phi'(y)(\alpha_n) + \Phi'(y)(\alpha_m)$ to obtain

$$\begin{aligned} \langle A\alpha_n - d_n, \alpha_n - \alpha_m + \Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\rangle \le 0, \\ \langle A\alpha_m - d_m, \alpha_m - \alpha_n + \Phi'(y)(\alpha_n) - \Phi'(y)(\alpha_m)\rangle \le 0. \end{aligned}$$

Adding these inequalities, we find

$$\langle A(\alpha_n - \alpha_m) - (d_n - d_m), \alpha_n - \alpha_m + \Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\rangle \le 0,$$

which implies

$$\begin{aligned} C_a \|\alpha_n - \alpha_m\|_V^2 &\le \langle (d_n - d_m), \alpha_n - \alpha_m\rangle + \langle A(\alpha_m - \alpha_n) - (d_m - d_n), \Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\rangle \\ &\le \|d_n - d_m\|_{V^*} \|\alpha_n - \alpha_m\|_V + C_b \|\alpha_m - \alpha_n\|_V \|\Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\|_V \\ &\quad + \|d_m - d_n\|_{V^*} \|\Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\|_V \\ &\le C \|d_n - d_m\|_{V^*} + C_b C \|\Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\|_V \\ &\quad + \|d_m - d_n\|_{V^*} \|\Phi'(y)(\alpha_m) - \Phi'(y)(\alpha_n)\|_V , \end{aligned}$$

and this tends to zero since $\{d_n\}$ is a Cauchy sequence in V^* and the weak convergence of $\{\alpha_n\}$ in V implies the strong convergence of $\{\Phi'(y)(\alpha_n)\}$ in V. Hence $\{\alpha_n\}$ is a Cauchy sequence and we indeed have (still for a subsequence) the strong convergence

$$\alpha_n \to \alpha \quad \text{in } V.$$

Now, in the inequality for α_n, choose the test function $v_n := v - \Phi'(y)(\alpha) + \Phi'(y)(\alpha_n)$ where v is such that $v \in \mathcal{K}^y(\alpha)$. It follows that $v_n \to v$ in V. This allows us to pass to the limit and we get

$$\langle A\alpha - d, \alpha - v\rangle \leq 0 \quad \forall v \in \mathcal{K}^y(\alpha)$$

and it remains to be seen that $\alpha \in \mathcal{K}^y(\alpha)$.

This is easy to do: the strong convergence of α_n in V implies that $\alpha_n \to \alpha$ pointwise q.e. and we know that $\alpha_n - \Phi'(y)(\alpha_n) \leq 0$ on $\{y = \Phi(y)\} \setminus A_n$ where A_n is a set of capacity zero. Utilising the fact that a countable union of sets of capacity zero has capacity zero, we find $\alpha - \Phi'(y)(\alpha) \leq 0$ q.e. on $\{y = \Phi(y)\}$. This shows that α solves the desired QVI. □

The continuity result in the next proposition strengthens the previous lemma and is crucial for several results that we need in §5.3 for strong stationarity.

Proposition 3.18. *Suppose that*

$$\Phi'(y)\colon V \to V \text{ is Lipschitz with Lipschitz constant } C_L \text{ satisfying } C_L < C_a/C_b. \tag{30}$$

Then solutions to the QVIs (25) *and* (29) *are unique and furthermore, under also the assumptions of the previous lemma,* $d \mapsto \alpha(d)$ *is continuous from* V^* *to* V*.*

Proof. Consider two solutions of (25):

$$\alpha_1 \in \mathcal{K}^y(\alpha_1) : \langle A\alpha_1 - d, \alpha_1 - v_1\rangle \leq 0 \quad \forall v_1 \in \mathcal{K}^y(\alpha_1),$$
$$\alpha_2 \in \mathcal{K}^y(\alpha_2) : \langle A\alpha_2 - d, \alpha_2 - v_2\rangle \leq 0 \quad \forall v_2 \in \mathcal{K}^y(\alpha_2).$$

Take $v_1 = \alpha_2 - \Phi'(y)(\alpha_2) + \Phi'(y)(\alpha_1)$ and a similar ansatz for v_2 and we end up with

$$\langle A\alpha_1 - d, \alpha_1 - \alpha_2 - \Phi'(y)(\alpha_1) + \Phi'(y)(\alpha_2)\rangle \leq 0,$$
$$\langle A\alpha_2 - d, \alpha_2 - \alpha_1 - \Phi'(y)(\alpha_2) + \Phi'(y)(\alpha_1)\rangle \leq 0.$$

Adding and manipulating leads to

$$C_a \|\alpha_1 - \alpha_2\|_V^2 \leq \langle A(\alpha_1 - \alpha_2), \Phi'(y)(\alpha_1) - \Phi'(y)(\alpha_2)\rangle \leq C_b C_L \|\alpha_1 - \alpha_2\|_V^2,$$

which gives $\alpha_1 = \alpha_2$ under the assumption of the lemma. The continuity is a result of applying the subsequence principle to the result of the previous lemma to deduce that the whole sequence converges (since the limiting inequality has a unique solution). The same argument with the correct modifications proves the result for (29). □

4 Existence of optimal controls

We now address the optimal control problem (2). The function space context requires $V \subset H \subset V^*$ to be a Gelfand triple of Hilbert spaces and $U \hookrightarrow H$ to be a given Hilbert space. Given a desired state $y_d \in H$, define $J\colon H \times U \to \mathbb{R}$ by

$$J(y, u) := \frac{1}{2}\|y - y_d\|_H^2 + \frac{\nu}{2}\|u\|_U^2,$$

and with $U_{ad} \subseteq U$ not necessarily bounded, we consider the problem (2) which we recall here:

$$\min_{\substack{u \in U_{ad}\\ y \in \mathbf{Q}(u)}} J(y, u).$$

Theorem 4.1. *Let $U \overset{c}{\hookrightarrow} V^*$ and let $U_{ad} \subset U$ be a weakly sequentially closed[4] set and let* (5) *and* (6) *hold. Then there exists an optimal control $u^* \in U_{ad}$ and associated state $y^* \in \mathbf{Q}(u^*)$ to the problem* (2)*.*

Proof. Let $u_n \in U_{ad}$ be an infimising sequence with $y_n \in \mathbf{Q}(u_n)$ (this exists by Theorem 2.3), i.e.,

$$J(y_n, u_n) \to \inf_{\substack{u \in U_{ad}, \\ y \in \mathbf{Q}(u)}} J(y, u).$$

Then u_n and y_n are bounded in U and V respectively (the latter arises from (5)) and therefore, there exists a subsequence such that

$$\begin{aligned} u_{n_j} &\rightharpoonup u^* \quad \text{in } U, \\ y_{n_j} &\rightharpoonup y^* \quad \text{in } V. \end{aligned}$$

By assumption, u^* also belongs to U_{ad}. Since the y_n are solutions of QVIs, we have the following estimate

$$\left\|y_{n_j} - y_{n_k}\right\|_V \leq C \left(\left\|u_{n_j} - u_{n_k}\right\|_{V^*} + \left\|\Phi(y_{n_j}) - \Phi(y_{n_k})\right\|_V\right).$$

In the limit, the first term on the right-hand side vanishes due to the compact embedding, and the second term vanishes too because Φ is completely continuous. Thus $\{y_{n_j}\}$ is Cauchy in V and it has a strong limit in V to y^*. Taking an arbitrary $v \in V$ such that $v \leq \Phi(y^*)$, we set $v_{n_j} := v - \Phi(y^*) + \Phi(y_{n_j})$ and use this as a test function in the QVI for y_{n_j} in which we can pass to the limit and find $y^* \in \mathbf{Q}(u^*)$. To see that this pair is optimal, we observe that (dispensing with the subsequence notation now)

$$J(y^*, u^*) \leq \liminf_{n \to \infty} J(y_n, u_n) \leq \lim_{n \to \infty} J(y_n, u_n) = \min_{\substack{u \in U_{ad} \\ y \in \mathbf{Q}(u)}} J(y, u).$$

□

Regarding regularity of the optimal control, see Theorem 5.4. In general there is no uniqueness for the optimal control and state regardless of whether $\mathbf{Q}$ is single valued or not.

Remark 4.2. *For optimal control problems with more general quasi-variational constraints one might need to assume Mosco convergence properties of the constraint sets. Given the structure of our unilaterally constrained compliant obstacle problem, this is obtained due in part to the complete continuity assumption on Φ.*

For the rest of the paper, we will just take $U \equiv H$ for simplicity.

5 Stationarity

In this section, we shall derive three forms of necessary conditions satisfied by optimal controls and states. Let us first define some concepts of stationarity which are motivated by analogous concepts from the VI case and also by the results that we shall obtain later.

[4]That is, if $u_n \rightharpoonup u$ in U with $u_n \in U_{ad}$, then $u \in U_{ad}$.

Let (y, u) be a solution of (2). Certain sets associated to the lower-level QVI problem in (2) are important in stating stationarity conditions. Recalling the notation ξ in Proposition 2.1, let us formally define then the following sets:

$$\begin{aligned}\mathcal{A} &:= \{y = \Phi(y)\} \text{ is the \textit{active} (or coincidence) set,}\\ \mathcal{I} &:= \{y < \Phi(y)\} \text{ is the \textit{inactive} set,}\\ \mathcal{A}_s &:= \{\xi > 0\} \text{ is the \textit{strongly active} set,}\\ \mathcal{B} &:= \{y = \Phi(y)\} \cap \{\xi = 0\} \text{ is the \textit{biactive} set.}\end{aligned}$$

These definitions are merely heuristic due to the low regularity of ξ, see for example [26, Definitions 2.4.1 and 2.4.2] for a rigorous way to define these objects.

We say that $(y, u) \in V \times H$ is a *C-stationarity* point of (2) if (y, u) is a solution of (2) and there exists $(p, \xi, \lambda) \in V \times V^* \times V^*$ such that

$$y + \lambda + A^* p = y_d, \tag{31a}$$

$$Ay - u + \xi = 0, \tag{31b}$$

$$u \in U_{ad} : (\nu u - p, u - v) \leq 0 \quad \forall v \in U_{ad}, \tag{31c}$$

$$\xi \geq 0 \text{ in } V^*, \quad y \leq \Phi(y), \quad \langle \xi, y - \Phi(y)\rangle = 0, \tag{31d}$$

$$\langle \xi, p^+\rangle = \langle \xi, p^-\rangle = 0 \tag{31e}$$

$$\langle \lambda, p\rangle \geq 0, \quad \langle \lambda, y - \Phi(y)\rangle = 0, \tag{31f}$$

$$\langle \lambda, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ on } \Omega \setminus \mathcal{I}. \tag{31g}$$

The function p is said to be the *adjoint state* and λ is the *Lagrange multiplier* associated to the adjoint state equation (31a). Note that we use the condition (31e) in lieu of the more commonly seen condition

$$p = 0 \text{ a.e. in } \{\xi > 0\}$$

due to the low regularity of ξ.

The condition (31g) is in practice difficult to check due to the fact that in general, λ possesses only the low V^* regularity. Therefore, one looks for a weaker concept. In the first instance, for an *almost C-stationarity* point, (31g) is replaced by

$$\langle \lambda, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ on } \Omega \setminus \mathcal{I},\ v|_{\mathcal{I}} \in H^1_0(\mathcal{I}).$$

More generally, an $\mathcal{E}$*-almost C-stationarity* point, the concept of which was introduced by Hintermüller and Kopacka in [31, 30], satisfies (31a)–(31f) but now (31g) is replaced with

$$\forall \tau > 0, \exists E^\tau \subset \mathcal{I} \text{ with } |\mathcal{I} \setminus E^\tau| \leq \tau : \langle \lambda, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ on } \Omega \setminus E^\tau.$$

This is a condition that arises from an application of Egorov's theorem as we shall see later.

Now, in the other direction, a point which satisfies (31a)–(31d) and additionally

$$\begin{aligned} p \geq 0 &\quad \text{q.e. on } \mathcal{B} \text{ and } p = 0 \text{ q.e. on } \mathcal{A}_s,\\ \langle \lambda, v\rangle \geq 0 &\quad \forall v \in V : v \geq 0 \text{ q.e. on } \mathcal{B} \text{ and } v = 0 \text{ q.e. on } \mathcal{A}_s,\end{aligned}$$

is called a *strong stationarity* point, which is typically the most stringent notion of stationarity possible and requires differentiability of the control-to-state map to be obtainable.

In the proceeding sections, we will show that there exist $\mathcal{E}-$almost C-stationarity and strong stationarity points under various assumptions. We will, however, first start in §5.1 with the so-called *Bouligand stationarity* which is a primal condition and is defined below. It also requires differentiability of $\mathbf{Q}$.

For sections §5.2–5.3, we will work in the setting of $V = H_0^1(\Omega)$ with $H = L^2(\Omega)$ (as before) and we take U_{ad} to have the form

$$U_{ad} = \{u \in H : u_a \leq u \leq u_b \text{ a.e. in } \Omega\} \tag{32}$$

for given functions $u_a, u_b \in H$.

5.1 Bouligand stationarity

In the case where $\mathbf{Q}$ is directionally differentiable from the results of §3, we have the following Bouligand stationarity (or B-stationarity) characterisation of the optimal control, see [41, §5] and [42, Lemma 3.1] for the VI case. To start, define the *radial cone* of U_{ad} at u^* by

$$\mathcal{R}_{U_{ad}}(u^*) = \{h \in H : \exists s^* > 0 \text{ s.t. } u^* + sh \in U_{ad} \quad \forall s \in [0, s^*]\},$$

and the *tangent cone* by

$$\mathcal{T}_{U_{ad}}(u^*) := \overline{\mathcal{R}_{U_{ad}}(u^*)}.$$

Proposition 5.1 (Bouligand stationarity). *Let (u^*, y^*) be a minimiser of* (2) *and let the local assumptions* (17)*,* (23) *and* (30) *hold at y^*. If the assumptions of Theorem 3.12 hold, then*

$$(\alpha_h, y^* - y_d) + \nu(u^*, h) \geq 0 \quad \forall h \in \mathcal{T}_{U_{ad}}(u^*), \tag{33}$$

whereas if instead the assumptions of Theorem 3.15 hold, the above inequality holds for all

$$h \in \overline{\mathcal{R}_{U_{ad}}(u^*) \cap (H_+ - G)}.$$

The term α_h above is the directional derivative associated to the perturbation $y^s \in \mathbf{Q}(u^* + sh)$ given through Theorem 3.12 or Theorem 3.15 respectively.

Proof. Consider the case in which Theorem 3.12 is applicable. By definition of the minimiser, we have $J(y_s, u^* + sh) \geq J(y^*, u^*)$ for any admissible direction h, $s \geq 0$ and any $y_s \in \mathbf{Q}(u^* + sh)$, and it is clear that every h in the radial cone of U_{ad} at u^* is an admissible direction. Writing this inequality out, we get

$$\begin{aligned} 0 &\leq \|y_s - y_d\|_H^2 + \nu \|u^* + sh\|_U^2 - \|y^* - y_d\|_H^2 - \nu \|u^*\|_U^2 \\ &= \|y_s\|_H^2 - \|y^*\|_H^2 + 2(y^* - y_s, y_d) + \nu s^2 \|h\|_U^2 + 2\nu s(u^*, h). \end{aligned}$$

We select y_s as given by Theorem 3.12 after having initially selected $y^* \in \mathbf{Q}(u^*)$, which satisfies $y_s = y^* + s\alpha_h + o(s)$ where α_h is the directional derivative (uniquely determined thanks to Proposition 3.18) and o is a remainder term. This leads to

$$\begin{aligned} 0 &\leq \|y^* + s\alpha_h + o(s)\|_H^2 - \|y^*\|_H^2 - 2(s\alpha_h + o(s), y_d) + \nu s^2 \|h\|_U^2 + 2\nu s(u^*, h) \\ &= \|s\alpha_h + o(s)\|_H^2 + 2(s\alpha_h + o(s), y^* - y_d) + \nu s^2 \|h\|_U^2 + 2\nu s(u^*, h) \\ &= s^2 \left\|\alpha_h + s^{-1}o(s)\right\|_H^2 + 2(s\alpha_h + o(s), y^* - y_d) + \nu s^2 \|h\|_U^2 + 2\nu s(u^*, h). \end{aligned}$$

Dividing by s and sending to zero, the above yields

$$0 \leq 2(\alpha_h, y^* - y_d) + 2\nu(u^*, h) \quad \forall h \in \mathcal{R}_{U_{ad}}(u^*),$$

and by density and the continuity result of Proposition 3.18, also for $h \in \mathcal{T}_{U_{ad}}(u^*)$.

In the setting where Theorem 3.15 is applied, the above displayed inequality only holds for all $h \in \mathcal{R}_{U_{ad}}(u^*) \cap (H_+ - G)$ since we need the direction to be bounded from below by a function G. Then taking the closure in H again leads to the stated inequality. □

5.2 $\mathcal{E}$-almost C-stationarity

As we specified above, we shall take $V = H_0^1(\Omega)$ and U_{ad} as in (32) from now on. In this section we will show $\mathcal{E}$-almost C-stationarity for the optimal pair by passing to the limit in the stationarity system satisfied by the optimal pair of the PDE regularisation of the QVI. Recall the notations and framework of §2.3 where we studied the convergence of solutions of certain PDEs to a solution of the associated QVI. Consider for each $\rho > 0$ the penalisation of (2):

$$\min_{u \in U_{ad}} J(y_\rho, u) \quad \text{s.t.} \quad Ay_\rho + \frac{1}{\rho} m_\rho(y_\rho - \Phi(y_\rho)) = u, \tag{34}$$

or equivalently, recalling the map $\mathbf{P}_\rho$ from §2.3,

$$\min_{\substack{u \in U_{ad} \\ y_\rho \in \mathbf{P}_\rho(u)}} J(y_\rho, u).$$

We shall first check that this minimisation problem suitably approximates (2).

Lemma 5.2. *Let* (6) *hold and suppose that* $\mathbf{Q}$ *is single-valued. Then there exist optimal pairs* (y_ρ^*, u_ρ^*) *of* (34) *and an optimal pair* (y^*, u^*) *of* (2) *such that*

$$(y_\rho^*, u_\rho^*) \to (y^*, u^*) \text{ in } V \times H.$$

Proof. Let (y_ρ^*, u_ρ^*) denote an optimal pair of (34), which must satisfy

$$J(y_\rho^*, u_\rho^*) \leq J(w_\rho, u) \quad \forall u \in U_{ad}, \quad \forall w_\rho \in \mathbf{P}_\rho(u). \tag{35}$$

Given any fixed $\tilde{u} \in U_{ad}$, we pick a subsequence $\{\tilde{y}_{\rho_n}\}$ such that $\mathbf{P}_{\rho_n}(\tilde{u}) \ni \tilde{y}_{\rho_n} \to \tilde{y}$ where $\tilde{y} \in \mathbf{Q}(\tilde{u})$; this is possible by Theorem 2.11. The inequality (35) implies that $J(y_{\rho_n}^*, u_{\rho_n}^*)$ is bounded above by $J(\tilde{y}_{\rho_n}, \tilde{u})$ which in turn is bounded uniformly in ρ_n because $\tilde{y}_{\rho_n}$ is bounded in V by the estimate of Lemma 2.9. Furthermore, by Lemma 2.9,

$$\|y_{\rho_n}^*\|_V \leq C_a^{-1} \left(\|u_{\rho_n}^*\|_{V^*} + \frac{C\epsilon}{\rho} \right)$$

hence for another subsequence (which we shall relabel)

$$\begin{aligned} u_{\rho_n}^* &\rightharpoonup u^* \quad \text{in } U_{ad}, \\ y_{\rho_n}^* &\rightharpoonup y^* \quad \text{in } V, \end{aligned}$$

for some (u^*, y^*) that we need to show is an optimal pair. By following steps 3 and 4 in the proof of Theorem 2.11, $y^*_{\rho_n} \to y^* = \mathbf{Q}(u^*)$ in V (since $u^*_{\rho_n} \to u^*$ in V^*). Hence (y^*, u^*) is a feasible point of (2). Then observe that for $(\hat{y}, \hat{u})$ being any optimal point of (2),

$$J(\hat{y}, \hat{u}) \leq J(y^*, u^*) \leq \liminf_{n\to\infty} J(y^*_{\rho_n}, u^*_{\rho_n}) \leq \limsup_{n\to\infty} J(y^*_{\rho_n}, u^*_{\rho_n}) \leq \limsup_{n\to\infty} J(w^*_{\rho_n}, \hat{u}) \quad \forall w^*_{\rho_n} \in \mathbf{P}_{\rho_n}(\hat{u})$$

with the last inequality by (35). Now it becomes necessary for $\mathbf{Q}$ to be single-valued since then, $\hat{y} = \mathbf{Q}(\hat{u})$ and it must be the case that we can select a sequence $\{w^*_{\rho_n}\}$ such that $w^*_{\rho_n} \in \mathbf{P}_{\rho_n}(\hat{u})$ and $w^*_{\rho_n} \rightharpoonup \hat{y}$ in V (by Theorem 2.11), and since the convergence is strong in H (for a subsequence), we find

$$J(\hat{y}, \hat{u}) \leq J(y^*, u^*) \leq \lim_{n\to\infty} J(y^*_{\rho_n}, u^*_{\rho_n}) \leq J(\hat{y}, \hat{u}).$$

Because $J(\hat{y}, \hat{u})$ is the minimal value and hence is either independent of $(\hat{y}, \hat{u})$ or uniquely determined by $(\hat{y}, \hat{u})$, the subsequence principle shows that $J(y^*_\rho, u^*_\rho) \to J(\hat{y}, \hat{u})$ (for the entire sequence). Furthermore, the above inequality shows that (y^*, u^*) is optimal and we get $u^*_\rho \to u^*$ in H since we have weak convergence and the convergence of the norm. □

To derive stationarity conditions for the regularised problem (34), we check the Zowe–Kurcyusz constraint qualification [62] (see also the Robinson constraint qualification [52]).

Lemma 5.3. *Suppose that*

$$\exists \epsilon > 0 : \Phi \colon V \to V^* \text{ is continuously Fréchet differentiable on } B_\epsilon(y^*), \tag{36}$$
$$\exists \epsilon > 0 : \Phi'(z)(v)v \leq C_P v^2 \text{ a.e. in } \Omega \quad \forall z \in B_\epsilon(y^*), \forall v \in V, \text{ where } C_P < 1. \tag{37}$$

Then, for ρ sufficiently small and any optimal point (y^*_ρ, u^*_ρ) of (34), there exists $p^*_\rho \in V$ such that

$$y^*_\rho + \frac{1}{\rho} m'_\rho(y^*_\rho - \Phi(y^*_\rho))(I - \Phi'(y^*_\rho))(p^*_\rho) + A^* p^*_\rho = y_d,$$
$$\langle \nu u^*_\rho - p^*_\rho, u^*_\rho - v \rangle \leq 0 \qquad \forall v \in U_{ad}.$$

Proof. We introduce the following notation:

$$X := V \times H, \quad F(x) := J(y, u), \quad C := V \times U_{ad}, \quad g(x) = g(y, u) := Ay + \frac{1}{\rho} m_\rho(y - \Phi(y)) - u,$$
$$C(x_\rho) := \{k(v - y^*_\rho, h - u^*_\rho) : v \in V, h \in U_{ad}, k \geq 0\}, \quad x_\rho = (y^*_\rho, u^*_\rho).$$

We must check that $g'(x_\rho)C(x_\rho) - K(g(x_\rho)) = V^*$, but since $\tilde{C} := V \times \{0\} \subset C(x_\rho)$, it suffices to verify $g'(x_\rho)\tilde{C} = V^*$. Observing that

$$g'(x_\rho)(y, 0) := Ay + \frac{1}{\rho} m'_\rho(y^*_\rho - \Phi(y^*_\rho))(y - \Phi'(y^*_\rho)(y)),$$

it follows that we need existence for

$$Az + \frac{1}{\rho} m'_\rho(y^*_\rho - \Phi(y^*_\rho))(I - \Phi'(y^*_\rho))(z) = f. \tag{38}$$

As Φ is Fréchet differentiable, $\Phi'(y^*_\rho) \colon V \to V^*$ is a bounded linear operator. Furthermore, by (37),

$$\frac{1}{\rho} \int_\Omega m'_\rho(y^*_\rho - \Phi(y^*_\rho))(I - \Phi'(y^*_\rho))(z) z \geq 0 \quad \forall z \in V$$

so that the elliptic operator is coercive. Thus the Lax–Milgram theorem allows us to conclude existence for the PDE. Then applying, for example, [59, Theorem 6.3], we get the existence of $p^*_\rho \in V$ such that for all $k \geq 0$,

$$\langle y^*_\rho - y_d + \frac{1}{\rho} m'_\rho(0, y^*_\rho - \Phi(y^*_\rho))(I - \Phi'(y^*_\rho))p^*_\rho + A^* p^*_\rho, k(c_1 - y^*_\rho)\rangle \geq 0 \qquad \forall c_1 \in V,$$
$$(\nu u^*_\rho - p^*_\rho, k(c_2 - u^*_\rho)) \geq 0 \qquad \forall c_2 \in U_{ad}.$$

As $c_1 \in V$ can be chosen arbitrarily, we find the stated result. □

Assumption (37) is sufficient to guarantee that (38) has a solution but perhaps not necessary.

Now the object is to pass to the limit which we shall do in the next theorem. In it, we will use the following fact. Since we have shown that $y^*_\rho \to y^*$ in V, whenever ρ is sufficiently small, we obtain that $y^*_\rho \in B_\epsilon(y^*)$ and hence for such ρ the assumption (37) is applicable and thus

$$\Phi'(y^*_\rho)(p^*_\rho)p^*_\rho \leq C_P (p^*_\rho)^2 \quad \text{a.e. in } \Omega.$$

Theorem 5.4 ($\mathcal{E}$-almost C-stationarity)**.** *Let* (6) *and the local assumptions* (36) *and* (37) *hold and suppose that* $\mathbf{Q}$ *is single-valued. Then there exists an* $\mathcal{E}$*-almost C-stationarity point* (y^*, u^*) *for* (2)*, i.e.,* (y^*, u^*) *is an optimal point and there exists* $(p^*, \xi^*, \lambda^*) \in V \times V^* \times V^*$ *such that*

$$\begin{aligned}
y^* + \lambda^* + A^* p^* &= y_d,\\
Ay^* - u^* + \xi^* &= 0,\\
u \in U_{ad} : (\nu u^* - p^*, u^* - v) &\leq 0 \quad \forall v \in U_{ad},\\
\xi^* \geq 0 \text{ in } V^*, \quad y^* \leq \Phi(y^*), \quad \langle \xi^*, y^* - \Phi(y^*)\rangle &= 0,\\
\langle \lambda^*, p^*\rangle \geq 0, \quad \langle \lambda^*, y^* - \Phi(y^*)\rangle &= 0,\\
p^* &= 0 \text{ a.e. in } \{\xi^* > 0\},
\end{aligned}$$
$$\forall \tau > 0, \exists E^\tau \subset \{y^* < \Phi(y^*)\} \text{ with } |\{y^* < \Phi(y^*)\} \setminus E^\tau| \leq \tau : \langle \lambda^*, v\rangle = 0 \quad \forall v \in V : v = 0 \text{ on } \Omega \setminus E^\tau.$$

In addition, $(y^*, u^*, p^*, \xi^*, \lambda^*)$ *can be characterised as a limit of the following subsequences (which we have relabelled):*

$$\begin{aligned}
y^*_\rho &\to y^* && \text{in } V\\
u^*_\rho &\to u^* && \text{in } H\\
p^*_\rho &\rightharpoonup p^* && \text{in } V\\
\rho^{-1} m_\rho(y^*_\rho - \Phi(y^*_\rho)) &\to \xi^* && \text{in } V^*\\
\rho^{-1} m'_\rho(y^*_\rho - \Phi(y^*_\rho))(\mathrm{I} - \Phi'(y^*_\rho))p^*_\rho &\rightharpoonup \lambda^* && \text{in } V^*
\end{aligned} \tag{39}$$

where $(y^*_\rho, u^*_\rho, p^*_\rho)$ *are as in Lemma 5.3.*

Furthermore, if $u_a, u_b \in V$ *then the optimal control has the regularity* $u^* \in V$*.*

Proof. The proof is similar to that of [31, Theorem 3.4]. Proposition 2.1 directly gives the fourth line in the system. If we test the equation satisfied by the adjoint p^*_ρ from the last lemma with itself and use assumption (37), we easily find for a subsequence (relabelled here) the convergence

$$p^*_\rho \rightharpoonup p^* \quad \text{in } V.$$

Using the equivalence of the VI relating u_ρ^* and p_ρ^* in the previous lemma to a projection (see [31, Theorem 2.3] or [37, §II.3]), thanks to the strong convergence in H of p_ρ^*, we find that

$$\begin{aligned} u_\rho^* &= \frac{1}{\nu}p_\rho^* + \max\left(0, u_a - \frac{p_\rho^*}{\nu}\right) - \max\left(0, \frac{p_\rho^*}{\nu} - u_b\right) \\ &\to \frac{1}{\nu}p^* + \max\left(0, u_a - \frac{p^*}{\nu}\right) - \max\left(0, \frac{p^*}{\nu} - u_b\right) = u^*. \end{aligned}$$

It follows that $u^* \in V$ if u_a and u_b belong to V. Define

$$\begin{aligned} \lambda_\rho^* &:= \frac{1}{\rho}m_\rho'(y_\rho^* - \Phi(y_\rho^*))(\mathrm{I} - \Phi'(y_\rho^*))p_\rho^* = y_d - y_\rho^* - A^*p_\rho^*, \\ \xi_\rho^* &:= \frac{1}{\rho}m_\rho(y_\rho^* - \Phi(y_\rho^*)) = u_\rho^* - Ay_\rho^*, \end{aligned}$$

which, since their right-hand sides converge, satisfy the following convergences both in V^*:

$$\lambda_\rho^* \rightharpoonup \lambda^* := y_d - y^* - A^*p^* \qquad \text{and} \qquad \xi_\rho^* \to \xi^* := u^* - Ay^*.$$

Test the adjoint equation in Lemma 5.3 with p_ρ^* and use (37) to see that

$$\langle A^*p_\rho^*, p_\rho^*\rangle + (y_\rho^* - y_d, p_\rho^*) = -\frac{1}{\rho}\int m_\rho'(y_\rho^* - \Phi(y_\rho^*))(I - \Phi'(y_\rho^*))(p_\rho^*)^2 \le 0.$$

Taking the limit inferior of this, recalling the definition of λ^* and using the weak lower semicontinuity, we obtain

$$\langle \lambda^*, p^*\rangle = -\langle A^*p^*, p^*\rangle + (y_d - y^*, p^*) \ge 0.$$

Observe that since m_ρ' vanishes on $(-\infty, 0]$,

$$\langle \lambda_\rho^*, (y_\rho^* - \Phi(y_\rho^*))^-\rangle = \frac{1}{\rho}\int_\Omega m_\rho'(y_\rho^* - \Phi(y_\rho^*))(I - \Phi'(y_\rho^*))(p_\rho^*)(y_\rho^* - \Phi(y_\rho^*))^- = 0,$$

which, due to the continuity of $\max(0, \cdot)\colon V \to V$, implies that

$$\langle \lambda^*, (y^* - \Phi(y^*))^-\rangle = 0$$

and since $y^* \le \Phi(y^*)$, we have shown that $\langle \lambda, y^* - \Phi(y^*)\rangle = 0$.

Finally, since $y_\rho^* \to y^*$ in V, $y_\rho \to y^*$ pointwise a.e. in Ω for a subsequence that we do not relabel. Take $x \in \Omega$ such that $y^*(x) - \Phi(y^*)(x) < 0$, then there exists a ρ_0 such that if $\rho \le \rho_0$, then

$$y_\rho(x) - \Phi(y_\rho)(x) \le \frac{1}{2}(y^*(x) - \Phi(y^*)(x)) < 0$$

and hence $m_\rho'(y_\rho(x) - \Phi(y_\rho)(x)) = 0$ for $\rho \le \rho_0$. This gives $\lambda_\rho^*(x) = 0$ for all such ρ. That is, $\lambda_\rho^* \to 0$ pointwise a.e. on $\{y^* < \Phi(y^*)\}$, and then applying Egorov's theorem gives the final statement of the system.

For the remaining statement, let us introduce the sets

$$M_1(\rho) := \{0 \le y_\rho^* - \Phi(y_\rho^*) < \epsilon\} \qquad \text{and} \qquad M_2(\rho) := \{y_\rho^* - \Phi(y_\rho^*) \ge \epsilon\}.$$

Since $\langle \xi_\rho, y_\rho^* - \Phi(y_\rho^*)\rangle \to \langle \xi, y - \Phi(y)\rangle = 0$, we find (after recalling the definition of m_ρ),

$$\begin{aligned}(\xi_\rho, y_\rho^* - \Phi(y_\rho^*)) &= \frac{1}{\rho}\int_\Omega m_\rho(y_\rho^* - \Phi(y_\rho^*))(y_\rho^* - \Phi(y_\rho^*)) \\ &= \frac{1}{\rho}\int_{M_1(\rho)} \frac{(y_\rho^* - \Phi(y_\rho^*))^3}{2\epsilon} + \frac{1}{\rho}\int_{M_2(\rho)} \left(y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2}\right)(y_\rho^* - \Phi(y_\rho^*)) \\ &\to 0,\end{aligned} \tag{40}$$

and as both integrands in (40) are non-negative, each integral must individually converge to zero too. Hence

$$\left\| \frac{\chi_{M_1(\rho)}(y_\rho^* - \Phi(y_\rho^*))^{\frac{3}{2}}}{\sqrt{\rho\epsilon}} \right\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \left\| \frac{\chi_{M_2(\rho)}(y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2})}{\sqrt{\rho}} \right\|_{L^2(\Omega)} \to 0, \tag{41}$$

where for the second convergence we used the fact that $y_\rho^* - \Phi(y_\rho^*) \geq y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2} \geq 0$. We calculate

$$\begin{aligned}\langle \xi_\rho^*, p_\rho^*\rangle &= \frac{1}{\rho}\int_{M_1(\rho)} \frac{(y_\rho^* - \Phi(y_\rho^*))^2}{2\epsilon} p_\rho^* + \frac{1}{\rho}\int_{M_2(\rho)} \left(y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2}\right) p_\rho^* \\ &= \frac{1}{2}\int_\Omega \chi_{M_1(\rho)} \frac{(y_\rho^* - \Phi(y_\rho^*))^{3/2}}{\sqrt{\rho\epsilon}} \frac{(y_\rho^* - \Phi(y_\rho^*))^{1/2}}{\sqrt{\rho\epsilon}} \chi_{M_1(\rho)} p_\rho^* \\ &\quad + \int_\Omega \frac{\chi_{M_2(\rho)}\left(y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2}\right)}{\sqrt{\rho}} \frac{\chi_{M_2(\rho)} p_\rho^*}{\sqrt{\rho}} \\ &\leq \frac{1}{2}\left\| \chi_{M_1(\rho)} \frac{(y_\rho^* - \Phi(y_\rho^*))^{3/2}}{\sqrt{\rho\epsilon}} \right\| \left\| \frac{(y_\rho^* - \Phi(y_\rho^*))^{1/2}}{\sqrt{\rho\epsilon}} \chi_{M_1(\rho)} p_\rho^* \right\| \\ &\quad + \left\| \frac{\chi_{M_2(\rho)}\left(y_\rho^* - \Phi(y_\rho^*) - \frac{\epsilon}{2}\right)}{\sqrt{\rho}} \right\| \left\| \frac{\chi_{M_2(\rho)} p_\rho^*}{\sqrt{\rho}} \right\|.\end{aligned} \tag{42}$$

Now, using (41), the first factor in each term above converges to zero and hence the above right-hand side will converge to zero if we are able to show that the second factor in each term remains bounded. Since λ_ρ^* and p_ρ^* are bounded, so is their duality product, and therefore

$$\begin{aligned}C &\geq |\langle \lambda_\rho^*, p_\rho^*\rangle| \\ &= \frac{1}{\rho}\left| \int_\Omega m_\rho'(y_\rho^* - \Phi(y_\rho^*))(I - \Phi'(y_\rho^*))(p_\rho^*)p_\rho^* \right| \\ &= \frac{1}{\rho}\left| \int_{M_1(\rho)} \frac{y_\rho^* - \Phi(y_\rho^*)}{\epsilon}(I - \Phi'(y_\rho^*))(p_\rho^*)p_\rho^* + \int_{M_2(\rho)} (I - \Phi'(y_\rho^*))(p_\rho^*)p_\rho^* \right| \\ &\geq \frac{1 - C_P}{\rho}\int_{M_1(\rho)} \frac{y_\rho^* - \Phi(y_\rho^*)}{\epsilon}(p_\rho^*)^2 + \frac{1 - C_P}{\rho}\int_{M_2(\rho)} (p_\rho^*)^2 \\ &= \frac{1 - C_P}{\rho}\int_\Omega \chi_{M_1(\rho)} \frac{y_\rho^* - \Phi(y_\rho^*)}{\epsilon}(p_\rho^*)^2 + \frac{1 - C_P}{\rho}\int_\Omega \chi_{M_2(\rho)}(p_\rho^*)^2.\end{aligned}$$

Furthermore, both of the terms on the right-hand side are individually bounded uniformly in ρ as the integrands are non-negative. This implies from (42) that

$$\langle \xi^*, p^*\rangle = 0.$$

Replacing p^*_ρ by $(p^*_\rho)^+$ in (42) and in the above calculation, we also obtain in the same way (after using the fact that $v_n \rightharpoonup v$ in V implies that $v_n^+ \rightharpoonup v^+$ in V)

$$\langle \xi^*, (p^*)^+ \rangle = 0.$$

□

5.3 Strong stationarity

For the sake of completeness, we give strong stationarity conditions for (2). After providing some background and context, we reduce this section to the essence of the statement of the result since a similar result has recently been obtained in [61] whilst this work was under preparation.

Strong stationarity for the VI obstacle problem in the absence of constraints on the control was the focus of the classical works by Mignot [41, Theorem 5.2] and Mignot–Puel [42]. The approach in the latter work is as follows. By using the results on the differentiability of the solution map associated to VIs of Mignot [41], the Bouligand stationarity condition (for example, see Proposition 5.1 with $U_{ad} = H$) reads

$$(\alpha_h, y^* - y_d) + \nu(u^*, h) \geq 0 \quad \forall h \in H$$

where α_h denotes the directional derivative of the solution map with respect to the direction h. The key idea of Mignot and Puel in [42] is to use the fact that the optimal control u^* in fact belongs to V (this is a regularity result in certain situations; otherwise one needs to simply assume this) and to extend, by continuity, the above inequality to

$$(\alpha_h, y^* - y_d) + \nu\langle u^*, h\rangle \geq 0 \quad \forall h \in V^* \tag{43}$$

so that the set of feasible directions has been enlarged to V^*. Then, by writing the duality product in (43) as $\langle AA^{-1}h, \nu u^*\rangle$ and using properties of the projection operator with respect to the bilinear form generated by A onto the critical cone, it is shown [42, Theorem 3.3] that this inequality is equivalent to a strong stationarity system. Our theory of differentiability for QVIs [3] (which was for non-negative sources and directions) could not be immediately used to obtain strong stationarity by arguing in this fashion since the setting of [3] would have forced U_{ad} to be selected such that $U_{ad} \subset H_+$. This is why the development of the results of §2 and §3 are crucial.

The presence of control constraints complicates the derivation of strong stationarity conditions. In the VI setting, by using the above-mentioned technique of Mignot and Puel of enlarging the set of feasible directions onto the dual space in combination with a fine analysis of the various resulting objects and sets, strong stationarity conditions for VI optimal control problems subject to box constraints were obtained by Wachsmuth in [60]. The author also showed that certain restrictions are required on the control bounds in order to obtain a positive answer for strong stationarity, and counterexamples were given showing that violating those conditions can lead to a lack of strong stationarity. These necessary conditions (which are stated in (47)–(49) below) in the context of admissible sets as in (32) are implied [60, Lemma 5.3] by the condition

$$u_a, u_b \in H^1(\Omega) \text{ with } u_a < 0 \leq u_b \text{ q.e. on } \Omega, \tag{44}$$

which in turn implies that the control space must allow for negative functions, meaning that one ultimately needs existence and directional differentiability results for QVIs with source terms and directions that may be strictly negative[5].

[5]This requirement meant that the differentiability theory of [3] for non-negative sources and directions could not be directly applied.

Remark 5.5. *For source terms $f \in U_{ad}$ with U_{ad} satisfying* (44)*, $\mathbf{Q}(f)$ is well defined through Theorems 2.3 or 2.11 or Theorem 2.6 by taking a lower bound function $G \geq -u_a$. The derivatives for directions belonging to U_{ad} also exist by either of the two theorems in §3.*

We will address the case where Theorem 3.12 is applicable[6] so that the resulting directional derivative of $\mathbf{Q}$ has the form (25). Let (y^*, u^*) be an optimal pair of (2). As in [42], we make the fundamental assumption that $u^* \in V$ and we refer to Theorem 5.4 from the previous section for the satisfaction of this assumption. Let us take U_{ad} as stated in (32) where we include the possibility of taking $u_a = -\infty$ and $u_b = \infty$, in which case the problem becomes one with no constraints and we can argue as in [42]. Outside of this case, we can argue as in [60]. Let the assumptions of Proposition 5.1 hold under the regime of Theorem 3.12 and denote by $j\colon H \to V^*$ the inclusion map through the Riesz isomorphism. Then, as done in [60], the Bouligand stationarity condition (33) can be extended to

$$(\alpha_h, y^* - y_d) + \nu\langle h, u^*\rangle \geq 0 \quad \forall h \in \overline{j\mathcal{T}_{U_{ad}}(u^*)}^{V^*}.$$

Observe that we needed the continuity in V^* of $h \mapsto \alpha_h$ assured by Proposition 3.18 to do this. This is starting point of the steps leading to the strong stationarity conditions in [60] for the VI case.

Defining the (quasi-closed) coincidence sets

$$U_a := \{x \in \Omega : u^*(x) = u_a(x)\} \qquad \text{and} \quad U_b := \{x \in \Omega : u^*(x) = u_b(x)\}$$

and arguing identically to the proof of [60, Lemma 4.3], we obtain the following sign conditions on u^*:

$$\begin{aligned} &u^* = 0 \text{ q.e. on } \mathcal{A}_s(y^*) \cap (\Omega \setminus (U_a \cup U_b)),\\ &u^* \leq 0 \text{ q.e. on } \mathcal{A}_s(y^*) \cap U_b,\\ &u^* \geq 0 \text{ q.e. on } (\mathcal{A}_s(y^*) \cap U_a) \cup (\mathcal{B}(y^*) \cap (\Omega \setminus U_b)) \end{aligned}$$

where $\mathcal{B}(y^*) = \mathcal{A}(y^*) \setminus \mathcal{A}_s(y^*)$ is the *biactive set*.

Let $\operatorname{cap}(A)$ denote the capacity of a Borel subset A of Ω with respect to $H^1_0(\Omega)$ (see [17, Definition 6.47]). We have the following strong stationarity characterisation, the proof of which involves modifications of [60] and is omitted.

Theorem 5.6 (Strong stationarity)**.** *Let (y^*, u^*) be an optimal point of* (2) *with $u^* \in V$. Suppose that*

$$\Phi\colon V \to V \text{ is Frèchet differentiable at } y^* \tag{45}$$

and let (6)*, the local assumptions*[7] (15)*,* (17)*,* (22)*,* (23)*,* (30)*,*

$$(\mathrm{I} - \Phi'(y^*))\colon V \to V \text{ is invertible}, \tag{46}$$

$$\operatorname{cap}(U_a \cap \mathcal{B}(y^*)) = 0, \tag{47}$$

$$u_b \geq 0 \text{ q.e. on } \mathcal{B}(y^*), \tag{48}$$

$$u^* = 0 \text{ q.e. on } \mathcal{A}_s(y^*), \tag{49}$$

[6]The case where Theorem 3.15 is applied instead (with resulting directional derivative satisfying (29)) requires additional work which we do not consider in this paper; the complications arise from Proposition 5.1 where we see that the set of admissible directions for the first-order condition is not a tangent cone but rather the closure of a radial cone intersected with another set.

[7]These, of course, should be evaluated at y^*.

and Assumption 3.1 hold. Then there exists $p^* \in V$, $\xi^*, \lambda^* \in V^*$ *such that*

$$\begin{aligned}
y^* + (\mathrm{I} - \Phi'(y^*)^*)\lambda^* + A^*p^* &= y_d, \\
Ay^* - u^* + \xi^* &= 0, \\
u^* \in U_{ad} : (\nu u^* - p^*, u^* - v) &\le 0 \quad \forall v \in U_{ad}, \\
\xi^* \ge 0 \text{ in } V^*, \quad y^* \le \Phi(y^*), \quad \langle \xi^*, y^* - \Phi(y^*)\rangle &= 0, \\
p^* &\ge 0 \quad \text{q.e. on } \mathcal{B}(y^*) \text{ and } p^* = 0 \text{ q.e. on } \mathcal{A}_s(y^*), \\
\langle \lambda^*, v\rangle &\ge 0 \quad \forall v \in V : v \ge 0 \text{ q.e. on } \mathcal{B}(y^*) \\
&\quad\ \text{and } v = 0 \text{ q.e. on } \mathcal{A}_s(y^*).
\end{aligned}$$

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