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Extreme at-the-money skew in a local volatility model

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Abstract

We consider a local volatility model, with volatility taking two possible values, depending on the value of the underlying with respect to a fixed threshold. When the threshold is taken at-the-money, we establish exact pricing formulas and compute short-time asymptotics of the implied volatility surface. We derive an exact formula for the at-the-money implied volatility skew, which explodes as $T^{-1/2}$, reproducing the empirical "steep short end of the smile". This behavior does not depend on the precise choice of the parameters, but simply follows from the "regime-switch" of the local volatility at-the-money.

1 Introduction

We assume that, under the pricing measure, a stock price follows the local volatility (LV) model:

$$dS_t = S_t \sigma_{loc}(S_t) dW_t, \tag{1}$$

where the volatility takes positive values σ_-, σ_+, switching accordingly to a fixed threshold $R > 0$:

$$\sigma_{loc}(x) = \sigma_- \mathbf{1}_{x<R} + \sigma_+ \mathbf{1}_{x \geq R}. \tag{2}$$

Strong existence and uniqueness of solutions to such equation follow from [30]. In this sense, this LV model is regime-switching, non-trivial only at the threshold level, where the volatility is a discontinuous function of the underlying asset. The main result of the present paper is that, when the discontinuity is taken at-the-money (ATM), i.e. $S_0 = R$, the model reproduces the empirical behavior close to maturity of the implied volatility skew, the so-called "steep short end of the smile". Indeed, in this model the implied volatility skew explodes as $T^{-1/2}$, which is remarkably close to empirical observations: the empirical skew explodes as a power law with negative exponent between -0.3 and -0.5.

To prove the blow-up of the skew, we first obtain exact explicit pricing formulas (cf. Theorem 1), which are interesting in their own right since they can be implemented directly, improving in this sense the results in [20]. Moreover, they allow us to obtain short time asymptotics of the implied volatility surface and the following ATM expansion of the implied volatility in short time:

$$\sigma_{BS}(T,k) \approx \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} + \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \frac{k}{\sqrt{T}} + \frac{\sigma_+ - \sigma_-}{2\sigma_+\sigma_-} \left(\frac{\sigma_+ - \sigma_-}{2(\sigma_+ + \sigma_-)} - \operatorname{sgn} k \right) \frac{k^2}{T}, \tag{3}$$

T being the maturity, $k = \log(K/S_0)$ the log-moneyness (K is the strike and S_0 the underlying at $t = 0$). We notice that the first order term explodes in T as $T^{-1/2}$. More precisely, in Theorem 2, we show that the volatility surface is differentiable in 0 wrt the log-moneyness, find an explicit expression for the derivative and prove that

$$\lim_{T \downarrow 0} \sqrt{T} \partial_k \sigma_{BS}(T,0) = \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}. \tag{4}$$

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Notice that the implied volatility $\sigma_{BS}(T,k)$ remains bounded between σ_+ and σ_-, nevertheless allowing the skew to blow-up. In case $\sigma_- > \sigma_+$ the implied volatility, for fixed maturity, is decreasing in the log-moneyness. This is consistent with the smiles produced by stock indices on the interval of observed strikes (in particular for large maturities). For extreme strikes, the implied volatility tends to a constant. Even if not completely accurate, flat (constant) extrapolation is a common usage in practice. The first coefficient of the development is the harmonic mean of σ_+, σ_-, which plays the role of spot volatility (which does not exist) or "effective volatility at the discontinuity". Notice that for this specific model the implied volatility is the harmonic mean of the LV in a different sense from the classic result in [7, 22]. See also next Remark 5 for a connection with the Skew Brownian motion.

In the present paper, we prove the blow-up of the skew for the specific volatility function in (2). Anyway, more generally, this result may suggest that the blow-up of the skew, in pure LV models, could be associated with a LV function non-smooth ATM. In support of this, we consider a continuous but not differentiable version of (2) which behaves as x^r ATM, with $0 < r < 1$ (cf. Remark 6). Simulations suggest that the implied volatility skew may explode as $T^{(1-r)/2}$. This "smoothed" version is hard to handle analitically, but offers more flexibility for calibrating the exponent of the power-law explosion.

The blow-up of the implied volatility skew. The power-law explosion of the skew in short time is one well documented stylized fact, to which much attention has been devoted in recent years by the mathematical finance community.

From the marginals-mimicking results by Dupire and Gyongy [14, 26] follows that a LV function depending on time and underlying $\sigma_{loc}(t,S)$ can reproduce any implied volatility smile, even though the dynamics of such models are highly unrealistic. In general, the relation between local and implied volatility is well understood [7, 22]. In the short end, from the one-half rule ($\partial_S \sigma_{loc}(t,0) \approx 2\partial_k \sigma_{BS}(t,0)$ for $t \downarrow 0$) follows that the explosion of the implied volatility skew implies that the LV skew explodes as well (if $\partial_S \sigma_{loc}(t,0)$ exists); therefore, the LV surface cannot be smooth ATM uniformly up to time 0, if the associated implied volatility displays exploding skew. On the contrary, (4) shows that a simple form of the LV as in (2) (remark that there is no dependence on t) produces the blow-up of the skew, with the correct dependence on T, as a consequence of the non-smoothness of the LV function ATM. This is also reminiscent of [23], where it is shown on simulations that a path dependent volatility (PDV) can reproduce a large forward skew. In this approach the volatility is written as a product of a "local" part and a "path dependent" part. The large skew comes from the path dependent part, when this involves a step function computed on past values of the stock. So, in PDV models and (1), extreme skews seems to have a similar origin, especially when the past stock value is taken not too far in the past.

Stochastic volatiliy (SV) models have several advantages on LV models and produce more realistic dynamics. Anyway, standard diffusive SV models with continuous paths predict a constant ATM implied volatility skew on the short end [21], in contrast with market data. Combining the two classes in the so-called stochastic-local volatility (SLV) models (introduced in [37, 44]), one is able to fit market implied volatility surfaces, including the steep short end of the smile. These fits are based on numerical methods, such as calibration via particle methods [25] or other more classical numerical methods [15, 48]. In forthcoming work [5], we combine the ideas of SLV with those of this paper, and present a numerical case study of a family of parsimonious models, which we call SSV (Stepping Stoch Vol), which offer the ability to calibrate to extreme skews and the remaining volatility surface.

A different approach to reproduce the steepness of the implied volatility is to consider different (and richer) processes as SV, for example adding jumps. In many models with jumps, the blow-up is of order $T^{-1/2}$, but there is no limit smile since the implied volatility blows up off-the-money (see [21, 24, 40] and the related work [39, 49]).

More recently, the capability of reproducing the exploding skew has been one of the reasons behind the succes of rough volatility models [1, 3, 4, 18, 19]. These path-continuous SV models are “rough” in Hölder sense, since the noise driving the volatility typically is a Fractional Brownian motion with Hurst exponent $H < 1/2$. In these models, the skew explodes as $T^{H-1/2}$, which is worth comparing with the continuous versions of (2) in Remark 6, producing (on simulations) skew with arbitrary blow-up exponent. The extreme behavior in (4) is obtained in the limit $H \downarrow 0$ (cf. [42]).

From the modeling point of view, one main feature of such models is their non-Markovianity. On one hand, temporal dependence is a reason of great interest; one the other hand, the fact that these are processes with memory rules out all pricing methods based on PDEs and heat kernels. The same memory property is costly when pricing via Monte Carlo methods, and efficient simulation algorithms are currently being investigated [3, 6, 2].

Discontinuous models for local volatility. Several aspects of models with discontinuous local volatiliy function have already been considered. Pricing formulas for model (1) are derived in [20], but they are quite involved and hard to implement directly, so approximations based on Black & Scholes formulas are instead used. The proofs in [20] are based on random walks approximations of the Skew Brownian Motion, whereas here we work directly with the pricing PDE associated with model (1). The multi-tile case (piecewise constant LV but with more than one discontinuity) is considered in [38]. A semi-analytic approach for calibration based on Green function and Laplace–Carson transform is proposed. Anyway, the actual implementation is done in the one or two-tile cases, which coincide again with (1). In [38][Formula (20)] it is interestingly remarked that for this model the standard short time approximation of the implied volatility as the harmonic mean of the LV does not work. Pricing in a similar model with mean reversion is considered in [11].

Option prices could as well be computed via numerical simulations. We point out that the presence of a discontinuity in the coefficients of the SDE slows down the convergence of discrete approximation schemes, affecting numerical pricing methods. For numerical methods concerning SDEs with discontinuous coefficients we refer to [13, 16, 17, 33, 34].

Local volatilities with thresholds or regime-switch have been considered also under the historical measure, for example with the aim of reproducing leverage effects. Model (1) is a continuous time version of SETAR (self-exciting threshold-auto-regressiv) models (see [10, 47] and the bibliography there). Estimation of similar continuous time models on empirical time series of prices can be found on [35, 41]. Results on the convergence of statistical estimators for the volatility in (1) are proved in [36]. Threshold models for interest rate have also been proposed [12, 43].

The analitical techniques used in the present paper are based on Laplace transforms. In particular, we compute the Laplace transform of the price directly, using resolvent kernels, without first computing the transform of the underlying. We refer the reader to [9, 27, 32] for the theoretical background.

Outline. In Section 2 we compute the pricing formulas for model (1): in Section 2.1 we find the Laplace transforms of option prices using resolvent kernels, in Section 2.2 we invert such transforms to obtain explicit expressions for the option prices; in Section 3 we consider the volatility surface: in Section 3.1 we study its short time asymptotics, in Section 3.2 we prove the explosion of the implied volatility skew.

2 Pricing

Let us denote with $S_0, K, R > 0$ respectively initial condition (deterministic), strike and threshold level. We assume that, under the pricing measure, prices follow the LV model (1). Let us consider the European call on S with maturity T and strike price K:

$$\mathcal{P}(T,K,S_0) = \mathbb{E}[(S_T - K)_+]. \tag{5}$$

with $(\cdot)_+$ denoting the positive part. The following theorem gives explicit pricing formulas for $\mathcal{P}$ in case $S_0 = R$ (regime-switch ATM). A similar formula also hold in case $K = R$, but case $S_0 = R$ will be the case of interest for us. Indeed, we can price under this hypothesis call options for all strikes and maturities and derive the corresponding volatility surface. The regime-switch at-the money will produce the short-end explosion of the implied volatility skew. Let us set

$$\sigma(x) = \sigma_- \mathbf{1}_{x<0} + \sigma_+ \mathbf{1}_{x\geq 0}. \tag{6}$$

We have the following formula for prices of call options.

Theorem 1 (Pricing formula)**.** *Let S as in* (1)*, with $\sigma_+ \neq \sigma_-$, $t \geq 0$. The price $\mathcal{P}$ in* (5) *is given by:*

1 if $S_0 = R$ (threshold at the initial condition)

$$\mathcal{P}(T,K,S_0) = (S_0 - K)_+ + \sqrt{S_0 K}\, \theta\big(T, \log(K/S_0), \sigma_+, \sigma_-\big) \tag{7}$$

2 if $K = R$

$$\mathcal{P}(T,K,S_0) = (S_0 - K)_+ + \sqrt{S_0 K}\, \theta\big(T, \log(S_0/K), \sigma_+, \sigma_-\big) \tag{8}$$

where

$$\begin{aligned}\theta(t,q,\sigma_+,\sigma_-) = {} & \frac{1}{\sqrt{2}} \frac{\sigma_+^2 \sigma_-^2}{(\sigma_- - \sigma_+)(\sigma_- + \sigma_+)} \\ & \times \left[\frac{1}{\sigma_+} G\left(t, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_+^2}{8}\right) - \frac{1}{\sigma_-} G\left(t, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_-^2}{8}\right)\right]\end{aligned}$$

with $\sigma(\cdot)$ in (6) *and*

$$\begin{aligned}G(t,a,b,c) = \frac{1}{\pi}\Bigg[& \int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z+b-c}} \frac{(b+z)tc + (\exp(-(b+z)t) - 1)(c-b-z)}{(b+z)^2} dz \\ & + \int_0^{b-c} \frac{\exp(-a\sqrt{b-c-z})}{\sqrt{z}} \frac{(c+z)tc - z(\exp(-(c+z)t) - 1)}{(c+z)^2} dz \Bigg].\end{aligned}$$

Notice that this formula splits the claim $(S_0 - K)_+$ and and the "variation part" $\sqrt{S_0 K}\theta(\cdot)$. The explicit form of θ is key in the computation of implied volatility (cf. also with Remark 2).

2.1 Laplace transform of the price via resolvent kernels

We start obtaining a PDE for the price. We recall the reduced-variabe framework of [8] or [7][Equation (6)] (holding for general LV models). Let $l = \log K;\ y = \log S_0;\ \rho = \log R$. We have that

$$\mathcal{P}(t,l,y) = e^l u(t, y-l)$$

where u satisfies $u(0,x) = (e^x - 1)_+$ and

$$\partial_t u(t,x) = \frac{\sigma_{loc}(x)}{2}(\partial_{xx} - \partial_x)u(t,x),$$

with σ_{loc} as in (2), with a slight abuse of notation since here the discontinuity is in $\rho - l$. In this setting it is more convenient to have the discontinuity in 0, so we apply a translation of $-\rho + l$ and obtain

$$\mathcal{P}(t,l,y) = e^l u(t, y-\rho)$$

where u satisfies the same PDE, but with discontinuity in 0, and $u(0,x) = e^{\rho-l}(e^x - e^{l-\rho})_+$. This is equivalent to the following differential problem.

Problem 1 (The option pricing PDE)**.** *The price of the call option* (5)*, with log-initial condition y, log-threshold ρ, log-strike l is given by*

$$\mathcal{P}(t,K,S_0) = \mathcal{P}(t,l,y) = e^{\rho} u(t, y-\rho) \tag{9}$$

with u solution to

$$\partial_t u(t,x) = \frac{\sigma(x)}{2}(\partial_{xx} - \partial_x)u(t,x), \tag{10}$$

where σ is given in (6)*, with initial condition*

$$u(0,x) = (e^x - e^{l-\rho})_+. \tag{11}$$

Expression (9) for the price holds for all y, ρ, l. In what follows, we find explicit formulas for the Laplace transform of u in two special cases. Let us denote with $\mathcal{L}_\lambda f = \int_0^\infty e^{-\lambda t} f(t)dt$ the Laplace transform of a function f and with $\mathcal{L}_t^{-1}$ it inverse. From now on, σ is the function given in (6).

Lemma 1 (Laplace transform of the price)**.** *The following formulas hold:*

1 *For fixed $q \in \mathbb{R}$, let $u_q(t,x)$ be solution to* (10) *with initial condition*

$$u_q(0,x) = (e^x - e^q)_+.$$

Then the Laplace transform of u in $x = 0$ is given by

$$\mathcal{L}_\lambda\big(u_q(\cdot,0)\big) = \int_0^\infty e^{-\lambda t} u_q(t,0)dt = \frac{1}{\lambda}\left((1-e^q)_+ + 2\exp\left(\frac{q}{2}\right)\frac{\exp\left(-\frac{|q|}{2}\sqrt{1+\frac{8\lambda}{\sigma(q)^2}}\right)}{\sqrt{1+\frac{8\lambda}{\sigma_+^2}} + \sqrt{1+\frac{8\lambda}{\sigma_-^2}}}\right). \tag{12}$$

2 *Let* $u(t,x)$ *be solution to* (10)*, with initial condition*

$$u(0,x) = (e^x - 1)_+.$$

Then the Laplace transform of u *is given by*

$$\mathcal{L}_\lambda\big(u(\cdot,x)\big) = \int_0^\infty e^{-\lambda t} u(t,x)dt = \frac{1}{\lambda}\left((e^x - 1)_+ + 2\exp\left(\frac{x}{2}\right)\frac{\exp\left(-\frac{|x|}{2}\sqrt{1+\frac{8\lambda}{\sigma(x)^2}}\right)}{\sqrt{1+\frac{8\lambda}{\sigma_-^2}}+\sqrt{1+\frac{8\lambda}{\sigma_-^2}}}\right) \tag{13}$$

Remark 1*.* Notice that Case 1 is applicable when $y = \rho$ and we have $q = l - \rho$. Case 2 is applicable when $l = \rho$ and we have $x = y - \rho$.

Proof. We start noticing that the differential Problem 1 can be seen from the point fo view of the resolvent kernel [32]. Let us set $Y_t = \log(S_t) - \rho$ (therefore, $Y_0 = y - \rho$). We have

$$\mathcal{P}(t,l,y) = \mathbb{E}[(e^{Y_t+\rho} - e^l)_+] = e^\rho \mathbb{E}[(e^{Y_t} - e^{l-\rho})_+],$$

where the dynamics of Y is given by

$$dY_t = \sigma(Y_t)dW_t - \frac{\sigma(Y_t)^2}{2}dt \tag{14}$$

and the generator of Y is

$$L = \frac{\sigma(x)^2}{2}(\partial_{xx} - \partial_x). \tag{15}$$

For a function $f : \mathbb{R} \to \mathbb{R}$, let us denote $\mathbb{E}_z[f(Y_t)]$ the expectation with initial condition $Y_0 = z$. Let us define the *resolvent kernel* of Y as the Laplace transform

$$r(\lambda,x,y) = \mathcal{L}_\lambda(p(\cdot,x,y)) = \int_0^\infty p(t,x,y)\exp(-\lambda t)dt \tag{16}$$

of the transition density p of Y (which exists and is explicit, see for example [20]). We are going to compute here the resolvent kernel r following the method in [32], which will allow us to compute Laplace transforms such as

$$\mathcal{L}_\lambda \mathbb{E}_z[f(Y_\cdot)] = \int_0^\infty e^{-\lambda t}\mathbb{E}_z[f(Y_t)]dt = \int_{\mathbb{R}} r(\lambda,z,y)f(y)dy := R_\lambda f(z). \tag{17}$$

Since $u(t,z) := \mathbb{E}_z[f(Y_t)]$ solves (10), this formulation is equivalent to Problem 1.

The resolvent techniques that we are going to apply are generally stated for f vanishing at infinity, which is clearly not the case here. Anyway, the law of Y_t has upper Gaussian bounds [20, 32] which imply that all the expectations in the proofs are finite. Therefore the method applies (this justification could be made formal truncating f and taking limits).

Since the Markov process Y is one-dimensional, it can be caracterized by its speed measure and scale function [9, 27, 46]. These quantities can be computed from the generator of Y in (15):

$$m(x) = \frac{2}{\sigma^2(x)}\exp(-x); \quad S(x) = \exp(x) - 1. \tag{18}$$

Much information on Y can be read also from its so-called *minimal functions*. Let us consider (15) and the following equation

$$(L-\lambda)u = 0 \tag{19}$$

For fixed $\lambda > 0$, the set of solutions to (19) is a two-dimensional vector space. There exist two continuous, positive functions $\psi_\lambda, \varphi_\lambda$ solution to (19) with ψ_λ increasing from 0 to ∞ and φ_λ decreasing from ∞ to 0, $\psi_\lambda(0) = \varphi_\lambda(0) = 1$, called the minimal functions. When σ in (15) has the specific form (6) (piecewise constant, discontinous at 0) these two functions can be computed explicitly. Imposing the fact that $\psi_\lambda, \varphi_\lambda$ are, separately on $\mathbb{R}_+, \mathbb{R}_-$, linear combinations of the minimal functions of (19) in the case of constant coefficients (which are explicit and can be found in [32][Example 2.2]), and imposing continuity of $\psi_\lambda, \varphi_\lambda, \psi'_\lambda, \varphi'_\lambda$ at 0, we get explicit expressions for $\psi_\lambda, \varphi_\lambda$. Let us set $\Delta_\pm = \sqrt{1+8\lambda/\sigma_\pm^2}$. We have

$$\psi_\lambda(x) = \begin{cases} \exp\left(\frac{x}{2}(1+\Delta_-\right) & \text{if } x<0 \\ \frac{1}{2}\left(1+\frac{\Delta_-}{\Delta_+}\right)\exp\left(\frac{x}{2}\left(1+\Delta_+\right)\right) + \frac{1}{2}\left(1-\frac{\Delta_-}{\Delta_+}\right)\exp\left(\frac{x}{2}\left(1-\Delta_+\right)\right) & \text{if } x\geq 0 \end{cases} \tag{20}$$

and

$$\varphi_\lambda(x) = \begin{cases} \frac{1}{2}\left(1-\frac{\Delta_+}{\Delta_-}\right)\exp\left(\frac{x}{2}\left(1+\Delta_-\right)\right) + \frac{1}{2}\left(1+\frac{\Delta_+}{\Delta_-}\right)\exp\left(\frac{x}{2}\left(1-\Delta_-\right)\right) & \text{if } x<0 \\ \exp\left(\frac{x}{2}(1-\Delta_+\right) & \text{if } x\geq 0 \end{cases} \tag{21}$$

The Wronskian of the diffusion is defined as

$$Wr_\lambda = \varphi_\lambda(x)\frac{\partial_x\psi_\lambda(x)}{\partial_x S(x)} - \psi_\lambda(x)\frac{\partial_x\varphi_\lambda(x)}{\partial_x S(x)},$$

where S is the scale function in (18). Direct computations give (notice $\partial_x S(x) = \exp(x)$)

$$Wr_\lambda = \frac{\Delta_+ + \Delta_-}{2}. \tag{22}$$

Notice that the Wronskian does not depend on x. The resolvent kernel in (16) can be computed as

$$r(\lambda,x,y) = \frac{m(y)}{Wr_\lambda}\begin{cases}\psi_\lambda(x)\varphi_\lambda(y) & \text{if } x<y \\ \varphi_\lambda(x)\psi_\lambda(y) & \text{if } x>y\end{cases} \tag{23}$$

where m is the speed measure (18).

We compute now $R_\lambda f(0)$ in (17) for the function $f(x) = (e^x - e^q)_+$. We have

$$\begin{aligned} R_\lambda f(0) &= \int_{\mathbb{R}} r(\lambda,0,y)(\exp(y)-\exp(q))_+ dy \\ &= \int_q^\infty r(\lambda,0,y)\exp(y)dy - \exp(q)\int_q^\infty r(\lambda,0,y)dy. \end{aligned} \tag{24}$$

For $0 = x \leq q$,

$$\int_q^\infty r(\lambda,0,y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda}\int_q^\infty \varphi_\lambda(y)\exp(-y)dy,$$
$$\int_q^\infty r(\lambda,0,y)\exp(y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda(y)}\int_q^\infty \varphi_\lambda(y)dy.$$

Explicit computations using (20), (21), (22), (23), (24) give

$$R_\lambda f(0) = \frac{2}{\lambda}\frac{\exp(\frac{q}{2}(1-\Delta_+))}{\Delta_+ + \Delta_-}. \tag{25}$$

For $0 = x > q$,

$$\int_q^\infty r(\lambda,0,y)dy = \frac{1}{Wr_\lambda}\left(\frac{2}{\sigma_-^2}\int_q^0 \psi_\lambda(y)\exp(-y)dy + \frac{2}{\sigma_+^2}\int_0^\infty \varphi_\lambda(y)\exp(-y)dy\right),$$
$$\int_q^\infty r(\lambda,0,y)\exp(y)dy = \frac{1}{Wr_\lambda}\left(\frac{2}{\sigma_-^2}\int_q^0 \psi_\lambda(y)dy + \frac{2}{\sigma_+^2}\int_0^\infty \varphi_\lambda(y)dy\right).$$

Explicit computations as before give now

$$R_\lambda f(0) = \frac{1}{\lambda}\left(\frac{2\exp(\frac{q}{2}(1-\Delta_-))}{\Delta_+ + \Delta_-} + 1 - \exp(q)\right). \tag{26}$$

Putting together (25) and (26) and recalling (17) and $\Delta_\pm = \sqrt{1+\frac{8\lambda}{\sigma_\pm^2}}$ we have (12).

We compute now $R_\lambda f(x)$ in (17) for the function $f(x) = (e^x - 1)_+$. We have

$$R_\lambda f(x) = \int_{\mathbb{R}} r(\lambda,x,y)(\exp(y)-1)_+ dy = \int_0^\infty r(\lambda,x,y)\exp(y)dy - \int_0^\infty r(\lambda,x,y)dy. \tag{27}$$

For $x \geq 0$,

$$\int_0^\infty r(\lambda,x,y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda}\left(\varphi_\lambda(x)\int_0^x \psi_\lambda(y)\exp(-y)dy + \psi_\lambda(x)\int_x^\infty \varphi_\lambda(y)\exp(-y)dy\right),$$
$$\int_0^\infty r(\lambda,x,y)\exp(y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda}\left(\varphi_\lambda(x)\int_0^x \psi(\lambda,y)dy + \psi_\lambda(x)\int_x^\infty \varphi_\lambda(y)dy\right)$$

From (20), (21), (22), (23),(27) we get

$$R_\lambda f(x) = \frac{1}{\lambda}\left(\frac{2\exp(\frac{x}{2}(1-\Delta_+))}{\Delta_+ + \Delta_-} + \exp(x) - 1\right). \tag{28}$$

For $x < 0$, since $x < y$;

$$\int_0^\infty r(\lambda,x,y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda}\psi_\lambda(x)\int_0^\infty \varphi_\lambda(y)\exp(-y)dy$$
$$\int_0^\infty r(\lambda,x,y)\exp(y)dy = \frac{2}{\sigma_+^2}\frac{1}{Wr_\lambda(y)}\psi_\lambda(x)\int_0^\infty \varphi_\lambda(y)dy.$$

As before, we get

$$R_\lambda f(x) = \frac{2}{\lambda}\frac{\exp(\frac{x}{2}(1+\Delta_-)}{\Delta_+ + \Delta_-}. \tag{29}$$

We put together (28) and (29) and we obtain (13). □

2.2 Inversion of Laplace transforms

In this section we use the Laplace transforms in Lemma 1 to obtain formulas for the price (5). We remark that, from the application of a classic Tauberian theorem, we directly obtain that, in both the cases considered in the theorem,

$$\mathcal{P}(t,l,y) \xrightarrow{t\downarrow 0} (e^y - e^l)_+, \qquad \mathcal{P}(t,l,y) \xrightarrow{t\to\infty} e^y.$$

(the short time result was to be expected from the PDE). Exponential Tauberian theorems [29] also give some information on the speed of convergence, but to compute exactly the explosion of the the skew we actually need to invert the Laplace transforms in Lemma 1. This will complete the proof of Theorem 1.

Proof. (of Theorem 1.) Let us set, for $q \in \mathbb{R}, \sigma_\pm, \lambda > 0$,

$$D(\lambda, q, \sigma_+, \sigma_-) = 2\frac{\exp\left(-\frac{|q|}{2}\sqrt{1+\frac{8\lambda}{\sigma(q)^2}}\right)}{\sqrt{1+\frac{8\lambda}{\sigma_+^2}} + \sqrt{1+\frac{8\lambda}{\sigma_-^2}}}. \tag{30}$$

Recall (9). In case $R = S_0$, also $\rho = y$ so we have

$$\begin{aligned}\mathcal{L}_\lambda \mathcal{P}(\cdot, K, S_0) = e^\rho \mathcal{L}_\lambda u_{l-\rho}(\cdot, 0) &= \frac{(e^y - e^l)_+}{\lambda} + e^{\frac{y+l}{2}}\frac{D(\lambda, l-y, \sigma_+, \sigma_-)}{\lambda} \\ &= \frac{(S_0 - K)_+}{\lambda} + \sqrt{S_0 K}\frac{D(\lambda, \log(K/S_0), \sigma_+, \sigma_-)}{\lambda}.\end{aligned}$$

From basic properties of Laplace transform,

$$\mathcal{P}(t, K, S_0) = (S_0 - K)_+ + \sqrt{S_0 K}\mathcal{L}_t^{-1}\left(\frac{D(\lambda, \log(K/S_0), \sigma_+, \sigma_-)}{\lambda}\right).$$

In case $R = K$ analogous computations give

$$\mathcal{P}(t, K, S_0) = (S_0 - K)_+ + \sqrt{S_0 K}\mathcal{L}_t^{-1}\left(\frac{D(\lambda, \log(S_0/K), \sigma_+, \sigma_-)}{\lambda}\right).$$

Now, to prove Theorem 1, we only need to show that

$$\theta(t, q, \sigma_+, \sigma_-) = \mathcal{L}_t^{-1}\left(\frac{D(\lambda, q, \sigma_+, \sigma_-)}{\lambda}\right). \tag{31}$$

We can rewrite D in (30) as

$$D(\lambda, q, \sigma_+, \sigma_-) = \frac{\exp\left(-\frac{|q|}{2}\sqrt{1+\frac{8\lambda}{\sigma^2(q)}}\right)}{4\lambda\frac{(\sigma_- - \sigma_+)(\sigma_- + \sigma_+)}{\sigma_+^2\sigma_-^2}}\left(\sqrt{1+\frac{8\lambda}{\sigma_+^2}} - \sqrt{1+\frac{8\lambda}{\sigma_-^2}}\right).$$

Some standard manipulations give

$$\begin{aligned}\frac{1}{\lambda}D(\lambda, q, \sigma_+, \sigma_-) &= \frac{\sigma_+^2\sigma_-^2}{(\sigma_- - \sigma_+)(\sigma_- + \sigma_+)} \\ &\times\left[\frac{\sigma_+}{8\sqrt{2}}h_2\Big(\lambda, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_+^2}{8}\Big) + \frac{1}{\sqrt{2}\sigma_+}h_1\Big(\lambda, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_+^2}{8}\Big)\right. \\ &\left.- \frac{\sigma_-}{8\sqrt{2}}h_2\Big(\lambda, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_-^2}{8}\Big) - \frac{1}{\sqrt{2}\sigma_-}h_1\Big(\lambda, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_-^2}{8}\Big)\right]\end{aligned}$$

where we have denoted, for $i \in \mathbb{N}$, $\lambda, a, b, c > 0$

$$h_i(\lambda, a, b, c) = \frac{\exp(-a\sqrt{\lambda + b})}{\lambda^i \sqrt{\lambda + c}}.$$

Now, let us also denote the inverse Laplace transform of h_i as

$$g_i(t, a, b, c) = \mathcal{L}_t^{-1} h_i(\cdot, a, b, c). \tag{32}$$

These functions can be explicitly computed. Recall the following well known facts on Laplace transforms:

if $F(\lambda)$ is Laplace transform of $f(t)$, then $F(a + b\lambda)$ is Laplace transform of $\frac{e^{-a/b}}{b} f(t/b)$;

if $F(\lambda)$ is Laplace transform of $f(t)$, then $\frac{1}{\lambda}F(\lambda)$ is Laplace transform of $\int_0^t f(u)du$.

The following inverse Laplace transform formula is a modification of [45][Formula 1. page 154]. Remark that in the formula in [45] there is one wrong sign (corresponding to $-a\sqrt{b - c + z}$ which is instead $-a\sqrt{b - c - z}$).

$$\begin{aligned} &g_0(t, a, b, c) \\ &= \frac{1}{\pi}\left[\int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z + b - c}} \exp(-(b + z)t)dz + \int_0^{b-c} \frac{\exp(-a\sqrt{b - c - z} - (z + c)t)}{\sqrt{z}} dz\right] \end{aligned}$$

Integrating in t we also obtain

$$\begin{aligned} g_1(t, a, b, c) = \frac{1}{\pi}\Bigg[&\int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z + b - c}} \frac{1 - \exp(-(b + z)t)}{b + z} dz \\ &+ \int_0^{b-c} \frac{\exp(-a\sqrt{b - c - z})}{\sqrt{z}} \frac{1 - \exp(-(c + z)t)}{c + z} dz\Bigg] \end{aligned} \tag{33}$$

and integrating again we obtain

$$\begin{aligned} g_2(t, a, b, c) = \frac{1}{\pi}\Bigg[&\int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z + b - c}} \frac{\exp(-(b + z)t) - 1 + t(b + z)}{(b + z)^2} dz \\ &+ \int_0^{b-c} \frac{\exp(-a\sqrt{b - c - z})}{\sqrt{z}} \frac{\exp(-(c + z)t) - 1 + t(c + z)}{(c + z)^2} dz\Bigg]. \end{aligned} \tag{34}$$

We can compute now $\theta(t, l, \sigma_+, \sigma_-)$ in (31), and regrouping terms we obtain

$$\begin{aligned} \theta(t, q, \sigma_+, \sigma_-) = &\frac{1}{\sqrt{2}} \frac{\sigma_+^2 \sigma_-^2}{(\sigma_- - \sigma_+)(\sigma_- + \sigma_+)} \\ &\times \left[\frac{1}{\sigma_+} G\left(t, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_+^2}{8}\right) - \frac{1}{\sigma_-} G\left(t, \frac{|q|}{\sigma(q)}\sqrt{2}, \frac{\sigma^2(q)}{8}, \frac{\sigma_-^2}{8}\right)\right], \end{aligned} \tag{35}$$

where we have denoted with G

$$\begin{aligned} G(t, a, b, c) &= c g_2(t, a, b, c) + g_1(t, a, b, c) \\ &= \frac{1}{\pi}\Bigg[\int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z + b - c}} \frac{(b + z)tc + (\exp(-(b + z)t) - 1)(c - b - z)}{(b + z)^2} dz \\ &+ \int_0^{b-c} \frac{\exp(-a\sqrt{b - c - z})}{\sqrt{z}} \frac{(c + z)tc - z(\exp(-(c + z)t) - 1)}{(c + z)^2} dz\Bigg]. \end{aligned}$$

This concludes the proof of Theorem 1, once we recall Lemma 1, definitions (30), (31), and that $e^l = K$, $e^y = S_0$, $e^\rho = R$. $\square$

Remark 2. When $q = 0$, which corresponds to the ATM case $K = S_0$, G has the following simpler expression

$$G(t, 0, b, c) = \sqrt{\frac{t}{\pi}} \exp(-ct) + \Big(\frac{1}{2\sqrt{c}} + t\sqrt{c}\Big) \operatorname{erf}(\sqrt{tc}) \Big), \tag{36}$$

for all $b, c > 0$. As a consequence, the pricing formula in Theorem 1 also has a simpler expression.

This can be proved as follows: using identity

$$\frac{\sqrt{c}}{\pi} \int_0^\infty \frac{\exp(-(c+z)t)}{\sqrt{z}(c+z)} dz = \operatorname{erfc}(\sqrt{tc}), \tag{37}$$

one can show

$$\begin{aligned} G(t, 0, b, c) &= \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{z}} \frac{(c+z)tc - z(\exp(-(c+z)t) - 1)}{(c+z)^2} dz \\ &= \frac{1}{\sqrt{c}} \operatorname{erf}(\sqrt{ct}) + \sqrt{c} \int_0^t \operatorname{erf}(\sqrt{cs}) ds. \end{aligned}$$

Then, integrating by parts and changing variable setting $\sqrt{cs} = x$, one gets (36).

In the following lemma we look at the behavior of prices (more specifically: price variation θ) in short time, in the "central limit theorem" regime $\sqrt{t}$. When compared to rough volatiliy models, where the CLT regime holds at $t^{1/2-H}$ (cf. [4]), we find again the analogy with the $H \downarrow 0$ case.

Lemma 2 (Central limit theorem for price variation)**.** *Let* $\gamma \neq 0$ *be a fixed constant. Then*

$$\lim_{t \downarrow 0} \frac{1}{\sqrt{t}} \theta(t, \gamma\sqrt{t}, \sigma_+, \sigma_-) = \frac{\sigma_+ \sigma_-}{\sigma_+ + \sigma_-} \left(\frac{\sqrt{2}}{\sqrt{\pi}} \exp\Big(-\frac{|\gamma|^2}{2\sigma(\gamma)^2} \Big) - \frac{|\gamma|}{\sigma(\gamma)} \operatorname{erfc}\Big(\frac{|\gamma|}{\sigma(\gamma)\sqrt{2}} \Big) \right)$$

Proof. We fix $\gamma, b, c > 0$. We recall (33) and change the integration variable via $u = \sqrt{tz}$. We find

$$\begin{aligned} g_1(t, \gamma\sqrt{t}, b, c) = \frac{2\sqrt{t}}{\pi} \Bigg[&\int_0^\infty \frac{u \cos \gamma u}{\sqrt{u^2 + t(b-c)}} \frac{1 - \exp(-bt - u^2)}{bt + u^2} du \\ &+ \int_0^{b-c} \exp(-\gamma\sqrt{t(b-c) - u^2}) \frac{1 - \exp(-ct - u^2)}{ct + u^2} du \Bigg]. \end{aligned}$$

Taking the limit for $t \downarrow 0$ we obtain

$$\frac{1}{\sqrt{t}} g_1(t, \gamma\sqrt{t}, b, c) \xrightarrow{t\downarrow 0} \frac{2}{\pi} \int_0^\infty \cos(\gamma u) \frac{1 - \exp(-u^2)}{u^2} du = \frac{2\exp(-\gamma^2/4)}{\sqrt{\pi}} - |\gamma| \operatorname{erfc}(|\gamma|/2).$$

We recall now (34), and rescale again the integration variable multiplying with $\sqrt{t}$. This time we obtain

$$t^{-3/2} g_2(t, \gamma\sqrt{t}, b, c) \xrightarrow{t\downarrow 0} \frac{2}{\pi} \int_0^\infty \cos(\gamma u) \frac{\exp(-u^2) - 1 + u^2}{u^4} du < \infty.$$

Therefore, since $G(t, a, b, c) = c g_2(t, a, b, c) + g_1(t, a, b, c)$,

$$\frac{1}{\sqrt{t}} G(t, \gamma\sqrt{t}, b, c) \xrightarrow{t\downarrow 0} \frac{2}{\pi} \int_0^\infty \cos(\gamma u) \frac{1 - \exp(-u^2)}{u^2} du = \frac{2\exp(-\gamma^2/4)}{\sqrt{\pi}} - |\gamma| \operatorname{erfc}(|\gamma|/2).$$

Recalling now (35) we obtain the statement. □

Remark 3 (Relation with Black & Scholes model). The price process S in (1) is a generalization of the Black & Scholes model, which we recover when taking $\sigma_+ = \sigma_-$ in (2). In particular, Lemma 1 also holds for this choice of the parameters. On the other hand, in the proof of Theorem 1, we have supposed $\sigma_+ \neq \sigma_-$. With $\sigma_+ = \sigma_- =: \bar{\sigma} > 0$, the proof of Theorem 1 does not apply directly, but a similar result can be obtained via some minor adaptations. Indeed one can prove that setting

$$\tilde{\theta}(t,q,\bar{\sigma}) = \mathcal{L}_t^{-1}\left(\frac{1}{\lambda}D(\lambda,q,\bar{\sigma},\bar{\sigma})\right), \tag{38}$$

the following holds for $q \in \mathbb{R}$:

$$\tilde{\theta}(t,q,\bar{\sigma}) = \frac{\bar{\sigma}}{2\sqrt{2}}J\left(t,\frac{|q|}{\bar{\sigma}}\sqrt{2},\frac{\bar{\sigma}^2}{8}\right), \tag{39}$$

with, for $a \geq 0, b > 0$

$$J(t,a,b) = \frac{1}{\pi}\int_0^\infty \frac{\cos a\sqrt{z}}{\sqrt{z}}\frac{1-\exp(-(b+z)t)}{b+z}dz. \tag{40}$$

For $q = 0$ the follwing simpler expression holds:

$$\tilde{\theta}(t,0,\bar{\sigma}) = \operatorname{erf}\left(\bar{\sigma}\sqrt{\frac{t}{8}}\right), \tag{41}$$

again using (37). As in the proof of Theorem 1, we obtain the following pricing formula (under Black & Sholes model):

$$\begin{aligned}\mathcal{P}_{BS}(t,K,S_0) = \mathcal{P}_{BS}(t,l,y) &= (e^y - e^l)_+ + \exp\left(\frac{y+l}{2}\right)\tilde{\theta}(t,l-y,\bar{\sigma}) \\ &= (S_0 - K)_+ + \sqrt{S_0 K}\tilde{\theta}(t, log(K/S_0),\bar{\sigma})\end{aligned} \tag{42}$$

Remark that this is a reformulation of the Black & Scholes formula. Differently from θ, the function $\tilde{\theta}$ depends on the second variable (the log-moneyness) only through its absolute value, so that (42) is analogous to both (7) and (8).

Concerning Lemma 2, the rescaling $u = \sqrt{tz}$ in the integration in (40) gives

$$\lim_{t\downarrow 0}\frac{1}{\sqrt{t}}\tilde{\theta}(t,\gamma\sqrt{t},\bar{\sigma}) = \frac{\bar{\sigma}}{2}\left(\frac{\sqrt{2}}{\sqrt{\pi}}\exp\Big(-\frac{|\gamma|^2}{2\bar{\sigma}^2}\Big) - \frac{|\gamma|}{\bar{\sigma}}\operatorname{erfc}\Big(\frac{|\gamma|}{\bar{\sigma}\sqrt{2}}\Big)\right), \tag{43}$$

which coincides with the result in Lemma 2 when taking $\sigma_+ = \sigma_- = \bar{\sigma}$.

3 Implied volatility

In this section, we analyze the implied volatility surface produced by model (1). To do so, we suppose in this section $S_0 = R$ (discontinuity/regime-switch ATM), which allow us to apply Theorem 1-1. In particular, our aim is to understand the short-time behavior of the implied volatility skew.

For fixed K, S_0 and time to expiry t, the implied volatilty $\sigma_{BS}(t,K,S_0)$ is the constant $\bar{\sigma}$ such that the price of a call in the Black & Scholes model with volatility $\bar{\sigma}$ coincides with the call price in model (1):

$$\mathcal{P}_{BS}(t,K,S_0) = \mathcal{P}(t,K,S_0).$$

We recall the notation for the log-moneyness $k = \log(K/S_0)$. Since $\mathcal{P}_{BS}(t,K,S_0), \mathcal{P}(t,K,S_0)$ satisfy equations (7), (42), the implied voaltility $\sigma_{BS}(t,K,S_0) = \sigma_{BS}(t,k)$ is the constant $\bar\sigma$ such that the following equation is satisfied:

$$\tilde\theta(t,k,\bar\sigma) = \theta(t,k,\sigma_+,\sigma_-). \tag{44}$$

Now, reacalling (40), (32), this reads

$$\begin{aligned} &\frac{\bar\sigma}{2\sqrt{2}} J\Big(t, \frac{|k|}{\bar\sigma}\sqrt{2}, \frac{\bar\sigma^2}{8}\Big) \\ &= \frac{1}{\sqrt{2}} \frac{\sigma_+^2\sigma_-^2}{\sigma_-^2-\sigma_+^2}\left[\frac{1}{\sigma_+} G\Big(t, \frac{|k|}{\sigma(k)}\sqrt{2}, \frac{\sigma^2(k)}{8}, \frac{\sigma_+^2}{8}\Big) - \frac{1}{\sigma_-} G\Big(t, \frac{|k|}{\sigma(k)}\sqrt{2}, \frac{\sigma^2(k)}{8}, \frac{\sigma_-^2}{8}\Big)\right], \end{aligned} \tag{45}$$

meaning that the implied volatility $\sigma_{BS}(t,k)$ is $\bar\sigma$ which solves such equation. Recall that (40) is defined in order to have (39) and (38). Of course, the standard method is to compute the price using (7), and then use one of the classic algorithms (e.g. Newton-Raphson) to recover implied volatility from the price, but (45) allows to obtain asymptotics on the behavior of σ_{BS} in short time. From the point of view of the numerical implementation, the two methods are equivalent.

Remark 4 (Implied volatility at-the-money.). When $K = S_0$ (so $k = 0$) the simpler expressions in Remark (36) and (41) hold. We obtain

$$\begin{aligned} \sigma_{BS}(t,0) = \sqrt{\frac{8}{t}}\,\mathrm{erf}^{-1}\Bigg(\frac{\sigma_+^2\sigma_-^2}{4(\sigma_-^2-\sigma_+^2)}\Bigg(&\frac{\sqrt{8t}}{\sigma_+\sqrt{\pi}}\exp\Big(-\frac{t\sigma_+^2}{8}\Big) - \frac{\sqrt{8t}}{\sigma_-\sqrt{\pi}}\exp\Big(-\frac{t\sigma_-^2}{8}\Big) \\ &- \Big(\frac{4}{\sigma_-^2}+t\Big)\mathrm{erf}\sqrt{\frac{t\sigma_-^2}{8}} + \Big(\frac{4}{\sigma_+^2}+t\Big)\mathrm{erf}\sqrt{\frac{t\sigma_+^2}{8}}\Bigg)\Bigg). \end{aligned} \tag{46}$$

From the expression above, using the asymptotic developments of the erf function, one can see that $\sigma_{BS}(t) \uparrow \frac{2\sigma_-\sigma_+}{\sigma_-+\sigma_+}$ for $t \downarrow 0$ and $\sigma_{BS}(t) \downarrow \min(\sigma_-,\sigma_+)$ for $t \uparrow \infty$. Indeed, for $x \to 0$, $\mathrm{erf}(x) \approx \frac{2}{\sqrt{\pi}}x$, so for $t \downarrow 0$

$$\lim_{t\downarrow 0}\sigma_{BS}(t,0) = \lim_{t\downarrow 0}\sqrt{\frac{8}{t}}\frac{\sqrt{\pi}}{2}\frac{\sigma_+^2\sigma_-^2}{4(\sigma_-^2-\sigma_+^2)}\Bigg(-\frac{4}{\sigma_-^2}\frac{2}{\sqrt{\pi}}\sqrt{\frac{t\sigma_-^2}{8}} + \frac{2}{\sqrt{\pi}}\frac{4}{\sigma_+^2}\sqrt{\frac{t\sigma_+^2}{8}}\Bigg) = \frac{2\sigma_+\sigma_-}{\sigma_-+\sigma_+}.$$

For $x \to \infty$, $1-\mathrm{erf}(x) = \mathrm{erfc}(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}}$. From this asymptotic, with some computations, we get

$$\lim_{t\to\infty}\sigma_{BS}(t,0) = \lim_{t\to\infty}\sqrt{-\frac{8}{t}\log\Big(\exp\Big(-\frac{t\sigma_-^2}{8}\Big)+\exp\Big(-\frac{t\sigma_+^2}{8}\Big)\Big)} = \min(\sigma_+,\sigma_-).$$

3.1 Short time asymptotics

For fixed γ let us define the function

$$f_\gamma : v \to v\left(\frac{\sqrt{2}}{\sqrt{\pi}}\exp\Big(-\frac{|\gamma|^2}{2v^2}\Big) - \frac{|\gamma|}{v}\,\mathrm{erfc}\Big(\frac{|\gamma|}{v\sqrt{2}}\Big)\right),$$

which is strictly increasing and therefore injective from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let $v_{BS}(\gamma)$ be the unique positive solution in v of the following equation, for fixed γ:

$$f_\gamma(v) = \frac{2\sigma_+\sigma_-}{\sigma_++\sigma_-}\left(\frac{\sqrt{2}}{\sqrt{\pi}}\exp\Big(-\frac{|\gamma|^2}{2\sigma(\gamma)^2}\Big) - \frac{|\gamma|}{\sigma(\gamma)}\,\mathrm{erfc}\Big(\frac{|\gamma|}{\sigma(\gamma)\sqrt{2}}\Big)\right) \tag{47}$$

We see, formally, that because of (45) and the asymptotic behavior of θ in Lemma 2 and $\tilde{\theta}$ in (43), for fixed $\gamma \in \mathbb{R}$ the following limit holds:

$$\lim_{t \downarrow 0} \sigma_{BS}(t, \gamma\sqrt{t}) = v_{BS}(\gamma). \tag{48}$$

This relation suggest that the limit skew may blow-up as $1/\sqrt{t}$, fact that we will consider in next Theorem 2. In Figure 1 we see approximation (48). From (47) we obtain easily that $v_{BS}(\cdot)$ is continuous in

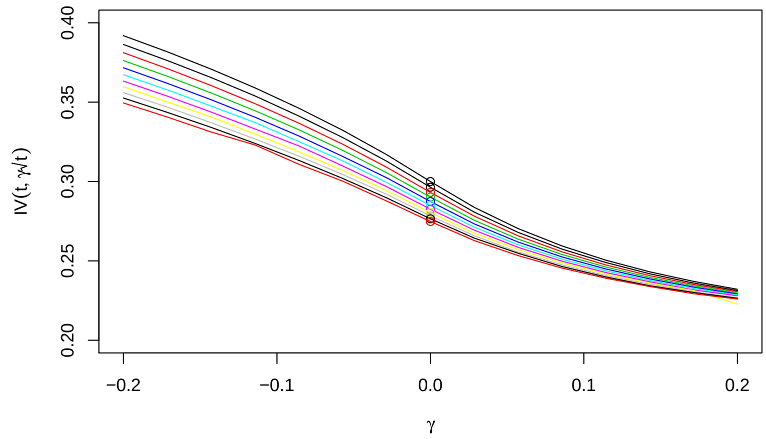

Figure 1: For fixed $\sigma_- = 0.6, \sigma_+ = 0.2$, we see $\gamma \to v_{BS}(\gamma)$ and $\gamma \to \sigma_{BS}(t, \gamma\sqrt{t})$, for fixed $t = 10, 20, \ldots, 100$. The “limit smile” $v_{BS}(\gamma)$ is almost indistinguishable from $\gamma \to \sigma_{BS}(t, \gamma\sqrt{t})$, exact solution to (45), for fixed, small $t > 0$. This numerical experiment also suggest that $t \to \sigma_{BS}(t, \gamma\sqrt{t})$ is decreasing in t. We also guess that both $t \to \sigma_{BS}(t, \gamma\sqrt{t})$ and $t \to \sigma_{BS}(t, k)$ converge to $\min(\sigma_+, \sigma_-)$ for $t \to \infty$.

0, with

$$v_{BS}(0) = \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}.$$

This implies that for all $k \in \mathbb{R}$

$$\lim_{t \downarrow 0} \sigma_{BS}(t, k) = \sigma_- \mathbf{1}_{k<0} + \sigma_+ \mathbf{1}_{k>0} + \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} \mathbf{1}_{k=0}.$$

Let us write $o_\gamma(1)$ for functions going to 0 as $\gamma \downarrow 0$. Differentiating (47) wrt γ we obtain

$$\frac{\sqrt{2}}{\sqrt{\pi}} \partial_\gamma v_{BS}(\gamma) - \operatorname{sgn}(\gamma) + o_\gamma(1) = -\frac{2\sigma(-\gamma)}{\sigma_+ + \sigma_-} \operatorname{sgn}(\gamma) + o_\gamma(1),$$

so that

$$\partial_\gamma v_{BS}(\gamma) = \frac{\sqrt{\pi}}{\sqrt{2}} \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} + o_\gamma(1).$$

Similar computations also give

$$\frac{\sqrt{2}}{\sqrt{\pi}} \partial_{\gamma\gamma} v_{BS}(\gamma) + \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{v(\gamma)} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} \frac{1}{\sigma(\gamma)^2} + o_\gamma(1)$$

so that, after some manipulations,

$$\partial_{\gamma\gamma} v_{BS}(0\pm) = \frac{\sigma_+ - \sigma_-}{2\sigma_+\sigma_-}\left(\frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \mp 2\right).$$

Notice that, while $\partial_k v_{BS}$ is continuous in 0, $\partial_{kk} v_{BS}$ assumes two different values in 0^+ and 0^-.

We obtain the following development in $\gamma = 0$ of v:

$$v_{BS}(\gamma) \approx \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} + \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}\frac{\sqrt{\pi}}{\sqrt{2}}\gamma + \frac{\sigma_+ - \sigma_-}{2\sigma_+\sigma_-}\left(\frac{\sigma_+ - \sigma_-}{2(\sigma_+ + \sigma_-)} - \operatorname{sgn} k\right)\gamma^2, \tag{49}$$

Recalling the rescaling $k = \gamma\sqrt{t}$ in (48), we get expansion (3).

Remark 5 (Relation with Skew Brownian motion). Process (14) is strictly connected with the Skew Brownian motion (SBM) (such connection is a key point in [20]). We can use this relation to get some insights on the coefficients appearing in (49), (3). Let ξ be the process solution to $d\xi_t = \sigma(\xi_t)dW_t$; $\xi_0 = 0$ (remark the absence of the drift here when compared to (14)). The process $\xi_t/\sigma(\xi_t)$ is a SBM X_t, solution to

$$X_t = W_t + \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} L_t^0(X),$$

where W is a Brownian motion and $L_t^0(X)$ is the local time at 0 of X [31, 36]. The coefficient in front of the local time is the one quantifying the "asymmetry" of the process, and we find it again in the ATM skew of the implied volatility. Let $L_t^0(\xi)$ be the local time at 0 (the discontinuity level) of ξ. Then, the following relation holds:

$$L_t^0(\xi) = \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} L_t^0(X) \overset{\text{law}}{=} \frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} L_t^0(W) \overset{\text{law}}{=} L_t^0\left(\frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-} W\right).$$

The local time at 0 of ξ is in law the same as the one of a Brownian Motion with volatility $\frac{2\sigma_+\sigma_-}{\sigma_+ + \sigma_-}$. This confirms this constant as the "effective volatility" at the discontinuity.

3.2 Implied volatility skew and blow-up in short time

The following theorem gives an explicit formula for the ATM implied volatility skew of model (1) for all positive t. From this formula follows the short time limit result on the blow-up of the skew.

For $b, c \geq 0$ we set

$$R(t, b, c) = \frac{1}{\pi}\int_c^b \sqrt{\left(\frac{b}{u} - 1\right)\left(1 - \frac{c}{u}\right)}\frac{\exp(-ut)}{u}du. \tag{50}$$

Theorem 2 (ATM skew)**.** *The volatility surface is differentiable in ∂_k at 0 for all $t > 0$. The ATM implied volatility skew has the following form:*

$$\partial_k \sigma_{BS}(t, 0) = \frac{1}{\sqrt{t}}\sqrt{\frac{\pi}{2}}\exp\left(\frac{\sigma_{BS}^2(t,0)t}{8}\right)\frac{2\sigma_+\sigma_-}{|\sigma_+ - \sigma_-|(\sigma_- + \sigma_+)} R\left(t, \frac{\sigma_+^2}{8}, \frac{\sigma_-^2}{8}\right)$$

where R is given in (50) *and $\sigma_{BS}(t, 0)$ in* (46)*. Moreover,*

$$\lim_{t \downarrow 0} \sqrt{t}\partial_k \sigma_{BS}(t, 0) = \frac{\sqrt{\pi}}{\sqrt{2}}\frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-}.$$

Remark 6 (Skew explosion in continuous local volatility)*.* Looking at Theorem 2, one may wonder if there exist a continuous LV function, depending only on the underlying S, such that the associated implied volatility surface displays exploding skew. Recall the one-half rule $\partial_S \sigma_{loc}(t,0) \approx 2\partial_k \sigma_{BS}(t,0)$ (see [7, 22]). Since we assume that the LV function does not depend on t, it will be possible to obtain the blow up of the skew only if $\sigma_{loc}(S)$ is not differentiable. Being discontinuous, the step function looks like an extreme case, where the implied volatility skew explodes as $t^{-1/2}$. Moreover, it would be desirable to have a parameter to calibrate the power law explosion of the skew, as for rough volatility models, where $\partial_k \sigma_{BS}(t,0) \sim t^{H-1/2}$ (see [3, 4, 19, 18] and cf. with [42] for the limit case $H \downarrow 0$).

Therefore, we shall look at LV functions which are not differentiable at 0, but smoother that the step function, a good candidate being a function which admits a finite *fractional derivative* ∂_S^r at $S = 0$ for some $0 < r < 1$, but not the derivative ∂_S . For the short-time ATM skew, the behavior of the LV far from 0 should not matter: let us set, for fixed $0 < r < 1$,

$$\sigma_{loc}^{[r]}(x) = \sigma_0 + \operatorname{sgn}(R - S_t)\Big(|R - S_t|^r \wedge \frac{\sigma_0}{2}\Big) \tag{51}$$

where $\sigma_0 > 0$ is the spot volatility and $R = S_0$. The choice above produces negative skew, and positive skew would be obtained taking $\operatorname{sgn}(S_t - R)$ instead of $\operatorname{sgn}(R - S_t)$.

Let $\sigma_{BS}^{[r]}(t,k)$ the implied volatility associated to model (51). It is reasonable to expect, in short time,

$$\partial_k \sigma_{BS}^{[r]}(t,0) \sim t^{-f(r)}.$$

When $r \approx 1$, $\sigma_{loc}^{[r]}$ is almost a differentiable function, for which the one-half rule tell us that the skew does not explode, so $f(1-) = 0$ is to be expected. When $r \approx 0$, $\sigma_{loc}^{[r]}$ looks like a step function, so the extreme case $f(0+) = 1/2$ is to be expected. The simplest guess it to take f linear (as in the rough volatility case), which would give $f(x) = (1-r)/2$.

In Figure 2 we display the skew on Monte Carlo simulations of model (51) and the power-law fit $Ct^{-(1-r)/2}$; this suggests that the skew may explode in t as a power law of exponent $-\frac{1-r}{2}$.

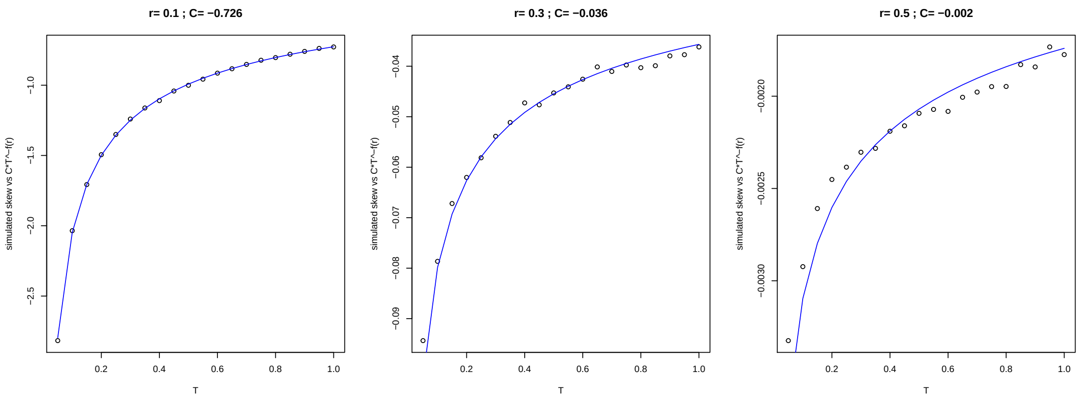

Figure 2: Implied volatility skew $\partial_k \sigma_{BS}^{[r]}(T,0)$ computed on Monte Carlo simulations of model (51) compared with $CT^{-(1-r)/2}$. For fixed $\sigma_0 = 0.3$, the exponent $-\frac{1-r}{2}$ is chosen a priori, the constant C chosen to fit the simulations.

Proof. Let us start with some preliminary computations. Recall (40). We have, for all, fixed $b > 0$

$$J(t,a,b) \xrightarrow{a \downarrow 0} J(t,0,b) = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{z}} \frac{1 - \exp(-(b+z)t)}{b+z} dz = \frac{\operatorname{erf}\sqrt{bt}}{\sqrt{b}} \tag{52}$$

(cf. (37)). Again from (40) we can write

$$\partial_a J(t,a,b) = -\frac{1}{\pi}\int_0^\infty \sin(a\sqrt{z})\frac{1-\exp(-(b+z)t)}{b+z}dz \xrightarrow{a\downarrow 0} -1. \tag{53}$$

Similarly,

$$\begin{aligned}\partial_b J(t,a,b) &= \frac{1}{\pi}\int_0^\infty \frac{\cos(a\sqrt{z})}{\sqrt{z}}\frac{\exp(-(b+z)t)(1+(b+z)t)-1}{(b+z)^2}dz \\ &\xrightarrow{a\downarrow 0} \frac{1}{\pi}\int_0^\infty \frac{\exp(-(b+z)t)(1+(b+z)t)-1}{\sqrt{z}(b+z)^2}dz = \partial_b\left(\frac{\mathrm{erf}(\sqrt{bt})}{\sqrt{bt}}\right)\end{aligned} \tag{54}$$

With an integration by parts one can see that

$$\frac{1}{\sqrt{t}}\partial_b\left(\frac{\mathrm{erf}(\sqrt{bt})}{\sqrt{b}}\right) = \frac{1}{b}\left(\frac{\exp(-bt)}{\sqrt{\pi}} - \frac{1}{2}\frac{\mathrm{erf}(\sqrt{bt})}{\sqrt{bt}}\right)$$

so that, using (52),

$$\lim_{a\downarrow 0}\frac{J(t,a,b)+2b\partial_b J(t,a,b)}{\sqrt{2t}} = \frac{\sqrt{2}}{\sqrt{\pi}}\exp(-bt). \tag{55}$$

Recall now (32), from which we get

$$\begin{aligned}&\partial_a G(t,a,b,c) \\ &= -\frac{1}{\pi}\Bigg[\int_0^\infty \sin(a\sqrt{z})\frac{\sqrt{z}}{\sqrt{z+b-c}}\frac{(b+z)tc+(\exp(-(b+z)t)-1)(c-b-z)}{(b+z)^2}dz \\ &+\int_0^{b-c}\exp\big(-a\sqrt{b-c-z}\big)\frac{\sqrt{b-c-z}}{\sqrt{z}}\frac{(c+z)tc-z(\exp(-(c+z)t)-1)}{(c+z)^2}dz\Bigg].\end{aligned} \tag{56}$$

Since $\lim_{a\downarrow 0}\sin(a\sqrt{z}) = 0$, the integrable factors multiplying the $\sin(\cdot)$ function give a vanishing contribution in the limit, using dominate convergence. So

$$\begin{aligned}&\lim_{a\downarrow 0}\int_0^\infty \sin(a\sqrt{z})\frac{\sqrt{z}}{\sqrt{z+b-c}}\frac{(b+z)tc+(\exp(-(b+z)t)-1)(c-b-z)}{(b+z)^2}dz \\ &= \lim_{a\downarrow 0}\int_0^\infty \sin(a\sqrt{z})\frac{\sqrt{z}}{\sqrt{z+b-c}}\frac{(b+z)tc+b+z}{(b+z)^2}dz = \lim_{a\downarrow 0}\int_0^\infty \sin(a\sqrt{z})\frac{tc+1}{b+z}dz \\ &= \pi(tc+1).\end{aligned} \tag{57}$$

In case $b=c$, the second integral in (56) vanishes and

$$\lim_{a\downarrow 0}\partial_a G(t,a,b,c) = -tb-1 = -tc-1. \tag{58}$$

We suppose now that $b>c$, and compute the limit for $a\downarrow 0$ of the second integral:

$$\begin{aligned}&\lim_{a\downarrow 0}\int_0^{b-c}\exp\big(-a\sqrt{b-c-z}\big)\frac{\sqrt{b-c-z}}{\sqrt{z}}\frac{(c+z)tc-z(\exp(-(c+z)t)-1)}{(c+z)^2}dz \\ &= t\int_0^{b-c}\frac{\sqrt{b-c-z}}{\sqrt{z}}dz + \int_0^{b-c}\sqrt{(b-c-z)z}\frac{1-\exp(-(c+z)t)-(c+z)t}{(c+z)^2}dz.\end{aligned}$$

With the change of variable $c+z=u$ we have

$$\int_0^{b-c}\frac{\sqrt{b-c-z}}{\sqrt{z}}dz = \int_c^b\sqrt{\frac{b-u}{u-c}}du = \frac{t\pi(b-c)}{2}$$

and

$$\int_0^{b-c} \sqrt{(b-c-z)z}\frac{1-\exp(-(c+z)t)-(c+z)t}{(c+z)^2}dz$$
$$= \int_c^b \sqrt{\Big(\frac{b}{u}-1\Big)\Big(1-\frac{c}{u}\Big)}\frac{1-\exp(-ut)-ut}{u}du.$$

Now,

$$t\int_c^b \sqrt{\Big(\frac{b}{u}-1\Big)\Big(1-\frac{c}{u}\Big)}du = \pi\bigg(\frac{c+b}{2}-\sqrt{bc}\bigg)t$$

and

$$\int_c^b \sqrt{\Big(\frac{b}{u}-1\Big)\Big(1-\frac{c}{u}\Big)}\frac{du}{u} = \pi\bigg(\frac{c+b}{2\sqrt{bc}}-1\bigg). \tag{59}$$

Therefore,

$$\int_c^b \sqrt{\Big(\frac{b}{u}-1\Big)\Big(1-\frac{c}{u}\Big)}\frac{1-\exp(-ut)-ut}{u}du$$
$$= \pi\bigg(\frac{c+b}{2\sqrt{bc}}-1-\bigg(\frac{c+b}{2}-\sqrt{bc}\bigg)t-R(t,b,c)\bigg).$$

Recalling also (58), we have

$$\lim_{a\downarrow 0}\partial_a G(t,a,b,c) = -\frac{c+b}{2\sqrt{bc}} - t\sqrt{bc} + R(t,b,c).$$

Suppose now $c > b$. In this case, it is not clear from (32) that G is a real function. For this choice of the parameters, with standard manipulations G can be rewritten as

$$\begin{aligned} G(t,a,b,c) = \frac{1}{\pi}\bigg[&\int_{c-b}^{\infty} \frac{\cos a\sqrt{z}}{\sqrt{z+b-c}}\frac{(b+z)tc+(\exp(-(b+z)t)-1)(c-b-z)}{(b+z)^2}dz \\ &+\int_0^{c-b} \frac{\sin a\sqrt{z}}{\sqrt{-z-b+c}}\frac{(b+z)tc+(\exp(-(b+z)t)-1)(c-b-z)}{(b+z)^2}dz\bigg] \end{aligned} \tag{60}$$

In this form, one can see that G is a real function. Completely analogous computations can be made starting from (60). One can show that

$$\begin{aligned} &\partial_a G(t,a,b,c) \\ &= \frac{1}{\pi}\bigg[-\int_{c-b}^{\infty} \sin(a\sqrt{z})\frac{\sqrt{z}}{\sqrt{z+b-c}}\frac{(b+z)tc+(\exp(-(b+z)t)-1)(c-b-z)}{(b+z)^2}dz \\ &+\int_0^{c-b} \cos\big(a\sqrt{z}\big)\frac{\sqrt{z}}{\sqrt{c-b-z}}\frac{(b+z)tc+(c-b-z)(\exp(-(b+z)t)-1)}{(b+z)^2}dz\bigg] \end{aligned} \tag{61}$$

and

$$\begin{aligned} &\lim_{a\downarrow 0}\int_0^{c-b} \cos\big(a\sqrt{z}\big)\frac{\sqrt{z}}{\sqrt{c-b-z}}\frac{(b+z)tc+(c-b-z)(\exp(-(b+z)t)-1)}{(b+z)^2}dz \\ &= \pi\left(tc+1-\sqrt{bc}-\frac{b+c}{2\sqrt{cb}}+R(t,c,b)\right). \end{aligned} \tag{62}$$

From (57), (61), (62) (clearly it does not matter if the lower integration bound in the first integral is 0 or $c-b$) we get

$$\lim_{a\downarrow 0}\partial_a G(t,a,b,c) = -\frac{c+b}{2\sqrt{bc}} - t\sqrt{bc} + R(t,c,b).$$

Therefore, for all b, c,

$$\lim_{a\downarrow 0}\partial_a G(t,a,b,c) = -\frac{c+b}{2\sqrt{bc}} - t\sqrt{bc} + R\big(t,\max(b,c),\min(b,c)\big). \tag{63}$$

In $k\neq 0$, σ_{BS} is defined by (45). Differentiating wrt k we obtain

$$\begin{aligned}\frac{\partial_k\sigma_{BS}(t,k)}{2\sqrt{2}}J(t,\cdot,\cdot) &+ \left(\frac{\operatorname{sgn}(k)}{2} - \frac{|k|\partial_k\sigma_{BS}(t,k)}{2\sigma_{BS}(t,k)}\right)\partial_a J(t,\cdot,\cdot)\\ &+\frac{\sigma_{BS}^2(t,k)}{8\sqrt{2}}\partial_k\sigma_{BS}(t,k)\partial_b J(t,\cdot,\cdot)\\ = \frac{1}{\sqrt{2}}\frac{\sigma_+^2\sigma_-^2}{(\sigma_- -\sigma_+)(\sigma_- +\sigma_+)}&\left[\frac{1}{\sigma_+}\partial_a G(t,\cdot,\cdot,\cdot) - \frac{1}{\sigma_-}\partial_a G(t,\cdot,\cdot,\cdot)\right]\frac{\operatorname{sgn}(k)\sqrt{2}}{\sigma(k)}\end{aligned}$$

so that the derivative $\partial_k\sigma_{BS}(t,k)$ can be expressed as

$$\begin{aligned}&\partial_k\sigma_{BS}(t,k)\\ &= \frac{-\operatorname{sgn}(k)\sqrt{2}\partial_a J(t,\cdot,\cdot) + \frac{2\sigma_+^2\sigma_-^2}{(\sigma_- -\sigma_+)(\sigma_- +\sigma_+)}\left[\frac{1}{\sigma_+}\partial_a G(t,\cdot,\cdot,\cdot) - \frac{1}{\sigma_-}\partial_a G(t,\cdot,\cdot,\cdot)\right]\frac{\operatorname{sgn}(k)\sqrt{2}}{\sigma(k)}}{J(t,\cdot,\cdot) - \frac{|k|\sqrt{2}}{\sigma_{BS}(t,k)}\partial_a J(t,\cdot,\cdot) + \frac{\sigma_{BS}^2(t,k)}{4}\partial_b J(t,\cdot,\cdot)}.\end{aligned}$$

We can now take the limits for $k\downarrow 0$, and from (52),(54), (53), (63), (55) we have

$$\begin{aligned}&\partial_k\sigma_{BS}(t,0+)\\ &= \frac{-\partial_a J\left(t,0+,\frac{\sigma_{BS}(t,0)^2}{8}\right) + \frac{2\sigma_+\sigma_-}{(\sigma_- -\sigma_+)(\sigma_- +\sigma_+)}\left[\frac{\sigma_-}{\sigma_+}\partial_a G\big(t,0+,\frac{\sigma_+^2}{8},\frac{\sigma_+^2}{8}\big) - \partial_a G\big(t,0+,\frac{\sigma_+^2}{8},\frac{\sigma_-^2}{8}\big)\right]}{\frac{1}{\sqrt{2}}\left(J\big(t,0,\frac{\sigma_{BS}(t,0)^2}{8}\big) + \frac{\sigma_{BS}^2(t,0)}{4}\partial_b J\big(t,0,\frac{\sigma_{BS}(t,0)^2}{8}\big)\right)}\\ &= \frac{1}{\sqrt{t}}\frac{\frac{2\sigma_+\sigma_-}{(\sigma_+-\sigma_-)(\sigma_-+\sigma_+)}R\left(t,\frac{\max(\sigma_+,\sigma_-)^2}{8},\frac{\min(\sigma_+,\sigma_-)^2}{8}\right)}{\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\sigma_{BS}(t,0)^2}{8}t\right)} = \frac{1}{\sqrt{t}}\frac{\frac{2\sigma_+\sigma_-}{|\sigma_+-\sigma_-|(\sigma_-+\sigma_+)}R\left(t,\frac{\sigma_+^2}{8},\frac{\sigma_-^2}{8}\right)}{\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\sigma_{BS}(t,0)^2}{8}t\right)}\end{aligned}$$

Analogously, after some explicit computations, for $k\uparrow 0$ we have

$$\begin{aligned}&\partial_k\sigma_{BS}(t,0-)\\ &= \frac{\partial_a J\left(t,0+,\frac{\sigma_{BS}(t,0)^2}{8}\right) - \frac{2\sigma_+\sigma_-}{(\sigma_- -\sigma_+)(\sigma_- +\sigma_+)}\left[\partial_a G\big(t,0+,\frac{\sigma_-^2}{8},\frac{\sigma_+^2}{8}\big) - \frac{\sigma_+}{\sigma_-}\partial_a G\big(t,0+,\frac{\sigma_-^2}{8},\frac{\sigma_-^2}{8}\big)\right]}{J\big(t,0,\frac{\sigma_{BS}(t,0)^2}{8}\big) + \frac{\sigma_{BS}^2(t,0)}{4}\partial_b J\big(t,0,\frac{\sigma_{BS}(t,0)^2}{8}\big)}\\ &= \partial_k\sigma_{BS}(t,0+).\end{aligned}$$

Therefore the derivative exists also in $k=0$ and has the value given in the statement. The blow-up as $t^{-1/2}$ follows from

$$\lim_{t\downarrow 0}R(t,b,c) = \frac{1}{\pi}\int_c^b\sqrt{\left(\frac{b}{u}-1\right)\left(1-\frac{c}{u}\right)}\frac{du}{u} = \operatorname{sgn}(b-c)\left(\frac{c+b}{2\sqrt{bc}}-1\right)$$

(cf. (59)) and substitution $b=\sigma_+^2/8, c=\sigma_-^2/8$. □

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