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Stochastic unfolding and homogenization

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Abstract

The notion of periodic two-scale convergence and the method of periodic unfolding are prominent and useful tools in multiscale modeling and analysis of PDEs with rapidly oscillating periodic coefficients. In this paper we are interested in the theory of stochastic homogenization for continuum mechanical models in form of PDEs with random coefficients, describing random heterogeneous materials. The notion of periodic two-scale convergence has been extended in different ways to the stochastic case. In this work we introduce a stochastic unfolding method that features many similarities to periodic unfolding. In particular it allows to characterize the notion of stochastic two-scale convergence in the mean by mere convergence in an extended space. We illustrate the method on the (classical) example of stochastic homogenization of convex integral functionals, and prove a stochastic homogenization result for an non-convex evolution equation of Allen-Cahn type. Moreover, we discuss the relation of stochastic unfolding to previously introduced notions of (quenched and mean) stochastic two-scale convergence. The method descibed in the present paper extends to the continuum setting the notion of discrete stochastic unfolding, as recently introduced by the second and third author in the context of discrete-to-continuum transition.

1 Introduction

Homogenization theory deals with the derivation of effective, macroscopic models for problems that involve two or more length-scales. Typical examples are continuum mechanical models for microstructured materials that give rise to boundary value problems or evolutionary problems for partial differential equations with coefficients that feature rapid, spatial oscillations. The first results in homogenization theory were motivated by a mechanics problem which was about the determination of the macroscopic behavior of linearly elastic composites with periodic microstructure, see Hill [41]. In the mathematical community early contributions in the 70s came from the *French school* (e.g. see [10] for an early standard reference, and [68, 57] for Tartar and Murat's notion of H-convergence), the *Russian school* (e.g. Zhikov, Kozlov and Oleinik, see [72]), and from the *Italian school* for variational problems (e.g. Marcellini [49], Spagnolo [67] for G-convergence, and De Giorgi and Franzoni for Γ-convergence [27]). In the 80s and later homogenization was intensively studied for a variety of models from continuum mechanics including non-convex integral functionals and applications to non-linear elasticity (e.g. Müller [56, 28] and Braides [14]), or the topic of effective flow through porous media (e.g. see Hornung et al. [6, 44] and Allaire [2]). Most results in homogenization theory discuss problems with periodic microstructure, and specific analytic tools for periodic homogenization of linear (or monotone) operators are developed, including the notions of two-scale convergence and

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periodic unfolding [62, 3, 70, 20], which by now are standard tools in multiscale modeling and analysis. In the last decade considerable interest in applied mathematics emerged in understanding random heterogeneous materials, i.e. materials whose properties on a small length-scale are only described on a statistical level, such as polycrystalline composites, foams, or biological tissues, see [69] for a standard reference. Although the first results in stochastic homogenization were already obtained in the 70s and 80s for linear elliptic equations and convex minimization problems, see [63, 46, 24], the theory in the stochastic case is still less developed as in the periodic case and object of various recent studies, e.g. regarding error estimates and regularity properties (see [32, 33, 31, 29, 30, 8, 7], or modeling of random heterogeneous materials [73, 1, 17, 42, 38, 39, 11, 61]. With the present paper we contribute to the latter. In particular, we introduce a *stochastic unfolding method* that shares many similarities to periodic unfolding and two-scale convergence with the intention to systematize and simplify the process of lifting results from periodic homogenization to the stochastic case. We illustrate this by reconsidering stochastic homogenization of convex integral functionals and by proving a new stochastic homogenization result for semilinear gradient flows of Allen-Cahn type. In order to put the notion into perspective, in the following we recall the concepts of two-scale convergence and periodic unfolding.

For problems with periodic coefficients, the notion of (periodic) two-scale convergence was introduced in [62] and further developed in [3, 48]. Two-scale convergence refines weak convergence in L^p-spaces: The two-scale limit captures not only the averaged behavior of an oscillating sequence (as opposed to the weak limit), but also oscillations on a prescribed small scale ε. In particular, let $Q \subset \mathbb{R}^d$ and $\square = [0,1)^d$, a sequence $(u_\varepsilon) \subset L^p(Q)$ two-scale converges to $u \in L^p(Q \times \square)$ (as $\varepsilon \to 0$) if

$$\lim_{\varepsilon \to 0} \int_Q u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_Q \int_\square u(x,y)\varphi(x,y)dydx,$$

for all $\varphi \in L^q(Q; C_\#(\square))$. Here $C_\#(\square)$ denotes the space of continuous and $\square$-periodic functions and $p, q \in (1, \infty)$ are dual exponents.

In [6] in the specific context of homogenization of flow through porous media Arbogast et al. introduced a dilation operator to resolve oscillations on a prescribed scale of weakly converging sequences; it turned out that the latter yields a characterization of two-scale convergence (see [12, Proposition 4.6]). In a similar spirit, Cioranescu et al. introduced in [20, 21] the periodic unfolding method as a systematic approach to homogenization. The key object of this method is a linear isometry $\mathcal{T}_\varepsilon^p : L^p(Q) \to L^p(Q \times \square)$ (the periodic unfolding operator) which invokes a change of scales and allows (at the expense of doubling the dimension) to use standard weak and strong convergence theorems in L^p-spaces to capture the microscopic behavior of oscillatory sequences. It turned out that the method is well-suited for periodic multiscale problems, e.g. see [19, 34, 55, 70, 59, 36, 47]. Moreover, the unfolding method allows to rephrase two-scale convergence: Applied to an oscillatory sequence $(u_\varepsilon) \subset L^p(Q)$, the unfolded sequence $(\mathcal{T}_\varepsilon^p u_\varepsilon)$ weakly converges in $L^p(Q \times \square)$ if and only if (u_ε) two-scale converges, and the corresponding limits are the same. We refer to [55] where this perspective on two-scale convergence is investigated and applied in the context of evolutionary problems.

Motivated by the idea of (periodic) two-scale convergence, in [13] the notion of stochastic two-scale convergence in the mean was introduced suited for homogenization problems that invoke random coefficients, see also [5]. In stochastic homogenization typically random coefficients of the form $a(\omega,x)=a_0(\tau_x\omega)$ (for $x\in\mathbb{R}^d$) are considered where ω stands for a "random configuration" and a_0 is defined on a probability space $(\Omega,\mathcal{F},P)$ that is equipped with a measure preserving action $\tau_x:\Omega\to\Omega$, see Section 2.1. A sequence $(u_\varepsilon)\subset L^p(\Omega\times Q)$ (where $Q\subset\mathbb{R}^d$ denotes a continuum domain) is said to two-scale converge in the mean to some $u\in L^p(\Omega\times Q)$ if

$$\lim_{\varepsilon\to 0}\int_\Omega\int_Q u_\varepsilon(\omega,x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega,x)dxdP(\omega)=\int_\Omega\int_Q u(\omega,x)\varphi(\omega,x)dxdP(\omega)$$

for all $\varphi\in L^q(\Omega\times Q)$ satisfying suitable measurability conditions.

Motivated by the concept of the periodic unfolding method, in [61] the second and third author developed a stochastic unfolding method for a discrete-to-continuum analysis of discrete models of random heterogeneous materials. In the present work, we extend the concept to problems defined on continuum domains $Q\subset\mathbb{R}^d$. In particular, we introduce a stochastic unfolding operator $\mathcal{T}_\varepsilon:L^p(\Omega\times Q)\to L^p(\Omega\times Q)$ which is an isometric isomorphism (see Section 2.2). It displays similar properties as the periodic unfolding operator; in particular, weak convergence of the unfolded sequence $(\mathcal{T}_\varepsilon u_\varepsilon)$ is equivalent to stochastic two-scale convergence in the mean, and – as in the periodic case – we recover a compactness statement for two-scale limits of gradients.

A first example that we treat via stochastic unfolding is the classical problem of stochastic homogenization of convex integral functionals. As in the periodic case, the proof of the homogenization theorem via unfolding is merely based on elementary properties of the unfolding operator and on (semi-)continuity of convex functionals (with suitable growth assumption). The second example we consider is homogenization for gradient flows driven by λ-convex energies. In particular, we consider an Allen-Cahn type equation with random and oscillating coefficients, yet with a non-convexity only acting on statistically averaged quantities. The homogenization procedure follows the abstract strategy for evolutionary Γ-convergence of gradients systems, see [52] and the references therein (we provide more references in Section 3). On the one hand, the example illustrates that the stochastic unfolding method yields a short and rather elementary argument for stochastic homogenization of the specific problem, on the other hand, the example points out certain limitations of the method (e.g. due to the failure of Rellich-type compactness properties in the extended space of random fields).

An alternative and finer "quenched" notion of stochastic two-scale convergence was introduced by Zhikov and Piatnitski [73]. In a very general setting, they introduced two-scale convergence on random measures as a generalization of periodic two-scale convergence as presented in [71]. In this work, we restrict to the simplest case where the random measure is the Lebesgue measure. The concept of stochastic two-scale convergence in [73] is based on Birkhoff's ergodic theorem. Although the definition of (quenched) stochastic two-scale convergence, which we recall in Section 4, and two-scale convergence in the mean look quite similar, it is non-trivial to derive quenched convergence from mean convergence (while the opposite direction in most cases is straight forward). In this paper we investigate this issue and provide some tools that allow to draw conclusions on quenched homogenization from mean homogenization, as we illustrate at the example of convex integral functionals. For the analysis we appeal to Young measures generated by stochastically two-scale convergent sequences in the mean and in particular establish a compactness result (see Theorem 4.11 and Lemma 4.14). Moreover, we exploit a recent lower semicontinuity result of convex integral functionals w.r.t. quenched stochastic two-scale convergence that has been recently obtained by the first author and Nesenenko in [39].

Structure of the paper. In Section 2 we introduce the standard setting for stochastic homogenization, introduce the notion of stochastic unfolding and derive the most significant properties of the unfolding operator. In the following Section 3 two examples of the homogenization procedure via stochastic unfolding are presented. Namely, Section 3.1 is dedicated to homogenization of convex functionals and in Section 3.2 homogenization for Allen-Cahn type gradient flows is provided. In Section 4 we discuss the relations of stochastic unfolding and quenched stochastic two-scale convergence. Section 2 and 3.1, which contain the basic concepts and the application to convex homogenization, are self-contained and require only basic input from functional analysis. Section 3.2 and Section 4 require some advanced tools from analysis and measure theory.

2 Stochastic unfolding and properties

2.1 Description of random media - a functional analytic framework

To fix ideas we consider for a moment the setup of Papanicolaou and Varadhan [63] for homogenization of elliptic operators of the form $-\nabla\cdot a(\frac{x}{\varepsilon})\nabla$ with a coefficient field $a:\mathbb{R}^d\to\mathbb{R}^{d\times d}$. In the stochastic case the coefficients are assumed to be random and thus a can be viewed as a family of random variables $\{a(x)\}_{x\in\mathbb{R}^d}$. A minimal requirement for stochastic homogenization of such operators is that the distribution of the coefficient field is *stationary* and *ergodic*. Stationarity means that the coefficients are statistically homogeneous (i.e. for any finite set of points $x_1,\ldots,x_n\in\mathbb{R}^d$ the joint distribution of the *shifted* random variables $a(x_1+z),\ldots,a(x_n+z)$ is independent of $z\in\mathbb{R}^d$), while *ergodicity* (see below for the precise definition) is an assumption that ensures a separation of scales in the sense that long-range correlations of the coefficients become negligible in the large scale limit, e.g. $\mathrm{cov}[\fint_{B+z}a,\fint_B a]\to 0$ as $z\to\infty$. In [63], Papanicolaou and Varadhan introduced a (by now standard) setup that allows to phrase these conditions in the following functional analytic framework (see also [45]):

Assumption 2.1. *Let $(\Omega,\mathcal{F},P)$ denote a probability space with a countably generated σ-algebra, and let $\tau=\{\tau_x\}_{x\in\mathbb{R}^d}$ denote a group of measurable, bijections $\tau_x:\Omega\to\Omega$ such that*

(i) (group property). $\tau_0=Id$ and $\tau_{x+y}=\tau_x\circ\tau_y$ for all $x,y\in\mathbb{R}^d$,

(ii) (measure preserving). $P(\tau_xA)=P(A)$ for all $A\in\mathcal{F}$ and $x\in\mathbb{R}^d$,

(iii) (measurability). $(\omega,x)\mapsto\tau_x\omega$ is $\mathcal{F}\otimes\mathcal{L}$-measurable ($\mathcal{L}$ denotes the Lebesgue-σ-algebra on $\mathbb{R}^d$).

From now on we assume that $(\Omega,\mathcal{F},P,\tau)$ satisfies these assumptions and we write $\langle\cdot\rangle:=\int_\Omega\cdot\,dP$ as shorthand for the expectation.

In the functional analytic setting, a random coefficient field is described by a map $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with the interpretation that $a(\omega,\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\omega \in \Omega$ sampled according to P yields a realization of the random coefficient field. Likewise, solutions to an associated PDE with physical domain $Q \subset \mathbb{R}^d$ might be considered as *random* functions, i.e. quantities defined on the product $\Omega \times Q$. In this paper we denote by $L^p(\Omega)$ and $L^p(Q)$ (with $Q \subset \mathbb{R}^d$ open) the usual Banach spaces of p-integrable functions defined on $(\Omega, \mathcal{F}, P)$ and Q, respectively. We introduce function spaces for functions defined on $\Omega \times Q$ as follows: For closed subspaces $X \subset L^p(\Omega)$ and $Y \subset L^p(Q)$ (resp. $Y \subset W^{1,p}(Q)$) we denote by $X \otimes Y$ the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^{n} \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in $L^p(\Omega; L^p(Q))$ (resp. $L^p(\Omega; W^{1,p}(Q))$). Since the probability space is countably generated, $L^p(\Omega)$ (with $1 \le p < \infty$) is separable, and thus we have $L^p(\Omega) \otimes L^p(Q) = L^p(\Omega \times Q) = L^p(\Omega; L^p(Q))$ up to isometric isomorphisms. We therefore simply write $L^p(\Omega \times Q)$ instead of $L^p(\Omega) \otimes L^p(Q)$.

In the functional analytic setting and in view of the measure preserving property of τ, the requirement of stationarity can be rephrased as the assumption that the coefficient field can be written in the form $a(\omega, x) = a_0(\tau_x \omega)$ for some measurable map $a_0 : \Omega \to \mathbb{R}^{d\times d}$. The transition from a_0 to a conserves measurability. As usual we denote by $\mathcal{B}(Q)$ (resp. $\mathcal{L}(Q)$) the Borel (resp. Lebesgue)-σ-algebra on $Q \subset \mathbb{R}^d$. The proof of the following lemma is obvious and therefore we do not present it.

Lemma 2.2 (Stationary extension). *Let $\varphi : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then $S\varphi : \Omega \times Q \to \mathbb{R}$, $S\varphi(\omega, x) := \varphi(\tau_x \omega)$ defines a $\mathcal{F} \otimes \mathcal{L}(Q)$-measurable function – called the stationary extension of φ. Moreover, if Q is bounded, for all $1 \le p < \infty$ the map $S : L^p(\Omega) \to L^p(\Omega \times Q)$ is a linear injection satisfying*

$$\|S\varphi\|_{L^p(\Omega\times Q)} = |Q|^{\frac{1}{p}} \|\varphi\|_{L^p(\Omega)}.$$

The assumption of ergodicity can be phrased as follows: We say $(\Omega, \mathcal{F}, P, \tau)$ is *ergodic* (shorter $\langle\cdot\rangle$ is ergodic), if

$$\text{every shift invariant } A \in \mathcal{F} \text{ (i.e. } \tau_x A = A \text{ for all } x \in \mathbb{R}^d\text{) satisfies } P(A) \in \{0, 1\}\,.$$

In this case the celebrated ergodic theorem of Birkhoff applies, which we recall in the following form:

Theorem 2.3 (Birkhoff's ergodic Theorem [25, Theorem 10.2.II]). *Let $\langle\cdot\rangle$ be ergodic and $\varphi : \Omega \to \mathbb{R}$ be integrable. Then for P-a.e. $\omega \in \Omega$ it holds: $S\varphi(\omega, \cdot)$ is locally integrable and for all open, bounded sets $Q \subset \mathbb{R}^d$ we have*

$$\lim_{\varepsilon\to 0} \int_Q S\varphi(\omega, \tfrac{x}{\varepsilon})\, dx = |Q| \langle \varphi \rangle\,. \tag{1}$$

Furthermore, if $\varphi \in L^p(\Omega)$ with $1 \le p \le \infty$, then for P-a.e. $\omega \in \Omega$ it holds: $S\varphi(\omega, \cdot) \in L^p_{loc}(\mathbb{R}^d)$, and provided $p < \infty$ it holds $S\varphi(\omega, \frac{\cdot}{\varepsilon}) \rightharpoonup \langle \varphi \rangle$ weakly in $L^p_{loc}(\mathbb{R}^d)$ as $\varepsilon \to 0$.

Basic examples for stationary and ergodic systems include the random checkerboard (e.g. see [58, Example 2.12]), Gaussian random fields (e.g. see [58, Example 2.13]). We remark that the setting for periodic homogenization fits as well into this framework. In particular, $\Omega = \square$ equipped with the Lebesque-σ-algebra and the Lebesgue measure, and the shift $\tau_x y = y + x \mod 1$ defines a system satisfying Assumption 2.1 and ergodicity.

2.2 Stochastic unfolding operator and two-scale convergence in the mean

In the following we introduce the stochastic unfolding operator, which is a key object in this paper. It is a linear, ε-parametrized, isometric isomorphism $\mathcal{T}_\varepsilon$ on $L^p(\Omega\times Q)$ where $Q\subset\mathbb{R}^d$ denotes an open set which we think of as the domain of a PDE.

Lemma 2.4. *Let $\varepsilon>0$, $1<p<\infty$, $q:=\frac{p}{p-1}$, and $Q\subset\mathbb{R}^d$ be open. There exists a unique linear isometric isomorphism*

$$\mathcal{T}_\varepsilon : L^p(\Omega\times Q)\to L^p(\Omega\times Q)$$

such that

$$\forall u\in L^p(\Omega)\overset{a}{\otimes}L^p(Q)\ :\qquad (\mathcal{T}_\varepsilon u)(\omega,x)=u(\tau_{-\frac{x}{\varepsilon}}\omega,x)\qquad a.e.\ in\ \Omega\times Q.$$

Moreover, its adjoint is the unique linear isometric isomorphism $\mathcal{T}_\varepsilon^*:L^q(\Omega\times Q)\to L^q(\Omega\times Q)$ that satisfies $(\mathcal{T}_\varepsilon^* u)(\omega,x)=u(\tau_{\frac{x}{\varepsilon}}\omega,x)$ a.e. in $\Omega\times Q$ for all $u\in L^q(\Omega)\overset{a}{\otimes}L^q(Q)$.

For the proof see Section 2.4.

Definition 2.5 (Unfolding operator and two-scale convergence in the mean)**.** *The operator $\mathcal{T}_\varepsilon:L^p(\Omega\times Q)\to L^p(\Omega\times Q)$ of Lemma 2.4 is called the stochastic unfolding operator. We say that a sequence $(u_\varepsilon)\subset L^p(\Omega\times Q)$ weakly (strongly) two-scale converges in the mean in $L^p(\Omega\times Q)$ to $u\in L^p(\Omega\times Q)$ if (as $\varepsilon\to 0$)*

$$\mathcal{T}_\varepsilon u_\varepsilon\to u\qquad weakly\ (strongly)\ in\ L^p(\Omega\times Q).$$

In this case we write $u_\varepsilon\overset{2s}{\rightharpoonup}u$ (resp. $u_\varepsilon\overset{2s}{\to}u$) in $L^p(\Omega\times Q)$.

To motivate the definition, let $u_\varepsilon\in H^1_0(Q)$ denote a (distributional) solution to $-\nabla\cdot a_\varepsilon(x)\nabla u_\varepsilon=f$ in Q, where a_ε is a family of uniformly elliptic, random coefficient fields of the form $a_\varepsilon(\omega,x)=a_0(\tau_{\frac{x}{\varepsilon}}\omega)$. The main difficulty in homogenization of this PDE is the passage to the limit $\varepsilon\to 0$ in the product $a_\varepsilon\nabla u_\varepsilon$, since both factors in general only weakly converge. The stochastic unfolding operator $\mathcal{T}_\varepsilon$ turns this expression into a product of a strongly and a weakly convergent sequence in $L^2(\Omega\times Q)$: Indeed, we have $\mathcal{T}_\varepsilon(a_\varepsilon\nabla u_\varepsilon)=a_0(\mathcal{T}_\varepsilon\nabla u_\varepsilon)$ and thus it remains to characterize the limit of $\mathcal{T}_\varepsilon\nabla u_\varepsilon$, as will be done in the next section. Since $\mathcal{T}_\varepsilon$ is an isometry, we obtain the following properties (which resemble the key properties of the periodic unfolding method). The below lemma is obtained using the isometry property of $\mathcal{T}_\varepsilon$ and the usual properties of weak and strong convergence in $L^p(\Omega\times Q)$ and therefore we do not present its proof.

Lemma 2.6 (Basic properties)**.** *Let $p\in(1,\infty)$ and $Q\subset\mathbb{R}^d$ be open. Consider sequences $(u_\varepsilon)\subset L^p(\Omega\times Q)$ and $(v_\varepsilon)\subset L^q(\Omega\times Q)$.*

(i) (Boundedness and lower-semicontinuity of the norm). If $u_\varepsilon\overset{2s}{\rightharpoonup}u$, then $\sup_{\varepsilon\in(0,1)}\|u_\varepsilon\|_{L^p(\Omega\times Q)}<\infty$ and $\|u\|_{L^p(\Omega\times Q)}\le\liminf_{\varepsilon\to0}\|u_\varepsilon\|_{L^p(\Omega\times Q)}$.

(ii) (Compactness of bounded sequences). If $\limsup_{\varepsilon\to0}\|u_\varepsilon\|_{L^p(\Omega\times Q)}<\infty$, then there exists a subsequence ε' and $u\in L^p(\Omega\times Q)$ such that $u_{\varepsilon'}\overset{2s}{\rightharpoonup}u$ in $L^p(\Omega\times Q)$.

(iii) (Characterization of strong two-scale convergence). $u_\varepsilon \xrightarrow{2s} u$ *if and only if* $u_\varepsilon \xrightharpoonup{2s} u$ *in* $L^p(\Omega\times Q)$ *and* $\|u_\varepsilon\|_{L^p(\Omega\times Q)} \to \|u\|_{L^p(\Omega\times Q)}$.

(iv) (Products of strongly and weakly two-scale convergent sequences). If $u_\varepsilon \xrightharpoonup{2s} u$ *in* $L^p(\Omega\times Q)$ *and* $v_\varepsilon \xrightarrow{2s} v$ *in* $L^q(\Omega\times Q)$*, then*

$$\left\langle \int_Q u_\varepsilon(\omega,x)v_\varepsilon(\omega,x)dx \right\rangle \to \left\langle \int_Q u(\omega,x)v(\omega,x)dx \right\rangle.$$

Remark 2.7. The stochastic unfolding operator enjoys many similarities to the periodic unfolding operator, however we would like to point out one considerable difference. Namely, in the periodic case if a sequence $(u_\varepsilon)\subset L^p(Q)$ satisfies $u_\varepsilon \to u$ strongly in $L^p(Q)$, it follows that $\mathcal{T}_\varepsilon^p u_\varepsilon \to u$ strongly in $L^p(Q\times\Box)$ (see e.g. [55, Proposition 2.4]). In the stochastic case, this does not hold in general, specifically even for a fixed function $u\in L^p(\Omega\times Q)$, in general it does not hold $\mathcal{T}_\varepsilon u \rightharpoonup u$. However, if $\langle\cdot\rangle$ is ergodic, using Proposition 2.11 below, it follows that for a sequence $(u_\varepsilon)\subset L^p(\Omega)\otimes W^{1,p}(Q)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega\times Q)$ it holds that $u_\varepsilon \xrightharpoonup{2} \langle u\rangle$. In this respect, stochastic two-scale convergence might be viewed as an ergodic theorem for weakly convergent sequences.

For homogenization of variational problems (in particular, convex integral functionals) the following transformation and (lower semi-)continuity properties are convenient.

Proposition 2.8. *Let* $p\in(1,\infty)$ *and* $Q\subset\mathbb{R}^d$ *be open and bounded. Let* $V:\Omega\times Q\times\mathbb{R}^m\to\mathbb{R}$ *be such that* $V(\cdot,\cdot,F)$ *is* $\mathcal{F}\otimes\mathcal{L}(Q)$*-measurable for all* $F\in\mathbb{R}^m$ *and* $V(\omega,x,\cdot)$ *is continuous for a.e.* $(\omega,x)\in\Omega\times Q$*. Also, we assume that there exists* $C>0$ *such that for a.e.* $(\omega,x)\in\Omega\times Q$

$$|V(\omega,x,F)|\le C(1+|F|^p),\quad \textit{for all } F\in\mathbb{R}^m.$$

(i) We have

$$\forall u\in L^p(\Omega\times Q)^m \quad \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u(\omega,x))dx \right\rangle = \left\langle \int_Q V(\omega,x,\mathcal{T}_\varepsilon u(\omega,x))dx \right\rangle. \quad (2)$$

(ii) If $u_\varepsilon \xrightarrow{2s} u$ *in* $L^p(\Omega\times Q)^m$*, then*

$$\lim_{\varepsilon\to 0}\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u_\varepsilon(\omega,x))dx \right\rangle = \left\langle \int_Q V(\omega,x,u(\omega,x))dx \right\rangle.$$

(iii) We additionally assume that for a.e. $(\omega,x)\in\Omega\times Q$, $V(\omega,x,\cdot)$ *is convex. Then, if* $u_\varepsilon \xrightharpoonup{2s} u$ *in* $L^p(\Omega\times Q)^m$,

$$\liminf_{\varepsilon\to 0}\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u_\varepsilon(\omega,x))dx \right\rangle \ge \left\langle \int_Q V(\omega,x,u(\omega,x))dx \right\rangle.$$

(For the proof see Section 2.4.)

Remark 2.9 (A technical remark about measurability). The stochastic unfolding operator $\mathcal{T}_\varepsilon$ is defined as a linear operator on the Banach space $L^p(\Omega\times Q)$, which is convenient since this prevents us from (fruitless) discussions on measurability properties. The elements of $L^p(\Omega\times Q)$ are strictly speaking not functions but equivalence classes of functions that coincide a.e. in $\Omega\times Q$. Thus, a representative function $\tilde{u}$ in $L^p(\Omega\times Q)$ is measurable w.r.t. the completion of the product σ-algebra $\mathcal{F}\otimes\mathcal{L}(Q)$, and thus the map $(\omega,x)\mapsto\tilde{u}(\tau_x\omega,x)$ might not be measurable. However, if $\tilde{u}$ is $\mathcal{F}\otimes\mathcal{L}(Q)$-measurable (e.g. if $\tilde{u}\in L^p(\Omega)\overset{a}{\otimes}L^p(Q)$), then $\tilde{u}_\varepsilon(\omega,x):=\tilde{u}(\tau_{\frac{x}{\varepsilon}}\omega,x)$ is $\mathcal{F}\otimes\mathcal{L}(Q)$-measurable. In particular, since $L^p(\Omega)\overset{a}{\otimes}L^p(Q)$ is dense in $L^p(\Omega\times Q)$, for any $u\in L^p(\Omega\times Q)$ we can find a representative-$\mathcal{F}\otimes\mathcal{L}(Q)$ measurable function $\tilde{u}:\Omega\times Q\to\mathbb{R}$ and we have $\mathcal{T}_\varepsilon u=\tilde{u}_\varepsilon$ a.e. in $\Omega\times Q$.

Remark 2.10 (Comparison to the notion of [13]). The notion of weak two-scale convergence in the mean of Definition 2.5, i.e. the weak convergence of the unfolded sequence, coincides with the convergence introduced in [13] (see also [5]). More precisely, for a bounded sequence $(u_\varepsilon)\subset L^p(\Omega\times Q)$ we have $u_\varepsilon\overset{2s}{\rightharpoonup}u$ in $L^p(\Omega\times Q)$ (in the sense of Definition 2.5) if and only if u_ε stochastically 2-scale converges in the mean to u in the sense of [13], i.e.

$$\lim_{\varepsilon\to0}\left\langle\int_Q u_\varepsilon(\omega,x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega,x)dx\right\rangle=\left\langle\int_Q u(\omega,x)\varphi(\omega,x)dx\right\rangle,\tag{3}$$

for any $\varphi\in L^q(\Omega\times Q)$ that is admissible (in the sense that the transformation $(\omega,x)\mapsto\varphi(\tau_{\frac{x}{\varepsilon}}\omega,x)$ is well-defined). Indeed, with help of $\mathcal{T}_\varepsilon$ (and its adjoint) we might rephrase the integral on the left-hand side in (3) as

$$\left\langle\int_Q u_\varepsilon(\mathcal{T}_\varepsilon^*\varphi)\,dx\right\rangle=\left\langle\int_Q(\mathcal{T}_\varepsilon u_\varepsilon)\varphi dx\right\rangle,\tag{4}$$

which proves the equivalence.

2.3 Two-scale limits of gradients

As for periodic homogenization via periodic unfolding or two-scale convergence, also in the stochastic case it is important to understand the interplay of the unfolding operator and the gradient operator and to characterize two-scale limits of gradient fields. As a motivation we first recall the periodic case. A standard result states that for any bounded sequence in $W^{1,p}(Q)$ we can extract a subsequence such that u_ε weakly converges in $W^{1,p}(Q)$ to a single scale function $u\in W^{1,p}(Q)$ and ∇u_ε weakly two-scale converges to a two-scale limit of the form $\nabla u(x)+\chi(x,y)$, where χ is a vector field in $L^p(Q)\otimes L^p_{\mathsf{per}}(\Box)$ and $L^p_{\mathsf{per}}(\Box)$ denotes the space of locally p-integrable, $\Box$-periodic functions on $\mathbb{R}^d$, and χ is mean-free and curl-free w.r.t. $y\in\Box=:[0,1)^d$. Since $\Box$ is compact, such vector fields can be represented with help of a periodic potential field, i.e. there exists $\varphi\in L^p(Q,W^{1,p}_{\mathsf{per}}(\Box))$ s.t. $\chi(x,y)=\nabla_y\varphi(x,y)$ for a.e. (x,y). A helpful example to have in mind is the following $u_\varepsilon(x):=\varepsilon\varphi(\frac{x}{\varepsilon})\eta(x)$ with $\eta\in W^{1,p}(Q)$ and $\varphi\in C^\infty_{\mathsf{per}}(\Box)$. Then a direct calculation shows that $\nabla u_\varepsilon(x)=\nabla_y\varphi(\frac{x}{\varepsilon})\eta(x)+O(\varepsilon)$, which obviously two-scale converges to $\nabla_y\varphi(y)\eta(x)$.

In the stochastic case the torus of the periodic case (which is above represented by $\Box$) is replaced by the probability space Ω and periodic functions (e.g. φ above) are conceptually replaced by stationary functions, i.e. functions of the form $S\varphi(\omega,x)=\varphi(\tau_x\omega)$ with $\varphi:\Omega\to\mathbb{R}$ measurable. To proceed further, we need to introduce an analogon of the gradient

∇_y and its domain $W^{1,p}_{\mathsf{per}}(\square)$ in the stochastic setting. As illustrated below, the shift-group τ together with standard concepts from functional analysis lead to a horizontal gradient D and the space $W^{1,p}(\Omega)$. With help of these objects we prove, as in the periodic case, that any bounded sequence in $L^p(\Omega)\otimes W^{1,p}(Q)$ admits (up to extraction of a subsequence) a weak two-scale limit u and the sequence of gradients converges weakly two-scale to a limit of the form $\nabla u + \chi$ where χ is D-curl-free w.r.t. ω. A difference to the periodic case to be pointed out is that χ in general does not admit a representation by means of a stationary potential.

In order to implement the above philosophy we require some input from functional analysis, which we recall from the original work by Papanicolaou and Varadhan [63]. We consider the group of isometric operators $\left\{U_x : x \in \mathbb{R}^d\right\}$ on $L^p(\Omega)$ defined by $U_x\varphi(\omega) = \varphi(\tau_x\omega)$. This group is strongly continuous (see [45, Section 7.1]). For $i = 1, ..., d$, we consider the 1-parameter group of operators $\{U_{he_i} : h \in \mathbb{R}\}$ ($\{e_i\}$ being the usual basis of $\mathbb{R}^d$) and its infinitesimal generator $D_i : \mathcal{D}_i \subset L^p(\Omega) \to L^p(\Omega)$

$$D_i\varphi = \lim_{h\to 0} \frac{U_{he_i}\varphi - \varphi}{h},$$

which we refer to as *horizontal* derivative. D_i is a linear and closed operator and the associated domain $\mathcal{D}_i$ is dense in $L^p(\Omega)$. We set $W^{1,p}(\Omega) = \cap_{i=1}^d \mathcal{D}_i$ and define for $\varphi \in W^{1,p}(\Omega)$ the horizontal gradient as $D\varphi = (D_1\varphi, ..., D_d\varphi)$. In this manner, we obtain a linear, closed and densely defined operator $D : W^{1,p}(\Omega) \to L^p(\Omega)^d$, and we denote by

$$L^p_{\mathsf{pot}}(\Omega) := \overline{\mathcal{R}(D)} \subset L^p(\Omega)^d$$

the closure of the range of D in $L^p(\Omega)^d$. We denote the adjoint of D by $D^* : \mathcal{D}^* \subset L^q(\Omega)^d \to L^q(\Omega)$ which is a linear, closed and densely defined operator ($\mathcal{D}^*$ is the domain of D^*). Note that $W^{1,q}(\Omega)^d \subset \mathcal{D}^*$ and for all $\varphi \in W^{1,p}(\Omega)$ and $\psi \in W^{1,q}(\Omega)$ $(i = 1, ..., d)$ we have the integration by parts formula

$$\langle D_i\varphi\psi\rangle = -\langle \varphi D_i\psi\rangle,$$

and thus $D^*\psi = -\sum_{i=1}^d D_i\psi_i$ for $\psi \in W^{1,q}(\Omega)^d$. We define the subspace of shift invariant functions in $L^p(\Omega)$ by

$$L^p_{\mathsf{inv}}(\Omega) = \left\{\varphi \in L^p(\Omega) : U_x\varphi = \varphi \quad \text{for all } x \in \mathbb{R}^d\right\},$$

and denote by $P_{\mathsf{inv}} : L^p(\Omega) \to L^p_{\mathsf{inv}}(\Omega)$ the conditional expectation with respect to the σ-algebra of shift invariant sets $\left\{A \in \mathcal{F} : \tau_x A = A \text{ for all } x \in \mathbb{R}^d\right\}$. It is a contractive projection and for $p = 2$ it coincides with the orthogonal projection onto $L^2_{\mathsf{inv}}(\Omega)$.

Proposition 2.11 (Compactness). *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open. Let (u_ε) be a bounded sequence in $L^p(\Omega) \otimes W^{1,p}(Q)$. Then, there exist $u \in L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)$ and $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$ such that (up to a subsequence)*

$$u_\varepsilon \overset{2s}{\rightharpoonup} u \text{ in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \overset{2s}{\rightharpoonup} \nabla u + \chi \text{ in } L^p(\Omega \times Q)^d. \tag{5}$$

If, additionally, $\langle\cdot\rangle$ is ergodic, then $u = P_{\mathsf{inv}}u = \langle u\rangle \in W^{1,p}(Q)$ and $\langle u_\varepsilon\rangle \rightharpoonup u$ weakly in $W^{1,p}(Q)$.

We remark that the above result is already established in [13] in the context of two-scale convergence in the mean in the L^2-space setting. We recapitulate its proof from the perspective of stochastic unfolding, see section 2.4.

Remark 2.12. Since closed, convex subsets of a Banach space are also weakly closed, for any sequence (u_ε) that satisfies the assumption of Proposition 2.11 and $\mathcal{T}_\varepsilon u_\varepsilon \in X$ where $X \subset L^2(\Omega \times Q)$ is closed and convex, the two-scale limit from Proposition 2.11 satisfies $u \in X$. This is useful to study problems with boundary conditions.

Lemma 2.13 (Nonlinear recovery sequence). *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open. For $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$ and $\delta > 0$, there exists a sequence $g_{\delta,\varepsilon}(\chi) \in L^p(\Omega) \otimes W^{1,p}(Q)$ such that*

$$\|g_{\delta,\varepsilon}(\chi)\|_{L^p(\Omega\times Q)} \leq \varepsilon C(\delta), \quad \limsup_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}(\chi) - \chi\|_{L^p(\Omega\times Q)^d} \leq \delta.$$

(For the proof see Section 2.4.)

Proposition 2.14 (Linear recovery sequence). *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open, bounded and C^1. For $\varepsilon > 0$ there exists a linear operator $\mathcal{G}_\varepsilon : L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q) \to L^p(\Omega) \otimes W^{1,p}_0(Q)$, that is uniformly bounded in ε, with the property that for any $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$*

$$\mathcal{G}_\varepsilon \chi \overset{2s}{\to} 0 \text{ in } L^p(\Omega\times Q), \quad \nabla \mathcal{G}_\varepsilon \chi \overset{2s}{\to} \chi \text{ in } L^p(\Omega \times Q)^d.$$

(For the proof see Section 2.4.)

Remark 2.15. If $Q \subset \mathbb{R}^d$ is open, bounded and C^1, using Proposition 2.14, we obtain a mapping

$$\big(L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)\big) \times \big(L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)\big) \ni (u,\chi) \mapsto u_\varepsilon(u,\chi) := u + \mathcal{G}_\varepsilon \chi \in L^p(\Omega) \otimes W^{1,p}(Q)$$

which is linear, uniformly bounded in ε and it satisfies (for all (u,χ))

$$u_\varepsilon(u,\chi) \overset{2s}{\to} u \text{ in } L^p(\Omega\times Q), \quad \nabla u_\varepsilon(u,\chi) \overset{2s}{\to} \nabla u + \chi \text{ in } L^p(\Omega\times Q). \tag{6}$$

In the case that Q is merely open, we can use the nonlinear construction from Lemma 2.13. Specifically, for $(u,\chi) \in (L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)) \times \big(L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)\big)$ we define $u_{\delta,\varepsilon}(u,\chi) = u + g_{\delta,\varepsilon}(\chi)$. Using Attouch's diagonal argument, we find a sequence $u_\varepsilon(u,\chi) = u_{\delta(\varepsilon),\varepsilon}$ which satisfies (6). We remark that in both cases, the recovery sequence u_ε matches the boundary conditions of the function u (see constructions in Section 2.4).

We conclude this section with some basic facts from functional analysis used in the proof of Proposition 2.11.

Remark 2.16. Let $1 < p < \infty$ be fixed.

(i) $\langle \cdot \rangle$ is ergodic $\Leftrightarrow L^p_{\mathsf{inv}}(\Omega) \simeq \mathbb{R} \Leftrightarrow P_{\mathsf{inv}} f = \langle f \rangle$.

(ii) The following orthogonality relations hold (for a proof see [15, Section 2.6]): Identify the dual space $L^p(\Omega)^*$ with $L^q(\Omega)$, and define for a set $A \subset L^q(\Omega)$ its orthogonal complement $A^\perp \subset L^p(\Omega)$ as $A^\perp = \{\varphi \in L^p(\Omega) : \langle \varphi\psi \rangle = 0 \text{ for all } \psi \in A\}$. Then

$$\mathcal{N}(D) = \mathcal{R}(D^*)^\perp, \quad L^p_{\mathsf{pot}}(\Omega) = \overline{\mathcal{R}(D)} = \mathcal{N}(D^*)^\perp. \tag{7}$$

Above, $\mathcal{N}(\cdot)$ denotes the kernel and $\mathcal{R}(\cdot)$ the range of an operator.

2.4 Proofs

Proof. We first define $\mathcal{T}_\varepsilon$ on $\mathscr{A} := \{\psi(\omega,x) = \varphi(\omega)\eta(x) \;:\; \varphi \in L^p(\Omega),\, \eta \in L^p(Q)\} \subset L^p(\Omega\times Q)$ by setting $(\mathcal{T}_\varepsilon\psi)(\omega,x) = (S\varphi)(\omega,-\frac{x}{\varepsilon})\eta(x)$ for all $\psi = \varphi\eta \in \mathscr{A}$. In view of Lemma 2.2 $(\mathcal{T}_\varepsilon\psi)$ is $\mathcal{F}\otimes\mathcal{L}(Q)$-measurable, and

$$\left\langle \int_Q |\mathcal{T}_\varepsilon\psi|^p\,dx \right\rangle = \int_Q \Big(\int_\Omega |S\varphi(\omega,-\tfrac{x}{\varepsilon})|^p\,dP(\omega)\Big)|\eta(x)|^p\,dx = \|\varphi\|^p_{L^p(\Omega)}\|\eta\|^p_{L^p(Q)} = \|\psi\|^p_{L^p(\Omega\times Q)}.$$

Since $\operatorname{span}(\mathscr{A})$ is dense in $L^p(\Omega\times Q)$, $\mathcal{T}_\varepsilon$ extends to a linear isometry from $L^p(\Omega\times Q)$ to $L^p(\Omega\times Q)$. We define a linear isometry $\mathcal{T}_{-\varepsilon} : L^q(\Omega\times Q)\to L^q(\Omega\times Q)$ analogously as $\mathcal{T}_\varepsilon$ with ε replaced by $-\varepsilon$. Then for any $\varphi \in L^p(\Omega)\overset{a}{\otimes} L^p(Q)$ and $\psi \in L^q(\Omega)\overset{a}{\otimes} L^q(Q)$ we have (thanks to the measure preserving property of τ):

$$\begin{aligned}\left\langle \int_Q (\mathcal{T}_\varepsilon\varphi)\psi\,dx \right\rangle &= \int_Q\int_\Omega \varphi(\tau_{-\frac{x}{\varepsilon}}\omega,x)\psi(\omega,x)\,dP(\omega)\,dx \\ &= \int_Q\int_\Omega \varphi(\omega,x)\psi(\tau_{\frac{x}{\varepsilon}}\omega,x)\,dP(\omega)\,dx = \left\langle \int_Q \varphi(\mathcal{T}_{-\varepsilon}\psi)\right\rangle dx.\end{aligned}$$

Since these functions are dense in $L^p(\Omega\otimes Q)$ and $L^q(\Omega\otimes Q)$, respectively, we conclude that $\mathcal{T}^*_\varepsilon = \mathcal{T}_{-\varepsilon}$.

It remains to argue that $\mathcal{T}_\varepsilon$ and $\mathcal{T}^*_\varepsilon$ are surjective. Since $\mathcal{T}^*_\varepsilon$ is an isometry, it follows that $\mathcal{T}_\varepsilon$ is surjective (see [15, Theorem 2.20]). Analogously, $\mathcal{T}^*_\varepsilon$ is as well surjective. □

Proof of Proposition 2.8. We first note that V is a *Charathéodory integrand* (which is defined as a function satisfying the measurability and continuity assumptions given in the statement of the proposition) and therefore it follows that V is $\mathcal{F}\otimes\mathcal{L}(Q)\otimes\mathcal{B}(\mathbb{R}^{d\times d})$-measurable. For fixed $\varepsilon>0$, the mapping $(\omega,x)\mapsto(\tau_{\frac{x}{\varepsilon}}\omega,x)$ is $\mathcal{F}\otimes\mathcal{L}(Q)$-$\mathcal{F}\otimes\mathcal{L}(Q)$-measurable and therefore $(\omega,x,F)\mapsto V(\tau_{\frac{x}{\varepsilon}}\omega,x,F)$ defines as well a Charathéodory integrand (with same measurability as V). As a result of these facts, for any function $u\in L^p(\Omega\times Q)^m$ it follows that $(\omega,x)\mapsto V(\omega,x,u(\omega,x))$ and $(\omega,x)\mapsto V(\tau_{\frac{x}{\varepsilon}}\omega,x,u(\omega,x))$ define measurable functions with respect to the completion of $\mathcal{F}\otimes\mathcal{L}(Q)$. Additionally, these functions are integrable thanks to the growth assumptions on V. Thus all the integrals in the statement of the proposition are well-defined.

(i) We first argue that it suffices to prove that

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u(\omega,x))dx \right\rangle = \left\langle \int_Q V(\omega,x,\mathcal{T}_\varepsilon u(\omega,x))dx \right\rangle \quad \text{for all } u\in L^p(\Omega)\overset{a}{\otimes} L^p(Q)^m. \tag{8}$$

Indeed, for any $u\in L^p(\Omega\times Q)^m$ we can find a sequence $u_k\in L^p(\Omega)\overset{a}{\otimes} L^p(Q)^m$ such that $u_k\to u$ strongly in $L^p(\Omega\times Q)^m$, and by passing to a subsequence (not relabeled) we may additionally assume that $u_k\to u$ pointwise a.e. in $\Omega\times Q$. By continuity of V in its last variable, we thus have $V(\tau_{\frac{x}{\varepsilon}}\omega,x,u_k(\omega,x))\to V(\tau_{\frac{x}{\varepsilon}}\omega,x,u(\omega,x))$ for a.e. $(\omega,x)\in\Omega\times Q$. Since $|V(\tau_{\frac{x}{\varepsilon}}\omega,x,u_k(\omega,x))|\le C(1+|u_k(\omega,x)|^p)$ a.e. in $\Omega\times Q$, the dominated convergence theorem by Vital implies that $\lim_{k\to\infty}\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u_k(\omega,x))dx\right\rangle = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,u(\omega,x))dx\right\rangle$. In the same way we conclude that

$$\lim_{k\to\infty}\left\langle \int_Q V(\omega,x,\mathcal{T}_\varepsilon u_k(\omega,x))dx\right\rangle = \left\langle \int_Q V(\omega,x,\mathcal{T}_\varepsilon u(\omega,x))dx\right\rangle,$$

and thus (8) extends to general $u \in L^p(\Omega \times Q)^m$.

It is left to show (8). Let $u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)^m$. By Fubini's theorem, the measure preserving property of τ, and the transformation $\omega \mapsto \tau_{-\frac{x}{\varepsilon}}\omega$ in the second equality below, it follows

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dx \right\rangle = \int_Q \left\langle V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x)) \right\rangle dx = \int_Q \left\langle V(\omega, x, u(\tau_{-\frac{x}{\varepsilon}}\omega, x)) \right\rangle dx.$$

Since $u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)$, we have $u(\tau_{-\frac{x}{\varepsilon}}\omega, x) = \mathcal{T}_\varepsilon u(\omega, x)$, and thus the right-hand side equals $\left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u(\omega, x))dx \right\rangle$, which completes the proof of (i).

(ii) By part (i) we get $\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x))dx \right\rangle$. Since $\mathcal{T}_\varepsilon u_\varepsilon \to u$ strongly in $L^p(\Omega \times Q)^m$ (by assumption), using the growth conditions of V and the dominated convergence theorem, it follows (similarly as in the proof of part (i)) that $\lim_{\varepsilon\to 0} \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, u(\omega, x))dx \right\rangle$.

(iii) We note that the functional $L^p(\Omega \times Q)^m \ni u \mapsto \left\langle \int_Q V(\omega, x, u(\omega, x))dx \right\rangle$ is convex and lower semi-continuous, therefore it is weakly lower semi-continuous (see [15, Corollary 3.9]). Combining this fact with the transformation formula from (i) and the weak convergence $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u$ (by assumption), the claim follows. □

Before stating the proof of Proposition 2.11, we present some auxiliary lemmas.

Lemma 2.17. *Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$.*

(i) If $\varphi \in \left\{D^*\psi : \psi \in W^{1,q}(\Omega)^d\right\}^\perp$, then $\varphi \in L^p_{\mathsf{inv}}(\Omega)$.

(ii) If $\varphi \in \left\{\psi \in W^{1,q}(\Omega)^d : D^*\psi = 0\right\}^\perp$, then $\varphi \in L^p_{\mathsf{pot}}(\Omega)$.

Proof. (i) First, we note that

$$\varphi \in L^p_{\mathsf{inv}}(\Omega) \quad \Leftrightarrow \quad U_{he_i} U_y \varphi = U_y \varphi \quad \text{for all } y \in \mathbb{R}^d, h \geq 0, i = 1, ..., d.$$

We consider $\varphi \in \left\{D^*\psi : \psi \in W^{1,q}(\Omega)^d\right\}^\perp$ and we show that $\varphi \in L^p_{\mathsf{inv}}(\Omega)$ using the above equivalence. Let $\psi \in W^{1,q}(\Omega)$ and $i \in \{1, ..., d\}$. Then, by the group property we have $U_{-he_i}\psi - \psi = \int_0^t U_{-te_i} D_i^* \psi dt$ and therefore

$$\langle (U_{he_i}\varphi - \varphi)\psi \rangle = \langle \varphi(U_{-he_i}\psi - \psi) \rangle = \left\langle \varphi \int_0^t U_{-te_i} D_i^* \psi dt \right\rangle = \int_0^h \langle \varphi D_i^*(U_{-te_i}\psi) \rangle \, dt.$$

Since $U_{-te_i}\psi \in W^{1,q}(\Omega)$ for any $t \in [0, h]$, we obtain $\langle \varphi D_i^*(U_{-te_i}\psi) \rangle = 0$ and thus $U_{he_i}\varphi = \varphi$. Furthermore, for any $y \in \mathbb{R}^d$, we have $\langle (U_{he_i} U_y \varphi - U_y \varphi)\psi \rangle = \langle (U_{he_i}\varphi - \varphi) U_{-y}\psi \rangle = 0$ by the same argument.

(ii) In view of $L^p_{\mathsf{pot}}(\Omega) = \mathcal{N}(D^*)^\perp$ (see (7)), it is sufficient to prove density of the set $\left\{\varphi \in W^{1,q}(\Omega)^d : D^*\varphi = 0\right\}$ in $\mathcal{N}(D^*)$. This follows by an approximation argument as in [45], Section 7.2. Let $\varphi \in \mathcal{N}(D^*)$ and we define for $t > 0$

$$\varphi^t(\omega) = \int_{\mathbb{R}^d} p_t(y)\varphi(\tau_y \omega)dy, \quad \text{where } p_t(y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4t}}.$$

Then the claimed density follows, since $\varphi^t \in W^{1,q}(\Omega)^d$, $D^*\varphi^t = 0$ for any $t > 0$ and $\varphi^t \to \varphi$ strongly in $L^q(\Omega)^d$. The last statement can be seen as follows. By the continuity property of U_x, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\langle |\varphi(\tau_y\omega) - \varphi(\omega)|^q\rangle \leq \varepsilon$ for any $y \in B_\delta(0)$. It follows that

$$\begin{aligned}\left\langle|\varphi^t - \varphi|^q\right\rangle &= \left\langle\left|\int_{\mathbb{R}^d} p_t(y)\left(\varphi(\tau_y\omega) - \varphi(\omega)\right) dy\right|^q\right\rangle \\ &\leq \int_{\mathbb{R}^d} p_t(y)\left\langle|\varphi(\tau_y\omega) - \varphi(\omega)|^q\right\rangle dy \\ &= \int_{B_\delta} p_t(y)\left\langle|\varphi(\tau_y\omega) - \varphi(\omega)|^q\right\rangle dy + \int_{\mathbb{R}^d\setminus B_\delta} p_t(y)\left\langle|\varphi(\tau_y\omega) - \varphi(\omega)|^q\right\rangle dy.\end{aligned}$$

The first term on the right-hand side of the above inequality is bounded by ε as well as the second term for sufficiently small $t > 0$. □

Lemma 2.18. *Let $u_\varepsilon \in L^p(\Omega)\otimes W^{1,p}(Q)$ be such that $u_\varepsilon \xrightharpoonup{2s} u$ in $L^p(\Omega\times Q)$ and $\varepsilon\nabla u_\varepsilon \xrightharpoonup{2s} 0$ in $L^p(\Omega\times Q)^d$. Then $u \in L^p_{\mathsf{inv}}(\Omega)\otimes L^p(Q)$.*

Proof. Consider a sequence $v_\varepsilon = \varepsilon\mathcal{T}_\varepsilon^*(\varphi\eta)$ such that $\varphi \in W^{1,q}(\Omega)$ and $\eta \in C_c^\infty(Q)$. Note that $\mathcal{T}_\varepsilon v_\varepsilon = \varepsilon\varphi\eta$ and we have ($i = 1, ..., d$)

$$\left\langle\int_Q \partial_i u_\varepsilon v_\varepsilon dx\right\rangle = \left\langle\int_Q \mathcal{T}_\varepsilon\partial_i u_\varepsilon \mathcal{T}_\varepsilon v_\varepsilon dx\right\rangle = \left\langle\int_Q \mathcal{T}_\varepsilon\partial_i u_\varepsilon \varepsilon\varphi\eta dx\right\rangle \to 0.$$

Moreover, it holds that $\partial_i v_\varepsilon = \mathcal{T}_\varepsilon^*(D_i\varphi\eta + \varepsilon\varphi\partial_i\eta)$ and therefore

$$\begin{aligned}\left\langle\int_Q \partial_i u_\varepsilon v_\varepsilon dx\right\rangle &= -\left\langle\int_Q u_\varepsilon\partial_i v_\varepsilon dx\right\rangle = -\left\langle\int_Q u_\varepsilon\mathcal{T}_\varepsilon^*(D_i\varphi\eta + \varepsilon\varphi\partial_i\eta)dx\right\rangle \\ &= -\left\langle\int_Q \mathcal{T}_\varepsilon u_\varepsilon D_i\varphi\eta + \varepsilon\mathcal{T}_\varepsilon u_\varepsilon\varphi\partial_i\eta dx\right\rangle.\end{aligned}$$

The last expression converges to $-\left\langle\int_Q uD_i\varphi\eta dx\right\rangle$ as $\varepsilon \to 0$. As a result of this, $\langle u(x)D_i\varphi\rangle = 0$ for almost every $x \in Q$ and therefore $u \in L^p_{\mathsf{inv}}(\Omega)\otimes L^p(Q)$ by Lemma 2.17 (i). □

Lemma 2.19. *Let u_ε be a bounded sequence in $L^p(\Omega)\otimes W^{1,p}(Q)$. Then there exists $u \in L^p_{inv}(\Omega)\otimes W^{1,p}(Q)$ such that*

$$u_\varepsilon \xrightharpoonup{2s} u \text{ in } L^p(\Omega\times Q), \quad P_{inv}u_\varepsilon \xrightharpoonup{2s} u \text{ in } L^p(\Omega\times Q), \quad P_{inv}\nabla u_\varepsilon \xrightharpoonup{2s} \nabla u \text{ in } L^p(\Omega\times Q)^d.$$

Proof. Step 1. $P_{inv}\circ\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon\circ P_{inv} = P_{inv}$.

The second equality holds clearly. To show that $P_{\mathsf{inv}}\circ\mathcal{T}_\varepsilon = P_{\mathsf{inv}}$, we consider $v \in L^p(\Omega\times Q)$, $\varphi \in L^q(\Omega)$ and $\eta \in L^q(Q)$. We have

$$\begin{aligned}\left\langle\int_Q (P_{\mathsf{inv}}\mathcal{T}_\varepsilon v)(\varphi\eta)dx\right\rangle &= \left\langle\int_Q (\mathcal{T}_\varepsilon v)P_{\mathsf{inv}}^*(\varphi\eta)dx\right\rangle \\ &= \left\langle\int_Q vP_{\mathsf{inv}}^*(\varphi\eta)dx\right\rangle = \left\langle\int_Q (P_{\mathsf{inv}}v)(\varphi\eta)dx\right\rangle,\end{aligned}$$

where we use the fact that $\mathcal{T}_\varepsilon^* P_{\mathsf{inv}}^* = P_{\mathsf{inv}}^*$ since the adjoint P_{inv}^* of P_{inv} satisfies $\mathcal{R}(P_{\mathsf{inv}}^*) \subset L^q_{\mathsf{inv}}(\Omega)$. The claim follows by an approximation argument since $L^q(\Omega) \overset{a}{\otimes} L^q(Q)$ is dense in $L^q(\Omega \times Q)$.

Step 2. Convergence of $P_{\mathsf{inv}} u_\varepsilon$.

P_{inv} is bounded and it commutes with ∇ and therefore

$$\limsup_{\varepsilon\to 0} \left\langle \int_Q |P_{\mathsf{inv}} u_\varepsilon|^p + |\nabla P_{\mathsf{inv}} u_\varepsilon|^p dx \right\rangle \leq \infty.$$

As a result of this and with help of 2.6 (ii) and Lemma 2.18, it follows that $P_{\mathsf{inv}} u_\varepsilon \overset{2s}{\rightharpoonup} v$ and $\nabla P_{\mathsf{inv}} u_\varepsilon \overset{2s}{\rightharpoonup} w$ (up to a subsequence), where $v \in L^p_{\mathsf{inv}}(\Omega) \otimes L^p(Q)$ and $w \in L^p_{\mathsf{inv}}(\Omega) \otimes L^p(Q)^d$. Let $\varphi \in W^{1,q}(\Omega)$ and $\eta \in C_c^\infty(Q)$. On the one hand, we have

$$\left\langle \int_Q (\partial_i P_{\mathsf{inv}} u_\varepsilon) \mathcal{T}_\varepsilon^*(\varphi\eta) dx \right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon(\partial_i P_{\mathsf{inv}} u_\varepsilon)(\varphi\eta) dx \right\rangle \to \left\langle \int_Q w_i \varphi\eta dx \right\rangle.$$

On the other hand,

$$\left\langle \int_Q (\partial_i P_{\mathsf{inv}} u_\varepsilon) \mathcal{T}_\varepsilon^*(\varphi\eta) dx \right\rangle = -\frac{1}{\varepsilon} \left\langle \int_Q (P_{\mathsf{inv}} u_\varepsilon)(D_i\varphi\eta) dx \right\rangle - \left\langle \int_Q (P_{\mathsf{inv}} u_\varepsilon)(\varphi\partial_i\eta) dx \right\rangle.$$

The first term on the right-hand side vanishes since $P_{\mathsf{inv}} u_\varepsilon(\cdot, x) \in L^p_{\mathsf{inv}}(\Omega)$ for almost every $x \in Q$ and by (7). The second term converges to $-\left\langle \int_Q v\varphi\partial_i\eta dx \right\rangle$ as $\varepsilon \to 0$. Consequently, we obtain $w = \nabla v$ and therefore $v \in L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)$.

Step 3. Convergence of u_ε.

Since u_ε is bounded, by Lemma 2.18 there exists $u \in L^p_{\mathsf{inv}}(\Omega) \otimes L^p(Q)$ such that $u_\varepsilon \overset{2s}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$. Also, P_{inv} is a linear and bounded operator which, together with Step 1, implies that $P_{\mathsf{inv}} u_\varepsilon \rightharpoonup u$. Using this, we conclude that $u = v$. □

Proof of Proposition 2.11. Lemma 2.19 implies that $u_\varepsilon \overset{2s}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$ (up to a subsequence), where $u \in L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)$. Moreover, it follows that there exists $v \in L^p(\Omega \times Q)^d$ such that $\nabla u_\varepsilon \overset{2s}{\rightharpoonup} v$ in $L^p(\Omega \times Q)^d$ (up to another subsequence). We show that $\chi := v - \nabla u \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$.

Let $\varphi \in W^{1,q}(\Omega)^d$ with $D^*\varphi = 0$ and $\eta \in C_c^\infty(Q)$. We have

$$\left\langle \int_Q \nabla u_\varepsilon \cdot \mathcal{T}_\varepsilon^*(\varphi\eta) dx \right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon \nabla u_\varepsilon \cdot \varphi\eta dx \right\rangle \to \left\langle \int_Q v \cdot \varphi\eta dx \right\rangle. \tag{9}$$

On the other hand,

$$\begin{aligned}\left\langle \int_Q \nabla u_\varepsilon \cdot \mathcal{T}_\varepsilon^*(\varphi\eta) dx \right\rangle &= -\left\langle \int_Q u_\varepsilon \sum_{i=1}^d \mathcal{T}_\varepsilon^*(\frac{1}{\varepsilon} D_i\varphi\eta + \varphi_i\partial_i\eta) dx \right\rangle \\ &= \left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon)(D^*\varphi\eta) dx \right\rangle - \left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon) \sum_{i=1}^d \varphi_i\partial_i\eta dx \right\rangle.\end{aligned} \tag{10}$$

Above, the first term on the right-hand side vanishes by assumption and the second converges to $\left\langle \int_Q \nabla u \cdot \varphi\eta \right\rangle$ as $\varepsilon \to 0$. Using (10), (9) and Lemma 2.17 (ii) we complete the proof. □

Proof of Lemma 2.13. For $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$ and $\delta > 0$, by definition of the space $L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$ and by density of $\mathcal{R}(D)$ in $L^p_{\mathsf{pot}}(\Omega)$, we find $g_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta_i^\delta$ with $\varphi_i^\delta \in W^{1,p}(\Omega)$ and $\eta_i^\delta \in C_c^\infty(Q)$ such that

$$\|\chi - Dg_\delta\|_{L^p(\Omega\times Q)^d} \leq \delta.$$

We define $g_{\delta,\varepsilon} = \varepsilon \mathcal{T}_\varepsilon^{-1} g_\delta$ and note that $g_{\delta,\varepsilon} \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ and $\nabla g_{\delta,\varepsilon} = \mathcal{T}_\varepsilon^{-1} D g_\delta + \mathcal{T}_\varepsilon^{-1} \varepsilon \nabla g_\delta$. As a result of this and with help of the isometry property of $\mathcal{T}_\varepsilon^{-1}$, the claim of the lemma follows. □

Proof of Proposition 2.14. For $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)$ we define $\mathcal{G}_\varepsilon \chi = v_\varepsilon$ as the unique weak solution in $W_0^{1,p}(Q)$ to the equation (for P-a.e. $\omega \in \Omega$)

$$-\Delta v_\varepsilon(\omega) = -\nabla \cdot (\mathcal{T}_\varepsilon^{-1} \chi(\omega)). \tag{11}$$

Above and further in this proof, we use the notation $u(\omega) := u(\cdot, \omega) \in L^p(Q)$ for functions $u \in L^p(\Omega \times Q)$.

By Poincaré's inequality and the Calderon-Zygmund estimate, we obtain

$$\|v_\varepsilon(\omega)\|_{L^p(Q)} \leq C\|\nabla v_\varepsilon(\omega)\|_{L^p(Q)^d} \leq C\|\mathcal{T}_\varepsilon^{-1}\chi(\omega)\|_{L^p(Q)^d},$$

and therefore

$$\|v_\varepsilon\|_{L^p(\Omega\times Q)} \leq C\|\nabla v_\varepsilon\|_{L^p(\Omega\times Q)^d} \leq C\|\chi\|_{L^p(\Omega\times Q)^d}.$$

Using Lemma 2.13, we find a sequence $g_{\delta,\varepsilon} \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ such that

$$\|g_{\delta,\varepsilon}(\chi)\|_{L^p(\Omega\times Q)} \leq \varepsilon C(\delta), \quad \limsup_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}(\chi) - \chi\|_{L^p(\Omega\times Q)^d} \leq \delta.$$

Note that $v_\varepsilon(\omega) - g_{\delta,\varepsilon}(\omega) \in W_0^{1,p}(Q)$ (for P-a.e. $\omega \in \Omega$) and it is the unique weak solution to

$$-\Delta(v_\varepsilon(\omega) - g_{\delta,\varepsilon}(\omega)) = -\nabla \cdot (\mathcal{T}_\varepsilon^{-1}\chi(\omega) - \nabla g_{\delta,\varepsilon}(\omega)).$$

As before, we have

$$\|v_\varepsilon - g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)} \leq C\|\nabla v_\varepsilon - \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d} \leq C\|\chi - \mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d}. \tag{12}$$

Therefore, using the isometry property of $\mathcal{T}_\varepsilon$, we obtain

$$\begin{aligned}\|\mathcal{T}_\varepsilon \nabla v_\varepsilon - \chi\|_{L^p(\Omega\times Q)^d} &\leq \|\nabla v_\varepsilon - \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d} + \|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon} - \chi\|_{L^p(\Omega\times Q)^d} \\ &\leq C\|\chi - \mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d}.\end{aligned}$$

Consequently, first letting $\varepsilon \to 0$ and then $\delta \to 0$ we obtain that $\nabla v_\varepsilon \overset{2s}{\to} \chi$ in $L^p(\Omega \times Q)^d$. Furthermore, using (12) we obtain that $v_\varepsilon \overset{2s}{\to} 0$ in $L^p(\Omega\times Q)$ which completes the proof. □

3 Applications to homogenization in the mean

In this section we apply the stochastic unfolding method to homogenization problems. We discuss the classical homogenization problem of convex integral functionals and derive a homogenization result for an evolutionary gradient system. We refer to [61] where a similar analysis has been conducted in a discrete-to-continuum setting for convex integral functionals and for an evolutionary rate-independent system.

The treatment of integral functionals is a well-known topic in stochastic homogenization and previous results typically rely on the subadditive ergodic theorem (see e.g. [24, 60]) or on the notion of quenched stochastic two-scale convergence (see [39] and Section 4). The analysis via unfolding is less involved than these methods since it merely relies on lower semi-continuity of convex functionals and weak compactness properties of „unfolded" sequences of functions in $L^p(\Omega \times Q)$. On the other hand, the method we present yields weaker results than other procedures, namely convergence for solutions is obtained in a statistically averaged sense (see Proposition 3.5), whereas the analysis based on the subadditive ergodic theorem (e.g. [60]) yields convergence for every typical realization of the medium and it even allows to consider non-convex functionals. We refer to a recent study [11] for an investigation of homogenization of non-convex integral functionals by a two-scale Γ-convergence approach.

The second part of this section is dedicated to the analysis of an evolutionary problem, a gradient system which corresponds to an Allen-Cahn type equation. A significant number of mathematical models can be phrased in the setting of evolutionary gradient systems which are formulated variationally, with the help of an energy and a dissipation functional (see Section 3.2 for a specific example). We refer to [4, 53] for the abstract theory of gradient systems. Typically, the asymptotic analysis of sequences of gradient systems (so called evolutionary Γ-convergence [52]) relies merely on Γ-convergence properties of the underlying two functionals. For various general strategies for such problems we refer to [65, 26, 53, 52]. In [47] a gradient system driven by a non-convex (Cahn-Hilliard type) energy is considered and a periodic homogenization result is established using periodic unfolding. In this study, we consider a related random model and derive a homogenization result based on the stochastic unfolding procedure (see Section 3.2).

3.1 Convex integral functionals

Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ be open, bounded and Lipschitz. We consider $V : \Omega \times Q \times \mathbb{R}^{d\times d} \to \mathbb{R}$ and the following set of assumptions.

(A1) $V(\cdot, \cdot, F)$ is measurable w.r.t. the product σ-algebra $\mathcal{F} \otimes \mathcal{L}(Q)$ for all $F \in \mathbb{R}^{d\times d}$.

(A2) $V(\omega, x, \cdot)$ is convex for a.e. $(\omega, x) \in \Omega \times Q$.

(A3) There exists a $C > 0$ such that

$$\frac{1}{C}|F|^p - C \leq V(\omega, x, F) \leq C(|F|^p + 1)$$

for a.e. $(\omega, x) \in \Omega \times Q$ and every $F \in \mathbb{R}^{d\times d}$.

(A4) There exists $b : \mathbb{R} \to \mathbb{R}$ positive, continuous and with $b(0) = 0$ such that

$$|V(\omega, x_1, F_1) - V(\omega, x_2, F_2)| \leq b(|x_1 - x_2|)$$

for P-a.e. $\omega \in \Omega$ and every $x_1, x_2 \in Q$, $F_1, F_2 \in \mathbb{R}^{d\times d}$.

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(A5) $V \geq 0$ and it holds that for a.e. $(\omega, x) \in \Omega \times Q$, $V(\omega, x, \cdot)$ is uniformly convex with modulus $(\cdot)^p$, i.e. there exists $C > 0$ (independent of (ω, x)) such that for all $F, G \in \mathbb{R}^{d\times d}$ and $t \in [0,1]$

$$V(\omega, x, tF + (1-t)G) \leq tV(\omega, x, F) + (1-t)V(\omega, x, G) - (1-t)tC|F-G|^p.$$

Below we use the shorthand notation $\nabla^s u = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ and $\chi^s = \frac{1}{2}\left(\chi + \chi^T\right)$. We consider problems with homogeneous Dirichlet boundary conditions and energy functional

$$\mathcal{E}_\varepsilon : L^p(\Omega) \otimes W_0^{1,p}(Q)^d \to \mathbb{R}, \quad \mathcal{E}_\varepsilon(u) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla^s u(\omega, x))dx \right\rangle. \tag{13}$$

Under the assumptions $(A1)-(A3)$, in the limit $\varepsilon \to 0$ we obtain the following functional

$$\begin{aligned} &\mathcal{E}_0 : \left(L^p_{\mathsf{inv}}(\Omega) \otimes W_0^{1,p}(Q)^d\right) \times \left(L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d\right), \\ &\mathcal{E}_0(u, \chi) = \left\langle \int_Q V(\omega, x, \nabla^s u(\omega, x) + \chi^s(\omega, x))dx \right\rangle. \end{aligned} \tag{14}$$

Theorem 3.1 (Two-scale homogenization)**.** *Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ open, bounded and Lipschitz. Assume $(A1)-(A3)$.*

(i) (Compactness) Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)^d$ be such that $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. There exist $(u, \chi) \in \left(L^p_{inv}(\Omega) \otimes W_0^{1,p}(Q)^d\right) \times \left(L^p_{pot}(\Omega) \otimes L^p(Q)^d\right)$ and a subsequence (not relabeled) such that

$$u_\varepsilon \overset{2s}{\rightharpoonup} u \text{ in } L^p(\Omega \times Q)^d, \quad \nabla u_\varepsilon \overset{2s}{\rightharpoonup} \nabla u + \chi \text{ in } L^p(\Omega \times Q)^{d\times d}. \tag{15}$$

(ii) (Liminf inequality) If the above convergence holds for the whole sequence, then

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u, \chi).$$

(iii) (Limsup inequality) Let $(u, \chi) \in \left(L^p_{inv}(\Omega) \otimes W_0^{1,p}(Q)^d\right) \times \left(L^p_{pot}(\Omega) \otimes L^p(Q)^d\right)$. There exists a sequence $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)^d$ such that

$$u_\varepsilon \overset{2s}{\to} u \text{ in } L^p(\Omega \times Q)^d, \quad \nabla u_\varepsilon \overset{2s}{\to} \nabla u + \chi \text{ in } L^p(\Omega \times Q)^{d\times d}, \quad \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi).$$

(For the proof see Section 3.3.)

Corollary 3.2. *Assume the same assumptions as in Theorem 3.1. Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)^d$ be a minimizer of $\mathcal{E}_\varepsilon$. Then there exists a subsequence (not relabeled), $u \in L^p_{inv}(\Omega) \times W_0^{1,p}(Q)^d$, and $\chi \in L^p_{pot}(\Omega) \otimes L^p(Q)^d$ such that $u_\varepsilon \overset{2s}{\rightharpoonup} u$ in $L^p(\Omega \times Q)^d$, $\nabla u_\varepsilon \overset{2s}{\rightharpoonup} \nabla u + \chi$ in $L^p(\Omega \times Q)^{d\times d}$, and*

$$\lim_{\varepsilon\to 0} \min \mathcal{E}_\varepsilon = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0.$$

(For the proof see Section 3.3.)

Remark 3.3. If $V(\omega, x, \cdot)$ is strictly convex the minimizers are unique and the convergence in the above corollary holds for the entire sequence.

Remark 3.4. We might consider the perturbed energy functional $\mathcal{I}_\varepsilon(\cdot) = \mathcal{E}_\varepsilon(\cdot) + \langle l_\varepsilon, \cdot\rangle_{(L^p)^*, L^p}$ with $l_\varepsilon \overset{2}{\to} l$ in $L^q(\Omega \times Q)$. As in Corollary 3.2, minimizers of $\mathcal{I}_\varepsilon$ converge in the above two-scale sense (up to a subsequence) to minimizers of $\mathcal{I}_0(\cdot) = \mathcal{E}_0(\cdot) + \langle P_{\mathsf{inv}} l, \cdot\rangle_{(L^p)^*, L^p}$.

If we additionally assume that $\langle\cdot\rangle$ is ergodic and $(A4)$, the limit functional reduces to a single-scale energy

$$\mathcal{E}_{\text{hom}} : W_0^{1,p}(Q)^d \to \mathbb{R}, \quad \mathcal{E}_{\text{hom}}(u) = \int_Q V_{\text{hom}}(x, \nabla u(x))dx,$$

where the homogenized integrand V_{hom} is given for $x \in \mathbb{R}^d$ and $F \in \mathbb{R}^{d\times d}$ by

$$V_{\text{hom}}(x, F) = \inf_{\chi \in L^p_{\mathsf{pot}}(\Omega)^d} \langle V(\omega, x, F^s + \chi^s(\omega))\rangle . \tag{16}$$

Theorem 3.5 (Ergodic case). *Assume $(A1) - (A4)$ and $\langle\cdot\rangle$ is ergodic.*

(i) Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)^d$ be such that $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. There exist $u \in W_0^{1,p}(Q)^d$ and a subsequence (not relabeled) such that

$$\begin{gathered} u_\varepsilon \overset{2s}{\rightharpoonup} u \text{ in } L^p(\Omega \times Q)^d, \quad \langle u_\varepsilon\rangle \to u \text{ strongly in } L^p(Q)^d, \\ \langle \nabla u_\varepsilon\rangle \rightharpoonup \nabla u \text{ weakly in } L^p(Q)^{d\times d}. \end{gathered} \tag{17}$$

Moreover,

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_{\text{hom}}(u).$$

(ii) Let $u \in W_0^{1,p}(Q)^d$. There exists a sequence $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)^d$ such that

$$u_\varepsilon \overset{2s}{\to} u \text{ in } L^p(\Omega \times Q)^d, \quad \langle \nabla u_\varepsilon\rangle \to \nabla u \text{ strongly in } L^p(Q)^{d\times d}, \quad \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_{\text{hom}}(u).$$

(For the proof see Section 3.3.)

We consider problems with an additional strong convexity assumption and consequently obtain that the whole sequence of unique minimizers of $\mathcal{E}_\varepsilon$ converges strongly in the usual strong topology of $L^2(\Omega \times Q)$ to the unique minimizer of $\mathcal{E}_{\text{hom}}$:

Proposition 3.6. *Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ open, bounded and Lipschitz. Assume $(A1) - (A5)$. $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\text{hom}}$ admit unique minimizers $u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}(Q)^d$ and $u \in W_0^{1,p}(Q)$, respectively. We have*

$$u_\varepsilon \to u \text{ in } L^p(\Omega \times Q)^d, \quad \langle \nabla u_\varepsilon\rangle \rightharpoonup \nabla u \text{ weakly in } L^p(Q)^{d\times d}.$$

(For the proof see Section 3.3.)

3.2 Allen-Cahn type equations

In this section we provide a homogenization result for an evolutionary gradient system. The system is defined on a state space $\mathscr{B} = L^2(\Omega\times Q)$ and with the help of two functionals - a dissipation potential $\mathcal{R}_\varepsilon$ and an energy functional $\mathcal{E}_\varepsilon$. The dissipation potential $\mathcal{R}_\varepsilon : \mathscr{B} \to [0,\infty)$ is given by

$$\mathcal{R}_\varepsilon(u) = \frac{1}{2}\left\langle \int_Q r(\tau_{\frac{x}{\varepsilon}}\omega)|u(\omega,x)|^2 dx \right\rangle,$$

where $r : \Omega \to \mathbb{R}$. The energy functional $\mathcal{E}_\varepsilon : \mathscr{B} \to \mathbb{R}\cup\{\infty\}$ is defined as follows. For $u \in L^2(\Omega)\otimes H^1(Q)$,

$$\mathcal{E}_\varepsilon(u) = \left\langle \int_Q a(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u(\omega,x)\cdot\nabla u(\omega,x) + b(\tau_{\frac{x}{\varepsilon}}\omega)|u(\omega,x)|^2 + f(\tau_{\frac{x}{\varepsilon}}\omega, \langle u(\omega,x)\rangle)dx \right\rangle,$$

and we extend $\mathcal{E}_\varepsilon$ by ∞ to the whole $\mathscr{B}$. Above, $a : \Omega \to \mathbb{R}^{d\times d}$, $b : \Omega \to \mathbb{R}$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$. We consider the following assumptions:

(B1) r, a, b are measurable and there exist $C_1, C_2 > 0$ such that for P-a.e. $\omega\in\Omega$ it holds $r(\omega), b(\omega) \in [C_1, C_2]$. Moreover, $a \in L^\infty(\Omega)^{d\times d}$ and there exists $C > 0$ such that

$$a(\omega)F\cdot F \geq C|F|^2 \quad \text{for } P\text{-a.e. } \omega\in\Omega \text{ and all } F\in\mathbb{R}^d.$$

(B2) $f(\cdot,x)$ is measurable for all $x\in\mathbb{R}$ and $f(\omega,\cdot)$ is continuous for P-a.e. $\omega\in\Omega$. There exist $\lambda\in\mathbb{R}$, $C>0$ and $p<2^*$ ($2^*=\infty$ for $d=1,2$ and $2^* = \frac{2d}{d-2}$ for $d\geq 3$) such that for P-a.e. $\omega\in\Omega$

$$\begin{aligned} &f(\omega,\cdot) \text{ is } \lambda\text{-convex, i.e.} \quad x\mapsto f(\omega,x) - \frac{\lambda}{2}x^2 \text{ is convex,}\\ &-C \leq f(\omega,x) \leq C(|x|^p+1) \quad \text{for all } x\in\mathbb{R}. \end{aligned}$$

We remark that the above assumptions imply that $u\mapsto \mathcal{E}_\varepsilon(u) - \Lambda\mathcal{R}_\varepsilon(u)$ is convex, where $\Lambda := \frac{\lambda}{C_1}$.

We consider the following gradient system in the energy dissipation principle formulation (see [52] for equivalent formulations of gradient systems): Let $T>0$. It is said that $u\in AC([0,T];\mathscr{B})$ is a solution to the gradient system associated with $(\mathcal{R}_\varepsilon, \mathcal{E}_\varepsilon)$ (shorter u satisfies EDP_ε (energy dissipation principle)) if

$$\begin{aligned} \mathcal{E}_\varepsilon(u(T)) + \int_0^T \mathcal{R}_\varepsilon(\dot{u}(t)) + \mathcal{R}^*_\varepsilon(\xi(t))dt \leq \mathcal{E}_\varepsilon(u(0)),\\ \xi(t)\in\partial_F\mathcal{E}_\varepsilon(u(t)) \quad \text{for a.e. } t\in(0,T]. \end{aligned} \tag{18}$$

Above $\mathcal{R}^*_\varepsilon : \mathscr{B}^* \to [0,\infty)$, $\mathcal{R}^*_\varepsilon(\xi) = \sup_{u\in\mathscr{B}}\Big(\langle\xi,u\rangle_{\mathscr{B}^*,\mathscr{B}} - \mathcal{R}_\varepsilon(u)\Big)$ denotes the Legendre-Fenchel conjugate of $\mathcal{R}_\varepsilon$. Moreover, $\partial_F\mathcal{E}_\varepsilon(u)$ denotes the Fréchet subdifferential of $\mathcal{E}_\varepsilon$ at the point $u\in\mathscr{B}$ and it is defined as

$$\partial_F\mathcal{E}_\varepsilon(u) = \Big\{\xi\in\mathscr{B}^* : \mathcal{E}_\varepsilon(u) \leq \mathcal{E}_\varepsilon(w) + \langle\xi, u-w\rangle_{\mathscr{B}^*,\mathscr{B}} - \Lambda\mathcal{R}_\varepsilon(u-w) \quad \text{for all } w\in\mathscr{B}\Big\}.$$

Remark 3.7. Gradient systems driven by Λ-convex energies (in a suitable setting) have unique solutions (see [22]). In particular, if $(B1)-(B2)$ hold and for an initial data $u_\varepsilon^0 \in L^2(\Omega)\otimes H^1(Q)$, there exists $u_\varepsilon \in C^{Lip}([0,T],\mathscr{B})$, a unique solution to EDP_ε with $u_\varepsilon(0)=u_\varepsilon^0$ (see e.g. [22, Theorem 3.2]).

As $\varepsilon\to 0$, we derive a limit gradient system which is described in the following. The state space for the effective model is $\mathscr{B}_0 := L^2_{\mathsf{inv}}(\Omega)\otimes L^2(Q)$. The effective dissipation potential $\mathcal{R}_{\hom} : \mathscr{B}_0 \to [0,\infty)$ is given by

$$\mathcal{R}_{\hom}(u) = \left\langle \int_Q r(\omega)|u(\omega,x)|^2 dx \right\rangle.$$

The energy functional $\mathcal{E}_{\hom} : \mathscr{B}_0 \to \mathbb{R}\cup\{\infty\}$ is defined as

$$\begin{aligned}\mathcal{E}_{\hom}(u) = \inf_{\chi\in L^2_{pot}(\Omega)\otimes L^2(Q)} \Bigg\langle \int_Q & a(\omega)\left(\nabla u(\omega,x)+\chi(\omega,x)\right)\cdot\left(\nabla u(\omega,x)+\chi(\omega,x)\right) \\ & + b(\omega)|u(\omega,x)|^2 + f(\omega,\langle u(\omega,x)\rangle)dx \Bigg\rangle\end{aligned} \tag{19}$$

for $u\in L^2_{\mathsf{inv}}(\Omega)\otimes H^1(Q)$ and $\mathcal{E}_{\hom}=\infty$ otherwise. We remark that $u\mapsto \mathcal{E}_{\hom}(u)-\Lambda\mathcal{R}_{\hom}(u)$ is convex. We say that $u\in AC([0,T],\mathscr{B}_0)$ is a solution to the gradient system associated with $(\mathcal{R}_{\hom},\mathcal{E}_{\hom})$ (shorter satisfies EDP_0) if it satisfies inequality (18) with $\mathscr{B}$, $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ replaced by $\mathscr{B}_0$, $\mathcal{E}_{\hom}$ and $\mathcal{R}_{\hom}$, respectively.

Remark 3.8. If $(B1)-(B2)$ hold and for initial data $u^0\in L^2_{\mathsf{inv}}(\Omega)\otimes H^1(Q)$, there exists $u\in C^{Lip}([0,T],\mathscr{B}_0)$, a unique solution to EDP_0 with $u(0)=u^0$ (see [22, Theorem 3.2]).

The following homogenization result is based on a general strategy for evolutionary Γ-convergence of abstract gradient systems presented in [52, 53]. We remark that an important ingredient (which allows to consider non-convex energy functionals) in this theory is a compactness assumption for solutions $u_\varepsilon(t)$ (w.r.t. the strong topology of $\mathscr{B}$). However, in our model a priori bounds do not lead to compactness, namely the uniform bounds we obtain in the space $L^2(\Omega)\otimes H^1(Q)$ do not attain convergent subsequences in $\mathscr{B}$ (but merely weakly convergent subsequences). In contrast, in deterministic homogenization of similar problems (e.g. [47]) the compact Sobolev embedding $H^1(Q)\subset L^p(Q)$ with $p<2^*$ is critically used. In the stochastic case, we only have $L^2(\Omega)\otimes H^1(Q)\subset L^2(\Omega)\otimes L^p(Q)$ continuously. We remedy this issue, by restricting the analysis to a special class of problems in which the non-convex term in the energy acts only on the statistical average $\langle u_\varepsilon(t)\rangle\in H^1(Q)$ of the solution and in this manner we are able to exploit the compact Sobolev embedding for passing to the limit in the non-convex part of the energy.

Theorem 3.9 (Evolutionary Γ-convergence)**.** *Let $(B1)-(B2)$ hold and consider $u^0\in L^2_{inv}(\Omega)\otimes H^1(Q)$, $u^0_\varepsilon\in L^2(\Omega)\otimes H^1(Q)$ such that*

$$u^0_\varepsilon\to u^0 \text{ strongly in } L^2(\Omega\times Q),\quad \mathcal{E}_\varepsilon(u^0_\varepsilon)\to\mathcal{E}_{\hom}(u^0)\quad \text{(well-prepared initial data).}$$

Then $u_\varepsilon\in C^{Lip}([0,T],\mathscr{B})$, the unique solution to EDP_ε with $u_\varepsilon(0)=u^0_\varepsilon$, satisfies: For all $t\in[0,T]$

$$u_\varepsilon(t)\overset{2}{\rightharpoonup} u(t) \text{ in } L^2(\Omega\times Q),\quad P_{inv}\nabla u_\varepsilon(t)\rightharpoonup \nabla u(t) \text{ weakly in } L^2(\Omega\times Q)^d.$$

where $u\in C^{Lip}([0,T],\mathscr{B}_0)$ is the unique solution to EDP_0 with $u(0)=u^0$. Moreover, for any $t\in[0,T]$

$$\mathcal{E}_\varepsilon(u_\varepsilon(t))\to\mathcal{E}_{\hom}(u(t)).$$

(For the proof see Section 3.3.)

Remark 3.10 (Ergodic case). If we additionally assume that $\langle\cdot\rangle$ is ergodic, the limit system is driven by deterministic functionals. In particular, the limit is described by a state space $\tilde{\mathscr{B}}_0 = H^1(Q)$, dissipation potential

$$\tilde{\mathcal{R}}_{\mathrm{hom}}(u) = \int_Q \langle r\rangle\, |u(x)|^2 dx,$$

and energy functional

$$\tilde{\mathcal{E}}_{\mathrm{hom}}(u) = \int_Q a_{\mathrm{hom}} \nabla u(x) \cdot \nabla u(x) + \langle b\rangle\, |u(x)|^2 + f_{\mathrm{hom}}(u(x)) dx,$$

where a_{hom} and f_{hom} are defined as: $a_{\mathrm{hom}} F \cdot F = \inf_{\chi \in L^2_{pot}(\Omega)} \langle a(\omega)(F + \chi(\omega)) \cdot (F + \chi(\omega))\rangle$ for $F \in \mathbb{R}^d$, and let $f_{\mathrm{hom}}(x) = \langle f(\omega, x)\rangle$ for $x \in \mathbb{R}$.

3.3 Proofs

Proof of Theorem 3.1. (i) The Poincaré-Korn inequality and the growth conditions of V imply that u_ε is bounded in $L^p(\Omega) \otimes W^{1,p}(Q)$. By Proposition 2.11 there exist $u \in L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}(Q)^d$ and $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d$ with the claimed convergence (up to a subsequence). From $\mathcal{T}_\varepsilon u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}_0(Q)^d$ for every $\varepsilon > 0$, we conclude that $u \in L^p_{\mathsf{inv}}(\Omega) \otimes W^{1,p}_0(Q)^d$ (cf. Remark 2.12).

(ii) The claim follows from Proposition 2.8 (iii).

(iii) The existence of a strongly two-scale convergent sequence $u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}_0(Q)$ from Remark 2.15. Furthermore, the convergence of the energy $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}_0(u, \chi)$ follows from Proposition 2.8 (ii). □

Proof of Corollary 3.2. The statement follows by a standard argument from Γ-convergence: Since u_ε is a minimizer we conclude that $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(0) < \infty$. Hence, by Theorem 3.1 there exists $u \in L^p_{\mathsf{inv}}(\Omega) \times W^{1,p}_0(Q)^d$ and $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d$ such that $u_\varepsilon \overset{2s}{\rightharpoonup} u$ in $L^p(\Omega \times Q)^d$, $\nabla u_\varepsilon \overset{2s}{\rightharpoonup} \nabla u + \chi$ in $L^p(\Omega \times Q)^{d\times d}$, and

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u, \chi).$$

Let (u_0, χ_0) denote the minimizer of $\mathcal{E}_0$. Then by Theorem 3.1 (iii) there exists a recovery sequence v_ε s.t. $\mathcal{E}_\varepsilon(v_\varepsilon) \to \mathcal{E}_0(u_0, \chi_0)$, and thus

$$\min \mathcal{E}_0 = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \geq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \min \mathcal{E}_\varepsilon \geq \mathcal{E}_0(u, \chi) \geq \min \mathcal{E}_0,$$

and thus (u, χ) is a minimizer of $\mathcal{E}_0$ and $\mathcal{E}_\varepsilon(u_\varepsilon) = \min \mathcal{E}_\varepsilon \to \min \mathcal{E}_0 = \mathcal{E}_0(u, \chi)$. □

Before presenting the proof of Theorem 3.5, we provide an auxiliary result. The argument of the following Korn inequality in $L^p(\Omega)$ is similar as the proof for the case $p = 2$ in [40].

Lemma 3.11. *There exists $C > 0$ such that*

$$\langle |\chi|^p \rangle \le C \langle |\chi^s|^p \rangle \quad \textit{for every } \chi \in L^p_{\mathsf{pot}}(\Omega)^d.$$

Proof. We prove the claim for $\chi = D\varphi$ for $\varphi \in W^{1,p}(\Omega)^d$ and the general case follows by density.

Recall, the stationary extension of $D\varphi$ is given by $SD\varphi(\omega, x) = D\varphi(\tau_x\omega)$ and we have $\nabla S\varphi(\omega, x) = SD\varphi(\omega, x)$. Let $R > 0$, $K > 0$ and $\eta_R \in C_c^\infty(B_{R+K})$ be a cut-off function satisfying $\eta = 1$ in B_R, $0 \le \eta \le 1$ and $|\nabla \eta_R| \le \frac{2}{K}$. Using stationarity of P, we obtain

$$\langle |D\varphi|^p \rangle = \left\langle \fint_{B_R} |\nabla S\varphi|^p dx \right\rangle = \left\langle \fint_{B_R} |\nabla(\eta_R S\varphi)|^p dx \right\rangle \le \left\langle \frac{1}{|B_R|} \int_{\mathbb{R}^d} |\nabla(\eta_R S\varphi)|^p dx \right\rangle.$$

Using this and Korn's inequality in $L^p(\mathbb{R}^d)$,

$$\begin{aligned}
\langle |D\varphi|^p \rangle &\le 2 \left\langle \frac{1}{|B_R|} \int_{\mathbb{R}^d} |\nabla^s(\eta_R S\varphi)|^p \right\rangle \\
&= 2 \left\langle \fint_{B_R} |\nabla^s S\varphi|^p dx \right\rangle + \frac{2}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\nabla^s(\eta_R S\varphi)|^p dx \right\rangle.
\end{aligned}$$

The first term on the right-hand side of the above inequality equals $2 \langle |D^s\varphi|^p \rangle$ and therefore to conclude the proof, it is sufficient to show that the second term vanishes in the limit $R \to \infty$. We have

$$\begin{aligned}
\frac{1}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\nabla^s(\eta_R S\varphi)|^p dx \right\rangle &\le \frac{1}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\nabla(\eta_R S\varphi)|^p dx \right\rangle \\
&\le \frac{C}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\eta_R|^p |\nabla S\varphi|^p + |\nabla \eta_R|^p |S\varphi|^p dx \right\rangle \\
&\le \frac{C}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\nabla S\varphi|^p dx \right\rangle + \frac{C}{|B_R| K^p} \left\langle \int_{B_{R+K}\setminus B_R} |S\varphi|^p dx \right\rangle.
\end{aligned} \tag{20}$$

For the first term on the right-hand side, we have

$$\begin{aligned}
\frac{C}{|B_R|} \left\langle \int_{B_{R+K}\setminus B_R} |\nabla S\varphi|^p dx \right\rangle &= \frac{C|B_{R+K}|}{|B_R|} \left\langle \fint_{B_{R+K}} |\nabla S\varphi|^p dx \right\rangle - C \left\langle \fint_{B_R} |\nabla S\varphi|^p dx \right\rangle \\
&= C \langle |D\varphi|^p \rangle \left(\frac{|B_{R+K}|}{|B_R|} - 1 \right)
\end{aligned}$$

and as $R \to \infty$ the last expression vanishes. Similarly, the second term on the right-hand side of (20) vanishes as $R \to \infty$. $\square$

For the proof of Theorem 3.5 we apply Castaing's measurable selection lemma in the following form:

Lemma 3.12 (See [16])**.** *Let X be a complete separable metric space, $(\mathcal{S}, \sigma)$ a measurable space and $f : \mathcal{S} \to P(X)$ a multifunction. Further, assume that for all $x \in \mathcal{S}$, $f(x)$ is nonempty and closed in X, and for any closed $G \subset X$ we have*

$$\{x \in S : f(x) \cap G \neq \varnothing\} \in \sigma.$$

Then f admits a measurable selection, i.e. there exists $\tilde{f} : \mathcal{S} \to X$ measurable with $\tilde{f}(x) \in f(x)$.

Proof of Theorem 3.5. (i) According to Theorem 3.1 (i) there exist $u \in W_0^{1,p}(Q)$ and $\chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d$ such that (using Proposition 2.11) u_ε satisfies the claimed convergences. Furthermore, we have

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u,\chi) \geq \mathcal{E}_{\mathrm{hom}}(u).$$

(ii) We show that there exists $\chi \in L^p_{pot}(\Omega)\otimes L^p(Q)$ such that $\mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u)$ as follows. By Lemma 3.11 and the direct method of calculus of variations, the cell problem (16) has a solution for every $(x,F) \in Q \times \mathbb{R}^{d\times d}$, thus

$$f : Q \times \mathbb{R}^{d\times d} \ni (x,F) \mapsto \left\{\chi \in L^p_{\mathsf{pot}}(\Omega)^d : \langle V(\omega, x, F^s + \chi^s(\omega))\rangle = V_{\mathrm{hom}}(x,F)\right\}$$

defines a multifunction. We equip $Q\times\mathbb{R}^{d\times d}$ with the Borel sigma algebra $\mathcal{B}(Q\times\mathbb{R}^{d\times d})$ and we verify the assumptions of Castaing's measurable selection theorem (see Lemma 3.12): $f(x,F)$ is nonempty and closed for any $(x,F) \in Q\times\mathbb{R}^{d\times d}$, and if $G \subset L^p_{\mathsf{pot}}(\Omega)^d$ is a closed ball, it holds

$$f^-(G) := \left\{(x,F) \in Q\times\mathbb{R}^{d\times d} : f(x,F)\cap G \neq \emptyset\right\} \in \mathcal{B}(Q\times\mathbb{R}^{d\times d}),$$

as can be seen as follows. Consider a sequence $(x_j, F_j) \in f^-(G)$ such that $x_j \to x$ and $F_j \to F$. There exists $\chi_j \in f(x_j,F_j)\cap G$ and up to a subsequence (not relabeled) it satisfies $\chi_j \rightharpoonup \chi_*$ weakly in L^p_{pot} and $\chi^* \in G$. Let $\chi \in f(x,F)$, it holds

$$\begin{aligned}\langle V(\omega, x, F^s + \chi^s_*(\omega))\rangle &\leq \liminf_{j\to\infty} \left\langle V(\omega, x_j, F_j^s + \chi_j^s)\right\rangle \\ &\leq \liminf_{j\to\infty} \left\langle V(\omega, x_j, F_j^s + \chi^s)\right\rangle = V_{\mathrm{hom}}(x,F).\end{aligned}$$

The first inequality is obtained using the continuity assumption $(A4)$. As a result of this, $\chi_* \in f(x,F)\cap G$ and therefore $f^-(G)$ is a closed set and therefore it is measurable. Further, we consider the case $G \subset L^p_{\mathsf{pot}}(\Omega)$ is a closed set. Since $L^p(\Omega)$ is separable, the closed set $G \subset L^p(\Omega)^{d\times d}$ can be represented as a countable intersection of countable unions of closed balls. Therefore, $f^-(G) \in \mathcal{B}(Q\times\mathbb{R}^{d\times d})$. Hence we may apply Theorem 3.12 and obtain a measurable function $\widetilde{f} : Q\times\mathbb{R}^{d\times d} \to L^p_{\mathsf{pot}}(\Omega)^d$ such that $\widetilde{f}(x,F) \subset f(x,F)$.

We define $\chi(x) = \widetilde{f}(x, \nabla u(x))$ which is measurable by the properties of $\widetilde{f}$ and therefore χ defines an element in $L^p_{\mathsf{pot}}(\Omega)^d \otimes L^p(Q)$ with $\mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u)$. Thus, by Theorem 3.1 (iii) there exists a strongly two-scale convergent sequence $u_\varepsilon \in L^p(\Omega)\times W_0^{1,p}(Q)^d$ such that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u).$$

Since χ is mean-free, the convergence for $\langle \nabla u_\varepsilon\rangle$ follows. □

Proof of Proposition 3.6. Uniqueness of minimizers follows by the uniform convexity assumption on the integrand V. As in the proof of Theorem 3.5 (ii), we select $\chi \in L^p_{\mathsf{pot}}(\Omega)\otimes L^p(Q)^d$ such that $V_{\mathrm{hom}}(x, \nabla u(x)) = \langle V(\omega, x, \nabla^s u(x) + \chi^s(\omega,x))\rangle$. Theorem 3.1 (iii) implies that there exists a sequence $v_\varepsilon \in L^p(\Omega)\otimes W_0^{1,p}(Q)^d$ such that $v_\varepsilon \overset{2s}{\to} u$ in $L^p(\Omega\times Q)^d$ and $\mathcal{E}_\varepsilon(v_\varepsilon) \to \mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u)$. By triangle's inequality we have $\|u_\varepsilon - u\|_{L^p(\Omega\times Q)} \leq \|u_\varepsilon - v_\varepsilon\|_{L^p(\Omega\times Q)} + \|v_\varepsilon - u\|_{L^p(\Omega\times Q)}$. By the isometry property of $\mathcal{T}_\varepsilon$ and strong two-scale convergence of v_ε we have $\|v_\varepsilon - u\|_{L^p(\Omega\times Q)} = \|\mathcal{T}_\varepsilon(v_\varepsilon - u)\|_{L^p(\Omega\times Q)} = \|\mathcal{T}_\varepsilon v_\varepsilon - u\|_{L^p(\Omega\times Q)} \to 0$.

Furthermore, the Poincaré-Korn inequality $\|u_\varepsilon - v_\varepsilon\|^p_{L^p(\Omega\times Q)} \le C\|\nabla^s u_\varepsilon - \nabla^s v_\varepsilon\|^p_{L^p(\Omega\times Q)}$ (for a generic constant C that is independent of ε but might change from line to line), the uniform convexity of V in form of $\frac{C}{4}\|\nabla^s u_\varepsilon - \nabla^s v_\varepsilon\|^p_{L^p(\Omega\times Q)} \le \frac{1}{2}\mathcal{E}_\varepsilon(u_\varepsilon) + \frac{1}{2}\mathcal{E}_\varepsilon(u_\varepsilon) - \mathcal{E}_\varepsilon(\frac{1}{2}(u_\varepsilon + v_\varepsilon))$, and the minimality of u_ε, yield the estimate

$$\|u_\varepsilon - v_\varepsilon\|^p_{L^p(\Omega\times Q)} \le C\mathcal{E}_\varepsilon(v_\varepsilon) - \mathcal{E}_\varepsilon(\tfrac{1}{2}(u_\varepsilon + v_\varepsilon)).$$

Since $\mathcal{E}_\varepsilon(v_\varepsilon) \to \mathcal{E}_{\mathrm{hom}}(u)$ and $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\frac{1}{2}(u_\varepsilon + v_\varepsilon)) \ge \mathcal{E}(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u)$, we conclude that the right-hand side converges to 0. Thus, $u_\varepsilon \to u$ in $L^p(\Omega\times Q)$, and the convergence of the gradient follows using Proposition 2.11. □

Proof of Theorem 3.9. This proof follows the general strategy outlined in [52, Theorem 3.3] (see also [53]) with slight modifications regarding compactness issues.

Step 1. A priori estimates and compactness.

The assumptions on the initial data (well-preparedness) imply that there exists $C > 0$ such that $\mathcal{E}_\varepsilon(u^0_\varepsilon) \le C$. This means that the right-hand side of (18) (for u_ε) is bounded (uniformly in ε), consequently using the growth conditions in $(B1)-(B2)$ we obtain that

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{L^2(\Omega)\otimes H^1(Q)} + \|u_\varepsilon\|_{H^1([0,T];\mathscr{B})} + \|\xi_\varepsilon\|_{L^2([0,T],\mathscr{B})} \le C. \tag{21}$$

Here and below, we identify $\xi_\varepsilon(t) \in \mathscr{B}^*$ with its Riesz-representative in $\mathscr{B}$. In this respect, $\mathcal{R}^*_\varepsilon$ might be indentified with the functional $\mathcal{R}^*_\varepsilon(\xi) = \left\langle \int_Q r^{-1}(\tau_{\frac{x}{\varepsilon}}\omega)(\xi(\omega,x))^2 dx \right\rangle$ (not relabeled) defined on $\mathscr{B}$.

With help of the estimate (21) we extract a subsequence (not relabeled) such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } H^1([0,T];\mathscr{B}), \quad \xi_\varepsilon \rightharpoonup \xi \text{ weakly in } L^2([0,T];\mathscr{B}).$$

Moreover, since (21) implies a uniform estimate for u_ε in the $C^{0,\frac{1}{2}}([0,T],\mathscr{B})$ norm, the Arzelá-Ascoli theorem implies that there exists a subsequence (not relabeled) such that for all $t \in [0,T]$ we have

$$u_\varepsilon(t) \rightharpoonup u(t) \text{ weakly in } \mathscr{B}.$$

Furthermore, weak lower semi-continuity of the $\mathscr{B}$-norm and the uniform estimate for u_ε in $C^{0,\frac{1}{2}}([0,T],\mathscr{B})$ yield $u \in C^{0,\frac{1}{2}}([0,T];\mathscr{B})$. It holds that $u(0) = u^0$ since by assumption $u^0_\varepsilon \to u^0$. Note that (21) and Jensen's inequality imply that $\sup_{t\in[0,T]} \|\langle u_\varepsilon(t)\rangle\|_{H^1(Q)} \le C$ and therefore the compact Sobolev embedding implies that for all $t \in [0,T]$, $\langle u_\varepsilon(t)\rangle \to \langle u(t)\rangle$ strongly in $L^p(Q)$.

Step 2. Passage to the limit $\varepsilon \to 0$.

First, we remark that the conditional expectation P_{inv} is a contraction on $L^s(\Omega)$ for any $s \in [1,\infty]$. As a result of this and since r is positive, it holds $\left\langle \int_Q (P_{\mathsf{inv}}\mathcal{T}^*_\varepsilon r)(\omega,x)(w(\omega,x))^2 dx \right\rangle \le \mathcal{R}_\varepsilon(w)$ for any $w \in \mathscr{B}$. Furthermore, we have $P_{\mathsf{inv}}\mathcal{T}^*_\varepsilon r = P_{\mathsf{inv}} r$ (cf. proof of Lemma 2.19) and appealing to Jensen's inequality for P_{inv}, the above inequality yields $\mathcal{R}_{\mathrm{hom}}(P_{\mathsf{inv}}w) = \left\langle \int_Q (P_{\mathsf{inv}}r)(\omega)((P_{\mathsf{inv}}w)(\omega,x))^2 dx \right\rangle \le \mathcal{R}_\varepsilon(w)$ for any $w \in \mathscr{B}$ (where the first equality is obtained using an approximation argument). The analogous inequality holds as well for $\mathcal{R}_\varepsilon^*$. As a result of this, inequality (18) (for u_ε) implies

$$\mathcal{E}_\varepsilon(u_\varepsilon(T)) + \int_0^T \mathcal{R}_{\mathrm{hom}}(P_{\mathsf{inv}}\dot{u}_\varepsilon(t)) + \mathcal{R}_{\mathrm{hom}}^*(P_{\mathsf{inv}}\xi_\varepsilon(t))dt \leq \mathcal{E}_\varepsilon(u_\varepsilon(0)). \tag{22}$$

The right-hand side of the above inequality converges to $\mathcal{E}_{\mathrm{hom}}(u(0))$ using the well-preparedness of the initial data. We note that the functional $\overline{u} \in L^2([0,T],\mathscr{B}_0) \mapsto \int_0^T \mathcal{R}_{\mathrm{hom}}(\overline{u}(t))dt$ is weakly l.s.c. since it is convex and strongly l.s.c. (the same holds if we replace $\mathcal{R}_{\mathrm{hom}}$ by $\mathcal{R}_{\mathrm{hom}}^*$) and therefore we conclude that

$$\liminf_{\varepsilon\to 0} \int_0^T \mathcal{R}_{\mathrm{hom}}(P_{\mathsf{inv}}\dot{u}_\varepsilon(t)) + \mathcal{R}_{\mathrm{hom}}^*(P_{\mathsf{inv}}\xi_\varepsilon(t))dt \geq \int_0^T \mathcal{R}_{\mathrm{hom}}(\partial_t(P_{\mathsf{inv}}u)(t)) + \mathcal{R}_{\mathrm{hom}}^*(P_{\mathsf{inv}}\xi(t))dt,$$

where we use that P_{inv} is a linear and bounded operator and thus $P_{\mathsf{inv}}\dot{u}_\varepsilon \rightharpoonup \partial_t(P_{\mathsf{inv}}u)$ weakly in $L^2([0,T];\mathscr{B})$ (P_{inv} and $\dot{(\cdot)} =: \partial_t(\cdot)$ commute) and $P_{\mathsf{inv}}\xi_\varepsilon \rightharpoonup P_{\mathsf{inv}}\xi$ weakly in $L^2([0,T];\mathscr{B})$.

Moreover, we have $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(T)) = \lim_{\varepsilon'\to 0}\mathcal{E}_{\varepsilon'}(u_{\varepsilon'}(T))$ for a subsequence ε'. Using the uniform bound in (21) and Proposition 2.11 we extract a further subsequence ε'' such that

$$u_{\varepsilon''}(T) \overset{2s}{\rightharpoonup} P_{\mathsf{inv}}u(T), \quad \nabla u_{\varepsilon''}(T) \overset{2s}{\rightharpoonup} \nabla P_{\mathsf{inv}}u(T) + \chi,$$

for some $\chi \in L^2_{pot}(\Omega)\otimes L^2(Q)$. We have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(T)) &\geq \liminf_{\varepsilon''\to 0}\left\langle \int_Q a(\tau_{\frac{x}{\varepsilon''}}\omega)\nabla u_{\varepsilon''}(T)(\omega,x)\cdot\nabla u_{\varepsilon''}(T)(\omega,x)dx\right\rangle \\ &+ \liminf_{\varepsilon''\to 0}\left\langle \int_Q b(\tau_{\frac{x}{\varepsilon''}}\omega)|u_{\varepsilon''}(T)(\omega,x)|^2 dx\right\rangle + \liminf_{\varepsilon''\to 0}\left\langle \int_Q f(\omega,\langle u_{\varepsilon''}(T)(\omega,x)\rangle)dx\right\rangle.\end{aligned}$$

The third term on the right-hand side of equals $\left\langle \int_Q f(\omega,\langle u(T)(\omega,x)\rangle)dx\right\rangle$. This follows from the strong convergence of $\langle u(T)\rangle$, the continuity and growth assumptions of f and by the dominated convergence theorem (cf. proof Proposition 2.8). For the other terms we apply Proposition 2.8 (ii) to obtain that $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(T)) \geq \mathcal{E}_{\mathrm{hom}}(P_{\mathsf{inv}}u(T))$. Collecting all the previous estimates for the terms in (22) and using the shorthand $v := P_{\mathsf{inv}}u$, we obtain

$$\mathcal{E}_{\mathrm{hom}}(v(T)) + \int_0^T \mathcal{R}_{\mathrm{hom}}(\dot{v}(t)) + \mathcal{R}_{\mathrm{hom}}^*(P_{\mathsf{inv}}\xi(t))dt \leq \mathcal{E}_{\mathrm{hom}}(v(0)).$$

Step 4. Weak-weak closedness of the subdifferential.

In this part we show that $P_{\mathsf{inv}}\xi(t) \in \partial_F\mathcal{E}_{\mathrm{hom}}(v(t))$ for a.e. $t\in[0,T]$. Commonly, such closedness of the subdifferential is proved using strong convergence of $u_\varepsilon(t)$ (missing in our case) and Mosco convergence of the energy $\mathcal{E}_\varepsilon$. In order to show the above property, we borrow an argument from the analysis of evolutionary rate-independent systems (see [55, 54, Proposition 4.5]). Namely, we show that the construction of suitable joint recovery sequences yields the above closedness property.

First, we show that for $\xi_\varepsilon,\xi\in\mathscr{B}$ (recall that we identify elements in $\mathscr{B}$ and $\mathscr{B}^*$):

$$\text{If } \xi_\varepsilon \in \partial_F\mathcal{E}_\varepsilon(u_\varepsilon(t)) \text{ and } \xi_\varepsilon \rightharpoonup \xi \text{ weakly in } \mathscr{B}, \text{ then } \xi \in \partial_F\mathcal{E}_{\mathrm{hom}}(v(t)). \tag{23}$$

We assume that $\xi_\varepsilon \in \partial_F \mathcal{E}_\varepsilon(u_\varepsilon(t))$ and $\xi_\varepsilon \rightharpoonup \xi$ weakly in $L^2(\Omega\times Q)$, and we consider a subsequence (not relabeled) such that $u_\varepsilon(t) \overset{2}{\rightharpoonup} v(t)$ and $\nabla u_\varepsilon(t) \overset{2}{\rightharpoonup} \nabla v(t)+\chi$ for some $\chi \in L^2_{pot}(\Omega)\otimes L^2(Q)$ (using Proposition 2.11). It follows that

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(w) + \langle \xi_\varepsilon, u_\varepsilon - w\rangle_{\mathscr{B}^*,\mathscr{B}} - \Lambda\mathcal{R}_\varepsilon(u_\varepsilon - w) \quad \text{for all } w\in\mathscr{B}. \tag{24}$$

We consider an arbitrary $\tilde{w} \in L^2_{\mathsf{inv}}(\Omega)\otimes H^1(Q)$ and denote by $\chi_{\tilde{w}} \in L^2_{pot}(\Omega)\otimes L^2(Q)$, a minimizer to

$$L^2_{pot}(\Omega)\otimes L^2(Q) \ni \chi \mapsto \left\langle \int_Q a(\omega)(\nabla\tilde{w}(\omega,x)+\chi(\omega,x))\cdot(\nabla\tilde{w}(\omega,x)+\chi(\omega,x))dx \right\rangle.$$

Using Remark 2.15, we are able to construct a sequence $\tilde{w}_\varepsilon \in L^2(\Omega)\otimes H^1(Q)$ such that

$$\tilde{w}_\varepsilon \overset{2}{\to} v(t)-\tilde{w} \text{ in } \mathscr{B}, \quad \nabla\tilde{w}_\varepsilon \overset{2}{\to} \nabla v(t) - \tilde{w} + \chi - \chi_{\tilde{w}} \text{ in } \mathscr{B}^d.$$

Furthermore, we define the joint recovery sequence as $w_\varepsilon = u_\varepsilon(t) - \tilde{w}_\varepsilon$. We set $w = w_\varepsilon$ in (24), to obtain

$$0 \leq \mathcal{E}_\varepsilon(w_\varepsilon) - \mathcal{E}_\varepsilon(u_\varepsilon(t)) + \langle \xi_\varepsilon, u_\varepsilon(t) - w_\varepsilon\rangle_{\mathscr{B}} - \Lambda\mathcal{R}_\varepsilon(u_\varepsilon(t) - w_\varepsilon). \tag{25}$$

Note that by construction $u_\varepsilon(t) - w_\varepsilon \to v(t) - \tilde{w}$ strongly and in two-scales and therefore the sum of the third and fourth terms on the right-hand side of the above inequality converges to $\langle P_{\mathsf{inv}}\xi, v(t)-\tilde{w}\rangle - \Lambda\mathcal{R}_{\mathrm{hom}}(v(t)-\tilde{w})$. The first two terms are treated as follows. Since the first two terms in the energy are quadratic we obtain

$$\begin{aligned}
&\mathcal{E}_\varepsilon(w_\varepsilon) - \mathcal{E}_\varepsilon(u_\varepsilon(t)) \\
=\ &\left\langle \int_Q a(\omega)\left(\mathcal{T}_\varepsilon(\nabla w_\varepsilon - \nabla u_\varepsilon(t))\right)\cdot\left(\mathcal{T}_\varepsilon(\nabla w_\varepsilon + \nabla u_\varepsilon(t))\right)\right\rangle \\
&+ \left\langle \int_Q b(\omega)\left(w_\varepsilon - u_\varepsilon(t)\right)\left(w_\varepsilon + u_\varepsilon(t)\right)dx + f(\omega,\langle w_\varepsilon\rangle) - f(\omega,\langle u_\varepsilon(t)\rangle)dx \right\rangle.
\end{aligned}$$

We remark that the first and the second terms on the right-hand side above are by construction products of strongly and weakly converging sequence. As a result of this and with the help of the facts that $\langle w_\varepsilon\rangle \to \langle \tilde{w}\rangle$ and $\langle u_\varepsilon\rangle \to \langle v(t)\rangle$ strongly in $L^p(Q)$, we are able to pass to the limit in the above inequality

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(w_\varepsilon) - \mathcal{E}_\varepsilon(u_\varepsilon(t)) \leq \mathcal{E}_{\mathrm{hom}}(\tilde{w}) - \mathcal{E}_{\mathrm{hom}}(v(t)).$$

Collecting the previous statements for (25), we obtain

$$0 \leq \mathcal{E}_{\mathrm{hom}}(\tilde{w}) - \mathcal{E}_{\mathrm{hom}}(v(t)) + \langle P_{\mathsf{inv}}\xi, v(t)-\tilde{w}\rangle_{\mathscr{B}_0^*,\mathscr{B}_0} - \Lambda\mathcal{R}_{\mathrm{hom}}(v(t)-\tilde{w}).$$

This proves (23).

Second, we refer to [64, Theorem 3.2] (see also Section 4.5.1) to obtain that there exists a parametrized measure μ_t on $\mathscr{B}$ such that the weak limit ξ of ξ_ε satisfies $\xi(t) = \int_{\mathscr{B}} \eta d\mu_t(\eta)$. Moreover, the measure μ_t is concentrated on the set of weak cluster points of $\xi_\varepsilon(t)$. For any weak cluster point ξ of the sequence $\xi_\varepsilon(t)$, using (23), it holds $\xi \in \partial_F\mathcal{E}_{\mathrm{hom}}(v(t))$. As a result of this and with the help of the fact that $\partial_F\mathcal{E}_{\mathrm{hom}}(v(t))$ is a convex set, we conclude that $\xi(t) = \int_{\mathscr{B}} \eta d\mu_t(\eta) \in \partial_F\mathcal{E}_{\mathrm{hom}}(v(t))$.

Step 5. Convergence for the whole sequence and convergence of the energy.

As a result of the previous steps, we obtain that up to a subsequence for all $t \in [0,T]$, $u_\varepsilon(t) \overset{2}{\rightharpoonup} v(t) = P_{\mathsf{inv}}u(t)$ and $P_{\mathsf{inv}}\nabla u_\varepsilon(t) \rightharpoonup \nabla v(t)$ weakly in $\mathscr{B}^d$, where $v \in C^{Lip}([0,T],\mathscr{B}_0)$ is the unique solution to EDP_0 with $v(0) = u^0$. Using the uniqueness of solutions to the limit problem and by a standard contradiction argument, we obtain that the convergence holds for the whole sequence.

The above procedure of passing to the limit in inequality (18) can be repeated if we replace T in the inequality by an arbitrary $t \in (0,T]$. We remark that using the chain rule for $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\text{hom}}$ (see e.g. [52, Theorem 3.2]) it follows that the inequalities in the formulations of EDP_ε and EDP_0 (where T is replaced by t) are equalities. Using this and the fact that the liminf inequalities hold separately for $\mathcal{E}_\varepsilon(u_\varepsilon(t))$ and $\int_0^t \mathcal{R}_\varepsilon(\dot{u}_s) + \mathcal{R}_\varepsilon^*(\xi_\varepsilon(s))ds$, we obtain that for any $t \in [0,T]$, $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon(t)) = \mathcal{E}_{\text{hom}}(v(t))$. This concludes the proof. □

4 Quenched stochastic two-scale convergence and relation to stochastic unfolding

In this section, we recall the concept of quenched stochastic two-scale convergence (cf. [73]) and study its relation to stochastic unfolding. The notion of quenched stochastic two-scale convergence is based on the individual ergodic theorem, see Theorem 2.3. We thus assume throughout this section that

$$(\Omega, \mathcal{F}, P, \tau) \text{ satisfies Assumption 2.1 and } P \text{ is ergodic.}$$

Moreover, throughout this section we fix exponents $p \in (1,\infty)$, $q := \frac{p}{p-1}$, and an open and bounded domain $Q \subset \mathbb{R}^d$. We denote by $(\mathscr{B}^p, \|\cdot\|_{\mathscr{B}^p})$ the Banach space $L^p(\Omega\times Q)$ and the associated norm, and we write $(\mathscr{B}^p)^*$ for the dual space. For the definition of quenched two-scale convergence we need to specify a suitable space of test-functions in $\mathscr{B}^q$ that is countably generated. To that end we fix sets $\mathscr{D}_\Omega$ and $\mathscr{D}_Q$ such that

- $\mathscr{D}_\Omega$ is a countable set of bounded, measurable functions on $(\Omega,\mathcal{F})$ that contains the identity $\mathbf{1}_\Omega \equiv 1$ and is dense in $L^1(\Omega)$ (and thus in $L^r(\Omega)$ for any $1 \le r < \infty$).
- $\mathscr{D}_Q \subset C(\overline{Q})$ is a countable set that contains the identity $\mathbf{1}_Q \equiv 1$ and is dense in $L^1(Q)$ (and thus in $L^r(Q)$ for any $1 \le r < \infty$).

We denote by $\mathscr{A} := \{\varphi(\omega,x) = \varphi_\Omega(\omega)\varphi_Q(x) \,:\, \varphi_\Omega \in \mathscr{D}_\Omega, \varphi_Q \in \mathscr{D}_Q\}$ the set of simple tensor products (a countable set), and by $\mathscr{D}_0$ the $\mathbb{Q}$-linear span of $\mathscr{A}$, i.e.

$$\mathscr{D}_0 := \Big\{ \sum_{j=1}^m \lambda_j\varphi_j \,:\, m \in \mathbb{N},\, \lambda_1,\ldots,\lambda_m \in \mathbb{Q},\, \varphi_1,\ldots,\varphi_m \in \mathscr{A} \Big\}.$$

We finally set $\mathscr{D} := \text{span}\mathscr{A} = \text{span}\mathscr{D}_0$ and denote by $\overline{\mathscr{D}} := \text{span}(\mathscr{D}_Q)$ (the span of $\mathscr{D}_Q$ seen as a subspace of $\mathscr{D}$), and note that $\mathscr{D}$ and $\mathscr{D}_0$ are dense subsets of $\mathscr{B}^q$, while the closure of $\overline{\mathscr{D}}$ in $\mathscr{B}^q$ is isometrically isomorph to $L^q(Q)$. Let us anticipate that $\mathscr{D}$ serves as our space of test-functions for stochastic two-scale convergence. As opposed to two-scale convergence in the mean, "quenched" stochastic two-scale convergence is defined relative to a fixed "admissible" realization $\omega_0 \in \Omega$. Throughout this section we denote by Ω_0 the set of admissible realizations; it is a set of full measure determined by the following lemma:

Lemma 4.1. *There exists a measurable set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ s.t. for all $\varphi, \varphi' \in \mathscr{A}$, all $\omega_0 \in \Omega_0$, and $r \in \{p, q\}$ we have with $(\mathcal{T}_\varepsilon^* \varphi)(\omega, x) := \varphi(\tau_{\frac{x}{\varepsilon}}\omega, x)$,*

$$\limsup_{\varepsilon\to 0} \|(\mathcal{T}_\varepsilon^*\varphi)(\omega_0, \cdot)\|_{L^r(Q)} \leq \|\varphi\|_{\mathscr{B}^r}$$

$$\text{and} \qquad \lim_{\varepsilon\to 0} \int_Q \mathcal{T}_\varepsilon^*(\varphi\varphi')(\omega_0, x)dx = \left\langle \int_Q (\varphi\varphi')(\omega_0, x)\, dx \right\rangle.$$

Proof. This is a simple consequence of Theorem 2.3 and the fact that $\mathscr{A}$ is countable. □

For the rest of the section Ω_0 is fixed according to Lemma 4.1.

4.1 Definition and basic properties

The idea of quenched stochastic two-scale convergence is similar to periodic two-scale convergence: We associate with a bounded sequence $(u_\varepsilon) \subset L^p(Q)$ and $\omega_0 \in \Omega_0$, a sequence of linear functionals (U_ε) defined on $\mathscr{D}$. We can pass (up to a subsequence) to a pointwise limit U, which is again a linear functional on $\mathscr{D}$ and which (thanks to Lemma 4.1) can be uniquely extended to a bounded linear functional on $\mathscr{B}^q$. We then define the *weak quenched ω_0-two-scale limit* of (u_ε) as the Riesz-representation $u \in \mathscr{B}^p$ of $U \in (\mathscr{B}^q)^*$.

Definition 4.2 (quenched two-scale limit, cf. [73, 38])**.** *Let (u_ε) be a sequence in $L^p(Q)$, and let $\omega_0 \in \Omega_0$ be fixed. We say that u_ε converges (weakly, quenched) ω_0-two-scale to $u \in \mathscr{B}^p$, and write $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$, if the sequence u_ε is bounded in $L^p(Q)$, and for all $\varphi \in \mathscr{D}$ we have*

$$\lim_{\varepsilon\to 0} \int_Q u_\varepsilon(x)(\mathcal{T}_\varepsilon^*\varphi)(\omega_0, x)\, dx = \int_\Omega \int_Q u(x, \omega)\varphi(\omega, x)\, dx\, dP(\omega). \tag{26}$$

Lemma 4.3 (Compactness)**.** *Let (u_ε) be a bounded sequence in $L^p(Q)$ and $\omega_0 \in \Omega_0$. Then there exists a subsequence (still denoted by ε) and $u \in \mathscr{B}^p$ such that $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$ and*

$$\|u\|_{\mathscr{B}^p} \leq \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(Q)}, \tag{27}$$

and $u_\varepsilon \rightharpoonup \langle u \rangle$ weakly in $L^p(Q)$.

(For the proof see Section 4.1.1).

For our purpose it is convenient to have a metric characterization of two-scale convergence.

Lemma 4.4 (Metric characterization)**.** *(i) Let $\{\varphi_j\}_{j\in\mathbb{N}}$ denote an enumeration of $\mathscr{A}_1 := \{\frac{\varphi}{\|\varphi\|_{\mathscr{B}^q}} : \varphi \in \mathscr{D}_0\}$. The vector space $\mathrm{Lin}(\mathscr{D}) := \{U : \mathscr{D} \to \mathbb{R}$ linear$\}$ endowed with the metric*

$$d(U, V; \mathrm{Lin}(\mathscr{D})) := \sum_{j\in\mathbb{N}} 2^{-j} \frac{|U(\varphi_j) - V(\varphi_j)|}{|U(\varphi_j) - V(\varphi_j)| + 1}$$

is complete and separable.

(ii) Let $\omega_0 \in \Omega_0$. Consider the maps

$$J_\varepsilon^{\omega_0} : L^p(Q) \to \mathrm{Lin}(\mathscr{D}), \qquad (J_\varepsilon^{\omega_0} u)(\varphi) := \int_Q u(x)(\mathcal{T}_\varepsilon^* \varphi)(\omega_0, x)\,dx,$$
$$J_0 : \mathscr{B}^p \to \mathrm{Lin}(\mathscr{D}), \qquad (J_0 u)(\varphi) := \left\langle \int_Q u\varphi \right\rangle.$$

Then for any bounded sequence u_ε in $L^p(Q)$ and any $u \in \mathscr{B}^p$ we have $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$ if and only if $J_\varepsilon^{\omega_0} u_\varepsilon \to J_0 u$ in $\mathrm{Lin}(\mathscr{D})$.

(For the proof see Section 4.1.1).

Remark 4.5. Convergence in the metric space $(\mathrm{Lin}(\mathscr{D}), d(\cdot,\cdot,\mathrm{Lin}(\mathscr{D}))$ is equivalent to pointwise convergence. $(\mathscr{B}^q)^*$ is naturally embedded into the metric space by means of the restriction $J : (\mathscr{B}^q)^* \to \mathrm{Lin}(\mathscr{D})$, $JU = U|_{\mathscr{D}}$. In particular, we deduce that for a bounded sequences (U_k) in $(\mathscr{B}^q)^*$ we have $U_k \overset{*}{\rightharpoonup} U$ if and only if $JU_k \to JU$ in the metric space. Likewise, $\mathscr{B}^p$ (resp. $L^p(Q)$) can be embedded into the metric space $\mathrm{Lin}(\mathscr{D})$ via J_0 (resp. $J_\varepsilon^{\omega_0}$ with $\varepsilon > 0$ and $\omega_0 \in \Omega_0$ arbitrary but fixed), and for a bounded sequence (u_k) in $\mathscr{B}^p$ (resp. $L^p(Q)$) weak convergence in $\mathscr{B}^p$ (resp. $L^p(Q)$) is equivalent to convergence of $(J_0 u_k)$ (resp. $(J_\varepsilon^{\omega_0} u_k)$) in the metric space.

Lemma 4.6 (Strong convergence implies quenched two-scale convergence)**.** *Let (u_ε) be a strongly convergent sequence in $L^p(Q)$ with limit $u \in L^p(Q)$. Then for all $\omega_0 \in \Omega_0$ we have $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$.*

(For the proof see Section 4.1.1).

Definition 4.7 (set of quenched two-scale cluster points)**.** *For a bounded sequence (u_ε) in $L^p(Q)$ and $\omega_0 \in \Omega_0$ we denote by $\mathscr{CP}(\omega_0, (u_\varepsilon))$ the set of all ω_0-two-scale cluster points, i.e. the set of $u \in \mathscr{B}^p$ with $J_0 u \in \bigcap_{k=1}^\infty \overline{\{J_\varepsilon^{\omega_0} u_\varepsilon \,:\, \varepsilon < \frac{1}{k}\}}$ where the closure is taken in the metric space $\big(\mathrm{Lin}(\mathscr{D}), d(\cdot,\cdot;\mathrm{Lin}(\mathscr{D}))\big)$.*

We conclude this section with two elementary results on quenched stochastic two-scale convergence:

Lemma 4.8 (Approximation of two-scale limits)**.** *Let $u \in \mathscr{B}^p$. Then for all $\omega_0 \in \Omega_0$, there exists a sequence $u_\varepsilon \in L^p(Q)$ such that $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$ as $\varepsilon \to 0$.*

(For the proof see Section 4.1.1).

Similar to the slightly different setting in [38] one can prove the following result:

Lemma 4.9 (Two-scale limits of gradients)**.** *Let (u_ε) be a sequence in $W^{1,p}(Q)$ and $\omega_0 \in \Omega_0$. Then there exist a subsequence (still denoted by ε) and functions $u \in W^{1,p}(Q)$ and $\chi \in L^p_{pot}(\Omega) \otimes L^p(Q)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(Q)$ and*

$$u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u \quad \text{and} \quad \nabla u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} \nabla u + \chi \qquad \text{as } \varepsilon \to 0\,.$$

4.1.1 Proofs

Proof of Lemma 4.3. Set $C_0 := \limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(Q)}$ and note that $C_0 < \infty$. By passing to a subsequence (not relabeled) we may assume that $C_0 = \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(Q)}$. Fix $\omega_0 \in \Omega_0$. Define linear functionals $U_\varepsilon \in \mathrm{Lin}(\mathscr{D})$ via

$$U_\varepsilon(\varphi) := \int_Q u_\varepsilon(x)(\mathcal{T}_\varepsilon^*\varphi)(\omega_0, x)\,dx.$$

Note that for all $\varphi \in \mathscr{A}$, $(U_\varepsilon(\varphi))$ is a bounded sequence in $\mathbb{R}$. Indeed, by Hölder's inequality and Lemma 4.1,

$$\limsup_{\varepsilon\to 0} |U_\varepsilon(\varphi)| \le \limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(Q)} \|\mathcal{T}_\varepsilon^*\varphi(\omega_0,\cdot)\|_{L^q(Q)} \le C_0\|\varphi\|_{\mathscr{B}^q}. \tag{28}$$

Since $\mathscr{A}$ is countable we can pass to a subsequence (not relabeled) such that $U_\varepsilon(\varphi)$ converges for all $\varphi \in \mathscr{A}$. By linearity and since $\mathscr{D} = \mathrm{span}(\mathscr{A})$, we conclude that $U_\varepsilon(\varphi)$ converges for all $\varphi \in \mathscr{D}$, and $U(\varphi) := \lim_{\varepsilon\to 0} U_\varepsilon(\varphi)$ defines a linear functional on $\mathscr{D}$. In view of (28) we have $|U(\varphi)| \le C_0\|\varphi\|_{\mathscr{B}^q}$, and thus U admits a unique extension to a linear functional in $(\mathscr{B}^q)^*$. Let $u \in \mathscr{B}^p$ denote its Riesz-representation. Then $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$, and

$$\|u\|_{\mathscr{B}^p} = \|U\|_{(\mathscr{B}^q)^*} \le C_0 = \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(Q)}.$$

Since $\mathbf{1}_\Omega \in \mathscr{D}_\Omega$ we conclude that for all $\varphi_Q \in \mathscr{D}_Q$ we have

$$\int_Q u_\varepsilon(x)\varphi_Q(x)\,dx = U_\varepsilon(\mathbf{1}_\Omega\varphi_Q) \to U(\mathbf{1}_\Omega\varphi_Q) = \left\langle \int_Q u(\omega,x)\varphi_Q(x)\,dx \right\rangle = \int_Q \langle u(x)\rangle\,\varphi_Q(x)\,dx.$$

Since (u_ε) is bounded in $L^p(Q)$, and $\mathscr{D}_Q \subset L^p(Q)$ is dense, we conclude that $u_\varepsilon \rightharpoonup \langle u\rangle$ weakly in $L^p(Q)$. □

Proof of Lemma 4.4. (i) Argument for completeness: If (U_j) is a Cauchy sequence in $\mathrm{Lin}(\mathscr{D})$, then for all $\varphi \in \mathscr{A}_1$, $(U_j(\varphi))$ is a Cauchy sequence in $\mathbb{R}$. By linearity of the U_j's this implies that $(U_j(\varphi))$ is Cauchy in $\mathbb{R}$ for all $\varphi \in \mathscr{D}$. Hence, $U_j \to U$ pointwise in $\mathscr{D}$ and it is easy to check that U is linear. Furthermore, $U_j \to U$ pointwise in $\mathscr{A}_1$ implies $U_j \to U$ in the metric space.

Argument for separability: Consider the (injective) map $J : (\mathscr{B}^q)^* \to \mathrm{Lin}(\mathscr{D})$ where $J(U)$ denotes the restriction of U to $\mathscr{D}$. The map J is continuous, since for all $U, V \in (\mathscr{B}^q)^*$ and $\varphi \in \mathscr{A}_1$ we have $|(JU)(\varphi)-(JV)(\varphi)| \le \|U-V\|_{(\mathscr{B}^q)^*}\|\varphi\|_{\mathscr{B}^q} = \|U-V\|_{(\mathscr{B}^q)^*}$ (recall that the test functions in $\mathscr{A}_1$ are normalized). Since $(\mathscr{B}^q)^*$ is separable (as a consequence of the assumption that $\mathcal{F}$ is countably generated), it suffices to show that the range $\mathcal{R}(J)$ of J is dense in $\mathrm{Lin}(\mathscr{D})$. To that end let $U \in \mathrm{Lin}(\mathscr{D})$. For $k \in \mathbb{N}$ we denote by $U_k \in (\mathscr{B}^q)^*$ the unique linear functional that is equal to U on the the finite dimensional (and thus closed) subspace $\mathrm{span}\{\varphi_1, \ldots, \varphi_k\} \subset \mathscr{B}^q$ (where $\{\varphi_j\}$ denotes the enumeration of $\mathscr{A}_1$), and zero on the orthogonal complement in $\mathscr{B}^q$. Then a direct calculation shows that $d(U, J(U_k); \mathrm{Lin}(\mathscr{D})) \le \sum_{j>k} 2^{-j} = 2^{-k}$. Since $k \in \mathbb{N}$ is arbitrary, we conclude that $\mathcal{R}(J) \subset \mathrm{Lin}(\mathscr{D})$ is dense.

(ii) Let u_ε denote a bounded sequence in $L^p(Q)$ and $u \in \mathscr{B}^p$. Then by definition, $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$ is equivalent to $J_\varepsilon^{\omega_0} u_\varepsilon \to J_0 u$ pointwise in $\mathscr{D}$, and the latter is equivalent to convergence in the metric space $\mathrm{Lin}(\mathscr{D})$.

□

Lemma 4.6. This follows from Hölder's inequality and Lemma 4.1, which imply for all $\varphi \in \mathscr{A}$ the estimate

$$\limsup_{\varepsilon\to 0} \int_Q |(u_\varepsilon(x) - u(x))\mathcal{T}_\varepsilon^* \varphi(\omega_0, x)|\, dx$$

$$\leq \limsup_{\varepsilon\to 0} \left(\|u_\varepsilon - u\|_{L^p(Q)} \left(\int_Q |\mathcal{T}_\varepsilon^* \varphi(\omega_0, x)|^q \, dx \right)^{\frac{1}{q}} \right) = 0.$$

□

Proof of Lemma 4.8. Since $\mathscr{D}$ (defined as in Lemma 4.4) is dense in $\mathscr{B}^p$, for any $\delta > 0$ there exists $v_\delta \in \mathscr{D}_0$ with $\|u - v_\delta\|_{\mathscr{B}^p} \leq \delta$. Define $v_{\delta,\varepsilon}(x) := \mathcal{T}_\varepsilon^* v_\delta(\omega_0, x)$. Let $\varphi \in \mathscr{D}$. Since v_δ and φ (resp. $v_\delta\varphi$) are by definition linear combinations of functions (resp. products of functions) in $\mathscr{A}$, we deduce from Lemma 4.1 that $(v_{\delta,\varepsilon})_\varepsilon$ is bounded in $L^p(Q)$, and that

$$\int_Q v_{\delta,\varepsilon} \mathcal{T}_\varepsilon^* \varphi(\omega_0, x) = \int_Q \mathcal{T}_\varepsilon^*(v_\delta\varphi)(\omega_0, x) \to \left\langle \int_Q v_\delta \varphi \right\rangle.$$

By appealing to the metric characterization, we can rephrase the last convergence as $d(J_\varepsilon^{\omega_0} v_{\delta,\varepsilon}, J_0 v_\delta; \mathrm{Lin}(\mathscr{D})) \to 0$. By the triangle inequality we have

$$d(J_\varepsilon^{\omega_0} v_{\delta,\varepsilon}, J_0 u; \mathrm{Lin}(\mathscr{D})) \quad \leq \quad d(J_\varepsilon^{\omega_0} v_{\delta,\varepsilon}, J_0 v_\delta; \mathrm{Lin}(\mathscr{D})) + d(J_0 v_\delta, J_0 u; \mathrm{Lin}(\mathscr{D})).$$

The second term is bounded by $\|v_\delta - u\|_{\mathscr{B}^p} \leq \delta$, while the first term vanishes for $\varepsilon \downarrow 0$. Hence, there exists a diagonal sequence $u_\varepsilon := v_{\delta(\varepsilon),\varepsilon}$ (bounded in $L^p(Q)$) such that there holds $d(J_\varepsilon^{\omega_0} u_\varepsilon, J_0 u; \mathrm{Lin}(\mathscr{D})) \to 0$. The latter implies $u_\varepsilon \overset{2s}{\rightharpoonup}_{\omega_0} u$ by Lemma 4.4. □

4.2 Comparison to stochastic two-scale convergence in the mean via Young measures

In this paragraph we establish a relation between quenched two-scale convergence and two-scale convergence in the mean (in the sense of Definition 2.5). The relation is established by Young measures: We show that any bounded sequence u_ε in $\mathscr{B}^p$ – viewed as a functional acting on test-functions of the form $\mathcal{T}_\varepsilon^*\varphi$ – generates (up to a subsequence) a Young measure on $\mathscr{B}^p$ that (a) concentrates on the quenched two-scale cluster points of u_ε, and (b) allows to represent the two-scale limit (in the mean) of u_ε.

Definition 4.10. *We say $\boldsymbol{\nu} := \{\nu_\omega\}_{\omega\in\Omega}$ is a Young measure on $\mathscr{B}^p$, if for all $\omega \in \Omega$, ν_ω is a Borel probability measure on $\mathscr{B}^p$ (equipped with the weak topology) and*

$$\omega \mapsto \nu_\omega(B) \quad \textit{is measurable for all } B \in \mathcal{B}(\mathscr{B}^p),$$

where $\mathcal{B}(\mathscr{B}^p)$ denotes the Borel-σ-algebra on $\mathscr{B}^p$ (equipped with the weak topology).

Theorem 4.11. *Let u_ε denote a bounded sequence in $\mathscr{B}^p$. Then there exists a subsequence (still denoted by ε) and a Young measure $\boldsymbol{\nu}$ on $\mathscr{B}^p$ such that for all $\omega_0 \in \Omega_0$,*

$$\nu_{\omega_0} \text{ is concentrated on } \mathscr{CP}\left(\omega_0, \left(u_\varepsilon(\omega_0,\cdot)\right)\right),$$

and

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|^p_{\mathscr{B}^p} \geq \int_\Omega \left(\int_{\mathscr{B}^p} \|v\|^p_{\mathscr{B}^p} \, d\nu_\omega(v) \right) dP(\omega).$$

Moreover, we have

$$u_\varepsilon \stackrel{2s}{\rightharpoonup} u \qquad \text{where } u := \int_\Omega \int_{\mathscr{B}^p} v \, d\nu_\omega(v) dP(\omega).$$

Finally, if there exists $v : \Omega \to \mathscr{B}^p$ measurable and $\nu_\omega = \delta_{v(\omega)}$ for P-a.e. $\omega \in \Omega$, then up to extraction of a further subsequence (still denoted by ε) we have

$$u_\varepsilon(\omega) \stackrel{2s}{\rightharpoonup}_\omega v(\omega) \qquad \text{for } P\text{-a.e. } \omega \in \Omega.$$

(For the proof see Section 4.2.1).

In the opposite direction we observe that quenched two-scale convergence implies two-scale convergence in the mean in the following sense:

Lemma 4.12. *Consider a family $\{(u^\omega_\varepsilon)\}_{\omega\in\Omega}$ of sequences (u^ω_ε) in $L^p(Q)$ and suppose that:*

(a) There exists $u \in \mathscr{B}^p$ s.t. for P-a.e. $\omega \in \Omega$ we have $u^\omega_\varepsilon \stackrel{2s}{\rightharpoonup}_\omega u$.

(b) There exists a sequence $(\tilde{u}_\varepsilon)$ s.t. $u^\omega_\varepsilon(x) = \tilde{u}_\varepsilon(\omega, x)$ for a.e. $(\omega, x) \in \Omega \times Q$.

(c) There exists a bounded sequence (χ_ε) in $L^p(\Omega)$ such that $\|u^\omega_\varepsilon\|_{L^p(Q)} \leq \chi_\varepsilon(\omega)$ for a.e. $\omega \in \Omega$.

Then $\tilde{u}_\varepsilon \stackrel{2s}{\rightharpoonup} u$ weakly two-scale (in the mean).

(For the proof see Section 4.2.1).

To compare homogenization of convex integral functionals w.r.t. stochastic two-scale convergence in the mean and in the quenched sense, we appeal to the following result.

Definition 4.13 (Quenched two-scale normal integrand)**.** *A function $h : \Omega \times Q \times \mathbb{R}^d \to \mathbb{R}$ is called a quenched two-scale normal integrand, if for all $\xi \in \mathbb{R}^d$, $h(\cdot,\cdot,\xi)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and for a.e. $(\omega, x) \in \Omega \times Q$, $h(\omega, x, \cdot)$ is lower semicontinuous, and for P-a.e. $\omega_0 \in \Omega_0$ and sequence (u_ε) in $L^p(Q)$ the following implication holds:*

$$u_\varepsilon \stackrel{2s}{\rightharpoonup}_{\omega_0} u \qquad \Rightarrow \qquad \liminf_{\varepsilon\to 0} \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega_0, x, u_\varepsilon(x))dx \geq \int_\Omega \int_Q h(\omega, x, u(\omega, x))\, dx\, dP(\omega).$$

Lemma 4.14. *Let h denote a quenched two-scale normal integrand, let (u_ε) denote a bounded sequence in $\mathscr{B}^p$ that generates a Young measure $\boldsymbol{\nu}$ on $\mathscr{B}^p$ in the sense of Theorem 4.11. Suppose that $h_\varepsilon : \Omega \to \mathbb{R}$, $h_\varepsilon(\omega) := -\int_Q \min\left\{0, h(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))\right\} dx$ is uniformly integrable. Then*

$$\liminf_{\varepsilon\to 0} \int_\Omega \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))\, dx\, dP(\omega)$$
$$\geq \int_\Omega \int_{\mathscr{B}^p} \left(\int_\Omega \int_Q h(\tilde{\omega}, x, v(\tilde{\omega}, x))\, dx\, dP(\tilde{\omega}) \right) d\nu_\omega(v)\, dP(\omega) \quad (29)$$

(For the proof see Section 4.2.1).

4.2.1 Proof of Theorem 4.11 and Lemmas 4.14 and 4.12

We first recall some notions and results of Balder's theory for Young measures [9]. Throughout this section $\mathscr{M}$ is assumed to be a separable, complete metric space with metric $d(\cdot,\cdot;\mathscr{M})$.

Definition 4.15.
- *We say a function $s:\Omega\to\mathscr{M}$ is measurable, if it is $\mathcal{F}-\mathcal{B}(\mathscr{M})$-measurable where $\mathcal{B}(\mathscr{M})$ denotes the Borel-σ-algebra in $\mathscr{M}$.*
- *A function $h:\Omega\times\mathscr{M}\to(-\infty,+\infty]$ is called a normal integrand, if h is $\mathcal{F}\otimes\mathcal{B}(\mathscr{M})$-measurable, and for all $\omega\in\Omega$ the function $h(\omega,\cdot):\mathscr{M}\to(-\infty,+\infty]$ is lower semicontinuous.*
- *A sequence s_ε of measurable functions $s_\varepsilon:\Omega\to\mathscr{M}$ is called tight, if there exists a normal integrand h such that for every $\omega\in\Omega$ the function $h(\omega,\cdot)$ has compact sublevels in $\mathscr{M}$ and $\limsup_{\varepsilon\to0}\int_\Omega h(\omega,s_\varepsilon(\omega))\,dP(\omega)<\infty$.*
- *A Young measure in $\mathscr{M}$ is a family $\boldsymbol{\mu}:=\{\mu_\omega\}_{\omega\in\Omega}$ of Borel probability measures on $\mathscr{M}$ such that for all $B\in\mathcal{B}(\mathscr{M})$ the map $\Omega\ni\omega\mapsto\mu_\omega(B)\in\mathbb{R}$ is $\mathcal{F}$-measurable.*

Theorem 4.16 ([9, Theorem I]). *Let $s_\varepsilon:\Omega\to\mathscr{M}$ denote a tight sequence of measurable functions. Then there exists a subsequence, still indexed by ε, and a Young measure $\boldsymbol{\mu}:\Omega\to\mathscr{M}$ such that for every normal integrand $h:\Omega\times\mathscr{M}\to(-\infty,+\infty]$ we have*

$$\liminf_{\varepsilon\to0}\int_\Omega h(\omega,s_\varepsilon(\omega))\,dP(\omega)\geq\int_\Omega\int_{\mathscr{M}}h(\omega,\xi)d\mu_\omega(\xi)dP(\omega)\,, \tag{30}$$

provided that the negative part $h_\varepsilon^-(\cdot)=|\min\{0,h(\cdot,s_\varepsilon(\cdot))\}|$ is uniformly integrable. Moreover, for P-a.e. $\omega\in\Omega_0$ the measure μ_ω is supported in the set of all cluster points of $s_\varepsilon(\omega)$, i.e. in $\bigcup_{k=1}^\infty\overline{\{s_\varepsilon(\omega)\,:\,\varepsilon<\frac{1}{k}\}}$ (where the closure is taken in $\mathscr{M}$).

In order to apply the above theorem we require an appropriate metric space in which two-scale convergent sequences and their limits embed:

Lemma 4.17. *(i) We denote by $\mathscr{M}$ the set of all triples (U,ε,r) with $U\in\mathrm{Lin}(\mathscr{D})$, $\varepsilon\geq0$, $r\geq0$. $\mathscr{M}$ endowed with the metric*

$$d((U_1,\varepsilon_1,r_1),(U_2,\varepsilon_2,r_2);\mathscr{M}):=d(U_1,U_2;\mathrm{Lin}(\mathscr{D}))+|\varepsilon_1-\varepsilon_2|+|r_1-r_2|$$

is a complete, separable metric space.

(ii) For $\omega_0\in\Omega_0$ we denote by $\mathscr{M}^{\omega_0}$ the set of all triples $(U,\varepsilon,r)\in\mathscr{M}$ such that

$$U=\begin{cases}J_\varepsilon^{\omega_0}u & \text{for some } u\in L^p(Q) \text{ with } \|u\|_{L^p(Q)}\leq r \text{ in the case } \varepsilon>0,\\ J_0u & \text{for some } u\in\mathscr{B}^p \text{ with } \|u\|_{\mathscr{B}^p}\leq r \text{ in the case } \varepsilon=0.\end{cases} \tag{31}$$

Then $\mathscr{M}^{\omega_0}$ is a closed subspace of $\mathscr{M}$.

(iii) Let $\omega_0\in\Omega_0$, and $(U,\varepsilon,r)\in\mathscr{M}^{\omega_0}$. Then the function u in the representation (31) of U is unique, and

$$\begin{cases}\|u\|_{L^p(Q)}=\sup\limits_{\varphi\in\overline{\mathscr{D}},\ \|\varphi\|_{\mathscr{B}^q}\leq1}|U(\varphi)| & \text{if } \varepsilon>0,\\ \|u\|_{\mathscr{B}^p}=\sup\limits_{\varphi\in\mathscr{D},\ \|\varphi\|_{\mathscr{B}^q}\leq1}|U(\varphi)| & \text{if } \varepsilon=0.\end{cases} \tag{32}$$

(iv) *For $\omega_0 \in \Omega_0$ the function $\|\cdot\|_{\omega_0} : \mathscr{M}^{\omega_0} \to [0,\infty)$,*

$$\|(U,\varepsilon,r)\|_{\omega_0} := \begin{cases} \Big(\sup_{\varphi\in\overline{\mathscr{D}},\ \|\varphi\|_{\mathscr{B}^q}\le 1} |U(\varphi)|^p + \varepsilon + r^p \Big)^{\frac{1}{p}} & \text{if } (U,\varepsilon,r)\in\mathscr{M}^{\omega_0},\ \varepsilon>0, \\ \Big(\sup_{\varphi\in\mathscr{D},\ \|\varphi\|_{\mathscr{B}^q}\le 1} |U(\varphi)|^p + r^p \Big)^{\frac{1}{p}} & \text{if } (U,\varepsilon,r)\in\mathscr{M}^{\omega_0},\ \varepsilon=0, \end{cases}$$

is lower semicontinuous on $\mathscr{M}^{\omega_0}$.

(v) *For all (u,ε) with $u\in L^p(Q)$ and $\varepsilon>0$ we have $s:=(J_\varepsilon^{\omega_0}u,\varepsilon,\|u\|_{L^p(Q)})\in\mathscr{M}^{\omega_0}$ and $\|s\|_{\omega_0} = \big(2\|u\|^p_{L^p(Q)}+\varepsilon\big)^{\frac{1}{p}}$. Likewise, for all (u,ε) with $u\in\mathscr{B}^p$ and $\varepsilon=0$ we have $s=(J_0u,\varepsilon,\|u\|_{\mathscr{B}^p})$ and $\|s\|_{\omega_0}=2^{\frac{1}{p}}\|u\|_{\mathscr{B}^p}$.*

(vi) *For all $R<\infty$ the set $\{(U,\varepsilon,r)\in\mathscr{M}^{\omega_0} : \|(U,\varepsilon,r)\|_{\omega_0}\le R\}$ is compact in $\mathscr{M}$.*

(vii) *Let $\omega_0\in\Omega_0$ and let u_ε denote a bounded sequence in $L^p(Q)$. Then the triple $s_\varepsilon := (J_\varepsilon^{\omega_0}u_\varepsilon,\varepsilon,\|u_\varepsilon\|_{L^p(Q)})$ defines a sequence in $\mathscr{M}^{\omega_0}$. Morever, we have $s_\varepsilon\to s_0$ in $\mathscr{M}$ as $\varepsilon\to 0$ if and only if $s_0=(J_0u,0,r)$ for some $u\in\mathscr{B}^p$, $r\ge\|u\|_{\mathscr{B}^p}$, and $u_\varepsilon\overset{2s}{\rightharpoonup}_{\omega_0}u$.*

Proof. (i) This is a direct consequence of Lemma 4.4 (i) and the fact that the product of complete, separable metric spaces remains complete and separable.

(ii) Let $s_k := (U_k,\varepsilon_k,r_k)$ denote a sequence in $\mathscr{M}^{\omega_0}$ that converges in $\mathscr{M}$ to some $s_0=(U_0,\varepsilon_0,r_0)$. We need to show that $s_0\in\mathscr{M}^{\omega_0}$. By passing to a subsequence, it suffices to study the following three cases: $\varepsilon_k>0$ for all $k\in\mathbb{N}_0$, $\varepsilon_k=0$ for all $k\in\mathbb{N}_0$, and $\varepsilon_0=0$ while $\varepsilon_k>0$ for all $k\in\mathbb{N}$.

Case 1: $\varepsilon_k>0$ for all $k\in\mathbb{N}_0$.
W.l.o.g. we may assume that $\inf_k\varepsilon_k>0$. Hence, there exist $u_k\in L^p(Q)$ with $U_k=J^{\omega_0}_{\varepsilon_k}u_k$. Since $r_k\to r$, we conclude that (u_k) is bounded in $L^p(\Omega)$. We thus may pass to a subsequence (not relabeled) such that $u_k\rightharpoonup u_0$ weakly in $L^p(Q)$, and

$$\|u_0\|_{L^p(Q)} \le \liminf_k \|u_k\|_{L^p(Q)} \le \lim_k r_k = r_0. \tag{33}$$

Moreover, $U_k\to U$ in the metric space $\mathrm{Lin}(\mathscr{D})$ implies pointwise convergence on $\mathscr{D}$, and thus for all $\varphi_Q\in\mathscr{D}_Q$ we have $U_k(\mathbf{1}_\Omega\varphi_Q)=\int_Q u_k\varphi_Q\to\int_Q u_0\varphi_Q$. We thus conclude that $U_0(\mathbf{1}_\Omega\varphi_Q)=\int_Q u_0\varphi_Q$. Since $\mathscr{D}_Q\subset L^q(Q)$ dense, we deduce that $u_k\rightharpoonup u_0$ weakly in $L^p(Q)$ for the entire sequence. On the other hand the properties of the shift τ imply that for any $\varphi_\Omega\in\mathscr{D}_\Omega$ we have $\varphi_\Omega(\tau_{\frac{\cdot}{\varepsilon_k}}\omega_0)\to\varphi_\Omega(\tau_{\frac{\cdot}{\varepsilon_0}}\omega_0)$ in $L^q(Q)$. Hence, for any $\varphi_\Omega\in\mathscr{D}_\Omega$ and $\varphi_Q\in\mathscr{D}_Q$ we have

$$U_k(\varphi_\Omega\varphi_Q)=\int_Q u_k(x)\varphi_Q(x)\varphi_\Omega(\tau_{\frac{x}{\varepsilon_k}}\omega_0)\,dx\to\int_Q u_0(x)\varphi_Q(x)\varphi_\Omega(\tau_{\frac{x}{\varepsilon_0}}\omega_0)\,dx = J^{\omega_0}_{\varepsilon_0}(\varphi_\Omega\varphi_Q)$$

and thus (by linearity) $U_0=J^{\omega_0}_{\varepsilon_0}u_0$.

Case 2: $\varepsilon_k=0$ for all $k\in\mathbb{N}_0$.
In this case there exist a bounded sequence u_k in $\mathscr{B}^p$ with $U_k=J_0u_k$ for $k\in\mathbb{N}$. By passing to a subsequence we may assume that $u_k\rightharpoonup u_0$ weakly in $\mathscr{B}^p$ for some $u_0\in\mathscr{B}^p$ with

$$\|u_0\|_{\mathscr{B}^p}\le\liminf_k\|u_{\varepsilon_k}\|_{\mathscr{B}^p}\le r_0. \tag{34}$$

This implies that $U_k = J_0 u_k \to J_0 u_0$ in $\mathrm{Lin}(\mathscr{D})$. Hence, $U_0 = J_0 u_0$ and we conclude that $s_0 \in \mathscr{M}$.

Case 3: $\varepsilon_k > 0$ for all $k \in \mathbb{N}$ and $\varepsilon_0 = 0$.
There exists a bounded sequence u_k in $L^p(Q)$. Thanks to Lemma 4.3, by passing to a subsequence we may assume that $u_k \overset{2s}{\rightharpoonup}_{\omega_0} u_0$ for some $u \in \mathscr{B}^p$ with

$$\|u_0\|_{\mathscr{B}^p} \le \liminf_k \|u_k\|_{L^p(Q)} \le r_0. \tag{35}$$

Furthermore, Lemma 4.4 implies that $J^{\omega_0}_{\varepsilon_k} u_k \to J_0 u_0$ in $\mathrm{Lin}(\mathscr{D})$, and thus $U_0 = J_0 u_0$. We conclude that $s_0 \in \mathscr{M}$.

(iii) We first argue that the representation (31) of U by u is unique. In the case $\varepsilon > 0$ suppose that $u, v \in L^p(Q)$ satisfy $U = J^{\omega_0}_\varepsilon u = J^{\omega_0}_\varepsilon v$. Then for all $\varphi_Q \in \mathscr{D}_Q$ we have $\int_Q (u - v)\varphi_Q = J^{\omega_0}_\varepsilon u(\mathbf{1}_\Omega \varphi_Q) - J^{\omega_0}_\varepsilon v(\mathbf{1}_\Omega \varphi_Q) = U(\mathbf{1}_\Omega \varphi_Q) - U(\mathbf{1}_\Omega \varphi_Q) = 0$, and since $\mathscr{D}_Q \subset L^q(Q)$ dense, we conclude that $u = v$. In the case $\varepsilon = 0$ the statement follows by a similar argument from the fact that $\mathscr{D}$ is dense $\mathscr{B}^q$.

To see (32) let u and U be related by (31). Since $\overline{\mathscr{D}}$ (resp. $\mathscr{D}$) is dense in $L^q(Q)$ (resp. $\mathscr{B}^q$), we have

$$\begin{cases} \|u\|_{L^p(Q)} = \sup\limits_{\varphi \in \overline{\mathscr{D}},\ \|\varphi\|_{\mathscr{B}^q} \le 1} |\int_Q u\varphi\, dx\, dP| = \sup\limits_{\varphi \in \overline{\mathscr{D}},\ \|\varphi\|_{\mathscr{B}^q} \le 1} |U(\varphi)| & \text{if } \varepsilon > 0, \\ \|u\|_{\mathscr{B}^p} = \sup\limits_{\varphi \in \mathscr{D},\ \|\varphi\|_{\mathscr{B}^q} \le 1} |\int_\Omega \int_Q u\varphi\, dx\, dP| = \sup\limits_{\varphi \in \mathscr{D},\ \|\varphi\|_{\mathscr{B}^q} \le 1} |U(\varphi)| & \text{if } \varepsilon = 0. \end{cases}$$

(iv) Let $s_k = (U_k, \varepsilon_k, r_k)$ denote a sequence in $\mathscr{M}^{\omega_0}$ that converges in $\mathscr{M}$ to a limit $s_0 = (U_0, \varepsilon_0, r_0)$. By (ii) we have $s_0 \in \mathscr{M}^{\omega_0}$. For $k \in \mathbb{N}_0$ let u_k in $L^p(Q)$ or $\mathscr{B}^p$ denote the representation of U_k in the sense of (31). We may pass to a subsequence such that one of the three cases in (ii) applies and (as in (ii)) either u_k weakly converges to u_0 (in $L^p(Q)$ or $\mathscr{B}^p$), or $u_k \overset{2s}{\rightharpoonup}_{\omega_0} u_0$. In any of these cases the claimed lower semicontinuity of $\|\cdot\|_{\omega_0}$ follows from $\varepsilon_k \to \varepsilon_0$, $r_k \to r_0$, and (32) in connection with one of the lower semicontinuity estimates (33) – (35).

(v) This follows from the definition and duality argument (32).

(vi) Let s_k denote a sequence in $\mathscr{M}^{\omega_0}$. Let u_k in $L^p(Q)$ or $\mathscr{B}^p$ denote the (unique) representative of U_k in the sense of (31). Suppose that $\|s_k\|_{\omega_0} \le R$. Then (r_k) and (ε_k) are bounded sequences in $\mathbb{R}_{\ge 0}$, and $\sup_k \|u_k\| \le \sup_k r_k < \infty$ (where $\|\cdot\|$ stands short for either $\|\cdot\|_{L^p(Q)}$ or $\|\cdot\|_{\mathscr{B}^p}$). Thus we may pass to a subsequence such that $r_k \to r_0$, $\varepsilon_k \to \varepsilon_0$, and one of the following three cases applies:

- Case 1: $\inf_{k \in \mathbb{N}_0} \varepsilon_k > 0$. In that case we conclude (after passing to a further subsequence) that $u_k \rightharpoonup u_0$ weakly in $L^p(Q)$, and thus $U_k \to U_0 = J^{\omega_0}_{\varepsilon_0} u_0$ in $\mathrm{Lin}(\mathscr{D})$.
- Case 2: $\varepsilon_k = 0$ for all $k \in \mathbb{N}_0$. In that case we conclude (after passing to a further subsequence) that $u_k \rightharpoonup u_0$ weakly in $\mathscr{B}^p(Q)$, and thus $U_k \to U_0 = J_0 u_0$ in $\mathrm{Lin}(\mathscr{D})$.
- Case 3: $\varepsilon_k > 0$ for all $k \in \mathbb{N}$ and $\varepsilon_0 = 0$. In that case we conclude (after passing to a further subsequence) that $u_k \overset{2s}{\rightharpoonup}_{\omega_0} u_0$, and thus $U_k \to U_0 = J_0 u_0$ in $\mathrm{Lin}(\mathscr{D})$.

In all of these cases we deduce that $s_0 = (U_0, \varepsilon_0, r_0) \in \mathscr{M}^{\omega_0}$, and $s_k \to s_0$ in $\mathscr{M}$.

(vii) This is a direct consequence of (ii) – (vi), and Lemma 4.4.

□

Now we are in position to prove Theorem 4.11

Proof of Theorem 4.11. Let $\mathscr{M}$, $\mathscr{M}^{\omega_0}$, $J_\varepsilon^{\omega_0}$ etc. be defined as in Lemma 4.17.

Step 1. (Identification of (u_ε) with a tight $\mathscr{M}$-valued sequence). Since $u_\varepsilon \in \mathscr{B}^p$, by Fubini's theorem, we have $u_\varepsilon(\omega,\cdot) \in L^p(Q)$ for P-a.e. $\omega \in \Omega$. By modifying u_ε on a null-set in $\Omega \times Q$ (which does not alter two-scale limits in the mean), we may assume w.l.o.g. that $u_\varepsilon(\omega,\cdot) \in L^p(Q)$ for all $\omega \in \Omega$. Consider the measurable function $s_\varepsilon : \Omega \to \mathscr{M}$ defined as

$$s_\varepsilon(\omega) := \begin{cases} \left(J_\varepsilon^\omega u_\varepsilon(\omega,\cdot), \varepsilon, \|u_\varepsilon(\omega,\cdot)\|_{L^p(Q)}\right) & \text{if } \omega \in \Omega_0 \\ (0,0,0) & \text{else.} \end{cases}$$

We claim that (s_ε) is tight. To that end consider the integrand $h : \Omega \times \mathscr{M} \to (-\infty, +\infty]$ defined by

$$h(\omega, (U,\varepsilon,r)) := \begin{cases} \|(U,\varepsilon,r)\|_\omega^p & \text{if } \omega \in \Omega_0 \text{ and } (U,\varepsilon,r) \in \mathscr{M}^\omega, \\ +\infty & \text{else.} \end{cases}$$

From Lemma 4.17 we deduce that h is a normal integrand and $h(\omega,\cdot)$ has compact sublevels for all $\omega \in \Omega$. Moreover, for all $\omega_0 \in \Omega_0$ we have $s_\varepsilon(\omega_0) \in \mathscr{M}^{\omega_0}$ and thus $h(\omega_0, s_\varepsilon(\omega_0)) = 2\|u_\varepsilon(\omega_0,\cdot)\|^p_{L^p(Q)} + \varepsilon$. Hence,

$$\int_\Omega h(\omega, s_\varepsilon(\omega))\, dP(\omega) = 2\|u_\varepsilon\|^p_{\mathscr{B}^p} + \varepsilon.$$

We conclude that (s_ε) is tight.

Step 2. (Compactness and definition of $\boldsymbol{\nu}$). By appealing to Theorem 4.16 there exists a subsequence (still denoted by ε) and a Young measure $\boldsymbol{\mu}$ that is generated by (s_ε). Let $\boldsymbol{\mu_1}$ denote the first component of $\boldsymbol{\mu}$, i.e. the Young measure on $\mathrm{Lin}(\mathscr{D})$ characterized for $\omega \in \Omega$ by

$$\int_{\mathrm{Lin}(\mathscr{D})} f(\xi)\, d\mu_{1,\omega}(\xi) = \int_{\mathscr{M}} f(\xi_1)\, d\mu_\omega(\xi),$$

for all $f : \mathrm{Lin}(\mathscr{D}) \to \mathbb{R}$ continuous and bounded, where $\mathscr{M} \ni \xi = (\xi_1,\xi_2,\xi_3) \to \xi_1 \in \mathrm{Lin}(\mathscr{D})$ denotes the projection to the first component. By Balder's theorem, μ_ω is concentrated on the limit points of $(s_\varepsilon(\omega))$. By Lemma 4.17 we deduce that for all $\omega \in \Omega_0$ any limit point $s_0(\omega)$ of $s_\varepsilon(\omega)$ has the form $s_0(\omega) = (J_0 u, 0, r)$ where $0 \le r < \infty$ and $u \in \mathscr{B}^p$ is a ω-two-scale limit of a subsequence of $u_\varepsilon(\omega,\cdot)$. Thus, $\mu_{1,\omega}$ is supported on $\{J_0 u : u \in \mathscr{CP}(\omega, (u_\varepsilon(\omega,\cdot))\}$ which in particular is a subset of $(\mathscr{B}^q)^*$. Since $J_0 : \mathscr{B}^p \to (\mathscr{B}^q)^*$ is an isometric isomorphism (by the Riesz-Frechét theorem), we conclude that $\boldsymbol{\nu} = \{\nu_\omega\}_{\omega\in\Omega}$, $\nu_\omega(B) := \mu_{1,\omega}(J_0 B)$ (for all Borel sets $B \subset \mathscr{B}^p$ where $\mathscr{B}^p$ is equipped with the weak topology) defines a Young measure on $\mathscr{B}^p$, and for all $\omega \in \Omega_0$, ν_ω is supported on $\mathscr{CP}(\omega, (u_\varepsilon(\omega,\cdot))$.

Step 3. (Lower semicontinuity estimate). Note that $h:\Omega\times\mathscr{M}\to[0,+\infty]$,

$$h(\omega,(U,\varepsilon,r)):=\begin{cases}\sup_{\varphi\in\overline{\mathscr{D}},\,\|\varphi\|_{\mathscr{B}^q}\le 1}|U(\varphi)|^p & \text{if } \omega\in\Omega_0 \text{ and } (U,\varepsilon,r)\in\mathscr{M}^\omega,\varepsilon>0,\\ \sup_{\varphi\in\mathscr{D},\,\|\varphi\|_{\mathscr{B}^q}\le 1}|U(\varphi)|^p & \text{if } \omega\in\Omega_0 \text{ and } (U,\varepsilon,r)\in\mathscr{M}^\omega,\varepsilon=0,\\ +\infty & \text{else.}\end{cases}$$

defines a normal integrand, as can be seen as in the proof of Lemma 4.17. Thus Theorem 4.16 implies that

$$\liminf_{\varepsilon\to0}\int_\Omega h(\omega,s_\varepsilon(\omega))\,dP(\omega)\ge\int_\Omega\int_{\mathscr{M}}h(\omega,\xi)\,d\mu_\omega(\xi)dP(\omega).$$

In view of Lemma 4.17 we have $\sup_{\varphi\in\overline{\mathscr{D}},\,\|\varphi\|_{\mathscr{B}^q}\le1}|(J^\omega_\varepsilon u_\varepsilon)(\omega,\cdot))(\varphi)|=\|u_\varepsilon(\omega,\cdot)\|_{L^p(Q)}$ for $\omega\in\Omega_0$, and thus the left-hand side turns into $\liminf_{\varepsilon\to0}\|u_\varepsilon\|^p_{\mathscr{B}^p}$. Thanks to the definition of $\boldsymbol{\nu}$ the right-hand side turns into $\int_\Omega\int_{\mathscr{B}^p}\|v\|^p_{\mathscr{B}^p}\,d\nu_\omega(v)dP(\omega)$.

Step 4. (Identification of the two-scale limit in the mean). Let $\varphi\in\mathscr{D}_0$. Then $h:\Omega\times\mathscr{M}\to[0,+\infty]$,

$$h(\omega,(U,\varepsilon,r)):=\begin{cases}U(\varphi) & \text{if } \omega\in\Omega_0,\ (U,\varepsilon,r)\in\mathscr{M}^\omega,\\ +\infty & \text{else.}\end{cases}$$

defines a normal integrand. Since $h(\omega,s_\varepsilon(\omega))=\int_Q u_\varepsilon(\omega,x)\mathcal{T}^*_\varepsilon\varphi(\omega,x)\,dx$ for P-a.e. $\omega\in\Omega$, we deduce that $|h(\cdot,s_\varepsilon(\cdot))|$ is uniformly integrable. Thus, (30) applied to $\pm h$ and the definition of $\boldsymbol{\nu}$ imply that

$$\begin{aligned}\lim_{\varepsilon\to0}\int_\Omega\int_Q u_\varepsilon(\omega,x)(\mathcal{T}^*_\varepsilon\varphi)(\omega,x)\,dx\,dP(\omega)&=\lim_{\varepsilon\to0}\int_\Omega h(\omega,s_\varepsilon(\omega))\,dP(\omega)\\&=\int_\Omega\int_{\mathscr{B}^p}h(\omega,v)\,d\nu_\omega(v)\,dP(\omega)\\&=\int_\Omega\int_{\mathscr{B}^p}\left\langle\int_Q v\varphi\right\rangle d\nu_\omega(v)\,dP(\omega).\end{aligned}$$

Set $u:=\int_\Omega\int_{\mathscr{B}^p}v\,d\nu_\omega(v)dP(\omega)\in\mathscr{B}^p$. Then Fubini's theorem yields

$$\lim_{\varepsilon\to0}\int_\Omega\int_Q u_\varepsilon(\omega,x)(\mathcal{T}^*_\varepsilon\varphi)(\omega,x)\,dx\,dP(\omega)\;=\;\left\langle\int_Q u\varphi\right\rangle.$$

Since $\operatorname{span}(\mathscr{D}_0)\subset\mathscr{B}^q$ dense, we conclude that $u_\varepsilon\overset{2s}{\rightharpoonup}u$.

Step 5. Recovery of quenched two-scale convergence. Suppose that ν_ω is a delta distribution on $\mathscr{B}^p$, say $\nu_\omega=\delta_{v(\omega)}$ for some measurable $v:\Omega\to\mathscr{B}^p$. Note that $h:\Omega\times\mathscr{M}\to[0,+\infty]$,

$$h(\omega,(U,\varepsilon,r)):=-d(U,J_0v(\omega);\operatorname{Lin}(\mathscr{D}))$$

is a normal integrand and $|h(\cdot,s_\varepsilon(\cdot))|$ is uniformly integrable. Thus, (30) yields

$$\begin{aligned}&\limsup_{\varepsilon\to0}\int_\Omega d(J^\omega_\varepsilon u_\varepsilon(\omega,\cdot),J_0v(\omega);\operatorname{Lin}(\mathscr{D}))\,dP(\omega)\\=\;&-\liminf_{\varepsilon\to0}\int_\Omega h(\omega,s_\varepsilon(\omega))\,dP(\omega)\\\le\;&-\int_\Omega\int_{\mathscr{B}^p}h(\omega,J_0v)\,d\nu_\omega(v)\,dP(\omega)=-\int_\Omega h(\omega,J_0v(\omega))\,dP(\omega)=0.\end{aligned}$$

Thus, there exists a subsequence (not relabeled) such that $d(J^\omega_\varepsilon u_\varepsilon(\omega,\cdot), J_0 v(\omega); \mathrm{Lin}(\mathscr{D})) \to 0$ for a.e. $\omega \in \Omega_0$. In view of Lemma 4.4 this implies that $u_\varepsilon \xrightarrow{2s}_\omega v(\omega)$ for a.e. $\omega \in \Omega_0$. □

Proof of Lemma 4.14. Step 1. Representation of the functional by a lower semicontinuous integrand on $\mathscr{M}$.
For all $\omega_0 \in \Omega_0$ and $s = (U, \varepsilon, r) \in \mathscr{M}^{\omega_0}$ we write $\pi^{\omega_0}(s)$ for the unique representation u in $\mathscr{B}^p$ (resp. $L^p(Q)$) of U in the sense of (31). We thus may define for $\omega \in \Omega_0$ and $s \in \mathscr{M}^{\omega_0}$ the integrand

$$\overline{h}(\omega_0, s) := \begin{cases} \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega, x, (\pi^{\omega_0}s)(x))\,dx & \text{if } s = (U,\varepsilon,s) \text{ with } \varepsilon > 0, \\ \int_\Omega \int_Q h(\omega, x, (\pi^{\omega_0}s)(x))\,dx\,dP(\omega) & \text{if } s = (U,\varepsilon,s) \text{ with } \varepsilon = 0. \end{cases}$$

We extend $\overline{h}(\omega_0,\cdot)$ to $\mathscr{M}$ by $+\infty$, and define $\overline{h}(\omega,\cdot) \equiv 0$ for $\omega \in \Omega \setminus \Omega_0$. We claim that $\overline{h}(\omega,\cdot) : \mathscr{M} \to (-\infty, +\infty]$ is lower semicontinuous for all $\omega \in \Omega$. It suffices to consider $\omega_0 \in \Omega_0$ and a convergent sequence $s_k = (U_k, \varepsilon_k, r_k)$ in $\mathscr{M}^{\omega_0}$. For brevity we only consider the (interesting) case when $\varepsilon_k \downarrow \varepsilon_0 = 0$. Set $u_k := \pi^{\omega_0}(s_k)$. By construction we have

$$\overline{h}(\omega_0, s_k) = \int_Q h(\tau_{\frac{x}{\varepsilon_k}}\omega_0, u_k(\omega_0, x))\,dx,$$

and

$$\overline{h}(\omega_0, s_0) = \int_\Omega \int_Q h(\omega, x, u_0(\omega, x))\,dx\,dP(\omega).$$

Since $s_k \to s_0$ and $\varepsilon_k \to 0$, Lemma 4.17 (vi) implies that $u_k \xrightarrow{2s}_{\omega_0} u_0$, and since h is assumed to be a quenched two-scale normal integrand, we conclude that $\liminf_k \overline{h}(\omega_0, s_k) \geq \overline{h}(\omega_0, s_0)$, and thus $\overline{h}$ is a normal integrand.

Step 2. Conclusion.
As in Step 1 of the proof of Theorem 4.11 we may associate with the sequence (u_ε) a sequence of measurable functions $s_\varepsilon : \Omega \to \mathscr{M}$ that (after passing to a subsequence that we do not relabel) generates a Young measure $\boldsymbol{\mu}$ on $\mathscr{M}$. Since by assumption u_ε generates the Young measure $\boldsymbol{\nu}$ on $\mathscr{B}^p$, we deduce that the first component $\boldsymbol{\mu_1}$ satisfies $\nu_\omega(B) = \mu_\omega(J_0 B)$ for any Borel set B. Applying (30) to the integrand $\overline{h}$ of Step 1, yields

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \int_\Omega \int_Q h(\tau_{\frac{x}{\varepsilon}}\omega_0, u_\varepsilon(\omega_0, x))\,dx\,dP(\omega) \\ =\; &\liminf_{\varepsilon\to 0} \int_\Omega \overline{h}(\omega, s_\varepsilon(\omega))\,dP(\omega) \\ \geq\; &\int_\Omega \int_{\mathscr{M}} \overline{h}(\omega, \xi)\,d\mu_\omega(\xi)\,dP(\omega) \\ =\; &\int_\Omega \int_{\mathscr{B}^p} \Big(\int_\Omega \int_Q h(\tilde\omega, x, v(\tilde\omega, x))\,dx\,dP(\tilde\omega) \Big)\,d\nu_\omega(v)\,dP(\omega). \end{aligned}$$

□

Proof of Lemma 4.12. By (b) and (c) the sequence $(\tilde u_\varepsilon)$ is bounded in $\mathscr{B}^p$ and thus we can pass to a subsequence such that $(\tilde u_\varepsilon)$ generates a Young measure $\boldsymbol{\nu}$. Set $\tilde u := \int_\Omega \int_{\mathscr{B}^p} v\,d\nu_\omega(v)\,dP(\omega)$ and note that Theorem 4.11 implies that $\tilde u_\varepsilon \xrightarrow{2s} \tilde u$ weakly two-scale in the meam. On the other hand the theorem implies that ν_ω concentrates on the quenched two-scale cluster points of (u_ε^ω) (for a.e. $\omega \in \Omega$). Hence, in view of (a) we conclude that for a.e. $\omega \in \Omega$ the measure ν_ω is a Dirac measure concentrated on u, and thus $\tilde{u} = u$ a.e. in $\Omega \times Q$. $\square$

4.3 Quenched homogenization of convex functionals

In this section we demonstrate how to lift homogenization results w.r.t. two-scale convergence in the mean to quenched statements at the example of a convex minimization problem. Throughout this section we assume that $V : \Omega \times Q \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is a convex integrand satisfying the assumptions $(A1) - (A3)$ of Section 3.1. For $\omega \in \Omega$ we define $\mathcal{E}_\varepsilon^\omega : W_0^{1,p}(Q) \to \mathbb{R}$,

$$\mathcal{E}_\varepsilon^\omega(u) := \int_Q V\left(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla^s u(x)\right)\, dx,$$

and recall from Section 3.1 the definition (13) of the averaged energy $\mathcal{E}_\varepsilon$ and the definition (14) of the two-scale limit energy $\mathcal{E}_0$. The goal of this section is to relate two-scale limits of “mean”-minimizers, i.e. functions $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ that minimize $\mathcal{E}_\varepsilon$, with limits of “quenched”-minimizers, i.e. families $\{u_\varepsilon(\omega)\}_{\omega\in\Omega}$ of minimizers to $\mathcal{E}_\varepsilon^\omega$ in $W_0^{1,p}(Q)$.

Theorem 4.18. *Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ be a minimizer of $\mathcal{E}_\varepsilon$. Then there exists a subsequence such that $(u_\varepsilon, \nabla u_\varepsilon)$ generates a Young measure $\boldsymbol{\nu}$ in $\mathscr{B} := (\mathscr{B}^p)^{d+d^2}$ in the sense of Theorem 4.11, and for P-a.e. $\omega \in \Omega$, ν_ω concentrates on the set $\left\{ (u, \nabla u + \chi) : \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0 \right\}$ of minimizers of the limit functional. Moreover, if $V(\omega, x, \cdot)$ is strictly convex for all $x \in Q$ and P-a.e. $\omega \in \Omega$, then the minimizer u_ε of $\mathcal{E}_\varepsilon$ and the minimizer (u, χ) of $\mathcal{E}_0$ are unique, and for P-a.e. $\omega \in \Omega$ we have (for a not relabeled subsequence)*

$$u_\varepsilon \rightharpoonup u \text{ weakly in } W^{1,p}(Q), \qquad u_\varepsilon(\omega,\cdot) \overset{2s}{\rightharpoonup}_\omega u, \qquad \nabla u_\varepsilon(\omega,\cdot) \overset{2s}{\rightharpoonup}_\omega \nabla u + \chi,$$
$$\text{and } \min \mathcal{E}_\varepsilon^\omega = \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega,\cdot)) \to \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0.$$

Remark 4.19 (Identification of quenched two-scale cluster points). If we combine Theorem 4.18 with the identification of the support of the Young measure in Theorem 4.11 we conclude the following: There exists a subsequence such that $(u_\varepsilon, \nabla u_\varepsilon)$ two-scale converges in the mean to a limit of the form $(u_0, \nabla u_0 + \chi_0)$ with $\mathcal{E}_0(u_0, \chi_0) = \min \mathcal{E}_0$, and for a.e. $\omega \in \Omega$ the set of quenched ω-two-scale cluster points $\mathscr{CP}(\omega, (u_\varepsilon(\omega,\cdot), \nabla u_\varepsilon(\omega,\cdot)))$ is contained in $\left\{ (u, \nabla u + \chi) : \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0 \right\}$. In the strictly convex case we further obtain that $\mathscr{CP}(\omega, (u_\varepsilon(\omega,\cdot), \nabla u_\varepsilon(\omega,\cdot))) = \{(u, \nabla u + \chi)\}$ where (u, χ) is the unique minimizer to $\mathcal{E}_0$. Note, however, that our argument (that extracts quenched two-scale limits from the sequnece of “mean” minimizers) involves an exceptional P-null-set that a priori depends on the selected subsequence. This is in contrast to the classical result in [24] which is based on a subadditive ergodic theorem and states that there exists a set of full measure Ω' such that for all $\omega \in \Omega'$ the minizer u_ε^ω to $\mathcal{E}_\varepsilon^\omega$ weakly converges in $W^{1,p}(Q)$ to the deterministic minimizer u of the reduced functional $\mathcal{E}_{\text{hom}}$ for any sequence $\varepsilon \to 0$.

In the proof of Theorem 4.18 we combine homogenization in the mean in form of Theorem 3.1, the connection to quenched two-scale limits via Young measures in form of Theorem 4.11, and a recent result by Nesenenko and the first author that states that V is a quenched two-scale normal integrand:

Lemma 4.20 ([39, Lemma 5.1])**.** *V is a quenched two-scale normal integrand in the sense of Definition 4.13.*

Proof of Theorem 4.18. Step 1. (Identification of the support of $\boldsymbol{\nu}$).

Since u_ε is a sequence of minimizers, by Corollary 3.2 there exists a subsequence (not relabeled) and minimizers $(u,\chi) \in W_0^{1,p}(Q) \times (L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d)$ of $\mathcal{E}_0$ such that that $u_\varepsilon \overset{2s}{\rightharpoonup} u$ in $L^p(\Omega \times Q)^d$, $\nabla u_\varepsilon \overset{2s}{\rightharpoonup} \nabla u + \chi$ in $L^p(\Omega \times Q)^{d\times d}$, and

$$\lim_{\varepsilon\to 0} \min \mathcal{E}_\varepsilon = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0. \tag{36}$$

In particular, the sequence $(u_\varepsilon, \nabla u_\varepsilon)$ is bounded in $\mathscr{B}$. By Theorem 4.11 we may pass to a further subsequence (not relabeled) such that $(u_\varepsilon, \nabla u_\varepsilon)$ generates a Young measure $\boldsymbol{\nu}$ on $\mathscr{B}$. Since ν_ω is supported on the set of quenched ω-two-scale cluster points of $(u_\varepsilon(\omega,\cdot), \nabla u_\varepsilon(\omega,\cdot))$, we deduce from Lemma 4.9 that the support of ν_ω is contained in $\mathscr{B}_0 := \{\xi = (\xi_1,\xi_2) = (u', \nabla u' + \chi') \,:\, u' \in W_0^{1,p}(Q),\, \chi \in L^p_{\mathsf{pot}}(\Omega) \otimes L^p(Q)^d\}$ which is a closed subspace of $\mathscr{B}$. Moreover, thanks to the relation of the generated Young measure and stochastic two-scale convergence in the mean, we have $(u,\chi) = \int_\Omega \int_{\mathscr{B}_0} (\xi_1, \xi_2 - \nabla \xi_1)\, \nu_\omega(d\xi)\, dP(\omega)$. By Lemma 4.20, V is a quenched two-scale normal integrand and thus Lemma 4.14 implies that

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \int_\Omega \int_{\mathscr{B}} \Big(\int_\Omega \int_Q V(\tilde{\omega}, x, \xi_2)\, dx\, dP(\tilde{\omega}) \Big)\, \nu_\omega(d\xi)\, dP(\omega).$$

In view of (36) and the fact that ν_ω is supported in $\mathscr{B}_0$, we conclude that

$$\min \mathcal{E}_0 \geq \int_\Omega \int_{\mathscr{B}_0} \mathcal{E}_0(\xi_1, \xi_2 - \nabla\xi_1)\, \nu_\omega(d\xi)\, dP(\omega) \geq \min \mathcal{E}_0 \int_\Omega \int_{\mathscr{B}_0} \nu_\omega(d\xi) dP(\omega).$$

Since $\int_\Omega \int_{\mathscr{B}_0} \nu_\omega(d\xi) dP(\omega) = 1$, we have $\int_\Omega \int_{\mathscr{B}_0} |\mathcal{E}_0(\xi_1, \xi_2 - \nabla \xi_1) - \min \mathcal{E}_0|\, \nu_\omega(d\xi)\, dP(\omega) = 0$, and thus we conclude that for P-a.e. $\omega \in \Omega_0$, ν_ω concentrates on $\{(u, \nabla u + \chi) \,:\, \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0\}$.

Step 2. (The strictly convex case).

The uniqueness of u_ε and (u,χ) is clear. From Step 1 we thus conclude that $\nu_\omega = \delta_\xi$ where $\xi = (u, \nabla u + \chi)$. Theorem 4.11 implies that $(u_\varepsilon(\omega,\cdot), \nabla u_\varepsilon(\omega,\cdot)) \overset{2s}{\rightharpoonup}_\omega (u, \nabla u + \chi)$ (for P-a.e. $\omega \in \Omega$). By Lemma 4.20, V is a quenched two-scale normal integrand and thus for P-a.e. $\omega \in \Omega$,

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega,\cdot)) \geq \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0.$$

On the other hand, since $u_\varepsilon(\omega,\cdot)$ minimizes $\mathcal{E}_\varepsilon^\omega$, we deduce by a standard argument that for P-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \min \mathcal{E}_\varepsilon^\omega = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\omega(u_\varepsilon(\omega,\cdot)) = \mathcal{E}_0(u,\chi) = \min \mathcal{E}_0.$$

□

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