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Lower and upper bounds for the number of limit cycles on a cylinder

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Abstract

We consider autonomous systems with cylindrical phase space. Lower and upper bounds for the number of limit cycles surrounding the cylinder can be obtained by means of an appropriate Dulac-Cherkas function. We present different possibilities to improve these bounds including the case that the exact number of limit cycles can be determined. These approaches are based on the use of several Dulac-Cherkas functions or on applying some factorized Dulac function.

1 Introduction

Consider the planar autonomous differential system

$$\begin{aligned} \frac{dx}{dt} &= P(x, y), \\ \frac{dy}{dt} &= Q(x, y), \end{aligned} \tag{1.1}$$

where the functions $P, Q : \mathbb{R}^2 \to \mathbb{R}$ are 2π-periodic in the first variable. Under this assumption we can identify the phase space of (1.1) with the cylinder $\mathbb{Z} := \S^1 \times \mathbb{R}$, where $\S^1$ is the unit circle.
The most difficult problem in the qualitative investigation of autonomous differential systems is the localization and the estimate of the number of limit cycles.

In the case of a cylindrical phase space we have to distinguish two kinds of limit cycles. A limit cycle of system (1.1) which does not surround $\mathbb{Z}$ is called a limit cycle of the first kind, otherwise it is called a limit cycle of the second kind. Whereas the existence of a limit cycle of the first kind of system (1.1) requires the existence of an equilibrium point, a limit cycle of the second kind can exist without the existence of any equilibrium point [1, 2, 3]. For the study of limit cycles of the first kind, the methods for planar autonomous systems can be applied (see e.g. [3]). In particular, a well-known way to get an upper bound for the number of limit cycles of the first kind in planar systems is to check whether the criteria of I. Bendixson and H. Dulac [12] can be applied.
The method of the Dulac function has been extended by L. Cherkas [4]. The type of functions he has introduced is called Dulac-Cherkas function nowerdays [9]. The existence of a Dulac-Cherkas function has the following advantages over a Dulac function: it guarantees that all limit cycles are hyperbolic (there is no multiple limit cycle), it provides some annuli containing a unique limit cycle (approximate localization of a limit cycle), it yields a simple criterion to determine the stability of limit cycles and provides lower and upper bounds for their maximum number. These functions have been applied by L. Cherkas and his coauthors also for the investigation of limit cycles of the second kind [5, 6, 7, 10, 11].
The fundamental importance of a Dulac-Cherkas function consists in the fact that its zero-level set defines curves which are crossed transversally by the trajectories of the corresponding system. We denote these curves in what follows as transversal curves. By this way, the cylindrical phase space is

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divided into doubly connected regions, where we have to distinguish between interior regions whose boundaries consist of transversal curves and which contain a unique limit cycle, and two outer regions, where only one boundary of these regions is a transversal curve and which contain at most one limit cycle. To be able to determine the exact number of limit cycles we have to investigate the existence of a limit cycle in the two outer regions. The main contribution of this paper is to show that the existence of a unique limit cycle in the outer regions can be established either by means of the existence of additional Dulac-Cherkas functions or by factorized Dulac functions. Thus, we are able to present results on the exact number of limit cycles, where we restrict ourselves in this paper on limit cycles of the second kind.

The paper is organized as follows: In Section 2 we introduce basic definitions and properties related to Dulac and Dulac-Cherkas functions and their applications to derive lower and upper bounds for the number of limit cycles of the second kind, to estimate their location and to characterize their hyperbolicity and stability. In Section 3 we present approaches to improve the derived lower and upper bounds for the number of limit cycles of the second kind by the existence of additional Dulac-Cherkas functions defined either in different regions (multi-step approach) or in the same region (two-step approach). In Section 4 we present a new way for the improvement of derived bounds for the number of limit cycles of the second kind by means of a factorized Dulac function.

2 Preliminaries

The estimate of the number of limit cycles in some given region depends also on the structure of the region itself. Hence, our first assumption reads

(A_0). Let $\mathcal{G}$ be an open bounded doubly connected region on $\mathcal{Z}$ whose boundary consists of two simple closed curves Δ_u and Δ_l surrounding $\mathcal{Z}$. We suppose that Δ_u is located above Δ_l, that is, Δ_u is the upper boundary and Δ_l is the lower boundary of $\mathcal{G}$.

We denote by $C^1_{2\pi}(\mathcal{G}, \mathbb{R})$ the space of continuously differentiable functions mapping $\mathcal{G}$ into $\mathbb{R}$ and which are 2π-periodic in the first variable. For the following we assume

(A_1). The functions P and Q belong to the space $C^1_{2\pi}(\mathcal{G}, \mathbb{R})$.

(A_2). $\mathcal{G}$ does not contain an equilibrium point of (1.1).

Assumption (A_2) implies that any closed orbit of system (1.1) completely located in $\mathcal{G}$ must surround the cylinder $\mathcal{Z}$. That means that any limit cycle of system (1.1) in $\mathcal{G}$ is a limit cycle of the second kind which we denote by Γ^{II}. Our goal is to determine or at least to estimate the number of limit cycles of the second kind of system (1.1) in $\mathcal{G}$. We denote this number by $\sharp\Gamma^{II}(\mathcal{G})$. The vector field defined by system (1.1) is denoted by X.
A known tool to estimate the number $\sharp\Gamma^{II}(\mathcal{G})$ is the Dulac function.

Definition 2.1. *A function $D \in C^1_{2\pi}(\mathcal{G}, \mathbb{R})$ is called a Dulac function of system (1.1) in $\mathcal{G}$ if $div(DX)$ does not change sign in $\mathcal{G}$.*

The following result is well-known [3].

Theorem 2.2. *Suppose the assumptions (A_0) – (A_2) are satisfied. If there is a Dulac function of system (1.1) in the region $\mathcal{G}$ then it holds $\sharp\Gamma^{II}(\mathcal{G}) \leq 1$.*

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The concept of the Dulac function has been generalized by L. Cherkas [4]. For this new class of functions we introduced in [9] the name Dulac-Cherkas function.

Definition 2.3. *Suppose the assumptions (A_0) and (A_1) are satisfied. A function $\Psi \in C^1_{2\pi}(\mathcal{G}, \mathbb{R})$ is called a Dulac-Cherkas function of system (1.1) in $\mathcal{G}$ if*

(i). The set

$$\mathcal{W} := \{(x, y) \in \mathcal{G} : \Psi(x, y) = 0\}$$

does not contain a curve which is a trajectory of system (1.1).

(ii). There is a real number $k \neq 0$ such that

$$\Phi(x, y, k) := (grad\Psi, X) + k\Psi divX \geq 0 \; (\leq 0) \quad \forall (x, y) \in \mathcal{G}, \tag{2.1}$$

where the set

$$\mathcal{V}_k := \{(x, y) \in \mathcal{G} : \Phi(x, y, k) = 0\}$$

has measure zero.

For $k = 1$ the definition of a Dulac-Cherkas function coincides with the definition of a Dulac function. If Ψ is a Dulac-Cherkas function of system (1.1) in $\mathcal{G}$, then $|\Psi|^{1/k}$ is a Dulac function of (1.1) in $\mathcal{G} \setminus \mathcal{W}$. For the following results we introduce the assumption

(A_3). There is a Dulac-Cherkas function Ψ of system (1.1) in $\mathcal{G}$ with $k < 0$ such that the set $\mathcal{W}$ consists of $l \geq 1$ simple closed curves $w_1, ..., w_l$ surrounding the cylinder $\mathcal{Z}$ (we call them ovals) and which do not meet each other as well as the boundaries Δ_u and Δ_l of $\mathcal{G}$.

Remark 2.4. *If we consider the function Φ on any oval w_i of the set $\mathcal{W}$, then we get from (2.1)*

$$\Phi(x, y, k)_{|w_i} = (grad\Psi, X)_{|w_i} = \frac{d\Psi}{dt}_{|w_i} \geq 0 \; (\leq 0), \tag{2.2}$$

where d/dt denotes the differentiation along system (1.1). The condition (i) in Definition 2.3 implies

$$\frac{d\Psi}{dt}_{|w_i} \not\equiv 0, \tag{2.3}$$

and we can conclude that any trajectory of (1.1) which meets any oval w_i will cross it for increasing or decreasing t.

Concerning the location of these ovals on the cylinder $\mathcal{Z}$ we assume that the oval w_i is located over the oval w_{i+1}. The doubly connected subregion of $\mathcal{G}$ bounded by w_i and w_{i+1} is denoted by $\mathcal{Z}_i$, $i = 1, ..., l-1$, the region bounded by Δ_u and w_1 is denoted by $\mathcal{Z}_0$, and the region bounded by w_l and Δ_l is denoted by $\mathcal{Z}_l$ which are the outer regions (see Figure 1).

The following result is also known [7].

Theorem 2.5. *Suppose that the assumptions $(A_0) - (A_3)$ are valid. Then it holds*

(i). Each region $\mathcal{Z}_i, 1 \leq i \leq l-1$, contains a unique limit cycle Γ_i^{II} of the second kind of system (1.1). Γ_i^{II} is hyperbolic, it is stable (unstable) if $\Phi(x, y, k)\Psi(x, y) > 0 \; (< 0)$ in $\mathcal{Z}_i$.

(ii). The regions $\mathcal{Z}_0$ and $\mathcal{Z}_l$ may contain a unique limit cycle of the second kind which is hyperbolic.

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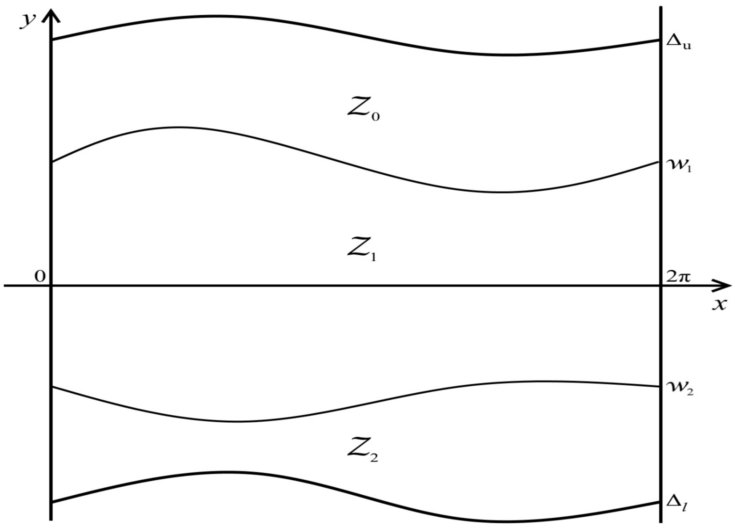

Figure 1: Location of the ovals w_i in the region $\mathcal{G}$ for $l = 2$

This theorem implies immediately

Corollary 2.6. *Suppose the assumptions $(A_0) - (A_3)$ to be valid. Then the estimate*

$$l - 1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l + 1 \tag{2.4}$$

holds.

Remark 2.7. *Under the assumptions $(A_0) - (A_3)$ any improvement of the estimate (2.4) is connected with the existence or absence of a limit cycle of the second kind in the regions $\mathcal{Z}_0$ and/or $\mathcal{Z}_l$.*

In the following sections we present different approaches to improve the estimate (2.4).

3 Improvement of the estimate (2.4) by means of additional Dulac-Cherkas functions

For what follows we assume that the assumptions $(A_0) - (A_3)$ are satisfied. We want to establish conditions for the existence of a limit cycle of the second kind in $\mathcal{Z}_0$ and/or in $\mathcal{Z}_l$.
By Remark 2.4 we can conclude that any trajectory of system (1.1) that meets an oval w_i of the set $\mathcal{W}$ will cross w_i for increasing or decreasing t. Therefore, appropriate Dulac-Cherkas functions can be used to construct doubly-connected regions to which the Poincaré-Bendixson theorem can be applied.

Theorem 3.1. *Suppose that the assumptions $(A_0) - (A_3)$ to be valid. Additionally, we assume the existence of a second Dulac-Cherkas function Ψ_0 of system (1.1) in some doubly connected subregion $\tilde{\mathcal{Z}}_0$ of $\mathcal{Z}_0$ whose boundaries surround $\mathcal{Z}$ such that the corresponding set $\mathcal{W}_0 := \{(x, y) \in \tilde{\mathcal{Z}}_0 : \Psi_0(x, y) = 0\}$ consists of exactly one oval v_0 and where the ovals v_0 and w_1 form the boundaries of the doubly connected region $\mathcal{Z}_{00}$ to which the Poincaré-Bendixson theorem can be applied. Then it holds*

$$l \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l + 1. \tag{3.1}$$

Proof. Under the assumptions $(A_0) - (A_3)$, the estimate (2.4) holds and the region $\mathcal{Z}_0$ contains at most one limit cycle of the second kind. By the Poincaré-Bendixson theorem, there exists at least one limit cycle of the second kind in $\mathcal{Z}_{00}$. Thus, the estimate (3.1) is valid. □

We consider the following simple example to illustrate Theorem 3.1

$$\begin{aligned}\frac{dx}{dt} &= f(y),\\ \frac{dy}{dt} &= 3 + 8y - 3y^2 - 2y^3 \equiv g(y)\end{aligned} \tag{3.2}$$

in the region $\mathcal{G} := \{(x, y) \in \mathcal{Z} : |y| < M\}$, where M is any positive number satisfying $M \geq 4$ and $f \in C^1([-M, M], \mathbb{R})$ is positive. Hence, the assumptions $(A_0) - (A_2)$ are fulfilled.
The polynomial g has three roots $y_2 < y_1 < y_0$, where $y_2 \approx -2.75448$, $y_1 \approx -0.341263$, $y_0 \approx 1.59574$. Thus, system (3.2) has three limit cycles Γ_i^{II} of the second kind

$$\Gamma_i^{II} := \{(x, y) \in \mathcal{Z} : y = y_i\}, \quad i = 1, 2, 3.$$

We define the function $\Psi \in C^1(\mathcal{G}, \mathbb{R})$ by

$$\Psi(x, y) := y^2 - 1$$

which satisfies for $k = -1$ according to (2.1) the relation

$$\Phi(x, y, k) \equiv k\Psi(x, y)g'(y) + \frac{\partial \Psi}{\partial y} g(y) = 2(y^4 + y^2 + 4) > 0. \tag{3.3}$$

Thus, Ψ is a Dulac-Cherkas function of system (3.2) in $\mathcal{G}$, the corresponding set $\mathcal{W}$ consists of the ovals

$$w_1 := \{(x, y) \in \mathcal{G} : y = 1\}, \quad w_2 := \{(x, y) \in \mathcal{G} : y = -1\}.$$

Hence, (A_3) is satisfied. Using the sets

$$\mathcal{Z}_0 := \{(x, y) \in \mathcal{G} : 1 < y < 4\}, \quad \mathcal{Z}_1 := \{(x, y) \in \mathcal{G} : -1 < y < 1\}, \quad \mathcal{Z}_2 := \{(x, y) \in \mathcal{G} : -4 < y < -1\}$$

and the property $\Psi(x, y) < 0$ for $(x, y) \in \mathcal{Z}_1$ we get from Theorem 2.5 and Corollary 1.6

$$1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq 3,$$

where $\mathcal{Z}_1$ contains a unique limit cycle of the second kind which is hyperbolic and unstable, $\mathcal{Z}_0$ and $\mathcal{Z}_2$ contain each at most one limit cycle of the second kind which is hyperbolic and unstable.

To prove the existence of an unique limit cycle in the region $\mathcal{Z}_0$ we construct a function Ψ_0 satisfying the conditions of the Theorem 3.1. For this purpose we put

$$\Psi_0(x, y) := y - 3.$$

Under the condition $k_0 = -1/3$ we get from (2.1) the relation

$$\Phi_0(x, y, k_0) \equiv k_0\Psi_0(x, y)g'(y) + \frac{\partial \Psi_0}{\partial y} g(y) = 11 - \frac{2}{3}y - 7y^2 \tag{3.4}$$

which has two roots $y = \bar{y}_2 \approx -1.30209$ and $y = \bar{y}_1 \approx 1.20685$. Thus, Ψ_0 is a Dulac-Cherkas function of system (3.2) in $\tilde{\mathcal{Z}}_0 \subset \mathcal{Z}_0$, where

$$\tilde{\mathcal{Z}}_0 := \{(x, y) \in \mathcal{G} : \bar{y}_1 < y < 4\},$$

and the corresponding set $\mathcal{W}_0$ consists of the unique oval

$$v_0 := \{(x, y) \in \mathcal{Z} : y = 3\}.$$

From (3.3) we obtain the relation

$$\frac{d\Psi}{dt}_{|w_1} = \Phi_{|w_1} > 0.$$

Since Ψ is increasing with y on w_1 we can conclude that the trajectories of system (3.2) intersecting the oval w_1 enter for increasing t the region

$$\mathcal{Z}_{00} := \{(x, y) \in \mathcal{Z} : 1 < y < 3\} \subset \mathcal{Z}_0,$$

see Figure 2, where $\mathcal{Z}_0$ is shaded, $\mathcal{Z}_{00}$ is grey colored.

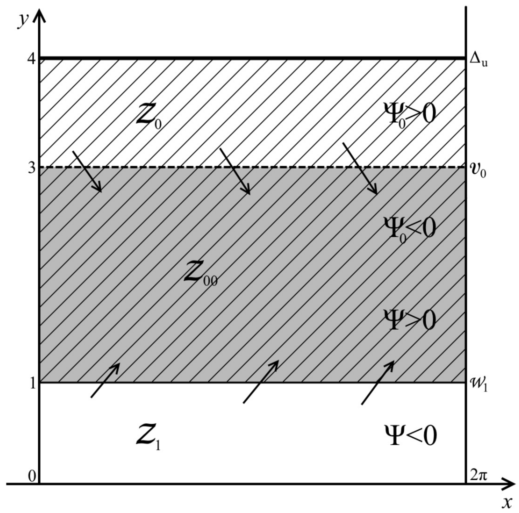

Figure 2: Location of the ovals w_1 and v_0 in the region $\mathcal{Z}_0$

From (3.4) we get the relation

$$\frac{d\Psi_0}{dt}_{|v_0} = \Phi_{0|v_0} < 0.$$

Since Ψ_0 is always increasing with y we can conclude that the trajectories of system (3.2) intersecting v_0 enter for increasing t the region $\mathcal{Z}_{00}$, see Figure 2. Therefore, the Poincaré-Bendixson theorem can be applied to the region $\mathcal{Z}_{00}$. Hence, $\mathcal{Z}_{00}$ contains at least one limit cycle of the second kind. From our conclusion above we obtain that $\mathcal{Z}_0$ contains an unique limit cycle of the second kind which is hyperbolic and stable. Thus, we have

$$2 \le \sharp\Gamma^{II}(\mathcal{G}) \le 3.$$

The following theorems can be proved in the same way.

Theorem 3.2. *Suppose that the assumptions $(A_0) - (A_3)$ are valid. Additionally, we assume the existence of a second Dulac-Cherkas function Ψ_l of system (1.1) in some doubly connected subregion $\tilde{\mathcal{Z}}_l$ of $\mathcal{Z}_l$ whose boundaries surround $\mathcal{Z}$ such that the corresponding set $\mathcal{W}_l := \{(x,y) \in \tilde{\mathcal{Z}}_l : \Psi_l(x,y) = 0\}$ consists of exactly one oval v_l and where the ovals v_l and w_l form the boundaries of the doubly connected region $\mathcal{Z}_{ll}$ to which the Poincaré-Bendixson theorem can be applied. Then the estimate (3.1) is valid.*

Theorem 3.3. *If the assumptions of the Theorem 3.1 and of Theorem 3.2 are fulfilled simultaneously, then it holds*

$$\sharp\Gamma^{II}(\mathcal{G}) = l + 1. \tag{3.5}$$

Now we illustrate Theorem 3.2 by means of system (3.2) in the same way as it has been done above for Theorem 3.1. We construct the function $\Psi_2 \in C^1(\mathcal{G}, \mathbb{R})$ by putting

$$\Psi_2(x,y) := y + 3.$$

Using $k_2 = -1/3$ we get from (2.1) the relation

$$\Phi_2(x,y,k_2) \equiv k_2 \Psi_2(x,y) g'(y) + \frac{\partial \Psi_2}{\partial y} g(y) = -5 + \frac{34}{3} y + 5y^2 \tag{3.6}$$

which has two roots $y = \underline{y}_2 \approx -2.64477$ and $y = \underline{y}_1 \approx 0.378105$. Thus, Ψ_2 is a Dulac-Cherkas function of system (3.2) in $\tilde{\mathcal{Z}}_2 \subset \mathcal{Z}_2$, where

$$\tilde{\mathcal{Z}}_2 := \{(x,y) \in \mathcal{Z} : -4 < y < \underline{y}_2\},$$

and the corresponding set $\mathcal{W}_2$ consists of the unique oval

$$v_2 := \{(x,y) \in \tilde{\mathcal{Z}}_2 : y = -3\}.$$

From (3.3) we obtain the relation

$$\frac{d\Psi}{dt}_{|w_2} = \Phi_{|w_2} > 0.$$

Since Ψ is increasing with y on w_2 we can conclude that the trajectories of system (3.2) intersecting w_2 enter for increasing t the region

$$\mathcal{Z}_{22} := \{(x,y) \in \mathcal{Z} : -3 < y < -1\} \subset \mathcal{Z}_2.$$

From (3.6) we get the relation

$$\frac{d\Psi_2}{dt}_{|v_2} = \Phi_{2|v_2} > 0.$$

Since Ψ_2 is always increasing with y we can conclude that the trajectories of system (3.2) intersecting v_2 enter for increasing t the region $\mathcal{Z}_{22}$. Therefore, the Poincaré-Bendixson theorem can be applied to the region $\mathcal{Z}_{22}$, and $\mathcal{Z}_{22}$ contains at least one limit cycle of the second kind.

From our conclusion above we obtain that $\mathcal{Z}_2$ contains an unique limit cycle of the second kind which is hyperbolic and stable. Finally, according to Theorem 3.3 it holds

$$\sharp\Gamma^{II}(\mathcal{G}) = 3.$$

From our investigations it follows that the estimate (2.4) can be improved by means of the existence of two Dulac-Cherkas functions defined in different regions. The exact number can be determined by means of three Dulac-Cherkas functions defined in different regions. In the paper [8] the authors determine the exact number of limit cycles of the second kind in $\mathcal{G}$ by means of an additional Dulac-Cherkas function defined in the same region $\mathcal{G}$. Their result can be formulated as follows.

Theorem 3.4. *Suppose the assumptions $(A_0) - (A_3)$ to be valid. Additionally, we assume the existence of a second Dulac-Cherkas function Ψ_1 of system (1.1) in $\mathcal{G}$ with $k_1 < 0$ such that the corresponding set $\mathcal{W}_1$ consists of $l + 2$ ovals. Then it holds*

$$\sharp\Gamma^{II}(\mathcal{G}) = l + 1.$$

In what follows we present a theorem which contains Theorem 3.4 as a special case. We note that our proof differs essentially from the proof of Theorem 3.4 in [8].

Theorem 3.5. *Suppose that the assumptions $(A_0) - (A_3)$ are valid. Additionally, we assume the existence of a second Dulac-Cherkas function Ψ_1 of system (1.1) in $\mathcal{G}$ with $k_1 < 0$ such that the corresponding set $\mathcal{W}_1$ consists of $l + m \geq 0$ ovals, $m = \pm 1, \pm 2$. Then it holds*

(i.) in case $m = -2$ we have $\sharp\Gamma^{II}(\mathcal{G}) = l - 1$, that is, in both regions $\mathcal{Z}_0$ and $\mathcal{Z}_l$ there is no limit cycle of the second kind,

(ii.) in case $m = -1$ we have $l - 1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l$, that is, in at most one of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_l$ there is a unique limit cycle of the second kind,

(iii.) in case $m = 1$ we have $l \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l + 1$, that is, in at least one of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_l$ there is a unique limit cycle of the second kind,

(iv.) in case $m = 2$ we have $\sharp\Gamma^{II}(\mathcal{G}) = l + 1$, that is, in both regions $\mathcal{Z}_0$ and $\mathcal{Z}_l$ there is a unique limit cycle of the second kind.

Proof. By Theorem 2.5, the assumptions $(A_0) - (A_3)$ imply

$$l - 1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l + 1. \tag{3.7}$$

If we replace l by $l + m$ we get from (3.7)

$$l + m - 1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq l + m + 1. \tag{3.8}$$

Setting $m = \pm 1, \pm 2$ in (3.8) and taking into account that both inequalities (3.7) and (3.8) have to be valid, the proof is complete. □

From this theorem we get the corollary.

Corollary 3.6. *Suppose that the assumptions $(A_0) - (A_3)$ are valid. Furthermore, we assume the existence of two additional Dulac-Cherkas functions Ψ_1 and Ψ_2 of system (1.1) in $\mathcal{G}$ with negative k_1 and k_2 such that the corresponding sets $\mathcal{W}_1$ and $\mathcal{W}_2$ consist of $l + 1$ and $l - 1$ ovals, respectively. Then it holds*

$$\sharp\Gamma^{II}(\mathcal{G}) = l.$$

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We illustrate Corollary 3.6 by means of system (3.2). As the first Dulac-Cherkas function we use

$$\Psi_1(x,y) := y^2 - 1,$$

where $\mathcal{W}_1$ consists of two ovals. We define Ψ_2 as the product

$$\Psi_2(x,y) := \Psi_1(x,y)(y^2-9) = (y^2-1)(y^2-9)$$

such that $\mathcal{W}_2$ consists of four ovals. The corresponding expression Φ_2 reads

$$\begin{aligned}&\Phi_2(x,y,\tilde{k}) = 216\tilde{k} - (240+216\tilde{k})y - (480+456\tilde{k})y^2 + \\ &(288+240\tilde{k})y^3 + (256+264\tilde{k})y^4 - (48+24\tilde{k})y^5 - (32+24\tilde{k})y^6.\end{aligned} \tag{3.9}$$

Putting $\tilde{k} = -1$ the function Φ_2 can be rewritten as

$$\Phi_2(x,y,\tilde{k}) = -216 - 24y - 24y^2 + 48y^3 - 8y^4 - 24y^5 - 8y^6.$$

It can be shown that this polynomial is always negative. Then from item (iv) of Theorem 3.5 it follows

$$\sharp\Gamma^{II}(\mathcal{G}) = 3.$$

All improvements of the estimate (2.4) in this section are based on applying appropriate Dulac-Cherkas functions. In the next section we present another approach based on factorized Dulac functions.

4 Improvement of the estimate (2.4) by means of factorized Dulac functions

Let χ_1 and χ_2 be functions of the space $C^1_{2\pi}(\mathcal{G},\mathbb{R})$. For the following we introduce the sets

$$\mathcal{U}_i := \{(x,y)\in\mathcal{G} : \chi_i(x,y) = 0\}, \quad i = 1,2.$$

We denote by $\mathcal{U}$ the set $\mathcal{U} := \mathcal{U}_1 \cup \mathcal{U}_2$ and define the function $D : \mathcal{G}\setminus\mathcal{U} \to \mathbb{R}^+$ by

$$D(x,y,k_1,k_2) := |\chi_1(x,y)|^{k_1}|\chi_2(x,y)|^{k_2}, \tag{4.1}$$

where k_1 and k_2 are real numbers. Thus, D belongs to the class $C^1_{2\pi}(\mathcal{G}\setminus\mathcal{U},\mathbb{R}^+)$, and system (1.1) and the system

$$\frac{dx}{dt} = D(x,y,k_1,k_2)P(x,y), \quad \frac{dy}{dt} = D(x,y,k_1,k_2)Q(x,y) \tag{4.2}$$

have in $\mathcal{G}\setminus\mathcal{U}$ the same topological structure of their trajectories. Our goal is to derive conditions such that D is a Dulac function in some region of $\mathcal{G}\setminus\mathcal{U}$. For the divergence of the vector field defined by the system (4.2) we get from (4.1) in the region $\mathcal{G}\setminus\mathcal{U}$

$$div(DX) = |\chi_1|^{k_1-1}|\chi_2|^{k_2-1}\, sgn\chi_1\, sgn\chi_2\Big(\chi_1\chi_2 divX + k_1\chi_2(grad\chi_1, X) + k_2\chi_1(grad\chi_2, X)\Big). \tag{4.3}$$

For the following we introduce the function $\Theta : \mathcal{G} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned}\Theta(x,y,k_1,k_2) :=& \chi_1\chi_2 divX + k_1\chi_2(grad\chi_1, X) + k_2\chi_1(grad\chi_2, X)\\ \equiv& \chi_1\chi_2 divX + k_1\chi_2\frac{d\chi_1(x,y)}{dt} + k_2\chi_1\frac{d\chi_2(x,y)}{dt}.\end{aligned} \tag{4.4}$$

Using this function we can rewrite (4.3) as

$$div(DX) = |\chi_1|^{k_1-1}|\chi_2|^{k_2-1}\, sgn\chi_1\, sgn\chi_2\ \Theta. \tag{4.5}$$

This relation implies that a change of the sign of $div(DX)$ in any connected subregion of $\mathcal{G} \setminus \mathcal{U}$ is determined by a change of the sign of $\Theta(x,y,,k_1,k_2)$. Thus, we have the result

Lemma 4.1. *Suppose the assumptions (A_0) and (A_1) are valid. Additionally we suppose*

(C_1). *There are functions $\chi_1,\chi_2 \in C^1_{2\pi}(\mathcal{G},\mathbb{R})$ and real numbers k_1,k_2 such that the function Θ satisfies*

$$\Theta(x,y,k_1,k_2) < 0\ (>0) \text{ for } (x,y) \in \mathcal{G}. \tag{4.6}$$

Then it holds in any connected subregion of $\mathcal{G} \setminus \mathcal{U}$

$$div(D(x,y,k_1,k_2)X(x,y)) \neq 0.$$

Now we are able to prove the following theorem.

Theorem 4.2. *Suppose the assumptions $(A_0) - (A_2)$ and (C_1) are satisfied. Let $\mathcal{G}_2 \subset \mathcal{G} \setminus \mathcal{U}$ be an open doubly connected region whose boundaries surround $\mathcal{Z}$ and where χ_1 and χ_2 do not change their sign. Then it holds*

(i). The function $D(x,y,k_1,k_2)$ defined in (4.1) is a positive Dulac function of system (1.1) in $\mathcal{G}_2$.

(ii). $\mathcal{G}_2$ contains at most one limit cycle of the second kind of system (1.1).

(iii). If $\mathcal{G}_2$ contains a limit cycle Γ^{II} of the second kind, then Γ^{II} is hyperbolic. Additionally, Γ^{II} is orbitally stable (unstable) if

$$\frac{\Theta(x,y,k_1,k_2)}{\chi_1(x,y)\chi_2(x,y)} < 0\ (>0) \quad \text{for} \quad (x,y) \in \mathcal{G}_2. \tag{4.7}$$

Proof. The assertion (i) follows from Lemma 4.1. Assertion (ii) is a consequence of assertion (i). To prove assertion (iii) we assume that Γ^{II} has the representation $x = x_p(t), y = y_p(t)$, where $x_p(t)$ and $y_p(t)$ are periodic functions with period T. It is well-known that the relation

$$\int_0^T divX(x_p(t),y_p(t))dt < 0 \quad (>0)$$

implies that Γ^{II} is hyperbolic and orbitally stable (unstable). From (4.4) we get

$$\int_0^T divX(x_p(t),y_p(t))dt = \int_0^T \frac{\Theta(x_p(t),y_p(t),k_1,k_2)}{\chi_1((x_p(t),y_p(t))\chi_2((x_p(t),y_p(t))}dt$$

$$-k_1 \int_0^T \frac{d}{dt} ln\chi_1(x_p(t), y_p(t)dt - k_2 \int_0^T \frac{d}{dt} ln\chi_2(x_p(t), y_p(t))dt$$
$$= \int_0^T \frac{\Theta(x_p(t), y_p(t), k_1, k_2)}{\chi_1((x_p(t), y_p(t))\chi_2((x_p(t), y_p(t))} dt.$$

Using the condition (4.7), we get that Γ^{II} is hyperbolic, additionally, Γ^{II} is orbitally stable (unstable). □

Assumption (C_1) has also consequences about the structure of the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ which in general describe curves in $\mathcal{G}$.

The following lemma says that the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ do not intersect.

Lemma 4.3. *Suppose the assumptions (A_0), (A_1) and (C_1) are valid. Then it holds*

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset.$$

Proof. Suppose there is a point $(x_0, y_0) \in \mathcal{G}$ such that $\chi_1(x_0, y_0) = \chi_2(x_0, y_0) = 0$. Then by (4.4) it holds $\Theta(x_0, y_0, k_1, k_2) = 0$, which contradicts assumption (C_1). □

The next lemma describes an important property of the curves $\mathcal{U}_1$ and $\mathcal{U}_2$.

Lemma 4.4. *Suppose the assumptions (A_0), (A_1) and (C_1) to be valid. Then the branches of the curves $\mathcal{U}_1$ and $\mathcal{U}_2$ are transversal curves with respect to the vector field X.*

Proof. If we consider the inequality (4.4) on the curve $\chi_1(x, y) = 0$ then we get

$$\Theta(x, y, k_1, k_2)_{|\chi_1(x,y)=0} = k_1 \Big(\chi_2(x, y) \frac{d\chi_1(x, y)}{dt}\Big)_{|\chi_1(x,y)=0} < 0 \ (> 0).$$

Taking into account Lemma 4.3 we get $\frac{d\chi_1(x,y)}{dt}_{|\chi_1(x,y)=0} \neq 0$, that is, the curve $\chi_1(x, y) = 0$ is crossed transversally by the trajectories of system (1.1). The same conclusion is valid for the curve $\chi_2(x, y) = 0$. □

Lemma 4.5. *Suppose the assumptions (A_0), (A_1) and (C_1) are valid. Then different branches of the curve $\mathcal{U}_1$ as well as different branches of the curve $\mathcal{U}_2$ do not meet.*

Proof. If we assume that there are two branches of the curve defined by $\chi_1(x, y) = 0$ which meet at the point (x_1, y_1), then according to the implicit function theorem it holds

$$\frac{\partial \chi_1}{\partial x}(x_1, y_1) = \frac{\partial \chi_1}{\partial y}(x_1, y_1) = 0.$$

Using these relations we have by (4.4)

$$\begin{aligned} &\Theta(x_1, y_1, k_1, k_2) = k_2\chi_2(x_1, y_1)\frac{d\chi_1(x_1, y_1)}{dt} = \\ &k_2\chi_2(x_1, y_1)\Big(\frac{\partial \chi_1}{\partial x}(x_1, y_1)P(x_1.y_1) + \frac{\partial \chi_1}{\partial y}(x_1, y_1)Q(x_1, y_1)\Big) = 0, \end{aligned} \tag{4.8}$$

which contradicts the inequality (4.6). An analogous conclusion holds for the curve defined by $\chi_2(x, y) = 0$. □

Lemma 4.6. *Suppose the assumptions (A_0), (A_1), (A_2) and (C_1) are valid. Then the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ do not contain simple closed curves not surrounding the cylinder $\mathcal{Z}$.*

Proof. Suppose the set $\mathcal{U}_1$ contains a simple closed curve $\mathcal{K}$ not surrounding $\mathcal{Z}$. By Lemma 4.5 and Lemma 4.4, $\mathcal{K}$ is a simple closed curve crossed transversally by the trajectories of system (1.1). Thus, the region bounded by $\mathcal{K}$ must contain an equilibrium point of system (1.1) which contradicts assumption (A_2). □

From the lemmata above we get the corollary.

Corollary 4.7. *Under the assumptions (A_0), (A_1), (A_2) and (C_1), we can distinguish two different types of the curves of the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{G}$:*

(i). simple ovals surrounding $\mathcal{Z}$ and which do not meet each other.

(ii). curves meeting the boundary of $\mathcal{G}$.

Since we are interested in estimating the number of limit cycles of the second kind in $\mathcal{G}$, we assume

(C_2). The set $\mathcal{U} := \mathcal{U}_1 \cup \mathcal{U}_2$ consists in $\mathcal{G}$ of n ovals surrounding $\mathcal{Z}$.

We denote by $v_1, ..., v_m$ the ovals of $\mathcal{U}$, where v_i is located above v_{i+1}. We denote by $\mathcal{Z}_i, 1 \le i \le n-1$, the open doubly connected region bounded by v_i and v_{i+1}, $\mathcal{Z}_0$ is the open doubly connected region bounded by Δ_u and v_1, $\mathcal{Z}_n$ is the open doubly connected region bounded by v_n and Δ_l.

Theorem 4.8. *Suppose the assumptions $(A_0), (A_1), (A_2)$ and (C_1) with $k_1 < 0, k_2 < 0$, and (C_2) are valid. Then it holds*

(i). Each region $\mathcal{Z}_i, 1 \le i \le n-1$, contains a unique limit cycle Γ_i^{II} of the second kind of system (1.1). Γ_i^{II} is hyperbolic and stable (unstable) if the inequality (4.7) is valid in $\mathcal{Z}_i$.

(ii). In each of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_n$ a unique hyperbolic limit cycle of the second kind could be located.

Proof. From the assumptions (C_2) and (A_2) we obtain that the regions $\mathcal{Z}_i, 0 \le i \le n$, are doubly connected regions in $\mathcal{G} \setminus \mathcal{U}$ whose boundaries surround $\mathcal{Z}$ and which do not contain an equilibrium point of (1.1). Hence, under our assumptions above, the assertion (ii) of Theorem 4.2 can be applied to these regions, and we get that in each region $\mathcal{Z}_i, 0 \le i \le n$, at most one limit cycle of the second kind is located.
In what follows we prove that any trajectory of system (1.1) which meets the upper or the lower boundary of the region $\mathcal{Z}_i, 1 \le i \le n-1$, either enters or leaves this region for increasing t. Hence, by using the Poincaré-Bendixson theorem, we can conclude that $\mathcal{Z}_i$ for $1 \le i \le n-1$ contains at least one limit cycle of the second kind. Using the conclusion before, we get that in the regions $\mathcal{Z}_i, 1 \le i \le n-1$, a unique limit cycle of the second kind is located.

Without loss of generality we can suppose that in assumption (C_2) the inequality

$$\Theta(x, y, k_1, k_2) > 0 \quad \text{for} \quad (x, y) \in \mathcal{G} \tag{4.9}$$

is valid.
First we assume that the upper and the lower boundary of the region $\mathcal{Z}_i$, also the ovals v_i and v_{i+1}, belong to the set $\mathcal{U}_1$. Since between the ovals v_i and v_{i+1} there is no other oval of the set $\mathcal{U}$, we can conclude that the function $\chi_2(x,y)$ takes the same sign on the ovals v_i and v_{i+1}. From (4.4) and (4.9) we obtain

$$k_1\big(\chi_2(x,y)\frac{d\chi_1(x,y)}{dt}\big)_{|v_i} > 0, \quad k_1\big(\chi_2(x,y)\frac{d\chi_1(x,y)}{dt}\big)_{|v_{i+1}} > 0. \tag{4.10}$$

Since $\chi_2(x,y)$ has on v_i and v_{i+1} the same sign, also $\frac{d\chi_1}{dt}$ has the same sign on v_i and v_{i+1}. Thus, the Poincaré-Bendixson theorem can be applied to the region $\mathcal{Z}_i$ and we obtain the existence of at least one limit cycle of the second kind in $\mathcal{Z}_i$. The same procedure can be applied to a region $\mathcal{Z}_i$ whose boundaries belong to the set $\mathcal{U}_2$.

Next we assume $v_i \in \mathcal{U}_1$ and $v_{i+1} \in \mathcal{U}_2$. From (4.4) and (4.9) we get

$$k_2\big(\chi_1(x,y)\frac{d\chi_2(x,y)}{dt}\big)_{|v_{i+1}} > 0, \quad k_2\big(\chi_1(x,y)\frac{d\chi_1(x,y)}{dt}\big)_{|v_i} > 0. \tag{4.11}$$

First we consider the case that $\chi_1(x,y)$ and $\chi_2(x,y)$ take negative values in the interior of $\mathcal{Z}_i$. Since k_1 and k_2 are negative, we get from (4.11) the relations

$$\frac{d\chi_2(x,y)}{dt}_{|v_{i+1}} > 0, \quad \frac{d\chi_1(x,y)}{dt}_{|v_i} > 0.$$

Thus, the trajectories of (1.1) leave the region $\mathcal{Z}_i$ for increasing t, and we can conclude that in $\mathcal{Z}_i$ at least one limit cycle of the second kind is located.
The case that $\chi_1(x,y)$ and $\chi_2(x,y)$ take positive values in the interior of $\mathcal{Z}_i$ can be treated analogously.
Now we consider the case $\chi_1(x,y) < 0$ and $\chi_2(x,y) > 0$ in $\mathcal{Z}_i$. From (4.11) we get

$$\frac{d\chi_2(x,y)}{dt}_{|v_{i+1}} > 0, \quad \frac{d\chi_1(x,y)}{dt}_{|v_i} < 0.$$

Thus, the trajectories of (1.1) enter the region $\mathcal{Z}_i$ for increasing t, and we can conclude that $\mathcal{Z}_i$ contains at least one limit cycle of the second kind. The case $\chi_1(x,y) > 0$ and $\chi_2(x,y) < 0$ in $\mathcal{Z}_i$ can be treated analogously.

By applying assertion (iii) of Theorem 4.2 we can establish the assertion on the hyperbolicity and stability of the unique limit cycle in the regions $\mathcal{Z}_i, 1 \le i \le n-1$. □

From Theorem 4.8 we get immediately

Corollary 4.9. *Under the assumptions of Theorem 4.8 it holds*

$$n-1 \le \sharp\Gamma^{II}(\mathcal{G}) \le n+1. \tag{4.12}$$

The proof of Theorem 4.8 is based on a decomposition of the region $\mathcal{G}$ into doubly connected subregions by means of the ovals defined by $\chi_1(x,y) = 0$ and $\chi_2(x,y) = 0$. Analogously to Theorem 3.5, the estimate (4.12) can be improved by means of another decomposition of $\mathcal{G}$ based on some functions $\tilde{\chi}_1, \tilde{\chi}_2 \in C^1_{2\pi}(\mathcal{G},\mathbb{R})$. For this reason we introduce the assumptions

$(\tilde{C}_1)$ There are functions $\tilde{\chi}_1, \tilde{\chi}_2 \in C^1_{2\pi}(\mathcal{G}, \mathbb{R})$ and real numbers $\tilde{k}_1, \tilde{k}_2$ with $\tilde{k}_1 < 0, \tilde{k}_2 < 0$ such that the function

$$\tilde{\Theta} := \tilde{\chi}_1\tilde{\chi}_2 divX + \tilde{k}_1\tilde{\chi}_2(grad\tilde{\chi}_1, X) + \tilde{k}_2\tilde{\chi}_1(grad\tilde{\chi}_2, X)$$

satisfies

$$\tilde{\Theta}(x, y, \tilde{k}_1, \tilde{k}_2) < 0 \ (> 0) \quad \forall (x, y) \in \mathcal{G}. \tag{4.13}$$

In analogy to the sets $\mathcal{U}_1, \mathcal{U}_2$ we introduce the sets

$$\tilde{\mathcal{U}}_i := \{(x, y) \in \mathcal{G} : \tilde{\chi}_i(x, y) = 0\}, \quad i = 1, 2.$$

($\tilde{C}_2$). The set $\tilde{\mathcal{U}} := \tilde{\mathcal{U}}_1 \cup \tilde{\mathcal{U}}_2$ consists in $\mathcal{G}$ of $n + m \geq 2$ ovals surrounding $\mathcal{Z}$, $m = \pm 1, \pm 2$.

Theorem 4.10. *Suppose the assumptions of Theorem 4.8 are valid. Additionally we assume the hypotheses $(\tilde{C}_1)$ and $(\tilde{C}_2)$ to be valid. Then it holds*

(i.) in case $m = -2$ we have $\sharp\Gamma^{II}(\mathcal{G}) = n - 1$, that is, in both regions $\mathcal{Z}_0$ and $\mathcal{Z}_n$ there is no limit cycle of the second kind,

(ii.) in case $m = -1$ we have $n - 1 \leq \sharp\Gamma^{II}(\mathcal{G}) \leq n$, that is, in at most one of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_n$ there is a unique limit cycle of the second kind,

(iii.) in case $m = 1$ we have $n \leq \sharp\Gamma^{II}(\mathcal{G}) \leq n + 1$, that is, in at least one of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_n$ there is a unique limit cycle of the second kind,

(iv.) in case $m = 2$ we have $\sharp\Gamma^{II}(\mathcal{G}) = n + 1$, that is, in both regions $\mathcal{Z}_0$ and $\mathcal{Z}_n$ there is a unique limit cycle of the second kind.

The proof of this theorem follows the same line as the proof of Theorem 3.5.

Now we illustrate Theorem 4.10 by means of system (3.2). First, we have to verify the assumptions of Theorem 4.10. The assumptions $(A_0) - (A_2)$ have been verified above. To satisfy the assumptions (C_1) and (C_2) we introduce

$$\chi_1(x, y) := y^2 - 1, \quad k_1 = -1, \quad \chi_2(x, y) := 1, \quad k_2 \in \mathbb{R}^-.$$

The corresponding function $\Theta(x, y)$ defined in (4.4) reads

$$\Theta(x, y) = 2(y^4 + 6)$$

and is always positive. Thus, the assumptions (C_1) is fulfilled. Since, the set $\mathcal{U}$ consists of two ovals we have $n = 2$ in the assumptions (C_2).
Now we introduce the functions

$$\tilde{\chi}_1(x, y) := y^2 - 1, \quad \tilde{\chi}_2(x, y) := y^2 - 9,$$

The corresponding expression $\tilde{\Theta}(x, y, \tilde{k}_1, \tilde{k}_2)$ reads

$$\tilde{\Theta}(x,y,\tilde{k}_1,\tilde{k}_2) = 72 + (-54 - 54\tilde{k}_1 - 6\tilde{k}_2)y + (-134 - 144\tilde{k}_1 - 16\tilde{k}_2)y^2 +$$
$$(60 + 60\tilde{k}_1 + 12\tilde{k}_2)y^3 + (68 + 52\tilde{k}_1 + 20\tilde{k}_2)y^4 + (-6 - 6\tilde{k}_1 - 6\tilde{k}_2)y^5 + (-6 - 4\tilde{k}_1 - 4\tilde{k}_2)y^6. \tag{4.14}$$

Putting $\tilde{k}_1 = -\frac{3}{2}$ and $\tilde{k}_2 = -\frac{1}{2}$ the expression (4.14) takes the form

$$\tilde{\Theta}(x,y,\tilde{k}_1,\tilde{k}_2) = 72 + 30y + 90y^2 - 36y^3 - 20y^4 + 6y^5 + 2y^6.$$

It can be shown that this polynomial is always positive. Hence, assumptions $(\tilde{C}_2)$ is valid. Since, the set $\tilde{\mathcal{U}}$ consists of four ovals we have $m = 2$ in assumptions $(\tilde{C}_2)$. Thus, according to item (iv) in Theorem 4.10 it follows

$$\sharp\Gamma^{II}(\mathcal{G}) = 3,$$

and we can conclude that each of the regions $\mathcal{Z}_0$ and $\mathcal{Z}_2$ contains an unique limit cycle of the second kind.

Summarizing our investigations we can state that the use of Dulac-Cherkas functions is a powerful tool for constructing transversal curves to get lower and upper bounds for the number of limit cycles of the second kind.

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