Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Exponential decay of covariances for the supercritical membrane model

Erwin Bolthausen [1], Alessandra Cipriani [2], Noemi Kurt[3]

submitted: September 15, 2016

[1] Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zurich
Switzerland
E-Mail: eb@math.uzh.ch

[2] Weierstrass Institute
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: cipriani@wias-berlin.de

[3] Technische Universität Berlin
MA 766
Strasse des 17. Juni 136
10623 Berlin
Germany
E-Mail: kurt@math.tu-berlin.de

No. 2301
Berlin 2016

2000 *Mathematics Subject Classification.* Primary 60K35; secondary 31B30, 39A12, 60K37, 82B41.

Key words and phrases. Membrane model; pinning; bilaplacian; decay of covariances.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

We consider the membrane model, that is the centered Gaussian field on $\mathbb{Z}^d$ whose covariance matrix is given by the inverse of the discrete Bilaplacian. We impose a $\delta-$pinning condition, giving a reward of strength ε for the field to be 0 at any site of the lattice. In this paper we prove that in dimensions $d \geq 5$ covariances of the pinned field decay at least exponentially, as opposed to the field without pinning, where the decay is polynomial. The proof is based on estimates for certain discrete weighted norms, a percolation argument and on a Bernoulli domination result.

Contents

1 Introduction **1**

2 The model and main results **3**
2.1 Basic notations 3
2.2 The membrane model and statement of the main result 5

3 General results on the membrane model **7**

4 Proof of the main result **9**
4.1 Sobolev norms 9
4.2 Trapping configurations under the Bernoulli law 16
4.3 Proof of Theorem 2.5 20

1 Introduction

Effective interface models are well-studied real-valued random fields, defined for instance on the lattice $\mathbb{Z}^d$, which predict the behavior of polymers and interfaces between two states of matter. The best known examples are the gradient models $\varphi = \{\varphi_x\}_{x\in\mathbb{Z}^d}$ which (in formal notation) are of the form

$$P(\mathrm{d}\,\varphi) := \frac{1}{Z}\exp\left[-H(\varphi)\right]\prod_x \mathrm{d}\,\varphi_x,$$

with the Hamitonian

$$H(\varphi) := \sum_{x,\,y\in\mathbb{Z}^d,\ \|x-y\|=1} V(\varphi_x - \varphi_y)$$

where $V : \mathbb{R} \to \mathbb{R}$ is the interaction function, satisfying $V(x) \to \infty$ for $\|x\| \to +\infty$. The measure has to be defined through a thermodynamic limit. In the case $V(x) := \beta x^2$, the model is Gaussian, but it is defined on the whole of $\mathbb{Z}^d$ only for $d \geq 3$. For lower dimensions, one has to restrict x to a finite set, and put boundary conditions. This is the so-called Gaussian free field which has attracted tremendous attention recently for $d = 2$. One simplifying feature of the free field is that the covariances of the model are given in terms of the Green's function of a standard random walk on the lattice, and many properties of the field can be derived from properties of the random walk. This has led to powerful techniques for analysing the model. The case where V is not quadratic is much more complicated. If V is convex, there is still a random walk representation of the correlation, the Helffer-Sjöstrand representation, but in the case of non-convex V, random walk techniques cannot be applied, and many of the very basic questions are still open. For a recent investigation, see Adams (2006).

The so-called massive free field has the Hamiltonian

$$H(\varphi) := \beta \sum_{x,y,\ \|x-y\|=1} (\varphi_x - \varphi_y)^2 + m \sum_x \varphi_x^2, \quad \beta,\, m > 0,$$

and it is a Gaussian field which is well-defined on the full lattice in any dimension, and has exponentially decaying covariances. This just comes from the fact that the covariances are given by the Green's function of a random walk with a positive killing rate (Friedli and Velenik, 2015, Theorem 8.46)

It is quite astonishing that an exponential decay of correlations, in physics jargon a positive mass, also appears when the free field Hamiltonian is perturbed by an arbitrary small attraction to the origin, for instance in the form

$$H(\varphi) := \beta \sum_{x,y,\ \|x-y\|=1} (\varphi_x - \varphi_y)^2 + a \sum_x \mathbf{1}_{[-b,b]}(\varphi_x) \tag{1.1}$$

with $a,\, b > 0$ (see Velenik (2006, Section 5)). A somewhat simpler case is that of so-called δ-pinning where the reference measure $\prod_x \mathrm{d}\,\varphi_x$ is replaced by $\prod_x (\mathrm{d}\,\varphi_x + \varepsilon\delta_0(\mathrm{d}\,\varphi_x))$, and which can be obtained from (1.1) by a suitable limiting procedure letting $b \to 0,\ a \to +\infty$. All the proofs we are aware of rely heavily on random walk representations.

Our main object here is to discuss similar properties for the δ-pinned membrane model which has the Hamiltonian

$$H(\varphi) := \frac{1}{2} \sum_{x,y,\ \|x-y\|=1} (\Delta\varphi_x)^2$$

where Δ is the discrete Laplace operator on functions $f : \mathbb{Z}^d \to \mathbb{R}$, defined by

$$\Delta f(x) := \frac{1}{2d} \sum_{y:\, \|y-x\|=1} (f(y) - f(x)). \tag{1.2}$$

We leave out the temperature parameter β as it just leads to a trivial rescaling of the field.

While the free field (see and for an overview Friedli and Velenik (2015, Chapter 8)) is used to model polymers or interfaces with a tendency to maintain a constant mean height, the the membrane model appears in physical and biological research to shape interfaces that tend to have constant curvature (Hiergeist and Lipowsky, 1997, Leibler, 1989, Lipowsky, 1995, Ruiz-Lorenzo et al., 2005). In solid state physics one often considers models with mixed gradient and Laplacian Hamiltonian, but we will not discuss such cases here. The two models share many common characteristics, for instance their variances are uniformly bounded in $\mathbb{Z}^d$ if the dimension is large enough, that is $d \geq 3$ for the gradient case resp. $d \geq 5$ for the membrane model, and have variances growing logarithmically in $d = 2$ resp. $d = 4$.

The main topic of the present paper is an investigation of the decay of correlations for the membrane model in dimensions $d \geq 5$. We restrict to the case of δ-pinning for technical reasons. We prove that the field becomes “massive”, i.e. has exponentially decaying correlation for any positive pinning parameter.

The main difficulty when compared with the proofs of similar results for the free field is the absence of useful random walk representations for the covariances and correlation inequalities. Random walk representations for gradient fields have been very important since the celebrated work Brydges et al. (1982). There is a variant of a random walk representation in the case of the membrane model, but only in the presence of particular boundary conditions, or in the case of the field on the whole lattice in the absence of boundary conditions. Results on the membrane model with pinning were shown in $(1+1)$ dimensions by Caravenna and Deuschel (2008) using a renewal type of argument which, however, is not applicable in higher dimensions.

Structure of the paper. The structure of the paper is as follows: in Section 2 we give precise definitions on the membrane model and the statement of our main theorem. We recall general results, including Bernoulli domination, in Section 3. In Section 4 we prove our main theorem.

Acknowledgements: We are grateful to Vladimir Maz'ya who gave a significant input to the present work by showing how to prove the exponential decay for the Bilaplacian in the continuous space with a sufficiently dense set of deterministic “traps” and appropriate boundary conditions. For more information on the analytic background the reader can consult Maz'ya (2003).

This work was performed in part during visits of the first author to the TU Berlin and WIAS Berlin, and of the last two authors to the University of Zurich. We thank these institutions for their hospitality. Francesco Caravenna, Jean-Dominique Deuschel and Rajat Subhra Hazra are acknowledged for feedback and helpful discussions.

2 The model and main results

2.1 Basic notations

We will work on the d-dimensional integer lattice $\mathbb{Z}^d$, and in the present paper our focus will be in $d \geq 5$, although the basic definition is well-posed in all dimensions. Also, some of the partial results which don't rely on the dimension restriction will be stated and proved in generality.

For $N \in \mathbb{N}$, let $V_N := [-N/2,\, N/2]^d \cap \mathbb{Z}^d$ and $V_N^{\mathrm{c}} := \mathbb{Z}^d \setminus V_N$.

For $x,\, y \in \mathbb{Z}^d$, $\mathrm{d}(x,\, y)$ is the graph distance between x and y on the lattice with nearest-neighbor bonds, i.e. the ℓ_1-norm of $x - y$. With $\|\cdot\|$, we denote the Euclidean norm.

We will use L as a generic positive constant which depends only on the dimension d, not necessarily the same at different occurencies, and also not necessarily the same within the same formula. The dependence on d will not be mentioned, but dependence on other parameters will be noted by writing $L(k)$ or $L(\varepsilon)$, for instance.

We will consider real valued random fields $\{\varphi_x\}_{x\in\mathbb{Z}^d}$. For $A\subset\mathbb{Z}^d$, we write $\mathcal{F}_A$ for the σ-field generated by the random variables $\{\varphi_x,\ x\in A\}$. To be definite, we can of course have all the measures constructed on $\mathbb{R}^{\mathbb{Z}^d}$, equipped with the product σ-field.

We will typically use $x,\ y$ for points in $\mathbb{Z}^d$. If we write $\sum_x$, this means summation over all $\mathbb{Z}^d$. We will use e exclusively for the $2d$ elements of $\mathbb{Z}^d$ which are neighbors of 0. To keep notations less heavy, $\sum_e$ means that we sum over all these elements, and similarly for other discrete differential operators we will introduce. For a function f on $\mathbb{Z}^d$, we write

$$D_e f(x) := f(x+e) - f(x).$$

We write ∇f for the vector $(D_e f)_e$, and $\nabla^2 f$ for the matrix $(D_e D_{e'} f)_{e,e'}$, and similarly for the higher order derivatives which are denoted by $\nabla^3,\ \nabla^4$ etc. Remark that $\nabla^k f(x)$ depends on all the values $f(y)$ with $\mathrm{d}(y,x)\leq k$. We write

$$\left\|\nabla^k f(x)\right\|^2 = \left\|\nabla^k f(x)\right\|_2^2 := \sum_{e_1,\dots,e_k} \left|D_{e_1}D_{e_2}\cdots D_{e_k} f(x)\right|^2.$$

We also define $\left\|\nabla^k f(x)\right\|_\infty := \sup_{e_1,\dots,e_k}|D_{e_1}\cdots D_{e_k}f(x)|$. The Laplacian in (1.2) can be rewritten as

$$\Delta f(x) := \frac{1}{2d}\sum_e D_e f(x).$$

Remark that although the right hand side looks like being a first order discrete derivative, it is of course a second order one through the presence of e and $-e$ in the summation. Namely, if we define only the positive coordinate directions as $\{e^{(1)},\ \dots,\ e^{(d)}\}$, then the alternative definition

$$\Delta f(x) = -\frac{1}{2d}\sum_{i=1}^d D_{e^{(i)}}D_{-e^{(i)}}f(x) \tag{2.1}$$

holds. For two square summable functions f, g on $\mathbb{Z}^d$, we write

$$\langle f, g\rangle := \sum_{x\in\mathbb{Z}^d} f(x)\, g(x).$$

Summation by parts leads to the following properties:

Lemma 2.1. *Let $f, g:\mathbb{Z}^d\to\mathbb{R}$ be square summable functions.*

a) For any e

$$\langle D_e f, g\rangle = \langle f, D_{-e} g\rangle. \tag{2.2}$$

b)

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle. \tag{2.3}$$

c)

$$\sum_e \langle D_e f, D_e g\rangle = -4d\,\langle f, \Delta g\rangle. \tag{2.4}$$

2.2 The membrane model and statement of the main result

Definition 2.2 (Sakagawa (2003), Velenik (2006), Kurt (2008)). *Let $W \neq \emptyset$ be a finite subset of $\mathbb{Z}^d$. The* membrane model *on W is the random field $\{\varphi_x\}_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ with zero boundary conditions outside W, whose distribution is given by*

$$P_W(\mathrm{d}\varphi) = \frac{1}{Z_W} \exp\left(-\frac{1}{2}\langle \Delta\varphi, \Delta\varphi\rangle\right) \prod_{x\in W} \mathrm{d}\varphi_x \prod_{x\in W^{\mathrm{c}}} \delta_0(\mathrm{d}\varphi_x), \tag{2.5}$$

where Z_W is the normalizing constant.

In the case $W := V_N$, we simply write P_N instead of P_{V_N}.

It is notationally convenient to define the field $\{\varphi_x\}$ for $x \in \mathbb{Z}^d$, but as $\varphi_x = 0$ for $x \notin W$, it is just a centered Gaussian random vector $\{\varphi_x\}_{x\in W}$. By (2.3), one has

$$\langle \Delta\varphi, \Delta\varphi\rangle = \left\langle \varphi, \Delta^2\varphi\right\rangle .$$

Remark that in the inner product on the left hand side, one cannot restrict the sum to W even if φ is 0 outside W. There is in fact a contribution from the points at distance 1 to W. In contrast, in the inner product on the right hand side, the sum is only over W. P_W, when regarded as a law of a $\mathbb{R}^W$-valued vector, has density proportional to

$$\exp\left(-\frac{1}{2}\left\langle \varphi, \Delta_W^2\varphi\right\rangle\right)$$

where $\Delta_W^2 = \left(\Delta^2(x,\, y)\right)_{\{x,\, y\in W\}}$ is the the restriction of the Bilaplacian to W. Actually, in order to make (2.5) meaningful, one needs that Δ_W^2 is positive definite. This follows from the maximum principle for Δ. In fact $\langle \Delta\varphi, \Delta\varphi\rangle > 0$ holds for all φ which do not vanish identically, and are 0 on W^{c}. This proves the positive definiteness of Δ_W^2.

The covariances of the membrane model are given as

$$G_W(x,\, y) := \mathrm{cov}_{P_W}(\varphi_x, \varphi_y) = \left(\Delta_W^2\right)^{-1}(x,\, y), \quad x,\, y \in W, \tag{2.6}$$

It is convenient to extend G_W to $x,\, y \in \mathbb{Z}^d$ by setting the entries to 0 outside $W \times W$. For $x \in W$, the function $\mathbb{Z}^d \ni y \mapsto G_W(x,\, y)$ is the unique solution of the boundary value problem (Kurt, 2009)

$$\begin{cases} \Delta^2 G_W(x,\, y) = \delta_{x,y}, & y \in W \\ G_W(x,\, y) = 0, & y \notin W \end{cases} .$$

For $d \geq 5$ the weak limit $P := \lim_{N\to\infty} P_N$ exits (Sakagawa, 2003, Section II). Under P, the canonical coordinates $\{\varphi_x\}_{x\in\mathbb{Z}^d}$ form a centered Gaussian random field with covariance given by

$$G(x,\, y) = \Delta^{-2}(x,\, y) = \sum_{z\in\mathbb{Z}^d} \Delta^{-1}(x,z)\Delta^{-1}(z,y) = \sum_{z\in\mathbb{Z}^d} \Gamma(x,z)\Gamma(z,y),$$

where Γ is the Green's function of the discrete Laplacian on $\mathbb{Z}^d$. In particular observe that

$$G(0,\, 0) < +\infty. \tag{2.7}$$

The matrix Γ has a representation in terms of the simple random walk $(S_m)_{m\geq 0}$ on $\mathbb{Z}^d$ given by

$$\Gamma(x,\, y) = \sum_{m\geq 0} \mathrm{P}_x[S_m = y]$$

(P_x is the law of S starting at x). This entails that

$$G(x,\, y) = \sum_{m\geq 0}(m+1)\mathrm{P}_x[S_m = y] = \mathrm{E}_{x,\, y}\left[\sum_{\ell,\, m=0}^{\infty} \mathbf{1}_{\{S_m = \tilde{S}_\ell\}}\right]$$

where S and $\tilde{S}$ are two independent simple random walks starting at x and y respectively. Γ and G are translation invariant. Using the above above representation one can easily derive the following property of the covariance:

Lemma 2.3 (Sakagawa (2003, Lemma 5.1))**.** *For $d \geq 5$ there exists a constant $\kappa_d > 0$*

$$\lim_{\|x\|\to\infty} \frac{G(0,x)}{\|x\|^{4-d}} = \kappa_d$$

In other words, as $\|x-y\| \to +\infty$, the covariance between φ_x and φ_y decays like $\kappa_d\|x-y\|^{4-d}$ in the supercritical dimensions.

For $d = 4$, $\lim_{N\to+\infty} P_N$ does not exist, and in fact, $\mathrm{var}_{P_N}(\varphi_0) \to +\infty$. It is known that $G_N(x,\, y)$ behaves in first order as $\gamma_4(\log N - \log\|x-y\|)$ for some $\gamma_4 \in (0,\, +\infty)$, if x and y are not too close to the boundary of V_N, see Cipriani (2013, Lemma 2.1).

Definition 2.4 (Pinned membrane model)**.** *Let $\varepsilon > 0$. The* membrane model on W with pinning of strength ε *is defined as*

$$P_W^\varepsilon(\mathrm{d}\varphi) = \frac{1}{Z_W^\varepsilon}\exp\left(-\frac{1}{2}\langle\Delta\varphi, \Delta\varphi\rangle\right)\prod_{x\in W}(\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x))\prod_{x\in W^c}\delta_0(\mathrm{d}\varphi_x) \tag{2.8}$$

where Z_W^ε is the normalizing constant

$$Z_W^\varepsilon := \int \exp\left(-\frac{1}{2}\langle\Delta\varphi, \Delta\varphi\rangle\right)\prod_{x\in W}(\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x))\prod_{x\in W^c}\delta_0(\mathrm{d}\varphi_x).$$

In case $W = V_N$, we write P_N^ε and Z_N^ε instead.

Our main result shows that for any positive pinning strength ε the correlations between two points decay exponentially in the distance.

Theorem 2.5 (Decay of covariances, supercritical case)**.** *Let $d \geq 5$ and $\varepsilon > 0$. Then there exist $C, \eta > 0$ depending on ε and d, but not on N, such that*

$$|E_N^\varepsilon[\varphi_x\varphi_y]| \leq C\mathrm{e}^{-\eta\|x-y\|}$$

whenever $x,\, y \in V_N$.

Remark 2.6. A more natural statement would be that $P^{\varepsilon} := \lim_{N\to\infty} P_N^{\varepsilon}$ has exponentially decaying covariances. Unfortunately, we do not know if this limit exists. The proof in Bolthausen and Velenik (2001) of the existence of the weak limit in the gradient case uses correlation inequalities which are not valid in the membrane case.

Remark 2.7 (Outlook on the case $d = 4$). The restriction to $d \geq 5$ is coming from a domination of the measure ν_N^{ε} defined in (3.1) from below by a Bernoulli measure which is true in a strong sense only for $d \geq 5$. The other steps of the proof do not depend on this dimension restriction in an essential way. The method we apply here would give for $d = 4$ an estimate of $|E_N^{\varepsilon}[\varphi_x\varphi_y]|$ in the form $\exp\left[-\eta\left\|x-y\right\|(\log N)^{-\alpha}\right]$ with some $\eta,\ \alpha > 0$. This is of course disappointing as for fixed $x,\ y$, one would not get decay properties which are uniformly in N, and one would also not get boundedness of the variances $\mathrm{var}_{P_N^{\varepsilon}}(\varphi_0)$. We remark also that with techniques similar to those of the present paper, albeit less refined, Bolthausen et al. (2016) show stretched exponential decay of covariances in $d \geq 4$.

We however expect that with some weaker domination properties, as the one used in Bolthausen and Velenik (2001) for $d = 2$, one could prove exponential decay also for the membrane model in $d = 4$. However, the proofs used in Bolthausen and Velenik (2001) rely again on correlation inequalities, so a proof eludes us.

It is well possible that exponential decay of correlations is true also for lower dimensions $d = 2, 3$, but we do not know of a method which could successfully be applied.

3 General results on the membrane model

Let $B \subset A \Subset \mathbb{Z}^d$. As the Hamiltonian of the membrane model is represented through an interaction of range 2, the conditional distribution of $\{\varphi_x\}_{x\in B}$ under P_A given $\mathcal{F}_{A\setminus B}$ depends only on $\{\varphi_y\}_{y\in\partial_2 B\cap A}$, where $\partial_2 B := \{y \notin B : \mathrm{d}(y, B) \leq 2\}$. As the measures are Gaussian, $E_A\left[\varphi_x|\mathcal{F}_{A\setminus B}\right]$ is a linear combination of the variables $\{\varphi_y\}_{y\in\partial_2 B\cap A}$.

From general properties of Gaussian distributions, one easily gets the following result.

Proposition 3.1 (Cipriani (2013, Lemma 2.2)). *Let A be a finite subset of $\mathbb{Z}^d$, and $B \subset A$, and let $\{\varphi_x\}_{x\in\mathbb{Z}^d}$ be the membrane model under the measure P_A. Let further $\{\varphi'_x\}_{x\in B}$ be independent of $\{\varphi_x\}$ and distributed according to P_B, i.e. with 0-boundary conditions outside B. Then $\{\varphi_x\}_{x\in B}$ has the same distribution under P_A as $\left\{E_A\left[\varphi_x|\mathcal{F}_{A\setminus B}\right] + \varphi'_x\right\}_{x\in B}$.*

Corollary 3.2. *Let $B \subset A$ be finite subsets of $\mathbb{Z}^d$, and $x_1, \ldots, x_k \in B,\ \lambda_1, \ldots, \lambda_k \in \mathbb{R}$, then*

$$\mathrm{var}_{P_B}\left(\sum\nolimits_{i=1}^{k}\lambda_i\varphi_{x_i}\right) \leq \mathrm{var}_{P_A}\left(\sum\nolimits_{i=1}^{k}\lambda_i\varphi_{x_i}\right).$$

Proof. By the previous proposition, $\sum_{i=1}^{k}\lambda_i\varphi_{x_i}$ has under P_A the same law as

$$E_A\left[\sum\nolimits_{i=1}^{k}\lambda_i\varphi_{x_i}|\mathcal{F}_{A\setminus B}\right] + \sum\nolimits_{i=1}^{k}\lambda_i\varphi'_{x_i}$$

where $\{\varphi'_x\}_{x\in B}$ is independent of the first summand and distributed according to P_B. From that, the claim follows. □

For $A \subset \mathbb{Z}^d$ we write $P_W^A := P_{W \setminus A}$, i.e. the membrane model with 0-boundary conditions on both W^c and on A. We also write G_W^A for the corresponding covariance matrix. If $A = \emptyset$, then $P_W^\emptyset = P_W$. Again, we just use the index N if $W = V_N$.

Corollary 3.3. *Let $A \subset \mathbb{Z}^d$, and $d \geq 5$. Then the weak limit $P^A := \lim_{N \to +\infty} P_N^A$ exists, and it is a centered Gaussian field, with covariances*

$$G^A(x, y) = \lim_{N \to +\infty} G_N^A(x, y), \quad x, y \in \mathbb{Z}^d.$$

Proof. By Corollary 3.2, $G_N^A(x,x) \uparrow G^A(x,x) < +\infty$ for all x, as $N \to +\infty$. The finiteness comes from $G_N^A(x,x) \leq G_N(x,x) \leq G(x,x) < +\infty$ (recall (2.7)). So $\left\{P_N^A\right\}_N$ is a tight sequence. But for $x, y \in \mathbb{Z}^d$, also $\lim_{N \to +\infty} \mathrm{var}_{P_N^A}(\varphi_x + \varphi_y)$ exists, and therefore $\lim_{N \to +\infty} G_N^A(x, y)$ exists. This implies the statement of the corollary. □

Bernoulli domination. A key step of our proof is that the environment of pinned points can be compared with Bernoulli site percolation. Expanding $\prod_{x \in W} (\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x))$ in (2.8), one has, for any measurable function $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$,

$$E_W^\varepsilon(f) = \frac{1}{Z_W^\varepsilon} \int f(\varphi) \exp\left(-\frac{1}{2}\langle \Delta\varphi, \Delta\varphi\rangle\right) \prod_{x \in W} (\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x)) \prod_{x \in W^c} \delta_0(\mathrm{d}\varphi_x) =$$
$$= \sum_{A \subset W} \varepsilon^{|A|} \frac{Z_W^A}{Z_W^\varepsilon} E_W^A(f),$$

where $Z_W^A := Z_{W \setminus A}$ i.e.

$$P_W^\varepsilon = \sum_{A \subset W} \zeta_W^\varepsilon(A) P_W^A.$$

with

$$\zeta_W^\varepsilon(A) := \varepsilon^{|A|} \frac{Z_W^A}{Z_W^\varepsilon}, \tag{3.1}$$

which is a probability measure on $\mathcal{P}(W)$, the set of subsets of W. We will often use $\mathcal{A}$ or $\mathcal{A}_W$ to denote a $\mathcal{P}(W)$-valued random variable with this distribution, so that we can write

$$E_W^\varepsilon[\varphi_x \varphi_y] = \sum_{A \subset V_N} \zeta_W^\varepsilon(A) G_W^A(x, y) = E_{\zeta_W^\varepsilon}\left(G_W^{\mathcal{A}}(x, y)\right). \tag{3.2}$$

Lemma 3.4. *In $d \geq 5$ there exist constants $0 < C_-, C_+ < \infty$ depending only on the dimension such that for every $w \in W$ and $E \subset W \setminus \{w\}$*

$$C_- \leq \frac{Z_W^{E \cup \{w\}}}{Z_W^E} \leq C_+. \tag{3.3}$$

Proof. The proof follows the ideas of Velenik (2006, Section 5.3). $Z_W^{E \cup \{w\}} / Z_W^E$ is the density at 0 of the distribution of φ_w under the law P_W^E, i.e.

$$\frac{Z_W^{E \cup \{w\}}}{Z_W^E} = \frac{1}{\sqrt{2\pi G_W^E(w, w)}}.$$

As

$$0 < G_{\{w\}}(w,w) \leq G_W^E(w,w) \leq G(w,w) = G(0,0) < +\infty,$$

the claim follows. □

Remark 3.5. For $d = 2,\,3,\,4$, one has a similar upper bound for $Z_W^{E\cup\{w\}}/Z_W^E$, but the lower bound depends on W, as $G(0,0) = +\infty$. For $d = 4$, one has

$$\frac{Z_N^{E\cup\{w\}}}{Z_N^E} \geq \frac{C_-}{\sqrt{\log N}}.$$

We control now the pinning measure ζ_N^ε through dominations by Bernoulli product measures.

Definition 3.6 (Strong stochastic domination)**.** *Given two probability measures μ and ν on the set $\mathcal{P}(W)$, $|W| < +\infty$, we say that μ* dominates ν strongly stochastically *if for all $x \in W$, $E \subset W \setminus \{x\}$,*

$$\mu(A:\ x \in A \,|\, A \setminus \{x\} = E) \geq \nu(A:\ x \in A \,|\, A \setminus \{x\} = E). \tag{3.4}$$

When this holds we write $\mu \succ \nu$.

Let $\mathbb{P}_W^\rho$ be the Bernoulli site percolation measure on W with intensity $\rho \in [0,1]$. We regard this as a probability measure on $\mathcal{P}(W)$.

Proposition 3.7. *Let $d \geq 5$ and $\varepsilon > 0$. Then*

$$\mathbb{P}_W^{\rho_-(d,\varepsilon)} \prec \zeta_W^\varepsilon \prec \mathbb{P}_W^{\rho_+(d,\varepsilon)}$$

with

$$\rho_\pm(d,\varepsilon) := \frac{C_\pm(d)\,\varepsilon}{1 + C_\pm(d)\,\varepsilon} \in (0,1) \tag{3.5}$$

where C_-, C_+ are defined in Lemma 3.4.

Proof. For $x,\ E$ as in Definition 3.6

$$\zeta_W^\varepsilon(A:\ x \in A \,|\, A \setminus \{x\} = E) = \left[1 + \frac{Z_{W\setminus E}}{Z_{W\setminus(E\cup\{x\})}}\right]^{-1}.$$

This proves the claim. □

4 Proof of the main result

4.1 Sobolev norms

A crucial role of the proof uses a Sobolev-type norm $\|\cdot\|_{A,E}$ depending on subsets $A,\ E \subset \mathbb{Z}^d$. Given A, let

$$\widehat{A} := \{x \in A : x + e \in A,\ \forall\, e\}.$$

$\widehat{A}$ is the subset of "interior" points of A. For $f : \mathbb{Z}^d \to \mathbb{R}$ and $A,\ E \subset \mathbb{Z}^d$, let

$$\|f\|^2_{A,\,E} := \sum_{x\in E} \frac{f(x)^2}{1+\mathrm{d}(x,\ \widehat{A})^{2d+3}} + \sum_{x\in E} \frac{\|\nabla f\,(x)\|^2}{1+\mathrm{d}(x,\ \widehat{A})^{d+2}} + \sum_{x\in E} \left\|\nabla^2 f\,(x)\right\|^2. \tag{4.1}$$

If $\widehat{A} = \emptyset$, then we put $\mathrm{d}(x,\ \widehat{A}) = +\infty$ by convention, and $\|f\|^2_{A,\,E} = \sum_{x\in E} \left\|\nabla^2 f\,(x)\right\|^2$. We note the following two facts:

1. $\|f\|^2_{A,\,E}$ is defined for $f :\ E \cup \partial_2 E \to \mathbb{R}$.
2. If E_1 and E_2 are disjoint then

$$\|f\|^2_{A,\,E_1\cup E_2} = \|f\|^2_{A,\,E_1} + \|f\|^2_{A,\,E_2}.$$

We now bound the $\|\cdot\|^2_{A,\,\mathbb{Z}^d}$ norm of a function vanishing on A by second derivates only.

Lemma 4.1. *Let f be a function which is identically zero on A. Then*

$$\|f\|^2_{A,E} \le L \sum_{x\in E} \left\|\nabla^2 f\,(x)\right\|^2.$$

Proof. There is nothing to prove when $\widehat{A} = \emptyset$, so we assume $\widehat{A} \neq \emptyset$.

We first show that the first summand on the right hand side of (4.1) is dominated by a multiple of the second, and afterwards that the second is dominated by the third.

If $x \in \mathbb{Z}^d$, we choose a nearest-neighbor path ψ_x of shortest length to $\widehat{A}$, that is, $\psi_x = (x_0 = x, x_1, \ldots, x_k)$ with $x_k \in \widehat{A}$. As f is 0 on A, one has

$$f(x) = \sum_{l=1}^{k} (f(x_l) - f(x_{l-1})).$$

We can choose the collection $\{\psi_x\}$ of paths in such a way that the same bond is not used for two different end points in $\widehat{A}$. More formally: if $x, x' \in \mathbb{Z}^d$ with paths $\psi_x = (x, x_1, \ldots, x_k)$, $\psi_{x'} = (x', x'_1, \ldots, x'_{k'})$ have the property that there exists a bond b which belongs to both paths, then $x_k = x'_{k'}$. This can be achieved by choosing an enumeration $\{x_n\}$ of $\mathbb{Z}^d$, and constructing the paths recursively with this property.

By Cauchy-Schwarz,

$$f(x)^2 \le |\psi_x| \sum_{l=1}^{k} (f(x_l) - f(x_{l-1}))^2 = \mathrm{d}(x, \widehat{A}) \sum_{l=1}^{\mathrm{d}(x,\widehat{A})} (f(x_l) - f(x_{l-1}))^2,$$

and thus, exchanging the order of summation between points x and paths ψ_x,

$$\begin{aligned} \sum_x \frac{f\,(x)^2}{1+\mathrm{d}(x,\widehat{A})^{2d+3}} &\le \sum_x \frac{\mathrm{d}(x,\widehat{A})}{1+\mathrm{d}(x,\widehat{A})^{2d+3}} \sum_{l=1}^{\mathrm{d}(x,\widehat{A})} (f(x_l) - f(x_{l-1}))^2 \\ &\le \sum_z \|\nabla f\,(z)\|^2 \sum_{x : z\in\psi_x} \frac{\mathrm{d}(x,\widehat{A})}{1+\mathrm{d}(x,\widehat{A})^{2d+3}}. \end{aligned} \tag{4.2}$$

For $z \in \mathbb{Z}^d$ write $R_{z,k} := \{x \in \mathbb{Z}^d : \mathrm{d}(x, \widehat{A}) = k$ and $z \in \psi_x\}$. Observe that due to our convention, every $x \in \mathbb{Z}^d$ with $z \in \psi_x$ satisfies $\mathrm{d}(x, \widehat{A}) \geq \mathrm{d}(z, \widehat{A})$, and ψ_x and ψ_z end at the same point in $\widehat{A}$. Thus there exists a constant $c_1 = c_1(d)$ such that $|R_{z,k}| \leq c_1(k - \mathrm{d}(z, \widehat{A}))^{d-1} \leq c_1 k^{d-1}$. Therefore

$$\sum_{x:z\in\psi_x} \frac{\mathrm{d}(x, \widehat{A})}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}} \leq \sum_{k=\mathrm{d}(z,\widehat{A})}^{\infty} \frac{|R_{z,k}|}{1 + k^{2d+2}}$$
$$\leq L \sum_{k=\mathrm{d}(z,\widehat{A})}^{\infty} \frac{1}{1 + k^{d+3}} \leq L \frac{1}{1 + \mathrm{d}(z, \widehat{A})^{d+2}}. \tag{4.3}$$

Thus we have, plugging (4.3) in (4.2),

$$\sum_x \frac{f(x)^2}{1 + \mathrm{d}(x,\, \widehat{A})^{2d+3}} \geq L \sum_x \frac{1}{1 + \mathrm{d}(x,\, \widehat{A})^{d+2}} \left\| \nabla f\,(x) \right\|^2.$$

It remains to prove that the right hand side is bounded by some multiple of $\sum_x \left\| \nabla^2 f\,(x) \right\|^2$. If ψ_x is the same as above, we have

$$\nabla f(x) = \sum_{l=1}^{k} (\nabla f(x_l) - \nabla f(x_{l-1})),$$

because $\nabla f(x_k) = 0$ for $x_k \in \widehat{A}$. Thus by the same arguments as above, we get

$$\left\| \nabla f\,(x) \right\|^2 \leq \mathrm{d}(x, \widehat{A}) \sum_{l=1}^{\mathrm{d}(x,\widehat{A})} \left| \nabla \left[f(x_l) - f(x_{l-1}) \right] \right|^2,$$

and

$$\sum_x \frac{\left\| \nabla f\,(x) \right\|^2}{1 + \mathrm{d}(x, \widehat{A})^{d+2}} \leq \sum_{x \in \mathbb{Z}^d} \frac{\mathrm{d}(x, \widehat{A})}{1 + \mathrm{d}(x, \widehat{A})^{d+2}} \sum_{l=1}^{k} \left\| \nabla \left[f\,(x_l) - f\,(x_{l-1}) \right] \right\|^2$$
$$\leq L \sum_z \left\| \nabla^2 f\,(z) \right\|^2 \sum_{x:z\in\psi_x} \frac{1}{1 + \mathrm{d}(x, \widehat{A})^{d+1}}$$
$$\leq L \sum_z \left\| \nabla^2 f\,(x) \right\|^2 \left[\sup_{y\in\widehat{A}} \sum_x \frac{1}{1 + \mathrm{d}(x,\, y)^{d+1}} \right] \leq L \sum_z \left\| \nabla^2 f\,(x) \right\|^2.$$

□

For $k \geq 0$ and $E \subset \mathbb{Z}^d$ let

$$\upsilon_k\,(E) := \{x : \mathrm{d}\,(x, E) \leq k\}\,.$$

For $x,\, y \in \mathbb{Z}^d$ let $\Gamma_{x,\,y}$ be the set of non-intersecting nearest-neighbor paths

$$\psi = (x_0 = x, x_1, \ldots, x_n = y)\,,$$

and we write $\ell(\psi)$ for the length n. For such a ψ we define

$$\phi_A(\psi) := \sum_{i=0}^{n} q_A(x_i), \tag{4.4}$$

where

$$q_A(x) := \frac{1}{1 + \mathrm{d}\left(x, \widehat{A}\right)^{2d+3}}, \quad x \in \mathbb{Z}^d.$$

Define

$$\begin{aligned} \widehat{\mathrm{d}}_A(x,\, y) &:= \min\left\{\phi_A(\psi) : \psi \in \Gamma_{x,y}\right\}, \\ \widehat{\mathrm{d}}_A(0,\, 0) &:= 0. \end{aligned}$$

$\widehat{\mathrm{d}}_A$ may well be bounded, for instance if A is a finite set. In the cases we are interested in, it will however be unbounded. We will often just write $\widehat{\mathrm{d}}$ if it is clear from the context what set A is considered. Since $q_A(x) \le 1$ for any x, note also the bound $\widehat{\mathrm{d}}(x,\, y) \le \mathrm{d}(x,\, y)$ for all $x,\, y \in \mathbb{Z}^d$.

We define

$$C_n := \left\{x :\, \widehat{\mathrm{d}}(0, x) \le 10n\right\}. \tag{4.5}$$

C_n is connected in the usual graph structure of $\mathbb{Z}^d$, but the complement may be disconnected. If we want to emphasize the dependence of C_n on A, we write $C_{n,A}$.

Remark 4.2. Remark that $\upsilon_2(C_n) \cap \upsilon_2(C_{n+1}^{\mathrm{c}}) = \emptyset$. In fact, assuming that there is a $w \in \upsilon_2(C_n) \cap \upsilon_2(C_{n+1}^{\mathrm{c}})$, then there exist $w_1 \in C_n$, and $w_2 \in C_{n+1}^{\mathrm{c}}$ with $\mathrm{d}(w,\, w_i) \le 2$ for $i = 1,\, 2$. Hence $\widehat{\mathrm{d}}(w_1,\, w_2) \le \mathrm{d}(w_1,\, w_2) \le 4$, and by the triangle inequality, $w_1 \in C_n$ implies $\widehat{\mathrm{d}}(0,\, w_2) \le 10n + 4$ which contradicts $w_2 \notin C_{n+1}$.

We will need a monotonicity property in the dependence on A. First remark that if $A \subset A'$ then $\mathrm{d}\left(x, \widehat{A}\right) \ge \mathrm{d}\left(x, \widehat{A'}\right)$ for all x, and therefore

$$\widehat{\mathrm{d}}_A \le \widehat{\mathrm{d}}_{A'}. \tag{4.6}$$

Lemma 4.3. *For every n, there exists a function $\eta_n : \mathbb{Z}^d \to [0, 1]$ with the following properties:*

$$\eta_n = 0 \text{ on } \upsilon_2(C_n),\ \eta_n = 1 \text{ on } \upsilon_2\left(C_{n+1}^{\mathrm{c}}\right), \tag{4.7}$$

$$\|\nabla \eta_n(x)\|_\infty \le \frac{L}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}},\ \forall\, x \in \mathbb{Z}^d. \tag{4.8}$$

Proof. Let $f_1(x) := \widehat{\mathrm{d}}(x, \upsilon_2(C_n))$ and $f_2(x) := \widehat{\mathrm{d}}(x, \upsilon_2(C_{n+1}^{\mathrm{c}}))$. We define

$$\eta_n(x) := \frac{f_1(x)}{f_1(x) + f_2(x)}.$$

which evidently satisfies (4.7).

To prove (4.8), notice first that one can find an L large uniformly for all x with $\mathrm{d}(x, \widehat{A}) \le 4$, so let us consider x such that $\mathrm{d}(x, \widehat{A}) \ge 5$. We have from Remark 4.2 that

$$f_1(x) + f_2(x) \ge 1.$$

Then

$$|D_e \eta_n(x)| \le \frac{|D_e f_1(x)|}{f_1(x) + f_2(x)} + f_1(x+e) \left| \frac{1}{f_1(x) + f_2(x)} - \frac{1}{f_1(x+e) + f_2(x+e)} \right|. \tag{4.9}$$

We see that

$$\begin{aligned} |D_e f_1(x)| &\le q_A(x) + q_A(x+e) \\ &\le \frac{2}{1 + \min\left\{\mathrm{d}\left(x, \widehat{A}\right), \mathrm{d}\left(x+e, \widehat{A}\right)\right\}^{2d+3}} \\ &\le \frac{L}{1 + \mathrm{d}\left(x, \widehat{A}\right)^{2d+3}}, \end{aligned}$$

as we assumed $\mathrm{d}(x, \widehat{A}) \ge 5$. The same estimate is true also for $|D_e f_2(x)|$.

The second summand in (4.9) is bounded above by $|D_e f_1(x)| + |D_e f_2(x)|$, so the claim follows. □

Corollary 4.4. *For all $x \in \mathbb{Z}^d$ it holds that*

a) for all $k \ge 2$ there exists $L = L(k) > 0$ such that

$$\left\|\nabla^k \eta_n(x)\right\|_\infty \le \frac{L}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}}. \tag{4.10}$$

b) For all e neighbors of the origin and $k \ge 1$ there exists $L = L(k) > 0$ such that

$$\left\|\nabla^k \eta_n(x+e)\right\|_\infty \le \frac{L}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}}, \ \forall x \in \mathbb{Z}^d, \ \forall e \tag{4.11}$$

Proof.

a) (4.8) implies that also higher order derivatives can be estimated by the same bound with a changed L because the supremum norm of higher order discrete derivatives can be estimated by the first order ones.

b) Again this holds by an estimate with first order derivatives and the fact that

$$\left|\mathrm{d}(x+e, \widehat{A}) - \mathrm{d}(x, \widehat{A})\right| \le 1.$$

□

Consider now an infinite set A with the property that $C_{n,A}$ is finite for all n. Given A with A^{c} finite, and $0 \notin A$, we consider the unique function h_A which satisfies $h_A(x) = 0$ on A, and for all $x \in A^{\mathrm{c}}$

$$\Delta^2 h_A(x) = \delta_0(x).$$

Lemma 4.5. *With the above notation, we have for $n \geq 1$*

$$\|h_A\|^2_{A,C^{\mathrm{c}}_{n+1}} \leq L\|h_A\|^2_{A,C_{n+1}\setminus C_n}.$$

It is important to emphasize that L depends neither on A nor on n.

Proof. Fix n, and let η_n be as in Lemma 4.3. We also drop the subscript A in h_A. We have with Lemma 4.1 and Lemma 2.1

$$\begin{aligned}\|h\|^2_{A,C^{\mathrm{c}}_{n+1}} = \|\eta_n h\|^2_{A,C^{\mathrm{c}}_{n+1}} \leq \|\eta_n h\|^2_{A,\mathbb{Z}^d} &\leq L\sum_{e,e'}\langle D_e D_{e'}\eta_n h, D_e D_{e'}\eta_n h\rangle \\ &= L\left\langle \eta_n h, \Delta^2(\eta_n h)\right\rangle. \end{aligned} \tag{4.12}$$

By an elementary computation, one has for any $f, g : \mathbb{Z}^d \to \mathbb{R}$ and $x \in \mathbb{Z}^d$

$$\Delta(fg)(x) = f(x)\Delta g(x) + \Delta f(x) g(x) + \frac{1}{2d}\sum_e D_e f(x) D_e g(x). \tag{4.13}$$

Applying this twice gives

$$\begin{aligned}\Delta^2(\eta_n h) &= \eta_n \Delta^2 h + \left(\Delta^2\eta_n\right)h + 2\Delta\eta_n\Delta h + \frac{1}{d}\sum_e (D_e\Delta\eta_n) D_e h \\ &\quad + \frac{1}{d}\sum_e (D_e\eta_n)(D_e\Delta h) + \frac{1}{4d^2}\sum_{e,e'}(D_{e'}D_e\eta_n) D_{e'}D_e h \\ &=: F_1 + F_2 + 2F_3 + \frac{1}{d}F_4 + \frac{1}{d}F_5 + \frac{1}{4d^2}F_6.\end{aligned}$$

Note that

$$\langle \eta_n h, F_1\rangle = \left\langle \eta_n h, \eta_n\Delta^2 h\right\rangle = 0, \tag{4.14}$$

as for $x \neq 0$ we have $\Delta^2 h(x) = 0$ and for $x = 0$ we have $\eta_n(0) = 0$. All the other terms contain derivatives of η_n. Therefore, every derivative of the function η_n will be non-zero only for points in $C_{n+1}\setminus C_n$. Since we have (4.12), we need to estimate $\langle \eta_n h, F_i\rangle$ for $i = 2, \ldots, 6$. Let us begin with $i = 2$: by Corollary 4.4

$$\begin{aligned}|\langle \eta_n h, F_2\rangle| \leq \sum_x \left|\Delta^2\eta_n(x)\right| h(x)^2 &= \sum_{x\in C_{n+1}\setminus C_n}\left|\Delta^2\eta_n(x)\right| h(x)^2 \\ &\leq \sum_{x\in C_{n+1}\setminus C_n}\frac{L}{1+\mathrm{d}(x,\widehat{A})^{2d+3}} h(x)^2 \leq L\|h\|^2_{A,C_{n+1}\setminus C_n}.\end{aligned} \tag{4.15}$$

Let us see now $i = 3$. With the Cauchy-Schwarz inequality we get

$$\begin{aligned} |\langle \eta_n h, F_3 \rangle| &\le L \left| \sum\nolimits_{x \in C_{n+1} \setminus C_n} \eta_n(x) h(x) \Delta \eta_n(x) \Delta h(x) \right| \\ &\le L \sqrt{\sum\nolimits_{x \in C_{n+1} \setminus C_n} (\Delta h(x))^2} \sqrt{\sum\nolimits_{x \in C_{n+1} \setminus C_n} (\Delta \eta_n(x))^2 h(x)^2} \\ &\le L \|h\|^2_{A, C_{n+1} \setminus C_n} \end{aligned} \tag{4.16}$$

using Corollary 4.4, (2.1) and the arithmetic-geometric mean inequality.

To estimate the part with F_4, we first observe that $D_e \Delta \eta_n(x)$ is 0 outside $C_{n+1} \setminus C_n$, and by Remark 4.4

$$|D_e \Delta \eta_n(x)| \le \frac{L}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}}.$$

Therefore, using the inequality of arithmetic and geometric means,

$$\begin{aligned} |\langle \eta_n h, F_4 \rangle| &\le L \sum_e \sqrt{\sum_{x \in C_{n+1} \setminus C_n} \frac{h(x)^2}{\left(1 + \mathrm{d}(x, \widehat{A})^{2d+3}\right)^2}} \sqrt{\sum_{x \in C_{n+1} \setminus C_n} \frac{D_e h(x)^2}{\left(1 + \mathrm{d}(x, \widehat{A})^{2d+3}\right)^2}} \\ &\le L \sum_e \sqrt{\sum_{x \in C_{n+1} \setminus C_n} \frac{h(x)^2}{1 + \mathrm{d}(x, \widehat{A})^{2d+3}}} \sqrt{\sum_{x \in C_{n+1} \setminus C_n} \frac{D_e h(x)^2}{1 + \mathrm{d}(x, \widehat{A})^{d+2}}} \\ &\le L \|h\|^2_{A, C_{n+1} \setminus C_n}. \end{aligned} \tag{4.17}$$

For the estimate of $\langle \eta_n h,\, F_5 \rangle$ we can use Lemma 4.3 and (2.1) again to say that, for a fixed direction e,

$$\begin{aligned} &|\langle \eta_n h,\, (D_e \eta_n)(D_e \Delta h) \rangle| \\ &\le \left| \sum_{x \in C_{n+1} \setminus C_n} \eta_n(x) h(x) D_e \eta_n(x) \Delta h(x) \right| \\ &+ \left| \sum_{x \in C_{n+1} \setminus C_n} \eta_n(x) h(x) D_e \eta_n(x) \Delta h(x+e) \right| \\ &\le \sqrt{\sum_{x \in C_{n+1} \setminus C_n} h(x)^2 D_e \eta_n(x)^2} \left[\sqrt{\sum_{x \in C_{n+1} \setminus C_n} \Delta h(x)^2} + \sqrt{\sum_{x \in C_{n+1} \setminus C_n} \Delta h(x+e)^2} \right] \\ &\le L \|h\|^2_{A, C_{n+1} \setminus C_n}. \end{aligned}$$

Summing over e yields

$$|\langle \eta_n h, F_5 \rangle| \le L \|h\|^2_{A, C_{n+1} \setminus C_n}. \tag{4.18}$$

It finally remains to show

$$|\langle \eta_n h, F_6 \rangle| \le L \|h\|^2_{A, C_{n+1} \setminus C_n} \tag{4.19}$$

which follows in the same way as (4.16).

Combining (4.14)-(4.19) proves the lemma. □

With these preparations, we can now prove that $\|h\|^2_{A,C^{\mathrm{c}}_{n+1}}$ decays exponentially.

Lemma 4.6. *Let $d \geq 1$, and let $A \subset \mathbb{Z}^d \setminus \{0\}$ be such that A^{c} is finite. There exist constants $c_1(d) > 0$ and $\delta(d) > 0$, independent of A, such that, for all $n \in \mathbb{N}$,*

$$\|h\|^2_{A,C^{\mathrm{c}}_{n+1}} \leq c_1 \mathrm{e}^{-\delta n} \|h\|^2_{A,\mathbb{Z}^d}.$$

Proof. From Lemma 4.5 we get

$$\|h\|^2_{A,C^{\mathrm{c}}_{n+1}} \leq L \|h\|^2_{A,C_{n+1}\setminus C_n} = L\left(\|h\|^2_{A,C^{\mathrm{c}}_n} - \|h\|^2_{A,C^{\mathrm{c}}_{n+1}}\right),$$

that is, iterating the argument,

$$\begin{aligned}\|h\|^2_{A,C^{\mathrm{c}}_{n+1}} &\leq \frac{L}{1+L}\|h\|^2_{A,C^{\mathrm{c}}_n} \leq \left(\frac{L}{1+L}\right)^{n-1}\|h\|^2_{A,C^{\mathrm{c}}_1} \\ &\leq \frac{1+L}{L}\left(\frac{L}{1+L}\right)^{n}\|h\|^2_{A,\mathbb{Z}^d},\end{aligned}$$

proving the claim. □

Corollary 4.7. *If $d \geq 5$, then, under the same conditions and notation as above*

$$\|h\|^2_{A,C^{\mathrm{c}}_n} \leq c_1 \mathrm{e}^{-\delta n}.$$

Proof. By Lemma 4.1

$$\|h\|^2_{A,\mathbb{Z}^d} \leq L \sum_{x\in\mathbb{Z}^d} \left\|\nabla^2 h(x)\right\| = L\left\langle h, \Delta^2 h\right\rangle = h(0) \leq G(0,\,0) < +\infty.$$

Plugging this in Lemma 4.6 concludes the proof. □

4.2 Trapping configurations under the Bernoulli law

In order to prove our main theorem, we have to obtain probabilistic properties of the sequence $\{C_{n,\mathcal{A}}\}_n$ where $\mathcal{A}$ is random and distributed according to ζ^{ε}. Using the Bernoulli domination, the key probabilistic estimates have to be done only for a Bernoulli measure instead of ζ^{ε}. Therefore, let $p \in (0,1)$ and $\mathbb{P}^p$ be the Bernoulli site percolation measure on the set of subsets of $\mathbb{Z}^d$ with parameter p. As p is fixed in this section, we leave it out in the notation. We write $\widehat{\mathcal{A}}$ for the set of interior points. Let $B_m(x) := \{y :\, \mathrm{d}(x,\,y) \leq m\}$.

Lemma 4.8. *For $m \in \mathbb{N}$, $x \in \mathbb{Z}^d$,*

$$\mathbb{P}\left(B_m(x) \cap \widehat{\mathcal{A}} = \emptyset\right) \leq \left(1 - p^{2d+1}\right)^{\left\lfloor \frac{2m+1}{3}\right\rfloor^d}.$$

Proof. It suffices to take $x = 0$ and write B_m for $B_m(0)$. Note that B_m is a hypercube of side length $2m+1$. Put $n := \lfloor (2m+1)/3 \rfloor$. We can place n^d pairwise disjoint boxes $B_1(x_j)$, $1 \le j \le n^d$ in B_m. As these boxes are disjoint, the events $\left\{x_j \in \widehat{\mathcal{A}}\right\}$ are independent and they have probability p^{2d+1}. Therefore

$$\mathbb{P}\left(B_m \cap \widehat{\mathcal{A}} = \emptyset\right) \le \mathbb{P}\left(x_j \notin \widehat{\mathcal{A}},\, \forall j \le n^d\right) = \left(1 - p^{2d+1}\right)^{\left\lfloor \frac{2m+1}{3} \right\rfloor^d}.$$

□

Lemma 4.9. *There exist λ, $K > 0$ and $n_0 \in \mathbb{N}$ depending only on the dimension d and p such that for all $n \ge n_0$ and all $N \ge Kn$*

$$\mathbb{P}\left(\sup_{x \in C_{n,\mathcal{A}}} \mathrm{d}(0,x) > Kn\right) = \mathbb{P}\left(\inf_{x:\mathrm{d}(x,0)>Kn} \widehat{\mathrm{d}}(0,x) \le 10n\right) \le \mathrm{e}^{-\lambda n} \tag{4.20}$$

Proof. The equality in (4.20) holds by the definition of C_n. Let us prove the inequality on the right-hand side of the above formula.

For $M \in \mathbb{N}$ we subdivide $\mathbb{Z}^d$ in boxes $B_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{Z}^d$, of side-length M :

$$B_{\mathbf{i}} := \left(\left[(i_1 - 1)M + 1,\, i_1 M\right] \times \cdots \times \left[(i_d - 1)M + 1,\, i_d M\right]\right) \cap \mathbb{Z}^d,$$

and

$$B_{\mathbf{i}}^0 := \left(\left[(i_1 - 1)M + 2,\, i_1 M - 1\right] \times \cdots \times \left[(i_d - 1)M + 2,\, i_d M - 1\right]\right) \cap \mathbb{Z}^d,$$

which is a box contained in $B_{\mathbf{i}}$. We define

$$\eta(\mathbf{i}) = \begin{cases} 1 & \text{if } B_{\mathbf{i}}^0 \cap \widehat{\mathcal{A}} \ne \emptyset \\ 0 & \text{if } B_{\mathbf{i}}^0 \cap \widehat{\mathcal{A}} = \emptyset \end{cases}.$$

The $\eta(\mathbf{i})$ are i.i.d. In order to estimate $\mathbb{P}(\eta(\mathbf{i}) = 0)$, we subdivide the box $B_{\mathbf{i}}^0$ into boxes Q_j of side-length 3, with possibly some small part remaining close to the boundary of $B_{\mathbf{i}}^0$. As the $B_{\mathbf{i}}^0$ have side-length $M-2$, we can place $\lfloor (M-2)/3 \rfloor^d$ of the Q-boxes without overlaps into $B_{\mathbf{i}}^0$. For a Q-box, the probability that the middle point and all its neighbors belong to $\mathcal{A}$ is p^{2d+1}. Therefore

$$\mathbb{P}(\eta(\mathbf{i}) = 0) \le \left(1 - p^{2d+1}\right)^{\left\lfloor \frac{M-2}{3} \right\rfloor^d}.$$

We choose $M = M(p,d)$ such that

$$\mathbb{P}(\eta(\mathbf{i}) = 0) \le \frac{1}{64d^2}. \tag{4.21}$$

For $x \in \mathbb{Z}^d$, we write $\mathbf{i}(x)$ for the index $\mathbf{i}$ such that $x \in B_{\mathbf{i}}$. Remark that $\mathbf{i}(0) = 0$. Remark that M depends on d and p only.

Given any self-avoiding nearest-neighbor path connecting x with 0, that is,

$$\psi = (x_0 = x, x_1, \ldots, x_k = 0)$$

we attach to it a renormalized nearest neighbor path $\overline{\psi} = (\mathbf{i}(x), \mathbf{i}_1, \ldots, \mathbf{i}_\ell = 0)$ in the following way. ψ starts at x which is inside a box $B_{\mathbf{i}(x)}$. Put $\mathbf{i}_0 := \mathbf{i}(x)$. When ψ for the first time leaves $B_{\mathbf{i}_0}$, it enters a box $B_{\mathbf{i}_1}$ with $\mathbf{i}_1$ being a neighbor of $\mathbf{i}_0$ in $\mathbb{Z}^d$. Then we wait for the next time in which ψ leaves $B_{\mathbf{i}_1}$ and enters a neighbor box $B_{\mathbf{i}_2}$. This path is not yet self-avoiding, but we can make it so by erasing successively all the loops. See Figure 1 for an example.

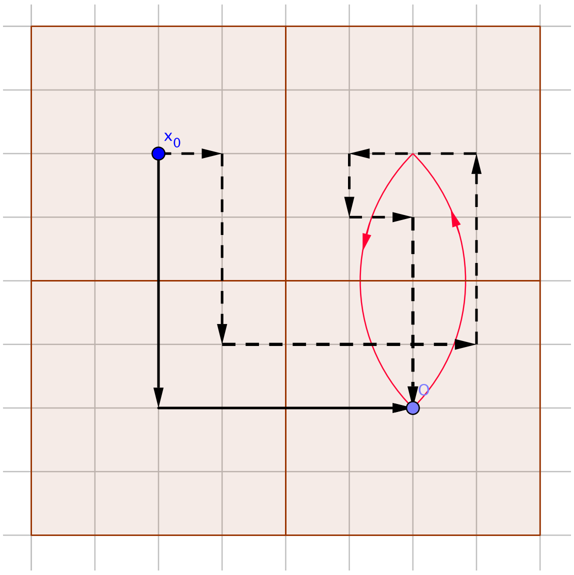

Figure 1: The dashed path represents ψ and the solid line the renormalised path $\overline{\psi}$ where the M-squares have side-length 4. Note we have erased the red loop from $\overline{\psi}$.

In this way, we proceed and obtain a path from $\mathbf{i}_0$ to 0, which we indicate as $\mathbf{i}_0 \longrightarrow \mathbf{0}$, of the form $(\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_\ell = 0)$. Evidently, we can define an injective mapping $\{0, \ldots, \ell\} \ni j \mapsto t_j \in \{0, \ldots, k\}$ with $x_{t_j} \in B_{\mathbf{i}_j}$ (for example assigning to each j the point x_{t_j} which is the center of the box $B_{\mathbf{i}_j}$). As this mapping is injective and the path ψ is self-avoiding, the x_{t_j}'s are different.

We attach to $\overline{\psi}$ a weight

$$\overline{\phi}\left(\overline{\psi}\right) = \left|\{j \leq \ell : \eta\left(\mathbf{i}_j\right) = 1\}\right|$$

that counts the number of large boxes in which a pinned point lies. From the construction one obtains that

$$\overline{\phi}\left(\overline{\psi}\right) = \sum_{j=0}^{\ell} \mathbf{1}_{\{\eta(\mathbf{i}_j)=1\}} \leq \sum_{j=0}^{\ell} \mathbf{1}_{\{\eta(\mathbf{i}_j)=1\}} \frac{2 + 2M^{2d+3}}{2 + 2\mathrm{d}\left(x_{t_j}, \widehat{A}\right)^{2d+3}}$$

as $\mathrm{d}\left(x, \widehat{A}\right) \leq M$ whenever $x \in B_{\mathbf{i}}$ with $B^0_{\mathbf{i}} \cap \widehat{A} \neq \emptyset$. Moreover, recalling (4.4), we can say that

$$\sum_{j=0}^{\ell} \frac{\mathbf{1}_{\{\eta(\mathbf{i}_j)=1\}}}{2 + 2\mathrm{d}\left(x_{t_j}, \widehat{A}\right)^{2d+3}} \leq \sum_{i=0}^{k} \frac{1}{2 + 2\mathrm{d}\left(x_i, \widehat{A}\right)^{2d+3}} \leq \phi\left(\psi\right)$$

and so

$$\overline{\phi}\left(\overline{\psi}\right) \le \left(2 + 2M^{2d+3}\right)\phi\left(\psi\right). \tag{4.22}$$

We have already fixed M above (depending only on d and p), and we choose now K as

$$K := \left\lceil 20M\left(2 + 2M^{2d+3}\right)\right\rceil. \tag{4.23}$$

If there exists x with $\mathrm{d}\left(0,\, x\right) > Kn$ and $\widehat{\mathrm{d}}\left(0, x\right) \le 10n$, then there exists a ψ from x to 0 with $\phi\left(\psi\right) \le 10n$, implying by means of (4.22) that there exists a path $\overline{\psi}$ from $\mathbf{i}\left(x\right)$ to 0 with weight

$$\overline{\phi}\left(\overline{\psi}\right) \le 10\left(2 + 2M^{2d+3}\right)n$$

and

$$\mathrm{d}\left(0,\, \mathbf{i}\right) > \frac{Kn}{M}.$$

Setting

$$m := \left\lfloor \frac{Kn}{M} \right\rfloor,$$

we see that by our choice (4.23)

$$\bigcup_{x:\,\mathrm{d}(0,x)>Kn} \left\{\widehat{\mathrm{d}}\left(0, x\right) \le 10n\right\} \subset \bigcup_{\mathbf{i}:\,\mathrm{d}(0,\mathbf{i})>m}\ \bigcup_{\overline{\psi}:\,\mathbf{i}\to\mathbf{0}} \left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \tag{4.24}$$

Fix $\mathbf{i}$ with $\mathrm{d}\left(0, \mathbf{i}\right) =: l > m$. A path $\overline{\psi} = \left(\mathbf{i}_0 = \mathbf{i}, \mathbf{i}_1, \dots, \mathbf{i}_r = 0\right)$ has length $r := \left|\overline{\psi}\right| \ge l$, hence

$$\left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \subset \left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{\left|\overline{\psi}\right|}{2}\right\}.$$

The number of paths of length $r \ge l$ on the lattice is bounded by $\left(2d\right)^r$. For every such path $\overline{\psi}$ the $\eta\left(\mathbf{i}_j\right)$ are i.i.d. with success probability (cf. (4.21))

$$\mathbb{P}\left(\eta\left(\mathbf{i}\right) = 1\right) =: \tau \ge 1 - \frac{1}{64d^2} > \frac{1}{2},$$

and therefore

$$\mathbb{P}\left(\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right) = \mathbb{P}\left(\left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \cap \left\{\widehat{\mathcal{A}} \ne \emptyset\right\}\right).$$

Therefore, for a fixed $\overline{\psi}$, the right-hand side above is bounded by the probability that a Bernoulli sequence of length r with success probability $1 - \tau$ has at least $r/2$ successes. This probability is bounded above by (see Arratia and Gordon (1989))

$$\exp\left[-rI\left(\frac{1}{2}\middle|\, 1 - \tau\right)\right],$$

where for $p_1, p_2 \in (0, 1)$ one defines

$$I\left(p_1 \middle|\, p_2\right) := p_1 \log\frac{p_1}{p_2} + \left(1 - p_1\right)\log\frac{1 - p_1}{1 - p_2}.$$

Hence

$$\mathbb{P}\left(\left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \cap \left\{\widehat{\mathcal{A}} \neq \emptyset\right\}\right) \le \exp\left[-rI\left(\frac{1}{2}\middle| 1-\tau\right)\right]$$
$$= \exp\left[-\frac{r}{2}\left(\log\frac{1}{2\left(1-\tau\right)} + \log\frac{1}{2\tau}\right)\right] \le \left(2\left(1-\tau\right)\right)^{r/2}\left(2\tau\right)^{r/2} \le \left(4\left(1-\tau\right)\right)^{r/2}.$$

Therefore, for $l > m$,

$$\mathbb{P}\left(\bigcup_{\overline{\psi}:\mathbf{i}\to\mathbf{0}}\left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \cap \left\{\widehat{\mathcal{A}} \neq \emptyset\right\}\right) \le \sum_{r=l}^{\infty}\left(2d\right)^r\left(4\left(1-\tau\right)\right)^{r/2}$$
$$\le \sum_{r\ge l}\left(2d\right)^r\left(4\frac{1}{64d^2}\right)^{r/2} = \sum_{r\ge l}\frac{1}{2^r} = \frac{1}{2^{l-1}},$$

and

$$\mathbb{P}\left(\bigcup_{\mathbf{i}:\mathrm{d}(0,\mathbf{i})>m}\ \bigcup_{\overline{\psi}:\mathbf{i}\to\mathbf{0}}\left\{\overline{\phi}\left(\overline{\psi}\right) \le \frac{m}{2}\right\} \cap \left\{\widehat{\mathcal{A}} \neq \emptyset\right\}\right) \le \sum_{l\ge m+1}\frac{\left|\left\{\mathbf{i}:\mathrm{d}\left(\mathbf{i},0\right)=l\right\}\right|}{2^{l-1}}$$
$$\le c_5\left(d\right)\sum_{l\ge m+1}\frac{l^{d-1}}{2^{l-1}} \le \left(\frac{2}{3}\right)^m$$

for large enough m. Together with (4.24), this gives

$$\mathbb{P}\left(\bigcup_{x:\mathrm{d}(0,x)>Kn}\left\{\widehat{\mathrm{d}}\left(0,x\right) \le 10n\right\}\right) \le \left(\frac{2}{3}\right)^{\frac{Kn}{M}}.$$

This concludes the proof of the lemma choosing $\lambda = \lambda(p,\, d) := -\left(K/M\right)\log(2/3)$. □

4.3 Proof of Theorem 2.5

We assume now $d \ge 5$. Let $x,\, y \in \mathbb{Z}^d$. We have to estimate $E_N^{\varepsilon}\left[\varphi_x\varphi_y\right]$. We may assume that $x,\, y \in V_N$, as otherwise the expression is 0. It is convenient to shift everything by x:

$$E_N^{\varepsilon}\left[\varphi_x\varphi_y\right] = E_{V_N+x}^{\varepsilon}\left[\varphi_0\varphi_{y-x}\right] = E_{\zeta_{V_N+x}^{\varepsilon}}\left(G_{\mathcal{A},\,V_N+x}(0,y-x)\mathbf{1}_{\{0\notin\mathcal{A}\}}\right)$$

where $\mathcal{A} \subset V_N + x$ is distributed according to $\zeta_{V_N+x}^{\varepsilon}$. Substituting $z := y - x$, we see that we have to estimate

$$\left|E_{\zeta_{V_N+x}^{\varepsilon}}\left(G_{\mathcal{A},\,V_N+x}(0,\,z)\mathbf{1}_{\{0\notin\mathcal{A}\}}\right)\right| \le E_{\zeta_{V_N+x}^{\varepsilon}}\left(\left|G_{\mathcal{A},\,V_N+x}(0,\,z)\right|\mathbf{1}_{\{0\notin\mathcal{A}\}}\right).$$

Let $\overline{\mathcal{A}} := \mathcal{A} \cup \left(V_N + x\right)^{\mathrm{c}}$. For a fixed realization of $\mathcal{A}$ with $0 \notin \mathcal{A}$, $G_{\mathcal{A},\,V_N+x}(0,\cdot)$ is $h_{\overline{\mathcal{A}}}$ restricted to $V_N + x$. Outside this set, $h_{\overline{\mathcal{A}}}$ is of course 0.

By Proposition 3.7, the distribution of $\overline{\mathcal{A}}$ under $\zeta^{\varepsilon}_{V_N+x}$ strongly dominates the Bernoulli law $\mathbb{P}^{\rho_-}$ where $\rho_- = \rho_-(d, \varepsilon)$ is defined by (3.5). The Bernoulli domination is proved there only for the configuration inside $V_N + x$, but as $\overline{\mathcal{A}}$ contains all the points outside $V_N + x$, the domination trivially extends to the measures on $\mathcal{P}\left(\mathbb{Z}^d\right)$.

Let $K = K(d, \varepsilon)$ be as defined in Lemma 4.9 with p there equal to ρ_-. Set

$$R_n := \left\{x \in \mathbb{Z}^d : Kn \leq \mathrm{d}(0, x) < K(n+1)\right\}.$$

We want to show that we can choose $\delta > 0$, depending on d, ε only, such that

$$\sup_{N,\, x} \zeta^{\varepsilon}_{V_N+x}\left(\sup_{z \in R_n} |G_{\mathcal{A},\, V_N+x}(0, z)| \geq \mathrm{e}^{-\delta n}\right) \leq L(\varepsilon)\, \mathrm{e}^{-\delta n}. \tag{4.25}$$

Having proved this, Theorem 2.5 follows, as $\sup_{z,\, x,\, N,\, A} |G_{A,\, V_N+x}(0, z)| \leq G(0, 0) < +\infty$ for $d \geq 5$ and therefore, if $z \in R_n$ for some n, by the law of total probability we get

$$\sup_{N,\, x} |E^{\varepsilon}_N[\varphi_x \varphi_{x+z}]| \leq \sup_{N,\, x} E_{\zeta^{\varepsilon}_{V_N+x}}\left(|G_{\mathcal{A},\, V_N+x}(0, z)|\, \mathbf{1}_{\{0 \notin \mathcal{A}\}}\right) \leq L(\varepsilon)\, \mathrm{e}^{-\delta n}.$$

In order to prove (4.25), set

$$\begin{aligned} X_n &:= \sup_{z \in R_n} |G_{\mathcal{A},\, V_N+x}(0, z)|, \\ Y_n &:= \|G_{\mathcal{A},\, V_N+x}(0, \cdot)\|_{\overline{\mathcal{A}}, R_n}, \\ \xi_n &:= \sqrt{1 + \sup_{x \in R_n} \mathrm{d}(x, \overline{\mathcal{A}})^{2d+3}}. \end{aligned}$$

Then

$$X_n \leq \sqrt{\sum_{z \in R_n} \left(G_{\mathcal{A},\, V_N+x}(0, z)\right)^2} \leq \xi_n Y_n. \tag{4.26}$$

To prove (4.25), we observe that for any $\delta' > 0$ and $n \geq n_0(\delta')$

$$\begin{aligned} \sup_{N,\, x} \zeta^{\varepsilon}_{V_N+x}\left(\xi_n Y_n \geq \mathrm{e}^{-\delta' n}\right) &= \sup_{N,\, x} \zeta^{\varepsilon}_{V_N+x}\left(\xi_n Y_n \geq \mathrm{e}^{-\delta' n},\, \xi_n < n^{2(d+2)}\right) \\ &\quad + \sup_{N,\, x} \zeta^{\varepsilon}_{V_N+x}\left(\xi_n Y_n \geq \mathrm{e}^{-\delta' n},\, \xi_n \geq n^{2(d+2)}\right) \\ &\leq \sup_{N, x} \zeta^{\varepsilon}_{V_N+x}\left(Y_n \geq \mathrm{e}^{-2\delta' n}\right) + \sup_{N, x} \zeta^{\varepsilon}_{V_N+x}\left(\xi_n \geq n^{2(d+2)}\right). \end{aligned} \tag{4.27}$$

Now define $2\delta' := \delta$ where δ appears in Corollary 4.7. Notice that

$$\begin{aligned} \zeta^{\varepsilon}_{V_N+x}\left(Y_n \geq \mathrm{e}^{-\delta n}\right) &= \zeta^{\varepsilon}_{V_N+x}\left(Y_n \geq \mathrm{e}^{-\delta n},\, R_n \subset C_n^{\mathrm{c}}\right) \\ &\quad + \zeta^{\varepsilon}_{V_N+x}\left(Y_n \geq \mathrm{e}^{-\delta n},\, R_n \cap C_n \neq \emptyset\right) \\ &\leq \zeta^{\varepsilon}_{V_N+x}\left(\overline{\mathcal{A}} = \emptyset\right) + \zeta^{\varepsilon}_{V_N+x}\left(\inf_{x : \mathrm{d}(x,0) > Kn} \widehat{\mathrm{d}}_{\overline{\mathcal{A}}}(0, x) \leq 10n\right). \end{aligned} \tag{4.28}$$

In the last inequality we have used Corollary 4.7. By means of the monotonicity property (4.6) and Bernoulli domination, the right-hand side above is dominated by

$$\mathbb{P}^{\rho^-}\left(\overline{\mathcal{A}}=\emptyset\right)+\mathbb{P}^{\rho^-}\left(\inf_{x:\mathrm{d}(x,0)>Kn}\widehat{\mathrm{d}}_{\overline{\mathcal{A}}}(0,x)\leq 10n\right),$$

where $\rho^- := \rho^-(d,\varepsilon)$. With λ as of Lemma 4.9 we can find $n=n(\lambda)$ large enough such that $\mathbb{P}^{\rho^-}\left(\overline{\mathcal{A}}=\emptyset\right)\leq\exp(-\lambda n)$ applying Lemma 4.8. We plug the result of Lemma 4.9 in (4.28) to get

$$\zeta^{\varepsilon}_{V_N+x}\left(Y_n\geq \mathrm{e}^{-\delta n}\right)\leq \mathrm{e}^{-\lambda n}. \tag{4.29}$$

We now look at the second summand of (4.27). For large enough n (depending on d, ε only)

$$\left\{\xi_n\geq n^{2(d+2)}\right\}\subset\left\{\sup_{x\in R_n}\mathrm{d}(x,\overline{\mathcal{A}})>n^2\right\}.$$

Using the monotonicity property (4.6), one has

$$\sup_{N,x}\zeta^{\varepsilon}_{V_N+x}\left(\sup_{x\in R_n}\mathrm{d}(x,\overline{\mathcal{A}})>n^2\right)\leq\mathbb{P}^{\rho^-}\left(\sup_{x\in R_n}\mathrm{d}(x,\mathcal{A})>n^2\right)$$

which evidently is of order $\exp\left[-L\times n^2\right]$ for large n. This, (4.29), (4.28) and (4.26) prove (4.25).

References

S. Adams. *Lectures on mathematical statistical mechanics*. Communications of the Dublin Institute for Advanced Studies: Theoretical Physics. Dublin Institute for Advanced Studies, 2006.

R. Arratia and L. Gordon. Tutorial on large deviations for the binomial distribution. *Bull. Math. Biol.*, 51(1):125–131, 1989. ISSN 0092-8240; 1522-9602/e. doi: 10.1007/BF02458840.

E. Bolthausen and Y. Velenik. Critical behavior of the massless free field at the depinning transition. *Comm. Math. Phys.*, 223(1):161–203, 2001.

E. Bolthausen, A. Cipriani, and N. Kurt. Fast decay of covariances under $\delta-$pinning in the critical and supercritical membrane model. *ArXiv e-prints*, Jan. 2016. URL http://arxiv.org/abs/1601.01513.

D. Brydges, J. Fröhlich, and T. Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982. URL http://projecteuclid.org/euclid.cmp/1103920749.

F. Caravenna and J.-D. Deuschel. Pinning and wetting transition for $(1+1)$-dimensional fields with Laplacian interaction. *Ann. Probab.*, 36(6):2388–2433, 2008. ISSN 0091-1798.

A. Cipriani. High points for the membrane model in the critical dimension. *Electron. J. Probab.*, 18:no. 86, 1–17, 2013. ISSN 1083-6489. doi: 10.1214/EJP.v18-2750. URL http://ejp.ejpecp.org/article/view/2750.

S. Friedli and Y. Velenik. Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction, 2015. URL http://www.unige.ch/math/folks/velenik/smbook/index.html.

C. Hiergeist and R. Lipowsky. Local contacts of membranes and strings. *Physica A: Statistical Mechanics and its Applications*, 244(1):164–175, 1997.

N. Kurt. *Entropic repulsion for a Gaussian membrane model in the critical and supercritical dimension*. PhD thesis, University of Zurich, 2008. URL https://www.zora.uzh.ch/6319/3/DissKurt.pdf.

N. Kurt. Maximum and entropic repulsion for a Gaussian membrane model in the critical dimension. *The Annals of Probability*, 37(2):687–725, 2009.

S. Leibler. Equilibrium statistical mechanics of fluctuating films and membranes. In *Statistical mechanics of membranes and surfaces*, pages 45–103. World Scientific Singapore, 1989.

R. Lipowsky. Generic interactions of flexible membranes. *Handbook of biological physics*, 1: 521–602, 1995.

V. Maz'ya. Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces. *Contemporary Mathematics*, 338:307–340, 2003.

J. J. Ruiz-Lorenzo, R. Cuerno, E. Moro, and A. Sánchez. Phase transition in tensionless surfaces. *Biophysical chemistry*, 115(2):187–193, 2005.

H. Sakagawa. Entropic repulsion for a Gaussian lattice field with certain finite range interactions. *J. Math. Phys.*, 44(7):2939–2951, 2003.

Y. Velenik. Localization and delocalization of random interfaces. *Probab. Surv.*, 3:112–169, 2006.