Weierstraß-Institut
für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Millions of Perrin pseudoprimes including a few giants

Holger Stephan

submitted: December 16, 2019

Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: holger.stephan@wias-berlin.de

No. 2657
Berlin 2019

2010 *Mathematics Subject Classification.* 11B37, 11B39, 11B50.

Key words and phrases. Pseudoprimes, recurrence sequences, fast algorithm, large numbers.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Millions of Perrin pseudoprimes including a few giants

Holger Stephan

Abstract

The calculation of many and large Perrin pseudoprimes is a challenge. This is mainly due to their rarity. Perrin pseudoprimes are one of the rarest known pseudoprimes. In order to calculate many such large numbers, one needs not only a fast algorithm but also an idea how most of them are structured to minimize the amount of numbers one have to test.

We present a quick algorithm for testing Perrin pseudoprimes and develop some ideas on how Perrin pseudoprimes might be structured. This leads to some conjectures that still need to be proved.

We think that we have found well over 90% of all 20-digit Perrin pseudoprimes. Overall, we have been able to calculate over 9 million Perrin pseudoprimes with our method, including some very large ones. The largest number found has 1436 digits. This seems to be a breakthrough, compared to the previously known just over 100,000 Perrin pseudoprimes, of which the largest have 20 digits.

In addition, we propose two sequences that do not provide any pseudoprimes up to 10^9 at all.

Contents

DOI 10.20347/WIAS.PREPRINT.2657 Berlin 2019

1 Introduction

To motivate that it makes sense to deal with primes, it is best to quote Gauss [3]:

> *The problem of distinguishing prime numbers from composite numbers, and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic. It has engaged the industry and wisdom of ancient and modern geometers to such an extent that it would be superfluous to discuss the problem at length. Nevertheless we must confess that all methods that have been proposed thus far are either restricted to very special cases or are so laborious and difficult that even for numbers that do not exceed the limits of tables constructed by estimable men, they try the patience of even the practiced calculator. And these methods do not apply at all to larger numbers.*

Prime numbers are a very serious issue. We prefer dealing with pseudoprimes. Pseudoprimes are numbers that behave similar to primes.

Sometimes it is a big challenge to compute all or at least many or some very large pseudoprimes of a given type.

In this paper, we introduce a quick algorithm for the calculation of Perrin pseudoprimes. This is nothing special, there are already many fast algorithms. Similar to primes, also for pseudoprimes it is difficult to guess their structure. Therefore, in order to calculate all of them there is nothing left but to test every single number. This strongly limits the size of the numbers. It turns out, however, that the structure can be guessed for most of the pseudoprimes. This very much limits the range of potential numbers to be tested and makes it possible to calculate millions of them and even very large ones.

We do the following **Notations:**

- The set of all primes is denoted by $\mathbb{P}$.
- $a|b$ means a divides b or b is divisible by a.
- We state some classical facts from number theory as theorems, omitting the proofs.

1.1 The Perrin sequence

Let us define a sequence (called Perrin sequence) P_n recursively:

$$\begin{aligned} P_0 &= 3 \\ P_1 &= 0 \\ P_2 &= 2 \\ P_n &= P_{n-2} + P_{n-3},\ n \geq 3 \end{aligned}$$

and calculate the first entries:

$$(P_n)_{n=0}^{\infty} = 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, ...$$

We observe: If n is prime, we have $n|P_n$ and that goes on for a long time.

Anyone seeing this sequence for the first time is certainly quite surprised, since it is believed that there is no simple algorithm for calculating the primes.

The recursion law of this sequence was found in 1899 by Edouard Lucas. This sequence with the initial values given above, was first used by Raoul Perrin [7, 8].

Probably many mathematicians and amateur mathematicians have tried to answer the question of whether this sequence really only produces primes. Considering that already the number P_{811} has 100 digits, one can imagine how difficult that has been.

The answer was not found until 1982, when Jeffrey Shallit (according to [8]) calculated the first two non-prime numbers – so-called Perrin pseudoprimes (PPP) – with a computer. Here they are: $271441 = 521 \cdot$ 521 and $904631 = 7 \cdot 13 \cdot 9941$. P_{271441} has 33150 digits. Today it is known that there are infinitely many Perrin pseudoprimes [1]. Nevertheless, they are very rare, which makes their finding still difficult.

In this paper, we develop an effective algorithm for calculating Perrin pseudoprimes and present some numerical results that constitute, to our knoledge, right now the world's largest collection of Perrin pseudoprimes including the largest PPP.

DOI 10.20347/WIAS.PREPRINT.2657 Berlin 2019

2 Pseudoprimes

2.1 Iff– and if–Theorems

There are two kinds of theorems dealing with primes that can be used to test a given number on whether it is a prime.

1) Theorems like: $p \in \mathbb{P}$ if and only if property $A(p)$ holds.

2) Theorems like: $p \in \mathbb{P}$, then property $A(p)$ holds.

Theorems of the first kind are, for example

- **Theorem:** $p \in \mathbb{P} \iff \forall k \in \mathbb{P}, k \leq \sqrt{p} : \ k \not| p$
- **Theorem (Wilson):** $p \in \mathbb{P} \iff p | 1 \cdot 2 \cdot 3 \cdots (p-1) + 1$
- **Theorem:**

$$p \in \mathbb{P} \iff p | \binom{p}{k} \quad \forall k = 1, ..., p-1 \tag{1}$$

These theorems allow for deterministic tests. If for a given number p the property $A(p)$ holds, then p is prime.

Unfortunately, algorithms based on deterministic testing have high complexity, so far.

Theorems of the second kind state: If for a given number p the property $A(p)$ holds, then p can be prime or not. This is useful, if p is prime with very high "probability". Testing $A(p)$ one can be "very sure" that p is prime. Typically such kind of probabilistic tests are much faster (have a lower complexity) than deterministic ones. Thus, it is useful to create tests with a very small equivalence gap, the gap between if and iff.

Numbers n that lie in this gap, i.e. $A(n)$ holds, but n is composite, are called pseudoprimes with respect to property A.

One example, following immediately from (1) is:

Theorem: $p \in \mathbb{P} \implies p \left| \sum_{k=1}^{p-1} a_k \binom{p}{k} \right.$ for some given integers a_k.

It is clear that looking at a linear combination of binomial coefficients instead of all coefficients in detail, we loose information. This is just the equivalence gap. Looking at a given linear combination of binomial coefficients is faster than looking at every one in detail. The idea is to choose such coefficients a_k so that the equivalence gap is small.

Here, we define some kind of probability (better frequency) for a pseudoprime test. Let $\pi(n)$ be the number of primes less than n and $P(n)$ the number of pseudoprimes less than n for a given pseudoprime test. By $W(n) = P(n)/\pi(n)$ we define the frequency of numbers incorrectly tested and call it error rate. Thus, the lower the error rate $W(n)$, the better the test.

Of course, it would be best if a test provided only a finite number of pseudoprimes. These would be calculated and stored in a database which allowed for a deterministic test, practically. Such a test is

not yet known. In contrast, until now, for many pseudoprime number type, it has been proved sooner or later that there are infinitely many ones.

2.2 Fermat and Carmichael pseudoprimes

The simplest pseudoprimes are Fermat pseudoprimes. They are consequences of Fermat's little

Theorem: Given an integer $z \geq 2$. If $p \in \mathbb{P}$ then $p|z^p - z$.

Conversely, if a number n for some z satisfies $n|z^n - z$ but $n \notin \mathbb{P}$, n is called Fermat$_z$ pseudoprime.

Best known is the special case $z = 2$:

Theorem: If $p \in \mathbb{P}$ then $p|2^p - 2$.

A number $n \notin \mathbb{P}$ with $n|2^n - 2$ is called Fermat$_2$ pseudoprime.

2.2.1 Fermat$_2$ pseudoprimes

Fermat's little Theorem for $z = 2$ is an easy consequence of Theorem 1.

Indeed, multiplying out $(a + b)^n$ with integers a, b we get

$$(a+b)^n \quad = \quad a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + b^n$$

Therefore, defining

$$f_n = (a+b)^n - a^n - b^n = \binom{n}{1} a^{n-1}b + \ldots + \binom{n}{n-1} ab^{n-1}$$

we obtain the

Theorem: If $p \in \mathbb{P}$ then $p|f_p$.

The special case ($a = b = 1$) yields Fermat's little Theorem to the base $z = 2$.

Let's calculate the first ones:

n	$2^n - 2$	$n\|2^n - 2$?	n is prime?
2	2	yes!	yes!
3	6	yes!	yes!
4	14	no!	no!
5	30	yes!	yes!
6	62	no!	no!
7	126	yes!	yes!
341	4479... (103 digits)	yes!	no! $341 = 11 \cdot 31$
561	7547... (169 digits)	yes!	no! $561 = 3 \cdot 11 \cdot 17$
645	1459... (195 digits)	yes!	no! $645 = 3 \cdot 5 \cdot 43$

Up to 100000 we have 78 pseudoprimes and 9592 primes. Thus, we have $W(10^5) = 0.00813178$.

2.2.2 Carmichael numbers

Instead of $z = 2$ we can consider Fermat$_z$ pseudoprimes with other bases. Maybe other bases provides fewer pseudoprimes? It turns out that $z = 2$ is one of the best bases. Moreover, there are non-primes n with $n|z^n - z$ for any base z, the so-called Carmichael numbers. 561 is the smallest one. The next ones are

Carmichael number	factors
561	$3 \cdot 11 \cdot 17$
1105	$5 \cdot 13 \cdot 17$
1729	$7 \cdot 13 \cdot 19$
2465	$5 \cdot 17 \cdot 29$
2821	$7 \cdot 13 \cdot 31$
6601	$7 \cdot 23 \cdot 41$
8911	$7 \cdot 19 \cdot 67$
10585	$5 \cdot 29 \cdot 73$

Carmichael number	factors
15841	$7 \cdot 31 \cdot 73$
29341	$13 \cdot 37 \cdot 61$
41041	$7 \cdot 11 \cdot 13 \cdot 41$
46657	$13 \cdot 37 \cdot 97$
52633	$7 \cdot 73 \cdot 103$
62745	$3 \cdot 5 \cdot 47 \cdot 89$
63973	$7 \cdot 13 \cdot 19 \cdot 37$
75361	$11 \cdot 13 \cdot 17 \cdot 31$

There are 16 Carmichael numbers up to 100000. Moreover, we have the following

Theorem: There are infinitely many Carmichael numbers [1].

2.3 General pseudoprimes

2.3.1 Sums of powers. Multinomial coefficients

Similar to binomial coefficients, there is a theorem for multinomial coefficients:

Theorem: $p \in \mathbb{P} \iff p|\frac{p!}{i!\ j!\ k!},\ \forall i,j,k$ with $0 < i + j + k = p$

From this, we conclude the following

Theorem: $p \in \mathbb{P} \implies p\left|\sum_{0<i+j+k=p} a_{ijk}\ \frac{p!}{i!\ j!\ k!}\right.$. for some integer coefficients a_{ijk}.

From this, multiplying out $(a + b + c)^n$ with integers a, b, c we conclude the

Theorem: Given a sequence

$$f_n = (a+b+c)^n - a^n - b^n - c^n \ = \sum_{0<i+j+k=n} \frac{n!}{i!\ j!\ k!}\, a^i b^j c^k$$

Then, $p \in \mathbb{P}$ implies $p|f_p$.

Similarly we get the

Theorem: Given integers $a_1, ..., a_k$. Build the sequence

$$f_n = (a_1 + a_2 + ... + a_k)^n - (a_1^n + a_2^n + ... + a_k^n) \tag{2}$$

Then, $p \in \mathbb{P}$ implies $p|f_p$.

The example $a_i = 1$ yields $f_n = k^n - k$, Fermat's little theorem in the general case.

Perrin's sequence is given in a recurrent way. Here, we recall the important connection between polynomials and recurrence sequences.

2.3.2 Polynomials and recurrence sequences

A linear recurrence sequence (or linear difference equation) of order k is a sequence $(h_n)_{n=0}^{\infty}$ defined in the following way:

Given k numbers $c_1, ..., c_k$ set

$$h_n = c_1 h_{n-1} + c_2 h_{n-2} + ... + c_k h_{n-k} \; . \tag{3}$$

Together with k initial conditions h_0, h_1, ..., h_{k-1} such a sequence is uniquely determined.

Obviously, if $c_1, ..., c_k$ and h_0, h_1, ..., h_{k-1}, are integers, then h_n is an integer for all n.

There is a remarkable connection between such sequences and polynomials of degree k. If we put $h_n = x^n$ and multiply by x^{k-n}, we get an algebraic equation for the roots of a polynomial formed from the coefficients of the sequence

$$Q(x) = -x^k + c_1 x^{k-1} + c_2 x^{k-2} + ... + c_{k-1} x + c_k \; . \tag{4}$$

This polynomial has k – in general complex – roots $x_1, ..., x_k$. For simplicity, we assume that the roots are different.

Set

$$g_n = b_1 x_1^n + b_2 x_2^n + ... + b_k x_k^n \; , \tag{5}$$

with some coefficients $b_1, ..., b_k$. Solve the system of k linear equations $h_i = g_i$, $i = 0, ..., k-1$ with respect to the unknown b_j. This is always uniquely solvable, because the corresponding matrix is the Vandermonde matrix (x_i^j). Its determinant does not vanish if the roots x_i are different, as required.

Theorem: For any $n \geq 0$ we have $g_n = h_n$.

This is easily proved, since we have $Q(x_i) = 0$ for $i = 1, ..., k$.

The opposite is also true:

Theorem: Given k different complex numbers $x_1, ..., x_k$ and k real numbers $b_1, ..., b_k$. Calculate the first entries $h_0, ..., h_{k-1}$ of some sequence (h_n) by the right-hand side of (5) and compile a polynomial (4) from it's roots $x_1, ..., x_k$

$$Q(x) = -(x - x_1) \cdots (x - x_k) = -x^k + (-1)^{k+1}(x_1 + ... + x_k)x^{k-1} + ...$$

Then, the sequence (3), given in a recurrent way is exactly the sequence (5), given explicitely.

Thus, we have a one-to-one correspondence between the linear recurrence sequence (3) and the sum of powers (5).

This can be applied to Perrin's sequence.

2.3.3 Perrin's sequence, given explicitely

Starting with the sequence

$$\begin{aligned} P_0 &= 3 \\ P_1 &= 0 \\ P_2 &= 2 \\ P_n &= P_{n-2} + P_{n-3},\ n \geq 3 \end{aligned}$$

at first, we compile the polynomial from the coefficients

$$Q(x) = -x^3 + x + 1$$

Its roots are

$$\begin{aligned} a &= 1.32472... \\ b &= -0.662359... + 0.56228...i \\ c &= -0.662359... - 0.56228...i \end{aligned}$$

Set $h_n = a^n + b^n + c^n$ (since $a + b + c = 0$). The first entries are

$$\begin{aligned} h_0 &= a^0 + b^0 + c^0 = 3 \\ h_1 &= a^1 + b^1 + c^1 = 0 \\ h_2 &= a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 0 - 2(-1) = 2 \end{aligned}$$

Thus, the sequences P_n and h_n coincide.

The theorem

$$p \in \mathbb{P} \implies p | P_p = a^p + b^p + c^p$$

does not follow from this, since a, b, c are not integers.

We have to answer two questions:

- When is $f_n = (a+b+c)^n - a^n - b^n - c^n$ an integer sequence?
- When does $(p \in \mathbb{P} \implies p | f_p)$ hold?

2.3.4 When is $f_n = (a+b+c)^n - a^n - b^n - c^n$ an integer?

For any n, the expression $(a+b+c)^n - a^n - b^n - c^n$ is a symmetric polynomial in a, b and c.

Theorem: Any symmetric polynomial can be expressed in terms of elementary symmetric polynomials.

Here, these are

$$A_1 = a + b + c,\ A_2 = ab + bc + ca,\ A_3 = abc$$

which are the coefficients of a polynomial with roots a, b, c.

Calculating, for example, the first entries, we get

$$\begin{aligned}
(a+b+c)^0 - a^0 - b^0 - c^0 &= -2 \\
(a+b+c)^1 - a^1 - b^1 - c^1 &= 0 \\
(a+b+c)^2 - a^2 - b^2 - c^2 &= 2A_2 \\
(a+b+c)^3 - a^3 - b^3 - c^3 &= 3A_1A_2 - 3A_3 \\
(a+b+c)^4 - a^4 - b^4 - c^4 &= 4A_1^2A_2 - 4A_1A_3 - 2A_2^2
\end{aligned}$$

Hence, f_n is integer if a, b, c are roots of a polynomial with integer coefficients.

2.3.5 When does $(p \in \mathbb{P} \implies p|f_p)$ hold?

We have

$$f_n = (a+b+c)^n - (a^n + b^n + c^n) = \sum_{0<i+j+k=n} \frac{n!}{i!\ j!\ k!}\ a^i b^j c^k$$

and $p \in \mathbb{P} \implies p|\frac{p!}{i!\ j!\ k!},\ \forall i, j, k$ with $0 < i + j + k = n$.

$\frac{n!}{i!\ j!\ k!}$ does not change by a permutation of i, j, k. It can be lifted out.

$$\sum_{0<i+j+k=n} \frac{n!}{i!\ j!\ k!}\ a^i b^j c^k = \sum_{0<i\le j\le k} \frac{n!}{i!\ j!\ k!} \sum_{\pi(i,j,k)} a^i b^j c^k$$

$\sum_{\pi(i,j,k)} a^i b^j c^k$ is again a symmetric polynomial and so it is an integer if a, b, c are roots of an polynomial with integer coefficients.

Hence, if a, b, c are roots of a polynomial with integer coefficients, and $f_n = (a+b+c)^n - (a^n + b^n + c^n)$, then $p \in \mathbb{P} \implies p|f_p$.

2.3.6 The recurrent calculation of the sequence

From the polynomial $Q(x)$ it is easy to compile the recurrent relation

$$g_n = a_1\ f_{n-1} + a_2\ f_{n-2} + a_3\ f_{k-3} + \ldots + a_k\ f_{n-k}$$

corresponding to the explicit expression

$$g_n = x_1^n + \ldots + x_k^n\ .$$

From this explicit expression we have to calculate the initial values $g_0, ..., g_{k-1}$. Then, we have

$$f_n = g_n - a_1^n \, .$$

Actually, this is practicable if $a_1 = 0$ (like in the Perrin case) or $a_1 = \pm 1$. In other cases, a_1^n increases rapidly and it is better to look on

$$f_n = (x_1^n + \ldots + x_k^n) - (x_1 + \ldots + x_k)^n$$

as on a sum of $k+1$ powers. This corresponds to a sequence of order $k+1$, having a corresponding polynomial with the $k+1$ roots $x_1, ..., x_k, a_1 = x_1 + \ldots + x_k$. This polynomial is

$$\begin{aligned} G(x) &= -(x - x_1) \cdots (x - x_1)(x - x_1 - \ldots - x_k) = Q(x)(x - a_1) = \\ &= -x^{k+1} + 2a_1 x^k + \sum_{i=1}^{k-1} (a_{i+1} - a_1 a_i) x^{k-i} - a_1 a_k \end{aligned}$$

2.3.7 The main theorem

Connecting the last facts together, we finally obtain the

Main Theorem: Given a polynomial of degree k

$$Q(x) = -x^k + a_1 \, x^{k-1} + a_2 \, x^{k-2} + a_3 \, x^{k-3} + \ldots + a_{k-1} \, x + a_k$$

with integer coefficients $a_i \in \mathbb{Z}$ and (maybe complex) roots $x_1, ..., x_k$. Then, the sequence

$$f_n = (x_1^n + \ldots + x_k^n) - (x_1 + \ldots + x_k)^n$$

is an integer sequence and it holds $\quad p \in \mathbb{P} \implies p | f_p$.

The sequence f_n can be calculated in a recurrent way from an order k-recurrent relation

$$g_n = a_1 \, f_{n-1} + a_2 \, f_{n-2} + a_3 \, f_{k-3} + \ldots + a_k \, f_{n-k}$$

by $f_n = g_n - a_1^n$ or directly from an order $(k+1)$-recurrent relation

$$f_n = 2a_1 f_{n-1} + \sum_{i=1}^{k-1} (a_{i+1} - a_1 a_i) f_{n-i-1} - a_1 a_k f_{n-k-1}$$

We can conclude that any polynomial with integer coefficients is cantidate to generate pseudoprimes.

3 Numerical algorithms

To calculate pseudoprimes, at first we have to calculate f_n by a recurrent or explicit expression and then we test whether $n|f_n$.

The recurrence relation seems to be very fast, with some additions for every number. Unfortunately, the entries f_n grow very fast. For the Perrin sequence we have $P_n \sim 1.32472...^n$ (the largest root). Thus, P_{271441} has 33150 decimal digits, $P_{99607901521441}$ – the 17-th Perrin pseudoprime has 12,164,524,642,561 decimal digits requiring $\sim$ 5 TByte to store it.

The same problem arises with the explicit expression. We have to calculate x_j^n considering a huge number of digits to get an integer in the end. But this is necessary to check the remainder of f_n when divided by n.

The only useful method is to carry out all operations modulo n. This will save us from the usage of the huge numbers f_n. We can still use the recurrence relation but for every new number we have to start at the very beginning of the sequence, since calculating f_n mod n, we cannot use the result to calculate f_{n+1} mod $n+1$.

Even doing so, this is still a problem if we want (and we want!) to deal with large numbers n having, say, 100 digits. Note, this is the number of digits of the index, not of the sequence member!

Thus, if $n = 10^{100}$ we need a fast algorithm for 10^{100} additions of numbers like 10^{100} (all done modulo n).

Clearly, this has to be an algorithm with logarithmic complexity. This can be done in pursuing following steps:

- We can calculate k entries of the sequence at once, using matrix powers.
- The n-th power of a matrix can be performed in $\log_b n$ operations using the decomposition of n with respect to a fixed basis and Horner's method.
- In some special cases – and the Perrin sequence is such a case – the calculation can be further simplified.

3.1 Matrix powers instead of additions

Given a recurrence sequence of order k

$$f_n = c_{k-1}f_{n-1} + c_{k-2}f_{n-2} + ... + c_0 f_{n-k} \tag{6}$$

with initial values

$$F_0 := (f_0, ..., f_{k-1}) \, . \tag{7}$$

The k-th entry

$$f_k \;\; = \;\; c_{k-1}f_{k-1} + c_{k-2}f_{k-2} + ... + c_0 f_0$$

is a linear combination of the initial values and so are all entries, for example the $k+1$-th entry

$$\begin{aligned} f_{k+1} &= c_{k-1}f_k + c_{k-2}f_{k-1} + ... + c_0 f_1 = \\ &= c_{k-1}(c_{k-1}f_{k-1} + c_{k-2}f_{k-2} + ... + c_0 f_0) + c_{k-2}f_{k-1} + c_{k-3}f_{k-2} + ... + c_0 f_1 = \\ &= (c_{k-1}^2 + c_{k-2})f_{k-1} + (c_{k-1}c_{k-2} + c_{k-3})f_{k-2} + ... + (c_{k-1}c_1 + c_0)f_1 + c_{k-1}c_0 f_0 \end{aligned}$$

Writing all the entries $F_1 := (f_k, ..., f_{2k-1})$ as linear combinations of $F_0 = (f_0, ..., f_{k-1})$, we can compile a matrix $\mathbf{A}$ and write $F_1 = \mathbf{A}F_0$, i.e.,

$$\begin{pmatrix} f_k \\ f_{k+1} \\ \vdots \\ f_{2k-1} \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_{k-1}c_0 & c_{k-1}c_1 + c_0 & \cdots & c_{k-1}^2 + c_{k-2} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{k-1} \end{pmatrix}$$

This is an equivalent description of (6), (7).

In the special case $k = 3$ we have

$$\begin{pmatrix} f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 \\ c_0c_2 & c_0 + c_1c_2 & c_2^2 + c_1 \\ c_0c_2^2 + c_0c_1 & c_1^2 + c_2^2c_1 + c_0c_2 & c_2^3 + 2c_1c_2 + c_0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$$

It follows $F_m = \mathbf{A}^m F_0$ for $F_m = (f_{mk}, f_{mk+1}, ..., f_{(m+1)k-1})$. Thus, if we want to know f_n, we have to divide n by k with remainder, i.e., to write $n = mk + i$ with $i = 0, ..., k-1$ and calculate $\mathbf{A}^m$. Instead of additions we have to calculate the power of a matrix. This can be done very effectively.

3.2 Horner's method instead of matrix powers

We have to calculate $\mathbf{A}^m$ for a given matrix $\mathbf{A}$. Let $m = a_0b^j + ... + a_{j-1}b + a_j$ be the decomposition of m to base b with $a_0 > 0$ and $b > a_i \geq 0$. Then, calculating the polynomial $a_0b^j + ... + a_{j-1}b + a_j$ with Horner's method, iteratively

$$a_0b^j + ... + a_{j-1}b + a_j = ((((a_0b)b + a_1)b + a_2)b + ... + a_j)$$

we conclude

$$\mathbf{A}^m = \mathbf{A}^{a_0b^j + ... + a_{j-1}b + a_j} = (((((\mathbf{I}^b\mathbf{A}^{a_0})^b\mathbf{A}^{a_1})^b\mathbf{A}^{a_2})^b \cdots)\mathbf{A}^{a_j})$$

The vector

$$(\mathbf{A}_0, \mathbf{A}_1, ..., \mathbf{A}_{b-1}) = (\mathbf{A}^0, \mathbf{A}^1, \mathbf{A}^2, ..., \mathbf{A}^{b-1})$$

can be calculated and stored in advance. The calculation runs especially effectively if b itself is a power of 2. For practicle purposes we used $b = 2, 4, 8$.

3.3 A fast algorithm for the Perrin sequence

The following algorithm was written in 1982 by Frank Bauernöppel and Uwe Kaufmann [2] in Berlin.

1st step: Given n. Set $n = 3m + i$, $i \in \{0, 1, 2\}$. Since we have

$$\begin{aligned} P_3 &= P_1 + P_0 \\ P_4 &= P_2 + P_1 \\ P_5 &= P_3 + P_2 = P_1 + P_0 + P_2 \end{aligned}$$

we can introduce a matrix

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and have

$$\begin{pmatrix} P_{3m} \\ P_{3m+1} \\ P_{3m+2} \end{pmatrix} = \mathbf{S}^m \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^m \cdot \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

2nd step: The power of $\mathbf{S}$ can be further simplified by using the square $\mathbf{S}^2$. Depending on whether m is even or odd, one have

$$\mathbf{S}^m = (\mathbf{S}^{\frac{m}{2}})^2, \quad 2|m \quad \text{or} \quad \mathbf{S}^m = (\mathbf{S}^{\frac{m-1}{2}})^2 \cdot \mathbf{S}, \quad 2 \not| m$$

The total power $\mathbf{S}^m$ can now be calculated iteratively by using the binary representation of m. Let $m = (\alpha_0, \alpha_1, \alpha_2, ..., \alpha_k, ...), \alpha_0 = 1$ be the dual number representation of m. We calculate iteratively matrices $\mathbf{S}_k$ in the following way:

$$\begin{aligned} \mathbf{S}_0 &= \mathbf{I} \\ \mathbf{S}_{k+1} &= \begin{cases} \mathbf{S}_k^2 & \text{if} \quad \alpha_k = 0 \\ \mathbf{S}_k^2 \cdot \mathbf{S} & \text{if} \quad \alpha_k = 1 \end{cases} \end{aligned}$$

Then, $\mathbf{S}^m = \mathbf{S}_{k_0}$ for some $k_0 < m$.

For example, we have

$$\mathbf{S}^{22} = \mathbf{S}^{10110_2} = ((((\mathbf{I}^2 \cdot \mathbf{S})^2 \cdot \mathbf{S})^2 \cdot \mathbf{S})^2)^2 = \mathbf{S}^{22}$$

For every 0 (the even digits) one has to square (operation Q), for every 1 (the odd digits) one has to square and then to multiply (operation QM).

3rd step: Observe that

$$\mathbf{S}^m = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^m = \begin{pmatrix} a & c & b \\ b & a+b & c \\ c & b+c & a+b \end{pmatrix}$$

Thus, one only has to remember the first column (a, b, c) and to know how this column changes when multiplying M and squaring Q.

Operation multiplying M:

$$\begin{pmatrix} a & c & b \\ b & a+b & c \\ c & b+c & a+b \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b & a+b+c & b+c \\ b+c & a+2b+c & a+b+c \\ a+b+c & a+2b+2c & a+2b+c \end{pmatrix}$$

Thus, $M : (a, b, c) \to (a+b, b+c, a+b+c)$.

Operation squaring Q:

$$\begin{pmatrix} a & c & b \\ b & a+b & c \\ c & b+c & a+b \end{pmatrix}^2 = \begin{pmatrix} a^2+2bc & b^2+2ac+2bc & 2ab+b^2+c^2 \\ 2ab+b^2+c^2 & a^2+2ab+b^2+2bc+c^2 & b^2+2ac+2bc \\ b^2+2ac+2bc & 2ab+2b^2+2ac+2bc+c^2 & a^2+2ab+b^2+2bc+c^2 \end{pmatrix}$$

Thus, $Q : (a, b, c) \to (a^2 + 2bc, b^2 + c^2 + 2ab, b^2 + 2ac + 2bc)$.

Furthermore, some numbers n can be excluded from the beginning, because we have

$$n \equiv 0 \bmod 4 \implies f_n \not\equiv 0 \bmod 4 \implies f_n \not\equiv 0 \bmod n \; .$$

The same happens for $n = 9, 14, ...$ Moreover, we have

$$n \equiv 0 \bmod 3 \; , \;\; n \not\equiv 0, 1, 3, 9 \bmod 13 \implies f_n \not\equiv 0 \bmod 3 \implies f_n \not\equiv 0 \bmod n$$

3.4 All steps combined

1 Decompose $n = 3m + i$, $i \in \{0, 1, 2\}$

2 Compute the dual representation D of m.

3 In D, replace every zero with Q and every 1 with QM and get the word W.

4 Set $(a, b, c) := (1, 0, 0)$ and, following the word W from left to right, perform the following operations modulo n:

$$\begin{aligned} M &: (a, b, c) := (a + b, b + c, a + b + c) \\ Q &: (a, b, c) := (a^2 + 2bc, b^2 + c^2 + 2ab, b^2 + 2ac + 2bc) \; . \end{aligned}$$

5 Finally, calculate

$$P_n \bmod n = \begin{cases} 3a + 2b & \text{for} \quad i = 0 \\ 3b + 2c & \text{for} \quad i = 1 \\ 2a + 2b + 3c & \text{for} \quad i = 2 \; . \end{cases}$$

For Example we test whether 19 divides P_{19}?

1 $19 = 3 \cdot 6 + 1$, $m = 6$, $i = 1$

2 Dual representation of 6: $D = 110$.

3 $W = QMQMQ$

4
$$\begin{aligned} & (a, b, c) = (1, 0, 0) \\ \overset{Q}{\implies} \; & (a, b, c) = (1, 0, 0) \\ \overset{M}{\implies} \; & (a, b, c) = (1, 0, 1) \\ \overset{Q}{\implies} \; & (a, b, c) = (1, 1, 2) \\ \overset{M}{\implies} \; & (a, b, c) = (2, 3, 4) \\ \overset{Q}{\implies} \; & (a, b, c) = (9, 18, 11) \end{aligned}$$

5 $(9, 18, 11) \overset{i=1}{\implies} 3 \cdot 18 + 2 \cdot 11 = 76 \equiv 0 \bmod 19$

Thus, we have $19 | P_{19}$ and therefore 19 can be a Perrin pseudoprime or a prime.

3.5 A *mathematica*-code for the algorithm

To deal with large integers we used *mathematica*. Of course, as an interpretive language it is slower than a compiled code. But that saved us the development of an own long integer operation package.

The following *mathematica*-code was used to check a given number n on whether $n|P_n$. The code outputs `True` if n is prime or a Perrin pseudoprime and `False` otherwise. We used *mathematica11.3* at a `Intel(R) Core(TM) i5-6500 CPU @ 3.20GHz`. To check the largest known 1436-digit PPP (see page 20) takes 0.18 seconds. Checking the largest Mersenne prime known in 1982 $2^{86243}-1$ takes 4 minutes. Though, at that time the computers were slower. Today, testing $2^{1398269}-1$, the 35-th Mersenne prime, found in 1996, takes a day.

```
PPP[n_] := (i = Mod[n, 3];
  k = Quotient[n, 3];
  lk = IntegerDigits[k, 2];
  b1 = 1; b2 = 0; b3 = 0;
  Do[ If[ lk[[j]] == 0,
          c1 = b1 * b1 + 2 * b2 * b3;
          c2 = b2 * b2 + b3 * b3 + 2 * b1 * b2;
          c3 = b2 * b2 + 2 * b1 * b3 + 2 * b2 * b3 ,
          a1 = b1 * b1 + 2 * b2 * b3;
          a2 = b2 * b2 + b3 * b3 + 2 * b1 * b2;
          a3 = b2 * b2 + 2 * b1 * b3 + 2 * b2 * b3;
          c1 = a1 + a2;
          c2 = a2 + a3;
          c3 = a1 + a2 + a3];
     b1 = Mod[c1, n]; b2 = Mod[c2, n]; b3 = Mod[c3, n],
   {j, 1, Length[lk]}];
   Which[i == 0, b = 3 * b1 + 2 * b2,
         i == 1, b = 3 * b2 + 2 * b3,
         i == 2, b = 2 * b1 + 2 * b2 + 3 * b3];
   Mod[b, n] == 0)
```

The Table on [6] can be tested with

```
ppp = << PPP-new-math;
Do[ If[ Not[ PPP[ ppp[[k1]] ] ] || PrimeQ[ ppp[[k1]] ],
      Print[ ppp[[k1]]," is not a PPP!" ] ], {k1, 1, Length[ppp]}]
```

Do not forget the semicolon, the list `ppp` is very large. It runs less than two hours.

4 How to reduce the number of candidates

It takes many weeks to calculate the 1700 PPP up to 10^{14} even with high performance algorithms and computers. One has to check every number (except a few ones like mentioned at page 14 that can be sorted out in advance). Thus, there is no hope, that one could calculate all PPPs, say, up to 10^{20} in the next years. Moreover, since they are very rare, if you take a random n, you will "never" get a PPP.

So, to calculate more PPPs, one must try to limit the set of potential candidates.

Dana Jacobsen tested other pseudoprimes, hoping that, for example many of the Fermat$_2$-PP are also PPPs. And indeed, she found 101994 PPPs up to $18446724258335155361 < 10^{20}$ [5].

It turns out that 510 of the 1700 PPPs less than 10^{14} are Fermat$_2$-PP, too.

4.1 The structure of most of the PPPs

Let's have a look at the first PPPs and factorize them:

$$
\begin{array}{rcll}
271441 & = & 521 \cdot 521 & = [1(521-1)+1] \cdot 521 \\
904631 & = & 7 \cdot 13 \cdot 9941 & \\
16532714 & = & 2 \cdot 11 \cdot 11 \cdot 53 \cdot 1289 & \\
24658561 & = & 19 \cdot 271 \cdot 4789 & \\
27422714 & = & 2 \cdot 11 \cdot 11 \cdot 47 \cdot 2411 & \\
27664033 & = & 3037 \cdot 9109 & = [3(3037-1)+1] \cdot 3037 \\
46672291 & = & 4831 \cdot 9661 & = [2(4831-1)+1] \cdot 4831 \\
102690901 & = & 5851 \cdot 17551 & = [3(5851-1)+1] \cdot 5851 \\
130944133 & = & 6607 \cdot 19819 & = [3(6607-1)+1] \cdot 6607 \\
196075949 & = & 5717 \cdot 34297 & = [6(5717-1)+1] \cdot 5717 \\
214038533 & = & 8447 \cdot 25339 & = [3(8447-1)+1] \cdot 8447 \\
517697641 & = & 6311 \cdot 82031 & = [13(6311-1)+1] \cdot 6311 \\
545670533 & = & 13487 \cdot 40459 & = [3(13487-1)+1] \cdot 13487 \\
801123451 & = & 8951 \cdot 89501 & = [10(8951-1)+1] \cdot 8951 \\
855073301 & = & 16883 \cdot 50647 & = [3(16883-1)+1] \cdot 16883 \\
903136901 & = & 17351 \cdot 52051 & = [3(17351-1)+1] \cdot 17351 \\
970355431 & = & 22027 \cdot 44053 & = [2(22027-1)+1] \cdot 22027
\end{array}
$$

We see that many of them have the structure $P = [k(p-1)+1] \cdot p$, with some $p \in \mathbb{P}$ and $k = 1, 2, 3, ...$ is a small number. Clearly, such numbers are never prime. Moreover, to calculate numbers P in the region of 10^{16}, it is sufficient to consider factors $\sim 10^8$. Thus, taking into account that we have 5761455 primes up to 10^8, we get all pseudoprimes of this structure up to $\sim 10^{16}$ for a given k in half an hour.

This was the starting point of a couple of ideas to reduce the amount of candidates to be tested. We list them here in their logical order.

1 Consider numbers $P = [k(p-1)+1]p, \quad p \in \mathbb{P}$
It was amazing that already $k = 3$ and $k = 2$ gives more than 50% of the 1700 known PPPs up to 10^{14}.

2 Next, we considered numbers like $P = [k_1(p-1)+1][k_2(p-1)+1], p \in \mathbb{P}$; gcd$(k_1, k_2) = 1$.

3 We saw that some PPPs of this structure were overlooked, because p must not be prime. Thus, we considered numbers like $P = [k_1(p-1)+1][k_2(p-1)+1], p \notin \mathbb{P}$, p odd.

4 Clearly, the next step were numbers of the form $P = [k_1(p-1)+1][k_2(p-1)+1][k_3(p-1)+1]$

5 and generally $P = \prod_{i=1}^{m} [k_i(p-1)+1]$. For $m > 3$ we get only a few new PPP's.

With this method, we calculated all PPP's with 2 factors for given $k_i < 100$, with 3 factors for given $k_i < 15$, and with 4 factors for $k_i < 10$ up to 10^{20}. More than 95% of the 1700 known PPPs up to 10^{14} have such a structure. Extrapolating this result, we assume that we know now 95% of the PPPs up to 10^{20}.

It was not possible to find such a PPP with 5 factors for months.

The largest PPPs have about 40 digits.

To calculate larger PPPs we used two different methods:

- Starting from a PPP with m factors, guess a PPP with $m+1$ factors with the same p and some k_{m+1} resulting form the other $k_1, ..., k_m$. For example, take k_{m+1} as a multiple of the least common multiple of the $k_1, ..., k_m$. In this way we could find some very large PPPs.
- Do we have to test all odd p? It turns out that only a few remainders of p with respect to 23 occur. In this way we could find millions of new PPPs up to 10^{24}.

4.2 The remainders of p

Since 23 is the discriminant of the corresponding polynomial of the Perrin sequence, we look at the remainders of p with respect to 23 in more detail. It turns out that for a given pair (k_1, k_2) we have only a few remainders instead of 23 possible ones.

For example:

- Take $(k_1, k_2) = (3, 1)$, we have the remainders $= (1, 2, 6, 9, 18)$
- Take $(k_1, k_2) = (2, 1)$, we have the remainders $= (1, 2, 13, 16, 18)$

The same holds for multiples of 23. Taking, for example, the number $23 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 690690$. We have

- For $(k_1, k_2) = (3, 1)$ only 14853 remainders (a proportion of 0.0215046),
- For $(k_1, k_2) = (2, 1)$ only 7425 remainders (a proportion of 0.0107501).

During our calculation we considered the remainders with respect to $23 \cdot 2 \cdot 3 = 138$.

Here is a collection of the remainders with respect to 138 for all pairs (k_1, k_2) with $k_1 = 5$ and $k_1 = 7$:

k_1	k_2	possible remainders with respect to 138
5	1	1, 25, 31, 55, 73, 121
5	2	1, 7, 15, 21, 25, 43, 61, 67, 93, 99, 117, 135
5	3	1, 9, 25, 43, 55, 63, 75, 93, 109, 117, 121, 135
5	4	1, 7, 31, 43, 67, 73
7	1	1, 13, 25, 29, 31, 35, 47, 59, 71, 77, 121, 127
7	2	1, 13, 25, 67, 97
7	3	1, 5, 11, 19, 25, 29, 47, 65, 71, 97, 103, 121
7	4	1, 11, 13, 19, 31, 47, 59, 65, 67, 77, 103, 113
7	5	1, 25, 31, 67, 121
7	6	1, 5, 13, 29, 47, 59, 67, 79, 97, 113, 121, 125

These remainders were found experimentally. For a given pair (k_1, k_2) we calculated some PPPs for any odd p, enough to be sure about the possible remainders. Having obtained these, we test the following p only with these remainders. That resulted in a strong speed-up.

Unfortunately, we have no idea how the remainders can be calculated in advance. We think this is an interesting problem for specialists, for example, in Carmichael numbers.

For PPPs with 3 factors we observed the following interesting experimental result:

Fix a pair (k_1, k_2) with $gcd(k_1, k_2) = 1$ and let be $R(k_1, k_2)$ the set of remainders of p. Then, the set of remainders $R(k_1, k_2, k_3)$ for a PPP with 3 factors is

$$R(k_1, k_2, k_3) = R(k_1, k_2) \cap R(k_1, k_3) \cap R(k_2, k_3)$$

Thus, the number of possible remainders decreases with the number of factors.

A similar result holds for PPPs with more than 3 factors. Again, we do not know how to prove this.

The remainder 1 with respect to multilpes of 23 contains in any set of remainders for any (k_i).

5 Numerical results

5.1 The state of the art

A current overview can be found in N.J.A. Sloanes famous OEIS (On-Line Encyclopedia of Integer Sequences) [8].

By now, all PPPs – 1700 – up to 10^{14} are known. Since we have 3204941750802 primes up to 10^{14}, using the Perrin prime test, a PPP occurs with probability $W(10^{14}) = 5.3043110^{-10}$. Thus, to check whether a given number less than 10^{14} is prime you can use the Perrin test and – if it is true – look at the table whether it is one of the 1700 PPPs. If not, it is prime.

The following table shows the probability $W(n)$ up to $n = 10^{14}$. We used [10] for the numbers of primes.

n	PPPs	primes	probability $W(n)$
10^8	7	5761455	$1.21497 * 10^{-6}$
10^9	17	50847534	$3.34333 * 10^{-7}$
10^{10}	42	455052511	$9.22970 * 10^{-8}$
10^{11}	116	4118054813	$2.84115 * 10^{-8}$
10^{12}	285	37607912018	$7.57819 * 10^{-9}$
10^{13}	649	346065536839	$1.87537 * 10^{-9}$
10^{14}	1700	3204941750802	$5.30431 * 10^{-10}$

5.2 Our results

We calculated **9261931** (by December 2019) PPPs that an be found in the database [6]. (Note, that the database is updated from time to time.)

We tried to find all PPPs up to 10^{20} and all with 2 factors and $(k_1, k_2) = (3, 1)$ and $(k_1, k_2) = (2, 1)$ up to 10^{22}. Of course there is a by-catch of many PPPs up to 10^{30}.

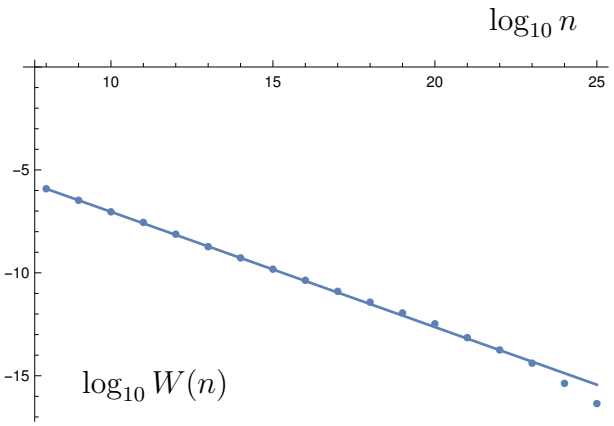
Moreover, we tried to find some very large ones using two methods:

At first, we constructed PPPs with $m + 1$ factors starting from a known ones with m factors.

Second, knowing that 1 is always a remainder with respect to multilpes of 23 for all p, we tested numbers of the form $n = p \cdot (k(p-1)+1)$. with $k = 2, 3$ and $p = 23 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots$ a multiple of 23 and the first primes This yields very large PPPs, for example the one on page 20.

5.2.1 Almost all PPPs

Having a look at the table above, we see that $\log W(n)$ behaves largely linearly. We extrapolate this and expect the following numbers of PPPs. The numbers up to 10^{20} are “almost all”, the numbers up to 10^{22} are “more than a half” of all PPPs.



n	expected PPPs	founded PPPs
10^{15}	4360	4409
10^{16}	11236	11972
10^{17}	29076	33045
10^{18}	75520	93001
10^{19}	196790	262236
10^{20}	514287	742759
10^{21}	1347560	1502883
10^{22}	3539332	3615622
10^{23}	9316050	7870747
10^{24}	24569601	7874995
10^{25}	64915566	7879187
10^{26}	171799266	7885930
10^{27}	455365341	7898184
10^{28}	1208691635	7920907
10^{29}	3212505576	7964655
10^{30}	8548808804	8049285

5.2.2 Huge PPPs

Collected by factors: We found

- 1 PPP with 14 factors.
- 13 PPPs with 13 factors.
- 64 PPPs with 12 factors.
- 113 PPPs with 11 factors.
- 176 PPPs with 10 factors.
- 481 PPPs with 9 factors.
- 1054 PPPs with 8 factors.
- 2591 PPPs with 7 factors.
- 7159 PPPs with 6 factors.
- 29529 PPPs with 5 factors.

Collected by digits: We found

- ~ 4000 PPPs with more than 80 decimal digits
- ~ 1600 PPPs with more than 100 decimal digits
- 36 PPPs with more than 500 decimal digits
- 6 PPPs with more than 1000 decimal digits
- The largest PPP has 1436 digits. Here it is:

$$
\begin{aligned}
&18035321462611916761117722236232083472759699645582838399320780305799331320503455414169754975085139912977737134345139244343056969038696305577\\
&43111655280495570411984946502205343010005990642808295636962616636416031291872175692516792403580436486465732201802954904379230282856981332645\\
&14824395059145601575922595755264427526752271957207656858256974013107809574559661619423944056021671501285297133896894324966949550607404821622\\
&41799440688227119569115149780154658697214069731091002775855856464349491580488972104215965030656524918332642168681287295417247876730157139466\\
&37668785783853194329163164641846731375014915492542184835196275906393289081430175473725289832102715305195730186544187568043060941506473248029\\
&71911093021576608385011562360146295403806806023545859256780350345290876633597339555433611318419806484463913249628542755559336822668585168661\\
&21772808574559897763091486703729929588033545309528093103113168706622588620016719546874263674058772991386018889417578678638141400547829210654\\
&64809042507529871911055497821274988821601125622936206329427570302736335840750535481387287278074059226975897195761344742618154360940114917188\\
&02634478573094785837424472290556578217205399591056639641175602504999332764161589563688660172193866529279002464507278753239327452377973281285\\
&82482904445605590952965642475889771515054226026285767270551613362219839163104142004738329396107955928417817443020309215814698240292630527201\\
&59887985344774794165426943949840690 1
\end{aligned}
$$

5.2.3 Some more information

- Our method found 1647 out of the known 1700 up to 10^{14}. Thus, 53 or $\sim 3\%$ left. We call them "sporadic PPPs".
- Dana Jacobsen's list of 101994 PPPs contains 699 that we could not find with our method.
- We found 742759 PPPs up to 10^{20}. If these compile 97% of all PPPs, then 22972 sporadic ones are left.
- Among the the first 10000 Carmichael numbers (taken from [9]) there are 16 PPPs:

$$\begin{aligned}
C_{1353} &= 7045248121 = 821 * 1231 * 6971 = \\
&= (2(411-1)+1) * (3(411-1)+1) * (17(411-1)+1) \\
C_{1375} &= 7279379941 = 211 * 3571 * 9661 \\
C_{2142} &= 24306384961 = 19 * 53 * 79 * 89 * 3433 \\
C_{2652} &= 43234580143 = 223 * 5107 * 37963 \\
C_{2837} &= 52437986833 = 23 * 463 * 1453 * 3389 \\
C_{2988} &= 60518537641 = 23 * 89 * 991 * 29833 \\
C_{3336} &= 80829302401 = 89 * 199 * 463 * 9857 \\
C_{3855} &= 118805562613 = 829 * 9109 * 15733 \\
C_{4125} &= 144377609419 = 1319 * 9227 * 11863 \\
C_{4322} &= 165321688501 = 101 * 271 * 691 * 8741 \\
C_{4342} &= 167385219121 = 83 * 6971 * 289297 \\
C_{5046} &= 254302215553 = 307 * 3673 * 225523 \\
C_{5731} &= 364573433665 = 5 * 7 * 23 * 37 * 997 * 12277 \\
C_{6743} &= 575687567521 = 11 * 19 * 79 * 137 * 307 * 829 \\
C_{6810} &= 588909469501 = 1871 * 16831 * 18701 = \\
&= 1871 * (9(1871-1)+1) * (10(1871-1)+1) \\
C_{7057} &= 652270080001 = 3361 * 9241 * 21001
\end{aligned}$$

Some of them, namely, C_{2142}, C_{2837}, C_{3336}, C_{4342}, C_{5731}, C_{6743} and C_{7057} we could not find with our method.

Note, that $C_{7057} = (4 * (841-1)+1) * (11 * (841-1)+1) * (25 * (841-1)+1)$ with $841 = 19^2$. We could not find it, since we restrict ourself to $k_i \leq 15$ for numbers with 3 factors.

5.2.4 Some conjectures

During the calculations, we were led to the following conjectures. We invite everyone to think about the proofs.

- Almost all PPPs have the structure $P = \prod_{i=1}^{m} [k_i(p-1)+1]$
- There are infinitely many of such type.
- The p has few remainders with respect to multiples of 23. They can be calculated theoretically in advance.

- If $\prod_{i=1}^{m}[k_i(p-1)+1]$ is a PPP, then with "high" probability $\prod_{i=1}^{m+1}[k_i(p-1)+1]$ is a PPP with $k_{m+1}=ck_m$. In such a way you can construct large PPPs.
- The set of remainders (with respect to multiples of 23) of p corresponding to given k_i with 3 (or more) factors are the intersection of the sets of remainders corresponding to fewer k_i, requiring $gcd(k_i,k_j)=1$.
- There are a particularly large number of PPPs if the k_i are prime, pairwise.
- If for some p the number with $\{k_2\cdot k_3,k_2,k_3\}$ is a PPP then so is the number with $\{k_2,k_3\}$.

6 Other promising polynomials for pseudoprimes

We tested polynomials of degree 3 and 4 with integer coefficients a_i with $|a_i|\leq 20$. Every corresponding sequences we tested for pseudoprimes up to 10^9. For polynomials of third order the Perrin sequence is indeed the rarest.

For polynomials of fourth order we find two polynomials without any pseudoprimes up to 10^9 at all. Here they are:

$$\begin{aligned} Q(x) &= -x^4+x^3-17x^2+0x+5 \\ R(x) &= -x^4+11x^3+x^2-12x+14 \end{aligned}$$

We have for $Q(x)$ the corresponding sequence

$$\begin{aligned} q_n &= q_{n-1}-17q_{n-2}+5q_{n-4} \\ q_0 &= 4 \\ q_1 &= 1 \\ q_2 &= -33 \\ q_3 &= -50 \end{aligned}$$

and the testing rule $n\in\mathbb{P}\Longrightarrow n|(q_n-1)$.

For $R(x)$ the sequence is

$$\begin{aligned} r_n &= 11r_{n-1}+r_{n-2}-12r_{n-3}+14r_{n-4} \\ r_0 &= 4 \\ r_1 &= 11 \\ r_2 &= 123 \\ r_3 &= 1328 \end{aligned}$$

and the testing rule is $n\in\mathbb{P}\Longrightarrow n|(r_n-11^n)$.

To avoid the term 11^n, it is better to consider

$$G(x)=Q(x)(x-11)=-x^5+22x^4-120x^3-23x^2+146x-154$$

instead of $R(x)$. This corresponds to the 5-th oder sequence

$$g_n = 22g_{n-1}-120g_{n-2}-23g_{n-3}+146g_{n-4}-154g_{n-5}$$

$$\begin{aligned} g_0 &= 3 \\ g_1 &= 0 \\ g_2 &= 2 \\ g_3 &= -3 \\ g_4 &= 14 \end{aligned}$$

with the testing rule $n \in \mathbb{P} \Longrightarrow n | g_n$.

References

[1] W. R. Alford, A. Granville, C. Pomerance, There are Infinitely Many Carmichael Numbers, Ann. Math. 139, 703-722, 1994.

[2] F. Bauernöppel, private communication

[3] C. F. Gauss, Article 329 of Disquisitiones Arithmeticae (1801)

[4] J. Grantham: There are infinitely many Perrin pseudoprimes. Journal of Number Theory. 130, Nr. 5, 2010, S. 1117-1128

[5] D. Jacobsen, http://ntheory.org/pseudoprimes.html

[6] H. Stephan, Perrin pseudoprimes. Data Sets, Weierstrass Institute Berlin (2019), http://doi.org/10.20347/WIAS.DATA.4

[7] https://en.wikipedia.org/wiki/Perrin_number

[8] https://oeis.org/search?q=perrin+pseudoprimes

[9] https://oeis.org/A002997/b002997.txt

[10] https://primes.utm.edu/howmany.html