Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

On a half-space radiation condition

Andreas Rathsfeld

submitted: December 20, 2019, revision: June 10, 2022

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: andreas.rathsfeld@wias-berlin.de

No. 2669
Berlin 2019

2010 *Mathematics Subject Classification.* 74J20, 76B15, 78A45.

Key words and phrases. Scattering problem, Dirichlet problem, Helmholtz equation, half space and rough surface, radiation condition.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

On a half-space radiation condition

Andreas Rathsfeld

Abstract

For the Dirichlet problem of the Helmholtz equation over the half space or above rough surfaces, a radiation condition is needed to guarantee a unique solution, which is physically meaningful. If the Dirichlet data is a general bounded continuous function, then the well-established Sommerfeld radiation condition, the angular spectrum representation, and the upward propagating radiation condition do not apply or require restrictions on the data, in order to define the involved integrals. In this paper a new condition based on a representation of the second derivative of the solution is proposed. The thrice differentiated half-space Green's function is integrable and the corresponding radiation condition applies to general bounded functions. The condition is checked for special functions like plane waves and point source solution. Moreover, the Dirichlet problem for the half plane is discussed. Note that such a "continuous" radiation condition is helpful e.g. if finite sections of the rough-surface problem are analyzed.

1 Introduction

⟨s0⟩

Throughout this paper we denote the points of the three-dimensional Euclidean space $\mathbb{R}^3$ by $\vec{x}$ and $\vec{y}$ with $\vec{x} := (x_1, x_2, x_3)^\top = (x', x_3)^\top$ and $\vec{y} := (y', y_3)^\top$, where $x' := (x_1, x_2)^\top \in \mathbb{R}^2$. For fixed numbers $x_{f,3}$, we define the half spaces $\mathbb{R}^3_{x_{f,3},+} := \{\vec{x} \in \mathbb{R}^3 : \ x_3 > x_{f,3}\}$ and $\mathbb{R}^3_+ := \mathbb{R}^3_{0,+}$ and the boundary planes $\mathbb{R}^3_{x_{f,3}} := \{(x', x_{f,3})^\top : \ x' \in \mathbb{R}^2\}$. We shall consider functions u defined on a perturbed half space Ω of $\mathbb{R}^3_+$, which are solutions of the Helmholtz equation $(\Delta + k^2 I)u = 0$ for a fixed wavenumber $k > 0$. We suppose that $\Omega = \Omega_F := \{\vec{x} \in \mathbb{R}^3 : \ x_3 > F(x')\}$ with a Lipschitz continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ s.t. $-h_F \leq F(x') < 0$ holds for all $x' \in \mathbb{R}^2$ (cf. Fig. 1). The number $h_F > 0$ is fixed.

The goal is to find a general radiation condition for Helmholtz solutions on Ω which are uniformly bounded on the planes $\mathbb{R}^3_{x_{f,3}}$, but, eventually, do not decay in the lateral directions, i.e. the directions of $\mathbb{R}^3_{x_{f,3}}$. If $\Phi(\vec{x}, \vec{y})$ is the Green's function for the Dirichlet problem of the Helmholtz equation over the upper three-dimensional half space $\mathbb{R}^3_+$, then, analogously to the upward propagating radiation

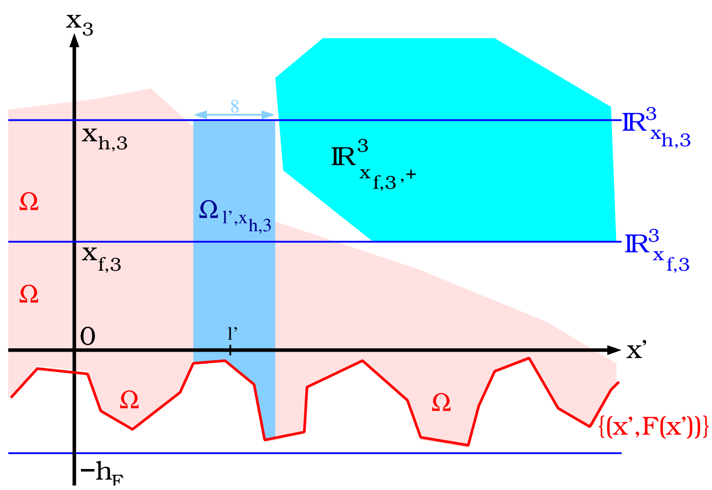

Figure 1: The geometry settings.

DOI 10.20347/WIAS.PREPRINT.2669 Berlin, December 20, 2019/rev. June 10, 2022

condition (UPRC) in the two-dimensional case (cf. [7]), a possible choice for the radiation condition would be to fix $x_{f,3} \geq 0$ and to require the condition (cf. [3], Chapt. 5.1.1 and [4, 5])

$$u(\vec{x}\,) = \int_{\mathbb{R}^2} \partial_{y_3} \Phi\big(\vec{x}, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y' \tag{1.1}$$ 1try

for all $\vec{x} \in \mathbb{R}^3_{x_{f,3},+}$. Note that this condition is equivalent to a representation as a superposition of outgoing generalized plane waves (cf. the angular spectrum representation in [2, 5, 7, 10]) and, in two dimensions, to the pole condition (cf. [2]). In three dimensions, the integral exists for functions from weighted L^2 or from more complicated spaces. Since we are interested in a class of solutions u containing plane-wave functions, we only know that the functions u restricted to the boundary plane $\mathbb{R}^3_{x_{f,3}}$ are smooth and uniformly bounded, and the existence of the integral in (1.1) is not guaranteed. Therefore, we formally differentiate twice to get

$$\partial^2_{x_3} u(\vec{x}\,) = \int_{\mathbb{R}^2} \partial^3_{y_3} \Phi\big(\vec{x}, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y', \quad \vec{x} \in \mathbb{R}^3_{x_{f,3},+}. \tag{1.2}$$ HSRC

This will be the main part of our radiation condition. We shall see that the kernel in (1.2) satisfies $\partial^3_{y_3}\Phi(\vec{x}, \vec{y}\,) = \mathcal{O}(|\vec{y}\,|^{-3})$ for $|\vec{y}\,| \to \infty$ and $\vec{y} \in \mathbb{R}^3_{x_{f,3}}$. Hence, the integral in (1.2) is well defined for any function u bounded and measurable over $\mathbb{R}^3_{x_{f,3}}$.

The first question is whether it is sufficient to require the equality in (1.2) over a subset of $\mathbb{R}^3_{x_{f,3},+}$. Suppose that $Op \subseteq \mathbb{R}^3_{x_{f,3},+}$ is a fixed open subset and that u is a solution of the Helmholtz equation. Then (1.2) is satisfied if and only if the equation in (1.2) holds for any $\vec{x} \in Op$. Indeed, on both sides of the equation we have solutions of the Helmholtz equation, and such analytic function coincide on $\mathbb{R}^3_{x_{f,3},+}$ if and only if they do on Op. On the other hand, it is not sufficient to require the equation for one or more planes $\mathbb{R}^3_{x_3}$ with $x_3 > x_{f,3}$. Indeed, the two sides of the equation might differ by the function $(x', x_3)^\top \mapsto \sin(k(x_3 - x_{f,3}))$ if they coincide only over $\mathbb{R}^3_{x_3}$ with $x_3 = x_{f,3} + l\pi/k$, $l = 1, 2, \cdots$. However, it would be sufficient to require the equation in (1.2) over $\mathbb{R}^3_{x_{l,3}}$ with $x_{l,3} > x_{f,3}$, $l = 1, 2$ if the homogeneous Dirichlet problem of the Helmholtz equation over the layer enclosed by $\mathbb{R}^3_{x_{1,3}}$ and $\mathbb{R}^3_{x_{2,3}}$ has the trivial solution only. In particular, this is the case for $|x_{1,3} - x_{2,3}| < \pi/k$ (cf. Sect. 9).

Now suppose function u is bounded and sufficiently smooth over $\mathbb{R}^3_{x_{f,3}}$. Note that, differentiating w.r.t. x_3 and taking the limit for $x_3 \to x_{f,3}$, the condition (1.2) implies

$$\partial^3_{x_3} u(\vec{x}\,) \;=\; \int_{\mathbb{R}^2} \partial_{x_3}\partial^3_{y_3} \Phi\big(\vec{x}, (y', x_{f,3})^\top\big)\, u\big((y', x_{f,3})^\top\big)\, \mathrm{d}y', \quad \vec{x} \in \mathbb{R}^3_{x_{f,3}}. \tag{1.3}$$ HSRCB

For Helmholtz solutions u, this condition (1.3) is even equivalent to (1.2). Indeed, condition (1.3) means that the Helmholtz solutions on the left- and right-hand side of (1.2) have the same Neumann data on the face $\mathbb{R}^3_{x_{f,3}}$. The Dirichlet data is the same due to the Helmholtz equation $\partial^2_{x_3} u = -\Delta_{x'} u - k^2 u$ and due to the limit relation (6.9) shown in the subsequent Sect. 6.2.2. Hence, the two sides of (1.2) coincide.

Next we look for additional requirements besides (1.2). For $l' \in \mathbb{Z}^2$ and $x_{h,3} > 0$, we introduce the finite cylindrical subdomains $\Omega_{l',x_{h,3}} := \{\vec{x} \in \Omega\colon\ |x' - l'| < 4\,,\ x_3 < x_{h,3}\}$ of Ω adjacent to the lower boundary. Clearly, for any $l' \in \mathbb{Z}^2$ and $x_{h,3} > 0$, we have to assume that u is in the Sobolev space $H^1(\Omega_{l',x_{h,3}})$. Since we analyze domains above rough surfaces, the condition should be invariant w.r.t. shifts in horizontal directions. So it is natural to require Sobolev norms independent of l'. Consequently, this uniform regularity condition enforces a uniform boundedness of the solutions in the horizontal directions, and we consider it as a radiation condition. However, it is not a condition of the kind, which selects “outgoing” modes from the set of “outgoing” and “incoming” plane-wave modes radiating into horizontal directions.

Unfortunately, this Sobolev regularity together with condition (1.2) still allows to add to u an unphysical solution of the form $u_{\mathrm{add}}(\vec{x}\,) := u_{\mathrm{2D}}(x')(x_3 - x_{f,3})$ with u_{2D} a solution of the two-dimensional Helmholtz equation $(\Delta_{x'} + k^2 I)u_{\mathrm{2D}} = 0$ (e.g. $u_{\mathrm{2D}}(x') = e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = k^2$). Adding such a function to u corresponds to adding the function $u_{\mathrm{2D}}(x')$ to the derivative $\partial_{x_3} u(\cdot, x_{3,f})$. In order to exclude such addends, we augment our radiation condition by the weak boundedness condition

$$\left| \frac{1}{x_3 - x_{f,3}} \int_{x_{f,3}}^{x_3} \frac{x_3 - t}{x_3 - x_{f,3}} u\big((x', t)^\top\big) \, \mathrm{d}t \right| \le c_u (x_3 - x_{f,3})^{1-\varepsilon_u}, \quad \forall x_3 > x_{f,3}, \quad \forall x' \in \mathbb{R}^2. \qquad (1.4) \; \boxed{\texttt{HSRC2}}$$

This condition (1.4) is nothing else than a restriction to an $\mathcal{O}\big((x_3 - x_{f,3})^{3-\varepsilon_u}\big)$ growth of the two-fold integral function $w(x', x_3) := \int_{x_{f,3}}^{x_3} (x_3 - t) u\big((x', t)^\top\big) \, \mathrm{d}t$, which is defined by the second order derivative $\partial_{x_3}^2 w = u$ and by $w(x', x_{f,3}) = \partial_{x_3} w(x', x_{f,3}) = 0$. If u is uniformly bounded or if it satisfies $|u(\vec{x}\,)| = \mathcal{O}(x_3^{1-\varepsilon_u})$ for $x_3 \to \infty$, then (1.4) is fulfilled.

Instead of requiring the weak boundedness in (1.4) for all $x' \in \mathbb{R}^2$, it is sufficient to require weak boundedness for all $x' \in S_{\mathrm{2D}}$, where $S_{\mathrm{2D}} \subset \mathbb{R}^2$ is a set with the property that, for any smooth 2D Helmholtz solution u_{2D} over $\mathbb{R}^2$, the vanishing $u_{\mathrm{2D}}(x') = 0, \; x' \in S_{\mathrm{2D}}$ implies $u_{\mathrm{2D}} \equiv 0$. In particular, S_{2D} can be a closed curve s.t. the Dirichlet problem for the Helmholtz equation in the interior of the curve admits a unique solution.

Altogether, we suggest the following outgoing radiation condition

Definition 1.1. *A solution u of $(\Delta + k^2 I)u = 0$ over Ω is said to satisfy the half-space radiation condition (HSRC) if there exist real numbers c_u, ε_u, $x_{h,3}$, and $x_{f,3}$ with $c_u > 0$, $\varepsilon_u > 0$, and $0 < x_{f,3} < x_{h,3}$ such that*

i) For any $l' \in \mathbb{Z}^2$, the restriction $u|_{\Omega_{l',x_{h,3}}}$ of u to the subdomain $\Omega_{l',x_{h,3}}$ is in the Sobolev space $H^1(\Omega_{l',x_{h,3}})$ and has a bounded norm $\|u|_{\Omega_{l',x_{h,3}}}\|_{H^1(\Omega_{l',x_{h,3}})} \le c_u$.

ii) The second order derivative $\partial_{x_3}^2 u$ admits the representation (1.2) (or, equivalently, the relation (1.3) is satisfied).

iii) The function u satisfies the weak boundedness estimate (1.4).

Alternatively to the weak boundedness (1.4), we can fix the derivative $\partial_{x_3} u$ over the plane $\mathbb{R}^3_{x_{f,3}}$ by $\partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}}} = D_t N u|_{\mathbb{R}_{x_{f,3}}}$ with a fixed linear Dirichlet-to-Neumann mapping $D_t N$, which might depend on a function class containing $u|_{\mathbb{R}_{x_{f,3}}}$. In general, this would lead to unique solutions, which are not necessarily bounded. For example, we can require

$$\partial_{x_3} u\big((x', x_{f,3})^\top\big) = -\int_{\mathbb{R}^2} \partial_{x_3}^2 \Phi\big((x', x_{f,3})^\top, (y', x_{f,3})^\top\big) \, u\big((y', x_{f,3})^\top\big) \, \mathrm{d}y', \quad x' \in \mathbb{R}^2. \qquad (1.5) \; \boxed{\texttt{HSRC2b}}$$

The existence of the integral in (1.5) can be shown under additional conditions on the behaviour of $u|_{\mathbb{R}^3_{x_{f,3}}}$ for $x' \to \infty$ (cf. e.g. the class AV_κ in (1.8)). To avoid hypersingular kernels (cf. the subsequent representation (2.10)), we can fix $\partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}}}$ by

$$\partial_{x_3} u\big((x', x_{f,3})^\top\big) = -\int_{\mathbb{R}^2} \partial_{x_3}^2 \Phi\big((x', x_{f,3})^\top, (y', x_{h,3})^\top\big) \, u\big((y', x_{h,3})^\top\big) \, \mathrm{d}y', \quad x' \in \mathbb{R}^2. \qquad (1.6) \; \boxed{\texttt{HSRC2c}}$$

In comparison to (1.5), in (1.6) there is no singularity in the kernel since $|(x',x_{f,3})^\top-(y',x_{h,3})^\top|\to 0$ is excluded. However, the more serious problem in (1.5) and (1.6) is that of the integrability at infinity.

For the equivalence of (1.4) and (1.5) and for the existence of a solution to the Dirichlet problem for the Helmholtz equation over $\mathbb{R}^3_+$, we have to restrict the functions u and the Dirichlet data, respectively. Here we use three special classes. For this, we introduce the space $C(\mathbb{R}^2)$ of continuous functions, the space $C_b(\mathbb{R}^2)$ of uniformly bounded and continuous functions. Furthermore, by $C_b^l(\mathbb{R}^2)$ we denote the space of all functions v, which together with all their derivatives upto order $l\geq 0$ are continuous and uniformly bounded. If $0<\kappa\leq 1$, then $C_b^{l,\kappa}(\mathbb{R}^2)$ is the space of all functions in $C_b^l(\mathbb{R}^2)$, for which the lth order derivatives are Hölder continuous with Hölder exponent κ. For $v\in C(\mathbb{R}^2)$, the average function is defined by

$$av(v,x',r) \quad := \quad \int_0^{2\pi} v\big(x'-r(\cos\phi,\sin\phi)^\top\big)\,\mathrm{d}\phi. \tag{1.7}$$ DefAve0

In other words, the value $av(v,x',r)$ is the average of v over the circle $\{x''\in\mathbb{R}^2:\ |x'-x''|=r\}$. Now we introduce the first class. For $\kappa>0$, we define the class AV_κ of functions with circular averages decaying as

$$AV_\kappa \quad := \quad \big\{v\in C_b^1(\mathbb{R}^2):\ \exists c_v>0 \text{ s.t. } |av(v,x',r)|<c_v r^{-\kappa},\ \forall r\geq 1,\ \forall x'\in\mathbb{R}^2\big\}\,. \tag{1.8}$$ AVkappa

The second class is the Dirichlet data set DD_τ with $0\leq\tau<1$ including all the sums $v=v_s+v_i$ of Helmholtz solutions v_s plus Helmholtz images v_i, i.e.,

$$\begin{aligned} DD_\tau \quad := \quad \big\{v\in C_b(\mathbb{R}^2):\ &\exists c_v>0,\ \exists v_s\in C_b^2(\mathbb{R}^2),\ \exists v_0\in C^2(\mathbb{R}^2) \text{ s.t.}\\ & v=v_s+v_i,\ (\Delta_{x'}+k^2I)v_s=0,\ v_i:=(\Delta_{x'}+k^2I)v_0, \text{ and}\\ & |v_0(x')|\leq c_v(1+|x'|)^\tau,\forall x'\in\mathbb{R}^2\big\}. \end{aligned} \tag{1.9}$$ DD

Finally, we introduce the third class of functions v characterized by the Fourier transforms $[\mathcal{F}v]$. For k the wave number and $\varepsilon>0$, we introduce annular domains $R_{k,\varepsilon}:=\{\xi'\in\mathbb{R}^2:\ k-\varepsilon<|\xi'|<k+\varepsilon\}$ and set

$$FC_k \quad := \quad \big\{v\in C_b^6(\mathbb{R}^2):\ \exists\varepsilon_v>0 \text{ s.t. } [\mathcal{F}v]|_{R_{k,\varepsilon_v}}\in L^2(R_{k,\varepsilon_v})\big\}\,. \tag{1.10}$$ DefFC

Unfortunately, the condition, characterizing functions as class members, will not be easy to check. For functions u with restrictions $v(x'):=u((x',x_{f,3})^\top)$ in one of the three classes, we get

⟨p0⟩ **Proposition 1.2.** *Suppose there is an $x_{f,3}$ with $0<x_{f,3}<x_{h,3}$ such that $v:=u|_{\mathbb{R}^3_{x_{f,3}}}$ is either in AV_κ, $\kappa>0$, or in DD_τ, $0\leq\tau<1$, or in FC_k. Then, the integral in (1.5) is meaningful and, in the radiation condition (HSRC), we can replace item iii) by the equivalent requirement*

iii') The derivative $\partial_{x_3}u$ restricted to the plane $\mathbb{R}^3_{x_{f,3}}$ fulfills (1.5).

Now suppose the integrals in (1.1) and (1.5) are well defined for the function u. Since (1.1) implies (1.2) and (1.5), (UPRC) implies (HSRC) with iii) replaced by iii'). Prop. 1.2 and its proof (cf. (7.11) or interprete the representation $[V_kv_0]$ in part iii) of the proof to Lemma 7.2 as (1.1) evaluated by partial integration for the differential operator $\partial_{x_3}^2=-(\Delta_{x'}+k^2I)$) means that, for $v:=u|_{\mathbb{R}^3_{x_{f,3}}}$ in DD_τ, AV_κ or FC_k, the condition (HSRC) is equivalent to the (UPRC).

In Sect. 7 we shall prove Prop. 1.2 and show that the (HSRC) is independent of the choice of $x_{h,3}$ and $x_{f,3}$. For the plausibility of the (HSRC), we remark:

i) Formula (1.2) can be considered to be a representation of $\partial_{x_3}^2 u$ as a superposition of generalized outgoing plan-wave solutions (cf. the right-hand side of (6.7)). Outgoing for the upper half plane means that the plane wave $u(\vec{x}) := e^{\mathbf{i}(\alpha x_1+\beta x_2+\gamma x_3)}$ with $\alpha,\beta\in\mathbb{R}$, $\gamma\in\mathbb{C}$, and $\alpha^2+\beta^2+\gamma^2=k^2$, is either a true plane-wave (i.e. $\alpha^2+\beta^2\leq k^2$) radiating into the upper half plane (i.e. $\gamma\geq 0$) or a generalized plane-wave (i.e. $\alpha^2+\beta^2>k^2$) decaying in the x_3 direction (i.e. $\Im m\,\gamma>0$).

ii) For any solution $u_{2\mathrm{D}}\in C_b^2(\mathbb{R}^2)$ of the two-dimensional equation $(\Delta_{x'}+k^2 I)u_{2\mathrm{D}}=0$, the function $u(\vec{x}) := u_{2\mathrm{D}}(x')$ satisfies the (HSRC), and $u(\vec{x}) := u_{2\mathrm{D}}(x')x_3$ does not (cf. Subsection 6.1). In particular, $u(\vec{x}) := J_0(k|x'|)$ fulfills the (HSRC) and cylindrical wave functions like $u(\vec{x}) := \frac{\mathbf{i}}{4}H_0^{(1)}(k|x'|)$ satisfy at least (1.2) (cf. Subsect. 6.1).

iii) Any generalized plane-wave solution $u(\vec{x}) := e^{\mathbf{i}(\alpha x_1+\beta x_2+\gamma x_3)}$ with parameters $\alpha,\beta\in\mathbb{R}$, $\gamma\in\mathbb{C}$, and $\alpha^2+\beta^2+\gamma^2=k^2$ satisfies the (HSRC) if and only if either $\gamma>0$ or $\Re e\,\gamma\geq 0$ together with $\Im m\,\gamma>0$ (cf. Subsect. 6.2). In other words, the (HSRC) is equivalent to the well-known radiation condition for quasiperiodic functions in the theory of gratings (cf. Corollary 6.1).

iv) Any solution, satisfying the classical Sommerfeld radiation condition, fulfills the (HSRC) (cf. Subsect. 6.3 and cf. [8] for Sommerfeld's condition on the half space). In other words, for $\vec{y}\notin\Omega$, functions $u(\vec{x}) := G(\vec{y},\vec{x})$ (cf. the subsequent (2.1)) and all their derivatives satisfy (HSRC), but "incoming" waves like $u(\vec{x}) := \overline{G(\vec{y},\vec{x})}$ do not.

v) For any solution $u_{2\mathrm{D}}$ of the two-dimensional Helmholtz equation in $\mathbb{R}^2$ satisfying the two-dimensional Sommerfeld radiation condition, the wave function $u(\vec{x}) := u_{2\mathrm{D}}((x_1,x_3)^\top)$ satisfies the radiation condition (HSRC). Moreover, for a source point $(y_1,y_3)^\top$ with $y_3<0$, the function $\vec{x}\mapsto G_{2\mathrm{D}}((y_1,y_3)^\top,(x_1,x_3)^\top)$, defined with the two-dimensional fundamental solution $G_{2\mathrm{D}}$ (cf. (2.2)), satisfies the radiation condition (HSRC) (cf. item iv) at the end of Sect. 8).

vi) Using the radiation condition (HSRC), uniqueness (cf. Prop. 8.1) and existence of the solution to the Dirichlet problem of the Helmholtz equation over the half plane $\mathbb{R}^3_+$ can be shown. However, special conditions are needed for the existence. For example, it is sufficient that the Dirichlet data is in DD_τ, AV_κ, or FC_k (cf. Prop. 8.3 and compare [6] for the two-dimensional case with continuous data over rough surfaces and [3], Sect. 5.1.1.1 for Dirichlet data from $L^2(\mathbb{R}^2)$, $H^{1/2}(\mathbb{R}^2)\subset H_k^{1/2}(\mathbb{R}^2)$, and a special subspace $X'\subset C_b(\mathbb{R}^2)+L^2(\mathbb{R}^2)$).

In many cases part iii) of the radiation condition (HSRC) is redundant in the sense that boundary conditions for the solution exclude additive terms $u_{\mathrm{add}}(\vec{x}) := u_{2\mathrm{D}}(x')(x_3-x_{f,3})$ and imply unique extension of $u|_{\mathbb{R}^3_{x_{f,3}}}$ to $\mathbb{R}^3_{x_{f,3},+}$ under the condition (1.2). E.g. for Dirichlet solutions, Prop. 4.2 claims the redundancy. Props. 4.3 and 4.3 present the redundancy for the Neumann and Robin condition, if the boundary $\partial\Omega$ is polyhedral. Restricting our consideration to Helmholtz solutions of the just mentioned boundary conditions, we expect a unique solution of the boundary value problem. However, the solution might not be bounded or weakly bounded. Then the solution of, e.g. the Dirichlet problem, is bounded if and only if the prescribed Dirichlet data is "physically meaningful". The additional condition part iii) of (HSRC) would automatically hold for "physically meaningful" Dirichlet data. Requiring part iii) of (HSRC) for "non-physical" Dirichlet data would lead to no solutions.

The plan of this paper is as follows. In Sect. 2 we present the formulas for the fundamental solution and its derivatives, show the asymptotics, and, using Green's formula, we derive a representation formula for the second order derivative of Helmholtz solutions. The weak boundedness of the potential based on the third order derivative of the fundamental solution is analyzed in Sect. 3. Using this potential and a second-order Taylor series expansion, we obtain a representation for the Helmholtz solution in Sect. 4 and the uniqueness of this representation either for Dirichlet problems over general domains Ω or for solutions with Neumann or Robin boundary condition over domains with polyhedral boundary. The representation requires the solution of a two-dimensional inhomogeneous Helmholtz equation over the full plane $\mathbb{R}^2$. So in Sect. 5 we give a series expansion for such solutions. For special important Helmholtz solutions over the perturbed half-space Ω, we show that the radiation condition (HSRC) is fulfilled in Sect. 6. In Sect. 7 we prove that condition (HSRC) is independent of the choice of the x_3 coordinates $x_{f,3}$ and $x_{h,3}$ and that the conditions iii) and iii') of the (HSRC) are equivalent. The solution of the Dirichlet problem for the Helmholtz equation over $\mathbb{R}^3_+$ is discussed in Sect. 8. Finally, in Sect. 9 we derive the uniqueness of the Dirichlet problem over a thin layer, which implies that it is sufficient to require condition ii) of (HSRC) over two planes with sufficiently small distance.

2 Fundamental solution, Green's function, and formula for $\partial^2_{x_3} u$

⟨s2⟩ 2.1 Fundamental solution and half-space Green's function

⟨s2.1⟩ Denote the points in $\mathbb{R}^3$ by $\vec{x}$ and $\vec{y}$ as in Sect. 1. For our positive real valued wave number k (even for complex valued k with $\Re e\, k > 0$ and $\Im m\, k \geq 0$), the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$ is given by

$$G(\vec{x},\vec{y}\,) \;=\; G(\vec{x}-\vec{y}\,) \;:=\; \frac{1}{4\pi}\frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{|\vec{x}-\vec{y}\,|}\,. \tag{2.1}$$ FS

Note that the corresponding two-dimensional function is given by

$$G_{\mathrm{2D}}(x',y') \;=\; G_{\mathrm{2D}}(x'-y') \;:=\; \frac{\mathbf{i}}{4} H^{(1)}_0(k|x'-y'|)\;, \tag{2.2}$$ 2DFS

where $H^{(1)}_0$ is the Hankel function of first kind and order zero.

Clearly, for $j=1,2$, the first and second order derivatives of the three-dimensional function are

$$\partial_{y_3} G(\vec{x},\vec{y}\,) \;=\; \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{(\mathbf{i}k)(y_3-x_3)}{|\vec{x}-\vec{y}\,|^2} - \frac{y_3-x_3}{|\vec{x}-\vec{y}\,|^3}\right\}, \tag{2.3}$$ DFS

$$\partial_{y_3}\partial_{y_j} G(\vec{x},\vec{y}\,), \;=\; \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}(y_3-x_3)(y_j-x_j)}{4\pi|\vec{x}-\vec{y}\,|^3}\left\{(\mathbf{i}k)^2 - \frac{2(\mathbf{i}k)}{|\vec{x}-\vec{y}\,|} - \frac{3}{|\vec{x}-\vec{y}\,|^2}\right\}, \tag{2.4}$$ DDJFS

$$\begin{aligned}\partial^2_{y_3} G(\vec{x},\vec{y}\,) \;=\; \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\Bigg\{&\frac{(\mathbf{i}k)}{|\vec{x}-\vec{y}\,|^2} - \frac{1}{|\vec{x}-\vec{y}\,|^3} + \frac{(\mathbf{i}k)^2(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^3}\\ &- \frac{3(\mathbf{i}k)(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^4} + \frac{3(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^5}\Bigg\}.\end{aligned} \tag{2.5}$$ DDFS

The third order derivative is of the form $\mathcal{O}_{x_3-y_3}(|\vec{x}-\vec{y}\,|^{-3})$ for $|\vec{x}-\vec{y}\,|\to\infty$. Indeed,

$$
\begin{aligned}
\partial^3_{y_3} G(\vec{x},\vec{y}\,) &= \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{3(\mathbf{i}k)^2(y_3-x_3)}{|\vec{x}-\vec{y}\,|^3}-\frac{9(\mathbf{i}k)(y_3-x_3)}{|\vec{x}-\vec{y}\,|^4}+\frac{(\mathbf{i}k)^3(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^4}\right. \\
&\qquad -\frac{6(\mathbf{i}k)^2(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^5}+\frac{9(y_3-x_3)}{|\vec{x}-\vec{y}\,|^5}+\frac{15(\mathbf{i}k)(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^6} \\
&\qquad \left.-\frac{15(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^7}\right\} \qquad (2.6) \\
&= \frac{(y_3-x_3)}{4\pi|\vec{x}-\vec{y}\,|^3}\left\{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}(\mathbf{i}k)^3\frac{(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|}+\mathcal{O}(1)\right\} \qquad (2.7)
\end{aligned}
$$

for $|\vec{x}-\vec{y}\,|\to\infty$. Note that the factor $(y_3-x_3)|\vec{x}-\vec{y}\,|^{-3}/4\pi$ is the double layer kernel for the Laplace equation. This kernel defines a uniformly bounded operator in the L^∞ space (cf. e.g. [9]).

For the fourth order derivative, we obtain

$$
\begin{aligned}
\partial^4_{y_3} G(\vec{x},\vec{y}\,) &= \frac{e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}}{4\pi}\left\{\frac{3(\mathbf{i}k)^2}{|\vec{x}-\vec{y}\,|^3}-\frac{9(\mathbf{i}k)}{|\vec{x}-\vec{y}\,|^4}+\frac{6(\mathbf{i}k)^3(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^4}+\frac{9}{|\vec{x}-\vec{y}\,|^5}\right. \\
&\qquad -\frac{36(\mathbf{i}k)^2(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^5}+\frac{(\mathbf{i}k)^4(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^5}+\frac{90(\mathbf{i}k)(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^6} \\
&\qquad -\frac{10(\mathbf{i}k)^3(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^6}-\frac{90(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^7}+\frac{45(\mathbf{i}k)^2(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^7} \\
&\qquad \left.-\frac{105(\mathbf{i}k)(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^8}+\frac{105(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^9}\right\}. \qquad (2.8)
\end{aligned}
$$

In order to enable the computation of finite-part integrals, we shortly look at the kernel behaviour for $|\vec{x}-\vec{y}\,|\to 0$. By the Taylor-series expansion of $e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}$ we get

$$
\begin{aligned}
G(\vec{x},\vec{y}\,) &= \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}\,|}+\mathcal{O}(1), \\
\partial_{y_3} G(\vec{x},\vec{y}\,) &= -\frac{1}{4\pi}\frac{(y_3-x_3)}{|\vec{x}-\vec{y}\,|^3}+\mathcal{O}(1)\,, \qquad (2.9) \\
\partial^2_{y_3} G(\vec{x},\vec{y}\,) &= -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}\,|^3}+\frac{1}{4\pi}\frac{3(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^5}+\mathcal{O}\left(\frac{1}{|\vec{x}-\vec{y}\,|}\right), \qquad (2.10) \\
\partial^3_{y_3} G(\vec{x},\vec{y}\,) &= \frac{1}{4\pi}\frac{9(y_3-x_3)}{|\vec{x}-\vec{y}\,|^5}-\frac{1}{4\pi}\frac{15(y_3-x_3)^3}{|\vec{x}-\vec{y}\,|^7}+\mathcal{O}\left(\frac{(y_3-x_3)}{|\vec{x}-\vec{y}\,|^3}+\frac{1}{|\vec{x}-\vec{y}\,|}\right), \\
\partial^4_{y_3} G(\vec{x},\vec{y}\,) &= \frac{1}{4\pi}\frac{3}{|\vec{x}-\vec{y}\,|^5}\left\{3-30\frac{(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^2}+35\frac{(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^4}\right\} \qquad (2.11) \\
&\qquad +\frac{1}{4\pi}\frac{3/2\;(\mathbf{i}k)^2}{|\vec{x}-\vec{y}\,|^3}\left\{-1+6\frac{(y_3-x_3)^2}{|\vec{x}-\vec{y}\,|^2}-5\frac{(y_3-x_3)^4}{|\vec{x}-\vec{y}\,|^4}\right\}+\mathcal{O}\left(\frac{1}{|\vec{x}-\vec{y}\,|}\right),
\end{aligned}
$$

which proves that the potential operators with kernels equal to $\partial^l_{y_3} G((x',x_{f,3})^\top,(y',x_{f,3})^\top)$ for $l=0,2$, with computation points $\vec{x}=(x',x_{f,3})^\top$, and with integration over the $\vec{y}=(y',x_{f,3})^\top$ s.t. $|x'-y'|<c$ are locally compact perturbations of the operators for the Laplace equation. Note that $\partial^l_{y_3} G((x',x_{f,3})^\top,(y',x_{f,3})^\top)=0$ for $l=1,3$. Altogether, the corresponding potentials of smooth and finitely supported layer functions $u|_{\mathbb{R}^3_{x_{f,3}}}$ computed at $\vec{x}\in\mathbb{R}^3_{x_{f,3},+}$ have well-defined limits for $x_3\to x_{f,3}$. These limits can be computed by the well-known jump relation including the values of the potential

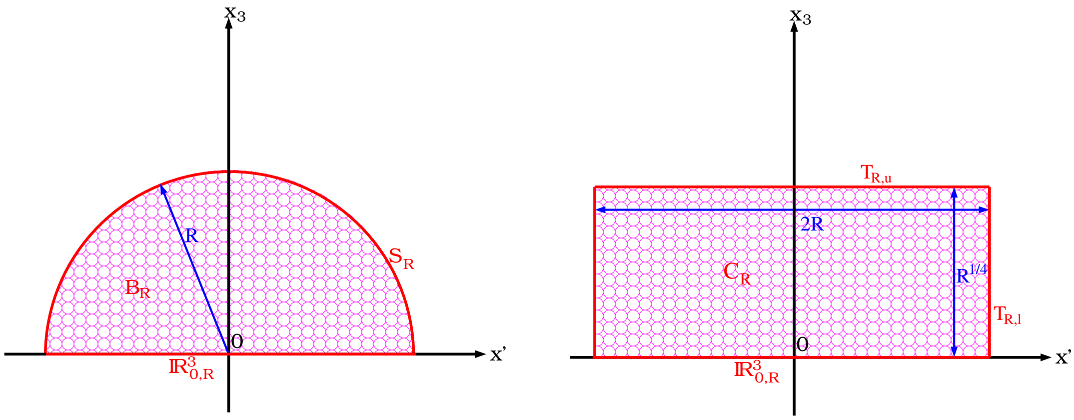

Figure 2: Half ball and cylinder.

integral at $(x', x_{x_{f,3}})^\top$. In case of the kernel $\partial^2_{y_3} G(\vec{x}, \vec{y})$, the potential integral is to be understood as a finite-part integral. For its definition, due to the Helmholtz equation, $\partial^2_{y_3} G(\vec{x}, \vec{y})$ can be replaced by $-(\Delta_{y'} + k^2 I) G(\vec{x}, \vec{y})$, and, by partial integration, the tangential operator $(\Delta_{y'} + k^2 I)$ can be moved to the layer function $u|_{\mathbb{R}^3_{x_{f,3}}}$ of the potential. In other words, we have to suppose that the layer function is twice continuously differentiable.

The Green's function for the Dirichlet problem over $\mathbb{R}^3_{x_{f,3},+}$ is given as

$$\Phi(\vec{x}, \vec{y}) = G(\vec{x}, \vec{y}) - G\big((x', 2x_{x_{f,3}} - x_3)^\top, \vec{y}\big) = G(\vec{x}, \vec{y}) - G\big(\vec{x}, (y', 2x_{x_{f,3}} - y_3)^\top\big) \,. \quad (2.12)\ \boxed{\mathrm{GF}}$$

We get vanishing boundary values $\Phi(\vec{x}, (y', x_{x_{f,3}})^\top) = 0$ and

$$\begin{aligned} \partial^3_{y_3} \Phi(\vec{x}, \vec{y}) &= \partial^3_{y_3} G(\vec{x}, \vec{y}) + \partial^3_{y_3} G\big(\vec{x}, (y', 2x_{x_{f,3}} - y_3)^\top\big) \,, \\ \partial^3_{y_3} \Phi\big(\vec{x}, (y', x_{x_{f,3}})^\top\big) &= 2\partial^3_{y_3} G\big(\vec{x}, (y', x_{x_{f,3}})^\top\big) \,, \end{aligned} \quad (2.13)\ \boxed{\mathrm{DGF}}$$

such that $\partial^3_{y_3} \Phi(\vec{x}, \vec{y}) = \mathcal{O}_{x_3 - y_3}(|\vec{x} - \vec{y}|^{-3})$ for $|\vec{x} - \vec{y}| \to \infty$ with $\vec{y} \in \mathbb{R}^3_{x_{f,3}}$, $\vec{x} \in \mathbb{R}^3_{x_3}$ follows from (2.7).

2.2 Representation formula (1.2) for the second order derivative

⟨ss2.2⟩ Next we recall the representation formula for the second order derivative $\partial^2_{x_3} u$ of the solution u to the Helmholtz equation $\Delta u + k^2 u = 0$ over $\mathbb{R}^3_+$ (cf. the subsequent (2.14)). To slightly simplify the subsequent formulas, we set $x_{f,3} = 0$ and consider the half space $\mathbb{R}^3_+$ instead of the general $\mathbb{R}^3_{x_{f,3},+}$. For large $R > 0$, we introduce the disc $\mathbb{R}^3_{0,R} := \{(x', 0)^\top \in \mathbb{R}^3 : \ |x'| \le R\}$ of radius R, the half ball $B_R := \{(x', x_3)^\top \in \mathbb{R}^3 : \ x_3 > 0, |\vec{x}| < R\}$ and $S_R := \{(x', x_3)^\top \in \mathbb{R}^3 : \ x_3 \ge 0, |\vec{x}| = R\}$ its upper spherical boundary. Further, we introduce the cylinder C_R of radius R and height $R^{1/4}$ given by $C_R := \{(x', x_3)^\top \in \mathbb{R}^3 : \ 0 < x_3 < R^{1/4}$ and $|x'| < R\}$ together with its lateral and upper boundary (cf. Fig. 2)

$$\begin{aligned} T_R \ := \ T_{R,l} \cup T_{R,u}, \quad T_{R,l} &:= \big\{(x', x_3)^\top \in \mathbb{R}^3 : \ 0 \le x_3 \le R^{1/4} \text{ and } |x'| = R)\big\} \,, \\ T_{R,u} &:= \big\{(x', x_3)^\top \in \mathbb{R}^3 : \ x_3 = R^{1/4} \text{ and } |x'| \le R\big\} \,. \end{aligned}$$

We consider either $\Omega_R := B_R$ and $\Sigma_R := S_R$ or $\Omega_R := C_R$ and $\Sigma_R := T_R$. By ν we denote the normal at the boundary $\mathbb{R}^3_{0,R} \cup \Sigma_R$ of Ω_R pointing into outward direction. We assume that condition i) in

(HSRC) is fulfilled. The symmetric Green's formula applied to u and $\vec{y}\mapsto\partial_{y_3}^2\Phi(\vec{x},\vec{y}\,)$ with fixed $\vec{x}\in\Omega_R$ leads to

$$\begin{aligned}&\int_{\mathbb{R}^3_{0,R}\cup\Sigma_R}\{\partial_\nu u\partial_{y_3}^2\Phi(\vec{x},\cdot)-u\partial_\nu\partial_{y_3}^2\Phi(\vec{x},\cdot)\}\\&\quad=\int_{\Omega_R}\left\{(\Delta+k^2I)u\partial_{y_3}^2\Phi(\vec{x},\cdot)-u(\Delta+k^2I)\partial_{y_3}^2\Phi(\vec{x},\cdot)\right\}.\end{aligned}$$

Using $\Phi(\vec{x},\cdot)\equiv 0$ over $\mathbb{R}^3_0$ and the Helmholtz equation, we get that the second order derivative $\partial_{y_3}^2\Phi(\vec{x},\cdot)=-\partial_{y_1}^2\Phi(\vec{x},\cdot)-\partial_{y_2}^2\Phi(\vec{x},\cdot)-k^2\Phi(\vec{x},\cdot)$ is zero over $\mathbb{R}^3_0$. From the Green's function property we obtain $(\Delta+k^2I)\partial_{y_3}^2\Phi(\vec{x},\cdot)=\partial_{y_3}^2(\Delta+k^2I)\Phi(\vec{x},\cdot)=\partial_{y_3}^2\delta_{\vec{x}}$. Thus, for a solution u of the Helmholtz equation $(\Delta+k^2I)u=0$, we arrive at

$$\partial_{x_3}^2u(\vec{x}\,) = \int_{\mathbb{R}^3_{0,R}}\partial_{y_3}^3\Phi(\vec{x},\cdot)u-\int_{\Sigma_R}\{\partial_\nu u\partial_{y_3}^2\Phi(\vec{x},\cdot)-u\partial_\nu\partial_{y_3}^2\Phi(\vec{x},\cdot)\},$$

$$\partial_{x_3}^2u(\vec{x}\,) = [V_ku](\vec{x}\,)-I_\infty, \tag{2.14}$$ RF

$$[V_ku](\vec{x}\,) := \int_{\mathbb{R}^2}\partial_{y_3}^3\Phi\big(\vec{x},(y',0)^\top\big)\,u\big((y',0)^\top\big)\,\mathrm{d}y', \tag{2.15}$$ DDLP

$$I_\infty := \lim_{R\to\infty}\int_{\Sigma_R}\left\{\partial_\nu u\partial_{y_3}^2\Phi(\vec{x},\cdot)-u\partial_\nu\partial_{y_3}^2\Phi(\vec{x},\cdot)\right\}.$$

Here $[V_ku]$ is the twice differentiated double layer potential on the right-hand side of (1.2). Note that any Helmholtz solution u satisfying i) of (HSRC) is uniformly bounded over $\mathbb{R}^3_0$ such that V_ku is well defined. Altogether, to get the representation in the radiation condition (1.2) for a solution u of the Helmholtz equation over $\mathbb{R}^3_+$, we only have to suppose condition i) of (HSRC) and to show that the limit I_∞ is zero.

3 Weak boundedness of the potential in (1.2)

⟨s3⟩ Consider a measurable function u bounded over $\mathbb{R}^3_0$ and consider the twice differentiated double layer potential $[V_ku](\vec{x}\,)$ defined by (2.15), where Φ is defined in (2.12). In other words, $[V_ku](\vec{x}\,)$ is given by the subsequent (3.1) with $x_{f,3}=0$. Without loss of generality, we fix $x'=(0,0)^\top$ and consider the behaviour of $[V_ku](\vec{x}\,)$ for $\vec{x}=(x',x_3)^\top$ with $x_3\to\infty$. Due to (2.13) we have to estimate

$$[V_ku](\vec{x}\,) = I_{x_3} := 2\int_{\mathbb{R}^2}\partial_{y_3}^3G\big(\vec{x},(y',x_{f,3})^\top\big)\,f(y')\mathrm{d}y',\quad f(y'):=u\big((y',x_{f,3})^\top\big), \tag{3.1}$$ IX30

$$I_{x_3} = 2\int_{\mathbb{R}^2}\partial_{y_3}^3G\big(\vec{x},(y',0)^\top\big)\,f(y')\mathrm{d}y',\qquad f(y'):=u\big((y',0)^\top\big). \tag{3.2}$$ IX3

Taking into account (2.7) and the boundedness of the integral of the double layer kernel (cf. [9]), we get $I_{x_3}=\frac{(\mathbf{i}k)^3}{2\pi}J_{x_3}+\mathcal{O}(1)$ with

$$\begin{aligned}J_{x_3} := \int_{\mathbb{R}^2}e^{\mathbf{i}k|\vec{x}-\vec{y}\,|}\frac{(0-x_3)^3}{|\vec{x}-\vec{y}\,|^4}f(y')\mathrm{d}y' &= -\int_{\mathbb{R}^2}e^{\mathbf{i}k\sqrt{x_3^2+|y'|^2}}\frac{x_3^3}{\{x_3^2+|y'|^2\}^2}f(y')\mathrm{d}y'\\&= -x_3\int_{\mathbb{R}^2}\frac{e^{\mathbf{i}kx_3\sqrt{1+|z'|^2}}}{\{1+|z'|^2\}^2}f(x_3z')\mathrm{d}z'.\end{aligned}$$

We substitute $z'=\sqrt{r^2-1}\,(\cos\phi,\sin\phi)^\top$ and $\mathrm{d}z'=r\mathrm{d}\phi\,\mathrm{d}r$ to get

$$J_{x_3} \;=\; -x_3\int_1^\infty \frac{e^{\mathbf{i}kx_3r}}{r^3}\int_0^{2\pi} f\left(x_3\sqrt{r^2-1}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right)\mathrm{d}\phi\,\mathrm{d}r.$$

The last integral is difficult to estimate. For bounded f, we get $|[V_ku](\vec{x}\,)|\le c|x_3|$. Here and in the following c stands for a generic positive constant, the value of which varies from instance to instance.

Next we go back to consider a general x' and prove that I_{x_3} of (3.2) fulfills at least the weak boundedness condition used in (1.4). We start with a weak estimate (cf. (1.4)) for I_{x_3} with f replaced by the Dirac delta $\delta_{\vec{y}}$ at $\vec{y}=(y',0)^\top$. For $x'\ne y'$, partial integration leads us to

$$\int_0^{x_3}(x_3-t)\partial_{y_3}^3 G\big((x',t)^\top,(y',0)^\top\big)\;\mathrm{d}t = x_3\,\partial_{y_3}^2 G\big((x',0)^\top,(y',0)^\top\big) \tag{3.3}$$
$$+\partial_{y_3}G\big((x',x_3)^\top,(y',0)^\top\big)-\partial_{y_3}G\big((x',0)^\top,(y',0)^\top\big)\,,$$

intDDDPHI

where we have used $\partial_{y_3}\partial_{y_3}^l G(\vec{x},\vec{y}\,)=-\partial_{x_3}\partial_{y_3}^l G(\vec{x},\vec{y}\,)$. The formulas in (2.3) and (2.5) together with

$$\begin{aligned} e^{\mathbf{i}k\left|(x',x_3)^\top-(y',0)^\top\right|} &= e^{\mathbf{i}k|x'-y'|}e^{\mathbf{i}kx_3^2/\{|(x',x_3)^\top-(y',0)^\top|+|x'-y'|\}}\\ &= e^{\mathbf{i}k|x'-y'|}+\mathcal{O}\left(\frac{x_3^2}{|(x',x_3)^\top-(y',0)^\top|}\right), \end{aligned}\tag{3.4}$$

cih1

$$\frac{1}{|(x',x_3)^\top-(y',0)^\top|^l} = \frac{1}{|x'-y'|^l} +\mathcal{O}\left(\frac{x_3^2}{|x'-y'|^{l+1}\,|(x',x_3)^\top-(y',0)^\top|}\right),\; l=2,3, \tag{3.5}$$

cih2

imply

$$\begin{aligned} &\int_0^{x_3}(x_3-t)\partial_{y_3}^3 G\big((x',t)^\top,(y',0)^\top\big)\;\mathrm{d}t\\ &= \frac{e^{\mathbf{i}k|x'-y'|}}{4\pi}\left\{\frac{(\mathbf{i}k)x_3}{|x'-y'|^2}-\frac{x_3}{|x'-y'|^3}\right\}\\ &\quad+\frac{e^{\mathbf{i}k\sqrt{x_3^2+|x'-y'|^2}}}{4\pi}\left\{\frac{(\mathbf{i}k)(-x_3)}{\sqrt{x_3^2+|x'-y'|^2}^2}-\frac{-x_3}{\sqrt{x_3^2+|x'-y'|^2}^3}\right\}\\ &= \mathcal{O}\left(\frac{x_3^3}{|x'-y'|^2\sqrt{x_3^2+|x'-y'|^2}}\right) = \mathcal{O}\left(\frac{x_3^{3-\epsilon_u}}{|x'-y'|^2\sqrt{x_3^2+|x'-y'|^2}^{1-\epsilon_u}}\right) \end{aligned}\tag{3.6}$$

newF1

for $|x'-y'|\to\infty$.

Now we come back to a general $f\in C_b^2(\mathbb{R}^2)$ and estimate

$$\begin{aligned} \int_0^{x_3}(x_3-t)I_t\mathrm{d}t &= 2\int_0^{x_3}(x_3-t)\int_{\mathbb{R}_0^3}\partial_{y_3}^3 G\big((x',t)^\top,(y',0)^\top\big)\,f(y')\mathrm{d}y'\mathrm{d}t\\ &= 2\int_{\mathbb{R}_0^3}\int_0^{x_3}(x_3-t)\partial_{y_3}^3 G\big((x',t)^\top,(y',0)^\top\big)\,f(y')\mathrm{d}t\mathrm{d}y'. \end{aligned}$$

We fix $\varepsilon>0$ and consider the case of $\vec{x}$ and $\vec{y}$ with $x_3\ge\varepsilon$. Surely, $y_3=x_{f,3}=0$ s.t. $|\vec{x}-\vec{y}\,|\ge\varepsilon$. Then the part of I_t for integration over y' with $|x'-y'\,|\le\varepsilon$ is bounded (cf. (2.6)), and the part I_t for integration over y' with $|x'-y'\,|\ge\varepsilon$ is the critical part. Due to (3.6), the corresponding part of the integral

$\int_0^{x_3}(x_3-t)I_t\mathrm{d}t$ is a convolution integral of f and an absolutely integrable convolution kernel bounded by $\mathcal{O}(x_3^{3-\varepsilon_u}[\varepsilon+|x'-y'|]^{3-\varepsilon_u})$. The result is uniformly bounded by $x_3^{3-\varepsilon_u}$.

Now we switch to the case $x_3<\varepsilon$. By the boundedness of I_{x_3} shown next, we even get a bound $\mathcal{O}(x_3)$ for $\int_0^{x_3}(x_3-t)I_t\mathrm{d}t$, and I_{x_3} is weakly bounded in the case $x_3<\varepsilon$ as well.

For the uniform boundedness in the strip with $0\le x_3\le\varepsilon$, we need the assumption $f\in C_b^2(\mathbb{R}^2)$. We choose a smooth cut-off function χ over $\mathbb{R}^2$ s.t. $\chi(z')=1$ for $|z'|\le 1$ and $\chi(z')=0$ for $|z'|\ge 2$. With this we split $I_{x_3}=I_{x_3}[f]$ into the sum of $I_{x_3}^a:=I_{x_3}[f(1-\chi(\cdot-x'))]$ and $I_{x_3}^b:=I_{x_3}[f\chi(\cdot-x')]$. Then the arguments for the case $x_3\ge\varepsilon$ prove that (2.7) implies the uniform boundedness of $I_{x_3}^a$. For $I_{x_3}^b$, we observe the equality $\partial_{y_3}^3 G(\vec{x},\vec{y})=-(\Delta_{y'}+k^2I)\partial_{y_3}G(\vec{x},\vec{y})$. Applying integration by parts we can move the operator $(\Delta_{y'}+k^2I)$ from the potential kernel to the layer function $f\chi(\cdot-x')$, and (2.9) implies the uniform boundedness of $I_{x_3}^b$. We even get

⟨BoundPot⟩ **Proposition 3.1.** *Suppose that f is continuous over $\mathbb{R}^2=\mathbb{R}_0^2$ and that there are constants $c_f>0$ and $0\le\tau_f<1$ such that $|f(x')|<c_f(1+|x'|)^{\tau_f}$ holds for any $x'\in\mathbb{R}^2$. Fix $\varepsilon>0$. Then there exist positive constants c_u and ε_u depending on f such that $[V_kf]$ satisfies the weak boundedness condition $1/(x_3-\varepsilon)\ |\int_\varepsilon^{x_3}(x_3-t)/(x_3-\varepsilon)[V_kf]((x',t)^\top)\mathrm{d}t|\le c_u(x_3-\varepsilon)^{1-\varepsilon_u}$. Additionally, if f is twice continuously differentiable with uniformly bounded derivatives, then the potential $[V_kf]$ satisfies the weak boundedness condition, i.e., (1.4) with u replaced by $[V_kf]$.*

For special functions u, i.e., for special f we can get more. Suppose $f\in AV_\kappa$ with $\kappa>1$ (cf. (1.8)) and assume $x_3>\varepsilon$ for a fixed positive ε. Then, by (1.7) and (2.7)

$$\begin{aligned}
V_k[f](\vec{x}) &= \int_{\mathbb{R}^2}\partial_{y_3}^3\Phi\left(\vec{x},(y',0)^\top\right)f(y')\mathrm{d}y'\\
&= \int_0^\infty\int_0^{2\pi}\partial_{y_3}^3\Phi\left((r\cos\phi,r\sin\phi,x_3)^\top,(0,0,0)^\top\right)f\left(x'-r(\cos\phi,\sin\phi)^\top\right)r\mathrm{d}\phi\mathrm{d}r\\
&= \int_0^\infty\partial_{y_3}^3\Phi\left((r,0,x_3)^\top,(0,0,0)^\top\right)av(f,x',r)r\mathrm{d}r,\\
|V_k[f](\vec{x})| &\le c+\int_1^\infty c\frac{1}{r}\,c_v r^{1-\kappa}\mathrm{d}r\le c.
\end{aligned}$$

4 Representation of the solution by the potential operator in (1.2)

⟨s3.5⟩ Suppose that u is a solution of the Helmholtz equation $\Delta u+k^2u=0$ over $\mathbb{R}_+^3$, that u as well as all derivatives upto order two are continuous on the closure of $\mathbb{R}_+^3$, and that the second derivative of u w.r.t. x_3 is given as $\partial_{x_3}^2u=[V_ku]$ by the right-hand side of (1.2). Then, due to the second order Taylor-series expansion, we get

$$u(\vec{x}) \;=\; f_1(x')+f_2(x')\,(x_3-x_{f,3})+\int_{x_{f,3}}^{x_3}(x_3-t)[V_ku]\big((x',t)^\top\big)\,\mathrm{d}t, \qquad (4.1)\ \boxed{\text{uR}}$$

where the functions $f_1(x'):=u((x',x_{f,3})^\top)$ and $f_2(x'):=\partial_{x_3}u((x',x_{f,3})^\top)$ are solutions of the two-dimensional inhomogeneous Helmholtz equations

$$\Delta_{x'}f_1(x')+k^2f_1(x') \;=\; -[V_ku]\big((x',x_{f,3})^\top\big)\,, \qquad (4.2)\ \boxed{\text{2DHa}}$$

$$\begin{aligned}
\Delta_{x'}f_2(x')+k^2f_2(x') &= -\partial_{x_3}[V_ku]\big((x',x_{f,3})^\top\big) \qquad (4.3)\ \boxed{\text{2DHb}}\\
&= \int_{\mathbb{R}^2}2\partial_{y_3}^4G\big((x',x_{f,3})^\top,(y',x_{f,3})^\top\big)\,u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y'.
\end{aligned}$$

DOI 10.20347/WIAS.PREPRINT.2669

Indeed, from $\Delta = \Delta_{x'} + \partial_{x_3}^2$ we conclude, for the Helmholtz solution u, that $(\Delta_{x'} + k^2 I)u = -\partial_{x_3}^2 u$ and $(\Delta_{x'} + k^2 I)\partial_{x_3} u = -\partial_{x_3}\partial_{x_3}^2 u$. For the Fourier series expansion of the general solution to equations (4.2) and (4.3), we refer to Lemma 5.1 and Remark 5.2. The last integral in (4.3) is defined as a finite-part integral.

⟨proRS⟩

Proposition 4.1. *For any function $u|_{\mathbb{R}^3_{x_{f,3}}}$ and the corresponding $[V_k u]$, the right-hand side of (4.1) satisfies the three-dimensional Helmholtz equation if and only if f_1 and f_2 are solutions of (4.2) and (4.3), respectively.*

Proof. Indeed, using the Taylor-series expansion for $[V_k u]$ and the inhomogeneous Helmholtz equation for the potential $[V_k u]$ in the form $(\Delta_{x'} + k^2 I)[V_k u] = -\partial_{x_3}^2 [V_k u]$, we get

$$\begin{aligned}
&(\Delta + k^2 I)\left(f_1(x') + f_2(x')(x_3 - x_{f_3}) + \int_{x_{f_3}}^{x_3} (x_3 - t)[V_k u]\big((x',t)^\top\big)\,\mathrm{d}t\right)\\
= \quad & (\Delta_x' + k^2 I) f_1(x') + (x_3 - x_{f_3})(\Delta_x' + k^2 I) f_2(x')\\
& + \int_{x_{f_3}}^{x_3} (x_3 - t)(\Delta_x' + k^2 I)[V_k u]\big((x',t)^\top\big)\,\mathrm{d}t + [V_k u]\big((x',x_3)^\top\big)\,.\\
= \quad & (\Delta_x' + k^2 I) f_1(x') + [V_k u]\big((x',0)^\top\big)\\
& + (x_3 - x_{f,3})\left[(\Delta_x' + k^2 I) f_2(x') + \partial_{x_3}[V_k u]\big((x',0)^\top\big)\right] + [V_k u]\big((x',x_3)^\top\big) -\\
& \left\{[V_k u]\big((x',0)^\top\big) + \partial_{x_3}[V_k u]\,\big((x',0)^\top\big)\,(x_3 - x_{f,3}) + \int_{x_{f,3}}^{x_3} (x_3 - t)\partial_{x_3}^2 [V_k u]\big((x',t)^\top\big)\,\mathrm{d}t\right\}\\
= \quad & (\Delta_x' + k^2 I) f_1(x') + [V_k u]\big((x',0)^\top\big)\\
& + (x_3 - x_{f,3})\left[(\Delta_x' + k^2 I) f_2(x') + \partial_{x_3}[V_k u]\big((x',0)^\top\big)\right]\,.
\end{aligned}$$

□

If a solution u is given over the plane $\mathbb{R}^3_{x_{f,3}}$, then the radiation condition should be formulated as an expression of $f_1 = u|_{\mathbb{R}^3_{x_{f,3}}}$, which in this sense is assumed to be known. If the function u is given over $\mathbb{R}^3_{x_{f,3}}$ and over $\mathbb{R}^3_{x_{d,3}}$ with, e.g., $x_{d,3} > x_{f,3}$, then f_2 is given by the Dirichlet-to-Neumann map

$$f_2(x') := (D_t N f_1)(x') := -2 \int_{\mathbb{R}^2} \partial_{y_3}^2 G\big((x',x_{f,3})^\top,(y',x_{f,3})^\top\big)\, u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y' \quad (4.4) \quad \text{DDtN}$$

$$= -\frac{1}{(x_{d,3} - x_{f,3})} u\big((x',x_{d,3})^\top\big) + \int_{\mathbb{R}^2} K_{DtN}(x',y') u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y', \quad (4.5) \quad \text{DTN}$$

$$K_{DtN}(x',y') := -2\partial_{y_3}^2 G\big((x',x_{d,3})^\top,(y',x_{d,3})^\top\big) + \frac{2}{(x_{d,3} - x_{f,3})}\partial_{y_3} G\big((x',x_{d,3})^\top,(y',x_{f,3})^\top\big)\,.$$

The integral in (4.4) does not exist for general functions u over $\mathbb{R}^3_{x_{f,3}}$. The integral in (4.5) does exist for $u|_{\mathbb{R}^3_{x_{f,3}}} \in C_b^2(\mathbb{R}^2)$. Namely, using the formulas (2.3), (2.5), (3.4), and (3.5), we get the kernel estimate $K_{DtN}(x',y') = \mathcal{O}(|x'-y'|^{-3})$ for $|x'-y'| \to \infty$. Clearly, $K_{DtN}(x',y')$ is locally a hypersingular kernel, and the integral in (4.5) exists as a finite-part integral for sufficiently smooth functions $x' \mapsto u((x',x_{f,3})^\top)$.

Unfortunately, the solutions of (4.2) and (4.3) are not unique. Even more, (1.2) and the representation with the right-hand side in (4.5) is fulfilled for $u(\vec{x}\,)$ replaced by the sum $u(\vec{x}\,) + u_{2\mathrm{D}}(x')(x_3 - x_{f,3})$ as well if only $(\Delta_{x'} + k^2 I)u_{2\mathrm{D}} = 0$. For instance, the functions $u_{2\mathrm{D}}(x') := e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = k^2$ or $u_{2\mathrm{D}}(x') := J_0(k\sqrt{x_1^2 + x_2^2})$ are solutions of the homogeneous Helmholtz equation

$(\Delta_{x'}+k^2 I)u_{2\mathrm{D}}=0$. Whereas f_1 for u in (4.1) might be uniquely determined as the argument for the radiation condition, the function f_2 is unique due to (1.4). Indeed, if f_2 leads to a solution in (4.1) bounded as in (1.4), then any different $\tilde{f}_2$ leads to a perturbation $\vec{x}\mapsto u(\vec{x})+[\tilde{f}_2-f_2](x')(x_3-x_{f,3})$ violating (1.4).

For Dirichlet solutions over Ω above general rough surfaces (cf. Sect. 1) satisfying the conditions i) and ii) of (HSRC), the representation (4.1) is unique without condition (1.4). In fact, the Dirichlet condition $u((x',F(x'))^\top)=v(x'),\ \forall x'\in\mathbb{R}^2$ with continuous Dirichlet data v enforces uniqueness:

⟨uni1⟩
Proposition 4.2. *If u_0 is a solution of the Dirichlet problem over Ω such that conditions i) and ii) of (HSRC) hold, then the second order derivative $\partial_{x_3}^2 u_0$ is fixed by condition (1.2). The general Helmholtz solution u with $\partial_{x_3}^2 u=\partial_{x_3}^2 u_0$ is given by (4.1) with $f_1=u|_{\Omega_{x_{f,3}}}$ and general solution $f_2\in C_b^l(\mathbb{R}^2)$ of (4.3). However, among these Helmholtz solutions, there is a unique solution satisfying the Dirichlet condition over the boundary of Ω, i.e., $u=u_0$.*

Proof. For two solutions u_a and u_b with the same restriction $u_a|_{\Omega_{x_{f,3}}}=u_b|_{\Omega_{x_{f,3}}}$ and the corresponding condition (HSRC) ii), we get the representation (4.1) with the function f_2 replaced by $f_{a,2}$ and $f_{b,2}$, respectively. The difference $u=u_a-u_b$ is $u(\vec{x})=(f_{a,2}(x')-f_{b,2}(x'))(x_3-x_{f,3})$ and satisfies the homogeneous Dirichlet condition. In other words, u is a Helmholtz solution over $\mathbb{R}^3$ and

$$0 \;=\; (f_{a,2}(x')-f_{b,2}(x'))\,(F(x')-x_{f,3})\,, \quad (\Delta_{x'}+k^2 I)\big(f_{a,2}(x')-f_{b,2}(x')\big) \;=\; 0,$$

where the real valued $(F(x')-x_{f,3})$ is uniformly bounded from above and below. Switching to real and imaginary parts, we get that $(F-x_{f,3})$ multiplied by a real valued Helmholtz solution is zero. If either the real or the imaginary parts of $f_{a,1}-f_{b,1}$ does not vanish identically, then $(F(x')-x_{f,3})$ is zero at least outside of a manifold of dimension one. In other words, the Lipschitz function $(F(x')-x_{f,3})$ vanishes, which contradicts to the uniform boundedness from below. We get $f_{a,2}=f_{b,2}$ and, by (4.1), $u_a=u_b$. □

Next we have a look at domains Ω_F with polyhedral boundary and Neumann's boundary condition. Over the boundary $\partial\Omega_F$ we introduce the normal field $\nu(x'):=(\nabla_{x'}F(x'),-1)^\top$ and the normal derivative $\partial_\nu u(x')=1/|\nu(x')|\ \nu(x')\cdot\nabla u(\vec{x})$. The boundary conditions $\partial_\nu u_a=g$ and $\partial_\nu u_b=g$ over $\partial\Omega_F$ lead us to $\nu(x')\cdot\nabla u=0$ over $\partial\Omega_F$ for the difference function $u:=u_a-u_b$. Introducing $f_2(x')=f_{a,2}(x')-f_{b,2}(x')$, we equivalently have $u(\vec{x})=f_2(x')(x_3-x_{f,3})$ and

$$(F(x')-x_{f,3})\nabla_{x'}F(x')\cdot\nabla_{x'}f_2(x') \;=\; f_2(x').$$

If $U\subset\mathbb{R}^2$ is a bounded domain such that the graph function F restricted to U is linear, then we denote the linear extension of $F|_U$ to the whole $\mathbb{R}^2$ by F_U. Since F_U and f_2 are analytic functions over $\mathbb{R}^2$, we arrive at

$$(F_U(x')-x_{f,3})\nabla_{x'}F_U\cdot\nabla_{x'}f_2(x') \;=\; f_2(x'). \tag{4.6}$$ ela1

Moreover, the scalar product on the left-hand side of (4.6) and its product by the first linear factor are both Helmholtz solutions. So either F_U is constant and $f_2\equiv 0$ or $\nabla_{x'}F_U\cdot\nabla_{x'}f_2(x')$ differentiated in the direction of $\nabla_{x'}F_U$ is zero. Hence, f_2 is linear in the direction of $\nabla_{x'}F_U$. Since f_2 is bounded, f_2 is a constant solution of the Helmholtz equation, and $f_2\equiv 0$ holds in any case. We obtain

⟨uni0000⟩
Proposition 4.3. *Suppose the boundary $\partial\Omega_F=\{(x',F(x'))^\top\colon x'\in\mathbb{R}^2\}$ contains a planar face. The general Helmholtz solution u with $\partial_{x_3}^2 u$ satisfying (1.2) is given by (4.1) with $f_1=u|_{\Omega_{x_{f,3}}}$ and general solution $f_2\in C_b^l(\mathbb{R}^2)$ of (4.3). However, among these Helmholtz solutions, there is at most one solution satisfying the Neumann condition over the boundary of Ω_F. Condition iii) is not needed for the Neumann problem.*

Next we choose a constant $a_R \neq 0$ and look at Robin's boundary condition. Again, for special cases, we can show $f_2 \equiv 0$ implying the redundancy of (HSRC) iii). Indeed, the conditions $\partial_\nu u_a + a_R u_a = g$ and $\partial_\nu u_b + a_R u_b = g$ lead us to $\nu(x')\cdot\nabla u + a_R|\nu(x')|u = 0$ for the difference $u := u_a - u_b$. We equivalently have

$$(F(x') - x_{f,3})\left(\nabla_{x'}F(x')\cdot\nabla_{x'}f_2(x') + a_R\sqrt{1+|\nabla_{x'}F(x')|^2}f_2(x')\right) \quad = \quad f_2(x').$$

Suppose $F|_U = F_U|_U$ for a domain U and a linear function $F_U(x') = F_U((0,0)^\top) + x'\cdot\nabla_{x'}F_U$. Then we get

$$(F_U(x') - x_{f,3})\left(\nabla_{x'}F_U\cdot\nabla_{x'}f_2(x') + a_R\sqrt{1+|\nabla_{x'}F_U|^2}f_2(x')\right) \quad = \quad f_2(x'). \qquad (4.7)$$ ela2

If $\nabla_{x'}F_U$ vanishes, then we get that f_2 is zero. If not, then, similarly to the Neumann case, the left hand side of (4.7) is a solution of the Helmholtz equation if and only if

$$\nabla_{x'}F_U\cdot\nabla_{x'}\left(\nabla_{x'}F_U\cdot\nabla_{x'}f_2(x') + a_R\sqrt{1+|\nabla_{x'}F_U|^2}f_2(x')\right) \quad = \quad 0.$$

Solving this ordinary differential equation over the line parallel to $\nabla_{x'}F_U$ containing the fixed point $y' \in \mathbb{R}^2$, we arrive at

$$\nabla_{x'}F_U\cdot\nabla_{x'}f_2\big(y' + t\nabla_{x'}F_U\big) \quad = \quad \nabla_{x'}F_U\cdot\nabla_{x'}f_2(y')\exp\left(-a_R\sqrt{1+|\nabla_{x'}F_U|^2}\,t\right).$$

The left-hand side is bounded such that $\nabla_{x'}F_U\cdot\nabla_{x'}f_2(y') = 0$. Hence, (4.7) turns in

$$\left[(F_U(x') - x_{f,3})a_R\sqrt{1+|\nabla_{x'}F_U|^2} - 1\right] f_2(x') \quad = \quad 0.$$

The Helmholtz solution $f_2(x')$ is zero for any x', at which the linear function $(F_U(x') - x_{f,3})$ is not zero. Thus the analytic function f_2 vanishes if the linear function F_U is not constant or if F_U is constant and $(F_U - x_{f,3})a_R \neq 1$.

Proposition 4.4. *Suppose the boundary $\partial\Omega_F = \{(x', F(x'))^\top : x' \in \mathbb{R}^2\}$ contains a planar face. Suppose either that this face is not parallel to the plane $\mathbb{R}^3_0$ or that $(F - x_{f,3})a_R \neq 1$ over this plane. The general Helmholtz solution u with $\partial^2_{x_3}u$ satisfying (1.2) is given by (4.1) with $f_1 = u|_{\Omega_{x_{f,3}}}$ and general solution $f_2 \in C^l_b(\mathbb{R}^2)$ of (4.3). However, among these Helmholtz solutions, there is at most one solution satisfying the Robin condition over the boundary of Ω_F. Condition iii) is not needed for the Robin boundary value problem.*

5 Solution of the inhomogeneous Helmholtz equation over $\mathbb{R}^2$

⟨ssol⟩

In this section, motivated by (4.2) and (4.3), we consider the solution of the two-dimensional Helmholtz equation $(\Delta_{x'} + k^2 I)v = f$ over the plane $\mathbb{R}^2$. Assuming the existence of a continuous v, we shall present the solution by a special Fourier series expansion (cf. the subsequent (5.2)). Analogously to (1.7), we introduce the modulated averages

$$av_m(v, x', r) \quad := \quad \int_0^{2\pi} e^{-\mathrm{i}m\phi}v\big(x' - r(\cos\phi, \sin\phi)^\top\big)\,\mathrm{d}\phi, \qquad (5.1)$$ DefMAve0

and use J_m and Y_m to denote the Bessel functions of the first and second kind, respectively. From the orthogonality of the functions $\phi \mapsto e^{\mathbf{i}m\phi}$ and from the Taylor series expansion of v at x', we conclude

$$\begin{aligned}
\text{æ}_m(v,x',r) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathbf{i}m\phi} \sum_{\alpha=0}^{|m|} \frac{\partial_{x_1}^{\alpha}\partial_{x_2}^{|m|-\alpha} v(x')\cos^{\alpha}(\phi)\sin^{|m|-\alpha}(\phi)}{\alpha!(|m|-\alpha)!}\,\mathrm{d}\phi\, r^{|m|} + o\big(r^{|m|}\big) \\
&= \big[(\partial_{x_1} \mp i\partial_{x_2})^{|m|} v\big](x') r^{|m|} + o\big(r^{|m|}\big)
\end{aligned}$$

for $r \to 0$. From Sects. 9.1.10-11 of [1], we infer

$$\begin{aligned}
J_{|m|}(t) &\sim \sum_{k=0}^{\infty} c_{|m|,|m|+2k}\, t^{|m|+2k}, \quad c_{|m|,|m|} = 2^{|m|}|m|!\,, \\
Y_{|m|}(t) &\sim \sum_{k=0}^{\infty} d_{|m|,-|m|+2k}\, t^{-|m|+2k} + \log t\, \frac{2}{\pi} J_{|m|}(t).
\end{aligned}$$

⟨l2Dsol⟩ **Lemma 5.1.** *We assume that $f \in C_b^l(\mathbb{R}^2)$ for any nonnegative integer l. Moreover, assume there is a twice continuously differentiable solution v of $(\Delta_{x'} + k^2 I)v = f$ in $C_b^l(\mathbb{R}^2)$ for $l \geq 0$. Fixing $x' \in \mathbb{R}^2$, the solution v admits the Fourier series expansion*

$$v\big(x' - r(\cos\phi, \sin\phi)^{\top}\big) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} v_m(x', kr) e^{-\mathbf{i}m\phi}, \tag{5.2}$$
FouExp

$$\begin{aligned}
v_m(x', s) &:= c_m(x') J_{|m|}(s) \\
&\quad + \frac{\pi}{2}\int_0^s t\,\big[J_{|m|}(t)Y_{|m|}(s) - Y_{|m|}(t)J_{|m|}(s)\big]\, k^{-2}\text{æ}_m\Big(f, x', \frac{t}{k}\Big)\mathrm{d}t, \\
c_{\pm|m|}(x') &:= \frac{1}{k^{|m|}}\big[(\partial_{x_1} \mp \mathbf{i}\partial_{x_2})^{|m|} v\big](x'),
\end{aligned} \tag{5.3}$$
FouCoe

where, substituting (5.3) into (5.2), the unknown derivatives $[(\partial_{x_1} \mp \mathbf{i}\partial_{x_2})^{|m|} v](x')$, $m \in \mathbb{Z}$ of the solution v only contribute to the part $1/(2\pi)\sum_{m\in\mathbb{Z}} c_m(x') J_{|m|}(s) e^{-\mathbf{i}m\phi}$ of the expansion (5.2). The latter part is a solution of the homogeneous Helmholtz equation.

Proof. In view of the Fourier series expansions

$$\begin{aligned}
f\big(x' - r(\cos\phi, \sin\phi)^{\top}\big) &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \text{æ}_m(f, x', r) e^{-\mathbf{i}m\phi}, \\
v\big(x' - r(\cos\phi, \sin\phi)^{\top}\big) &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \text{æ}_m(v, x', r) e^{-\mathbf{i}m\phi},
\end{aligned}$$

and the Fourier coefficient in (5.1), it is easy to see that v is an inhomogeneous Helmholtz solution if and only if (5.2) holds with

$$\left[\partial_r^2 + \frac{1}{r}\partial_r + [k^2 - m^2/r^2]I\right] \text{æ}_m(v, x', r) = \text{æ}_m(f, x', r), \quad m \in \mathbb{Z}$$

(cf. the representation of the Laplacian $\Delta_{x'}$ in the spherical coordinates $x' = r(\cos\phi, \sin\phi)^{\top}$). Setting $s := r/k$, the last condition is equivalent to

$$\left[\partial_s^2 + \frac{1}{s}\partial_s + [1 - m^2/s^2]I\right] \text{æ}_m(v, x', s/k) = k^{-2}\text{æ}_m(f, x', s/k), \quad m \in \mathbb{Z}.$$

In other words, $s \mapsto av_m(v,x',s/k)$ is the solution of an inhomogeneous Bessel equation. Since $J_{|m|}$ and $Y_{|m|}$ are two independent solutions of the homogeneous Bessel equation, since $J_{|m|}$ is smooth, since the Bessel function $Y_{|m|}$ is slightly singular at 0, and since the Wronskian determinant $W(t) := J_{|m|}(t)Y'_{|m|}(t) - Y_{|m|}(t)J'_{|m|}(t)$ satisfies $W(t) = 2/\pi t$ (cf. [1], Sect. 9.1.16), the method of the variation of constants implies (5.3). Note that a possible term $d_{|m|}(x')Y_{|m|}(s)$ must be set to zero in order to avoid unboundedness for $s \to 0$. □

So in view of Lemma 5.1, the big question is the following: For which functions f does there exist a solution $1/(2\pi)\sum_{m\in\mathbb{Z}} c_m(x')J_{|m|}(s)e^{-\mathrm{i}m\phi}$ of the homogeneous Helmholtz equation such that the sum of this and the particular solution of (5.2) and (5.3), defined with $c_m(x') = 0$, is in the space $C_b^l(\mathbb{R}^2)$ for $l > 2$.

⟨rFE⟩ **Remark 5.2.** *The asymptotics of the two functions J_m and Y_m (cf. [1], Sects. 9.2.1 and 9.2.2), the uniform boundedness of the averages $av_m(f,x',s/k)$, and further simple estimates imply the asymptotics $v_m(x',kr) = \mathcal{O}_m(r^{3/2})$ for $r \to \infty$. The proof of convergence for (5.2) without presuming the existence of a continuous solution v to $(\Delta_{x'} + k^2 I)v = f$ is difficult. Though, due to $f \in C_b^l$, the modulated average functions $av_m(f,x',t/k)$ decay as $\mathcal{O}(m^{-l})$ for $|m| \to 0$, the unboundedness w.r.t. m of the Bessel functions Y_m might lead to unbounded coefficients $v_m(x',s)$. Clearly, requiring convergence and $C_b^l(\mathbb{R}^2)$ boundedness for the series in (5.2) restricts the choice of the constants $c_m(x')$. For general $f \in C_b^l(\mathbb{R}^2)$, the $C_b^l(\mathbb{R}^2)$ boundedness of the series in (5.2) with suitable $c_m(x')$ is not guaranteed either.*

6 Radiation condition for special solutions

⟨s4?⟩

6.1 Radiation condition for tensor-product solutions

⟨ss4c⟩ Suppose $u(\vec{x}\,) = u_{2\mathrm{D}}(x')u_3(x_3)$ with a linear function u_3 and a solution $u_{2\mathrm{D}}$ of the two-dimensional Helmholtz equation $\Delta_{x'}u_{2\mathrm{D}} + k^2 u_{2\mathrm{D}} = 0$ over $\mathbb{R}^2$. Moreover suppose that $u_{2\mathrm{D}}$ and all first- and second-order derivatives are uniformly bounded. Clearly, u is a solution of the three-dimensional Helmholtz equation.

Without loss of generality, we may suppose $u_3(0) = 1$ such that $u|_{\mathbb{R}^3_0} = u_{2\mathrm{D}}$. We fix $\vec{x}$. For $j = 1, 2$, we obtain from the boundedness of $u_{2\mathrm{D}}$ and its derivatives and from the decay properties of the kernel functions in (2.3) and (2.4)

$$\int_{-R}^{R} \partial_{y_j}\big\{\partial_{y_j}\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,u_{2\mathrm{D}}(y') - \partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,\partial_{y_j}u_{2\mathrm{D}}(y')\big\}\,dy_j \;=\; \mathcal{O}\big(R^{-2}\big)\,,$$

$$\int_{-R}^{R} \Big\{\partial_{y_j}^2\partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,u_{2\mathrm{D}}(y') - \partial_{y_3}G\big(\vec{x}-(y',0)^\top\big)\,\partial_{y_j}^2u_{2\mathrm{D}}(y')\Big\}\,dy_j \;=\; \mathcal{O}\big(R^{-2}\big)\,,$$

if $R \to \infty$. Consequently,

$$\begin{aligned}
&\int_{-R}^{R}\int_{-R}^{R} \partial_{y_3}^3 G\big(\vec{x},(y',0)^\top\big)\, u_{2\mathrm{D}}(y')\, dy_1\, dy_2 \\
&= -\int_{-R}^{R}\int_{-R}^{R} (\Delta_{y'}+k^2 I)\partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, u_{2\mathrm{D}}(y')\, dy_1\, dy_2 \qquad (6.1) \\
&= -\int_{-R}^{R}\int_{-R}^{R} \partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, (\Delta_{y'}+k^2 I) u_{2\mathrm{D}}(y')\, dy_1\, dy_2 + \sum_{j=1}^{2}\int_{-R}^{R} \mathcal{O}\big(R^{-2}\big)\,\mathrm{d}y_j \\
&= \mathcal{O}\big(R^{-1}\big)\,.
\end{aligned}$$

PartInte

In other words, $[V_k u](\vec{x}\,) = 0$. Condition (1.2) is always fulfilled. However, the pair of radiation conditions (1.2)-(1.4) hold if and only if the linear function u_3 is a constant function.

6.2 Radiation condition for plane-wave functions, Fourier transform of the potential kernels

⟨ss4⟩

6.2.1 Radiation condition for plane waves

⟨sss4.1.1⟩

Now consider a plane-wave function $u(\vec{y}\,) = e^{\mathrm{i}(\alpha y_1+\beta y_2+\gamma y_3)}$ with $\alpha,\beta\in\mathbb{R}$ and $\alpha^2+\beta^2+\gamma^2 = k^2$. For the case $\alpha^2+\beta^2 = k^2$, we get $\gamma = 0$ and the results of Subsect. 6.1 apply. Thus we may suppose $\alpha^2+\beta^2 \neq k^2$ and $\gamma\neq 0$. We observe that $G(\vec{x},\vec{y}\,) = G(\vec{x}-\vec{y}\,)$ and $\partial_{y_3}^3\Phi(\vec{x},\vec{y}\,) = \Phi_{x_3,y_3}(x'-y')$ are convolution kernels. The exponential functions $y'\mapsto e^{\mathrm{i}(\alpha y_1+\beta y_2)}$ are eigenfunctions of the convolution and the eigenvalue is the value of the Fourier transform at $\xi' := (\alpha,\beta)\in\mathbb{R}^2$. Consequently, we have $[V_k u](\vec{x}\,) = e^{\mathrm{i}(\alpha x_1+\beta x_2)} g(x_3)$ with a special function $g(x_3)$ independent of x'. However, since $V_k u$ is a solution of the Helmholtz equation, we conclude

$$[V_k u](\vec{x}\,) \;=\; \begin{cases} e^{\mathrm{i}(\alpha x_1+\beta x_2)}\big(c_1 e^{\mathrm{i}\gamma x_3} + c_2 e^{-\mathrm{i}\gamma x_3}\big) & \text{if } \gamma\neq 0 \\ e^{\mathrm{i}(\alpha x_1+\beta x_2)}\big(c_1 + c_2 x_3\big) & \text{else} \end{cases} \qquad (6.2)$$

DLPPW

with special constants c_1 and c_2. The radiation condition (1.2) is fulfilled if and only if $c_1 = -\gamma^2$ and $c_2 = 0$. If $\Im m\,\gamma > 0$, then $e^{-\mathrm{i}\gamma x_3}$ increases exponentially for $x_3\to\infty$, and $c_2 = 0$ due to the weak boundedness estimates in Sect. 3. The explicit values of c_1 and c_2 can be computed by the Fourier transform of the convolution kernels. However, we prefer to argue using (2.14) with the choice $\Omega_R = C_R,\ \Sigma_R = T_R$.

If $\alpha^2+\beta^2 > k^2$ and if $\Im m\,\gamma > 0$, then $e^{\mathrm{i}\gamma x_3}$ decreases exponentially for $x_3\to\infty$. We get the estimate $|u(\vec{x}\,)|, |\partial_\nu u(\vec{x}\,)| \le c e^{-\Im m\,\gamma R^{1/4}}$ on $T_{R,u}$. The two involved kernel functions can be estimated by $|\partial_{y_3}^2 G(\vec{x},\vec{y}\,)|, |\partial_\nu\partial_{y_3}^2 G(\vec{x},\vec{y}\,)| \le c|\vec{x}-\vec{y}\,|^{-1}$ such that $|\partial_{y_3}^2\Phi(\vec{x},\vec{y}\,)|, |\partial_{y_3}^2\partial_\nu\Phi(\vec{x},\vec{y}\,)| \le c[R^{1/4}]^{-1}$. The area of $T_{R,u}$ is $\mathcal{O}(R^2)$. We arrive at

$$\int_{T_{R,u}} \{\partial_\nu u\partial_{y_3}^2\Phi(\vec{x},\cdot) - u\partial_\nu\partial_{y_3}^2\Phi(\vec{x},\cdot)\} \;=\; \mathcal{O}\Big(e^{-\Im m\,\gamma R^{1/4}} R^{7/4}\Big)\,. \qquad (6.3)$$

Es1

Now look at $T_{R,l}$. According to the formula (2.3), the Green's function differentiated w.r.t. y_3 can be estimated by the sum of the main terms $\mathcal{O}(|x_3\pm y_3|\,|\vec{x}-(y',\pm y_3)^\top|^{-2})$. Estimating the derivatives analogously, the two involved kernel functions satisfy $|\partial_{y_3}^2\Phi(\vec{x},\vec{y}\,)|, |\partial_{y'}\partial_{y_3}^2\Phi(\vec{x},\vec{y}\,)| \le cR^{1/4}R^{-2}$. The functions u and $\partial_\nu u$ are bounded and the area of $T_{R,l}$ is $\mathcal{O}(R^{1/4}R)$. We conclude

$$\int_{T_{R,l}} \{\partial_\nu u\partial_{y_3}^2\Phi(\vec{x},\cdot) - u\partial_\nu\partial_{y_3}^2\Phi(\vec{x},\cdot)\} \;=\; \mathcal{O}(R^{-1/2}). \qquad (6.4)$$

Es2

The estimates (6.3) and (6.4) together with $T_R = T_{R,l} \cup T_{R,u}$ yield $I_\infty = 0$. Consequently, the radiation condition (1.2) is satisfied and we get $c_1 = -\gamma^2,\ c_2 = 0$ in (6.2).

If $\alpha^2+\beta^2 > k^2$ and $\Im m\,\gamma < 0$, then we get the same $u|_{\mathbb{R}^3_0}$ as for the choice $\tilde\gamma = -\gamma$. From the just proved case for $\Im m\,\tilde\gamma > 0$, the formulas $c_1 = -\gamma^2,\ c_2 = 0$ in (6.2) for $\gamma = \tilde\gamma$ imply $c_1 = 0,\ c_2 = -\gamma^2$ in (6.2) for γ. Therefore, for $\Im m\,\gamma < 0$, the radiation condition (1.2) is not satisfied.

To compute the c_1 and c_2 for $k^2 > \alpha^2+\beta^2$, i.e. for real γ, we employ the principle of limited absorption. Choose a small $\varepsilon > 0$ and replace k by $k_\varepsilon := k + \mathrm{i}\varepsilon$. The corresponding fundamental solution $G_{k_\varepsilon}(\vec{x}-\vec{y}) = 1/(4\pi)\ e^{\mathrm{i}k|\vec{x}-\vec{y}|-\varepsilon|\vec{x}-\vec{y}|}/|\vec{x}-\vec{y}\,|$ is, in contrast to the case with real k, exponentially decaying. Then choosing $\gamma_\varepsilon := \sqrt{k_\varepsilon^2-\alpha^2-\beta^2}$ with $\Re e\,\gamma_\varepsilon > 0$ and $\Im m\,\gamma_\varepsilon > 0$ and following exactly the proof for the case $\alpha^2+\beta^2 > k^2$, we obtain the representation $\partial^2_{y_3} u_\varepsilon(\vec{x}\,) = [V_{k_\varepsilon} u_\varepsilon](\vec{x}\,)$ for the function $u_\varepsilon(\vec{x}\,) = e^{\mathbf{i}(\alpha x_1+\beta x_2+\gamma_\varepsilon x_3)}$. In this representation we consider the limit for the parameter $\varepsilon \to 0$. Due to the $\mathcal{O}(|\vec{x}-\vec{y}\,|^{-3})$ estimate for the kernel function in V_{k_ε}, Lebesgue's theorem on dominated convergence applies. We arrive at $\partial^2_{y_3} u(\vec{x}\,) = [V_k u](\vec{x}\,)$, where $u(\vec{x}\,) = e^{\mathbf{i}(\alpha x_1+\beta x_2+\gamma_0 x_3)}$ with $\gamma_0 := \lim \gamma_\varepsilon$ the solution of $\gamma_0^2 = k^2-\alpha^2-\ \beta^2$, for which $\gamma_0 > 0$. Thus $c_1 = -\gamma_0^2$ and $c_2 = 0$ holds in (6.2), and the radiation condition (1.2) holds for the plane wave with $\gamma > 0$. Furthermore, the equations $c_1 = 0$ and $c_2 = -\gamma^2$ hold in (6.2) if $\gamma = -\gamma_0 < 0$, and the radiation condition (1.2) does not hold for the plane wave with $\gamma < 0$.

⟨grating⟩

Corollary 6.1. *Any quasiperiodic solution of the Helmholtz equation in $\mathbb{R}^3_+$ satisfies the radiation condition (1.2)-(1.4) if and only if it satisfies the classical radiation condition, i.e., if it admits a Rayleigh series expansion into a sum of outgoing plane-wave modes (i.e. modes such that either $\alpha^2+\beta^2 > k^2$ and $\Im m\,\gamma > 0$ or $\alpha^2+\beta^2 \le k^2$ and $\gamma \ge 0$).*

6.2.2 Fourier transform of the potential kernel $x' \mapsto \partial^3_{x_3}\Phi(\vec{x},0)$ and the limit of the potential for $x_3 \to 0$

⟨s6.2.2⟩

As a consequence of Subsect. 6.2.1 we can fix formulas for the Fourier transform of the convolution kernel in V_k. Setting $u(\vec{x}\,) := e^{\mathbf{i}(\xi'\cdot x' + \sqrt{k^2-|\xi|^2}x_3)}$ and introducing the Fourier transform as

$$[\mathcal{F}f](\xi') \ := \ \int_{\mathbb{R}^2} e^{-\mathbf{i}x'\cdot\xi'} f(x')\mathrm{d}\xi', \quad [\mathcal{F}^{-1}g](x') \ = \ \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{\mathbf{i}x'\cdot\xi'} g(\xi')\mathrm{d}x', \tag{6.5}$$ FouTr

formula (6.2) and the computed values for the constants c_1 and c_2 lead us to

$$\begin{aligned} -\left(k^2-|\xi'|^2\right) e^{\mathbf{i}(\xi'\cdot x'+\sqrt{k^2-|\xi|^2}x_3)} \ &= \ [V_k u](\vec{x}\,) \\ &= \ \int_{\mathbb{R}^2} \partial^3_{y_3}\Phi\big(\vec{x},(y',0)^\top\big)\, e^{\mathbf{i}\xi'\cdot y'}\mathrm{d}y', \\ \sqrt{k^2-|\xi'|^2} \ &:= \ \begin{cases} \sqrt{k^2-|\xi'|^2} & \text{if } |\xi'| \le k \\ \mathrm{i}\sqrt{|\xi'|^2-k^2} & \text{if } |\xi'| > k \end{cases}. \end{aligned}$$

We introduce the function $G(\vec{x},\vec{y}\,) := G(\vec{x}-\vec{y}\,)$ and observe $G(\vec{x},\vec{y}\,) = G(x'-y', x_3-y_3)$ as well as $\partial^3_{y_3}\Phi(\vec{x},(y',0)) = -2\partial^3_{x_3}G(x'-y',x_3)$. Setting $x' = (0,0)^\top$, we arrive at

$$\begin{aligned} (k^2-|\xi'|^2)e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3} \ &= \ \int_{\mathbb{R}^2} 2\partial^3_{x_3}G(y',x_3)e^{-\mathbf{i}\xi'\cdot y'}\mathrm{d}y', \\ \left[\mathcal{F}\left(\partial^3_{y_3}\Phi\left((\cdot,x_3)^\top,(0,0,0)^\top\right)\right)\right](\xi') \ &= \ -\left[\mathcal{F}\left(2\partial^3_{x_3}G(\cdot,x_3)\right)\right](\xi') \\ &= \ -(k^2-|\xi'|^2)e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}. \end{aligned}$$

Now we derive a presentation of $[V_k u]$ by general plane-wave functions. Clearly, $[V_k u]((x',x_3)^\top)$ is the convolution of $u|_{\mathbb{R}^3_0}$ by the function $x' \mapsto -2\partial^3_{x_3} G(x',x_3)$. The just proved results imply

$$[V_k u]\big((x',x_3)^\top\big) \;=\; \Big[\mathcal{F}^{-1}\big\{m_{x_3}\big[\mathcal{F}(u|_{\mathbb{R}^3_0})\big]\big\}\Big](x'), \tag{6.6}$$
$$m_{x_3}(\xi') \;:=\; \big[\mathcal{F}\big\{-2\partial^3_{x_3} G(\cdot,x_3)\big\}\big](\xi') \;=\; -\big(k^2-|\xi'|^2\big)\, e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}.$$

NRF

In this generalized sense, (6.6) means

$$[V_k u](\vec{x}) \;=\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \big[\mathcal{F}(u|_{\mathbb{R}^3_0})\big](\xi')\,\partial^2_{x_3}\Big[e^{\mathbf{i}\left(\xi',\sqrt{k^2-|\xi'|^2}\right)^\top\cdot\vec{x}}\Big]\,\mathrm{d}\xi'. \tag{6.7}$$

PlRes

The Fourier transform can also be used to compute the limit of $[V_k u](\vec{x}\,)$ for $x_3\to 0$ if x' is fixed. Suppose $f:=u|_{\mathbb{R}^3_0}$ is a bounded function such that all derivatives upto order five are bounded. Choose a cut-off function $y'\mapsto\chi(y')$ of the same smoothness with χ identical to one in a neighbourhood of x' and identical to zero outside a larger neighbourhood of x'. Then $f\chi$ is in L^2 and its Fourier transform $\mathcal{F}(f\chi)(\xi')$ decays at infinity as $\mathcal{O}(|\xi'|^{-5})$. We get

$$[V_k u](\vec{x}\,) \;=\; \int_{\mathbb{R}^2}\partial^3_{y_3}\Phi\big(\vec{x},(y',0)^\top\big)\,(f\chi)(y')\,\mathrm{d}y' + \int_{\mathbb{R}^2}\partial^3_{y_3}\Phi\big(\vec{x},(y',0)^\top\big)\,(f[1-\chi])(y')\,\mathrm{d}y',$$

where the second term on the right-hand side tends to zero for $x_3\to 0$ due to that (2.6) allows the factors $(y_3-x_3)=-x_3$ and $(y_3-x_3)^3=-x_3^3$ to be pulled out of the integrals such that the remaining integrals are still bounded. By (6.6) we obtain

$$[V_k u](\vec{x}\,) \;=\; -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{\mathbf{i}x'\cdot\xi'}\,\big(k^2-|\xi'|^2\big)\,e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}\mathcal{F}(f\chi)(\xi')\,\mathrm{d}\xi' + o(1).$$

Again Lebesgue's theorem together with $\mathcal{F}(f\chi)(\xi')=\mathcal{O}(|\xi'|^{-5})$ for $|\xi'|\to\infty$ lead us to

$$\lim_{x_3\to 0}[V_k u](\vec{x}\,) \;=\; -\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{\mathbf{i}x'\cdot\xi'}\,\big(k^2-|\xi'|^2\big)\,\mathcal{F}(f\chi)(\xi')\,\mathrm{d}\xi',$$
$$[V_k u]\big((x',0)^\top\big) \;=\; -\Delta_{x'}f(x') - k^2 f(x') \;=\; -\Delta_{x'}u\big((x',0)^\top\big) - k^2 u\big((x',0)^\top\big)\,. \tag{6.8}$$

LaEq

Of course, by density arguments, (6.8) holds under reduced smoothness assumptions on u. Namely, it is sufficient to suppose that all the derivatives of u upto order two are bounded and continuous on $\mathbb{R}^3_0$. Due to the estimate $\partial^3_{y_3}G(\vec{x},\vec{y}\,)=O(|\vec{x}-\vec{y}\,|^{-3})$ for $|\vec{x}-\vec{y}\,|\to\infty$, we can fix $\vec{x}\in\mathbb{R}^3_0$ and can reduce the analysis to functions u which have a finite support. Computing classical limits of potential operators in the form of finite-part integrals, we obtain the same limits as in the smoother case considered before. Hence, if $u|_{\mathbb{R}^2_0}\in C^2_b(\mathbb{R}^2)$, then we get

$$[V_k u]\big((x',0)^\top\big) \;=\; -\big(\partial^2_{x_1}+\partial^2_{x_2}+k^2 I\big)\,u\big((x',0)^\top\big) \;=\; \partial^2_{x_3}u\big((x',0)^\top\big)\,. \tag{6.9}$$

BVF

In particular, if the function $f:=u|_{\mathbb{R}^3_0}$ is the restriction of a bounded Helmholtz solution in the half space $\{\vec{x}\in\mathbb{R}^3:\,-\varepsilon<x_3\}$, then u is sufficiently smooth and (6.9) holds. By the same arguments we get even more.

⟨limit⟩

Proposition 6.2. *Suppose the Hölder exponent κ satisfies $0<\kappa\le 1$. Then the limit relation (6.9) holds if $u|_{\mathbb{R}^2_0}\in C^{2,\kappa}(\mathbb{R}^2)$ and if there are constants $c>0$ and $0\le\tau<1$ such that $|u((x',0)^\top)|<c(1+|x'|)^\tau$ is true for any $x'\in\mathbb{R}^2$.*

Finally, we can derive a representation of the potential $V_k[u]$ based on the double layer integral applied to the derivative $\partial^2_{x_3} u$ (cf. the subsequent (6.10)). Indeed, under the conditions of Prop. 6.2, from the Helmholtz equation $\partial^2_{y_3} G(\vec{x},\vec{y}\,) = -(\Delta_{y'} + k^2 I) G(\vec{x},\vec{y}\,)$ and by the formula of partial integration $\int_{-R}^R [f''g - fg''] = \int_{-R}^R [f'g - fg']' = [f'g - fg'](R) - [f'g - fg'](-R)$, we get

$$\begin{aligned}
V_k[u](\vec{x}\,) &= \int_{\mathbb{R}^3_0} \partial^3_{y_3} G\big(\vec{x},(y',0)^\top\big)\, u\big((y',0)^\top\big)\, \mathrm{d}y' \\
&= -\lim_{R\to\infty} \int_{-R}^R \int_{-R}^R \left(\sum_{j=1}^2 \partial^2_{y_j} + k^2 I \right) \partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, u\big((y',0)^\top\big)\, \mathrm{d}y_1 \mathrm{d}y_2 \\
&= -\lim_{R\to\infty} \Bigg\{ \int_{-R}^R \int_{-R}^R \partial_{y_3} G\big(\vec{x},(y',0)^\top\big) \left(\sum_{j=1}^2 \partial^2_{y_j} + k^2 I \right) u\big((y',0)^\top\big)\, \mathrm{d}y_1 \mathrm{d}y_2 + \\
&\qquad \int_{-R}^R \Big\{ \mathcal{O}\big(\partial_{y_1}\partial_{y_3} G\big(\vec{x},(R,y_2,0)^\top\big)\, u\big((R,y_2,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_1}\partial_{y_3} G\big(\vec{x},(-R,y_2,0)^\top\big)\, u\big((-R,y_2,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_3} G\big(\vec{x},(R,y_2,0)^\top\big)\, \partial_{y_1} u\big((R,y_2,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_3} G\big(\vec{x},(-R,y_2,0)^\top\big)\, \partial_{y_1} u\big((-R,y_2,0)^\top\big)\big) \Big\} \mathrm{d}y_2 + \\
&\qquad \int_{-R}^R \Big\{ \mathcal{O}\big(\partial_{y_2}\partial_{y_3} G\big(\vec{x},(y_1,R,0)^\top\big)\, u\big((y_1,R,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_2}\partial_{y_3} G\big(\vec{x},(y_1,-R,0)^\top\big)\, u\big((y_1,-R,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_3} G\big(\vec{x},(y_1,R,0)^\top\big)\, \partial_{y_2} u\big((y_1,R,0)^\top\big)\big) \\
&\qquad\qquad + \mathcal{O}\big(\partial_{y_3} G\big(\vec{x},(y_1,-R,0)^\top\big)\, \partial_{y_2} u\big((y_1,-R,0)^\top\big)\big) \Big\} \mathrm{d}y_1 \Bigg\}.
\end{aligned}$$

Estimating the kernel derivatives by R^{-2} (cf. (2.3) and (2.4)) and $u, \partial_j u$ by a constant, we arrive at

$$\begin{aligned}
V_k[u](\vec{x}\,) &= \lim_{R\to\infty} \left\{ \int_{-R}^R \int_{-R}^R \partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, \partial^2_{y_3} u\big((y',0)^\top\big)\, \mathrm{d}y_1 \mathrm{d}y_2 + \mathcal{O}\left(\frac{1}{R}\right) \right\} \\
&= \lim_{R\to\infty} \int_{-R}^R \int_{-R}^R \partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, \partial^2_{y_3} u\big((y',0)^\top\big)\, \mathrm{d}y_1 \mathrm{d}y_2 \\
&=: \int_{\mathbb{R}^2} \partial_{y_3} G\big(\vec{x},(y',0)^\top\big)\, \partial^2_{y_3} u\big((y',0)^\top\big)\, \mathrm{d}y',
\end{aligned} \tag{6.10}$$

RLIM

which again proves the limit in Prop. 6.2 by the jump relation of the double layer potential over bounded surfaces.

6.3 Radiation condition for point-source functions

⟨ss5⟩

Suppose u is a Helmholtz solution on $\mathbb{R}^3_+$, which is bounded together with its derivatives upto order two on the closure of $\mathbb{R}^3_+$. Similarly to Sommerfeld's condition on the full space $\mathbb{R}^3$, we define

Definition 6.3. *We shall say that a function u on $\mathbb{R}^3_+$ satisfies the outgoing Sommerfeld half-space radiation condition if*

$$\sup_{\vec{x}\in\mathbb{R}^3_+:\,|\vec{x}\,|=r} r\,|\partial_\nu u(\vec{x}\,) - \mathbf{i}ku(\vec{x}\,)| \to 0,\ r\to\infty, \qquad \sup_{\vec{x}\in\mathbb{R}^3_+:\,|\vec{x}\,|\ge R} |\vec{x}\,|\,|u(\vec{x}\,)| < \infty. \tag{6.11}$$

SHSRC

It is well known that, for any fixed $\vec{y}\in\mathbb{R}^3$ with $y_3<0$, the Green's function $\mathbb{R}^3_+\ni\vec{x}\mapsto G(\vec{x},\vec{y}\,)$ and any of its derivatives w.r.t. $\vec{x}$ or $\vec{y}$ satisfy Sommerfeld's radiation condition. Hence, these point source functions also satisfies (6.11).

Suppose $\vec{y}=(y',y_3)^\top$ with $y_3<0$. We shall prove (1.2) for the point-source function $u(\vec{x}\,):=G(\vec{x},\vec{y}\,)$ using only the properties fixed in (6.11). Choosing $\Omega_R=B_R,\ \Sigma_R=S_R$ (cf. Sect. 2.2), we shall employ the representation (2.14). It remains to prove $I_\infty=0$. The estimates for this, however, are exactly the same as for the full space Sommerfeld condition. Indeed, the fundamental solution G satisfies (6.11) w.r.t. both arguments. If, for $\vec{x}$ and $\vec{y}$, we define the reflection points $\vec{x}_s:=(x',-x_3)^\top$ and $\vec{y}_s:=(y',-y_3)^\top$, respectively, then we obviously get $\partial^2_{y_3}G(\vec{x},\vec{y}_s)=\partial^2_{y_3}G(\vec{x}_s,\vec{y}\,)$. Consequently, the differentiated Green's function $\partial^2_{y_3}\Phi(\vec{x},\vec{y})=\partial^2_{y_3}G(\vec{x},\vec{y}\,)-\partial^2_{y_3}G(\vec{x}_s,\vec{y}\,)$ satisfies (6.11) w.r.t. $\vec{y}$. We get

$$\begin{aligned}\int_{\Sigma_R}\big\{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\big\} &= \int_{\Sigma_R}\big\{[\mathbf{i}k]u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\,[\mathbf{i}k]\partial^2_{y_3}\Phi(\vec{x},\cdot)\big\} \qquad (6.12)\ \boxed{\texttt{GFO}}\\ &\quad+\int_{\Sigma_R}o\big(R^{-2}\big) = \int_{\Sigma_R}o\big(R^{-2}\big) = o(1)\,,\\ I_\infty &= 0.\end{aligned}$$

Corollary 6.4. *Any solution of the Helmholtz equation over $\mathbb{R}^3_+$ satisfying the outgoing Sommerfeld half-space radiation condition (6.11) satisfies the (HSRC) too.*

For the "incoming" point-source $u(\vec{x}\,):=\overline{G(\vec{x},\vec{y}\,)}$ with $y_3<0$, we have (6.11) but with the term $\mathbf{i}ku(\vec{x}\,)$ replaced by $-\mathbf{i}ku(\vec{x}\,)$. Instead of (6.12), we arrive at

$$\begin{aligned}\int_{\Sigma_R}\big\{\partial_\nu u\partial^2_{y_3}\Phi(\vec{x},\cdot)-u\partial_\nu\partial^2_{y_3}\Phi(\vec{x},\cdot)\big\} &= 2(\mathbf{i}k)[I_{i,R}-J_{i,R}]+o(1)\,,\quad I_{i,R}:=\int_{\Sigma_R}\partial^2_{y_3}G(\vec{x},\cdot)u,\\ J_{i,R}:=\int_{\Sigma_R}\partial^2_{y_3}G(\vec{x},(z',-z_3)^\top)u(\vec{z})\mathrm{d}\vec{z} &= \int_{\Sigma_R}\partial^2_{y_3}G((x',-x_3)^\top,\cdot\,)u.\end{aligned}$$

Taking the asymptotically largest term from (2.5), we conclude

$$\begin{aligned}I_{i,R} &= \int_{\Sigma_R}(\mathbf{i}k)^2\frac{e^{\mathbf{i}k|\vec{x}-\vec{z}\,|}(z_3-x_3)^2}{4\pi|\vec{x}-\vec{z}\,|^3}\,\frac{e^{-\mathbf{i}k|\vec{y}-\vec{z}\,|}}{4\pi|\vec{y}-\vec{z}\,|}\mathrm{d}\vec{z}+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_{\Sigma_R}e^{\mathbf{i}k(\vec{x}-\vec{y}\,)\cdot\vec{z}/|\vec{z}\,|}\,\frac{z_3^2}{|\vec{z}\,|^4}\mathrm{d}\vec{z}+o(1).\end{aligned}$$

Switching to spherical coordinates, we get

$$\begin{aligned}I_{i,R} &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}k(\vec{x}-\vec{y}\,)\cdot(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)^\top}\cos^2\theta\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}k\sin\theta(x'-y')\cdot(\cos\phi,\sin\phi)^\top}\mathrm{d}\phi\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{16\pi^2}\int_0^{\pi/2}\int_0^{2\pi}e^{\mathbf{i}[k|x'-y'|\sin\theta]\cos\phi}\,\mathrm{d}\phi\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{8\pi}\int_0^{\pi/2}J_0(k|x'-y'|\sin\theta)\,e^{\mathbf{i}k(x_3-y_3)\cos\theta}\cos^2\theta\sin\theta\,\mathrm{d}\theta+o(1)\\ &= \frac{(\mathbf{i}k)^2}{8\pi}\int_0^1 e^{\mathbf{i}k(x_3-y_3)t}J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\,t^2\,\mathrm{d}t+o(1).\end{aligned}$$

A similar formula for $J_{i,R}$ lead us to

$$I_\infty \;=\; \frac{(\mathbf{i}k)^3}{4\pi}\int_0^1 \left[e^{\mathbf{i}k(x_3-y_3)t}-e^{\mathbf{i}k(-x_3-y_3)t}\right] J_0\Big(k|x'-y'|\sqrt{1-t^2}\Big)\, t^2\,\mathrm{d}t.$$

E.g. in the special case $x'=y'$, by integration by parts we arrive at

$$\begin{aligned}I_\infty &= \frac{(\mathbf{i}k)^3}{4\pi}\int_0^1 \left[e^{\mathbf{i}k(x_3-y_3)t}-e^{\mathbf{i}k(-x_3-y_3)t}\right] t^2\,\mathrm{d}t\\ &= \frac{(\mathbf{i}k)^3}{4\pi}\left\{e^{\mathbf{i}k(x_3-y_3)}\left[\frac{1}{ik(x_3-y_3)}-\frac{2}{[ik(x_3-y_3)]^2}+\frac{2}{[ik(x_3-y_3)]^3}\right]-\frac{2}{[ik(x_3-y_3)]^3}\right.\\ &\qquad\left.+e^{-\mathbf{i}k(x_3+y_3)}\left[\frac{1}{ik(x_3+y_3)}+\frac{2}{[ik(x_3+y_3)]^2}+\frac{2}{[ik(x_3+y_3)]^3}\right]-\frac{2}{[ik(x_3+y_3)]^3}\right\}.\end{aligned}$$

Hence, the limit I_∞ is not identically zero. In other words, for the “incoming” point-source, the radiation condition (1.2) is not fulfilled.

7 Condition (HSRC) independent of $x_{h,3}$ and $x_{f,3}$, and equivalence of the conditions (1.4) and (1.5)

⟨s4E⟩

7.1 Condition (HSRC) independent of $x_{h,3}$ and $x_{f,3}$

⟨ss4E.1⟩

For the dependence of (HSRC) on $x_{h,3}$ and $x_{f,3}$, we notice that the representation (4.1) together with the decay $\mathcal{O}(|\vec{x}-\vec{y}\,|^{-3})$ of the kernel functions easily imply the condition i) for any fixed $x_{h,3}$. Therefore, it is sufficient to check the dependence on $x_{f,3}$.

Note that, assuming the condition i) of (HSRC) we easily get the uniform boundedness of the function u in the layer $\{\vec{x}\colon x_{f,3}\le x_3\le x_{h,3}\}$. Using this fact, we easily get that (1.4) and, therewith, iii) is independent of $x_{f,3}$.

⟨LemHinr⟩

Lemma 7.1. *Suppose condition i), take $x_{f,3}$ and $x'_{f,3}$ with $0<x_{f,3}<x'_{f,3}<x_{h,3}$. Then condition ii) with $x_{f,3}$ implies ii) with $x'_{f,3}$.*

Proof. We only have to prove

$$\int_{\mathbb{R}^2} 2\partial^3_{y_3} G\big(\vec{x},(y',x_{f,3})^\top\big)\, u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y' = \int_{\mathbb{R}^2} 2\partial^3_{y_3} G\big(\vec{x},(y',x'_{f,3})^\top\big)\, u\big((y',x'_{f,3})^\top\big)\,\mathrm{d}y'. \quad (7.1)$$ EQN1

Substituting

$$\begin{aligned}u\big((y',x'_{f,3})^\top\big) &= u\big((y',x_{f,3})^\top\big) + (x'_{f,3}-x_{f,3})\partial_{x_3}u\big((y',x_{f,3})^\top\big)\\ &\quad + \int_{x_{f,3}}^{x'_{f,3}} (x'_{f,3}-z_3)\int_{\mathbb{R}^2} 2\partial^3_{y_3} G\big((y',z_3)^\top,(z',x_{f,3})^\top\big)\, u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'\mathrm{d}z_3\end{aligned}$$

on the right-hand side of (7.1), we first compute

$$\int_{\mathbb{R}^2}2\partial^3_{y_3}G\big(\vec{x},(y',x'_{f,3})^\top\big)\int_{x_{f,3}}^{x'_{f,3}}(x'_{f,3}-z_3)\int_{\mathbb{R}^2}2\partial^3_{y_3}G\big((y',z_3)^\top,(z',x_{f,3})^\top\big)u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'\mathrm{d}z_3\mathrm{d}y' =$$

$$\int_{\mathbb{R}^2}\int_{x_{f,3}}^{x'_{f,3}}(x'_{f,3}-z_3)\int_{\mathbb{R}^2}2\partial^3_{y_3}G\big(\vec{x},(y',x'_{f,3})^\top\big)\,2\partial^3_{y_3}G\big((y',z_3)^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y'\mathrm{d}z_3 u\big((z',x_{f,3})^\top\big)\mathrm{d}z'.$$

From the Fourier transform (6.6) we infer

$$
\begin{aligned}
&\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',x_{f,3}')^\top\big)\, 2\partial_{y_3}^3 G\big((y',z_3)^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y' \\
&= \int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big((x',x_3-x_{f,3}')^\top,(y',0)^\top\big)\, 2\partial_{y_3}^3 G\big((y',0)^\top,(z',x_{f,3}-z_3)^\top\big)\,\mathrm{d}y' \\
&= 2\partial_{y_3}^5 G\big((x',x_3-x_{f,3}')^\top,(z',x_{f,3}-z_3)^\top\big) \;=\; 2\partial_{y_3}^5 G\big(\vec{x},(z',x_{f,3}+x_{f,3}'-z_3)^\top\big)\,, \\
&\int_{x_{f,3}}^{x_{f,3}'}(x_{f,3}'-z_3)\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',x_{f,3}')^\top\big)\, 2\partial_{y_3}^3 G\big((y',z_3)^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y'\mathrm{d}z_3 \\
&= \int_{x_{f,3}}^{x_{f,3}'}(x_{f,3}'-z_3) 2\partial_{y_3}^5 G\big(\vec{x},(z',x_{f,3}+x_{f,3}'-z_3)^\top\big)\,\mathrm{d}z_3 \\
&= 2\partial_{y_3}^3 G\big(\vec{x},(z',x_{f,3})^\top\big) - 2\partial_{y_3}^3 G\big(\vec{x},(z',x_{f,3}')^\top\big) + (x_{f,3}'-x_{f,3}) 2\partial_{y_3}^4 G\big(\vec{x},(z',x_{f,3}')^\top\big)\,.
\end{aligned}
$$

Hence, the substitution for the right-hand side of (7.1), yields that (7.1) is equivalent to the equation

$$
\begin{aligned}
&\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',x_{f,3}')^\top\big)\,\partial_{y_3} u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y' \\
&+\int_{\mathbb{R}^2} 2\partial_{y_3}^4 G\big(\vec{x},(y',x_{f,3}')^\top\big)\, u\big((y',x_{f,3})^\top\big)\,\mathrm{d}y' \;=\; 0.
\end{aligned}
\tag{7.2}
$$

EQN2

This (7.2) is equivalent to $Te(x_{f,3})=0$, where

$$
\begin{aligned}
Te(z_3) \;:=\; &\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',x_{f,3}+x_{f,3}'-z_3)^\top\big)\,\partial_{z_3}\big[u\big((y',z_3)^\top\big)\big]\,\mathrm{d}y' \\
&-\int_{\mathbb{R}^2}\partial_{z_3}\big[2\partial_{y_3}^3 G\big(\vec{x},(y',x_{f,3}+x_{f,3}'-z_3)^\top\big)\big]\, u\big((y',z_3)^\top\big)\,\mathrm{d}y'.
\end{aligned}
$$

Choosing z_3 and z_3' with $x_{f,3}\le z_3\le z_3'$ and applying Green's identity to the Helmholtz solutions u and $(y',z_3)\mapsto 2\partial_{y_3}^3 G(\vec{x},(y',x_{f,3}+x_{f,3}'-z_3)^\top)$ over the layer enclosed by $\mathbb{R}^3_{z_3}$ and $\mathbb{R}^3_{z_3'}$, we obtain $Te(z_3)=Te(z_3')$, i.e., the function $z_3\mapsto Te(z_3)$ is constant for all $z_3\ge x_{f,3}$. Consequently, we only have to prove $Te(x_{f,3}+\varepsilon)=0$ for a small but fixed $\varepsilon>0$. Setting $y_{f,3}':=y_{f,3}'(\varepsilon):=x_{f,3}'-\varepsilon$ and $y_{f,3}:=y_{f,3}(\varepsilon):=x_{f,3}+\varepsilon$, we shall prove $Te(y_{f,3})=0$, i.e.

$$
\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',y_{f,3}')^\top\big)\,\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' + \int_{\mathbb{R}^2} 2\partial_{y_3}^4 G\big(\vec{x},(y',y_{f,3}')^\top\big)\, u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' = 0.
\tag{7.3}
$$

EQN2b

To treat the first term in (7.3), we introduce the disk $D_R^{x'}:=\{z'\in\mathbb{R}^2:\,|x'-z'|\le R\}$. Then, by the arguments leading to (6.10), we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',y_{f,3}')^\top\big)\,\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' \\
&= -\int_{D_R^{x'}} 2(\Delta_{y'}+k^2 I)\partial_{y_3} G\big(\vec{x},(y',y_{f,3}')^\top\big)\,\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' + \mathcal{O}\Big(\frac{1}{R}\Big) \\
&= -\int_{D_R^{x'}} 2\partial_{y_3} G\big(\vec{x},(y',y_{f,3}')^\top\big)\,(\Delta_{y'}+k^2 I)\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' + \mathcal{O}\Big(\frac{1}{R}\Big)
\end{aligned}
$$

such that (1.2) implies

$$\begin{aligned}
&\int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',y'_{f,3})^\top\big)\,\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' \\
&= -\int_{D_R^{x'}} 2\partial_{y_3} G\big(\vec{x},(y',y'_{f,3})^\top\big)\int_{\mathbb{R}^2} 2\partial_{y_3}^4 G\big((y',y_{f,3})^\top,(z',x_{f,3})^\top\big)\,u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'\mathrm{d}y' + \mathcal{O}\Big(\frac{1}{R}\Big) \\
&= -\int_{\mathbb{R}^2}\int_{D_R^{x'}} 2\partial_{y_3} G\big(\vec{x},(y',y'_{f,3})^\top\big)\,2\partial_{y_3}^4 G\big((y',y_{f,3})^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y' u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z' + \mathcal{O}\Big(\frac{1}{R}\Big).
\end{aligned}$$

We shall show that this expression is equal to $\int_{\mathbb{R}^2} 2\partial_{y_3}^4 G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\,u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'$ with $x''_{f,3} := x''_{f,3}(\varepsilon) := x_{f,3} + (y_{f,3} - x_{f,3}) + (x_3 - y'_{f,3}) = x_3 + (x_{f,3} - x'_{f,3}) + 2\varepsilon$. To get this equality, we even prove the generalized equation $I_m(\varepsilon, x_3) = 0$ for $m = 3, 4, 5$, where

$$\begin{aligned}
&I_m(\varepsilon,x_3) \\
&:= \int_{\mathbb{R}^2} 2\partial_{y_3}^m G\big(\vec{x},(y',y'_{f,3})^\top\big)\,\partial_{y_3} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y' \\
&\quad + \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+1} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\,u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z' \\
&= -\lim_{R\to\infty}\int_{\mathbb{R}^2}\int_{D_R^{x'}} 2\partial_{y_3}^{m-2} G\big(\vec{x},(y',y'_{f,3})^\top\big)\,2\partial_{y_3}^4 G\big((y',y_{f,3})^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y' u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z' \\
&\quad + \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+1} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\,u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'.
\end{aligned}$$

Note that $\partial_{x_3} I_m(\varepsilon,x_3) = -I_{m+1}(\varepsilon,x_3)$. Formally, it is clear that $I_m(\varepsilon,x_3) \equiv 0$. Indeed, interchange the limit $R\to\infty$ and the integration over $\mathbb{R}^2$ and use

$$\int_{\mathbb{R}^2} 2\partial_{y_3}^{m-2} G\big(\vec{x},(y',y'_{f,3})^\top\big)\,2\partial_{y_3}^4 G\big((y',y_{f,3})^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y' = 2\partial_{y_3}^{m+1} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\,,$$

which follows from switching to Fourier transforms (compare the Fourier transform in (6.6)) and using that convolutions turn into products. Unfortunately, the interchange of limit and integration might be not correct. However, for $m=5$ the convolution kernels $(x'-y') \mapsto 2\partial_{y_3}^{m-2} G(\vec{x},(y',y'_{f,3})^\top)$ and $(y'-z') \mapsto 2\partial_{y_3}^4 G((y',y_{f,3})^\top,(z',x_{f,3})^\top)$ are $L^1(\mathbb{R}^2)$ functions (cf. Sect. 2.1) such that a truncation to $D_{\mathbb{R}}^{x'}$ with $R\to\infty$ even leads to norm convergence of the truncated convolution operators. Hence, the interchange of limit and integration is correct and $I_5(\varepsilon,x_3) \equiv 0$. Moreover, we observe the estimate $\partial_{y_3} G(\vec{x},\vec{y}) - (y_3 - x_3)\partial_{y_3}^2 G(\vec{x},\vec{y}) = \mathcal{O}(|x'-y'|^{-3})$ for $|x'-y'|\to\infty$ (cf. (2.3) and (2.5)). Therefore, the convolution kernel $(x'-y') \mapsto \partial_{y_3} G(\vec{x},(y',y'_{f,3})^\top) - (y'_{f,3} - x_3)\partial_{y_3}^2 G(\vec{x},(y',y'_{f,3})^\top)$ is in $L^1(\mathbb{R}^2)$, and, analogously to $I_5(\varepsilon,x_3) \equiv 0$, we arrive at $I_3(\varepsilon,x_3) - (y'_{f,3} - x_3) I_4(\varepsilon,x_3) \equiv 0$. So we draw the first conclusions. By $I_5 = 0$, $\partial_{x_3} I_4 = -I_5$, we see that, w.r.t. x_3, the function I_4 is constant. We get $I_4(\varepsilon,x_3) = I_4(\varepsilon)$ and $I_3(\varepsilon,x_3) = (y'_{f,3}(\varepsilon) - x_3) I_4(\varepsilon)$.

To continue and to treat the second term on the left-hand side of (7.3), we introduce

$$
\begin{aligned}
&J_m(\varepsilon,x_3) \qquad\qquad (7.4)\ \texttt{defJ\_m}\\
&:= \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+1} G\big(\vec{x},(y',y'_{f,3})^\top\big)\, u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y'\\
&\quad - \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+1} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\, u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'\\
&= \lim_{R\to\infty} \int_{\mathbb{R}^2}\int_{D_R^{x'}} 2\partial_{y_3}^{m-1} G\big(\vec{x},(y',y'_{f,3})^\top\big)\, 2\partial_{y_3}^{3} G\big((y',y_{f,3})^\top,(z',x_{f,3})^\top\big)\,\mathrm{d}y' u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z'\\
&\quad - \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+1} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\, u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z', \quad m=2,3,4,\\
&K_{m+1}(\varepsilon,x_3)\\
&:= \int_{\mathbb{R}^2} 2\partial_{y_3}^{m} G\big(\vec{x},(y',y'_{f,3})^\top\big)\, \partial_{y_3}^{2} u\big((y',y_{f,3})^\top\big)\,\mathrm{d}y'\\
&\quad - \int_{\mathbb{R}^2} 2\partial_{y_3}^{m+2} G\big((x',x''_{f,3})^\top,(z',x_{f,3})^\top\big)\, u\big((z',x_{f,3})^\top\big)\,\mathrm{d}z' \;=\; 0, \quad m=3,4.
\end{aligned}
$$

Similarly to the analogous results for $I_m(\varepsilon,x_3)$, we obtain $\partial_{x_3}J_m(\varepsilon,x_3)=-J_{m+1}(\varepsilon,x_3)$, the equality $J_4(\varepsilon,x_3)=0$, and the relation $J_2(\varepsilon,x_3)-(y'_{f,3}-x_3)J_3(\varepsilon,x_3)=0$. Again we draw the first conclusions. By $J_4=0$ and $\partial_{x_3}J_3=-J_4$, we see that, w.r.t. x_3, the function J_3 is constant. We get $J_3(\varepsilon,x_3)=J_3(\varepsilon)$ and $J_2(\varepsilon,x_3)=(y'_{f,3}(\varepsilon)-x_3)J_3(\varepsilon)$. The functions I_m, J_m, and K_m are connected by $\partial_\varepsilon J_m(\varepsilon,x_3)=I_{m+1}(\varepsilon,x_3)-J_{m+1}(\varepsilon,x_3)$ and $\partial_\varepsilon I_m(\varepsilon,x_3)=K_{m+1}(\varepsilon,x_3)-I_{m+1}(\varepsilon,x_3)$, i.e., by $\partial_\varepsilon I_m(\varepsilon,x_3)=-I_{m+1}(\varepsilon,x_3)$. In particular, we have

$$
\begin{aligned}
\partial_\varepsilon I_4(\varepsilon,x_3) &= \partial_\varepsilon I_4(\varepsilon) \;=\; 0,\\
I_4(\varepsilon,x_3) &= I_4 \;=\; \text{const.},\\
I_3(\varepsilon,x_3) &= (y'_{f,3}(\varepsilon)-x_3)I_4, && (7.5)\ \texttt{EqUj9}\\
\partial_\varepsilon J_3(\varepsilon,x_3) &= I_4(\varepsilon,x_3)-0 \;=\; I_4,\\
J_3(\varepsilon,x_3) &= J_3(\varepsilon) \;=\; J_3(0)+I_4\varepsilon \;=\; I_4\varepsilon, && (7.6)\ \texttt{EqUj2}\\
J_2(\varepsilon,x_3) &= (y'_{f,3}(\varepsilon)-x_3)I_4\varepsilon, && (7.7)\ \texttt{EqUj10}
\end{aligned}
$$

where we have used $J_3(0)=0$, which follows from the definition (7.4) and from the kernel identity $\partial_{y_3}^4 G(\vec{x},(y',x'_{f,3})^\top)=\partial_{y_3}^4 G((x',x''_{f,3}(0))^\top,(y',x_{f,3})^\top)$. Now (7.7) shows a linear behaviour of J_2 w.r.t. x_3. On the other hand, this J_2 is weakly bounded by Prop. 3.1. Consequently, the function J_2 is at most constant, and we get $I_4=0$. Consequently, (7.5) and (7.6) imply $I_3(\varepsilon,x_3)\equiv 0$ and $J_3(\varepsilon,x_3)\equiv 0$, respectively. Finally, we arrive at $I_3(\varepsilon,x_3)+J_3(\varepsilon,x_3)\equiv 0$, which is equivalent to the equations (7.3) and (7.2), i.e., the condition ii) of (HSRC) with $x'_{f,3}$ holds. □

Now we look at the reverse direction and show that condition ii) of the (HSRC) with $x'_{f,3}$ implies ii) with $x_{f,3}$. We fix an $\varepsilon>0$ and get

$$
\begin{aligned}
\partial_{x_3}^2 u(\vec{x}\,) &= \int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big(\vec{x},(y',x_3-\varepsilon)^\top\big)\, u\big((y',x_3-\varepsilon)^\top\big)\,\mathrm{d}y' && (7.8)\ \texttt{CoEq}\\
&= \int_{\mathbb{R}^2} 2\partial_{y_3}^3 G\big((y',\varepsilon)^\top,(0,0,0)^\top\big)\, u\big((x'-y',x_3-\varepsilon)^\top\big)\,\mathrm{d}y',
\end{aligned}
$$

which is valid for any x_3 with $x_3-\varepsilon\geq x'_{f,3}$ by condition ii) of the (HSRC) with $x'_{f,3}$ and by Lemma 7.1. It is not hard to see (use something like (6.1)) that both sides of (7.8) are Helmholtz solutions. However, these analytic functions coincide over the whole domain of definition. In other words, (7.8) holds for $x_3-\varepsilon\geq x_{f,3}$. In particular, choosing $x_3-\varepsilon=x_{f,3}$, we get condition ii) of the (HSRC) with $x_{f,3}$.

7.2 Equivalence of the conditions (1.4) and (1.5)

Suppose the conditions i) and ii) of (HSRC) are satisfied. For the equivalence of (1.4) and (1.5), we observe that the restriction $v(x'):=u((x',x_{f,3})^\top)$ is in the space $C_b^l(\mathbb{R}^2)$, $l\geq 0$. This follows from condition i) of (HSRC) and from $x_{f,3}>\sup_{x'\in\mathbb{R}^2}|F(x')|$ (cf. Fig. 1). We retain the notation of the spaces AV_κ, DD_τ, and $FC_k(\mathbb{R}^2)$ from (1.8), (1.9), and (1.10), respectively. Furthermore, we suppose D_tN is defined by (4.4) for functions in AV_κ and, for functions in FC_k, use (4.4) and the Fourier transform to define the integral in (4.4). For functions $v\in DD_\tau$ we define $D_tNv:=-\partial_{x_3}[V_kv_0]|_{\mathbb{R}^3_{x_{f,3}}}$ (cf. (3.1)).

⟨lDtN⟩
Lemma 7.2. *If $v\in FC_k$ or if $v\in AV_\kappa$ with $\kappa>0$ or if $v=v_s+(\Delta_{x'}+k^2I)v_0\in DD_\tau$ with $0\leq\tau<1$ (cf. (1.9)), then D_tNv is a well-defined bounded function, which is a partial solution of (4.3) with u replaced by v. Moreover, the function*

$$u_p(\vec{x}\,) \;:=\; v(x')+(D_tNv)(x')\,(x_3-x_{f,3})+\int_{x_{f,3}}^{x_3}(x_3-t)[V_ku]\big((x',t)^\top\big)\,\mathrm{d}t \qquad (7.9)$$ uR_p

(cf. (1.4)) satisfies (1.2).

Proof. Again, for simplicity, we may suppose $x_{f,3}=0$. The proof will be split into three parts, each considering one of the three different function classes for v.

i) First we suppose $v\in FC_k$. Due to this we have $v\in C_b^6(\mathbb{R}^2)$. The multiplied and differentiated functions $x'\mapsto(1+|x'|^2)^{-1}\partial_{x_j}^l v(x')$, $l=0,1,\cdots,6$ are in the space $L^2(\mathbb{R}^2)$. Equivalently, the functions $x'\mapsto\partial_{x_j}^l[(1+|x'|^2)^{-1}v(x')]$, $l=0,1,\cdots,6$ are in $L^2(\mathbb{R}^2)$ such that there exists a function $v_L\in L^2(\mathbb{R}^2)$ with $v(x')=(1+|x'|^2)(I-\Delta_{x'})^{-3}v_L(x')$ and such that the Fourier transform of v (cf. (6.5)) is $[\mathcal{F}v](\xi')=(I-\Delta_{\xi'})(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')$. From $\mathcal{F}(-2\partial_{x_3}^2G((\cdot,0)^\top)=\sqrt{k^2-|\xi'|^2}$ (compare (6.6)), we obtain (cf. (4.4))

$$\begin{aligned}D_tNv(x') &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}(I-\Delta_{\xi'})\Big\{e^{\mathbf{i}x'\cdot\xi'}\sqrt{k^2-|\xi'|^2}\Big\}(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi' \qquad (7.10)\\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}e^{\mathbf{i}x'\cdot\xi'}\left\{(1+|x'|^2)\sqrt{k^2-|\xi'|^2}+\frac{\mathbf{i}\,x'\cdot\xi'+2}{\sqrt{k^2-|\xi'|^2}}\right.\\ &\qquad\left.+\frac{|\xi'|^2}{\sqrt{k^2-|\xi'|^2}^3}\right\}(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi'.\end{aligned}$$ newDTN

Similar formulas hold for the derivatives w.r.t. x', i.e., for

$$(\Delta_{x'}+k^2I)D_tNv(x') \;=\; \frac{1}{4\pi^2}\int_{\mathbb{R}^2}(I-\Delta_{\xi'})\Big\{e^{\mathbf{i}x'\cdot\xi'}\sqrt{k^2-|\xi'|^2}^3\Big\}(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi',$$

which is exactly $\partial_{x_3}[V_ku]((x',0)^\top)$ (for the Fourier transform of the kernel $\partial_{y_3}^4\Phi(\cdot,\vec{y}\,)$ compare (6.6)). Clearly, there are no troubles with integration for large $|\xi'|$ due to the factor $(1+|\xi'|^2)^{-3}$.

If we suppose that the Fourier transform of v vanishes over the annular domain $R_{k,\varepsilon}$, used in the definition of (1.10), then we get smooth and bounded values $D_tNv(x')$ and $(\Delta_{x'}+k^2I)D_tNv(x')$ for x' in bounded domains. Shifting the coordinate system, i.e. multiplying the Fourier transforms by exponentials of modulus one, does not change on the vanishing in $R_{k,\varepsilon}$. Repeating all the above arguments, we even get uniform boundedness over $\mathbb{R}^2$. In other words, $f_2:=D_tNv$ is a well-defined bounded function, which is a partial solution of (4.3) with u replaced by v. Now we turn to the estimate of the function u_p of (7.9). For the terms in (7.9) and for $x'\neq y'$, we take into account the definition of $[V_k u]$ (cf. (3.2)) and apply (3.3). From (2.3) and (2.9) as well as from the jump relation for the double layer kernel, we have $\partial_{y_3}\Phi((x',0)^\top,(y',0)^\top)=\delta_{x'}(y')$. Using this together with (3.3), we arrive at

$$
\begin{aligned}
u_p(\vec{x}\,) &:= v(x') + x_3[D_tNv](x') + \int_0^{x_3} (x_3-t)[V_k v]\big((x',t)^\top\big)\,\mathrm{d}t, \qquad (7.11)\ \boxed{\texttt{Trem}}\\
&= x_3[D_tNv](x') + \int_{\mathbb{R}^2} \big\{ x_3\partial_{y_3}^2\Phi\big((x',x_3)^\top,(y',0)^\top\big) + \partial_{y_3}\Phi\big((x',x_3)^\top,(y',0)^\top\big)\big\} v(y')\,\mathrm{d}y'\\
&= \int_{\mathbb{R}^2} \partial_{y_3}\Phi\big((x',x_3)^\top,(y',0)^\top\big) v(y')\,\mathrm{d}y'.
\end{aligned}
$$

This is the double layer integral with Green's function from (1.1). Though the integral on the second line of (7.11) is absolutely integrable, together with the definition of D_tNv by Fourier transform (cf. (7.10)) also the last integral must be defined by switching to Fourier transforms (compare (6.6)). We get

$$
\begin{aligned}
u_p(\vec{x}\,) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{\mathbf{i}x'\cdot\xi'} e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}x_3}[\mathcal{F}v](\xi')\,\mathrm{d}\xi'.\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} (I-\Delta_{\xi'})\Big\{ e^{\mathbf{i}x'\cdot\xi'} e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}x_3}\Big\}(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi'. \qquad (7.12)\ \boxed{\texttt{newRup}}
\end{aligned}
$$

Now the Taylor-series expansion at $x_3=0$ for the function $x_3\mapsto e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}x_3}/[|\xi'|^2-k^2]$ takes the form

$$
\frac{e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}}{|\xi'|^2-k^2} = \frac{1}{|\xi'|^2-k^2} - \frac{\mathbf{i}}{\sqrt{k^2-|\xi'|^2}}x_3 + \int_0^{x_3}(x_3-t)e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,t}\,\mathrm{d}t,
$$

and leads us to

$$
\begin{aligned}
\int_0^{x_3}(x_3-t)u_p\big((x',t)^\top\big)\,\mathrm{d}t &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}(I-\Delta_{\xi'})\Bigg\{ e^{-\mathbf{i}x'\cdot\xi'}\Bigg[\frac{e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}x_3}}{|\xi'|^2-k^2}\\
&\qquad -\frac{1}{|\xi'|^2-k^2} + \frac{\mathbf{i}}{\sqrt{k^2-|\xi'|^2}}x_3\Bigg]\Bigg\}\\
&\qquad (1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')\,\mathrm{d}\xi',
\end{aligned}
$$

where, again, the assumption $[\mathcal{F}v](\xi')=0,\ \xi'\in R_{k,\varepsilon}$ frees us from any trouble with the non-smoothness of $\sqrt{k^2-|\xi'|^2}$, and the factor $(1+|\xi'|^2)^{-3}$ guarantees integrability for large $|\xi'|$. Applying the two-dimensional Laplacian $\Delta_{\xi'}$ to the term in brackets, we get at most a factor x_3^2 or a factor $|x'|^2$ such that u_p satisfies the weak boundedness condition (1.4) for $|x'|<c$. Shifting the x' coordinates, i.e. multiplying the Fourier transforms by exponentials of modulus one, we get the same result for any x'. Hence the solution u_p of (7.9) satisfies (1.4).

Splitting a general v into a sum of two functions, one with a Fourier transform vanishing in the annular domain $R_{k,\varepsilon}$ and one with support contained in the domain $R_{k,2\varepsilon}$, it remains to proof the lemma for the latter case. This case, however, is completely analogous to the just finished case. The only difference is that we apply the assumptions of functions from FC_k (cf. (1.10)) on the annular domain. Thus

$[\mathcal{F}v](\xi') = (I - \Delta_{\xi'})(1+|\xi'|^2)^{-3}[\mathcal{F}v_L](\xi')$ turns to $[\mathcal{F}v](\xi') = [\mathcal{F}v](\xi')$ the L^2 function with support in $R_{k,2\varepsilon}$ and, e.g., (7.10) and (7.12) into

$$\begin{aligned} D_tNv(x') &= \frac{1}{4\pi^2}\int_{R_{k,\varepsilon}} e^{\mathbf{i}x'\cdot\xi'}\sqrt{k^2-|\xi'|^2}[\mathcal{F}v](\xi')\,\mathrm{d}\xi', \\ u_p(\vec{x}\,) &= \frac{1}{4\pi^2}\int_{R_{k,\varepsilon}} e^{\mathbf{i}x'\cdot\xi'}e^{\mathbf{i}\sqrt{k^2-|\xi'|^2}\,x_3}[\mathcal{F}v](\xi')\,\mathrm{d}\xi'. \end{aligned}$$

We finally get the uniform boundedness of u_p and all the assertions of Lemma 7.2 for $v\in FC_k$.

ii) Now we assume $v\in AV_\kappa$. The Dirichlet-to-Neumann map on the right-hand side of (1.5) is a convolution operator with kernel depending only on $|x'-y'|$. Using (2.5) and (1.7), it takes the form g

$$\begin{aligned} [D_tNv](x') &:= -2\int_{\mathbb{R}^2}\partial^2_{y_3}G\big((x',0)^\top,(y',0)^\top\big)\,v(y')\,\mathrm{d}y' \\ &= -2\int_{\mathbb{R}^2}\partial^2_{y_3}G\big((y',0)^\top,(0,0,0)^\top\big)\,v(x'-y')\,\mathrm{d}y' \\ &= \int_{y'\in\mathbb{R}^2:\,|y'|>1}\mathcal{O}\big(|y'|^{-2}\big)\,v(x'-y')\,\mathrm{d}y' + \mathcal{O}(1)\,, \\ |[D_tNv](x')| &\leq c\int_1^\infty r^{-1}\,|av(v,x',r)|\,\mathrm{d}r + \mathcal{O}(1)\,, \end{aligned}$$

where the term $\mathcal{O}(1)$ results from an integration of a finite-part integral for a sufficiently smooth function. The estimate of $av(v,x',r)$ in the definition (1.8) of AV_κ, implies the continuity and uniform boundedness of D_tNv. Analogously, from (7.11) we conclude

$$\begin{aligned} |u_p(\vec{x}\,)| &\leq c\int_1^\infty \frac{|x_3|}{\sqrt{|x_3|^2+r^2}^2}r\,|av(v,x',r)|\,\mathrm{d}r + \mathcal{O}(1) \\ &\leq c|x_3|^{1-\varepsilon}\int_1^\infty r^{\varepsilon-1}\,|av(v,x',r)|\,\mathrm{d}r + \mathcal{O}(1)\,, \quad 0<\varepsilon<\min\{1,\kappa/2\}, \end{aligned}$$

such that the estimate of $av(v,x',r)$ in (1.8), implies the continuity and weak boundedness of the solution u_p. If we apply $(\Delta_{x'}+k^2I)$ to D_tNv and $(\Delta+k^2I)$ to u_p, the convergence of the integrals follows easily since the differentiated kernel functions satisfy the same estimates as the original kernels. Using the two facts that $\vec{x}\mapsto\partial_{y_3}\Phi(\vec{x},\vec{y}\,)$ is a solution of the Helmholtz equation and that $(\Delta_{x'}+k^2I)\partial^2_{y_3}\Phi(\vec{x},\vec{y}\,) = -\partial^4_{y_3}\Phi(\vec{x},\vec{y}\,)$, we get that u_p is a Helmholtz solution and that v is a solution of (4.3) with u replaced by v.

iii) Finally, we assume $v\in DD_\tau$. Taking v_s, v_i, and v_0 in accordance with (1.9), we define the Helmholtz solution $u_{DD}((x',x_3)^\top) := v_s(x') - [V_kv_0]((x',x_3)^\top)$. Using Prop. 6.2, we get the boundary value

$$\begin{aligned} u_{DD}\big((x',0)^\top\big) &= v_s(x') - [V_kv_0]\big((x',0)^\top\big) \\ &= v_s(x') + \big[(\Delta_{x'}+k^2I)v_0\big]\,(x') \;=\; v_s(x')+v_i(x') \;=\; v(x'). \end{aligned}$$

Furthermore, using $[V_kv_s]=0$ (cf. Subsect. 6.1) and the fact that differentiation and convolution operator commute, we conclude

$$\begin{aligned} \partial^2_{x_3}u_{DD}(\vec{x}) &= -\partial^2_{x_3}[V_kv_0](\vec{x}) &&= (\Delta_{x'}+k^2I)[V_kv_0](\vec{x}) \\ &= [V_k(\Delta_{y'}+k^2I)v_0](\vec{x}) &&= [V_kv_i](\vec{x}) \;=\; [V_kv](\vec{x}) \end{aligned}$$

such that (1.2) is fulfilled. Together with Prop. 3.1 the radiation condition (HSRC) holds for u_{DD}. In other words, u_{DD} is a solution to the Dirichlet problem $u_{DD}(\vec{x}\,) = v(\vec{x}\,),\ \vec{x} \in \mathbb{R}^3_0$ of the Helmholtz equation satisfying the radiation condition (HSRC).

Setting $f_2 := D_tNv := \partial_{x_3} u_{DD}|_{\mathbb{R}^3_0}$ we get a well-defined solution of (4.3) with u replaced by v. Indeed,

$$\begin{aligned}(\Delta_{x'} + k^2 I) f_2 &= -(\Delta_{x'} + k^2 I)\partial_{x_3}[V_k v_0] \;=\; -\partial_{x_3}(\Delta_{x'} + k^2 I)[V_k v_0] \\ &= -\partial_{x_3}[V_k(\Delta_{y'} + k^2 I) v_0] \;=\; -\partial_{x_3}[V_k v_i] \;=\; -\partial_{x_3}[V_k v].\end{aligned}$$

Clearly, by the Taylor-series expansion we get that the function u_p of (7.9) is equal to u_{DD}. Moreover, it satisfies the weak boundedness (1.4) due to definition $u_{DD}((x', x_3)^\top) := v_s(x') - [V_k v_0]((x', x_3)^\top)$ and to Prop. 3.1. □

Now the equivalence of the conditions (1.4) and (1.5) is easy to show. The general solution of (4.3) is $f_{2,g} = D_tN\tilde{v} + f_{2,h}$ with $\tilde{v} := u|_{\mathbb{R}^3_0}$ and with a solution $f_{2,h}$ of the two-dimensional Helmholtz equation $(\Delta_{x'} + k^2 I) f_{2,h} = 0$. The solution $u_g = u$ in (4.1), defined with $f_1 := \tilde{v}$ and $f_2 := f_{2,g}$, takes the form $u_g(\vec{x}) = u_p(\vec{x}) + f_{2,h}(x')x_3$. This, however, fulfills (1.4) if and only if $f_{2,h} \equiv 0$, i.e., if and only if $\partial_{x_3} u_g = f_{2,g} = D_tN(u|_{\mathbb{R}^3_0})$.

8 Solution of the Dirichlet problem over the half space

⟨ss6⟩ Now fix a Dirichlet-data function $v \in C_b(\mathbb{R}^2)$. For a function u continuous on the closure of $\mathbb{R}^3_+$ and twice differentiable on $\mathbb{R}^3_+$, we consider the Dirichlet boundary value problem

$$\begin{aligned}\Delta u(\vec{x}\,) + k^2 u(\vec{x}\,) &= 0,\ \forall \vec{x} \in \mathbb{R}^3_+, \\ u\big((x', 0)^\top\big) &= v(x'),\ \forall x' \in \mathbb{R}^2, \\ &u \text{ satisfies (HSRC).}\end{aligned} \tag{8.1}$$

DiPr

⟨uni⟩ **Proposition 8.1.** *The solution of problem (8.1) is unique.*

Proof. For two solutions u_1 and u_2 the difference $u_d = u_1 - u_2$ satisfies the homogeneous problem (8.1), i.e., the problem with $v \equiv 0$. However, from the radiation condition we get the representation (1.2). Using the arguments in Subsect. 7.1, we can even suppose $x_{f,3} = 0$. We infer that $\partial^2_{x_3} u_d \equiv 0$. Hence, u_d is linear with respect to x_3 and $u_d(\vec{x}\,) = f_1(x') + f_2(x')x_3$. From the weak boundedness condition (1.4), we get $f_2 \equiv 0$ and, from the homogeneous Dirichlet condition, $f_1 \equiv 0$. Hence, $u_d \equiv 0$ and the two solutions u_1 and u_2 coincide. □

It is unclear to us, whether there exist solutions of (8.1) for any Dirichlet data in $C_b(\mathbb{R}^2)$. Even if only the items i) and ii) of the radiation condition (HSRC) are satisfied, then we get a necessary condition. Namely, there must exist solutions of (4.3) with $x_{f,3} > 0$. More precisely, we define $V_{k,0}$ as V_k replacing $x_{f,3}$ by zero, choose an $x_{f,3} > 0$ and consider (4.3) with V_k replaced by $V_{k,0}$. Clearly, the right-hand side of the modified (4.3) is given by

$$-\partial_{x_3}[V_{k,0}u]\big((x', x_{f,3})^\top\big) \;:=\; \int_{\mathbb{R}^2} 2\partial^4_{y_3} G\big((x', x_{f,3})^\top, (y', 0)^\top\big)\, u\big((y', 0)^\top\big)\, \mathrm{d}y'.$$

We do not know whether this modified (4.3) always has a solution. Therefore, by $DS = DS_{x_{f,3}}$ we denote a space of functions $v = u|_{\mathbb{R}^3_0} \in C^{3,\kappa}_b(\mathbb{R}^2) = C^{3,\kappa}_b(\mathbb{R}^3_0)$ with exponent $0 < \kappa \leq 1$ s.t. there is a solution f_2 of the modified (4.3) with $f_2 \in C_b^l(\mathbb{R}^2)$ for all integers $l \geq 0$. Unfortunately, the solution f_2 is not unique. The general solution of the modified (4.3) is the sum of the partial solution f_2 and a homogeneous solution of the two-dimensional Helmholtz equation (cf. Lemma 5.1). We easily obtain the formal result

⟨theory⟩

Proposition 8.2. *i) For $v = u|_{\mathbb{R}^3_0} \in C_b^{3,\kappa}(\mathbb{R}^2)$, there exists a solution of (8.1), possibly without condition (1.4), if and only if v is in the space DS. If $v \in DS$, then a particular solution is given by*

$$u(\vec{x}\,) \;=\; v(x') + [D_tNv]\big((x',0)^\top\big)\, x_3 + \int_0^{x_3} (x_3 - t)\, [V_{k,0}v]\big((x',t)^\top\big)\, \mathrm{d}t, \qquad (8.2) \; \boxed{\texttt{laRepr}}$$

$$[V_{k,0}v](\vec{x}\,) \;:=\; \int_{\mathbb{R}^2} \partial^3_{y_3} \Phi\big(\vec{x}, (y',0)^\top\big)\, v(y')\, \mathrm{d}y', \qquad (8.3) \; \boxed{\texttt{newDefVk}}$$

$$[D_tNv]\big((x',0)^\top\big) \;:=\; f_2(x') - \int_0^{x_{f,3}} [V_{k,0}v]\,\big((x',t)^\top\big)\;\mathrm{d}t \qquad (8.4) \; \boxed{\texttt{newDefDtN}}$$

with $f_2 \in C_b^l(\mathbb{R}^2),\ l \geq 0$ a solution of the modified (4.3) with right-hand side $-\partial_{x_3} V_k[v](x', x_{f,3})$ replaced by $-\partial_{x_3} V_{k,0}[v](x', x_{f,3})$ (cf. (8.3)).
ii) Suppose there is a linear operator $D_tN_2 : DS \to C_b^2(\mathbb{R}^2)$ mapping v to a solution f_2 of the modified (4.3). Define the mapping D_tN by (8.4) with $f_2 = D_tN_2v$. Moreover, for the current proposition, replace item iii) of (HSRC) by the alternative condition $\partial_{x_3} u|_{\mathbb{R}^3_{x_{f,3}}} = D_tN_2\, u|_{\mathbb{R}^3_0}$. Then, for any $v \in DS$, there exists a unique solution of (8.1), which takes the form (8.2).

Proof. Obviously, the function u defined in (8.2) fulfills the Dirichlet boundary condition $u|_{\mathbb{R}^3_0} = v$ and (HSCR) possibly without (1.4). If $v \in C_b^2(\mathbb{R}^2)$, then Prop. 6.2 implies $(\Delta_{x'} + k^2 I)v(x') = -[V_{k,0}v](x',0)$, and the inhomogeneous Helmholtz equation $(\Delta_{x'} + k^2 I) D_tN_2 v(x') = -\partial_{x_3}[V_{k,0}v](x',0)$ follows from the modified (4.3) for f_2 and from

$$\begin{aligned} \partial_{x_3}[V_{k,0}v](x', x_{f,3}) - \partial_{x_3}[V_{k,0}v](x',0) \;&=\; \int_0^{x_{f,3}} \partial^2_{x_3} [V_{k,0}v](x',t)\mathrm{d}t \\ &=\; -(\Delta_{x'} + k^2 I) \int_0^{x_{f,3}} [V_{k,0}v](x',t)\mathrm{d}t. \end{aligned}$$

Note that the condition $v \in C^{3,\kappa}$ guarantees the existence of the boundary value $\partial_{x_3}[V_{k,0}v](x',0)$ as a finite part integral (cf. (2.8) and (2.11)). In other words, the assumptions of Prop. 4.1 are fulfilled and u defined by (8.2) is a Helmholtz solution. Consequently, u is a solution of (8.1) and part i) of the proposition follows. Part ii) is obvious. □

To avoid the impractical assumptions in Prop. 8.2, we have to require a stronger condition on v. Using semigroup theory, existence can be shown e.g. for v the sum of an L^2 function and a function, the Fourier transform of which is a bounded Radon measure (cf. [3]).

⟨exi⟩

Proposition 8.3. *Suppose that v is either in $AV_\kappa,\ \kappa > 0$, or in $DD_\tau,\ 0 \leq \tau < 1$ or in FC_k. Then there exists a unique solution of (8.1). For $v \in AV_\kappa$ with $\kappa \geq 1$, this solution is even uniformly bounded .*

Proof. In the case of DD_τ, the function u_{DD} is the solution due to part iii) of the proof to Lemma 7.2. For the case of AV_κ and FC_k, the function u_p of (7.11) is the solution in accordance with the parts i) and ii) of the proof to Lemma 7.2. Alternatively, for $v \in AV_\kappa$ with $\kappa > 0$, we can use the double layer operator

$$\begin{aligned}
u(\vec{x}\,) &= \int_{\mathbb{R}^2} 2\partial_{y_3} G\left(\vec{x},(y',0)^\top\right) u\left((y',0)^\top\right) \mathrm{d}y' \\
&= \int_0^\infty 2\partial_{y_3} G\left(\vec{x},(x_1+r,x_2,0)^\top\right) av(u,x',r) r\mathrm{d}r, \\
|u(\vec{x}\,)| &\le c\int_0^\infty \frac{x_3}{r^2+x_3^2} r^{1-\kappa}\mathrm{d}r \;=\; cx_3^{1-\kappa}\int_0^\infty \frac{1}{s^2+1} s^{1-\kappa}\mathrm{d}s,
\end{aligned}$$

which implies the uniform boundedness for $\kappa \geq 1$ and the weak boundedness for $\kappa > 0$. □

Now we introduce the subspace DD_{-1} of DD_0 by the formula (cf. (1.9))

$$\begin{aligned}
DD_{-1} \;:=\; \Big\{ v\in C_b(\mathbb{R}^2):\ & \exists c_v>0,\ \exists v_s\in C_b^2(\mathbb{R}^2),\ \exists v_0\in C^2(\mathbb{R}^2) \ \text{s.t.} \\
& v = v_s+v_i,\ (\Delta_{x'}+k^2I)v_s=0,\ v_i := (\Delta_{x'}+k^2I)v_0, \text{ and} \\
& |v_0(x')| \le c_v(1+|x'|)^{-1}, \forall x'\in\mathbb{R}^2, \\
& |\partial_{x_j} v_0(x')| \le c_v(1+|x'|)^{-1}, j=1,2, \forall x'\in\mathbb{R}^2 \Big\}.
\end{aligned}$$

For example, the function v_0 could be like $v_0(x') = (1+|x'|^2)^{-1}\sin((1+x_1^2)^{-1}\sin(x_1^2))$, for which v_i is bounded but does not decay for $x_1\to\infty$.

Proposition 8.4. *Consider all the $v\in DD_{-1}$. Then the corresponding splitting $v=v_i+v_s$ is unique.*

Proof. We have to show that nontrivial Dirichlet data cannot satisfy both, the two-dimensional Helmholtz equation $(\Delta_{x'}+k^2I)v=0$ and the representation $v=(\Delta_{x'}+k^2I)v_0$. We shall suppose both and show $v\equiv 0$.

From the Helmholtz equation, we get, for any test function φ, that $\langle v, (\Delta_{x'}+k^2I)\varphi\rangle = 0$. Now we substitute the representation of v as the image of the Helmholtz operator and choose the test function $\varphi = \chi_l v_0$ with $\chi_l(x') := \chi_l^0(|x'|)$, where $\chi_l^0 \geq 0$ denotes a cut-off function with $\chi_l^0\equiv 1$ on the segment $[0,l]$ and $\chi_l^0\equiv 0$ on the exterior $[l+1,\infty)$ of the larger segment $[0,l+1]$. Surely, we may suppose that χ_l and its first and second order derivatives are uniformly bounded over $\mathbb{R}^2$ by a constant independent of l. The supports of the derivatives of χ_l are contained in the annular domain $\{x'\in\mathbb{R}^2: l\le|x'|\le l+1\}$. Then, due to the estimates for v_0 and $\partial_{x_j}v_0$,

$$\begin{aligned}
0 &= \left\langle (\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)\left[\chi_l v_0\right]\right\rangle \\
&= \left\langle \chi_l(\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)v_0\right\rangle \\
&\quad + \left\langle (\Delta_{x'}+k^2I)v_0, \left[v_0\Delta_{x'}\chi_l + 2\sum_{j=1}^2 \partial_{x_j}\chi_l\partial_{x_j}v_0\right]\right\rangle \\
&= \left\langle \chi_l(\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)v_0\right\rangle + \mathcal{O}(1).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\langle \chi_{L+1}(\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)v_0\right\rangle &= \left\langle \chi_1(\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)v_0\right\rangle + \\
&\quad \sum_{l=1}^L \left\langle [\chi_{l+1}-\chi_l](\Delta_{x'}+k^2I)v_0, (\Delta_{x'}+k^2I)v_0\right\rangle \\
&= \mathcal{O}(1).
\end{aligned}$$

If the last truncated sum of nonnegative terms is uniformly bounded, then the infinite sum is convergent, and we arrive at

$$\big\langle (\Delta_{x'} + k^2 I)v_0, (\Delta_{x'} + k^2 I)v_0 \big\rangle \;\;<\;\; \infty.$$

In other words, $x_2 \mapsto v(x') = (\Delta_{x'} + k^2 I)v_0(x')$ is square integrable over $\mathbb{R}^2$ and we may apply the Fourier transform $\mathcal{F}$. As a solution of the Helmholtz equation the square integrable Fourier transform $\mathcal{F}[v]$ satisfies $(k^2 - |\xi|^2)\mathcal{F}[v](\xi) = 0$. Thus $\mathcal{F}[v]$ and v are zero. □

The space DD_{-1} is algebraically the direct sum of the space of Helmholtz solutions plus the space of all images of the Helmholtz operator. If the metric of the function space corresponds to the uniform convergence over bounded subdomains, then the space of Helmholtz solutions is closed. However, the space of images is not. For example, the function $x' \mapsto e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha^2 + \beta^2 = k^2$ is the limit of functions $x' \mapsto e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$ with $\alpha^2 + \beta^2 \neq k^2$ (cf. the subsequent example i)). It would be nice to have an intrinsic description of the spaces DD_τ. Instead, we only recall important functions belonging to the spaces DD_0 and AV_κ:

i) The space DD_0 contains all exponential functions $x' \mapsto v(x') = e^{\mathbf{i}(\alpha x_1 + \beta x_2)}$, i.e., the traces of the plane-wave functions. For $\alpha^2 + \beta^2 = k^2$, the function $v = v_s$ is a two-dimensional Helmholtz solution and, for $\alpha^2 + \beta^2 \neq k^2$, the function v is an image $(\Delta_{x'} + k^2 I)v_0$ with $v_0 = \frac{1}{-\alpha^2 - \beta^2 + k^2}\, v$. If $\alpha^2 + \beta^2 \neq 0$, then the exponential function is in $AV_{1/2}$ due to the fact that $av(v, x', r)$ satisfies $av(v, x', r) = 2\pi\, e^{\mathbf{i}(\alpha x_1 + \beta x_2)} J_0(\sqrt{\alpha^2 + \beta^2}\, r)$.

ii) The space DD_0 contains all decaying functions $v \in C_b(\mathbb{R}^2)$, with $v(x') = \mathcal{O}(|x'|^{-3/2-\varepsilon})$ for $|x'| \to 0$ and fixed positive ε. Indeed, such a function is an image $v = (\Delta_{x'} + k^2 I)v_0$ with

$$v_0(x') \;\;=\;\; \frac{\mathbf{i}}{4} \int_{\mathbb{R}^2} H_0^{(1)}\big(k|x' - y'|\big) v(y')\, \mathrm{d}y'.$$

By the same argument, we even get $AV_{3/2+\varepsilon} \subset DD_0$. Obviously, the space AV_κ contains all decaying functions $v \in C_b^1(\mathbb{R}^2)$, with $v(x') = \mathcal{O}(|x'|^{-\kappa})$ for $|x'| \to 0$ and fixed positive κ.

iii) The space DD_0 contains all traces $y' \mapsto G(\vec{x}, (y', 0)^\top)$ of point source functions for fixed $\vec{x} \notin \mathbb{R}^3_+ \cup \mathbb{R}^3_0$. Indeed, such a trace is an image according to

$$\begin{aligned}
(\Delta_{y'} + k^2 I)\Big(e^{\mathbf{i}k|\vec{x} - (y',0)^\top|}\Big) \;&=\; (\mathbf{i}k)\frac{e^{\mathbf{i}k|\vec{x} - (y',0)^\top|}}{|\vec{x} - (y',0)^\top|} + k^2 \frac{e^{\mathbf{i}k|\vec{x} - (y',0)^\top|} x_3^2}{|\vec{x} - (y',0)^\top|^2} + (\mathbf{i}k)\frac{e^{\mathbf{i}k|\vec{x} - (y',0)^\top|} x_3^2}{|\vec{x} - (y',0)^\top|^3}, \\
G\big(\vec{x}, (y',0)^\top\big) \;&=\; (\Delta_{y'} + k^2 I)\frac{1}{4\pi(\mathbf{i}k)} \Bigg\{ e^{\mathbf{i}k|\vec{x} - (y',0)^\top|} \\
&\quad - \frac{\mathbf{i}x_3^2}{4} \int_{\mathbb{R}^2} H_0^{(1)}\big(k|y' - z'|\big) \left[\frac{k^2 e^{\mathbf{i}k|\vec{x} - (z',0)^\top|}}{|\vec{x} - (z',0)^\top|^2} + \frac{(\mathbf{i}k) e^{\mathbf{i}k|\vec{x} - (z',0)^\top|}}{|\vec{x} - (z',0)^\top|^3} \right] \mathrm{d}z' \Bigg\}.
\end{aligned}$$

Similarly, the traces of all the derivatives $y' \mapsto \partial_{\vec{x}}^{\alpha_x} \partial_{\vec{y}}^{\alpha_y} G(\vec{x}, (y', 0)^\top)$ with multi-indices α_x and α_y are contained in DD_0. These function belong to AV_1 in accordance with example ii).

iv) Now look at the traces $y' \mapsto G_{2D}((x_1, x_3)^\top, (y_1, 0)^\top)$ of $\vec{y} \mapsto G_{2D}((x_1, x_3)^\top, (y_1, y_3)^\top)$ for fixed $(x_1, x_3)^\top \notin \mathbb{R}^2_+ \cup \mathbb{R}^2_0$. Such a trace function is the image $(\Delta_{x'} + k^2 I)v_0$, where v_0 is the y_2-independent solution $(y_1, y_2)^\top \mapsto v_0((y_1, y_2)^\top) = v_0(y_1)$ of the ordinary differential equation

$(\partial_{y_1}^2 + k^2 I)v_0 = G_{2D}((x_1,x_3)^\top,(y_1,0)^\top)$. Namely, supposing $x_1=0$ without loss of generality, we obtain

$$\begin{aligned}
v_0(y_1) &= \int_0^{|y_1|} \frac{e^{\mathbf{i}k(|y_1|-t)} - e^{-\mathbf{i}k(|y_1|-t)}}{2\mathbf{i}k}\, \frac{\mathbf{i}}{4} H_0^{(1)}\Big(k\sqrt{t^2+x_3^2}\Big)\,\mathrm{d}t, \\
& \quad v_0(0) = v_0'(0) = 0, \\
v_0(y_1) &= \mathcal{O}(1) + \sqrt{\frac{2}{k\pi}}\, e^{\mathrm{i}\pi/4}\,\frac{\mathbf{i}}{4}\int_0^{|y_1|} \frac{e^{\mathbf{i}k(|y_1|-t)} - e^{-\mathbf{i}k(|y_1|-t)}}{2\mathbf{i}k}\, \frac{e^{-\mathbf{i}k\sqrt{t^2+x_3^2}}}{\sqrt{t^2+x_3^2}^{1/2}}\mathrm{d}t \\
&= \mathcal{O}(1) + \sqrt{\frac{2}{k\pi}}\, e^{\mathrm{i}\pi/4}\,\frac{\mathbf{i}}{4}\int_0^{|y_1|} \frac{e^{\mathbf{i}k(|y_1|-t)} - e^{-\mathbf{i}k(|y_1|-t)}}{2\mathbf{i}k}\, \frac{e^{-\mathbf{i}kt}}{\sqrt{t^2+x_3^2}^{1/2}}\mathrm{d}t \\
&= \mathcal{O}(1) - \sqrt{\frac{2}{k\pi}}\, e^{\mathrm{i}\pi/4}\,\frac{1}{8k}\, e^{-\mathbf{i}k|y_1|}\int_0^{|y_1|} \frac{1}{\sqrt{t^2+x_3^2}^{1/2}}\mathrm{d}t \;=\; \mathcal{O}\big(|y_1|^{1/2}\big)
\end{aligned}$$

for $|y_1|\to\infty$, due to the asymptotics of the Hankel function (cf. [1], Sect. 9.2.7). In other words, the traces $y'\mapsto G_{2D}((x_1,x_3)^\top,(y_1,0,0)^\top)$ belong to a “two-dimensional” version of $DD_{1/2}$. Clearly, the potential $V_{k,0}[v_0]$ is absolutely integrable since

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \partial_{y_3}^3 G\big(\vec{x},(y',0)^\top\big)\, v_0\big((y',0)^\top\big)\,\mathrm{d}y'\right| &\le c\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|\vec{x}-(y',0)^\top|^3}\mathrm{d}y_2 |y_1|^{1/2}\mathrm{d}y_1 \\
&\le c\int_{\mathbb{R}} \frac{1}{|(x_1,x_3)^\top-(y_1,0)^\top|^2}\,|y_1|^{1/2}\mathrm{d}y_1.
\end{aligned}$$

Repeating the arguments of part iii) of the proof to Lemma 7.2, we conclude that the fundamental solution $\vec{y}\mapsto G_{2D}((x_1,x_3)^\top,(y_1,0)^\top)$ with fixed $(x_1,x_3)^\top\notin\mathbb{R}_+^2\cup\mathbb{R}_0^2$ satisfies (1.2) and the condition (HSRC). By the same proof, even any derivative of this function satisfies condition (HSRC). However, any two-dimensional Helmholtz solution u_{2D}, satisfying the two-dimensional Sommerfeld radiation condition, is a superposition of such functions. In other words, any Helmholtz solution $\vec{y}\mapsto u_{\mathrm{2D}}((y_1,y_3)^\top)$ with a two-dimensional Helmholtz solution u_{2D} satisfying the two-dimensional Sommerfeld radiation condition, fulfills the condition (HSRC).

v) By definition DD_0 contains all solutions $u\in C_b^2(\mathbb{R}^2)$ of the two-dimensional Helmholtz equation. These function are contained in $AV_{1/2}$, since $av(v,x',r)=v(x')J_0(kr)$.

9 Uniqueness for the Dirichlet problem on thin layers

⟨sLay⟩ For a height $h_L>0$ and an index pair $l'\in\mathbb{Z}^2$, we introduce $\Omega_L:=\{\vec{x}\in\mathbb{R}^3:\ 0<x_3<h_L\}$ and the cylindrical domain $\Omega_{L,l'}:=\{\vec{x}\in\Omega_L:\ |x'-l'|<4\}$. We consider the Dirichlet problem

$$\begin{aligned}
\Delta u(\vec{x}\,) + k^2 u(\vec{x}\,) &= 0,\ \forall \vec{x}\in\Omega_L, \\
u\big((x',0)^\top\big) &= v_0(x'),\ \forall x'\in\mathbb{R}_0^3, \\
u\big((x',h_L)^\top\big) &= v_{h_L}(x'),\ \forall x'\in\mathbb{R}_{h_L}^3, \\
\sup_{l'\in\mathbb{Z}^2}\|u\|_{H^1(\Omega_{L,l'})} &< \infty
\end{aligned} \tag{9.1}$$ DiPrLa

with prescribed bounded and continuous Dirichlet data v_0 and v_{h_L}.

Lemma 9.1. *If the positive width h_L is less than π/k, then any solution of the Dirichlet problem (9.1) over the layer Ω_L of thickness h_L is unique.*

Proof. Of course, we have to prove that any solution of the homogeneous problem (9.1) is trivial. Suppose u is a solution of (9.1) with $v_0 \equiv 0$ and $v_{h_L} \equiv 0$. Then we extend u to a function over $\mathbb{R}^3$ by

$$u(\vec{x}\,) \;:=\; \begin{cases} u\big((x',z_3)^\top\big) & \text{if } x_3 = z_3 + (2m)\,h_L, \quad\;\; 0 \le z_3 \le h_L,\ m \in \mathbb{Z} \\ -u\big((x',h_L - z_3)^\top\big) & \text{if } x_3 = z_3 + (2m+1)\,h_L,\ 0 \le z_3 \le h_L,\ m \in \mathbb{Z} \end{cases},$$

which is $(2h_L)$ periodic w.r.t. x_3. Since this extended u and the normal derivatives $\partial_{x_3} u$ are continuous through the interface planes $\mathbb{R}^3_{mh_L}$, $m \in \mathbb{Z}$, the function u is a periodic Helmholtz solution over $\mathbb{R}^3$. Consequently, the modulated Fourier coefficients $\hat{v}_m$, defined by

$$\begin{aligned} u\big((x',x_3)^\top\big) &= \sum_{m\in\mathbb{Z}} \hat{u}_m(x') e^{\mathbf{i}\pi m x_3/h_L}, \qquad (9.2)\ \boxed{\texttt{FoSe}} \\ \hat{u}_m(x') &:= \int_0^{2h_L} u\big((x',x_3 - z_3)^\top\big)\, e^{\mathbf{i}\pi(x_3-z_3)m/h_L} \mathrm{d}z_3, \\ \hat{v}_m\big((x',x_3)^\top\big) &:= \int_0^{2h_L} e^{-\mathbf{i}\pi z_3 m/h_l} u\big((x',x_3 - z_3)^\top\big)\,\mathrm{d}z_3 \;=\; \hat{u}_m(x') e^{-\mathbf{i}\pi x_3 m/h_L}, \end{aligned}$$

are Helmholtz solution for any $m \in \mathbb{Z}$. In other words, the function $\hat{u}_m$ satisfies $(\Delta_{x'} - \varrho_m^2 I)\hat{u}_m = 0$ with $\varrho_m := \sqrt{(\pi m/h_L)^2 - k^2}$. The average $av(\hat{u}_m, x', r)$ defined in (1.7) satisfies the corresponding Bessel equation over the real half axis and is smooth at zero. Since the Bessel function $r \mapsto Y_0(\mathbf{i}\varrho_m r)$ is singular at zero (cf. [1], Sect. 9.1.89) and since the Bessel function $r \mapsto J_0(\mathbf{i}\varrho_m r)$ is unbounded for $r \to \infty$ (cf. [1], Sect. 9.2.1), the solution $av(\hat{u}_m, x', r)$ is zero. Using (9.2) and the differentiated (9.2), we get $av(u|_{\mathbb{R}^3_{x_3}}, x', r) = 0$ and $av(\partial_{x_3} u|_{\mathbb{R}^3_{x_3}}, x', r) = 0$. Taking the derivative of $av(u|_{\mathbb{R}^3_{x_3}}, x', r) = 0$, we additionally get $\partial_r av(u|_{\mathbb{R}^3_{x_3}}, x', r) = 0$. Now Green's identity yields, for any $\vec{x}$ with $0 < x_3 < h_L$

$$\begin{aligned} u(\vec{x}\,) &= \int_{\{\vec{y}:|y'-x'|<R, x_3 = h_L\}} \{\partial_{y_3} G(\vec{x},\cdot)u - G(\vec{x},\cdot)\partial_{y_3} u\} \\ &\quad - \int_{\{\vec{y}:|y'-x'|<R, x_3 = 0\}} \{\partial_{y_3} G(\vec{x},\cdot)u - G(\vec{x},\cdot)\partial_{y_3} u\} \\ &\quad + \int_{\{\vec{y}:|y'-x'|=R, 0<x_3<h_L\}} \{\partial_r G(\vec{x},\cdot)u - G(\vec{x},\cdot)\partial_r u\}\,. \end{aligned}$$

Substituting $y' = x' + r(\cos\phi, \sin\phi)^\top$, the differentiated fundamental solutions depend on r but not on ϕ such that all the integrals turn to integrals w.r.t. r and u, $\partial_{x_3} u$, $\partial_r u$ appears only via multiplication by the averages $av(u|_{\mathbb{R}^3_{x_3}}, x', r)$, $av(\partial_{x_3} u|_{\mathbb{R}^3_{x_3}}, x', r)$, $\partial_r av(u|_{\mathbb{R}^3_{x_3}}, x', r)$. Since these averages are zero, the integrals are zero, and we conclude $u(\vec{x}\,) = 0$. □

References

AbSt [1] M. Abramowitz and I.A. Stegun, eds., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Applied Mathematics Series **55**, Washington D.C., New York, United States Department of Commerce, National Bureau of Standards, Dover Publication, 1983.

ArHo [2] T. Arens and T. Hohage, On radiation conditions for rough surface scattering problems, *IMA J. Appl. Math.* **70** (2005), pp. 839–847.

Ce [3] M. Cessenat, *Mathematical methods in electromagnetism*, World Scientific, Singapore, 1996.

ChEl [4] S.N. Chandler-Wilde and J. Elschner, Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces, *SIAM J. Mathem. Analysis* **42** (2010), pp. 2554–2580.

ChMo [5] S.N. Chandler-Wilde and P. Monk, Existence, uniqueness and variational methods for scattering by unbounded rough surfaces, *SIAM J. Mathem. Analysis* **37** (2005), pp. 598–618.

ChRoZh [6] S.N. Chandler-Wilde, C.R. Ross, and B. Zhang, Scattering by infinite one-dimensional rough surfaces, *Proc. R. Soc. London Ser. A* **455** (1999), pp. 3767–3787.

ChZh [7] S.N. Chandler-Wilde and B. Zhang, A uniqueness result for scattering by infinite rough surfaces, *SIAM J. Appl. Math.* **58** (1998), pp. 1774–1790.

HuLuRa [8] G. Hu, W. Lu, and A. Rathsfeld, Time-harmonic acoustic scattering from locally perturbed periodic curves. *SIAM J. Appl. Math.* **81** (2021), pp. 2569–2595.

Kr [9] J. Král, *Integral operators in potential theory*, Lecture Notes in Mathematics, Vol. **823**, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

Sa [10] J.A. DeSanto, Scattering by rough surfaces, in *Scattering: Scattering and inverse scattering in pure and applied science*, R. Pike and P. Sabatier, eds., Academic Press, New York, 2002, pp. 15–36.