Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Discretisation and error analysis for a mathematical model of milling processes

Dietmar Hömberg[1], Oliver Rott[2], Kevin Sturm[3]

submitted: December 22, 2016

[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
and
Department of Mathematical Sciences
NTNU
Alfred Getz vei 1
7491 Trondheim
Norway
E-Mail: dietmar.hoemberg@wias-berlin.de

[2] Danieli Germany
Scherl 12
58540 Meinerzhagen
E-Mail: o.rott@germany.danieli.com

[3] Johann Radon Institute
Altenberger Str. 69
4040 Linz
Austria
E-Mail: kevin.sturm@ricam.oeaw.ac.at

No. 2364
Berlin 2016

2010 *Mathematics Subject Classification.* 35Q74, 65M15, 74H15.

Key words and phrases. Error estimates, high speed milling, finite elements.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We investigate a mathematical model for milling where the cutting tool dynamics is considered together with an elastic workpiece model. Both are coupled by the cutting forces consisting of two dynamic components representing vibrations of the tool and of the workpiece, respectively, at the present and previous tooth periods. We develop a numerical solution algorithm and derive error estimates both for the semi-discrete and the fully discrete numerical scheme. Numerical computations in the last section support the analytically derived error estimates.

1 Introduction

Milling is a process that allows the shaping of metal or other solids. Its basic components are a rotating cutter and a table on which the workpiece is mounted. The modelling of milling dynamics, the determination of stable cutting conditions and the design of more efficient milling machines are important research topics in production technology. Effective methods to predict stable processes have been developed in recent years (cf., e.g., [2, 5]). An essential part of these methods is an abstract dynamical model, represented by an ordinary differential equation. Adjusted to vibration measurement data it reproduces local characteristics of the actual milling system in terms of the dynamics at the tip of the cutter. The combination with a process model to describe the cutting forces leads to a delay-differential equation (DDE). The last decade has seen a number of approaches to identify efficiently stable machining parameters by means of bifurcation analysis of these DDE systems (cf., e.g., [7, 11]).

However, these methods provide only few detailed information about the dynamics of the entire process. Therefore, in [10] an improved model has been developed allowing for the inclusion of workpiece effects. In addition to the DDE model for the cutter the workpiece is accounted for by a thermoelastic material model. The coupling is realised through the cutting force. This approach allows for a refined stability analysis and will eventually lead an improved theoretical derivation of stable cutting conditions.

Considering workpiece effects in the dynamics of milling processes leads to an interesting novel mathematical model comprising of a PDE model to describe the workpiece mechanics together with a DDE for the machine dynamics coupled by a force condition with time dependent support on the boundary. In [10] it has been shown that the systems admits a unique weak solution.

In the present paper we present a numerical scheme for the workpiece coupled milling process and derive error estimates for its numerical solution. The paper is organized as follows: in the following section we describe the model and formulate the main convergence results. The error estimate for the semi-discretized problem is derived in Section 3, while the the fully discretized scheme is investigated in Section 4. The paper is concluded with some numerical results in Section 5.

2 Problem formulation and main results

2.1 Modelling and problem setting

Figure 1 depicts a schematic view of the milling process. The workpiece Ω is in contact with a cutting tool on the time-dependent contact boundary $\Gamma(t)$. The tip of the cutter is modelled as a two degree of freedom multibody system. More precisely, we assume $\Omega \subset \mathbf{R}^3$ to be a bounded Lipschitz domain. Let $\Gamma \subset \partial\Omega$ be a relatively open and non-empty part of the boundary and define the relatively closed set $\Gamma_0 := \partial D \setminus \Gamma$. We call Γ Neumann part of the boundary and Γ_0 Dirichlet part. We assume that the Neumann part Γ is decomposed into two disjoint parts

$$\Gamma = \Gamma(t) \cup \Gamma_R(t),$$

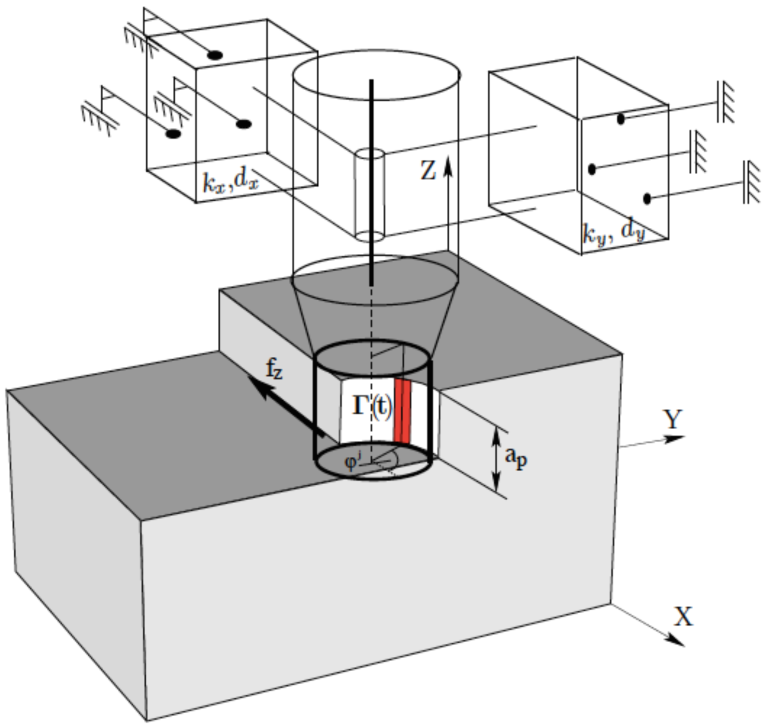


Figure 1: Main components of milling process model.

where the load transmission $\Gamma(t) \subset \Gamma$ is a measurable subset of the boundary Γ, whose evolution $t \mapsto \Gamma(t)$ in time is known and $\Gamma_R(t) := \partial\Omega \backslash (\Gamma_0 \cup \Gamma(t))$. To avoid technicalities, we may assume that the surface measure of the set $\Gamma(t)$ is never zero, which technically means that the cutter has several teeth such that at least one is always in contact with the workpiece.

Assumption 1. *There is a constant $\gamma > 0$ so that $\mu(\Gamma(t)) > \gamma$ for all $t \in [0, T]$.*

We assume elastic material behaviour for the workpiece Ω. Then the constitutive relation between strain σ and stress $\varepsilon(\mathbf{u}) = (\partial\mathbf{u} + \partial\mathbf{u}^\top)/2$ has the form

$$\sigma(\boldsymbol{x}, t) = A\varepsilon(\mathbf{u}(\boldsymbol{x}, t)) + \delta\varepsilon(\dot{\mathbf{u}}(\boldsymbol{x}, t)), \tag{1}$$

where A is the constant second order elasticity tensor with entries $A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}\right)$ for $i, j, k, l \in \{1, 2, 3\}$. The constants $\mu, \lambda > 0$ are the material dependent Lamé coefficients and $\delta > 0$ is a constant. We suppose that the body is homogeneous which is reflected in the fact that the mass density ϱ_0 of the body in the reference configuration is a positive constant. Furthermore, we assume that there are no body-forces so that the local form of the linearized momentum balance reads (see [3])

$$\varrho_0 \partial_{tt}\mathbf{u}(\boldsymbol{x}, t) - \operatorname{div}\left(\mathrm{A}\varepsilon(\mathbf{u}(\boldsymbol{x}, \mathrm{t})) + \delta\varepsilon(\dot{\mathbf{u}}(\boldsymbol{x}, \mathrm{t}))\right) = 0, \tag{2}$$

where ∂_{tt} denotes the second partial derivative with respect to time t.

In order to obtain a complete model we have to impose boundary and initial conditions. Since in milling machine the workpiece is usually fixed somewhere at the boundary it is natural to impose Dirichlet boundary conditions on Γ_0

$$\mathbf{u}(\boldsymbol{x}, t) = 0 \quad \text{on } \Gamma_0 \times [0, T]. \tag{3}$$

Next we introduce a simple model for the cutter that acts on the workpiece. The cutter is modeled as two degree of freedom oscillator described by:

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = N(t)(q(t) - q_\tau(t) - \bar{\mathbf{u}}(t) + \bar{\mathbf{u}}_\tau(t)) \quad \text{in } [0, T]. \tag{4}$$

Here, $q_\tau(t) := q(t-\tau)$, $\bar{\mathbf{u}}_\tau(t) := \bar{\mathbf{u}}(t-\tau)$ are delay terms and the mean of $\mathbf{u}$ at time $t \in [0,T]$ is defined by

$$\bar{\mathbf{u}}(t) := \frac{1}{|\Gamma(t)|} \int_{\Gamma(t)} \mathbf{u}(y,t)\, ds(y). \tag{5}$$

The cutting force on the right-hand side of (4) is based on the so-called uncut chip thickness, see, e.g., [1]. It consists of two dynamic components caused by vibrations of the tool and of the workpiece, respectively, at the present and previous tooth periods. For the derivation of the model including workpiece effects we refer to [10]. Note that the force has no static component since we assume zero deed velocity and just consider the dynamical effect of cutting forces between cutter and workpiece.

Assumption 2. *The matrices $M \in \mathbf{R}^{3\times 3}$ is invertible.*

In addition to the boundedness of the family $(\Gamma(t))_{t\in[0,T]}$ we assume.

Assumption 3 (Continuity of $\Gamma(t)$)**.** *For every $t' \in [0,T]$ the mappings*

$$t \mapsto |\Gamma(t') \setminus \Gamma(t)| \;\text{ and }\; t \mapsto |\Gamma(t) \setminus \Gamma(t')| \tag{6}$$

are continuous at t'.

Notice that the previous assumption implies the continuity of $t \mapsto |\Gamma(t)|$ on the interval $[0,T]$.

The surface traction exerted by the cutter on $\Gamma(t)$ is assumed to have the following form $F : \Gamma \to \mathbf{R}^3$,

$$F(\boldsymbol{x},t,q,q_\tau,\bar{\mathbf{u}},\bar{\mathbf{u}}_t) := \begin{cases} 0 & \text{if } (\boldsymbol{x},t) \in \Gamma_R(t)\times[0,T] \\ f(q,\bar{\mathbf{u}},q_\tau,\bar{\mathbf{u}}_t) & \text{if } (\boldsymbol{x},t) \in \Gamma(t)\times[0,T] \end{cases}. \tag{7}$$

Assumption 4. *The function f is assumed to be (globally) Lipschitz continuous with respect to all arguments.*

The function F has to obey Newton's second law: *actio = reactio*. To be more precise suppose $x \in \partial\Omega \setminus \Gamma_0$ and $x = q(t)$ then the surface force $F(x,q)$, by virtue of Newton's law, should be the negative of the force exerted from the cutter $q(t)$ to the mass point x. Again, for details, we refer the reader to [9]. Therefore, the *Neumann boundary* condition reads

$$\sigma(\boldsymbol{x},t)\nu(\boldsymbol{x}) = F(\boldsymbol{x},t,q,q_\tau,\bar{\mathbf{u}},\bar{\mathbf{u}}_t) \quad \text{on } \Gamma\times[0,T], \tag{8}$$

where $\nu(\cdot)$ is the outwarding unit normal along Γ. Let us summarize the equations of motion for $(\mathbf{u},q)$ describing the *cutter* and the *workpiece*, respectively. We seek $\mathbf{u} : \Omega\times[0,T] \to \mathbf{R}^3$ and $q : [0,T] \to \mathbf{R}^3$ satisfying

$$\left.\begin{array}{rcll} \varrho_0\ddot{\mathbf{u}}(\boldsymbol{x},t) - \operatorname{div}\sigma(\boldsymbol{x},\mathrm{t}) &=& 0 & \text{in } \Omega\times(0,T] \\ \mathbf{u}(\boldsymbol{x},t) &=& 0 & \text{on } \Gamma_0\times[0,T] \\ \sigma(\boldsymbol{x},t)\nu(\boldsymbol{x}) &=& f(\boldsymbol{x},t,q,q_\tau,\bar{\mathbf{u}},\bar{\mathbf{u}}_t) & \text{on } \Gamma\times[0,T] \end{array}\right\} \tag{9}$$

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = N(t)(q(t) - q_\tau(t) - \bar{\mathbf{u}}(t) + \bar{\mathbf{u}}_\tau(t)) \quad \text{in } [0,T]\} \tag{10}$$

with the initial conditions

$$\mathbf{u}(0) = g^0, \quad \text{and} \quad \dot{\mathbf{u}}(0) = h^0 \quad \text{in } \Omega, \tag{11a}$$

and

$$q = l^1 \quad \text{and} \quad \bar{\mathbf{u}} = l^2, \quad \text{on } [-\tau,0]. \tag{12}$$

The functions g^0, h^0, l^1 and l^2 are assumed to be given with

$$g^0 \in H^1_\Gamma(\Omega,\mathbf{R}^3), \qquad h^0 \in L_2(\Omega;\mathbf{R}^3), \tag{13}$$

$$l^1 \in W^{1,\infty}([-\tau,0],\mathbf{R}^3), \qquad l^2 \in C([-\tau,0];\mathbf{R}^3). \tag{14}$$

We assume $N \in C([0,T];\mathbf{R}^{3,3})$. The force $F(\boldsymbol{x},t)$ is not continuous since if we fix t then the function exhibits a discontinuity while passing from $\Gamma_R(t)$ to $\Gamma(t)$. Note that F depends not only on the spatial and time variable $\boldsymbol{x}$ resp. t but also on $\bar{\mathbf{u}}$ and q.

2.2 Notation

For Ω, Γ and Γ_0 as above and for $m \geq 1$, we define,

$$\begin{aligned}
C_c^\infty(\Omega, \mathbf{R}^m) &:= \{f|_\Omega : \ f \in C^\infty(\mathbf{R}^2, \mathbf{R}^m), \ \text{supp} f \cap \partial\Omega = \emptyset\} \\
C_\Gamma^\infty(\Omega, \mathbf{R}^m) &:= \{f|_\Omega : \ f \in C^\infty(\mathbf{R}^2, \mathbf{R}^m), \ \text{supp} f \cap \Gamma_0 = \emptyset\} \\
C_\Gamma(\Omega, \mathbf{R}^m) &:= \{f : \ f \in C(\bar{\Omega}, \mathbf{R}^m), \ f = 0 \ \text{ on} \, \Gamma_0\}.
\end{aligned}$$

In the scalar valued case, that is, $m = 1$, we omit the last argument, for instance, we write $C_c^\infty(\Omega) := C_c^\infty(\Omega, \mathbf{R}^1)$. For all finite integers $p, p' \geq 1$ with $1/p + 1/p' = 1$, we define the Sobolev space

$$W^1_{\Gamma,p}(\Omega, \mathbf{R}^d) = \overline{C_\Gamma^\infty(\Omega, \mathbf{R}^d)}^{W^1_p}, \qquad W^{-1}_{\Gamma,p}(\Omega, \mathbf{R}^d) := (W^1_{\Gamma,p'}(\Omega, \mathbf{R}^d))^*. \tag{15}$$

In case $\Gamma = \emptyset$ we write $\mathring{W}^1_p(\Omega, \mathbf{R}^d) := W^1_{\Gamma,p}(\Omega, \mathbf{R}^d)$. In the scalar valued case we set $W^1_{\Gamma,p}(\Omega) := W^1_{\Gamma,p}(\Omega, \mathbf{R}^1)$ and similarly for the other spaces. In case $p = 2$ we the use the notation $W^1_{\Gamma,2}(\Omega, \mathbf{R}^d) =: H^1_\Gamma(\Omega, \mathbf{R}^d)$ and in case $\Gamma = \emptyset$ also $\mathring{H}^1(\Omega, \mathbf{R}^d) := W^1_{\Gamma,2}(\Omega, \mathbf{R}^d)$.

2.3 Weak solutions and well-posedness

Let us first study the function $t \mapsto \bar{\mathbf{u}}(t)$.

Lemma 1. *Let Assumption 3 be satisfied and suppose that $\mathbf{u} \in C(0, T; H^1(\Omega, \mathbf{R}^3))$ then the map*

$$[0, T] \ni t \mapsto \bar{\mathbf{u}}(t) \in \mathbf{R}^3 \tag{16}$$

is continuous after possibly redefining $\bar{\mathbf{u}}$ on a set $\mathcal{A} \subset [0, T]$ of measure zero, i.e., $\bar{\mathbf{u}} \in C([0, T]; \mathbf{R}^3)$.

Proof. Due to Assumption 3 the mapping $[0, T] \ni t \mapsto |\Gamma(t)|$ is continuous. Thus it is sufficient to show that $[0, T] \ni t \mapsto \int_{\Gamma(t)} \mathbf{u}(t) \, ds$ is continuous. Indeed we have for a.e. $t, t' \in [0, T]$

$$\begin{aligned}
&\left| \int_{\Gamma(t)} \mathbf{u}(t) \, ds - \int_{\Gamma(t')} \mathbf{u}(t') \, ds \right| = \left| \int_{\Gamma(t)} \mathbf{u}(t) - \mathbf{u}(t') \, ds + \int_{\Gamma(t)} \mathbf{u}(t') \, ds - \int_{\Gamma(t')} \mathbf{u}(t') \, ds \right| \\
&\leq C \|\mathbf{u}(t) - \mathbf{u}(t')\|_{H^1} + \sqrt{|\Gamma(t) \setminus \Gamma(t')|} \|\mathbf{u}\|_{C([0,T];H^1)} + \sqrt{|\Gamma(t') \setminus \Gamma(t)|} \|\mathbf{u}\|_{C([0,T];H^1)}.
\end{aligned} \tag{17}$$

The last two terms on the right-hand side of the inequality are controlled by Assumption 6 and for the first note that $\mathbf{u} \in C([0, T]; H^1(\Omega, \mathbf{R}^3))$. ■

In what follows we use the abbreviation

$$f(t, \mathbf{u}, q) = f(q(t), \bar{\mathbf{u}}(t), q_\tau(t), \bar{\mathbf{u}}_\tau(t)) \tag{18}$$

It is convenient to introduce for every linear mapping $A : \mathbf{R}^{3,3} \to \mathbf{R}^{3,3}$ the bilinear form

$$\mathfrak{a}^A(\mathbf{u}, \boldsymbol{v}) = \int_\Omega A\varepsilon(\mathbf{u}) : \varepsilon(\boldsymbol{v}) \, dx, \quad \mathbf{u}, \boldsymbol{v} \in H^1(\Omega; \mathbf{R}^3). \tag{19}$$

In case $A = I$ is the identity we write $\mathfrak{a}(u, v)$. Since A is positive definite and symmetric, and due to Korn's inequality the bilinear form $\mathfrak{a}^A(\cdot, \cdot)$ is $H^1_\Gamma(\Omega, \mathbf{R}^3)$-coercive, continuous and symmetric. At first we give the definition of a weak solution of the governing equations (9) and (10).

Definition 2.1 (Weak solution). *A pair* $(\mathbf{u}, q) \in L_2(0,T; H^1_\Gamma(\Omega,\mathbf{R}^3)) \times L_2(0,T;\mathbf{R}^3)$ *with*

$$\dot{\mathbf{u}} \in L_2(0,T; H^1_\Gamma(\Omega,\mathbf{R}^3)) \text{ and } \ddot{\mathbf{u}} \in L_2(0,T; H^{-1}_\Gamma(\Omega,\mathbf{R}^3)), \tag{20}$$

and

$$\dot{q}, \ddot{q} \in L_2(0,T;\mathbf{R}^3),$$

is a weak solution of the initial-boundary value problem (9) *and* (10)*, if*

$$\langle \varrho_0 \ddot{\mathbf{u}}(t), \boldsymbol{v}\rangle_{H^{-1}_\Gamma, H^1_\Gamma} + \mathfrak{a}^A(\mathbf{u}(t), \boldsymbol{v}) + \delta\mathfrak{a}(\dot{\mathbf{u}}(t), \boldsymbol{v}) = f(t,\boldsymbol{z}) \cdot \int_{\Gamma(t)} \boldsymbol{v}\, ds, \tag{21}$$

for all $\boldsymbol{v} \in H^1_\Gamma(\Omega,\mathbf{R}^3)$ *and a.e.* $t \in (0,T)$*,*

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = N(t)(q(t) - q_\tau(t) - \bar{\mathbf{u}}(t) + \bar{\mathbf{u}}_\tau(t)), \tag{22}$$

for a.e. $t \in (0,T)$*, and*

$$\begin{aligned} &\mathbf{u}(0) = g^0 \quad \text{and} \quad \dot{\mathbf{u}}(0) = h^0 \text{ on } \Omega, \\ &q(s) = l^1(s) \quad \text{and} \quad \bar{\mathbf{u}}(s) = l^2(s) \quad \text{for a.e. } s \in [-\tau, 0]. \end{aligned} \tag{23}$$

We refer to [10] for the following result:

Lemma 2. *For every* $T > 0$ *there exists a unique weak solution* $(q, \mathbf{u})$ *to the equations* (9) *and* (10)*. Moreover, we have the following regularity*

$$\mathbf{u} \in L_\infty(0,T; H^1_\Gamma(\Omega,\mathbf{R}^3)), \quad \dot{\mathbf{u}} \in L_\infty(0,T; L_2(\Omega;\mathbf{R}^3)) \cap L_2(0,T; H^1_\Gamma(\Omega,\mathbf{R}^3)), \quad \ddot{\mathbf{u}} \in L_2(0,T; H^{-1}_\Gamma(\Omega,\mathbf{R}^3)),$$

$$q, \dot{q} \in L_\infty(0,T;\mathbf{R}^3), \quad \ddot{q} \in L_2(0,T;\mathbf{R}^3).$$

We now introduce a semi-discrete and a fully discrete scheme for the state system (25)-(27) and derive error estimates for the respective discretization.

2.4 Semi-discretization

We consider the following semi-discrete approximation of the solution to (25),(26):

Definition 2.2 (Semi-discrete scheme). *Let* $\mathcal{S}_h \subset H^1_\Gamma(\Omega,\mathbf{R}^3)$ *be a finite dimensional subspace. We call the pair* $(\mathbf{u}_h, q_h) \in L_2(0,T;\mathcal{S}_h) \times L_2(0,T;\mathbf{R}^3)$ *semi-discrete solution of* (9) *and* (10) *with respect to* $\mathcal{S}_h$ *if*

$$\dot{\mathbf{u}}_h, \ddot{\mathbf{u}}_h \in L_2(0,T;\mathcal{S}_h) \quad \text{and} \quad \dot{q}_h, \ddot{q}_h \in L_2(0,T;\mathbf{R}^3), \tag{24}$$

solves

$$\int_\Omega \varrho_0 \ddot{\mathbf{u}}_h(t) \cdot \boldsymbol{v}_h \, dx + \mathfrak{a}^A(\mathbf{u}_h(t), \boldsymbol{v}_h) + \delta\mathfrak{a}(\dot{\mathbf{u}}_h(t), \boldsymbol{v}_h) = f(t,\boldsymbol{z}) \cdot \int_{\Gamma(t)} \boldsymbol{v}_h \, ds, \tag{25}$$

for all $\boldsymbol{v}_h \in H^1_\Gamma(\Omega,\mathbf{R}^3)$ *and a.e.* $t \in (0,T)$*,*

$$M\ddot{q}_h(t) + D\dot{q}_h(t) + Kq_h(t) = N(t)(q_h(t) - (q_h)_\tau(t) - \bar{\mathbf{u}}_h(t) + (\bar{\mathbf{u}}_h)_\tau(t)), \tag{26}$$

for a.e. $t \in (0,T)$*, and*

$$\mathbf{u}_h(0) = \mathcal{R}_h(g^0) \quad \textit{and} \quad \dot{\mathbf{u}}_h(0) = \mathcal{P}_h(h^0) \textit{ on } \Omega,$$
$$q_h(s) = l_h^1(s) \quad \textit{and} \quad \bar{\mathbf{u}}(s) = l_h^2(s) \quad \textit{for a.e. } s \in [-\tau, 0]. \tag{27}$$

Here $\mathcal{P}_h : H^1_\Gamma(\Omega;\mathbf{R}^3) \to \mathcal{S}_h$ *and* $\mathcal{R}_h : L_2(\Omega;\mathbf{R}^3) \to \mathcal{S}_h$ *are projections from* $H^1_\Gamma(\Omega;\mathbf{R}^3)$ *and* $L_2(\Omega;\mathbf{R}^3)$ *respectively into* $\mathcal{S}_h$*, satisfying* $\lim_{h\downarrow 0} \|\mathcal{R}_h(g^0) - g^0\|_{L_2} = 0$ *and* $\lim_{h\downarrow 0} \|\mathcal{P}_h(h^0) - h^0\|_{H^1} = 0$*. The function* l_h^1, l_h^2 *satisfy* $\|l_h^1 - l^1\|_{L_\infty} \to 0$ *and* $\|l_h^2 - l^2\|_{L_\infty} \to 0$ *as* $h \searrow 0$*.*

To prove error estimates, we assume that our state system has the following additional regularity:

Assumption 5. *There is* $\epsilon > 0$ *so that* $\mathbf{u} \in H^2(0,T;H^{3/2-\epsilon}(\Omega;\mathbf{R}^3))$*.*

Recall the embedding $H^2(0,T;H^{3/2-\epsilon}(\Omega;\mathbf{R}^3)) \subset C^1([0,T];H^{3/2-\epsilon}(\Omega;\mathbf{R}^3))$.

Remark 2.3. *Under the regularity assumption on* $\mathbf{u}$ *the duality paring on the left hand side of* (25) *becomes an proper integral, i.e.,*

$$\langle \varrho_0 \ddot{\mathbf{u}}(t), \boldsymbol{v} \rangle_{H_\Gamma^{-1}, H_\Gamma^1} = \varrho_0 \int_\Omega \ddot{\mathbf{u}}(t) \cdot \boldsymbol{v} \, dx$$

for almost all $t \in [0,T]$ *and all* $\boldsymbol{v} \in H^1_\Gamma(\Omega, \mathbf{R}^3)$*.*

Remark 2.4. *Notice that we cannot expect the solution associated with the bilinear form* $\mathfrak{a}^A(\cdot,\cdot)$ *to belong to* $H^2(\Omega;\mathbf{R}^3)$ *due to the mixed boundary conditions. At best (cf. Grisvard [8]) we can expect* $\mathbf{u} \in H^s(\Omega;\mathbf{R}^3)$ *for some* $s < 3/2$*, which motivates the above regularity assumption.*

We obtain the following error estimate for the semi-discrete approximation:

Theorem 1. *Let* $(\mathbf{u}, q)$ *be the weak solution of* (9),(10)*, and let* $(\mathbf{u}_h, q_h)$ *be the corresponding Galerkin solution. Furthermore, suppose Assumption 7 holds true. Then there are constants* $C_1, C_2 > 0$*, independent of* h*, such that for all small* $h > 0$*:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L_\infty(H^1)} + \|\dot{\mathbf{u}} - \dot{\mathbf{u}}_h\|_{L_\infty(L_2)} + \|\dot{\mathbf{u}} - \dot{\mathbf{u}}_h\|_{L_2(H^1)} \le C_1 h. \tag{28}$$

and

$$\|q - q_h\|_{L_\infty} + \|\dot{q} - \dot{q}_h\|_{L_\infty} \le C_2 h. \tag{29}$$

The constants C_1, C_2 *depend on* $\mathbf{u}, g^0$ *and* h^0*.*

2.5 A fully discrete scheme of order two

We divide $[0,T]$ into $J \ge 1$ equidistant pieces of width $\kappa := T/J$,

$$0 = t_0 < t_1 < t_2 < \cdots < t_{J-1} < t_J = T,$$

where $t_n = n\kappa$, $n = 0,1,2,\ldots,J$. Assume that there is a smallest number $J' \in \mathbf{N}$ such that $J'\kappa = \tau$. We also set $t_{n-J'} := t_n - \tau$ for $n \le J'$. Let $(\mathbf{u}, q)$ denote the weak solution. In what follows, we use the abbreviations

$$\mathbf{u}^n := \mathbf{u}(t_n) \quad \text{on } \Omega \quad (n = 0,1,2,\ldots,J)$$

and

$$q^n := q(t_n) \quad (n = 0,1,2,\ldots,J).$$

For the initial data l^1, l^2 specified in (13),(14), we set

$$l_n^{1,\tau} := l^1(t_n - \tau) \quad \text{and} \quad l_n^{2,\tau} := l^2(t_n - \tau) \quad (n = 0, 1, 2, \dots, J'). \tag{30}$$

Moreover, we introduce the following notation

$$\begin{aligned} q^{n+1/2} &:= \frac{q^{n+1} + q^{n-1}}{2}, \qquad \partial_\kappa q := \frac{q^{n+1} - q^{n-1}}{2\kappa}, \\ \partial_\kappa^+ q^n &:= \frac{q^{n+1} - q^n}{\kappa}, \qquad \partial_\kappa^- q^n := \frac{q^n - q^{n-1}}{\kappa}, \\ \partial_\kappa^2 q^n &:= \frac{q^{n+1} - 2q^n + q^{n-1}}{\kappa^2}. \end{aligned} \tag{31}$$

Now we define a fully discrete approximation of the solution to (25),(26):

Definition 2.5. *(Fully discrete scheme) Find $U^0, \dots, U^J \in \mathcal{S}_h$ such that*

$$(\varrho_0 \partial_\kappa U^n, \boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(U^{n+1/2}, \boldsymbol{v}_h) + \delta\mathfrak{a}(\partial_\kappa U^n, \boldsymbol{v}_h) = f(Q^n, \bar{U}^n, Q^{n-J'}, \bar{U}^{n-J'}) \cdot \int_{\Gamma(t_n)} \boldsymbol{v}_h \, ds \tag{32}$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h, n = 1, 2, \dots, J-1$ and $Q^0, \dots, Q^J \in \mathbf{R}^3$ such that

$$M\partial_\kappa^2 Q^n + D\partial_\kappa Q^n + KQ^{n+1/2} = N(t_n)(Q^n - \bar{U}^n - Q^{n-J'} + \bar{U}^{n-J'}), \quad n = 1, 2, \dots, J-1, \tag{33}$$

where $Q^{n-J'}, \bar{U}^{n-J'}$ is prescribed for $n = 1, 2, \dots, J'$ by the initial data l_n^1, l_n^2, respectively.

Note that U^0, U^1 as well as Q^0, Q^1 have to be chosen before calculating $U^2, \dots, U^J$ respectively $Q^2, \dots, Q^J$ from the above schemes. In what follows we assume that U^1, Q^1 are good approximations of $\mathbf{u}(\kappa), q(\kappa)$, respectively and we set $U^0 := \mathbf{W}_h(0)$ and $Q^0 := q(0) = l_1(0)$, where $\mathbf{W}_h$ is defined as usual by (37).

The fully discrete scheme is a second order consistent scheme and as a second main result of this paper we show second order in time and $1/2 - \epsilon$ convergence in space.

Theorem 2. *Let $(\mathbf{u}, q)$ be the weak solution of* (25), (26) *and let U^n, Q^n be defined by the scheme* (32), (33)*. Suppose that Assumption 10 is fulfilled. Then exist constants $C_1, C_2 > 0$ independent of κ and h, so that*

$$\|\partial_\kappa \mathbf{u}^n - \partial_\kappa U^n\|_{L_2} \leq C_1(h^{1/2-\epsilon} + \kappa^2), \tag{34}$$

for $n = 1, 2, \dots, J-1$ and

$$\|q^n - Q^n\| + \|\mathbf{u}^n - U^n\|_{H^1} \leq C_2(h^{1/2-\epsilon} + \kappa^2), \tag{35}$$

for $n = 0, 1, 2, \dots, J-1.$

3 The semi-discrete problem – proof of Theorem 1

In what follows, we assume that $(\mathbf{u}, q)$ is the weak solution satisfying (25), (26) and $(\mathbf{u}_h, q_h)$ is the Galerkin solution with respect to $\mathcal{S}_h$. We split $\mathbf{u}(t) - \mathbf{u}_h(t)$ into two parts for fixed $t \in (0, \tau)$:

$$\mathbf{u}_h(t) - \mathbf{u}(t) = \underbrace{\mathbf{u}_h(t) - \mathbf{W}_h(t)}_{:=\boldsymbol{\zeta}_h(t)} + \underbrace{\mathbf{W}_h(t) - \mathbf{u}(t)}_{\boldsymbol{\eta}_h(t)}, \tag{36}$$

where the function $\mathbf{W}_h : [0, T] \to \mathcal{S}_h$ is defined for every t in $[0, T]$ by

$$\mathfrak{a}^A\left(\mathbf{W}_h(t) - \mathbf{u}(t), \boldsymbol{v}_h\right) = 0 \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h. \tag{37}$$

For each t the function $\mathbf{W}_h(t)$ is the projection of $\mathbf{u}(t)$ in $\mathcal{S}_h$ with respect to the inner product defined by $\mathfrak{a}^A(\cdot, \cdot)$. The existence of the function $\mathbf{W}_h(t)$ for every $t \in [0, T]$ is ensured by the Theorem of Lax-Milgram. We next show that $\mathbf{W}_h$ is differentiable with respect to t if $\mathbf{u}$ is.

Lemma 3. *Let $\boldsymbol{W}$ be the projection of $\mathbf{u}$ onto $\mathcal{S}_h$ defined in* (37).

(a) Suppose $\mathbf{u} \in W_2^k(0,T;H^1_\Gamma(\Omega,\mathbf{R}^3))$. Then $\boldsymbol{W}_h : [0,T] \to H^1_\Gamma(\Omega,\mathbf{R}^3)$ is k times almost everywhere differentiable when $H^1_\Gamma(\Omega,\mathbf{R}^3)$ is equipped with the weak topology. The kth derivative $\mathbf{W}_h^{(k)}(t) \in \mathcal{S}_h$ is the projection of the kth derivative of $\mathbf{u}(t)$, i.e.,

$$\mathfrak{a}^A(\boldsymbol{W}_h^{(k)}(t) - \mathbf{u}^{(k)}(t), \boldsymbol{v}_h) = 0 \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h. \tag{38}$$

(b) Suppose $\mathbf{u} \in C^k(0,T;H^1_\Gamma(\Omega,\mathbf{R}^3))$. Then $\boldsymbol{W}_h : [0,T] \to H^1_\Gamma(\Omega,\mathbf{R}^3)$ is k-times differentiable. The kth derivative $\mathbf{W}_h^{(k)}(t) \in \mathcal{S}_h$ is the projection of the kth derivative of $\mathbf{u}(t)$ and as in (a) given by (38).

(c) Moreover, if $\mathbf{u} \in H^k(0,T;H^1_\Gamma(\Omega,\mathbf{R}^3))$ we have for $i = 0,1,2,\ldots,k$,

$$\|\boldsymbol{W}_h^{(i)}(t) - \mathbf{u}^{(i)}(t)\|_{H^1} \le C \inf_{\boldsymbol{v}_h \in \mathcal{S}_h} \|\boldsymbol{v}_h - \mathbf{u}^{(i)}(t)\|_{H^1} \tag{39}$$

for a.e. $t \in [0,T]$, where $C > 0$.

Proof. (a) By definition of $\boldsymbol{W}_h$ we have for all $t \in (0,T)$ and all small Δt,

$$\mathfrak{a}^A\left(\frac{\boldsymbol{W}_h(t+\Delta t) - \boldsymbol{W}_h(t)}{\Delta t} - \frac{\mathbf{u}_h(t+\Delta t) - \mathbf{u}_h(t)}{\Delta t}, \boldsymbol{v}_h\right) = 0 \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h. \tag{40}$$

Hence testing (40) with $\boldsymbol{v}_h = (\boldsymbol{W}_h(t+\Delta t) - \boldsymbol{W}_h(t))/\Delta t$, using the coercivity of $\mathfrak{a}^A$ and applying Young's inequality yields for some constant $C > 0$,

$$\left\|\frac{\boldsymbol{W}_h(t+\Delta t) - \boldsymbol{W}_h(t)}{\Delta t}\right\|_{H^1} \le C \left\|\frac{\mathbf{u}_h(t+\Delta t) - \mathbf{u}_h(t)}{\Delta t}\right\|_{H^1}. \tag{41}$$

Now thanks to [6, p.286, Thm. 2] we have $\mathbf{u}_h(t+\Delta t) - \mathbf{u}_h(t) = \int_{\Delta t}^{t+\Delta t} \dot{\mathbf{u}}_h(s)\, ds$ and consequently (41) yields

$$\left\|\frac{\boldsymbol{W}_h(t+\Delta t) - \boldsymbol{W}_h(t)}{\Delta t}\right\|_{H^1} \le \frac{C}{\Delta t} \int_h^{t+\Delta t} \|\dot{\mathbf{u}}_h(s)\|_{H^1}\; ds. \tag{42}$$

At every Lebesgue point t of $s \mapsto \|\dot{\mathbf{u}}_h(s)\|_{H^1}$ the right hand side of (42) is bounded for all small h. It follows that for almost all t in $(0,T)$ the sequence $f_{\Delta t}(t) := \frac{\boldsymbol{W}_h(t+\Delta t) - \boldsymbol{W}_h(t)}{\Delta t}$ is bounded in $H^1_\Gamma(\Omega,\mathbf{R}^3)$ and hence for every null-sequence (Δt_k) there is a subsequence still indexed the same so that $(f_{\Delta t_k}(t))$ converges weakly to some element f in $H^1_\Gamma(\Omega,\mathbf{R}^3)$. As a result we may pass to the limit in (40) to obtain

$$\mathfrak{a}^A(f(t) - \dot{\mathbf{u}}(t), \boldsymbol{v}_h) = 0 \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h. \tag{43}$$

As the previous equation admits a unique solution we must have $f_{\Delta t}(t) \to f(t)$ weakly in $H^1_\Gamma(\Omega,\mathbf{R}^3)$ as $\Delta t \to 0$ for almost all t in $(0,T)$. This shows (a) in case $k = 1$ and the case $k \ge 2$ follows easily by induction.

(b) The proof is similar to the one of (a) and omitted.

(c) Finally equation (39) follows from Cea's Lemma; cf.[4]. ∎

Assumption 6 (Interpolation property of $\mathcal{S}_h$). *Let $1 \le m \le k < \infty$ be two integers. Suppose the family of spaces $\mathcal{S}_h \subset H^m(\Omega;\mathbf{R}^3) \cap H^1_\Gamma(\Omega,\mathbf{R}^3)$ has the property that for all $\mathbf{u} \in H^k(\Omega;\mathbf{R}^3) \cap H^1_\Gamma(\Omega,\mathbf{R}^3)$*

$$\inf_{\boldsymbol{v} \in \mathcal{S}_h} \|\mathbf{u} - \boldsymbol{v}\|_{H^m(\Omega;\mathbf{R}^3)} \le C h^{m-k} \|\mathbf{u}\|_{H^k(\Omega;\mathbf{R}^3)}. \tag{44}$$

The previous assumption only provides us with estimates on integer Sobolev spaces. However, [4, Theorem 14.4.2, p. 379] shows that Assumption 6 implies for $s < m$ and $m \le r \le k$,

$$\inf_{\boldsymbol{v}\in\mathcal{S}_h} \left(h^s\|\mathbf{u}-\boldsymbol{v}\|_{H^s(\Omega;\mathbf{R}^3)} + h^m\|\mathbf{u}-\boldsymbol{v}\|_{H^m(\Omega;\mathbf{R}^3)}\right) \le ch^r\|\mathbf{u}\|_{H^r(\Omega;\mathbf{R}^3)} \tag{45}$$

for all $\mathbf{u} \in H^r(\Omega;\mathbf{R}^3)$. Now (45) applied with $r = 3/2-\epsilon$, $s = 0$ and $m = 1$ yields for $i = 0,1,2$

$$\inf_{\boldsymbol{v}\in\mathcal{S}_h} \|\mathbf{u}^{(i)}(t)-\boldsymbol{v}\|_{H^1(\Omega;\mathbf{R}^3)} \le ch^{1/2-\epsilon}\|\mathbf{u}^{(i)}(t)\|_{H^{3/2-\epsilon}(\Omega;\mathbf{R}^3)}, \tag{46}$$

for a.e. $t \in (0,T)$. Combining (39) with the previous equation (46) shows for $i = 0,1,2$,

$$\|\mathbf{W}_h^{(i)}(t)-\mathbf{u}^{(i)}(t)\|_{H^1} \le ch^{1/2-\epsilon}\|\mathbf{u}^{(i)}(t)\|_{H^{3/2-\epsilon}(\Omega;\mathbf{R}^3)} \tag{47}$$

for a.e. $t \in (0,T)$.

We gather our findings in the following lemma.

Lemma 4. *Let Assumption 5 and 6 be satisfied. Then there is a positive constant c such that*

$$\|\boldsymbol{\eta}_h\|_{L_\infty(H^1)} + \|\dot{\boldsymbol{\eta}}_h\|_{L_\infty(H^1)} + \|\ddot{\boldsymbol{\eta}}_h\|_{L_2(H^1)} \le ch^{1/2-\epsilon}. \tag{48}$$

Proof. This follows at once from (47) and Assumption 5. ■

In view of the decomposition (36) we only need to find an estimate for $\boldsymbol{\zeta}_h$ in order to get an estimate for $\mathbf{u}-\mathbf{u}_h$, which we recall is our final goal.

Using the projection equation (37) and the definition of the weak solution $\mathbf{u}$ (see Definition 25), we find

$$\mathfrak{a}^A(\mathbf{W}_h(t),\boldsymbol{v}_h) \overset{(37)}{=} \mathfrak{a}^A(\mathbf{u}(t),\boldsymbol{v}_h) \overset{(25)}{=} -(\varrho_0\ddot{\mathbf{u}}(t),\boldsymbol{v}_h)_{L_2} + \int_{\Gamma(t)} f(t,\mathbf{u},q)\cdot\boldsymbol{v}_h\,ds - \delta\mathfrak{a}(\dot{\mathbf{u}}(t),\boldsymbol{v}_h) \tag{49}$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h$ and for a.e. $t \in (0,T)$. Consequently

$$\begin{aligned}&(\varrho_0\ddot{\mathbf{W}}_h(t),\boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(\mathbf{W}_h(t),\boldsymbol{v}_h) + \delta\mathfrak{a}(\dot{\mathbf{W}}_h(t),\boldsymbol{v}_h) = \\ &= (\varrho_0\ddot{\boldsymbol{\eta}}_h(t),\boldsymbol{v}_h)_{L_2} + \delta\int_\Omega \varepsilon(\dot{\boldsymbol{\eta}}_h(t)):\varepsilon(\boldsymbol{v}_h)\,dx + \int_{\Gamma(t)} f(t,\mathbf{u},q)\cdot\boldsymbol{v}_h\,ds\end{aligned} \tag{50}$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h$ and for a.e. $t \in (0,T)$. Now subtracting (50) from the Galerkin equation for $\mathbf{u}_h$ and then inserting $\dot{\boldsymbol{\zeta}}_h(t) \in \mathcal{S}_h$ as a test function (recall $\boldsymbol{\zeta}_h(t) = \mathbf{u}_h(t) - \mathbf{W}_h(t)$), we get

$$\begin{aligned}&\tfrac{\varrho_0}{2}\tfrac{d}{dt}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \mathfrak{a}^A(\boldsymbol{\zeta}_h(t),\dot{\boldsymbol{\zeta}}_h(t)) + \delta\mathfrak{a}(\dot{\boldsymbol{\zeta}}_h(t),\dot{\boldsymbol{\zeta}}_h(t)) = \\ &\underbrace{(f(t,\mathbf{u}_h,q_h)-f(t,\mathbf{u},q))\cdot\int_{\Gamma(t)}\dot{\boldsymbol{\zeta}}_h(t)\,ds}_{:=\mathfrak{B}(t)} - (\varrho_0\ddot{\boldsymbol{\eta}}_h(t),\dot{\boldsymbol{\zeta}}_h(t))_{L_2} - \delta\mathfrak{a}(\dot{\boldsymbol{\eta}}_h(t),\dot{\boldsymbol{\zeta}}_h(t))\end{aligned} \tag{51}$$

for a.e. $t \in (0,T)$. We get from Korn's inequality

$$\left.\begin{aligned}&\tfrac{\varrho_0}{2}\tfrac{d}{dt}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \tfrac{1}{2}\tfrac{d}{dt}\mathfrak{a}^A(\boldsymbol{\zeta}_h(t),\boldsymbol{\zeta}_h(t)) + \delta\alpha_K/2\|\dot{\boldsymbol{\zeta}}_h(t)\|_{H^1}^2 \\ &\le \mathfrak{B}(t) + c(\|\ddot{\boldsymbol{\eta}}_h(t)\|_{L_2}^2 + \|\dot{\boldsymbol{\eta}}_h(t)\|_{H^1}^2) + \tfrac{1}{2}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2\end{aligned}\right\} \tag{52}$$

for a.e. $t \in (0,T)$. Since f is Lipschitz continuous with constant $L > 0$, we obtain from Hölder's inequality and the trace theorem,

$$\begin{aligned}|\mathfrak{B}(t)| \le{}& L\sqrt{|\partial\Omega|}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{H^1}(|\bar{\boldsymbol{\eta}}_h(t)| + |\bar{\boldsymbol{\zeta}}_h(t)| + |q(t)-q_h(t)| + |q_\tau(t)-q_{h,\tau}(t)| \\ &+ |\bar{\boldsymbol{\eta}}_{h,\tau}(t)| + |\bar{\boldsymbol{\zeta}}_{h,\tau}(t)|),\end{aligned} \tag{53}$$

for a.e. $t \in (0,T)$. The next lemma shows how to handle the term $|q-q_h|$ on the interval $(0,\tau)$.

Lemma 5. *Let $(\mathbf{u}, q)$ and $(\mathbf{u}_h, q_h)$ be the weak and the Galerkin solution (with respect to $\mathcal{S}_h$), respectively. Then there exists a positive constant c, such that*

$$|q(t) - q_h(t)|^2 + |\dot{q}(t) - \dot{q}_h(t)|^2 \leq \mathfrak{R}_h + c \int_0^t |\bar{\mathbf{u}}(s) - \bar{\mathbf{u}}_h(s)|^2 \, ds \tag{54}$$

for a.e. $t \in (0, \tau)$. Here $\mathfrak{R}_h$ is independent of time and given by

$$\mathfrak{R}_h := c \left(|l_h^1(0) - l^1(0)|^2 + \int_{-\tau}^0 |l_h^1(s) - l^1(s)|^2 + |l_h^2(s) - l^2(s)|^2 + |\dot{q}(0) - \dot{q}_h(0)|^2 \right). \tag{55}$$

Proof. Introduce $(\hat{\mathbf{u}}, \hat{q}) = (\mathbf{u} - \mathbf{u}_h, q - q_h)$. Subtracting the differential equations for q and q_h (multiyplied by M^{-1} on both sides) and integrating over $(0, t) \subset (0, \tau)$ and using Young's inequality yields setting $D_M := M^{-1}D$ and $K_M := M^{-1}K$ and $N_M(t) := M^{-1}N(t)$,

$$\dot{\hat{q}}(t) + D_M \hat{q}(t) + K_M \int_0^t \hat{q}(s)\, ds = \int_0^t N_M(s)(\hat{\bar{\mathbf{u}}}_\tau(s) - \hat{q}_\tau(s) + \hat{q}(s) - \hat{\bar{\mathbf{u}}}(s))\, ds + D_M \hat{q}(0) + \dot{\hat{q}}(0) \tag{56}$$

for a.e. $t \in (0, \tau)$. Let us set $\hat{l}^1 := l_h^1 - l^1$ and $\hat{l}^2 := l_h^2 - l^2$ and recall $\hat{l}^1(t) = \hat{q}(t)$ and $\hat{l}^2(t) = \hat{\bar{\mathbf{u}}}(t)$ for $t \in [-\tau, 0]$. Then multiplying (56) with $\hat{q}$ and integrating over $(0, t) \subset (0, \tau)$ and using Young's inequality gives,

$$\frac{d}{dt}|\hat{q}(t)|^2 \leq C(|\hat{q}(t)|^2 + \int_0^t |\hat{q}(s)|^2 + |\hat{\bar{\mathbf{u}}}(s)|^2\, ds + \int_{-\tau}^0 |\hat{l}^1(s)|^2 + |\hat{l}^2(s)|^2\, ds + |\hat{q}(0)|^2 + |\dot{\hat{q}}(0)|^2) \tag{57}$$

for a.e $t \in (0, T)$. Gronwall's lemma in differential form yields

$$\begin{aligned} |\hat{q}(t)|^2 &\leq C(|\hat{q}(0)|^2 + |\dot{\hat{q}}(0)|^2 + \int_{-\tau}^0 |\hat{l}^1(s)|^2 + |\hat{l}^2(s)|^2\, ds) + C \int_0^t \left(\int_0^{s'} |\hat{q}(s)|^2 + |\hat{\bar{\mathbf{u}}}(s)|^2\, ds + \right) ds' \\ &\leq C \left(|\hat{q}(0)|^2 + |\dot{\hat{q}}(0)|^2 + \int_{-\tau}^0 |\hat{l}^1(s)|^2 + |\hat{l}^2(s)|^2\, ds + \int_0^t |\hat{q}(s)|^2 + |\hat{\bar{\mathbf{u}}}(s)|^2\, ds \right) \end{aligned} \tag{58}$$

for a.e $t \in (0, T)$. Another application of Gronwall's lemma in integral form yields

$$|\hat{q}(t)|^2 \leq C \left(|\hat{l}^1(0)|^2 + |\dot{\hat{q}}(0)|^2 + \int_{-\tau}^0 |\hat{l}^1(s)|^2 + |\hat{l}^2(s)|^2\, ds + \int_0^t |\hat{\bar{\mathbf{u}}}(s)|^2\, ds \right) \tag{59}$$

for a.e $t \in (0, T)$. Finally inequalities (59) and (56) show that also $|\dot{\hat{q}}(t)|^2$ is bounded by the right hand side of (59) and this finishes the proof. ∎

K: I improved the lemma, so now it is valid for M only invertible and no conditions on D and K.

Notice that the mean satisfies $|\bar{\mathbf{u}}(t)| \leq \gamma_1^{-1/2} \|\mathbf{u}(t)\|_{L_2(\Gamma(t);\mathbf{R}^3)}$ for almost every t in $(0, T)$. Hence Lemma 5 gives

$$|q(t) - q_h(t)|^2 \leq \mathfrak{R}_h + c \int_0^t \|\boldsymbol{\eta}_h(s)\|_{H^1}^2 + \|\boldsymbol{\zeta}_h(s)\|_{H^1}^2\, ds \quad \text{for a.e. } t \in (0, \tau). \tag{60}$$

Return now to inequality (52) and apply the estimates (60) and Young's inequality to (53),

$$|\mathfrak{B}(t)| \leq \frac{c}{\gamma} \left[\mathfrak{R}_h + \|\boldsymbol{\eta}_h(t)\|_{H^1}^2 + \|\boldsymbol{\zeta}_h(t)\|_{H^1}^2 + \int_0^t \|\boldsymbol{\eta}_h(s)\|_{H^1}^2 ds + \int_0^t \|\boldsymbol{\zeta}_h(s)\|_{H^1}^2 ds \right] + \gamma \|\dot{\boldsymbol{\zeta}}_h(t)\|_{H^1}^2 \tag{61}$$

for a.e. $t \in (0,\tau)$. Further, combining (52) and (61) and choosing $\gamma > 0$ sufficiently small gives

$$\frac{\varrho_0}{2}\frac{d}{dt}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \frac{1}{2}\frac{d}{dt}\mathfrak{a}(\boldsymbol{\zeta}_h(t),\boldsymbol{\zeta}_h(t)) + \alpha_K/4\|\dot{\boldsymbol{\zeta}}_h(t)\|_{H^1}^2 \le \frac{1}{2}\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2$$
$$+\|\boldsymbol{\zeta}_h(t)\|_{H^1}^2 + \int_0^t \|\boldsymbol{\zeta}_h(s)\|_{H^1}^2\,ds + c\bigg[\mathfrak{R}_h + \|\dot{\boldsymbol{\eta}}_h(t)\|_{H^1}^2 + \|\boldsymbol{\eta}_h(t)\|_{H^1}^2 \tag{62}$$
$$+\int_0^t \|\boldsymbol{\eta}_h(s)\|_{H^1}^2\,ds + \|\ddot{\boldsymbol{\eta}}_h(t)\|_{L_2}^2\bigg],$$

for a.e. $t \in (0,\tau)$. Now integrating the previous equation over $(0,t) \subset (0,T)$ yields

$$\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(t)\|_{H^1}^2 + \int_0^t \|\dot{\boldsymbol{\zeta}}_h(s)\|_{H^1}^2\,ds \le$$
$$c\bigg(\int_0^t \|\dot{\boldsymbol{\eta}}_h(s)\|_{H^1}^2 + \|\boldsymbol{\eta}_h(s)\|_{H^1}^2 + \|\ddot{\boldsymbol{\eta}}_h(s)\|_{L_2}^2\,ds + \mathfrak{R}_h \tag{63}$$
$$+\int_0^t \|\dot{\boldsymbol{\zeta}}_h(s)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(s)\|_{H^1}^2\,ds + \|\dot{\boldsymbol{\zeta}}_h(0)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(0)\|_{H^1}^2\bigg).$$

Now Gronwall's Lemma in integral form yields

$$\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(t)\|_{H^1}^2 + \textstyle\int_0^t \|\dot{\boldsymbol{\zeta}}_h(s)\|_{H^1}^2\,ds \le Cg(t), \tag{64}$$

for a.e. $t \in (0,\tau)$, where $C > 0$ is some constant and

$$g(t) := \int_0^t \|\dot{\boldsymbol{\eta}}_h(s)\|_{H^1}^2 + \|\boldsymbol{\eta}_h(s)\|_{H^1}^2 + \|\ddot{\boldsymbol{\eta}}_h(s)\|_{L_2}^2\,ds + \|\dot{\boldsymbol{\zeta}}_h(0)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(0)\|_{H^1}^2 + \mathfrak{R}_h.$$

Suppose that the following assumption is satisfied.

Assumption 7. *There holds*

$$\|\dot{\boldsymbol{\zeta}}_h(0)\|_{L_2} = \mathcal{O}(h), \quad \|\boldsymbol{\zeta}_h(0)\|_{H^1} = \mathcal{O}(h)$$

Then the function g can be estimated as follows

$$g(t) \le ch\left(\int_0^T \|\mathbf{u}(s)\|_{H^2(\Omega;\mathbf{R}^3)}^2 + \|\dot{\mathbf{u}}(s)\|_{H^2(\Omega;\mathbf{R}^3)}^2\,ds + 1\right).$$

We conclude from (64) that

$$\|\dot{\boldsymbol{\zeta}}_h(t)\|_{L_2}^2 + \|\boldsymbol{\zeta}_h(t)\|_{H^1}^2 + \int_0^t \|\dot{\boldsymbol{\zeta}}_h(s)\|_{H^1}^2 ds \le ch. \tag{65}$$

and hence taking the supremum on both sides yields

$$\|\dot{\boldsymbol{\zeta}}_h\|_{L_\infty(0,\tau;L_2)}^2 + \|\boldsymbol{\zeta}_h\|_{L_\infty(0,\tau;H^1)}^2 + \|\dot{\boldsymbol{\zeta}}_h\|_{L_2(0,\tau;H^1)}^2 \le ch. \tag{66}$$

Remark 3.1. *Although we have only shown the estimate* (66) *for times in* $(0,\tau)$ *it can be readily seen via a bootstrap argument that we have indeed*

$$\|\dot{\boldsymbol{\zeta}}_h\|_{L_\infty(0,T;L_2)}^2 + \|\boldsymbol{\zeta}_h\|_{L_\infty(0,T;H^1)}^2 + \|\dot{\boldsymbol{\zeta}}_h\|_{L_2(0,T;H^1)}^2 \le ch. \tag{67}$$

To see this suppose that (28),(29) *holds on* $(0,\tau)$ *and replace* $\mathfrak{R}_h$ *by*

$$\tilde{\mathfrak{R}}_h := c\left(|q_h(\tau) - q(\tau)|^2 + \int_0^\tau |q_h(s) - q(s)|^2 + |\bar{\mathbf{u}}_h(s) - \bar{\mathbf{u}}(s)|^2 + |\dot{q}(\tau) - \dot{q}_h(\tau)|^2\right). \tag{68}$$

Then it can be readily checked that $\tilde{\mathfrak{R}}_h \leq ch$ by using the estimates for $q_h - q$ and $\bar{\mathbf{u}}_h - \bar{\mathbf{u}}$ already shown for the interval $(0, \tau)$. A close inspection of the proof of Lemma 5 shows that the estimate (54) *still remains valid with $\mathfrak{R}_h$ replaced by $\tilde{\mathfrak{R}}_h$. So we can proceed as before and obtain*

$$\|\dot{\boldsymbol{\zeta}}_h\|^2_{L_\infty(0,2\tau;L_2)} + \|\boldsymbol{\zeta}_h\|^2_{L_\infty(0,2\tau;H^1)} + \|\dot{\boldsymbol{\zeta}}_h\|^2_{L_2(0,2\tau;H^1)} \leq ch \tag{69}$$

and consequently (28),(29) *on the interval $(0, 2\tau)$. Repeating these steps successively shows that* (28),(29) *must hold true on $(0, T)$.*

Remark 3.2. *Assumption 7 makes sense as we may estimate*

$$\begin{aligned} \|\dot{\boldsymbol{\zeta}}_h(0)\|_{L_2} = \|\dot{\mathbf{u}}_h(0) - \dot{\mathbf{W}}_h(0)\|_{L_2} &\leq \|\dot{\mathbf{u}}_h(0) - \dot{\mathbf{u}}(0)\|_{L_2} + \|\dot{\mathbf{u}}(0) - \dot{\mathbf{W}}_h(0)\|_{L_2} \\ &= \|\mathcal{R}_h(h^0) - h^0\|_{L_2} + \|h^0 - \dot{\mathbf{W}}_h(0)\|_{L_2}, \end{aligned} \tag{70}$$

where $\mathcal{R}_h$ is the projection defined in Definition 2.2. The first term constitutes the approximation of the initial data, whereas an estimate for the second is given by (47)*. The discussion of $\|\boldsymbol{\zeta}_h(0)\|_{H^1}$ is completely analogous.*

4 The fully discrete problem – proof of Theorem 2

At first we recall a discrete version of Gronwall's lemma that will be frequently used.

Lemma 6. *Let $N \in \mathbf{N}$ and suppose that the non-negative real numbers a_n, b_n, $0 \leq n \leq N$, satisfy*

$$a_n \leq \rho + \sum_{k=0}^{n-1} a_k b_k, \quad 0 \leq n \leq N. \tag{71}$$

Then

$$a_n \leq \rho \exp\left(\sum_{k=0}^{n-1} b_k\right), \quad 0 \leq n \leq N. \tag{72}$$

In particular, if $b_n = b$ for $0 \leq n \leq N$ then

$$a_n \leq \rho \exp nb. \tag{73}$$

As an immediate consequence of the previous lemma we obtain.

Lemma 7. *Let $N \in \mathbf{N}$ and suppose that the non-negative real numbers a_n, b_n, ρ_n, $0 \leq n \leq N$, and assume that ρ_n is non-decreasing. If*

$$a_n \leq \rho_n + \sum_{k=0}^{n-1} a_k b_k, \quad 0 \leq n \leq N \tag{74}$$

then

$$a_n \leq \rho_n \exp\left(\sum_{k=0}^{n-1} b_k\right), \quad 0 \leq n \leq N. \tag{75}$$

Let us first show that the system (33),(32) admits a unique solution.

Lemma 8. *The system* (33),(32) *admits a unique solution $U^0, \ldots, U^J \in \mathcal{S}_h$ and $Q^0, \ldots, Q^J \in \mathbf{R}^3$.*

Proof. We start with scheme (32),(33) and expand each function U^n in the basis $\{v_1, \ldots, v_m\}$ of $\mathcal{S}_h$

$$U^n(x) = \sum_{k=1}^{m} d_k^n v_k(x), \tag{76}$$

where $d_k^m \in \mathbf{R}$ are constant numbers and n runs from 0 to $J-1$. Note that κ indicates the approximating accuracy in time and h in space. With each U^n we associate a row vector $d^n := (d_1^n, \ldots, d_m^n)^\top$ containing the coefficients of the basis expansion (76). Now inserting (76) in (32) and selecting $\boldsymbol{v}_h = v_k,\ k = 1, 2, \ldots, J$, the scheme (32),(33) reads

$$
\begin{aligned}
&\hat{M}\partial_\kappa^2 d^n + \hat{K}d^{n+1/2} + \hat{D}\partial_\kappa d^n = \hat{F}(Q^n, d^n), \\
&M\partial_\kappa^2 Q^n + D\partial_\kappa Q^n + KQ^{n+1/2} = N(t_n)(Q^n - \bar{U}^n - Q^{n-J'} + \bar{U}^{n-J'}),
\end{aligned}
\tag{77}
$$

for $n = 1, 2, \ldots, J-1$. The components of $\hat{M}$, $\hat{D}$, $\hat{K}$ and $\hat{F}$ are

$$
\hat{M}_{ij} := \varrho_0(v_i, v_j)_{L_2}, \qquad \hat{K}_{ij} := \mathfrak{a}^A(v_i, v_j), \qquad \hat{D}_{ij} := \delta\mathfrak{a}(v_i, v_j), \tag{78}
$$

and

$$
\hat{F}_i(Q^n, d^n) = f(Q^n, \sum_{k=1}^m d_k^n \int_{\Gamma(t_n)} v_k\, ds, l_n^{1,\tau}, l_n^{2,\tau}) \cdot \int_{\Gamma(t_n)} v_i\, ds, \tag{79}
$$

where $i, j = 1, 2, \ldots, m$. Reordering the vectors d^{n-1}, d^n, d^{n+1} and Q^{n-1}, Q^n, Q^{n+1} in (77) yields:

$$
(\hat{M} + \frac{\kappa}{2}\hat{D} + \frac{\kappa^2}{2}\hat{K})d^{n+1} = -(\hat{M} - \frac{\kappa}{2}\hat{D} + \frac{\kappa^2}{2}\hat{K})d^{n-1} + 2\hat{M}d^n + \kappa^2\hat{F}(Q^n, d^n) \tag{80}
$$

$$
(M + \frac{\kappa}{2}D + \frac{\kappa^2}{2}K)Q^{n+1} = -(M - \frac{\kappa}{2}D + \frac{\kappa^2}{2}K)Q^{n-1} + 2MQ^n + \kappa^2 G(Q^n, d^n), \tag{81}
$$

for $n = 1, 2, \ldots, J-1$. Moreover,

$$
G(Q^n, d^n) := N(t_n)(Q^n - \sum_{k=1}^m d_k^n \frac{1}{|\Gamma(t_n)|} \int_{\Gamma(t_n)} v_k\, ds - q(t_n - \tau) + \bar{\mathbf{u}}(t_n - \tau)) \tag{82}
$$

for $n = 1, 2, \ldots, J-1$. Recall that the set of invertiable matrices in $\mathbf{R}^d$ are open. Thus since $\hat{M}$ and M are invertiable it follows that also $\hat{M} + \frac{\kappa}{2}\hat{D} + \frac{\kappa^2}{2}\hat{K}$ and $M + \frac{\kappa}{2}D + \frac{\kappa^2}{2}K$ are invertible for small $\kappa > 0$. As a consequence the system (80),(81) admits a unique solution provided κ is small enough. ■

Remark 4.1. *Note that once given the space $\mathcal{S}_h$, we have to compute the matrices $\hat{M}, \hat{D}$ and $\hat{K}$ only once. If d^0, d^1 and Q^0, Q^1 are given we can compute d^2 and Q^2 by solving the linear systems* (80) *and* (81)*. Also note that to compute $\bar{\mathbf{u}}(t_n - \tau)$ we only need to store the values of $u(x, t)$ on $\Gamma(t_n)$.*

Suppose we have computed $d^2, \ldots, d^J$ and $Q^2, \ldots, Q^J$. Furthermore, assume (u, q) is the weak solution of (9),(10). The following holds

$$
u(x, t_n) \approx \sum_{k=1}^m d_k^n v_k(x) \quad \text{and also} \quad q(t_n) \approx Q^n, \tag{83}
$$

where $n = 0, 1, \ldots, J$ and $x \in \Omega$. For that reason we define as before u_h to be the piecewise constant functions

$$
u_h(x, t) := \sum_{k=1}^m d_k^n v_k(x), \text{ if } t \in [t_n, t_{n+1}] \tag{84}
$$

and

$$
q_h(t) := Q^n \text{ if } t \in [t_n, t_{n+1}]. \tag{85}
$$

Let us first consider the local error $e_q^n := q(t_n) - Q^n$. We are going to derive an analogous equation to (33) for the points $q^n \in \mathbf{R}^3$, $n = 1, \ldots, J$.

As in the semi-discrete case, we need to assume a certain regularity of $(\mathbf{u}, q)$ to derive error estimates of the above scheme.

Assumption 8. *We assume that* $\mathbf{u} \in C^3([0,T]; H^{3/2-\epsilon}(\Omega; \mathbf{R}^3))$ *and* $q \in C^3([0,T]; \mathbf{R}^3)$.

Let us introduce the remainders

$$r_n^1 := \partial_\kappa^2 q^n - \ddot{q}(t_n), \qquad r_n^2 := \partial_\kappa q^n - \dot{q}(t_n), \qquad r_n^3 := q^{n+1/2} - q(t_n). \tag{86}$$

Not that $r_n^1, r_n^2 \in O(\kappa^2)$. They represent the consistency error of the time discretization for the ODE.

All constants c or C appearing in the following are independent of the subdivision of the interval $[-\tau, T]$.

Lemma 9. *Let* $(\mathbf{u}, q)$ *denote the weak solution of* (25),(26) *and* $U^0, \ldots, U^N$, $Q^0, \ldots, Q^N$ *the solution of* (32),(33), *respectively. There is a constant* $C > 0$, *so that*

$$|\partial_\kappa^+ e_q^k| + |e_q^{k+1}| + |e_q^k| \le C\left[(|\partial_\kappa^+ e_q^0| + |e_q^0| + |e_q^1|) + \kappa \sum_{n=1}^{J'} |e_{l_h^1}(t_{n-J'})| + |l_h^2(t_{n-J'})| + \sum_{n=1}^{k} \kappa(|\bar{e}_{\mathbf{u}}^n| + |r_n|)\right] \tag{87}$$

for $k = 0, 1, \ldots, J-1$, *where*

$$e_{l_h^1}(t_n) = l_h^1(t_n) - l^1(t_n) \quad \text{and} \quad e_{l_h^2}(t_n) = l_h^2(t_n) - l^2(t_n). \tag{88}$$

Proof. By definition of the remainders (86) equation (26) is equivalent to

$$\partial_\kappa^2 q^n + D_M \partial_\kappa q^n + K_M q^{n+1/2} + r_n = N_M(t_n)(q^n - q_\tau^n - \bar{\mathbf{u}}^n + \bar{\mathbf{u}}_\tau^n), \tag{89}$$

for $n = 1, 2, \ldots, J-1$, where r_n^i, $i = 1, 2, 3$ are the Taylor remainders in (86). Here, we introduced the remainder $r_n := r_n^1 + D_M r_n^2 + K_M r_n^3 \in O(\kappa)$. By assumption there is a constant $c > 0$ so that $r_{N,\text{max}} := \max_{l=0,\ldots,N-1} |r_l|$ satisfies

$$r_{N,\text{max}} \le \kappa^2 c \quad \text{for all } N \ge 1.$$

Recall the notation $D_M = M^{-1}D$, $K_M = M^{-1}K$ and $N_M(t) = M^{-1}N(t)$ and define the pointwise error

$$e_q^n := q(t_n) - Q^n, \quad \text{and} \quad e_{\mathbf{u}}^n := \mathbf{u}(t_n) - U^n.$$

Subtract (33) from (89) to obtain

$$\partial_\kappa^2 e_q^n + D_M \partial_\kappa e_q^n + K_M e_q^{n+1/2} = N_M(t_n)(e_q^n - \overline{e_{\mathbf{u}}}^n + \overline{e_{\mathbf{u}}}^{n-J'} - e_q^{n-J'}) - r_n, \tag{90}$$

for $n = 1, 2, \ldots, J-1$. In view of $\partial_\kappa^2 e_q^n = (\partial_\kappa^+ e_q^n - \partial_\kappa^+ e_q^{n-1})/\kappa$ we have

$$\sum_{n=1}^{k} \partial_\kappa^2 e_q^n = \frac{1}{\kappa}(\partial_\kappa^+ e_q^k - \partial_\kappa^+ e_q^0) \quad \text{and} \quad \sum_{n=1}^{k} \partial_\kappa e_q^n = \frac{1}{2\kappa}(e_q^{k+1} + e_q^k - e_q^1 - e_q^0). \tag{91}$$

Hence summing (90) over k and multiplying the result by 2κ yield

$$\begin{aligned} 2\partial_\kappa^+ e_q^k + D_M(e_q^{k+1} + e_q^k) + \kappa \sum_{n=1}^{k} K_M e_q^{n+1/2} =& 2\partial_\kappa^+ e_q^0 + D_M(e_q^1 + e_q^0) \\ &+ 2\sum_{n=1}^{k} \kappa N_M(t_n)(e_q^n - \overline{e_{\mathbf{u}}}^n + \overline{e_{\mathbf{u}}}^{n-J'} - e_q^{n-J'}) - \kappa r_n \end{aligned} \tag{92}$$

for $k = 1, \ldots, J'-1$ or equivalently

$$2\partial_\kappa^+ e_q^k + D_M(e_q^{k+1} + e_q^k) + \kappa K_M\left(\frac{e_q^k + e_q^{k-1}}{2}\right) = a_k \tag{93}$$

with

$$a_k := \kappa \sum_{n=1}^{k-1} K_M e_q^{n+1/2} - 2\partial_\kappa^+ e_q^0 + D_M(e_q^1 + e_q^0) + 2\sum_{n=1}^{k} \kappa N_M(t_n)(e_q^n - \overline{e_{\mathbf{u}}}^n + \overline{e_{\mathbf{u}}}^{n-J'} - e_q^{n-J'}) - \kappa r_n$$

On account of

$$e_q^{k+1} = \kappa \partial_\kappa^+ e_q^k + e_q^k \quad \text{and} \quad e_q^{k+1} = \sum_{l=0}^{k} \kappa \partial_\kappa^+ e_q^l + e_q^0 \tag{94}$$

we have

$$\begin{aligned} D_M(e_q^{k+1} + e_q^k) &= D_M\left(2e_q^0 + \kappa \sum_{l=0}^{k-1} \partial_\kappa^+ e_q^l\right) + \kappa D_M \partial_\kappa^+ e_q^k \\ \kappa K_M\left(\frac{e_q^k + e_q^{k-1}}{2}\right) &= \kappa K_M\left(e_q^0 + \frac{\kappa}{2}\sum_{l=0}^{k-2} \partial_\kappa^+ e_q^l\right) + \frac{\kappa^2}{2} K_M \partial_\kappa^+ e_q^{k-1}. \end{aligned} \tag{95}$$

Plugging these identities into (93) gives

$$(2I + \kappa D_M + \frac{\kappa^2}{2}K_M)\partial_\kappa^+ e_q^k + D_M\left(2e_q^0 + \kappa\sum_{l=0}^{k-1}\partial_\kappa^+ e_q^l\right) + \kappa K_M\left(e_q^0 + \frac{\kappa}{2}\sum_{l=0}^{k-2}\partial_\kappa^+ e_q^l\right) = a_k \tag{96}$$

for $k = 1, \ldots, J'-1$. Now according to [13, Lemma 2.8] there is a constant $c > 0$ so that for all sufficiently small κ we have

$$c|\zeta| \leq |(2I + \kappa D_M + \frac{\kappa^2}{2}K_M)\zeta| \quad \text{for all } \zeta \in \mathbf{R}^3. \tag{97}$$

Hence estimating (96) gives

$$|\partial_\kappa^+ e_q^k| \leq c\left(|e_q^0| + \kappa\sum_{n=0}^{k-1}|\partial_\kappa^+ e_q^n|\right) + |a_k| \tag{98}$$

for $k = 0, 1, \ldots, J'-1$. Applying the discrete version of Gronwall's lemma yields

$$|\partial_\kappa^+ e_q^k| \leq |a_k| \exp\left(\sum_{n=0}^{J'-1} c\kappa\right) \leq |a_k|\exp(c\tau). \tag{99}$$

Now using (94) we may estimate a_k as follows

$$|a_k| \leq c\kappa \sum_{n=0}^{k-1} |\partial_\kappa^+ e_q^n| + b_k, \quad k = 0, 1, \ldots, J'-1, \tag{100}$$

with

$$b_k := c\bigg(|\partial_\kappa^+ e_q^0| + |e_q^1| + |e_q^0| + \kappa\sum_{n=1}^{k} |\overline{e_{\mathbf{u}}}^n| + |e_{l_h^1}(t_{n-J'})| + |e_{l_h^2}(t_{n-J'})| + |r_n|\bigg). \tag{101}$$

Thus plugging (100) into (99) gives

$$|\partial_\kappa^+ e_q^k| \leq c\sum_{n=0}^{k-1} \kappa|\partial_\kappa^+ e_q^n| + cb_k, \quad k = 0, 1, \ldots, J'-1. \tag{102}$$

Another application of the discrete version of Gronwall's lemma gives $|\partial_\kappa^+ e_q^k| \leq cb_k$ for $k = 0, 1, \ldots, J'-1$ which is nothing but

$$|\partial_\kappa^+ e_q^k| \leq c\bigg(|\partial_\kappa^+ e_q^0| + |e_q^1| + |e_q^0| + \kappa\sum_{n=1}^{k} |\overline{e_{\mathbf{u}}}^n| + |e_{l_h^1}(t_{n-J'})| + |e_{l_h^2}(t_{n-J'})| + |r_n|\bigg) \tag{103}$$

for $k = 0, 1, \ldots, J' - 1$. In view of (94) also $|e_q^k|$ can be bounded by the right hand side of (103) which finishes the proof.

■

The previous lemma is the discrete analog of Lemma 5. In particular, we get from (87)

$$
\begin{aligned}
|q(t_k) - Q^k|^2 &\le \sum_{n=1}^{k} c\kappa |r_n|^2 + \kappa^2 c(\sum_{n=1}^{k} \|\mathbf{u}(t_k) - U^k\|_{H^1}^2 + \kappa^2 \sum_{n=1}^{J'} |e_{l_h^1}(t_{n-J'})|^2 + |e_{l_h^2}(t_{n-J'})|^2) \\
&\le c r_{N,\max}^2 + \kappa^2 c(\sum_{n=1}^{k} \|\boldsymbol{\eta}_h^n\|_{H^1}^2 + \|\boldsymbol{\zeta}_h^n\|_{H^1}^2 + \sum_{n=1}^{J'} |e_{l_h^1}(t_{n-J'})|^2 + |e_{l_h^2}(t_{n-J'})|^2)
\end{aligned}
\tag{104}
$$

for $k = 1, \ldots, J-1$, where $r_{N,\max} = \max_{l=0,\ldots,N-1} |r_l|$.

Now let us consider the local error $e_{\mathbf{u}}^n := \mathbf{u}(t_n) - U^n$ for $n = 0, 1, 2, \ldots, J$.

Assumption 9. *There are constants $c_1, c_2 > 0$ such that*

$$
\|\mathbf{u}(0) - U^0\|_{H^1} \le c_1 \kappa \quad \textit{and} \quad \|\mathbf{u}(\kappa) - U^1\|_{H^1} \le c_2 \kappa. \tag{105}
$$

To derive an asymptotic estimate we split the error at each time step t_n as follows

$$
\mathbf{u}(t_n) - U^n = \underbrace{\mathbf{u}(t_n) - \mathbf{W}_h(t_n)}_{=:\boldsymbol{\eta}_h^n} + \underbrace{\mathbf{W}_h(t_n) - U^n}_{=:\boldsymbol{\zeta}_h^n}, \tag{106}
$$

for $n = 0, 1, 2, \ldots, J$. Recall that $\mathbf{W}_h : [0, T] \to \mathcal{S}_h$ was defined in (37). As in the derivations above we use pointwise Taylor expansions of $\mathbf{u} : [0, T] \to H_\Gamma^{3/2-\epsilon}(\Omega, \mathbf{R}^3),\ t \mapsto \mathbf{u}(t)$. First define the remainders r_1^n, r_2^n and r_3^n

$$
r_1^n := \partial_\kappa^2 \mathbf{u}^n - \ddot{\mathbf{u}}(t_n), \quad r_2^n := (\partial_\kappa \mathbf{u}^n + \partial_\kappa \mathbf{u}^{n-1})/2 - \dot{\mathbf{u}}(t_n), \tag{107}
$$

and

$$
r_3^n := \mathbf{u}^{n+1/2} - \mathbf{u}(t_n), \tag{108}
$$

for $n = 1, 2, \ldots, J-1$. In view of Assumption 8 we get $r_i^n \in H^1(\Omega, \mathbf{R}^3),\ n = 1, 2, \ldots, J-1$. By definition of the function $\mathbf{u}$ and (107) it follows that $\mathbf{u}^0, \mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^J \in H_\Gamma^1(\Omega, \mathbf{R}^3)$ satisfy

$$
(\varrho_0 \partial_\kappa^2 \mathbf{u}^n, \boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(\mathbf{u}^{n+1/2}, \boldsymbol{v}_h) + \delta \mathfrak{a}^A(\partial_\kappa \mathbf{u}^n, \boldsymbol{v}_h) = R^n(\boldsymbol{v}_h) + f(q^n, \bar{\mathbf{u}}^n, q^{n-J'}, \bar{\mathbf{u}}_\tau^n) \cdot \int_{\Gamma(t_n)} \boldsymbol{v}_h \, ds \tag{109}
$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h$, and for $n = 1, 2, \ldots, J-1$. Here, we introduced the continuous (on H^1) functional $\boldsymbol{v}_h \mapsto R^n(\boldsymbol{v}_h) := -(\varrho_0 r_1^n, \boldsymbol{v}_h)_{L_2} - \mathfrak{a}^A(r_3^n, \boldsymbol{v}_h) - \delta \mathfrak{a}(r_2^n, \boldsymbol{v}_h)$ which satsfies $|R^n(\boldsymbol{v}_h)| \le c\|R^n\| \|\boldsymbol{v}_h\|_{H^1}$ and $\|R^n\| \le \tilde{c}\kappa$ for all n and all $\boldsymbol{v}_h \in \mathcal{S}_h$. The functional R_n represents the consistency error of our time discretization in time for the PDE. Now using the definition of the projection $\mathbf{W}_h^n$ and (109), we find

$$
\begin{aligned}
(\varrho_0 \partial_\kappa^2 \mathbf{W}_h^n, \boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(\mathbf{W}_h^{n+1/2}, \boldsymbol{v}_h) + \delta \mathfrak{a}(\partial_\kappa \mathbf{W}_h^n, \boldsymbol{v}_h) &= (\varrho_0 \partial_\kappa^2 \boldsymbol{\eta}_h^n, \boldsymbol{v}_h)_{L_2} + \delta \mathfrak{a}(\partial_\kappa \boldsymbol{\eta}_h^n, \boldsymbol{v}_h) \\
&\quad + R^n(\boldsymbol{v}_h) + f(q^n, \bar{\mathbf{u}}^n, q^{n-J'}, \bar{\mathbf{u}}^{n-J'}) \cdot \int_{\Gamma(t_n)} \boldsymbol{v}_h \, ds,
\end{aligned}
\tag{110}
$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h$ and $n = 1, 2, \ldots, J'-1$. On the other hand by definition of $U^0, \ldots, U^J \in \mathcal{S}_h$,

$$
(\varrho_0 \partial_\kappa U^n, \boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(U^{n+1/2}, \boldsymbol{v}_h) + \delta \mathfrak{a}(\partial_\kappa U^n, \boldsymbol{v}_h) = f(Q^n, \bar{U}^n, Q^{n-J'}, \bar{U}^{n-J'}) \cdot \int_{\Gamma(t_n)} \boldsymbol{v}_h \, ds \tag{111}
$$

for all $\boldsymbol{v}_h \in \mathcal{S}_h$ and $n = 1, 2, \ldots, J' - 1$. Hence subtracting (111) from (110) yields

$$(\varrho_0 \partial_\kappa^2 \boldsymbol{\zeta}_h^n, \boldsymbol{v}_h)_{L_2} + \mathfrak{a}^A(\boldsymbol{\zeta}_h^{n+1/2}, \boldsymbol{v}_h) + \delta \mathfrak{a}(\partial_\kappa \boldsymbol{\zeta}_h^n, \boldsymbol{v}_h) = (\varrho_0 \partial_\kappa^2 \boldsymbol{\eta}_h^n, \boldsymbol{v}_h)_{L_2} + \delta \mathfrak{a}(\partial_\kappa \boldsymbol{\eta}_h^n, \boldsymbol{v}_h) + R^n(\boldsymbol{v}_h) \\ +(f(q^n, \bar{\mathbf{u}}^n, q^{n-J'}, \bar{\mathbf{u}}^{n-J'}) - f(Q^n, \bar{U}^n, Q^{n-J'}, \bar{U}^{n-J'})) \cdot \int_{\Gamma(t_n)} \boldsymbol{v}_h \, ds. \tag{112}$$

For the next step we recall that $\partial_\kappa^2 \boldsymbol{\zeta}_h^n = (\partial_\kappa^+ \boldsymbol{\zeta}_h^n - \partial_\kappa^+ \boldsymbol{\zeta}_h^{n-1})/\kappa$ and $\partial_\kappa \boldsymbol{\zeta}_h^n = (\partial_\kappa^+ \boldsymbol{\zeta}_h^n - \partial_\kappa^+ \boldsymbol{\zeta}_h^n)/2$. Hence using $\boldsymbol{v}_h = \partial_\kappa \boldsymbol{\zeta}_h^n$ as a test function in (112) and summing up the result over $n = 1, \ldots, k \le J - 1$, we find

$$\begin{aligned} &\frac{1}{2\kappa} \varrho_0 \|\partial_\kappa^+ \boldsymbol{\zeta}_h^k\|_{L_2}^2 + \sum_{n=1}^k \delta \mathfrak{a}(\partial_\kappa \boldsymbol{\zeta}_h^n, \partial_\kappa \boldsymbol{\zeta}_h^n) + \frac{1}{4\kappa} \left(\mathfrak{a}^A(\boldsymbol{\zeta}_h^{k+1}, \boldsymbol{\zeta}_h^{k+1}) + \mathfrak{a}^A(\boldsymbol{\zeta}_h^k, \boldsymbol{\zeta}_h^k) \right) \\ =& \frac{1}{2\kappa} \varrho_0 \|\partial_\kappa^+ \boldsymbol{\zeta}_h^0\|_{L_2}^2 + \frac{1}{4\kappa} \left(\mathfrak{a}^A(\boldsymbol{\zeta}_h^0, \boldsymbol{\zeta}_h^0) + \mathfrak{a}^A(\boldsymbol{\zeta}_h^1, \boldsymbol{\zeta}_h^1) \right) + \sum_{n=1}^k (\varrho_0 \partial_\kappa^2 \boldsymbol{\eta}_h^n, \partial_\kappa \boldsymbol{\zeta}_h^n)_{L_2} \\ &+ \sum_{n=1}^k \delta \mathfrak{a}(\partial_\kappa \boldsymbol{\eta}_h^n, \partial_\kappa \boldsymbol{\zeta}_h^n) - \sum_{n=1}^k R^n(\partial_\kappa \boldsymbol{\zeta}_h^n) \\ &+ \sum_{n=1}^k \underbrace{(f(q^n, \bar{\mathbf{u}}^n, q^{n-J'}, \bar{\mathbf{u}}^{n-J'}) - f(Q^n, \bar{U}^n, Q^{n-J'}, \bar{U}^{n-J'}))}_{=:\mathfrak{B}^n} \cdot \int_{\Gamma(t_n)} \partial_\kappa \boldsymbol{\zeta}_h^n \, ds \end{aligned}$$

for $k = 1, 2, \ldots, J' - 1$. We next apply Korn's inequality to the left hand side and Youngs's inequality to the left hand side to shift the term $(\boldsymbol{\zeta}_h^{n+1} - \boldsymbol{\zeta}_h^{n-1})/2\kappa$ on the left-hand side

$$\|\partial_\kappa^+ \boldsymbol{\zeta}_h^k\|_{L_2}^2 + \|\boldsymbol{\zeta}_h^{k+1}\|_{H^1}^2 + \|\boldsymbol{\zeta}_h^k\|_{H^1}^2 + \sum_{n=1}^k \kappa \|\partial_\kappa \boldsymbol{\zeta}_h^n\|_{H^1}^2 \le \sum_{n=1}^k c\kappa^2 \left(\|\partial_\kappa^2 \boldsymbol{\eta}_h^n\|_{L_2}^2 + \|\partial_\kappa \boldsymbol{\eta}_h^n\|_{H^1}^2 \right) \\ + \sum_{n=1}^k \kappa^2 c |\mathfrak{B}^n|^2 + \sum_{n=1}^k c\kappa \|R_n\|^2 + \mathfrak{a}^A(\boldsymbol{\zeta}_h^1, \boldsymbol{\zeta}_h^1) + \mathfrak{a}^A(\boldsymbol{\zeta}_h^0, \boldsymbol{\zeta}_h^0) + \varrho_0 \|\partial_\kappa \boldsymbol{\zeta}_h^0\|_{L_2}^2 \tag{113}$$

for $k = 1, 2, \ldots, J' - 1$. For later reference notice that the consistency error $R_{N,\text{max}} = \max_{l=0,\ldots,N-1} \|R_l\|$ satisfies

$$R_{N,\text{max}} \le c\kappa^2 \quad \text{for all } N \ge 1$$

for some $c > 0$ which depends on the third derivative of $\mathbf{u}$. Therefore we may estimate as follows

$$\sum_{n=1}^k \kappa^2 \|R_n\|^2 \le c R_{N,\text{max}}^2.$$

It remains to estimate the term involving the Lipschitz continuous function f. Taking into account estimate (104), we get

$$\sum_{n=1}^k |\mathfrak{B}^n|^2 \le \sum_{n=1}^k c(\|\mathbf{u}^n - U^n\|_{H^1}^2 + |q^n - Q^n|^2 + |e_q^{n-J'}|^2 + |e_{\mathbf{u}}^{n-J'}|^2) \\ \le c r_{N,\text{max}}^2 + c \sum_{n=1}^k \|\boldsymbol{\zeta}_h^n\|_{H^1}^2 + \|\boldsymbol{\eta}_h^n\|_{H^1}^2 + \sum_{n=1}^{J'} c |e_{l_h^1}(t_{n-J'})|^2 + |e_{l_h^2}(t_{n-J'})|^2 \tag{114}$$

for $k = 1, 2, \ldots, J' - 1$. Notice that Assumption 9 ensures $|q^1 - Q^1| \le c\kappa$ and $\|\mathbf{u}(\kappa) - U^1\|_{H^1} \le c\kappa$.

Combining (114) and (113) we obtain

$$\begin{aligned}\|\partial_\kappa^+\boldsymbol{\zeta}_h^k\|_{L_2}^2+\|\boldsymbol{\zeta}_h^{k+1}\|_{H^1}^2+\|\boldsymbol{\zeta}_h^k\|_{H^1}^2\le{}& c(\kappa^2 r_{N,\max}^2+R_{N,\max}^2)+c\kappa^2\sum_{n=1}^{J'}|e_{l_h^1}(t_{n-J'})|^2+|e_{l_h^2}(t_{n-J'})|^2\\&+\sum_{n=1}^{k}c\kappa^2\left(\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2}^2+\|\partial_\kappa\boldsymbol{\eta}_h^n\|_{H^1}^2+\|\boldsymbol{\eta}_h^n\|_{H^1}^2\right)\\&+\|\boldsymbol{\zeta}_h^1\|_{H^1}^2+\|\boldsymbol{\zeta}_h^0\|_{H^1}^2+\varrho_0\|\partial_\kappa^+\boldsymbol{\zeta}_h^0\|_{L_2}^2\end{aligned}\tag{115}$$

for $k=1,2,\ldots,J'-1$. Choosing $c\kappa<1/2$ we may shift $c\kappa\|\zeta_h^k\|_{H^1}^2$ to the left hand side and get

$$\|\partial_\kappa^+\boldsymbol{\zeta}_h^k\|_{L_2}^2+\|\boldsymbol{\zeta}_h^{k+1}\|_{H^1}^2+\|\boldsymbol{\zeta}_h^k\|_{H^1}^2\le c\kappa\sum_{n=0}^{k-1}\|\partial_\kappa^+\boldsymbol{\zeta}_h^k\|_{L_2}^2+\|\boldsymbol{\zeta}_h^{k+1}\|_{H^1}^2+\|\boldsymbol{\zeta}_h^k\|_{H^1}^2+c_k\tag{116}$$

for $k=0,1,\ldots,J'-1$, where

$$\begin{aligned}c_k:={}&c(\kappa^2 r_{N,\max}^2+R_{N,\max}^2)+\sum_{n=0}^{k-1}c\kappa\|\boldsymbol{\zeta}_h^n\|_{H^1}^2+c\kappa^2\sum_{n=1}^{J'}|e_{l_h^1}(t_{n-J'})|^2+|e_{l_h^2}(t_{n-J'})|^2\\&+\sum_{n=1}^{k}c\kappa^2\left(\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2}^2+\|\partial_\kappa\boldsymbol{\eta}_h^n\|_{H^1}^2+\|\boldsymbol{\eta}_h^n\|_{H^1}^2\right)\\&+\|\boldsymbol{\zeta}_h^1\|_{H^1}^2+\|\boldsymbol{\zeta}_h^0\|_{H^1}^2+\varrho_0\|\partial_\kappa^+\boldsymbol{\zeta}_h^0\|_{L_2}^2.\end{aligned}\tag{117}$$

Consequently, an application of the discrete lemma of Gronwall yields

$$\begin{aligned}\|\partial_\kappa^+\boldsymbol{\zeta}_h^k\|_{L_2}^2+\|\boldsymbol{\zeta}_h^{k+1}\|_{H^1}^2+\|\boldsymbol{\zeta}_h^k\|_{H^1}^2\le{}&\sum_{n=1}^{k}\kappa^2\left(\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2}^2+\|\partial_\kappa\boldsymbol{\eta}_h^n\|_{H^1}^2+\|\boldsymbol{\eta}_h^n\|_{H^1}^2\right)\\&+c(\kappa^2 r_{N,\max}^2+R_{N,\max}^2)+c\kappa^2\sum_{n=1}^{J'}|e_{l_h^1}(t_{n-J'})|^2+|e_{l_h^2}(t_{n-J'})|^2\\&+\|\boldsymbol{\zeta}_h^1\|_{H^1}^2+\|\boldsymbol{\zeta}_h^0\|_{H^1}^2+\varrho_0\|\partial_\kappa^+\boldsymbol{\zeta}_h^0\|_{L_2}^2\end{aligned}\tag{118}$$

for $k=0,1,\ldots,J'-1$. To obtain a final estimate, we inspect the two terms on the right-hand side of the last inequality. Using Taylor's formula it can be readily checked that

$$\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2(\Omega,\mathbf{R}^3)}\le\kappa\|\ddot{\boldsymbol{\eta}}_h\|_{L_2(t_{n+1},t_{n-1};L_2(\Omega,\mathbf{R}^3))}\tag{119}$$

from whence we get

$$\sum_{n=1}^{J-1}\kappa\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2(\Omega,\mathbf{R}^3)}^2\le c\|\ddot{\boldsymbol{\eta}}_h\|_{L_2(0,T;L_2(\Omega;\mathbf{R}^3))}^2.\tag{120}$$

Now due to Assumption 8 we deduce from equation (47)

$$\int_0^t\|\ddot{\boldsymbol{\eta}}_h(s)\|_{H^1}^2ds\le C^2h^{1-2\epsilon}\|\ddot{\mathbf{u}}\|_{L_2(0,T;H^{3/2-\epsilon}(\Omega;\mathbf{R}^3))}^2\tag{121}$$

and thus we derive from (120)

$$\sum_{n=1}^{J-1}\kappa\|\partial_\kappa^2\boldsymbol{\eta}_h^n\|_{L_2}^2\le 4h^{1-2\epsilon}C^2\|\ddot{\mathbf{u}}\|_{L_2(0,T;H^{3/2-\epsilon}(\Omega;\mathbf{R}^3))}^2.\tag{122}$$

In a similar fashion we may show that

$$\|\boldsymbol{\eta}_h^n\|_{H^1} \le Ch^{1/2-\epsilon}\|\mathbf{u}\|_{L_2(0,T;H^{3/2-\epsilon}(\Omega;\mathbf{R}^3))}. \tag{123}$$

In order to estimate the other terms we pose the following natural assumption.

Assumption 10. *There are constants $c_1, c_2 > 0$, so that*

$$\|\boldsymbol{\zeta}_h^0\|_{H^1} \le c_1\kappa^2, \quad \|\boldsymbol{\zeta}_h^1\|_{H^1} \le c_2\kappa^2. \tag{124}$$

Recall that $J' \in \mathbf{N}$ is assumed to be the smallest integer such that $TJ' = \tau$. We have proved the following result:

Lemma 10. *Let $(\mathbf{u}, q)$ be the weak solution of* (25), (26) *and let U^n, Q^n be defined by the scheme* (32), (33). *Suppose that Assumption 10 is satisfied. Then we have*

$$\|\partial_\kappa^+\boldsymbol{\zeta}_h^k\|_{L_2} + \|\boldsymbol{\zeta}_h^{k+1}\|_{H^1} \le C(h^{1/2-\epsilon} + \kappa^2), \tag{125}$$

for $k = 1, 2, \dots, J' - 1$.

In view of $\|\mathbf{u}(t_n) - U^n\| \le \|\boldsymbol{\eta}_h^n\| + \|\boldsymbol{\zeta}_h^n\|$ equation (125) of the previous lemma proves Theorem 2 on the interval $[0, \tau]$, i.e. for $k = 1, \dots, J' - 1$. However using a bootstrap argument as in Remark 3.1 shows that Theorem 2 holds indeed on $[0, T]$.

Remark 4.2. *We conclude this section with some comments about the computation of U^0, U^1 and Q^0, Q^1. We set $U^0 := \mathbf{W}_h(0)$ where $\mathbf{W}_h(0)$ solves the following variational problem*

$$\mathbf{W}_h(0) \in \mathcal{S}_h : \quad a(\mathbf{W}_h(0), \boldsymbol{v}_h) = a(g^0, \boldsymbol{v}_h), \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h.$$

Furthermore, we take $U^1 := \mathbf{W}_h(0) + \kappa\dot{\mathbf{W}}_h(0) + \frac{1}{2}\kappa^2\ddot{\mathbf{W}}_h(0)$ whereas $\dot{\mathbf{W}}_h$ is a solution of

$$\dot{\mathbf{W}}_h \in \mathcal{S}_h : \quad a(\dot{\mathbf{W}}_h(0), \boldsymbol{v}_h) = a(h^0, \boldsymbol{v}_h), \quad \text{for all } \boldsymbol{v}_h \in \mathcal{S}_h, \tag{126}$$

and $\ddot{\mathbf{W}}_h$ is a solution of

$$\ddot{\mathbf{W}}_h \in \mathcal{S}_h : \quad a(\ddot{\mathbf{W}}_h(0), \boldsymbol{v}_h) = \frac{1}{\varrho_0} a(\operatorname{div}(\sigma(\cdot, 0)), \boldsymbol{v}_\mathrm{h}), \quad \text{for all } \boldsymbol{v}_\mathrm{h} \in \mathcal{S}_\mathrm{h}. \tag{127}$$

Note that $\operatorname{div}(\sigma(\cdot, 0))$ is completely determined by g^0 and h^0, but we require that the functions are smooth enough such that $\operatorname{div}(\sigma(\cdot, 0))$ makes sense. U^0 is simply the projection of $\mathbf{u}(0)$ onto $\mathcal{S}_h$ whereas U^1 is the projection of the Taylor expansion of $\mathbf{u}$ around zero up to the second derivative evaluated at κ. In case of the ODE we choose $Q^0 := q(0)$ and do a Taylor expansion around zero to get Q^1. Let $\hat{q}^0 := q(0)$ and $\hat{q}^1 := \dot{q}(0)$. We compute $\ddot{q}(0)$ from the ODE as follows:

$$\ddot{q}(0) = -M^{-1}D\dot{\hat{q}}(0) - M^{-1}K\hat{q}(0) + M^{-1}N(0)(q(0) - q_0(-\tau) - \bar{\mathbf{u}}(0) + \bar{\mathbf{u}}_0(-\tau)).$$

Finally, we define $Q^1 := \hat{q}(0) + \kappa\dot{\hat{q}}(0) + \frac{\kappa^2}{2}\ddot{\hat{q}}(0)$.

5 Numerical examples

In this chapter we present numerical simulations of the coupled model calculated with the fully discrete scheme (32),(33). The used code was developed by one of the authors as part of his PhD thesis [12] and is based on the semi-discretization explained above. We will use our fully discrete scheme (32),(33) and verify the convergence rates shown in Theorem 2. Since no analytic solution is available, we construct an auxiliary system for which a solution is known.

The space $\mathcal{S}_h$ is constructed using linear (Lagrange) finite elements. Assuming that $v \in H^{3/2-\epsilon}(\Omega; \mathbf{R}^3)$, we have the interpolation property

$$\|v - \mathcal{I}_h v\|_{H^1(\Omega;\mathbf{R}^3)} \leq ch^{1/2-\epsilon}\|v\|_{H^{3/2-\epsilon}(\Omega;\mathbf{R}^3)}, \tag{128}$$

where $\mathcal{I}_h$ denotes the usual (global) interpolation operator and $\mathcal{T}_h$ a triangulation of Ω, see [4] for definitions. The triangulation $\mathcal{T}_h$ consists of 3-*simplices* $K \subset \Omega$ and h is defined by[1] $h = max_{K\in\mathcal{T}_h} diam(K)$. The inequality shows that the interpolation property (128) (with $s = 1$ and $l = 0$) from the previous Chapter is satisfied. We define, as usual, the finite subspace

$$\mathcal{S}_h := \{v \in C(\overline{\Omega}; \mathbf{R}^3),\ v|_K \in \mathcal{P}^1(K; \mathbf{R}^3),\ K \in \mathcal{T}_h\}, \tag{129}$$

where $\mathcal{P}^1(K; \mathbf{R}^3)$ denotes the restriction of the space of polynomials of degree one.

5.1 An analytical solution

For numerically illustration of the above theory we consider a simplified situation, where the domain is a rectangular block with edge lengths $L_{1,2,3}$, i.e. $\Omega = [0, L_1] \times [0, L_2] \times [0, L_3]$. We assume that the load is transmitted on the whole face $x_2 = 0$, and consequently

$$\Gamma_N = \{(x_1, x_2, x_3)^\top \in \mathbf{R}^3 |\ x_2 = 0,\ 0 \leq x_1 \leq L_1,\ 0 \leq x_3 \leq L_3\}$$

is constant. Then we obtain

$$f(\tilde{q}(t), \bar{\tilde{\mathbf{u}}}(t), \tilde{q}(t-\tau), \bar{\tilde{\mathbf{u}}}(t-\tau)) = \frac{1}{L_1 L_3} N(t)(\tilde{q}(t) - \tilde{q}(t-\tau) - \bar{\tilde{u}}(t) + \bar{\tilde{u}}(t-\tau)). \tag{130}$$

For the other faces of the block we assume homogenous Dirichlet conditions. Since we have no analytic solution to (9) and (10) we proceed as follows. Suppose we are given two smooth enough functions $\tilde{u} : \overline{\Omega} \times [0, \tau] \to \mathbf{R}^3$ satisfying $\tilde{u}(x, t) = 0$ on $\Gamma_0 \times [0, T]$ and $\tilde{q} : [0, \tau] \to \mathbf{R}^3$ such that the following expressions make sense. Then define

$$f(x, t) := \partial_{tt}\tilde{u}(x, t) - \operatorname{div}(\hat{\sigma}(\tilde{\mathrm{u}}(\mathrm{x}, \mathrm{t}))), \tag{131}$$

$$g(x, t) := \hat{\sigma}(\tilde{u}(x, t)) \cdot \nu(x) + \frac{1}{L_1 L_3} N(t)(\tilde{q}(t) - \tilde{q}(t-\tau) - \bar{\tilde{u}}(t) + \bar{\tilde{u}}(t-\tau)) \tag{132}$$

and

$$\tilde{g}(t) := M\ddot{\tilde{q}}(t) + D\dot{\tilde{q}}(t) + K\tilde{q}(t) - N(t)(\tilde{q}(t) - \tilde{q}(t-\tau) - \bar{\tilde{u}}(t) + \bar{\tilde{u}}(t-\tau)). \tag{133}$$

One readily verifies that $u = \tilde{u}$ and $q = \tilde{q}$ solve

$$\begin{array}{rcll} \varrho_0\ddot{u}(x, t) - \operatorname{div}\sigma(\mathrm{x}, \mathrm{t}) & = & f(x, t) & \text{in } \Omega \times (0, T] \\ u(x, t) & = & 0 & \text{on } \Gamma_0 \times [0, T] \\ \sigma(x, t)\nu(x, t) & = & \frac{1}{L_1 L_3} N(t)(\tilde{q}(t) - \tilde{q}(t-\tau) & \\ & - & \bar{\tilde{u}}(t) - \bar{\tilde{u}}(t-\tau)) + g(x, t) & \text{on } \Gamma \times [0, T] \end{array} \tag{134}$$

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = N(t)(q(t) - q_\tau(t) - \bar{\mathbf{u}}(t) + \bar{\mathbf{u}}_t(-\tau)) + \tilde{g}(t) \quad \text{in } [0, T]. \tag{135}$$

Let Ω be the rectangular domain as described above. We prescribe the *displacement field* $\tilde{u} : \Omega \times [0, \tau] \to \mathbf{R}^3$ by

$$\tilde{u}(x_1, x_2, x_3, t) := x_1 x_3 (x_1 - L_1)(x_2 - L_2)(x_3 - L_3) \begin{pmatrix} A_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) \\ B_1 \sin(\omega_3 t) + B_2 \cos(\omega_4 t) \\ 0 \end{pmatrix} \tag{136}$$

[1] For $K \subset \mathbf{R}^3$ we define its *diameter* by $diam(K) := \sup_{x,y\in K} \|x - y\|$, here $\|\cdot\|$ denotes the euclidean norm on R^3.

λ in (kg/mm^2)	μ in (kg/mm^2)	ϱ_0 in kg	δ	m in (kg)	d_1, d_2, d_3 in (kg/s^2)	k_1, k_2, k_3 in $(kg * mm^2/s^3K)$
51083591.33	26315789.47	$2.70e-6$	$1.0e-2$	0.039930	462.6367147	1340049.648

Table 1: Cutter parameters and material parameters corresponding to aluminum.

ω_1 in $(1/s)$	ω_2 in $(1/s)$	ω_3 in $(1/s)$	ω_4 in $(1/s)$	A_1, A_2, B_1, B_2 in (mm)
$2\pi 100$	$2\pi 5$	$2\pi 5$	$2\pi 50$	1

Table 2: Data for analytic solution.

and the *cutter displacement* $\tilde{q} : [0, \tau] \to \mathbf{R}^3$ by

$$\tilde{q}(t) := \begin{pmatrix} C_1 \sin(\Omega_1 t) \\ D_1 \cos(\Omega_2 t) \\ E_1 \sin(\Omega_3 t) + E_2 \cos(\Omega_4 t) \end{pmatrix}. \tag{137}$$

Here, $\omega_i, \Omega_i, A_j, B_j, E_j$ $(i = 1, 2, 3, 4\ j = 1, 2)$ and C_1, D_1 are non-negative constants. The reason why we have chosen this particular form is that we want to see how good our scheme can approximate solutions with high frequency oscillations. Let $\lambda, \mu \in \mathbf{R}^+$ denote the Lamé constants. The outward unit normal vector along $\partial\Omega$ orthogonal to the $\{x_2 = 0\}$ plane has the simple form $\nu(x) = (0, -1, 0)^\top$. To calculate $\bar{\mathbf{u}}$, we parametrize

$$\Gamma_N = \{(x_1, x_2, x_3)^\top \in \mathbf{R}^3 |\ x_2 = 0,\ 0 \leq x_1 \leq L_1,\ 0 \leq x_3 \leq L_3\}$$

by $\xi : [0, L_1] \times [0, L_3] \to \mathbf{R}^3$

$$\xi(v, w) := ve_1 + we_3. \tag{138}$$

Furthermore, inserting the parametrization ξ into $u(x, t)$ the mean $\bar{\mathbf{u}}$ is given by

$$\bar{\tilde{u}}(t) = -L_1^2 L_2 L_3^2 \begin{pmatrix} A_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) \\ B_1 \sin(\omega_3 t) + B_2 \cos(\omega_4 t) \\ 0 \end{pmatrix}. \tag{139}$$

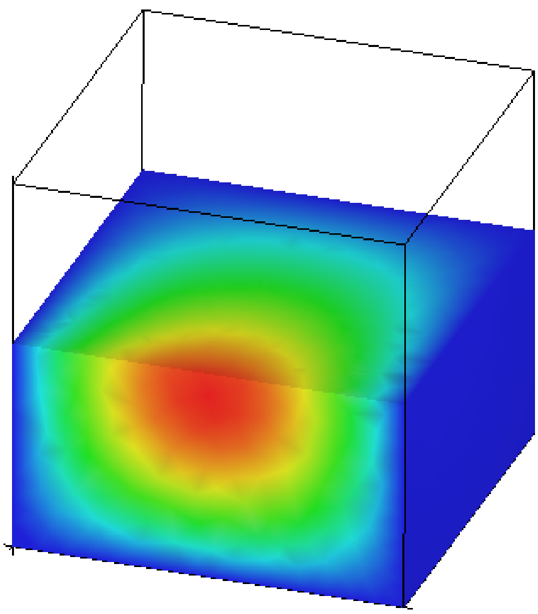


Figure 2: Simulation snapshot of $|u|$.

On the other hand we require $N(t)$ to be of the form

$$N(t) := \begin{pmatrix} P\sin(\gamma t) & P\cos(\gamma t) & 0 \\ -P\cos(\gamma t) & P\sin(\gamma t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{140}$$

where γ, P are non-negative constants. We can view the matrix N as an indicator for the strength of the coupling between the PDE and ODE. Let us assume further for M, D, K that $M = \text{diag}(m, m, m),\ D = \text{diag}(d_1, d_2, d_3)$ and $K = \text{diag}(k_1, k_2, k_3)$ for some constants $m, d_i, k_i \in \mathbf{R}^+\ i = 1, 2, 3$. Now we are in the position to calculate the corrections f, g and $\tilde{g}$ given by (131), (132) and (133), respectively. Tables 1, 2 contain a selection of parameters we used to compute the convergence results.

Remark 5.1. *It should be noticed that the auxiliary system* (134)*, in contrast to the former equations* (9) *and* (10)*, possesses additional terms $g(x,t), f(x,t)$ and $\tilde{g}(t)$. But as can be easily seen in the derivation of the error estimates for the fully discrete scheme* (32),(33)*, this doesn't affect investigations since we take differences $u - u_h,\ q - q_h$ and the inhomogeneities drop out.*

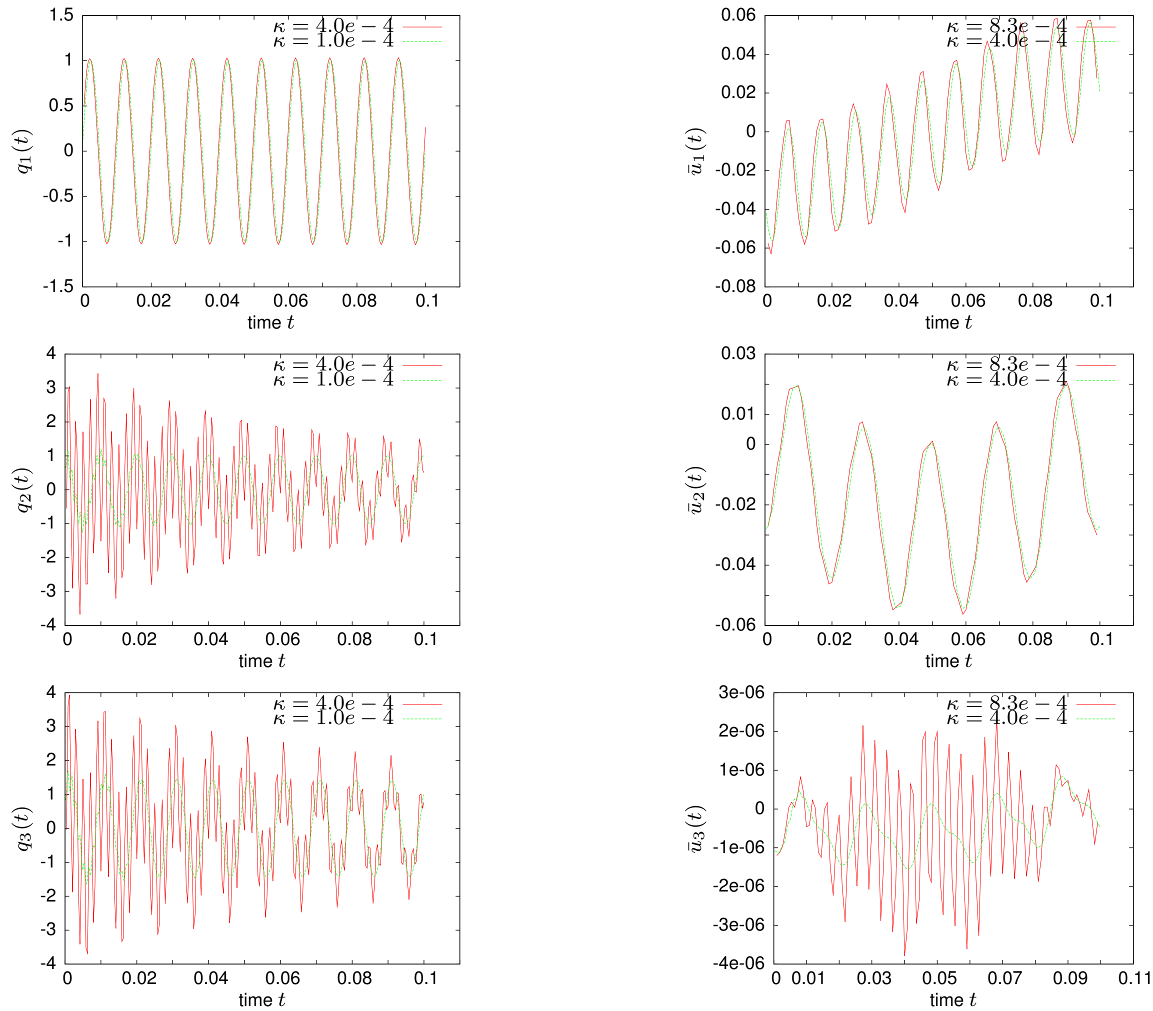


Figure 3: Temporal evolution of $q(t) = (q_1(t), q_2(t), q_3(t))^\top$ (left) and $\bar{\mathbf{u}}(t)$ (right), both for two different time discretizations.

To conclude, we summarize the numerical scheme to be solved iteratively:

$$(\hat{M} + \frac{\kappa}{2}\hat{D} + \frac{\kappa^2}{2}\hat{K})d^{n+1} = -(\hat{M} - \frac{\kappa}{2}\hat{D} + \frac{\kappa^2}{2}\hat{K})d^{n-1} + 2\hat{M}d^n + \kappa^2\hat{F}(Q^n, d^n), \tag{141}$$

and

$$(M + \frac{\kappa}{2}D + \frac{\kappa^2}{2}K)Q^{n+1} = -(M - \frac{\kappa}{2}D + \frac{\kappa^2}{2}K)Q^{n-1} + 2MQ^n + \kappa^2\hat{G}(Q^n, d^n), \tag{142}$$

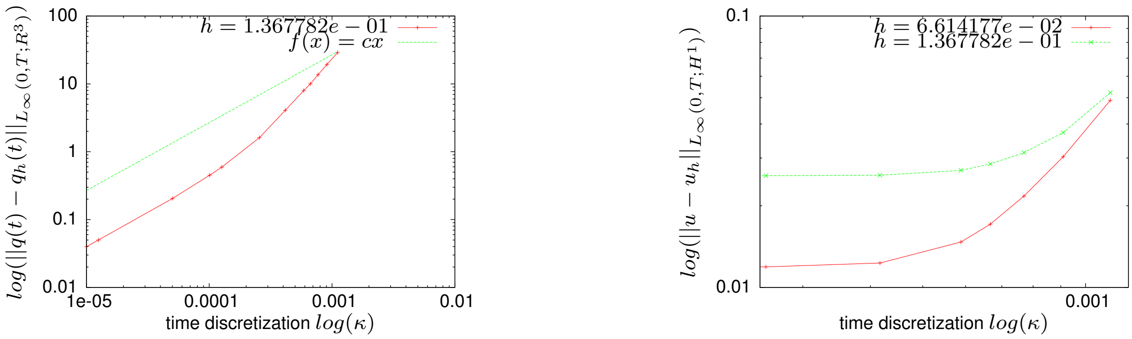
for $n = 1, 2, \ldots, J-1$. The functions $\hat{G}$ and $\hat{F}$ are defined by

$$\hat{G}(Q^n, d^n) := N(t_n)(Q^n - \sum_{k=1}^{m} d_k^n \frac{1}{|\Gamma_N|}\int_{\Gamma_N} v_k\, ds - l_n^{1,\tau} + l_n^{2,\tau}) + \tilde{g}(t_n)$$

and

$$\hat{F}_i(Q^n, d^n) := f(Q^n, \sum_{k=1}^{m} d_k^n \int_{\Gamma(t_n)} v_k\, ds, l_n^{1,\tau}, l_n^{2,\tau}) \cdot \int_{\Gamma_N} v_i\, ds$$
$$+ \int_{\Gamma_N} g(y, t_n) v_i\, ds + \int_{\Omega} f(x, t_n) v_i(x)\, dx,$$

for $i = 1, 2, \ldots, m$.



time κ	$\max_{s\in[0,T]} \|u(s) - u_h(s)\|_{H^1}$	$\|q_h - q\|_\infty$
$1.111111e-03$	$4.761151e-02$	$2.901541e+01$
$9.090909e-04$	$2.891132e-02$	$1.931800e+01$
$7.692308e-04$	$2.022721e-02$	$1.364647e+01$
$6.666667e-04$	$1.532741e-02$	$1.002739e+01$
$5.882353e-04$	$1.280207e-02$	$7.977786e+00$
$4.166667e-04$	$9.896003e-03$	$4.089229e+00$
$2.564103e-04$	$9.438067e-03$	$1.612259e+00$
$1.265823e-04$	$9.350392e-03$	$5.939188e-01$
$1.010101e-04$	$9.349263e-03$	$4.516434e-01$

Figure 4: Graphical error representation for cutter tip displacement (top left), workpiece displacement (top right), and a tabular representation for fixed space discretization $h = 6.614177e-02$ (bottom).

5.2 Numerical illustration of the convergence results

Figure 2 shows a snapshot of a numerical simulation at fixed time, the face in the foreground corresponds to $\{x_2 = 0\}$ where the load is transmitted. Figure 3 shows several oscillation periods for q and $\bar{u}$.

In the following we study the absolute errors $\|q - q_h\|_\infty$ and $\|u - u_h\|_{L_\infty(0,T;H^1)}$ for different time discretization κ and space discretization h. Here, $u = \tilde{u}$, $q = \tilde{q}$ are given by (136) and (137), respectively. The approximations $u_h(x,t), q_h(t)$ are computed with (141),(142). We have chosen $T = 0.1$ and set $\tau = T/J$ with $J \in \mathbf{N}$. All used parameters can be found in the tables at the end of the previous subsection. Note that we have (cf. (84),(85))

$$\|u - u_h\|_{L_\infty(0,T;H^1)} = \max_{n=0,1,\ldots,J} \|u(t_n) - U^n\|_{H^1} \tag{143}$$

and

$$\|q - q_h\|_\infty = \max_{n=0,1,\ldots,J} \|q(t_n) - Q^n\|. \tag{144}$$

Figure 4 depicts the error for cutter q and workpiece u displacements for fixed space discretization h and varying time step-size κ. Due to the additive decomposition of the error there is a threshold value depending on h such that the error remains invariant if we reduce κ below that threshold. Besides of this one can observe the predicted quadratic convergence behaviour.

Figure 5 shows the error in workpiece displacement for a fixed time step-size. The resulting linear convergence in h is better than the $h^{1/2} - \varepsilon$ rate predicted by Theorem 2, which is due to the higher regularity of the chosen explicit solution.

space h	$\max_{s\in[0,T]} \|u(s) - u_h(s)\|_{H^1}$
$2.875362e-01$	$5.507879e-02$
$2.308362e-01$	$4.424603e-02$
$1.367782e-01$	$2.570469e-02$
$1.107746e-01$	$2.059262e-02$
$6.614177e-02$	$1.186822e-02$

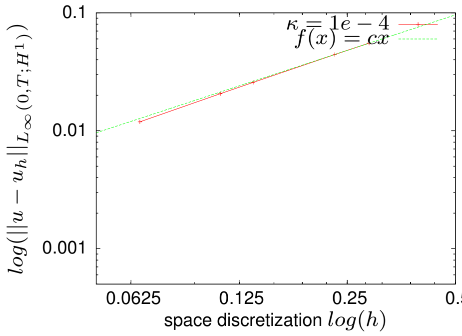


Figure 5: PDE error with fixed time discretization $\kappa = 1e-5$ and varying space discretization h.

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