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Planelike interfaces in long-range Ising models and connections with nonlocal minimal surfaces

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ABSTRACT. This paper contains three types of results:

- the construction of ground state solutions for a long-range Ising model whose interfaces stay at a bounded distance from any given hyperplane,
- the construction of nonlocal minimal surfaces which stay at a bounded distance from any given hyperplane,
- the reciprocal approximation of ground states for long-range Ising models and nonlocal minimal surfaces.

In particular, we establish the existence of ground state solutions for long-range Ising models with planelike interfaces, which possess scale invariant properties with respect to the periodicity size of the environment. The range of interaction of the Hamiltonian is not necessarily assumed to be finite and also polynomial tails are taken into account (i.e. particles can interact even if they are very far apart the one from the other).

In addition, we provide a rigorous bridge between the theory of long-range Ising models and that of nonlocal minimal surfaces, via some precise limit result.

CONTENTS

1. INTRODUCTION

In this paper we consider an Ising model whose Hamiltonian is obtained by the superposition of an energy of ferromagnetic type and a magnetic potential.

As customary, this model describes the equilibria of a discrete set of variables that represent magnetic dipole moments of atomic spins that can be in one of two states (which we denote by $+1$ or -1). These spins are arranged in a d-dimensional lattice (that we take to be $\mathbb{Z}^d$, with $d \geqslant 2$).

We consider the case in which the Hamiltonian depends periodically on the environment, that is, given $\tau \in \mathbb{N}$, both the ferromagnetic and the magnetic energy are invariant under integer translations of length τ. Of course, this type of periodicity assumption is very common in the statistical mechanics literature, especially in view of applications to crystals.

Differently from most of the existing literature, we take into account the possibility that the particle interaction is not finite-range, but it possesses a tail at infinity (in particular, tails with polynomial decays are taken into consideration).

We show that, if the magnetic potential averages to zero in the fundamental domain of such crystal, one can construct ground states in which the interface remains uniformly close to any given hyperplane. More precisely, fixed any hyperplane, we construct minimal interfaces that stay at a distance from the hyperplane of the same order of the periodicity size of the model.

We stress that the vicinity to the prescribed hyperplane is uniform in the whole of the space and that the hyperplane can have rational or irrational slope (the corresponding solutions will then have accordingly periodic and quasiperiodic features).

Of course, the fact that the oscillation of the interface is proved to be of the same order of the crystalline scale has clear physical relevance.

Furthermore, it provides an additional scale invariance that we can use to take suitable limits of the solution constructed.

More precisely, we will show that, if we scale appropriately the planelike ground states of the Ising model, we obtain in the limit a minimal solution for a nonlocal perimeter functional which has been intensively studied in the recent literature (in particular, in this way we show that there exist planelike nonlocal minimal surfaces).

To make the picture complete, we also show that any unique minimizer of the nonlocal perimeter problem can be approximated by ground states of the Ising model, thus providing a complete bridge between the long-range statistical mechanics framework and the geometric measure theory in nonlocal setting.

We recall that the construction of planelike solutions is a classical topic in several areas of pure and applied mathematics. This problem dates back, at least, to the construction of planelike geodesics on surfaces of genus greater than one, see [M24]. As pointed out in [H32], geodesics in higher dimensional manifolds fail, in general, to satisfy planelike conditions. Hence, the question of finding planelike solutions eventually led to the generalization of the notion of "orbits" with that of "invariant measures" in dynamical systems, which in turn gave a fundamental contribution to the birth of the Aubry-Mather (or weak KAM) theory, see [AD83, M89, M91].

In addition, in [M86] the problem of finding suitable planelike solutions was put in a new framework for elliptic partial differential equations, where the question of finding suitable analogues for hypersurfaces of minimal perimeter was also posed.

In turn, this question for minimal surfaces was successfully addressed in [CdlL01, AB01].

See also [CF96, RS04, V04, PV05, B08] for related results for elliptic partial differential equations, [T04, BV08] for additional results in Riemannian and sub-Riemannian settings, [CdlL05, dlLV07, dlLV10] for results in statistical mechanics, and [CV15, CV16] for results for fractional equations.

We now introduce the formal mathematical settings in which we work. Let $d \in \mathbb{N}$ with $d \geqslant 2$. We endow $\mathbb{Z}^d$ (and, more generally, $\mathbb{Q}^d$) with its natural ℓ^1 norm, that will be simply denoted by $|\cdot|$. For simplicity of exposition and rather uncharacteristically, we adopt this notation even for vectors in $\mathbb{R}^d$. Thus, we write

$$|i| = |i|_1 := \sum_{n=1}^{d} |i_n| \quad \text{for any } i \in \mathbb{R}^d.$$

Of course, for the vast majority of the arguments a different norm of $\mathbb{R}^d$ could be considered as well, with no significant changes in the computations.

We call any function $u : \mathbb{Z}^d \to \{-1, 1\}$ a *configuration*. Associated to any configuration u is its interface $\partial u \subset \mathbb{Z}^d$ defined by

$$\partial u := \Big\{ i \in \mathbb{Z}^d : u_i = 1 \text{ and there exists } j \in \mathbb{Z}^d \text{ such that } |i - j| = 1 \text{ and } u_j = -1 \Big\}.$$

Given a configuration u, we consider its (formal) Hamiltonian

$$H(u) := \sum_{i,j \in \mathbb{Z}^d} J_{ij} \left(1 - u_i u_j\right) + \sum_{i \in \mathbb{Z}^d} h_i u_i,$$

where $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, +\infty)$ satisfies

$$J_{ij} = J_{ji} \quad \text{for any } i, j \in \mathbb{Z}^d, \tag{1.1}$$

$$J_{ii} = 0 \quad \text{for any } i \in \mathbb{Z}^d, \tag{1.2}$$

$$J_{ij} \geqslant \lambda \quad \text{for any } i, j \in \mathbb{Z}^d \text{ such that } |i - j| = 1, \tag{1.3}$$

$$\sum_{j \in \mathbb{Z}^d} J_{ij} \leqslant \Lambda \quad \text{for any } i \in \mathbb{Z}^d, \tag{1.4}$$

for some $\Lambda \geqslant \lambda > 0$, while $h : \mathbb{Z}^d \to \mathbb{R}$ is such that

$$\sup_{i \in \mathbb{Z}^d} |h_i| \leqslant \mu, \tag{1.5}$$

$$\sum_{i \in F} h_i = 0, \tag{1.6}$$

for some $\mu > 0$ and with F denoting any fundamental domain of the quotient space $\mathbb{Z}^d / \tau \mathbb{Z}^d$, with $\tau \in \mathbb{N}$.

Sometimes, we will require J to fulfill the following stronger assumption, in place of (1.3) and (1.4):

$$\frac{\lambda}{|i - j|^{d+s}} \leqslant J_{ij} \leqslant \frac{\Lambda}{|i - j|^{d+s}} \quad \text{for any } i, j \in \mathbb{Z}^d \text{ with } i \neq j \text{ and for some } s \in (0, 1). \tag{1.7}$$

We point out that long-range Ising models like the ones described by the above requirements are well-studied in the literature (see for instance [DRAW02, CDR09, P12, BPR13] and references therein), with particular attention given to those taking into account *power-like* interactions as in (1.7). The array of models covered by our choice of parameters (namely, $s \in (0, 1)$) falls into the class of the so-called *weak* long-range interactions. Anyway, we stress that a wider generality (e.g. the case of (1.7) with $s \geqslant 1$) is already encompassed within the broader framework of hypotheses (1.3) and (1.4).

The periodicity of the medium is modeled by requiring that, given $\tau \in \mathbb{N}$,

$$J_{ij} = J_{i'j'} \quad \text{for any } i, j, i', j' \in \mathbb{Z}^d \text{ such that } i - i' = j - j' \in \tau \mathbb{Z}^d, \tag{1.8}$$

$$h_i = h_{i'} \quad \text{for any } i, i' \in \mathbb{Z}^d \text{ such that } i - i' \in \tau \mathbb{Z}^d. \tag{1.9}$$

Associated to the interaction kernel J, we consider the non-increasing function

$$\sigma(R) := \sup_{i \in \mathbb{Z}^d} \sum_{\substack{j \in \mathbb{Z}^d \\ |j - i|_\infty \geqslant R}} J_{ij}, \tag{1.10}$$

defined for any $R \in \mathbb{N}$. Note that we indicate with $| \cdot |_\infty$ the ℓ^∞ norm in $\mathbb{Z}^d$ and $\mathbb{R}^d$, that is

$$|i|_\infty := \sup_{n = 1, \dots, d} |i_n| \quad \text{for any } i \in \mathbb{R}^d. \tag{1.11}$$

Observe that σ quantifies the decay of the tails of J.

Given a set $\Gamma \subset \mathbb{Z}^d$, we introduce the restricted Hamiltonian H_Γ, defined on any configuration u by

$$\begin{aligned} H_\Gamma(u) &:= \sum_{(i,j)\in\mathbb{Z}^{2d}\setminus(\mathbb{Z}^d\setminus\Gamma)^2} J_{ij}(1-u_iu_j) + \sum_{i\in\Gamma} h_iu_i \\ &= \sum_{i\in\Gamma,\, j\in\Gamma} J_{ij}(1-u_iu_j) + 2\sum_{i\in\Gamma,\, j\in\mathbb{Z}^d\setminus\Gamma} J_{ij}(1-u_iu_j) + \sum_{i\in\Gamma} h_iu_i. \end{aligned}$$

Note that $H_\Gamma(u)$ is always well-defined when Γ is a finite set, as (1.4) is in force.

It will be useful to have a notation for the interaction energy involving two subsets of $\mathbb{Z}^d$. Given any two sets $\Gamma, \Omega \subseteq \mathbb{Z}^d$, we consider the restricted interaction term

$$I_{\Gamma,\Omega}(u) := \sum_{i\in\Gamma,\, j\in\Omega} J_{ij}(1-u_iu_j). \tag{1.12}$$

We also write

$$I_\Gamma(u) := I_{\Gamma,\Gamma}(u) + I_{\Gamma,\,\mathbb{Z}^d\setminus\Gamma}(u) + I_{\mathbb{Z}^d\setminus\Gamma,\,\Gamma}(u).$$

On the other hand, we indicate with B_Γ the part of the Hamiltonian H_Γ related to the magnetic field h. That is,

$$B_\Gamma(u) := \sum_{i\in\Gamma} h_iu_i. \tag{1.13}$$

With these notations, it holds that

$$H_\Gamma(u) = I_\Gamma(u) + B_\Gamma(u).$$

Definition 1.1. *We say that a configuration u is a* minimizer *for H in a set $\Gamma \subseteq \mathbb{Z}^d$ if it satisfies*

$$H_\Gamma(u) \leqslant H_\Gamma(v),$$

for any configuration v that agrees with u outside of Γ.

Remark 1.2. We point out that, although perhaps not immediately evident from the way the interaction term I is defined, the definition of minimizer is consistent with set inclusion. With this we mean that, given two sets $\Gamma \subseteq \Omega$, a minimizer in Ω is also a minimizer in Γ.

To see this, it suffices to observe that, if u and v are two configurations satisfying

$$u_i = v_i \quad \text{for any } i \in \mathbb{Z}^d \setminus \Gamma,$$

then

$$H_\Omega(u) - H_\Omega(v) = H_\Gamma(u) - H_\Gamma(v).$$

Of course, it is easy to check that such an identity is true for the magnetic term B. On the other hand, the computation of the interaction term is slightly more involved, due to the presence of a double summation. However, it becomes more apparent once one notices that $\left[\mathbb{Z}^{2d} \setminus (\mathbb{Z}^d \setminus \Gamma)^2\right] \subseteq \left[\mathbb{Z}^{2d} \setminus (\mathbb{Z}^d \setminus \Omega)^2\right]$ and

$$u_iu_j = v_iv_j \quad \text{for any } (i,j) \in \left[\mathbb{Z}^{2d} \setminus (\mathbb{Z}^d \setminus \Omega)^2\right] \setminus \left[\mathbb{Z}^{2d} \setminus (\mathbb{Z}^d \setminus \Gamma)^2\right].$$

Definition 1.3. *We say that a configuration u is a* ground state *for H if it is a minimizer for H in any finite set $\Gamma \subset \mathbb{Z}^d$.*

With this setting, we are in the position of stating our first result, which provides the existence of ground state solutions for long-range Ising models with interfaces that remain at a bounded distance from a given hyperplane (and, additionally, if J satisfies (1.7), such distance is of the same order of the size of periodicity of the medium):

Theorem 1.4. *Suppose that J and h satisfy assumptions (1.1), (1.2), (1.3), (1.4), (1.8) and (1.5) (1.6), (1.9), respectively. Then, there exist a small constant $\mu_0 > 0$, depending only on d, τ and λ, and a large constant $M > 0$, that may also depend on Λ and the function σ, for which, given any direction $\omega \in \mathbb{R}^d \setminus \{0\}$, we can find a ground state u_ω for H such that its interface ∂u_ω satisfies the inclusion*

$$\partial u_\omega \subset \left\{ i \in \mathbb{Z}^d : \frac{\omega}{|\omega|} \cdot i \in [0, M] \right\}, \tag{1.14}$$

provided that $\mu \leqslant \mu_0$.

More precisely, for any $i \in \mathbb{Z}^d$ with $\frac{\omega}{|\omega|} \cdot i \geqslant M$ we have that $u_{\omega,i} = -1$, and for any $i \in \mathbb{Z}^d$ with $\frac{\omega}{|\omega|} \cdot i \leqslant 0$ we have that $u_{\omega,i} = 1$.

Furthermore, if J satisfies (1.7), in addition to the conditions already specified, and h vanishes identically, then the constant M may be chosen of the form $M = M_0 \tau$, with $M_0 > 0$ depending only on d, s, λ and Λ.

In the case of finite-range periodic Ising models, the result in (1.14) was obtained in [CdlL05] (see in particular formula (2) and Theorem 2.1 there). We also point the reader's attention to the more recent [B14], where it is shown that such existence result fails when one considers coefficients that are only almost-periodic (i.e. that are the uniform limits of a family of periodic coefficients of increasing period).

We stress that the additional result that we obtain when J satisfies (1.7) plays for us a crucial role, since such scale invariance is the cornerstone to link the long-range Ising models to the nonlocal minimal surfaces (and this will be the content of the forthcoming Theorems 1.6 and 1.8).

In order to deal with nonlocal minimal surfaces in periodic media, it is convenient now to introduce the following auxiliary notation. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$ be a measurable function satisfying

$$K(x, y) = K(y, x) \quad \text{for a.e. } x, y \in \mathbb{R}^d, \tag{1.15}$$

and

$$\frac{\lambda}{|x - y|^{d+s}} \leqslant K(x, y) \leqslant \frac{\Lambda}{|x - y|^{d+s}} \quad \text{for a.e. } x, y \in \mathbb{R}^d, \tag{1.16}$$

for some exponent $s \in (0, 1)$ and for some constants $\Lambda \geqslant \lambda > 0$. We also assume K to be $\mathbb{Z}^d$-periodic, that is

$$K(x + z, y + z) = K(x, y) \quad \text{for any } z \in \mathbb{Z}^d \text{ and a.e. } x, y \in \mathbb{R}^d. \tag{1.17}$$

For any open set $\Omega \subseteq \mathbb{R}^d$ and any measurable function $u : \mathbb{R}^d \to \mathbb{R}$, we define

$$\mathscr{K}_K(u; \Omega) := \iint_{\mathscr{C}_\Omega} |u(x) - u(y)| K(x, y) \, dx \, dy,$$

where

$$\mathscr{C}_\Omega := \mathbb{R}^{2d} \setminus \left(\mathbb{R}^d \setminus \Omega \right)^2.$$

Given any two measurable sets $A, B \subseteq \mathbb{R}^d$, we also write

$$\mathscr{K}_K(u; A, B) := \int_A \int_B |u(x) - u(y)| K(x, y) \, dx \, dy, \tag{1.18}$$

so that, recalling (1.15), it holds

$$\mathscr{K}_K(u; \Omega) = \mathscr{K}_K(u; \Omega, \Omega) + 2\mathscr{K}_K(u; \Omega, \mathbb{R}^d \setminus \Omega).$$

The *K-perimeter* of a measurable set $E \subseteq \mathbb{R}^d$ inside Ω is defined by

(1.19) $$\mathrm{Per}_K(E;\Omega) := \mathcal{L}_K(E\cap\Omega, \Omega\setminus E) + \mathcal{L}_K(E\cap\Omega, \mathbb{R}^d\setminus(E\cup\Omega)) + \mathcal{L}_K(E\setminus\Omega, \Omega\setminus E),$$

where, for any two disjoint sets $A, B \subset \mathbb{R}^d$,

(1.20) $$\mathcal{L}_K(A,B) := \int_A\int_B K(x,y)\,dx\,dy.$$

We observe that

(1.21) $$\mathrm{Per}_K(E;\Omega) = \frac{1}{4}\mathscr{K}_K\left(\chi_E - \chi_{\mathbb{R}^d\setminus E};\Omega\right).$$

We recall that, when $K(x,y) := |x-y|^{-d-s}$, the nonlocal perimeter in (1.19) reduces to that introduced in [CRS10]. In this sense, the nonlocal perimeter in (1.19) is a natural notion of fractional perimeter in a non-homogeneous environment. For a basic presentation of nonlocal minimal surfaces (i.e. surfaces which locally minimize nonlocal perimeter functionals), see e.g. pages 97–126 in [BV16].

The concept of optimal set that we take into account here is rigorously described by the following definition:

Definition 1.5. *Given an open set $\Omega \subseteq \mathbb{R}^d$, a measurable set $E \subseteq \mathbb{R}^d$ is said to be a* minimizer *(or a* minimal surface[1]*) for Per_K in Ω if $\mathrm{Per}_K(E;\Omega) < +\infty$ and*

$$\mathrm{Per}_K(E;\Omega) \leqslant \mathrm{Per}_K(F;\Omega) \quad \text{for any measurable set } F\subseteq\mathbb{R}^d \text{ such that } F\setminus\Omega = E\setminus\Omega.$$

Furthermore, E is said to be a class A minimal surface *for Per_K if it is a minimizer for Per_K in every bounded open set $\Omega \subset \mathbb{R}^d$.*

By means of an argument similar to that presented in Remark 1.2 for the discrete setting, one can easily convince himself or herself that to verify that a set E is a class A minimal surface for Per_K it is enough to check that E minimizes the K-perimeter on each set of an exhaustion of $\mathbb{R}^d$ that consists of bounded subsets, e.g. concentric balls or cubes of increasing diameters.

In order to describe the similarity between the power-like long-range Ising model and the K-perimeter, we associate to each kernel K a specific family of systems of coefficients $J^{(\varepsilon)}$. Indeed, given $\varepsilon > 0$, we set for any $i, j \in \mathbb{Z}^d$

(1.22) $$J_{ij}^{(\varepsilon)} := \begin{cases} \varepsilon^{-d+s}\displaystyle\int_{Q_{\varepsilon/2}(\varepsilon i)}\int_{Q_{\varepsilon/2}(\varepsilon j)} K(x,y)\,dx\,dy & \text{if } i\neq j\\ 0 & \text{if } i=j.\end{cases}$$

As we will see in the forthcoming Lemma 5.1 in Section 5, the coefficients $J^{(\varepsilon)}$ satisfy assumptions (1.1), (1.2) and (1.7), uniformly in ε.

Related to $J^{(\varepsilon)}$ is then the Hamiltonian $H^{(\varepsilon)}$ with zero magnetic flux, defined on every finite set $\Gamma \subset \mathbb{Z}^d$ and any configuration u by

(1.23) $$H_\Gamma^{(\varepsilon)}(u) := \sum_{(i,j)\in\mathbb{Z}^{2d}\setminus(\mathbb{Z}^d\setminus\Gamma)^2} J_{ij}^{(\varepsilon)}(1-u_iu_j).$$

Moreover, to each configuration u, we associate its *extension* $\bar{u}_\varepsilon : \mathbb{R}^d \to \{-1,1\}$ defined a.e. by setting

(1.24) $$\bar{u}_\varepsilon(x) := u_i \quad \text{where } i\in\mathbb{Z}^d \text{ is the only site for which } x\in \mathring{Q}_{\varepsilon/2}(\varepsilon i).$$

[1]Here we adopt a partially misleading terminology, as the *boundary* ∂E, and not the set E, should be regarded as the minimal *surface*, in conformity with the classical geometrical notion of perimeter. However, we have $\mathrm{Per}_K(E;\Omega) = \mathrm{Per}_K(\mathbb{R}^d\setminus E;\Omega)$, for any set E, and thus no confusion should arise from this slightly improper notation.

Note that the above family of extensions allows us to understand configurations as characteristic functions in $\mathbb{R}^d$, via the embedding

$$\mathbb{Z}^d \longrightarrow \varepsilon\mathbb{Z}^d \hookrightarrow \mathbb{R}^d,$$

defined by

$$\mathbb{Z}^d \ni i \longmapsto \varepsilon i \in \mathbb{R}^d.$$

Clearly, the smaller the parameter ε is, the more densely the grid $\mathbb{Z}^d$ is embedded in $\mathbb{R}^d$, and so the closer the Hamiltonian $H^{(\varepsilon)}$ looks to the K-perimeter.

The following result addresses such similarity in a rigorous way, by showing that the limit of ground states for the long-range Ising models with Hamiltonians (1.23) produces a nonlocal minimal surface:

Theorem 1.6. *Suppose that K satisfies assumptions (1.15) and (1.16). Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be an infinitesimal sequence. For any $n \in \mathbb{N}$, let $u^{(n)}$ be a ground state for the Hamiltonian $H^{(\varepsilon_n)}$ and let $\bar{u}^{(n)} = \bar{u}^{(n)}_{\varepsilon_n}$ be its extension to $\mathbb{R}^d$, according to (1.24).*

Then, there exists a diverging sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that

$$\bar{u}^{(n_k)} \longrightarrow \chi_E - \chi_{\mathbb{R}^d\setminus E} \quad \text{a.e. in } \mathbb{R}^d, \text{ as } k \to +\infty,$$

where $E \subseteq \mathbb{R}^d$ is a class A minimal surface for Per_K.

By combining Theorems 1.4 and 1.6, we obtain the existence of planelike minimal surfaces, as stated in the following result:

Theorem 1.7. *Suppose that K satisfies assumptions (1.15), (1.16) and (1.17). Then, there exists a constant $M_0 > 0$, depending only on d, s, λ and Λ, for which, given any direction $\omega \in \mathbb{R}^d \setminus \{0\}$, we can construct a class A minimal surface E_ω for Per_K, such that*

$$\left\{ x \in \mathbb{R}^d : \frac{\omega}{|\omega|} \cdot x < -M_0 \right\} \subset E_\omega \subset \left\{ x \in \mathbb{R}^d : \frac{\omega}{|\omega|} \cdot x \leqslant M_0 \right\}. \tag{1.25}$$

The result in Theorem 1.7 here positively addresses a problem presented in [C09].

In the forthcoming paper [CV16], we plan to obtain the same result of Theorem 1.7 by a different method, namely by approaching nonlocal minimal surfaces by nonlocal phase transitions of Ginzburg-Landau-Allen-Cahn type: in this spirit, we may consider the nonlocal minimal surfaces as a natural “pivot”, which joins, in the limit, the Ginzburg-Landau-Allen-Cahn phase transitions and the Ising models in a rigorous way.

Also, as a partial counterpart to Theorem 1.6, we have the following result, which states that a unique minimizer of the nonlocal perimeter functional can be approximated by ground states of long-range Ising models:

Theorem 1.8. *Suppose that K satisfies assumptions (1.15) and (1.16). Let E be an open subset of $\mathbb{R}^d$ and suppose that it is a strict minimizer for Per_K in the cube[2] Q_R, with $R \geqslant 1$, that is $\mathrm{Per}_K(E;\Omega) < +\infty$ and*

$$\mathrm{Per}_K(E;\Omega) < \mathrm{Per}_K(F;\Omega) \quad \text{for any } F \subseteq \mathbb{R}^d \text{ such that } F \setminus \Omega = E \setminus \Omega \text{ and } F \neq E.$$

[2] Throughout the whole paper, Q_R denotes the closed cube of $\mathbb{R}^d$ having sides of length $2R$ and centered at the origin, i.e.

$$Q_R := \left\{ x \in \mathbb{R}^d : |x|_\infty \leqslant R \right\}.$$

We use the same notation for cubes in $\mathbb{Z}^d$. That is, for $\ell \in \mathbb{N} \cup \{0\}$, we write

$$Q_\ell := \left\{ i \in \mathbb{Z}^d : |i|_\infty \leqslant \ell \right\} = \left\{ -\ell, \dots, -1, 0, 1, \dots, \ell \right\}^d.$$

Cubes not centered at the origin are indicated with $Q_R(x) := x + Q_R$ and $Q_\ell(q) := q + Q_\ell$, with $x \in \mathbb{R}^d$ and $q \in \mathbb{Z}^d$.

Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be an infinitesimal sequence.

Then, for any $n \in \mathbb{N}$, there exists a minimizer $u^{(n)}$ for $H^{(\varepsilon_n)}$ in the[3] *cube $Q_{\lceil R/\varepsilon_n \rceil}$, such that, denoting by $\bar{u}^{(n)} = \bar{u}^{(n)}_{\varepsilon_n}$ its extension to $\mathbb{R}^d$ given by (1.24), it holds*

$$\bar{u}^{(n_k)} \longrightarrow \chi_E - \chi_{\mathbb{R}^d \setminus E} \quad \text{a.e. in } \mathbb{R}^d, \text{ as } k \to +\infty,$$

for some diverging sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers.

We remark that, in view of Theorems 1.6 and 1.8, there is a perfect correspondence between the ground states of the Ising model and the minimizers of the nonlocal perimeter, provided that the latter ones are unique.

To make this correspondence even more explicit, we may rephrase it through the language of Γ-convergence. We consider the topological space

$$\mathcal{X} := \Big\{ v \in L^\infty(\mathbb{R}^d) : \|v\|_{L^\infty(\mathbb{R}^d)} \leqslant 1 \Big\},$$

as endowed with the topology given by the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

For any $\varepsilon > 0$, we also introduce the subspace

$$\mathcal{X}_\varepsilon := \Big\{ v \in \mathcal{X} : v \text{ is constant on the cube } \mathring{Q}_{\varepsilon/2}(\varepsilon i), \text{ for any } i \in \mathbb{Z}^d \Big\}. \tag{1.26}$$

Also, given any bounded open set $\Omega \subset \mathbb{R}^d$, we consider the functionals $\mathscr{G}_K(\cdot;\Omega) : \mathcal{X} \to [0,+\infty]$ defined by

$$\mathscr{G}_K(v;\Omega) := \begin{cases} \mathscr{K}_K(v;\Omega) & \text{if } v|_\Omega = \chi_E - \chi_{\mathbb{R}^d \setminus E}, \text{ for some measurable } E \subseteq \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.27}$$

and $\mathscr{G}^{(\varepsilon)}_K(\cdot;\Omega) : \mathcal{X}_\varepsilon \to [0,+\infty]$ obtained by setting $\mathscr{G}^{(\varepsilon)}_K(\cdot;\Omega) := \mathscr{G}_K(\cdot;\Omega)|_{\mathcal{X}_\varepsilon}$.

Observe that, in view of identity (1.21), when v is globally the (modified) characteristic function of a set E, then $\mathscr{G}_K(v;\Omega)$ boils down to the K-perimeter of E inside Ω.

Notice that the map defined in (1.24) is actually a homeomorphism of the space of configurations (endowed with the standard pointwise convergence topology) onto the space $\mathcal{X}_\varepsilon$. Moreover, given any $\ell \in \mathbb{N}$, we observe that any configuration u, together with its extension $\bar{u}_\varepsilon \in \mathcal{X}_\varepsilon$ (as given by (1.24)), satisfies the Hamiltonian-energy relation

$$\varepsilon^{d-s} H^{(\varepsilon)}_{Q_\ell}(u) = \mathscr{K}_K(\bar{u}_\varepsilon, Q_R), \tag{1.28}$$

where $R = (\ell + 1/2)\varepsilon$. This identity completes the picture on the equivalence between the space of configurations with the associated Hamiltonian $H^{(\varepsilon)}$ and $\mathcal{X}_\varepsilon$ with the energy $\mathscr{K}_K$.

Thanks to this complete identification, it is legitimate to see the next result as an appropriate Γ-convergence formulation of the asymptotic relation intervening between the ε-Ising model (1.22)-(1.23) and the K-perimeter (1.19).

Theorem 1.9. *Suppose that K satisfies assumptions (1.15) and (1.16). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary.*[4]

Then, the family of functionals $\mathscr{G}^{(\varepsilon)}_K(\cdot,\Omega)$ Γ-converges to $\mathscr{G}_K(\cdot,\Omega)$, as $\varepsilon \to 0^+$. More precisely, we have

[3] As usual, we will denote by $\lceil x \rceil$ the smallest integer greater than or equal to x, and by $\lfloor x \rfloor$ the largest integer less than or equal to x.

[4] Actually, the Lipschitz regularity assumption on the boundary of Ω can be omitted for the deduction of the Γ-$\liminf$ inequality.

- $(\Gamma\text{-}\liminf$ *inequality*$)$*: for any* $u_\varepsilon \in \mathcal{X}_\varepsilon$ *converging to* $u \in \mathcal{X}$*, it holds*

$$\liminf_{\varepsilon\to 0^+} \mathscr{G}_K^{(\varepsilon)}(u_\varepsilon;\Omega) \geqslant \mathscr{G}_K(u;\Omega);$$

- $(\Gamma\text{-}\limsup$ *inequality*$)$*: for any* $u \in \mathcal{X}$*, there exists* $u_\varepsilon \in \mathcal{X}_\varepsilon$ *converging to* u *and such that*

$$\limsup_{\varepsilon\to 0^+} \mathscr{G}_K^{(\varepsilon)}(u_\varepsilon;\Omega) \leqslant \mathscr{G}_K(u;\Omega);$$

- $($*Compactness*$)$*: given any infinitesimal sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$*, if* $u_n \in \mathcal{X}_{\varepsilon_n}$ *satisfies*

$$\sup_{n\in\mathbb{N}} \mathscr{G}_K^{(\varepsilon_n)}(u_n;\Omega) \leqslant C,$$

for some $C \geqslant 0$*, then there exist a measurable set* $E \subseteq \Omega$ *and a diverging sequence* $\{n_k\}_{k\in\mathbb{N}}$ *of natural numbers such that* u_{n_k} *converges to* $\chi_E - \chi_{\mathbb{R}^d\setminus E}$ *a.e. in* Ω*, as* $k \to +\infty$*.*

The rest of the paper follows this organization: in Section 2 and 3 we give the proof of Theorem 1.4, by considering as a special case the one of power-like interactions with no magnetic term (which leads to additional, scale invariant, results).

Then, in Section 4, we present some ancillary results on nonlocal perimeter functionals. The link between Ising models and nonlocal minimal surfaces is discussed in Sections 5 and 7, where we give the proofs of Theorems 1.6 and 1.8, respectively. In between, in Section 6, we also prove Theorem 1.7, thus obtaining the existence of planelike nonlocal minimal surfaces as a byproduct of our analysis of the Ising model.

Finally, Section 8 is devoted to the proof of the Γ-convergence result given by Theorem 1.9.

2. Proof of Theorem 1.4 in the general setting

In this section we include the proof of Theorem 1.4 in the general case of J and h satisfying (1.1), (1.2), (1.3), (1.4), (1.8) and (1.5) (1.6), (1.9), respectively. The more specific scenario given by hypothesis (1.7) and $h = 0$, described in the latter claim of the statement of Theorem 1.4, will be considered in the next Section 3.

As the construction is rather involved, we split the argument into eight subsections.

First, we consider the case of a rational $\omega \in \mathbb{Q}^d \setminus \{0\}$. For any such direction, we build a ground state for H whose interface satisfies the inclusion (1.14), for some $M > 0$. As will be evident by following the steps of the construction, the constant M is indeed independent of the chosen direction ω. As a result, an approximation argument displayed in the conclusive Subsection 2.8 will show that Theorem 1.4 can be extended to general directions $\omega \in \mathbb{R}^d \setminus \{0\}$.

Although the existence of ground states will be eventually carried out in the generality announced in the statement of Theorem 1.4, we need to initially impose an additional condition on the interaction coefficients J. Throughout Subsections 2.1-2.6, we always assume that J satisfies

$$J_{ij} = 0 \quad \text{for any } i,j \in \mathbb{Z}^d \text{ such that } |i-j| > R, \tag{2.1}$$

for some $R > 0$. Assumption (2.1) allows us to avoid some technical complications related to the presence of tails in the interaction term of the Hamiltonian H. The estimates performed in the next subsections under hypothesis (2.1) will however turn out to be independent of the range of positivity $R > 0$. Therefore, in Subsection 2.7 we will be able to remove such assumption with the help of an easy limiting argument and thus recover the validity of Theorem 1.4 in its full generality.

2.1. **Constrained minimizers.** Let $\omega \in \mathbb{Q}^d \setminus \{0\}$ and $m \in \mathbb{N}$. We consider the $\mathbb{Z}$-modules

$$\mathscr{L}_\omega := \Big\{ i \in \tau\mathbb{Z}^d : \omega \cdot i = 0 \Big\},$$

and

$$\mathscr{L}_{m,\omega} := m\mathscr{L}_\omega.$$

We indicate with $\mathcal{F}_{m,\omega}$ any fundamental domain of the quotient space $\mathbb{Z}^d/\mathscr{L}_{m,\omega}$. Given any two real numbers $A < B$, we divide $\mathcal{F}_{m,\omega}$ into the three subregions

$$\begin{aligned} \mathcal{F}^{A,B}_{m,\omega} &:= \left\{ i \in \mathcal{F}_{m,\omega} : \frac{\omega}{|\omega|} \cdot i \in [A,B] \right\}, \\ \mathcal{F}^{A,-}_{m,\omega} &:= \left\{ i \in \mathcal{F}_{m,\omega} : \frac{\omega}{|\omega|} \cdot i < A \right\} \\ \text{and} \quad \mathcal{F}^{B,+}_{m,\omega} &:= \left\{ i \in \mathcal{F}_{m,\omega} : \frac{\omega}{|\omega|} \cdot i > B \right\}. \end{aligned}$$

A configuration u is said to be (m,ω)-*periodic* if

$$u_{i+k} = u_i \quad \text{for any } i \in \mathbb{Z}^d \text{ and any } k \in \mathscr{L}_{m,\omega}. \tag{2.2}$$

We denote by $\mathscr{P}_{m,\omega}$ the set of all (m,ω)-periodic configurations. Furthermore, we consider the class $\mathscr{A}^{A,B}_{m,\omega}$ of *admissible configurations*, defined by

$$\mathscr{A}^{A,B}_{m,\omega} := \Big\{ u \in \mathscr{P}_{m,\omega} : u_i = 1 \text{ for any } i \in \mathcal{F}^{A,-}_{m,\omega} \text{ and } u_i = -1 \text{ for any } i \in \mathcal{F}^{B,+}_{m,\omega} \Big\}.$$

Recalling the notation in (1.12) and (1.13), we introduce the auxiliary functional $G^{A,B}_{m,\omega}$, defined on any configuration u by

$$\begin{aligned} G^{A,B}_{m,\omega}(u) &:= I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d}(u) + B_{\mathcal{F}^{A,B}_{m,\omega}}(u) \\ &= \sum_{i \in \mathcal{F}_{m,\omega},\, j \in \mathbb{Z}^d} J_{ij}(1 - u_i u_j) + \sum_{i \in \mathcal{F}^{A,B}_{m,\omega}} h_i u_i. \end{aligned} \tag{2.3}$$

Observe that the interaction term of this functional differs from that of $H_{\mathcal{F}_{m,\omega}}$ for the fact that in $H_{\mathcal{F}_{m,\omega}}$ the interactions between the regions $\mathcal{F}_{m,\omega}$ and $\mathbb{Z}^d \setminus \mathcal{F}_{m,\omega}$ are counted twice. Also note that $G^{A,B}_{m,\omega}$ is well-defined on any configuration, as it involves a sum of only a finite number of terms, thanks to (2.1).

Moreover, we denote by $\mathscr{M}^{A,B}_{m,\omega}$ the subset of $\mathscr{A}^{A,B}_{m,\omega}$ composed by the minimizers of $G^{A,B}_{m,\omega}$. That is,

$$\mathscr{M}^{A,B}_{m,\omega} := \Big\{ u \in \mathscr{A}^{A,B}_{m,\omega} : G^{A,B}_{m,\omega}(u) \leqslant G^{A,B}_{m,\omega}(v) \text{ for any } v \in \mathscr{A}^{A,B}_{m,\omega} \Big\}.$$

Observe that $\mathscr{M}^{A,B}_{m,\omega}$ is non-empty, since $\mathscr{A}^{A,B}_{m,\omega}$ is made up of a finite number of configurations.

Now we introduce a couple of operations on the space of configurations. Given two configurations u, v we define their minimum $\min\{u,v\}$ and maximum $\max\{u,v\}$ by setting

$$\begin{aligned} \big(\min\{u,v\}\big)_i &:= \min\{u_i, v_i\}, \\ \big(\max\{u,v\}\big)_i &:= \max\{u_i, v_i\}, \end{aligned} \tag{2.4}$$

for any $i \in \mathbb{Z}^d$. Analogously, one defines the minimum and maximum of a finite number of configurations.

We present the following simple result which shows that the interaction energy (1.12) always decreases when considering minima and maxima.

Lemma 2.1. *Given any two subsets $\Gamma, \Omega \subseteq \mathbb{Z}^d$ and any two configurations u, v, it holds*

$$I_{\Gamma,\Omega}(\min\{u,v\}) + I_{\Gamma,\Omega}(\max\{u,v\}) \leqslant I_{\Gamma,\Omega}(u) + I_{\Gamma,\Omega}(v). \tag{2.5}$$

Proof. Simply write m and M for $\min\{u,v\}$ and $\max\{u,v\}$. We suppose that the right-hand side of (2.5) is finite, since otherwise the inequality is trivially satisfied.

Take $i \in \Gamma$ and $j \in \Omega$. Then, one of the following four situations necessarily occurs:

(i) $u_i \leqslant v_i$ and $u_j \leqslant v_j$;
(ii) $u_i < v_i$ and $u_j > v_j$;
(iii) $u_i > v_i$ and $u_j < v_j$;
(iv) $u_i \geqslant v_i$ and $u_j \geqslant v_j$.

If either (i) or (iv) is true, then u and v are equally ordered at both sites i and j. Hence, the identity

$$(1 - m_i m_j) + (1 - M_i M_j) = (1 - u_i u_j) + (1 - v_i v_j),$$

easily follows. Thus, we only need to inspect what happens when either (ii) or (iii) is verified. By symmetry, we may in fact restrict our attention to case (ii) only. In this case, we have $m_i = u_i = -1$, $M_i = v_i = 1$, $M_j = u_j = 1$ and $m_j = v_j = -1$. Therefore,

$$(1 - m_i m_j) + (1 - M_i M_j) = 0 < 2 + 2 = (1 - u_i u_j) + (1 - v_i v_j).$$

Consequently, both series on the left-hand side of (2.5) converge and the inequality follows. ☐

We conclude the subsection by investigating the relationship existing between the minimizers of the functionals $G^{A,B}_{m,\omega}$ and H. The following proposition shows that the periodic minimizers of $G^{A,B}_{m,\omega}$ just described are indeed minimizers of H with respect to perturbations supported inside $\mathcal{F}^{A,B}_{m,\omega}$.

Proposition 2.2. *Let $u \in \mathscr{M}^{A,B}_{m,\omega}$. Then, u is a minimizer for H in $\mathcal{F}^{A,B}_{m,\omega}$.*

Proof. Let v be a configuration that coincides with u outside $\mathcal{F}^{A,B}_{m,\omega}$. We claim that

$$\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u) \leqslant \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(v), \tag{2.6}$$

where, for any configuration w, we set[5] (recall notations (1.12) and (1.13))

$$\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(w) := I_{\mathcal{F}_{m,\omega},\mathcal{F}_{m,\omega}}(w) + 2I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(w) + B_{\mathcal{F}^{A,B}_{m,\omega}}(w). \tag{2.7}$$

To prove (2.6), we write $v = u + \varphi$, with $\varphi : \mathbb{Z}^d \to \{-2, 0, 2\}$ such that $\varphi_i = 0$ for any $i \in \mathbb{Z}^d \setminus \mathcal{F}^{A,B}_{m,\omega}$. We first restrict ourselves to the case in which φ has a sign, i.e.

$$\text{either } \varphi_i \geqslant 0 \text{ for any } i \in \mathbb{Z}^d, \text{ or } \varphi_i \leqslant 0 \text{ for any } i \in \mathbb{Z}^d. \tag{2.8}$$

Define $\tilde{v}$ and $\tilde{\varphi}$ as the (m,ω)-periodic extensions of $v|_{\mathcal{F}_{m,\omega}}$ and $\varphi|_{\mathcal{F}_{m,\omega}}$, respectively. That is,

$$\tilde{v}_{i+k} := v_i \quad \text{and} \quad \tilde{\varphi}_{i+k} := \varphi_i \quad \text{for any } i \in \mathcal{F}_{m,\omega} \text{ and } k \in \mathscr{L}_{m,\omega}. \tag{2.9}$$

Notice that $\tilde{v} \in \mathscr{A}^{A,B}_{m,\omega}$.

We now compare the functionals $G^{A,B}_{m,\omega}$ and $\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}$, when evaluated at u, v and u, $\tilde{v}$, respectively. We claim that

$$\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u) - \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(v) \leqslant G^{A,B}_{m,\omega}(u) - G^{A,B}_{m,\omega}(\tilde{v}). \tag{2.10}$$

To check the validity of (2.10), we begin by evaluating the contributions coming from the magnetic field. Recalling the definitions of v and $\tilde{v}$, we have

$$B_{\mathcal{F}^{A,B}_{m,\omega}}(u) - B_{\mathcal{F}^{A,B}_{m,\omega}}(v) = B_{\mathcal{F}^{A,B}_{m,\omega}}(u) - B_{\mathcal{F}^{A,B}_{m,\omega}}(\tilde{v}). \tag{2.11}$$

[5]Note that $\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}$ differs from $H_{\mathcal{F}_{m,\omega}}$ only with respect to the region over which the magnetic term B is extended. We take into account this slight modification, since $B_{\mathcal{F}_{m,\omega}}$ might not be well-defined even under assumption (1.6), as the set $\mathcal{F}_{m,\omega}$ is not finite.

We now address the interaction terms. Let $(i,j) \in \mathbb{Z}^{2d} \setminus (\mathbb{Z}^d \setminus \mathcal{F}_{m,\omega})^2$. If $i \in \mathcal{F}_{m,\omega}$, then $\tilde{v}_i = v_i$. Hence,

$$I_{\mathcal{F}_{m,\omega},\mathcal{F}_{m,\omega}}(u) - I_{\mathcal{F}_{m,\omega},\mathcal{F}_{m,\omega}}(v) = I_{\mathcal{F}_{m,\omega},\mathcal{F}_{m,\omega}}(u) - I_{\mathcal{F}_{m,\omega},\mathcal{F}_{m,\omega}}(\tilde{v}). \tag{2.12}$$

On the other hand, if $i \in \mathcal{F}_{m,\omega}$ and $j \in \mathbb{Z}^d \setminus \mathcal{F}_{m,\omega}$, then we can write $j = j' + k$, with $j' \in \mathcal{F}_{m,\omega}$ and $k \in \mathscr{L}_{m,\omega} \setminus \{0\}$ uniquely determined. Notice that $i - k \notin \mathcal{F}_{m,\omega}$, therefore $u_i = u_{i-k} = v_{i-k}$, due to (2.2). Therefore, using again (2.2) and (2.9),

$$\begin{aligned} 1 - v_i v_j &= 1 - \tilde{v}_i \tilde{v}_j + v_i(\tilde{v}_j - u_j) \\ &= (1 - \tilde{v}_i \tilde{v}_j) + v_i \varphi_{j'} \\ &= (1 - \tilde{v}_i \tilde{v}_j) + (1 - u_i u_{j'}) - (1 - u_i v_{j'}) + \varphi_i \varphi_{j'} \\ &= (1 - \tilde{v}_i \tilde{v}_j) + (1 - u_i u_j) - (1 - v_{i-k} v_{j'}) + \varphi_i \varphi_{j'}. \end{aligned}$$

Then, by taking advantage of (1.8) and (2.8), we have

$$\begin{aligned} I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(v) &= I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(\tilde{v}) + I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(u) \\ &\quad - \sum_{k \in \mathscr{L}_{m,\omega}\setminus\{0\}} \sum_{i,j' \in \mathcal{F}_{m,\omega}} \left[J_{(i-k)j'}(1 - v_{i-k} v_{j'}) - J_{i(j'+k)} \varphi_i \varphi_{j'} \right] \\ &\geqslant I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(\tilde{v}) + I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(u) - I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(v), \end{aligned}$$

that may be in turn rewritten as

$$2\left[I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(u) - I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(v) \right] \leqslant I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(u) - I_{\mathcal{F}_{m,\omega},\mathbb{Z}^d\setminus\mathcal{F}_{m,\omega}}(\tilde{v}).$$

By this, (2.12), (2.11) and the definition (2.7) of $\widetilde{H}^{A,B}_{m,\omega}$, claim (2.10) follows immediately.

As a consequence of (2.10), since $u \in \mathscr{M}^{A,B}_{m,\omega}$ and $\tilde{v} \in \mathscr{A}^{A,B}_{m,\omega}$, we deduce inequality (2.6) under the sign assumption (2.8) on φ.

In order to finish the proof of the proposition, we now only need to show that (2.8) is in fact unnecessary for the validity of (2.6). To do this, we consider a general $v = u + \varphi$ and define $\varphi_+ := \max\{\varphi, 0\}$ and $\varphi_- := \min\{\varphi, 0\}$. Both φ_+ and φ_- satisfy (2.8) and therefore

$$2\widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u) \leqslant \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u + \varphi_+) + \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u + \varphi_-).$$

But then, by Lemma 2.1, we have

$$\begin{aligned} \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u + \varphi_+) + \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u + \varphi_-) &= \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(\max\{u, v\}) + \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(\min\{u, v\}) \\ &\leqslant \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(u) + \widetilde{H}_{\mathcal{F}^{A,B}_{m,\omega}}(v), \end{aligned}$$

and (2.6) follows.

Thanks to (2.6), by arguing as in Remark 1.2 one can conclude the proof of Proposition 2.2. ☐

2.2. **The minimal minimizer.** We now select a specific element of $\mathscr{M}^{A,B}_{m,\omega}$ that will be proved to have further minimizing properties in the forthcoming subsections. To do this, we recall the definitions given in (2.4), and we introduce the main ingredient of this subsection and discuss its minimizing properties. We define the *minimal minimizer* $u^{A,B}_{m,\omega}$ as the minimum within the (finite) class $\mathscr{M}^{A,B}_{m,\omega}$. That is, we set

$$\left(u^{A,B}_{m,\omega}\right)_i := \min\left\{ u_i : u \in \mathscr{M}^{A,B}_{m,\omega} \right\},$$

for any $i \in \mathbb{Z}^d$. Clearly, $u^{A,B}_{m,\omega}$ belongs to the class $\mathscr{A}^{A,B}_{m,\omega}$ of admissible configurations. To check that $u^{A,B}_{m,\omega}$ is actually a minimizer, we first need an auxiliary lemma.

More precisely, by applying Lemma 2.1 to minimizers of $G^{A,B}_{m,\omega}$, we see that the operations of minimum and maximum are closed in the set $\mathscr{M}^{A,B}_{m,\omega}$. A thorough proof of this fact is contained in the next result.

Lemma 2.3. *Let* $u, v \in \mathscr{M}_{m,\omega}^{A,B}$. *Then,* $\min\{u, v\}, \max\{u, v\} \in \mathscr{M}_{m,\omega}^{A,B}$.

Proof. Recalling (2.3), by Lemma 2.1, one has

$$G_{m,\omega}^{A,B}(\min\{u, v\}) + G_{m,\omega}^{A,B}(\max\{u, v\}) \leqslant G_{m,\omega}^{A,B}(u) + G_{m,\omega}^{A,B}(v).$$

Moreover, since $\min\{u, v\}, \max\{u, v\} \in \mathscr{A}_{m,\omega}^{A,B}$, we easily deduce that

$$G_{m,\omega}^{A,B}(\min\{u, v\}), G_{m,\omega}^{A,B}(\max\{u, v\}) \geqslant G_{m,\omega}^{A,B}(u) = G_{m,\omega}^{A,B}(v),$$

and the thesis follows. □

By iterating Lemma 2.3, we finally obtain the minimality of the minimal minimizer $u_{m,\omega}^{A,B}$.

Corollary 2.4. $u_{m,\omega}^{A,B} \in \mathscr{M}_{m,\omega}^{A,B}$.

2.3. **The doubling property.** The minimal minimizer introduced in the previous subsection enjoys important geometrical properties. The first of such properties is often referred to in the literature as *no-symmetry-breaking* or *doubling property*. It asserts that the minimal minimizers $u_{m,\omega}^{A,B}$ corresponding to different multiplicities $m \in \mathbb{N}$ do in fact all coincide.

In order to prove this result, the following notation will be helpful. Given any $k \in \mathbb{Z}^d$, we define the translation $\mathcal{T}_k u$ of a configuration u along the vector k as

$$(\mathcal{T}_k u)_i := u_{i-k}, \tag{2.13}$$

for any $i \in \mathbb{Z}^d$.

Also, from now on, we drop reference to the multiplicity m when we deal with objects for which $m = 1$. That is, we write e.g. $\mathcal{F}_\omega, G_\omega^{A,B}, \mathscr{M}_\omega^{A,B}, u_\omega^{A,B}$ instead of $\mathcal{F}_{1,\omega}, G_{1,\omega}^{A,B}, \mathscr{M}_{1,\omega}^{A,B}, u_{1,\omega}^{A,B}$.

The doubling property for the minimal minimizer is proved in the following result.

Proposition 2.5. $u_{m,\omega}^{A,B} = u_\omega^{A,B}$, *for any* $m \in \mathbb{N}$.

Proof. Let $m \geqslant 2$. We define the configuration

$$v := \min\left\{\mathcal{T}_k u_{m,\omega}^{A,B} : k \in \mathscr{L}_\omega\right\}.$$

Clearly, $v \in \mathscr{A}_\omega^{A,B} \subset \mathscr{A}_{m,\omega}^{A,B}$. Furthermore, as $\mathcal{T}_k u_{m,\omega}^{A,B} \in \mathscr{M}_{m,\omega}^{A,B}$ for any $k \in \mathscr{L}_\omega$, by applying Lemma 2.3 we also obtain[6] that $v \in \mathscr{M}_{m,\omega}^{A,B}$. Since $u_\omega^{A,B} \in \mathscr{A}_{m,\omega}^{A,B}$, recalling the definition (2.3) of the functional $G_{m,\omega}^{A,B}$, we compute

$$G_\omega^{A,B}(v) = \frac{1}{m^{d-1}} G_{m,\omega}^{A,B}(v) \leqslant \frac{1}{m^{d-1}} G_{m,\omega}^{A,B}(u_\omega^{A,B}) = G_\omega^{A,B}(u_\omega^{A,B}). \tag{2.14}$$

Accordingly, by Corollary 2.4, we deduce that

$$v \in \mathscr{M}_\omega^{A,B} \tag{2.15}$$

and hence $u_\omega^{A,B} \leqslant v$, by definition of minimal minimizer. In particular, we conclude that

$$u_\omega^{A,B} \leqslant u_{m,\omega}^{A,B}. \tag{2.16}$$

To check the validity of the converse inequality it suffices to notice that, in light of (2.15), the first and the last terms of (2.14) are equal. Consequently, the middle inequality in (2.14) is indeed an identity and thus $u_\omega^{A,B} \in \mathscr{M}_{m,\omega}^{A,B}$. Therefore, $u_{m,\omega}^{A,B} \leqslant u_\omega^{A,B}$. This and (2.16) imply the desired result. □

[6] In this regard, observe that the family of configurations appearing in the definition of v is actually finite, thanks to the periodicity of $u_{m,\omega}^{A,B}$.

As a corollary of the doubling property and Proposition 2.2, we immediately deduce that the minimal minimizer is a local minimizer in the whole *strip*

$$\mathcal{S}_\omega^{A,B} := \Big\{ i \in \mathbb{Z}^d : \omega \cdot i \in [A,B] \Big\}. \tag{2.17}$$

Corollary 2.6. *The minimal minimizer $u_\omega^{A,B}$ is a minimizer for H in every finite subset Γ of $\mathcal{S}_\omega^{A,B}$.*

Proof. Given any finite $\Gamma \subset \mathcal{S}_\omega^{A,B}$, we may find a large enough $m \in \mathbb{N}$ and a fundamental region $\mathcal{F}_{m,\omega}$ for which $\Gamma \subseteq \mathcal{F}_{m,\omega}^{A,B}$. By Propositions 2.2 and 2.5, $u_\omega^{A,B} = u_{m,\omega}^{A,B}$ is a minimizer for H in $\mathcal{F}_{m,\omega}^{A,B}$ and the result follows by recalling Remark 1.2. □

2.4. **The Birkhoff property.** Here, we concentrate on another property of the minimal minimizer (that is also related to a similar feature in dynamical systems): the *Birkhoff property*. This trait essentially refers to a kind of discrete monotonicity of $u_\omega^{A,B}$.

Recalling the notation introduced in the previous subsection (in particular (2.13)), we may state the validity of the Birkhoff property for the minimal minimizer as follows.

Proposition 2.7. *Let $k \in \tau\mathbb{Z}^d$. Then,*

$$\begin{aligned} \mathcal{T}_k u_\omega^{A,B} &\leqslant u_\omega^{A,B} \quad \text{if } \omega \cdot k \leqslant 0,\\ \mathcal{T}_k u_\omega^{A,B} &\geqslant u_\omega^{A,B} \quad \text{if } \omega \cdot k \geqslant 0. \end{aligned} \tag{2.18}$$

Proof. We prove only the first inequality in (2.18), the second being completely analogous.

Let $k \in \tau\mathbb{Z}^d$ be such that $\omega \cdot k \leqslant 0$. Observe that $\mathcal{T}_k u_\omega^{A,B} \in \mathscr{A}_\omega^{A+\omega\cdot k,B+\omega\cdot k}$ and that, actually,

$$\mathcal{T}_k u_\omega^{A,B} = u_\omega^{A+\omega\cdot k,B+\omega\cdot k}. \tag{2.19}$$

Write $m := \min\{u_\omega^{A,B}, \mathcal{T}_k u_\omega^{A,B}\}$ and $M := \max\{u_\omega^{A,B}, \mathcal{T}_k u_\omega^{A,B}\}$. We have that $m \in \mathscr{A}_\omega^{A+\omega\cdot k,B+\omega\cdot k}$ and $M \in \mathscr{A}_\omega^{A,B}$. By arguing as in the proof of Lemma 2.3, we easily see that

$$G_\omega^{A,B}(m) \leqslant G_\omega^{A,B}(\mathcal{T}_k u_\omega^{A,B}). \tag{2.20}$$

We now claim that

$$m_i = \big(\mathcal{T}_k u_\omega^{A,B}\big)_i \quad \text{for any } i \in \mathcal{F}_\omega^{A,B} \,\Delta\, \mathcal{F}_\omega^{A+\omega\cdot k,B+\omega\cdot k}. \tag{2.21}$$

Indeed $\big(\mathcal{T}_k u_\omega^{A,B}\big)_i = -1$ for any $i \in \mathcal{F}_\omega^{B+\omega\cdot k,+} \supset \mathcal{F}_\omega^{A,B} \setminus \mathcal{F}_\omega^{A+\omega\cdot k,B+\omega\cdot k}$ and, on the other hand, $u_\omega^{A,B} = 1$ for any $i \in \mathcal{F}_\omega^{A,-} \supset \mathcal{F}_\omega^{A+\omega\cdot k,B+\omega\cdot k} \setminus \mathcal{F}_\omega^{A,B}$, which implies (2.21).

Recalling definitions (2.3) and (1.13) and using formulas (2.20) and (2.21), we conclude that

$$\begin{aligned} G_\omega^{A+\omega\cdot k,B+\omega\cdot k}(m) &- G_\omega^{A+\omega\cdot k,B+\omega\cdot k}(\mathcal{T}_k u_\omega^{A,B}) \\ &= G_\omega^{A,B}(m) - G_\omega^{A,B}(\mathcal{T}_k u_\omega^{A,B}) \\ &\quad + B_{\mathcal{F}_\omega^{A+\omega\cdot k,B+\omega\cdot k} \setminus \mathcal{F}_\omega^{A,B}}(m - \mathcal{T}_k u_\omega^{A,B}) - B_{\mathcal{F}_\omega^{A,B} \setminus \mathcal{F}_\omega^{A+\omega\cdot k,B+\omega\cdot k}}(m - \mathcal{T}_k u_\omega^{A,B}) \\ &\leqslant 0. \end{aligned}$$

Therefore, by (2.19), we have that $m \in \mathscr{M}_\omega^{A+\omega\cdot k,B+\omega\cdot k}$ and $\mathcal{T}_k u_\omega^{A,B} \leqslant m$, as $\mathcal{T}_k u_\omega^{A,B}$ is a minimal minimizer. The first inequality in (2.18) then follows. □

2.5. **An energy estimate.** We collect in this subsection a rather general proposition, that quantifies the energy of the minimizers of H inside large cubes. We stress that no periodicity of the coefficients is necessary for the validity of the results presented here. That is, (1.8) and (1.9) are not required to hold.

We begin by recalling the terminology adopted in footnote 2 at page 7 for cubes in $\mathbb{Z}^d$. Given $\ell \in \mathbb{N} \cup \{0\}$, we denote by Q_ℓ the cube having sides made up of $2\ell + 1$ sites and center located at the origin, i.e.

$$Q_\ell := \{-\ell, \dots, -1, 0, 1, \dots, \ell\}^d. \tag{2.22}$$

A general cube centered at $q \in \mathbb{Z}^d$ will be indicated with $Q_\ell(q) := q + Q_\ell$. We also write S_ℓ for the *boundary* of Q_ℓ, that is

$$\begin{aligned} S_\ell &:= Q_\ell \setminus Q_{\ell-1} \quad \text{if } \ell \geqslant 1, \\ S_0 &:= Q_0 = \{0\}. \end{aligned} \tag{2.23}$$

Again, $S_\ell(q) := q + S_\ell$.

In order to obtain the energy estimate, we plan to compare the Hamiltonian H_{Q_ℓ} of a minimizer in the cube Q_ℓ with that of a suitable competitor. Such auxiliary function will be modeled on the configuration $\psi^{(\ell)}$ defined by

$$\left(\psi^{(\ell)}\right)_i := \begin{cases} -1 & \text{if } i \in Q_\ell, \\ 1 & \text{if } i \in \mathbb{Z}^d \setminus Q_\ell. \end{cases}$$

Recalling (1.10), the following lemma provides an upper bound for the energy of $\psi^{(\ell)}$.

Lemma 2.8. *There exists a constant $C \geqslant 1$, depending only on d, μ and τ, for which*

$$H_{Q_\ell}(\psi^{(\ell)}) \leqslant C\ell^{d-1}\left(1 + \sum_{m=1}^{\ell+1} \sigma(m)\right).$$

Proof. First, observe that Q_ℓ may be written as the disjoint union of a possibly empty family $\mathcal{G}$ of fundamental domains for the quotient $\mathbb{Z}^d/\tau\mathbb{Z}^d$, leaving out at most N sites $\{i^{(n)}\}_{n=1}^N$. It is not hard to see that we can take $N \leqslant c_1\tau\ell^{d-1}$, for some dimensional constant $c_1 > 0$. Accordingly, recalling (1.13) and using (1.6) and (1.5), we have

$$B_{Q_\ell}(\psi^{(\ell)}) = -\sum_{F \in \mathcal{G}} \sum_{i \in F} h_i - \sum_{n=1}^{N} h_{i^{(n)}} \leqslant 0 + \mu N \leqslant c_1 \mu \tau \ell^{d-1}. \tag{2.24}$$

We now estimate the interaction term I_{Q_ℓ}. Recalling definition (1.10), we compute

$$\begin{aligned} I_{Q_\ell}(\psi^{(\ell)}) &= 4 \sum_{i \in Q_\ell, j \in \mathbb{Z}^d \setminus Q_\ell} J_{ij} = 4 \sum_{m=0}^{\ell} \sum_{i \in S_m} \sum_{|j|_\infty \geqslant \ell+1} J_{ij} \leqslant 4 \sum_{m=0}^{\ell} \sum_{i \in S_m} \sum_{|j-i|_\infty \geqslant \ell+1-m} J_{ij} \\ &\leqslant 8d \sum_{m=0}^{\ell} (2m+1)^{d-1} \sigma(\ell+1-m) \leqslant c_2 \ell^{d-1} \sum_{m=1}^{\ell+1} \sigma(m), \end{aligned} \tag{2.25}$$

for some dimensional constant $c_2 > 0$. The combination of (2.24) and (2.25) leads to the thesis. ☐

Now we show that each minimizer satisfies the same energy growth.

Proposition 2.9. *Let u be a minimizer for H in $Q_\ell(q)$, for some $q \in \mathbb{Z}^d$ and $\ell \in \mathbb{N}$. Then,*

$$H_{Q_\ell(q)}(u) \leqslant \bar{C}\ell^{d-1}\left(1 + \sum_{m=1}^{\ell} \sigma(m)\right), \tag{2.26}$$

for some constant $\bar{C} \geqslant 1$ depending only on d, μ and τ.

Proof. Without loss of generality, we may assume the center q to be the origin. Let $\psi^{(\ell)}$ be the configuration considered in Lemma 2.8 and define $v := \min\{u, \psi^{(\ell)}\}$, $w := \max\{u, \psi^{(\ell)}\}$. Observe that v and u agree outside of Q_ℓ. Consequently, the minimality of u implies that

$$H_{Q_\ell}(u) \leqslant H_{Q_\ell}(v). \tag{2.27}$$

Now we compare the energies of u and w. As u coincides with w in Q_ℓ, we have

$$I_{Q_\ell, Q_\ell}(u) = I_{Q_\ell, Q_\ell}(w) \quad \text{and} \quad B_{Q_\ell}(u) = B_{Q_\ell}(w). \tag{2.28}$$

On the other hand, by taking advantage of the computation (2.25),

$$I_{Q_\ell, \mathbb{Z}^d \setminus Q_\ell}(u) - I_{Q_\ell, \mathbb{Z}^d \setminus Q_\ell}(w) \leqslant 2 \sum_{i \in Q_\ell, j \in \mathbb{Z}^d \setminus Q_\ell} J_{ij} \leqslant \frac{c_1}{2} \ell^{d-1} \sum_{m=1}^{\ell+1} \sigma(m),$$

for some $c_1 > 0$. By this and (2.28), we conclude that

$$H_{Q_\ell}(u) \leqslant H_{Q_\ell}(w) + c_1 \ell^{d-1} \sum_{m=1}^{\ell+1} \sigma(m). \tag{2.29}$$

On the other hand, using Lemma 2.1 and (2.27), we see that

$$H_{Q_\ell}(v) + H_{Q_\ell}(w) \leqslant H_{Q_\ell}(u) + H_{Q_\ell}(\psi^{(\ell)}) \leqslant H_{Q_\ell}(v) + H_{Q_\ell}(\psi^{(\ell)}),$$

which gives that $H_{Q_\ell}(w) \leqslant H_{Q_\ell}(\psi^{(\ell)})$. This and Lemma 2.8 imply that

$$H_{Q_\ell}(w) \leqslant H_{Q_\ell}(\psi^{(\ell)}) \leqslant c_2 \ell^{d-1} \left(1 + \sum_{m=1}^{\ell+1} \sigma(m) \right),$$

for some $c_2 > 0$. This and (2.29) imply estimate (2.26). ☐

Remark 2.10. By inspecting the proofs of Lemma 2.8 and Proposition 2.9, it is clear that when the magnetic field h vanishes in $Q_\ell(q)$, the constant $\bar{C}$ appearing in (2.26) may be chosen to depend only on the dimension d.

2.6. **Unconstrained minimizers and ground states.** In this last subsection, we show that the minimal minimizer is actually a ground state, according to Definition 1.3, if the oscillation of its transition is chosen sufficiently large. This will finish the proof of Theorem 1.4 for the case of rational directions and truncated interactions.

From now on, we mostly restrict ourselves to the minimal minimizers that display a transition bounded in the strip $\mathcal{S}_\omega^{0,M}$, with $M > 0$ (recall (2.17)). For this reason, we slightly simplify our notation and denote with $\mathcal{F}_\omega^M, \mathscr{A}_\omega^M, \mathcal{S}_\omega^M, u_\omega^M, \ldots$ the quantities $\mathcal{F}_\omega^{0,M}, \mathscr{A}_\omega^{0,M}, \mathcal{S}_\omega^{0,M}, u_\omega^{0,M}, \ldots$

Our main goal is to show that the minimal minimizer u_ω^M becomes *unconstrained*, provided M is large enough. To do this, we need a few auxiliary results.

First, we present a technical lemma related to the quantity σ introduced in (1.10).

Lemma 2.11. *Set*

$$\Sigma(R) := \frac{1}{R} \sum_{m=1}^{R} \sigma(m), \tag{2.30}$$

for any $R \in \mathbb{N}$. Then, it holds

$$\lim_{R \to +\infty} \Sigma(R) = 0.$$

Proof. Let $\varepsilon > 0$ be any small number. In view of (1.4) and (1.8), we know that

$$\lim_{R\to+\infty} \sigma(R) = 0.$$

Hence, we may select $R_0 \in \mathbb{N}$ such that, for any $m \geqslant R_0$, it holds $\sigma(m) \leqslant \varepsilon/2$. Using again (1.4), we see that $\sigma(m) \leqslant \Lambda$, for any m. Hence, taking $R \geqslant 2\Lambda R_0/\varepsilon$, we have

$$\Sigma(R) = \frac{1}{R}\sum_{m=1}^{R_0} \sigma(m) + \frac{1}{R}\sum_{m=R_0+1}^{R} \sigma(m) \leqslant \frac{R_0}{R}\Lambda + \frac{R-R_0}{R}\frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the conclusion follows. □

Then, we have a rigidity result for configurations that satisfy the Birkhoff property and display *fat* plateaux.

We remark that in the remainder of the subsection we slightly modify the notation fixed in (2.22) and denote with $\mathcal{C}_\ell$ any cube of $\mathbb{Z}^d$ with sides composed by ℓ sites, i.e.

$$\mathcal{C}_\ell = \mathcal{C}_\ell(q) := q + \{0, 1, \ldots, \ell-1\}^d.$$

Note that now q denotes the lower vertex, instead of the center. The reference to q will be however often neglected.

Lemma 2.12. *Let u be a configuration satisfying the Birkhoff property with respect to ω, i.e. for which inequalities (2.18) are fulfilled. Assume that there exists a cube $\mathcal{C}_\tau(q)$ such that*

$$u_i = -1 \quad \text{for any } i \in \mathcal{C}_\tau(q).$$

Then,

$$u_i = -1 \quad \text{for any } i \in \mathbb{Z}^d \text{ such that } \frac{\omega}{|\omega|}\cdot i \geqslant \frac{\omega}{|\omega|}\cdot q + \sqrt{d}\tau.$$

Proof. See [CdlL05, Proposition 3.5]. □

With the aid of these lemmata and the energy estimate obtained in Subsection 2.5, we are now able to prove the key result of this subsection.

Proposition 2.13. *There exist two real numbers $\mu_0 > 0$, depending only on d, τ and λ, and $M_0 > 0$, that may also depend on Λ and the function σ, for which*

$$\left(u^M_\omega\right)_i = -1 \quad \text{for any } i \in \mathbb{Z}^d \text{ such that } \frac{\omega}{|\omega|}\cdot i \geqslant M - \sqrt{d}\tau,$$

provided $\mu \leqslant \mu_0$ and $M \geqslant M_0$.

Proof. For shortness, we write $u = u^M_\omega$. In view of Lemma 2.12 and Proposition 2.7, it suffices to show that

$$u_i = -1 \quad \text{for any } i \in \mathcal{C}_\tau(q), \text{ with } q \in \mathbb{Z}^d \text{ satisfying } \frac{\omega}{|\omega|}\cdot q \leqslant M - 2\sqrt{d}\tau. \tag{2.31}$$

In order to check the validity of claim (2.31), we first prove a weaker fact. Take

$$\mu \leqslant \mu_0 := \lambda\tau^{-d}, \tag{2.32}$$

where λ is given in (1.3). Consider the strip

$$\widehat{\mathcal{S}}^M_\omega := \mathcal{S}^{\frac{M}{8},\frac{7M}{8}}_\omega = \left\{ i \in \mathbb{Z}^d : \frac{\omega}{|\omega|}\cdot i \in \left[\frac{M}{8}, \frac{7M}{8}\right]\right\} \subset \mathcal{S}^M_\omega,$$

and a cube $\mathcal{C}_{N\tau} \subset \widehat{\mathcal{S}}_\omega^M$ of sides $N\tau$, with $N \in \mathbb{N}$. It is not hard to see that N can be taken in such a way that

$$\frac{M}{2} \leqslant N\tau \leqslant \frac{3M}{4}.$$

Divide the cube $\mathcal{C}_{N\tau}$ in a partition $\{\mathcal{C}_\tau^{(n)}\}_{n=1}^{N^d}$ of N^d non-overlapping, smaller cubes of sides τ. We claim that

$$\text{there exists an index } \bar{n} \in \left\{1, \ldots, N^d\right\} \text{ for which} \\ \text{either } u_i = -1 \text{ for any } i \in \mathcal{C}_\tau^{(\bar{n})} \text{ or } u_i = 1 \text{ for any } i \in \mathcal{C}_\tau^{(\bar{n})}. \tag{2.33}$$

To prove (2.33) we argue by contradiction and suppose that, for any $n = 1, \ldots, N^d$, we can find two sites $i^{(n)}, j^{(n)} \in \mathcal{C}_\tau^{(n)}$ at which $u_{i^{(n)}} = -1$ and $u_{j^{(n)}} = 1$. Observe that we can take $i^{(n)}$ and $j^{(n)}$ to be adjacent, i.e. such that $|i^{(n)} - j^{(n)}| = 1$. Using (1.3), (1.5) and (2.32), we compute

$$\begin{aligned} H_{\mathcal{C}_{N\tau}}(u) &\geqslant I_{\mathcal{C}_{N\tau},\mathcal{C}_{N\tau}}(u) + B_{\mathcal{C}_{N\tau}}(u) \geqslant \sum_{n=1}^{N^d} \sum_{i,j\in\mathcal{C}_\tau^{(n)}} J_{ij}(1-u_iu_j) + \sum_{i\in\mathcal{C}_{N\tau}} h_iu_i \\ &\geqslant \sum_{n=1}^{N^d} J_{i^{(n)}j^{(n)}}\left(1 - u_{i^{(n)}}u_{j^{(n)}}\right) - \sum_{i\in\mathcal{C}_{N\tau}} |h_i| \geqslant 2\lambda N^d - \mu\,(N\tau)^d \geqslant \lambda N^d. \end{aligned} \tag{2.34}$$

On the other hand, the energy estimate established in Proposition 2.9 (recall that u is a minimizer for H in $\mathcal{C}_{N\tau}$, thanks to Corollary 2.6) gives that

$$H_{\mathcal{C}_{N\tau}}(u) \leqslant c_1 \left(\frac{N\tau - 1}{2}\right)^{d-1} \sum_{m=1}^{\left\lfloor \frac{N\tau-1}{2}\right\rfloor+1} \sigma(m) \leqslant c_2 N^{d-1}\tau^{d-1} \sum_{m=1}^{N\tau} \sigma(m),$$

for some constants $c_1, c_2 > 0$. By comparing this with (2.34) and recalling definition (2.30), we find that

$$\Sigma(N\tau) \geqslant \frac{\lambda}{c_2\tau^d},$$

which clearly contradicts Lemma 2.11, if N (and hence M) is chosen sufficiently large. Therefore, claim (2.33) is true, provided we take $M \geqslant M_0$, with M_0 only depending on d, τ, λ, Λ and the function σ.

Denote by $\bar{q}$ the lower vertex of the cube $\mathcal{C}_\tau^{(\bar{n})}$, so that $\mathcal{C}_\tau^{(\bar{n})} = \mathcal{C}_\tau(\bar{q})$. As $\mathcal{C}_\tau(\bar{q}) \subset \mathcal{C}_{N\tau} \subset \widehat{\mathcal{S}}_\omega^M$, we have that $\omega \cdot \bar{q} \leqslant 7M|\omega|/8 \leqslant (M - 2\sqrt{d}\tau)|\omega|$, by possibly enlarging M_0. Hence, (2.31) follows from (2.33), once we rule out the possibility that

$$u_i = 1 \quad \text{for any } i \in \mathcal{C}_\tau(\bar{q}). \tag{2.35}$$

Assume by contradiction that (2.35) holds true. By applying Lemma 2.12 (to $-u$ instead of u, which has the Birkhoff property with respect to $-\omega$), we deduce that

$$u_i = 1 \quad \text{for any } i \in \mathbb{Z}^d \text{ such that } \frac{\omega}{|\omega|} \cdot i \leqslant \frac{\omega}{|\omega|} \cdot \bar{q} - \sqrt{d}\tau.$$

Again, by possibly taking a larger M_0, we see that the above fact is valid in particular for any site i satisfying $\omega \cdot i < \tau|\omega|$. Supposing with no loss of generality that $\omega_1 > 0$ (as one can relabel the axes and invert their orientation) and setting $k = (-\tau, 0, \ldots, 0) \in \tau\mathbb{Z}^d$, we have that $\omega \cdot k < 0$ and, for the observation made just before, $\mathcal{T}_k u \in \mathscr{A}_\omega^M$. On the one hand, Proposition 2.7 implies that $\mathcal{T}_k u \leqslant u$. On the other hand, using (1.6) one can check that $G_\omega^M(\mathcal{T}_k u) = G_\omega^M(u)$. Consequently, $\mathcal{T}_k u \in \mathscr{M}_\omega^M$ and $\mathcal{T}_k u \geqslant u$, by the fact that u is the minimal minimizer. By putting together these two inequalities, we end up with the identity $\mathcal{T}_k u = u$, which clearly cannot occur.

As a result, (2.35) is false and claim (2.31) plainly follows. The proof of the proposition is therefore complete. ☐

Corollary 2.14. *Let μ_0 and M_0 be as in Proposition 2.13. If $\mu \leqslant \mu_0$, then $u_\omega^{M_0} = u_\omega^{M_0+a}$ for any $a \in \tau\mathbb{N}$.*

Proof. Consider any $M = M_0 + n\tau$, with $n \in \mathbb{N} \cup \{0\}$. The claim of the corollary is then equivalent to show that

$$u_\omega^M = u_\omega^{M+\tau}. \tag{2.36}$$

To see that (2.36) holds true, first notice that $u_\omega^M \in \mathscr{A}_\omega^{M+\tau}$. Also,

$$u_\omega^{M+\tau} \in \mathscr{A}_\omega^M, \tag{2.37}$$

as one can easily check by applying Proposition 2.13 to $u_\omega^{M+\tau}$. Hence,

$$G_\omega^{M+\tau}(u_\omega^{M+\tau}) \leqslant G_\omega^{M+\tau}(u_\omega^M) \quad \text{and} \quad G_\omega^M(u_\omega^M) \leqslant G_\omega^M(u_\omega^{M+\tau}). \tag{2.38}$$

But then, for any $w \in \mathscr{A}_\omega^M$ it holds

$$G_\omega^{M+\tau}(w) - G_\omega^M(w) = B_{\mathcal{F}_\omega^{M+\tau} \setminus \mathcal{F}_\omega^M}(w) = - \sum_{i \in \mathcal{F}_\omega^{M+\tau} \setminus \mathcal{F}_\omega^M} h_i = 0, \tag{2.39}$$

where the last identity is true by virtue of hypothesis (1.6), since $\mathcal{F}_\omega^{M+\tau} \setminus \mathcal{F}_\omega^M$ may be written as a disjoint union of fundamental domains of $\mathbb{Z}^d/\tau\mathbb{Z}^d$.

In particular, (2.37) and (2.39) give that

$$G_\omega^{M+\tau}(u_\omega^{M+\tau}) = G_\omega^M(u_\omega^{M+\tau}).$$

Using this and the two inequalities in (2.38), we obtain that

$$G_\omega^M(u_\omega^{M+\tau}) = G_\omega^{M+\tau}(u_\omega^{M+\tau}) \leqslant G_\omega^{M+\tau}(u_\omega^M) = G_\omega^M(u_\omega^M) \leqslant G_\omega^M(u_\omega^{M+\tau}).$$

Hence, u_ω^M and $u_\omega^{M+\tau}$ belong to $\mathscr{M}_\omega^M \cap \mathscr{M}_\omega^{M+\tau}$ and (2.36) follows by the fact that they are both minimal minimizers. ☐

When used in combination with Corollary 2.6, the previous result ensures in particular that the energy H of $u_\omega^{M_0}$ is lower than that of any perturbation involving a finite number of sites that lie over the module $\{\omega \cdot i = 0\}$. For this reason, the minimal minimizer $u_\omega^{M_0}$ does not feel the upper constraint $\{\omega \cdot i = M_0|\omega|\}$ and extends its minimizing properties well beyond it.

In the next result, we show that the same happens for the lower constraint and that the minimal minimizer is therefore fully unconstrained.

Proposition 2.15. *Let μ_0 and M_0 be as in Proposition 2.13. If $\mu \leqslant \mu_0$, then $u_\omega^{M_0} \in \mathscr{M}_{m,\omega}^{-a,M_0+a}$ for any $m \in \mathbb{N}$ and any $a \in \tau\mathbb{N}$.*

Proof. First, we note that, by arguing as in the proof of Corollary 2.14, one may check that, given any four real numbers $A < B$ and $A' < B'$ such that $A - A', B - B' \in \tau\mathbb{Z}$, it holds

$$G_{m,\omega}^{A,B}(w) = G_{m,\omega}^{A',B'}(w),$$

for any $w \in \mathscr{A}_{m,\omega}^{A,B} \cap \mathscr{A}_{m,\omega}^{A',B'}$.

Take now any configuration $v \in \mathscr{A}_{m,\omega}^{-a,M_0+a}$ and let $k \in \tau\mathbb{Z}^d$ be a vector satisfying $\omega \cdot k \geqslant a|\omega|$ and $\omega \cdot k \in \tau|\omega|\mathbb{N}$. We have that

$$\mathcal{T}_k v \in \mathscr{A}_{m,\omega}^{-a+\omega\cdot k/|\omega|, M_0+a+\omega\cdot k/|\omega|} \subseteq \mathscr{A}_{m,\omega}^{M_0+b}$$

for some $b \in \tau\mathbb{N}$ with $b \geqslant a + \omega \cdot k/|\omega|$. By Corollary 2.14 and Proposition 2.5, we know that $u_\omega^{M_0} \in \mathscr{M}_{m,\omega}^{M_0+b}$ and thus $G_{m,\omega}^{M_0+b}(u_\omega^{M_0}) \leqslant G_{m,\omega}^{M_0+b}(\mathcal{T}_k v)$. But

$$G_{m,\omega}^{M_0+b}(\mathcal{T}_k v) = G_{m,\omega}^{-\frac{\omega}{|\omega|}\cdot k,\, M_0+b-\frac{\omega}{|\omega|}\cdot k}(v) = G_{m,\omega}^{-a,M_0+a}(v),$$

and

$$G_{m,\omega}^{M_0+b}(u_\omega^{M_0}) = G_{m,\omega}^{M_0}(u_\omega^{M_0}) = G_{m,\omega}^{-a,M_0+a}(u_\omega^{M_0}),$$

thanks to the opening remark. Consequently, $G_{m,\omega}^{-a,M_0+a}(u_\omega^{M_0}) \leqslant G_{m,\omega}^{-a,M_0+a}(v)$ and the proposition is proved. □

A simple consequence of this fact is that the minimal minimizer is indeed a ground state. A rigorous proof of this fact is contained in the following

Corollary 2.16. *Let μ_0 and M_0 be as in Proposition 2.13. If $\mu \leqslant \mu_0$, then $u_\omega^{M_0}$ is a ground state for H.*

Proof. The proof is analogous to that of Corollary 2.6.

Given a finite set $\Gamma \subset \mathbb{Z}^d$, we take $m \in \mathbb{N}$ and $a \in \tau\mathbb{N}$ sufficiently large to have $\Gamma \subseteq \mathcal{F}_{m,\omega}^{-a,M_0+a}$. By Proposition 2.15, the minimal minimizer $u_\omega^{M_0}$ belongs to the class $\mathscr{M}_{m,\omega}^{-a,M_0+a}$ and then, by Proposition 2.2, it is a minimizer for H in $\mathcal{F}_{m,\omega}^{-a,M_0+a}$. The conclusion follows by recalling Remark 1.2 and since Γ can be chosen arbitrarily. □

We point out that, in light of this last result, the proof of Theorem 1.4 is concluded, at least for rational directions $\omega \in \mathbb{Q}^d \setminus \{0\}$ and under the finite-range hypothesis (2.1) on J.

In the next two subsections, we show that assumption (2.1) might in fact be removed and that irrational directions can be dealt with an approximation procedure. After this, Theorem 1.4 will be proved in its full generality.

2.7. **Ground states for infinite-range interactions.** Here we address the proof of Theorem 1.4 for models allowing infinite-range interactions. That is, we show that planelike ground states exist for Hamiltonians H whose interaction coefficients J satisfy the summability condition (1.4), but not necessarily (2.1).

Let $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, +\infty)$ be any function satisfying assumptions (1.1), (1.3), (1.4) and (1.8). Let $\{R_n\}_{n\in\mathbb{N}}$ be an increasing sequence of positive real numbers, diverging to $+\infty$. For any $n \in \mathbb{N}$, we define a function $J^{(n)} : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, +\infty)$ by setting

$$J_{ij}^{(n)} := \begin{cases} J_{ij} & \text{if } |i-j| \leqslant R_n, \\ 0 & \text{if } |i-j| > R_n, \end{cases}$$

and the associated Hamiltonian $H^{(n)}$, on any finite set $\Gamma \subset \mathbb{Z}^d$ and any configuration u, as

$$H_\Gamma^{(n)}(u) := \sum_{(i,j)\in\mathbb{Z}^{2d}\setminus(\mathbb{Z}^d\setminus\Gamma)^2} J_{ij}^{(n)}(1 - u_i u_j) + \sum_{i\in\Gamma} h_i u_i.$$

Observe that $J^{(n)}$ still satisfies (1.1), (1.3), (1.4) and (1.8). Moreover, $J^{(n)}$ fulfills condition (2.1), with $R = R_n$.

Let now $\omega \in \mathbb{Q}^d\setminus\{0\}$ be a fixed direction. By the work done in the previous subsections, for any $n \in \mathbb{N}$ we can find a ground state $u^{(n)}$ for $H^{(n)}$ with interface $\partial u^{(n)}$ satisfying

$$\partial u^{(n)} \subset \left\{ i \in \mathbb{Z}^d : \frac{\omega}{|\omega|} \cdot i \in [0, M] \right\}, \tag{2.40}$$

for some $M > 0$ independent of n. We stress that the uniformity of M in n is crucial for the following arguments, and is a consequence of the fact that the constant M_0 found in Proposition 2.13 is independent of the range of positivity R of (2.1).

By Tychonoff's Theorem, we can find a subsequence of the $u^{(n)}$'s, that we still label in the same way, that converges to a new configuration u. As a matter of fact, for any finite set $\Gamma \subset \mathbb{Z}^d$, there exists a number $N \in \mathbb{N}$ such that

$$u_i^{(n)} = u_i \quad \text{for any } i \in \Gamma \text{ and any } n \geqslant N. \tag{2.41}$$

We claim that u is a planelike ground state for H. Obviously, (2.40) passes to the limit and the same estimate holds true for the interface of u. Therefore, we are only left to verify that u is a ground state for H.

To see this, let $\Gamma \subset \mathbb{Z}^d$ be a finite set and v be a configuration for which $v_i = u_i$ at any site $i \in \mathbb{Z}^d \setminus \Gamma$. Write $v = u + \varphi$, with $\varphi : \mathbb{Z}^d \to \{-2, 0, 2\}$ and set $v^{(n)} := u^{(n)} + \varphi$. From now on, we always assume n to be larger than the number N for which (2.41) is valid. By (2.41) and the fact that $\varphi_i = 0$ for any $i \in \mathbb{Z}^d \setminus \Gamma$, we see that $v^{(n)}$ attains only the values -1 and 1, at least for a large enough n. That is, $v^{(n)}$ is an admissible configuration and $v_i^{(n)} = u_i^{(n)}$ for any $i \in \mathbb{Z}^d \setminus \Gamma$. As $u^{(n)}$ is a minimizer for $H^{(n)}$ in Γ, we have that

$$H_\Gamma^{(n)}(u^{(n)}) \leqslant H_\Gamma^{(n)}(v^{(n)}). \tag{2.42}$$

To finish the proof, we must show that (2.42) yields an analogous inequality for u, v and H. For this, we first recall that $u_i^{(n)} = u_i$ and, thus, $v_i^{(n)} = v_i$ at any site $i \in \Gamma$. Moreover, up to taking a larger N, we have that $J_{ij}^{(n)} = J_{ij}$ for any $i, j \in \Gamma$, as Γ is finite. Accordingly,

$$\begin{aligned}\left| H_\Gamma(u) - H_\Gamma^{(n)}(u^{(n)}) \right| &= 2 \left| \sum_{i \in \Gamma, j \in \mathbb{Z}^d \setminus \Gamma} \left[J_{ij}(1 - u_i u_j) - J_{ij}^{(n)}(1 - u_i u_j^{(n)}) \right] \right| \\ &\leqslant 2 \sum_{i \in \Gamma} \left(\sum_{j \in \mathbb{Z}^d \setminus \Gamma} J_{ij} |u_j - u_j^{(n)}| + 2 \sum_{j \in \mathbb{Z}^d \setminus \Gamma} \left| J_{ij} - J_{ij}^{(n)} \right| \right).\end{aligned}$$

But then, since $J_{ij}^{(n)} \leqslant J_{ij}$, $J_{ij}^{(n)} \to J_{ij}$ and $u_i^{(n)} \to u_i$, for any $i, j \in \mathbb{Z}^d$, we are in position to apply the Dominated Convergence Theorem for Series and conclude that the right-hand side of the above inequality goes to 0 as $n \to +\infty$. Note that the summability hypothesis (1.4) and the finiteness of Γ are crucial for this argument to work. As the same reasoning can be made for the $v^{(n)}$'s, we obtain that

$$\lim_{n \to +\infty} H_\Gamma^{(n)}(u^{(n)}) = H_\Gamma(u) \quad \text{and} \quad \lim_{n \to +\infty} H_\Gamma^{(n)}(v^{(n)}) = H_\Gamma(v).$$

By this and (2.42), we conclude that u is a minimizer for H in Γ and, hence, a ground state.

2.8. **Irrational directions.** In this subsection, we complete the proof of Theorem 1.4 by showing that there exist planelike minimizers also in correspondence to irrational directions.

For a fixed irrational direction $\omega \in \mathbb{R}^d \setminus \mathbb{Q}^d$, we take a sequence $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{Q}^d \setminus \{0\}$ converging to ω. Associated to each ω_n, we consider the ground state $u^{(n)}$ for H constructed previously. We have

$$\partial u^{(n)} \subset \left\{ i \in \mathbb{Z}^d : \frac{\omega_n}{|\omega_n|} \cdot i \in [0, M] \right\}, \tag{2.43}$$

for some constant $M > 0$ independent of n.

The proof continues as in the preceding subsection. By Tychonoff's Theorem, $u^{(n)}$ converges, up to a subsequence, to a configuration u. Given any finite subset $\Gamma \subset \mathbb{Z}^d$, the sequence $u^{(n)}$ actually coincides with u on Γ, provided n is large enough (in dependence of Γ). Therefore, we deduce from (2.43) that

$$\partial u \subset \left\{ i \in \mathbb{Z}^d : \frac{\omega}{|\omega|} \cdot i \in [0, M] \right\}.$$

The proof of the fact that u is a ground state for H is analogous to that displayed in the previous subsection (and easier).

Theorem 1.4 is thus now proved completely (in the general setting).

3. Proof of Theorem 1.4 for power-like interactions with no magnetic term

In this section we show that when J satisfies assumption (1.7) and no magnetic field h is incorporated in the Hamiltonian H, the width M of the strip $\mathcal{S}_\omega^M$ appearing in the statement of Theorem 1.4 can be further specified. Indeed, we shall show that M can be chosen of the form $M = M_0 \tau$, for some $M_0 > 0$ only depending on d, s, λ and Λ.

To show the validity of this fact, we remark that it is enough to adapt to this specific setting the sole arguments contained in Subsection 2.6, since that is the only point of the proof displayed in Section 2 where the width M is made precise. As a first step toward this goal, we obtain some density estimates for the level sets of the minimizers of H inside cubes that intercepts their interfaces.

We stress that throughout the whole section, J is supposed to fulfill hypothesis (1.7) (in addition to (1.1), (1.2) and (1.8)) and the magnetic term h vanishes, i.e.

$$h_i = 0 \quad \text{for any } i \in \mathbb{Z}^d.$$

3.1. Density estimates.

Here, we collect some results that aim to quantify the size of the level sets of a non-trivial minimizer u. The main result is Proposition 3.3, where optimal density estimates are obtained.

We begin with a few auxiliary results. The first is a purely geometrical estimate, reminiscent of the one contained in [DNPV12, Lemma 6.1].

Lemma 3.1. *Let $\Gamma \subset \mathbb{Z}^d$ be any finite, non-empty set and $i \in \mathbb{Z}^d$. Then, it holds*

$$\sum_{j \in \mathbb{Z}^d \setminus \Gamma} \frac{1}{|i-j|_\infty^{d+s}} \geqslant c \, (\#\Gamma)^{-s/d}, \tag{3.1}$$

for some constant $c > 0$ depending only on s.

Proof. Take $\ell \in \mathbb{N}$ in such a way that

$$(2\ell - 1)^d \leqslant \#\Gamma < (2\ell + 1)^d, \tag{3.2}$$

and let $\Gamma^* \supset \Gamma$ be any set with cardinality $\#\Gamma^* = (2\ell + 1)^d$. Notice that

$$\begin{aligned} \#(Q_\ell(i) \setminus \Gamma^*) &= \#Q_\ell(i) - \#(\Gamma^* \cap Q_\ell(i)) \\ &= \#\Gamma^* - \#(\Gamma^* \cap Q_\ell(i)) \\ &= \#(\Gamma^* \setminus Q_\ell(i)), \end{aligned}$$

and hence

$$\sum_{j \in Q_\ell(i) \setminus \Gamma^*} \frac{1}{|i-j|_\infty^{d+s}} \geqslant \frac{\#(Q_\ell(i) \setminus \Gamma^*)}{\ell^{d+s}} = \frac{\#(\Gamma^* \setminus Q_\ell(i))}{\ell^{d+s}} \geqslant \sum_{j \in \Gamma^* \setminus Q_\ell(i)} \frac{1}{|i-j|_\infty^{d+s}}.$$

Thanks to the above inequality, we compute

$$\begin{aligned}
\sum_{j\in\mathbb{Z}^d\setminus\Gamma}\frac{1}{|i-j|_\infty^{d+s}} &\geqslant \sum_{j\in\mathbb{Z}^d\setminus\Gamma^*}\frac{1}{|i-j|_\infty^{d+s}}\\
&= \sum_{j\in Q_\ell(i)\setminus\Gamma^*}\frac{1}{|i-j|_\infty^{d+s}} + \sum_{j\in\mathbb{Z}^d\setminus(\Gamma^*\cup Q_\ell(i))}\frac{1}{|i-j|_\infty^{d+s}}\\
&\geqslant \sum_{j\in \Gamma^*\setminus Q_\ell(i)}\frac{1}{|i-j|_\infty^{d+s}} + \sum_{j\in\mathbb{Z}^d\setminus(\Gamma^*\cup Q_\ell(i))}\frac{1}{|i-j|_\infty^{d+s}}\\
&= \sum_{j\in\mathbb{Z}^d\setminus Q_\ell(i)}\frac{1}{|i-j|_\infty^{d+s}}.
\end{aligned} \tag{3.3}$$

On the other hand, recalling the notation on (2.23),

$$\sum_{j\in\mathbb{Z}^d\setminus Q_\ell(i)}\frac{1}{|i-j|_\infty^{d+s}} = \sum_{k\in\mathbb{Z}^d\setminus Q_\ell}\frac{1}{|k|_\infty^{d+s}} = \sum_{m=\ell+1}^{+\infty}\frac{\#S_m}{m^{d+s}} \geqslant \sum_{m=\ell+1}^{+\infty}\frac{1}{m^{1+s}} \geqslant \frac{(\ell+1)^{-s}}{s}.$$

Accordingly, by this, (3.3) and (3.2), we finally get

$$\sum_{j\in\mathbb{Z}^d\setminus\Gamma}\frac{1}{|i-j|_\infty^{d+s}} \geqslant \frac{(\ell+1)^{-s}}{s} = \frac{(2\ell-1)^{-s}}{s}\left(\frac{2\ell-1}{\ell+1}\right)^s \geqslant \frac{(\#\Gamma)^{-s/d}}{2^s s},$$

that is (3.1). □

As a corollary, we immediately deduce the following discrete, non-local isoperimetric-type inequality. See e.g. [FS08, FFMMM15, DCNRV15] for similar results and further applications in a fairly related continuous setting.

Corollary 3.2. *Let $\Gamma\subset\mathbb{Z}^d$ be any finite set. Then, it holds*

$$\sum_{i\in\Gamma,\, j\in\mathbb{Z}^d\setminus\Gamma}\frac{1}{|i-j|_\infty^{d+s}} \geqslant c\,(\#\Gamma)^{\frac{d-s}{d}},$$

for some constant $c>0$ depending only on s.

With this in hand, we may now head to the main result of this subsection: the density estimates.

Proposition 3.3. *Let u be a minimizer for H in $Q_\ell(q)$, for some $q\in\mathbb{Z}^d$ and $\ell\in\mathbb{N}$. If $q\in\partial u$, then*

$$\min\Big\{\#\left(\{u=-1\}\cap Q_\ell(q)\right), \#\left(\{u=1\}\cap Q_\ell(q)\right)\Big\} \geqslant \bar{c}\,\ell^d,$$

for some constant $\bar{c}>0$ depending only on d, s, λ and Λ.

Proof. Of course, we can assume $q=0$. We also restrict ourselves to check that

$$\#\left(\{u=1\}\cap Q_\ell\right) \geqslant \bar{c}\,\ell^d, \tag{3.4}$$

for some $\bar{c}>0$, the estimate for the set $\{u=-1\}\cap Q_\ell$ being completely analogous.

For $m=0,\dots,\ell$, we set

$$V_m := \{u=1\}\cap Q_m, \quad A_m := \{u=1\}\cap S_m,$$

and

$$v_m := \#V_m, \quad a_m := \#A_m.$$

We consider the configuration w defined by

$$w_i := \begin{cases} -1 & \text{if } i \in Q_m, \\ u_i & \text{if } i \in \mathbb{Z}^d \setminus Q_m. \end{cases}$$

By its definition, w coincides with u outside of Q_m. Hence, by the minimimality of u, we get

$$H_{Q_m}(u) \leqslant H_{Q_m}(w).$$

Since $h = 0$, we may rewrite this inequality as

$$\sum_{i,j \in Q_m} J_{ij}(1 - u_i u_j) + 2 \sum_{i \in Q_m, j \in \mathbb{Z}^d \setminus Q_m} J_{ij}(1 - u_i u_j) \leqslant 2 \sum_{i \in Q_m, j \in \mathbb{Z}^d \setminus Q_m} J_{ij}(1 + u_j),$$

and, rearranging its terms conveniently,

$$\sum_{i \in V_m, j \in Q_m \setminus V_m} J_{ij} + \sum_{i \in V_m, j \in \{u=-1\} \setminus Q_m} J_{ij} \leqslant \sum_{i \in V_m, j \in \{u=1\} \setminus Q_m} J_{ij}.$$

By adding to both sides the series

$$\sum_{i \in V_m, j \in \{u=1\} \setminus Q_m} J_{ij},$$

and taking advantage of (1.7), we then find

$$\sum_{i \in V_m, j \in \mathbb{Z}^d \setminus V_m} \frac{1}{|i-j|_\infty^{d+s}} \leqslant c_1 \sum_{i \in V_m, j \in \{u=1\} \setminus Q_m} \frac{1}{|i-j|_\infty^{d+s}}, \tag{3.5}$$

for some $c_1 > 0$.

Now we deal with the two sides of (3.5) separately. On the one hand, we apply Corollary 3.2 (with $\Gamma := V_m$) and obtain that

$$\sum_{i \in V_m, j \in \mathbb{Z}^d \setminus V_m} \frac{1}{|i-j|_\infty^{d+s}} \geqslant c_2 v_m^{\frac{d-s}{d}}, \tag{3.6}$$

for some $c_2 > 0$. On the other hand, we compute

$$\begin{aligned} \sum_{i \in V_m, j \in \{u=1\} \setminus Q_m} \frac{1}{|i-j|_\infty^{d+s}} &\leqslant \sum_{i \in V_m, j \in \mathbb{Z}^d \setminus Q_m} \frac{1}{|i-j|_\infty^{d+s}} = \sum_{n=0}^{m} \sum_{i \in A_n} \sum_{|j|_\infty \geqslant m+1} \frac{1}{|i-j|_\infty^{d+s}} \\ &\leqslant \sum_{n=0}^{m} a_n \sum_{|k|_\infty \geqslant m+1-n} \frac{1}{|k|_\infty^{d+s}} \leqslant 3^d d \sum_{n=0}^{m} a_n \sum_{r=m+1-n}^{+\infty} \frac{1}{r^{1+s}} \\ &\leqslant c_3 \sum_{n=0}^{m} (m+1-n)^{-s} a_n, \end{aligned}$$

for some $c_3 > 0$. The combination of this, (3.6) and (3.5) yields

$$v_m^{\frac{d-s}{d}} \leqslant c_4 \sum_{n=0}^{m} (m+1-n)^{-s} a_n,$$

for some $c_4 > 0$. We now sum up the above inequality on $m = 0, \ldots, \ell$. We get

$$\begin{aligned} \sum_{m=0}^{\ell} v_m^{\frac{d-s}{d}} &\leqslant c_4 \sum_{m=0}^{\ell} \sum_{n=0}^{m} (m+1-n)^{-s} a_n = c_4 \sum_{n=0}^{\ell} a_n \sum_{m=n}^{\ell} (m+1-n)^{-s} \\ &= c_4 \sum_{n=0}^{\ell} a_n \sum_{r=1}^{\ell+1-n} r^{-s} \leqslant c_5 \sum_{n=0}^{\ell} (\ell+1-n)^{1-s} a_n \leqslant c_5 (\ell+1)^{1-s} \sum_{n=0}^{\ell} a_n, \end{aligned}$$

that is

$$\sum_{m=0}^{\ell} v_m^{\frac{d-s}{d}} \leqslant c_5(\ell+1)^{1-s} v_\ell, \tag{3.7}$$

for some constant $c_5 > 0$.

We now claim that (3.7) implies the validity of (3.4), with

$$\bar{c} := \left[\frac{4^{-d-1+s}}{c_5(d+1-s)}\right]^{d/s}. \tag{3.8}$$

To see this, we argue by induction. Of course, the claim holds true for $\ell = 0, 1$, as $0 \in \partial u$. Therefore, we take $\ell \geqslant 2$ and assume that

$$v_m \geqslant \bar{c}\, m^d \quad \text{for any } m \in \{0, \ldots, \ell - 1\}.$$

Using (3.7) and (3.8), we have

$$\begin{aligned} v_\ell &\geqslant \frac{(\ell+1)^{s-1}}{c_5} \sum_{m=0}^{\ell-1} v_m^{\frac{d-s}{d}} \geqslant \frac{\bar{c}^{\frac{d-s}{d}}}{c_5} (\ell+1)^{s-1} \sum_{m=0}^{\ell-1} m^{d-s} \\ &\geqslant \frac{\bar{c}^{\frac{d-s}{d}}}{c_5(d+1-s)} (\ell+1)^{s-1} (\ell-1)^{d+1-s} \geqslant \frac{\bar{c}^{\frac{d-s}{d}}}{c_5(d+1-s)} (2\ell)^{s-1} \left(\frac{\ell}{2}\right)^{d+1-s} \\ &\geqslant \frac{\bar{c}^{\frac{d-s}{d}} 4^{-d-1+s}}{c_5(d+1-s)} \ell^d = \bar{c}\, \ell^d, \end{aligned}$$

that is our claim. Hence, the proof of the proposition is concluded. □

A first application of the estimates just proved is contained in the next corollary, that establishes a bound from below for the interaction energy of non-trivial minimizers.

Corollary 3.4. *Let u be a minimizer for H in $Q_\ell(q)$, for some $q \in \mathbb{Z}^d$ and $\ell \in \mathbb{N}$. If $q \in \partial u$, then*

$$I_{Q_\ell(q), Q_\ell(q)}(u) \geqslant c_\star \ell^{d-s}, \tag{3.9}$$

for some constant $c_\star > 0$ depending only on d, s, λ and Λ.

Proof. We simply apply hypothesis (1.7) and Proposition 3.3 to deduce that

$$\begin{aligned} I_{Q_\ell(q), Q_\ell(q)}(u) &\geqslant \frac{\lambda}{d^{d+s}} \sum_{i,j \in Q_\ell(q)} \frac{1 - u_i u_j}{|i-j|_\infty^{d+s}} \\ &\geqslant \frac{4\lambda}{(2d\ell)^{d+s}} \left[\#\left(\{u=-1\} \cap Q_\ell(q)\}\right)\right] \cdot \left[\#\left(\{u=1\} \cap Q_\ell(q)\}\right)\right] \\ &\geqslant c_\star \ell^{d-s}, \end{aligned}$$

for some $c_\star > 0$, as desired. □

Remark 3.5. The bound (3.9) can be seen as a counterpart to the estimate from *above* obtained in Proposition 2.9. More specifically, Corollary 3.4 shows that the energy estimate (2.26) gives an optimal bound for the energy of a non-trivial minimizer u in a cube $Q_\ell(q)$, as a function of ℓ. Indeed, notice that under hypothesis (1.7), we can make the choice

$$\sigma(R) = \frac{c_d \Lambda}{s} R^{-s},$$

for some dimensional constant $c_d > 0$. Thanks to this observation and recalling Remark 2.10, estimate (2.26) becomes in this setting just

$$H_{Q_\ell(q)}(u) \leqslant \bar{C} \ell^{d-s}, \tag{3.10}$$

for some constant $\bar{C} \geqslant 1$ depending only on d, s and Λ. As a result, both estimates (3.9) and (2.26) (in its form (3.10) just deduced) show the same dependence on ℓ.

We conclude the subsection with a result that sharpens the density estimates of Proposition 3.3: the so-called *clean ball condition*. We obtain it by applying both Corollary 3.4 and Proposition 3.3 itself.

Proposition 3.6. *Suppose that J satisfies condition (1.7) and that $h = 0$. Let u be a minimizer for H in $Q_\ell(q)$, for some $q \in \mathbb{Z}^d$ and $\ell \in \mathbb{N}$. If $q \in \partial u$, then there exist two sites $q_-, q_+ \in Q_\ell(q)$ and a constant $\kappa \in (0,1)$, depending only on d, s, λ and Λ, such that*

$$Q_{\lfloor \kappa \ell \rfloor}(q_-) \subseteq \{u = -1\} \cap Q_\ell(q) \quad \textit{and} \quad Q_{\lfloor \kappa \ell \rfloor}(q_+) \subseteq \{u = 1\} \cap Q_\ell(q).$$

Proof. We prove the statement concerning the level set $\{u = 1\} \cap Q_\ell(q)$, the other one being completely analogous. Moreover, we restrict ourselves to consider $\ell \geqslant \ell_0$, for a large value $\ell_0 \geqslant 2$ to be later specified, as for the case $\ell < \ell_0$ one can simply choose $\kappa = 1/\ell_0$ and $q_+ = q$.

Fix $k \in \mathbb{N}$, with

$$k \leqslant \frac{\ell}{2}, \tag{3.11}$$

and let $N \in \mathbb{N}$ be the only integer for which

$$(2k+1)N \leqslant 2\ell + 1 < (2k+1)(N+1). \tag{3.12}$$

In view of (3.12), there is a family $\mathcal{Q} = \{Q^{(n)}\}_{n=1}^{N^d}$ of N^d non-overlapping cubes $Q^{(n)} = Q_k^{(n)}(q^{(n)})$ each contained in $Q_\ell(q)$, having center $q^{(n)} \in Q_\ell(q)$ and sides composed by $2k+1$ sites. Observe that we may choose $\mathcal{Q}$ so that the union of its elements covers $Q_{\ell-k}(q)$. Let then $\widetilde{\mathcal{Q}} \subseteq \mathcal{Q}$ be the subfamily of $\mathcal{Q}$ made up of those cubes having non-empty intersection with the level set $\{u = 1\}$. That is,

$$\widetilde{\mathcal{Q}} := \Big\{ Q \in \mathcal{Q} : \text{there exists } i \in Q \text{ at which } u_i = 1 \Big\}.$$

Denoting by $\widetilde{N} \in \mathbb{N}$ the cardinality of $\widetilde{\mathcal{Q}}$, we claim that

$$\widetilde{N} \geqslant c_1 N^d, \tag{3.13}$$

for some $c_1 > 0$ independent of N and ℓ. To check (3.13), we simply apply the density estimate of Proposition 3.3 to the cube $Q_{\ell-k}(q)$ and compute

$$\bar{c}(\ell-k)^d \leqslant \#\left(\{u=1\} \cap Q_{\ell-k}(q)\right) \leqslant \sum_{n=1}^{N^d} \#\left(\{u=1\} \cap Q^{(n)}\right) \leqslant \widetilde{N}(2k+1)^d.$$

This, (3.12) and (3.11) then imply that

$$\frac{\bar{c}}{2^d}\,\ell^d \leqslant \bar{c}(\ell-k)^d \leqslant \frac{\widetilde{N}}{N^d}(2\ell+1)^d \leqslant \frac{\widetilde{N}}{N^d} 3^d \ell^d,$$

which gives (3.13).

We relabel the cubes of the family $\widetilde{\mathcal{Q}}$ in order to write $\widetilde{\mathcal{Q}} = \{\widetilde{Q}^{(n)}\}_{n=1}^{\widetilde{N}}$, with $\widetilde{Q}^{(n)} = Q_k(\widetilde{q}^{(n)})$, with $\widetilde{q}^{(n)} \in Q_\ell(q)$. To finish the proof, we shall show that we can find a cube $\widetilde{Q}^{(\bar{n})}$, for some $\bar{n} \in \{1, \ldots, \widetilde{N}\}$, such that $u_i = 1$ at any $i \in \widetilde{Q}^{(\bar{n})}$. For this, we argue by contradiction and in fact suppose that, for any $n \in \{1, \ldots, \widetilde{N}\}$, there exists a site $i^{(n)} \in \widetilde{Q}^{(n)}$ at which $u_{i^{(n)}} = -1$. By the definition of $\widetilde{\mathcal{Q}}$, it is then clear that there also exist sites $j^{(n)} \in \widetilde{Q}^{(n)} \cap \partial u$, for any $n \in \{1, \ldots, \widetilde{N}\}$.

Up to modifying the family $\widetilde{\mathcal{Q}}$ and reducing its cardinality $\widetilde{N}$ by a factor 3^d, we may also assume that $j^{(n)} = \widetilde{q}^{(n)}$. By applying Proposition 2.9, Corollary 3.4 and estimate (3.13), we then get

$$\bar{C}\ell^{d-s} \geqslant H_{Q_\ell(q)}(u) \geqslant \sum_{n=1}^{\widetilde{N}} I_{Q_k(\widetilde{q}^{(n)}),\, Q_k(\widetilde{q}^{(n)})}(u) \geqslant c_\star \widetilde{N} k^{d-s} \geqslant c_\star c_1 N^d k^{d-s},$$

that, combined with (3.12) and (3.11), yields

$$k \geqslant c_2 \ell,$$

for some $c_2 > 0$ independent of ℓ. But this leads to a contradiction, since we are free to take $k \in \{1, \ldots, \lfloor c_2\ell/2 \rfloor\}$ and $\ell \geqslant \ell_0 := 4/c_2$. □

We stress that the argument adopted in the above proof is a refined version of the one displayed in Proposition 2.13, in light of the now available density estimates and the optimal energy bound (3.9). Indeed, Proposition 3.6 is the main tool that will be used in the next subsection to improve the result of Proposition 2.13 and finish the proof of Theorem 1.4.

3.2. **Completion of the proof of Theorem 1.4.** As discussed at the beginning of the present section, to finish the proof of Theorem 1.4 we only need to show that in Proposition 2.13 we can take

$$M_0 := \bar{M}_0 \tau, \tag{3.14}$$

for some $\bar{M}_0 > 0$ depending only on d, s, λ and Λ.

From now on, we freely use the notation adopted in Section 2 with no further explanation.

In order to prove that Proposition 2.13 holds true with M_0 given by (3.14), it suffices to show that the minimal minimizer $u = u_\omega^M$ satisfies

$$u_i = -1 \text{ for any } i \in Q_{2d\tau}(\bar{q}), \text{ for some } \bar{q} \in \mathcal{S}_\omega^M \text{ such that } Q_{2d\tau}(\bar{q}) \subset \mathcal{S}_\omega^M, \tag{3.15}$$

provided $M \geqslant M_0$, with M_0 as in (3.14). Note that (3.15) is indeed stronger than the claim (2.31) that was proved in Proposition 2.13. By arguing as in the proof of Proposition 2.13, we can reduce (3.15) to the weaker claim that

$$\text{either } u_i = -1 \text{ for any } i \in Q_{2d\tau}(\bar{q}) \text{ or } u_i = 1 \text{ for any } i \in Q_{2d\tau}(\bar{q}). \tag{3.16}$$

To check (3.16), we first notice that there are a site $q \in \mathcal{S}_\omega^M$ and a dimensional constant $c_\star > 0$ such that $Q_{3\ell}(q) \subset \mathcal{S}_\omega^M$, with $\ell = \lfloor c_\star M \rfloor$. Now, either

$$Q_\ell(q) \cap \partial u \neq \varnothing, \tag{3.17}$$

or u is identically equal to -1 or 1 in the whole of $Q_\ell(q)$. By taking $M \geqslant M_0 := (4d\tau)/c_\star$, this latter fact would imply (3.16) and the proof would then be over. Therefore, we suppose that (3.17) is verified and, thus, that there exists a site $q_\star \in Q_\ell(q) \cap \partial u$.

By Corollary 2.6, the minimal minimizer u is a minimizer for H in $Q_\ell(q_\star) \subset Q_{2\ell}(q)$ and, hence, Proposition 3.6 implies that, say,

$$u_i = -1 \quad \text{for any } i \in Q_{\lfloor \kappa\ell \rfloor}(\bar{q}),$$

for some site $\bar{q} \in Q_\ell(q_\star)$ and some constant $\kappa \in (0,1)$, depending only on d, s, λ and Λ. But then, (3.16) follows once again by choosing $M \geqslant M_0 := (4d\tau)/(c_\star\kappa)$.

Claim (3.16) is thus fully proved and so is Theorem 1.4.

4. Interlude. Some simple facts about non local perimeter functionals

In this intermediate section, we present a couple of results regarding the set functions $\mathcal{L}_K$ and Per_K, introduced in (1.20) and (1.19), respectively.

Throughout most of the section, $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$ is a general non-negative kernel, not necessarily satisfying any of conditions (1.15) or (1.16). In particular, K is never required here to fulfill the periodicity assumption (1.17).

We begin by presenting a lemma that establishes the lower semicontinuity of $\mathcal{L}_K$ with respect to L^1 convergence. As a byproduct, we also obtain the lower semicontinuity of the K-perimeter functional.

Lemma 4.1. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of measurable sets in $\mathbb{R}^d$. Suppose that there exist two measurable sets $A, B \subseteq \mathbb{R}^d$ such that $A_n \to A$ and $B_n \to B$ in L^1_{loc}, as $n \to +\infty$. Then,*

$$\mathcal{L}_K(A, B) \leqslant \liminf_{n\to+\infty} \mathcal{L}_K(A_n, B_n). \tag{4.1}$$

In particular,

$$\mathrm{Per}_K(A; B) \leqslant \liminf_{n\to+\infty} \mathrm{Per}_K(A_n; B_n). \tag{4.2}$$

Proof. Let $\{n_k\}$ be a subsequence along which the $\liminf$ on the right-hand side of (4.1) is attained as a limit. By a standard diagonal argument and up to selecting a further subsequence (that we do not relabel), we have that $\chi_{A_{n_k}} \to \chi_A$ and $\chi_{B_{n_k}} \to \chi_B$ a.e. in $\mathbb{R}^d$, as $k \to +\infty$. Then, Fatou's Lemma implies that

$$\begin{aligned} \mathcal{L}_K(A, B) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_A(x)\chi_B(y)K(x,y)\,dx\,dy \\ &\leqslant \liminf_{k\to+\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{A_{n_k}}(x)\chi_{B_{n_k}}(y)K(x,y)\,dx\,dy \\ &= \lim_{k\to+\infty} \mathcal{L}_K(A_{n_k}, B_{n_k}) \\ &= \liminf_{n\to+\infty} \mathcal{L}_K(A_n, B_n), \end{aligned}$$

that is (4.1).

The validity of (4.2) follows at once from (4.1) after one notices that the convergences of A_n and B_n imply that

$$\begin{cases} A_n \cap B_n \longrightarrow A \cap B \\ A_n \setminus B_n \longrightarrow A \setminus B \\ B_n \setminus A_n \longrightarrow B \setminus A \\ \mathbb{R}^d \setminus (A_n \cup B_n) \longrightarrow \mathbb{R}^d \setminus (A \cup B) \end{cases} \qquad \text{in } L^1_{\mathrm{loc}},$$

as $n \to +\infty$. $\square$

Next is a simple computation that may be seen as a generalized Coarea Formula. See e.g. [V91] and the very recent [CSV16, L16] for similar results. More precisely, we recall (1.18) and we prove the following:

Lemma 4.2. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $u : \Omega \to \mathbb{R}$ a measurable function. Then,*

$$\mathscr{K}_K(u; \Omega, \Omega) = \int_{-\infty}^{+\infty} \mathscr{K}_K(\chi_{\{u>t\}}; \Omega, \Omega)\,dt. \tag{4.3}$$

Proof. First of all, notice that, for any $x, y \in \Omega$, we may write

$$|u(x) - u(y)| = \int_{-\infty}^{+\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| \, dt. \tag{4.4}$$

Indeed, notice that

$$\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y) = \begin{cases} 1, & \text{if } u(x) > t \geqslant u(y), \\ -1, & \text{if } u(y) > t \geqslant u(x), \\ 0, & \text{otherwise}. \end{cases}$$

From this, formula (4.4) easily follows.

Hence, by (4.4) and Fubini's Theorem, we simply obtain

$$\begin{aligned} \mathscr{K}_K(u; \Omega, \Omega) &= \int_\Omega \int_\Omega |u(x) - u(y)| K(x, y) \, dx \, dy \\ &= \int_\Omega \int_\Omega \left(\int_{-\infty}^{+\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| \, dt \right) K(x, y) dx \, dy \\ &= \int_{-\infty}^{+\infty} \left(\int_\Omega \int_\Omega |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| K(x, y) \, dx \, dy \right) dt, \end{aligned}$$

and (4.3) follows. □

We conclude the section with the following basic integrability result.

Lemma 4.3. *Suppose that K satisfies (1.16) and let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Then,*

$$K \in L^1(\Omega \times (\mathbb{R}^d \setminus \Omega)). \tag{4.5}$$

Proof. By using polar coordinates and (1.16), we compute

$$\begin{aligned} \int_\Omega \int_{\mathbb{R}^d \setminus \Omega} K(x, y) \, dx \, dy &\leqslant \Lambda \int_\Omega \int_{\mathbb{R}^d \setminus \Omega} \frac{dx \, dy}{|x - y|^{d+s}} \leqslant \Lambda \int_\Omega \left(\int_{\mathbb{R}^d \setminus B_{\operatorname{dist}(x, \partial\Omega)}} \frac{dz}{|z|^{d+s}} \right) dx \\ &= d\Lambda |B_1| \int_\Omega \left(\int_{\operatorname{dist}(x, \partial\Omega)}^{+\infty} \frac{dt}{t^{1+s}} \right) dx = \frac{d\Lambda |B_1|}{s} \int_\Omega \frac{dx}{\operatorname{dist}(x, \partial\Omega)^s}. \end{aligned}$$

Then, (4.5) follows, as the last integral is finite, due to the Lipschitzianity of $\partial\Omega$. This last fact may be for instance deduced from [M00, Lemma 3.32], applied with $u = 1$ there. □

5. From the Ising model to the K-perimeter. Proof of Theorem 1.6

In this section, we give a proof of Theorem 1.6. The argument is rather articulated and thus will be split into various lemmata, most of which deal with convergence issues.

Notice that throughout the section, we always assume the kernel K to satisfy assumptions (1.15) and (1.16), but not (1.17), in accordance with the hypotheses made in the statement of Theorem 1.6.

We begin by checking that the coefficients $J^{(\varepsilon)}$ yield a power-like interaction term, bounded independently of ε.

Lemma 5.1. *Given any $\varepsilon > 0$, the interaction $J^{(\varepsilon)}$ defined in (1.22) satisfies conditions (1.1) and (1.2). Moreover, it fulfills (1.7) uniformly in ε. That is,*

$$\frac{\lambda_\star}{|i - j|^{d+s}} \leqslant J_{ij}^{(\varepsilon)} \leqslant \frac{\Lambda_\star}{|i - j|^{d+s}} \quad \text{for any } i, j \in \mathbb{Z}^d \text{ with } i \neq j, \tag{5.1}$$

for some constants $\Lambda_\star \geqslant \lambda_\star > 0$ that depend only on λ, Λ, d and s.

Proof. The fact that $J^{(\varepsilon)}$ satisfies (1.1) and (1.2) is a simple consequence of its definition and hypotheses (1.15) on K. Thus, we focus on the proof of (5.1).

By changing variables, for $i \neq j$ we have

$$J_{ij}^{(\varepsilon)} = \varepsilon^{d+s} \int_{Q_{1/2}(i)} \int_{Q_{1/2}(j)} K(\varepsilon x, \varepsilon y)\, dx\, dy.$$

To obtain the left-hand side inequality in (5.1), we observe that, for $x \in Q_{1/2}(i)$ and $y \in Q_{1/2}(j)$, it holds

$$|x-y| \leqslant |i-j| + |x-i| + |y-j| \leqslant |i-j| + \sqrt{d} \leqslant 2\sqrt{d}\,|i-j|,$$

and hence, by (1.16),

$$J_{ij}^{(\varepsilon)} \geqslant \lambda \int_{Q_{1/2}(i)} \int_{Q_{1/2}(j)} \frac{dx\, dy}{|x-y|^{d+s}} \geqslant \frac{(2\sqrt{d})^{-d-s}\lambda}{|i-j|^{d+s}},$$

which gives the first inequality in (5.1).

On the other hand, to get the second inequality in (5.1), we deal with the two cases $|i-j|_\infty \geqslant 2$ and $|i-j|_\infty = 1$ separately. If $|i-j|_\infty \geqslant 2$, we recall the notation in (1.11) and we simply have

$$|x-y| = \left(\sum_{k=1}^{d} (x_k - y_k)^2 \right)^{1/2} \geqslant |x-y|_\infty \geqslant |i-j|_\infty - |x-i|_\infty - |y-j|_\infty \geqslant |i-j|_\infty - 1 \geqslant \frac{|i-j|_\infty}{2},$$

for any $x \in Q_{1/2}(i)$ and $y \in Q_{1/2}(j)$. Thus, using (1.16),

$$J_{ij}^{(\varepsilon)} \leqslant \Lambda \int_{Q_{1/2}(i)} \int_{Q_{1/2}(j)} \frac{dx\, dy}{|x-y|^{d+s}} \leqslant \frac{2^{d+s}\Lambda}{|i-j|^{d+s}},$$

which proves the second inequality in (5.1) in this case.

When instead $|i-j|_\infty = 1$, by applying twice Coarea Formula and using again (1.16), we compute

$$\begin{aligned}
J_{ij}^{(\varepsilon)} &\leqslant \Lambda \int_{Q_{1/2}(i)} \int_{Q_{1/2}(j)} \frac{dx\, dy}{|x-y|^{d+s}} \leqslant \Lambda \int_{Q_{1/2}} \int_{Q_1 \setminus Q_{1/2}} \frac{dx\, dy}{|x-y|_\infty^{d+s}} \\
&\leqslant \Lambda \int_{Q_{1/2}} \left(\int_{Q_2 \setminus Q_{\frac{1}{2} - |x|_\infty}} \frac{dz}{|z|_\infty^{d+s}} \right) dx = 2^d d \Lambda \int_{Q_{1/2}} \left(\int_{\frac{1}{2} - |x|_\infty}^{2} \frac{dt}{t^{1+s}} \right) dx \\
&\leqslant \frac{2^{d+s} d\Lambda}{s} \int_{Q_{1/2}} \frac{dx}{(1 - 2|x|_\infty)^s} = \frac{2^{2d+s} d^2 \Lambda}{s} \int_0^{1/2} \frac{t^{d-1}}{(1-2t)^s}\, dt \\
&\leqslant \frac{C_{d,s}\, \Lambda}{|i-j|^{d+s}},
\end{aligned}$$

for some constant $C_{d,s} > 0$ depending only on d and s. This completes the proof of the second inequality in (5.1). □

Now that we know from Lemma 5.1 that $J^{(\varepsilon)}$ is a well-behaved power-like interaction term, with ferromagnetic constants independent of ε, we may use the estimate contained in Proposition 2.9 (in its form (3.10)) to deduce uniform-in-ε bounds for the Hamiltonian $H^{(\varepsilon)}$ defined in (1.23). More precisely, if u is a minimizer for $H^{(\varepsilon)}$ in a cube Q_ℓ of sides $\ell \in \mathbb{N}$, then

$$H_{Q_\ell}^{(\varepsilon)}(u) \leqslant C \ell^{d-s}, \tag{5.2}$$

for some constant $C \geqslant 1$, depending only on d, s and Λ.

Moreover, recall that to any configuration u and any $\varepsilon > 0$ we associated an (a.e.) extension $\bar{u}_\varepsilon$ of u to $\mathbb{R}^d$, via definition (1.24). We now consider the measurable set

(5.3) $$E(u,\varepsilon) := \Big\{x \in \mathbb{R}^d : \bar{u}_\varepsilon(x) = 1\Big\}.$$

By the definitions of $E(u,\varepsilon)$ and $J^{(\varepsilon)}$, recalling (1.21) and (1.28), we see that the identities

(5.4) $$\mathrm{Per}_K(E(u,\varepsilon); Q_R) = \frac{1}{4}\,\mathscr{K}_K(\bar{u}_\varepsilon; Q_R) = \frac{\varepsilon^{d-s}}{4}\,H^{(\varepsilon)}_{Q_\ell}(u),$$

hold true for any $R = (\ell + 1/2)\varepsilon$, with $\ell \in \mathbb{N}$.

Formula (5.4) is crucial in building a rigorous bridge between the discrete setting of the Hamiltonian $H^{(\varepsilon)}$ and the continuous one given by Per_K. In particular, we will shortly use it, in combination with (5.2), to obtain a uniform bound for the K-perimeter.

Let now $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be an infinitesimal sequence and, for any $n \in \mathbb{N}$, let $u^{(n)}$ be the ground state for the Hamiltonian $H^{(\varepsilon_n)}$ considered in the statement of Theorem 1.6. Let $\bar{u}^{(n)} = \bar{u}^{(n)}_{\varepsilon_n}$ be the extension of $u^{(n)}$ to $\mathbb{R}^d$, defined as in (1.24), and $E_n := E(u^{(n)}, \varepsilon_n)$ be the corresponding measurable set introduced in (5.3).

It is not hard to see that (5.2) and (5.4) imply the following result:

Lemma 5.2. *There exists a constant $C \geqslant 1$, depending on d, s and Λ, but not on n, such that*
$$\mathrm{Per}_K(E_n; Q_R) \leqslant C R^{d-s},$$
for any $R \geqslant 1$.

Thanks to Lemma 5.2 and hypothesis (1.16), we know that the $W^{s,1}(Q_R)$ norm of χ_{E_n} is bounded uniformly in n, for any $R \geqslant 1$. Hence, by the compact embedding of $W^{s,1}(Q_R)$ into $L^{d/(d-s)}(Q_R)$ (see e.g. [DNPV12, Corollary 7.2]) and a standard diagonal argument (in n and R), we conclude that, up to a subsequence (that we omit in the notation), χ_{E_n} converges in L^1_{loc} and a.e. to χ_E, for some measurable set $E \subseteq \mathbb{R}^d$, as $n \to +\infty$.

In what follows, we show that E is a class A minimal surface for Per_K, thus completing the proof of Theorem 1.6.

To check this, we fix a cube Q_R with sides $R \geqslant 2$. Of course, it is enough to prove that E is a minimal surface for Per_K in each such cube. For any $n \in \mathbb{N}$, let $\ell_n \in \mathbb{N}$ be defined by

(5.5) $$\ell_n := \left\lfloor \frac{1}{2}\left(\frac{2R}{\varepsilon_n} - 1\right)\right\rfloor.$$

Also set
$$R_n := \left(\ell_n + \frac{1}{2}\right)\varepsilon_n,$$
and notice that

(5.6) $$R - \varepsilon_n < R_n \leqslant R.$$

In particular, $R_n \to R$ as $n \to +\infty$.

By taking advantage of Lemma 4.1 in Section 4 and (5.4), we have that

(5.7) $$\mathrm{Per}_K(E; Q_R) \leqslant \liminf_{n\to+\infty} \mathrm{Per}_K(E_n; Q_{R_n}) = \frac{1}{4}\liminf_{n\to+\infty} \varepsilon_n^{d-s} H^{(\varepsilon_n)}_{Q_{\ell_n}}(u^{(n)}).$$

Now, let F be a competitor for E in Q_R, i.e. a measurable set with $F\backslash Q_R = E\backslash Q_R$ and $\mathrm{Per}_K(F; Q_R) < +\infty$. In view of the following lemma, we may assume without loss of generality that the boundary of F is smooth inside Q_R.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $F \subset \mathbb{R}^d$ be a measurable set such that $\mathrm{Per}_K(F;\Omega) < +\infty$. Then, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of measurable subsets of $\mathbb{R}^d$ such that, for any $n \in \mathbb{N}$,*

$$\partial F_n \cap \overline{\Omega} \text{ is smooth,} \tag{5.8}$$

$$F_n \setminus \overline{\Omega} = F \setminus \overline{\Omega}, \tag{5.9}$$

and

$$\lim_{n\to+\infty} |F_n \Delta F| = 0, \tag{5.10}$$

$$\lim_{n\to+\infty} \mathrm{Per}_K(F_n;\Omega) = \mathrm{Per}_K(F;\Omega). \tag{5.11}$$

The proof of Lemma 5.3 is inspired by the one of the analogous result for the classical perimeter (see e.g. [G84, Theorem 1.24]) and is similar to those of [CSV16, Proposition 6.4] and [L16, Theorem 1.1]. As it is rather technical but by now sufficiently standard, we defer it to Appendix A.

For such competitor F and a given $n \in \mathbb{N}$, we consider the partition (up to a negligible set) of the cube Q_{R_n} into the family of open[7] subcubes

$$\mathcal{Q}_n := \left\{ \mathring{Q}_{\varepsilon_n/2}(\varepsilon_n i) : i \in Q_{\ell_n} \right\},$$

with ℓ_n as in (5.5), and its further subdivision into the three disjoint subfamilies

$$\begin{aligned} \mathcal{G}_n^+ &:= \left\{ Q \in \mathcal{Q}_n : Q \subset \mathring{F} \right\}, \\ \mathcal{G}_n^- &:= \left\{ Q \in \mathcal{Q}_n : Q \subset \mathbb{R}^d \setminus \bar{F} \right\} \\ \text{and} \quad \mathcal{B}_n &:= \left\{ Q \in \mathcal{Q}_n : Q \cap \partial F \neq \varnothing \right\} = \mathcal{Q}_n \setminus \left(\mathcal{G}_n^+ \cup \mathcal{G}_n^- \right). \end{aligned}$$

We also write

$$G_n^\pm := \bigcup_{Q \in \mathcal{G}_n^\pm} Q \quad \text{and} \quad B_n := \bigcup_{Q \in \mathcal{B}_n} Q. \tag{5.12}$$

We then define a configuration $v^{(n)}$ by setting

$$v_i^{(n)} := \begin{cases} 1 & \text{if } Q_{\varepsilon_n/2}(\varepsilon_n i) \in \mathcal{G}_n^+, \\ -1 & \text{if } Q_{\varepsilon_n/2}(\varepsilon_n i) \in \mathcal{G}_n^- \cup \mathcal{B}_n, \\ u_i^{(n)} & \text{if } i \in \mathbb{Z}^d \setminus Q_{\ell_n}, \end{cases}$$

and, as in (5.3), the corresponding set

$$F_n := \bigcup_{i \in \{v_i^{(n)}=1\}} Q_{\varepsilon_n/2}(\varepsilon_n i).$$

By definition, $v^{(n)}$ coincides with $u^{(n)}$ outside Q_{ℓ_n} and $F_n \setminus Q_{R_n} = E_n \setminus Q_{R_n}$. Notice that (5.6) implies that

$$F_n \setminus Q_R = E_n \setminus Q_R. \tag{5.13}$$

Moreover, by (5.6) and (5.4), we see that

$$\mathrm{Per}_K(F_n;Q_R) \geqslant \mathrm{Per}_K(F_n;Q_{R_n}) = \frac{\varepsilon_n^{d-s}}{4} H^{(\varepsilon_n)}_{Q_{\ell_n}}(v^{(n)}).$$

[7] As usual, $\mathring{Q}$ denotes the interior of Q.

Hence, by (5.7) and the minimality of $u^{(n)}$ in Q_{ℓ_n}, we deduce that

$$\operatorname{Per}_K(E;Q_R) \leqslant \liminf_{n\to+\infty} \operatorname{Per}_K(F_n;Q_R).$$

To conclude the proof of the minimality of E it now suffices to verify the validity of the following result:

Lemma 5.4. *There exists a diverging sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers for which*

$$\lim_{k\to+\infty} \operatorname{Per}_K(F_{n_k};Q_R) = \operatorname{Per}_K(F;Q_R). \tag{5.14}$$

Proof. Given any set Ω and any $\delta > 0$, we denote by $N_\delta^\Omega(\partial F)$ the δ-neighborhood of ∂F in Ω, that is

$$N_\delta^\Omega(\partial F) := \Big\{x \in \Omega : \operatorname{dist}(x,\partial F) \leqslant \delta\Big\}.$$

Since $\partial F \cap Q_R$ is smooth (recall Lemma 5.3), we have that

$$|N_\delta^{Q_R}(\partial F)| \leqslant C\delta,$$

for any small $\delta > 0$ and some constant $C > 0$ independent of δ. Moreover, recalling (5.12), we notice that

$$B_n \subseteq N_{\sqrt{d}\varepsilon_n}^{Q_R}(\partial F),$$

and thus

$$|B_n| \leqslant c_1\varepsilon_n, \tag{5.15}$$

for some $c_1 > 0$ independent of n.

After these preliminary considerations, we now head to the proof of (5.14). First of all, we observe that

$$F_n \longrightarrow F \quad \text{in } L^1_{\rm loc}, \text{ as } n \to +\infty.$$

Indeed, the convergence outside Q_{R_n} comes from the fact that $F_n \setminus Q_{R_n} = E_n \setminus Q_{R_n}$ and $E_n \to E$ in $L^1_{\rm loc}$. On the other hand, $(F_n \Delta F) \cap Q_{R_n} \subset B_n$ and the conclusion follows by (5.15).

Up to considering a suitable subsequence (that we neglect to keep track of in the notation), we also have that

$$\chi_{F_n} \longrightarrow \chi_F \quad \text{and} \quad \chi_{B_n} \longrightarrow 0 \quad \text{a.e. in } \mathbb{R}^d, \text{ as } n \to +\infty. \tag{5.16}$$

Concerning the inner contributions to the K-perimeters of F_n and F, we recall the notation in (5.12) and we compute

$$\begin{aligned} &|\mathcal{L}_K(F_n \cap Q_R, Q_R \setminus F_n) - \mathcal{L}_K(F \cap Q_R, Q_R \setminus F)| \\ &\quad \leqslant \Big|\mathcal{L}_K(G_n^+, G_n^- \cup (B_n \setminus F)) - \mathcal{L}_K(F \cap Q_R, Q_R \setminus F)\Big| + \mathcal{L}_K(G_n^+, B_n \cap F) \\ &\quad \leqslant \mathcal{L}_K((F \cap Q_R) \setminus G_n^+, Q_R \setminus F) + \mathcal{L}_K(G_n^+, B_n \cap F) \\ &\quad = \mathcal{L}_K(B_n \cap F, Q_R \setminus F) + \mathcal{L}_K(G_n^+, B_n \cap F). \end{aligned} \tag{5.17}$$

Now, on the one hand,

$$\mathcal{L}_K(B_n \cap F, Q_R \setminus F) = \int_{F\cap Q_R}\int_{Q_R\setminus F} \chi_{B_n}(x)K(x,y)\,dx\,dy,$$

so that, by taking advantage of the Lebesgue's Dominated Convergence Theorem, (5.16) and the fact that F has finite K-perimeter in Q_R, we deduce that

$$\lim_{n\to+\infty} \mathcal{L}_K(B_n \cap F, Q_R \setminus F) = 0. \tag{5.18}$$

On the other hand, we use hypothesis (1.16), a suitable change of variables and the Coarea Formula to obtain

$$\begin{aligned}\mathcal{L}_K(G_n^+, B_n\cap F) &\leqslant \Lambda \sum_{Q\in\mathcal{B}_n}\int_Q\int_{\mathbb{R}^d\setminus Q}\frac{dx\,dy}{|x-y|_\infty^{d+s}} = \Lambda\,(\#\mathcal{B}_n)\int_{Q_{\varepsilon_n/2}}\int_{\mathbb{R}^d\setminus Q_{\varepsilon_n/2}}\frac{dx\,dy}{|x-y|_\infty^{d+s}}\\ &= \frac{\Lambda|B_n|}{\varepsilon_n^s}\int_{Q_{1/2}}\int_{\mathbb{R}^d\setminus Q_{1/2}}\frac{dx\,dy}{|x-y|_\infty^{d+s}} \leqslant \frac{\Lambda|B_n|}{\varepsilon_n^s}\int_{Q_{1/2}}\left(\int_{\mathbb{R}^d\setminus Q_{\frac12-|x|_\infty}}\frac{dz}{|z|_\infty^{d+s}}\right)dx\\ &\leqslant c_2\,\frac{|B_n|}{\varepsilon_n^s},\end{aligned}$$

for some $c_2 > 0$ independent of n. By this and (5.15), we conclude that

$$\lim_{n\to+\infty}\mathcal{L}_K(G_n^+, B_n\cap F) = 0,$$

and thus, recalling (5.17) and (5.18),

$$\lim_{n\to+\infty}\mathcal{L}_K(F_n\cap Q_R, Q_R\setminus F_n) = \mathcal{L}_K(F\cap Q_R, Q_R\setminus F). \tag{5.19}$$

In regards to the outer contributions, using (5.13), we have

$$\begin{aligned}&|\mathcal{L}_K(F_n\setminus Q_R, Q_R\setminus F_n) - \mathcal{L}_K(F\setminus Q_R, Q_R\setminus F)|\\ &\quad\leqslant |\mathcal{L}_K(F_n\setminus Q_R, Q_R\setminus F) - \mathcal{L}_K(F\setminus Q_R, Q_R\setminus F)| + \mathcal{L}_K(E_n\setminus Q_R, B_n\cap F)\\ &\quad\leqslant \mathcal{L}_K((F_n\Delta F)\setminus Q_R, Q_R) + \mathcal{L}_K(\mathbb{R}^d\setminus Q_R, B_n)\\ &\quad= \int_{\mathbb{R}^d\setminus Q_R}\left(\int_{Q_R}\left(\chi_{F_n\Delta F}(x)+\chi_{B_n}(y)\right)K(x,y)\,dy\right)dx.\end{aligned}$$

Notice that, by (1.16), the kernel K belongs to $L^1(Q_R\times(\mathbb{R}^d\setminus Q_R))$, thanks to Lemma 4.3. Therefore, we can use (5.16) and the Lebesgue's Dominated Convergence Theorem once again to get

$$\lim_{n\to+\infty}\mathcal{L}_K(F_n\setminus Q_R, Q_R\setminus F_n) = \mathcal{L}_K(F\setminus Q_R, Q_R\setminus F). \tag{5.20}$$

Analogously, one also checks that

$$\lim_{n\to+\infty}\mathcal{L}_K(F_n\cap Q_R, \mathbb{R}^d\setminus(F_n\cup Q_R)) = \mathcal{L}_K(F\cap Q_R, \mathbb{R}^d\setminus(F\cup Q_R)).$$

By putting together this, (5.20) and (5.19), the thesis immediately follows. □

6. Planelike minimal surfaces for the K-perimeter. Proof of Theorem 1.7

Here, we address the validity of Theorem 1.7. Thanks to the link, established in Theorem 1.6, between the discrete structure of the Hamiltonian H and the continuous character of the perimeter Per_K, Theorem 1.7 is an almost immediate consequence of Theorem 1.4.

Proof of Theorem 1.7. Fix any direction $\omega \in \mathbb{R}^d\setminus\{0\}$. Let $\{\varepsilon_n\}$ be the infinitesimal sequence of positive numbers defined by setting $\varepsilon_n := 1/n$, for any $n\in\mathbb{N}$. Let $J^{(\varepsilon_n)}$ be the interaction kernel associated to ε_n introduced in (1.22) and observe that, thanks to (1.17), it satisfies the periodicity condition (1.8) with $\tau = n$. Moreover, Lemma 5.1 ensures that $J^{(\varepsilon_n)}$ also fulfills hypotheses (1.1), (1.2) and (1.7).

In view of this, we may deduce from Theorem 1.4 the existence of a ground state $u^{(n)}$ for the Hamiltonian $H^{(\varepsilon_n)}$ associated to $J^{(\varepsilon_n)}$ (see (1.23) for the precise definition) for which

$$\left\{i\in\mathbb{Z}^d : \frac{\omega}{|\omega|}\cdot i\leqslant 0\right\}\subset\left\{i\in\mathbb{Z}^d : u_i^{(n)} = 1\right\}\subset\left\{i\in\mathbb{Z}^d : \frac{\omega}{|\omega|}\cdot i\leqslant M_0 n\right\}, \tag{6.1}$$

for some constant $M_0 > 0$ independent of n.

But then, Theorem 1.6 implies that a subsequence of the extensions $\bar{u}^{(n)} = \bar{u}^{(n)}_{\varepsilon_n}$ of the $u^{(n)}$'s, as given by (1.24), converges in L^1_{loc} and a.e. in $\mathbb{R}^d$ to the characteristic function χ_{E_ω} of a class A minimal surface $E_\omega \subseteq \mathbb{R}^d$ for Per_K. Also, it can be readily checked from definition (1.24) that inclusion (6.1) implies the analogous

$$\left\{ x \in \mathbb{R}^d : \frac{\omega}{|\omega|} \cdot x \leqslant -M_0 \right\} \subset \left\{ x \in \mathbb{R}^d : \bar{u}^{(n)} = 1 \right\} \subset \left\{ x \in \mathbb{R}^d : \frac{\omega}{|\omega|} \cdot x \leqslant M_0 \right\},$$

up to possibly taking a larger M_0, still independent of ε. Hence, this and the convergence of the $\bar{u}^{(n)}$'s establish the validity of the planelike condition (1.25) for the set E_ω.

The proof of Theorem 1.7 is therefore complete. □

7. From the K-perimeter to the Ising model. Proof of Theorem 1.8

In this section we prove Theorem 1.8.

Similarly to what we did in the proof of Theorem 1.6, for any $n \in \mathbb{N}$ we consider the (almost) partition of $\mathbb{R}^d$ into the family

$$\mathcal{Q}_n := \left\{ \mathring{Q}_{\varepsilon_n/2}(\varepsilon_n i) : i \in \mathbb{Z}^d \right\}, \tag{7.1}$$

and we divide it into the two disjoint subfamilies

$$\mathcal{G}_n := \left\{ Q \in \mathcal{Q}_n : Q \subset E \right\} \quad \text{and} \quad \mathcal{Q}_n \setminus \mathcal{G}_n.$$

Write

$$G_n := \bigcup_{Q \in \mathcal{G}_n} Q,$$

and notice that $G_n \subseteq E$. We then define a configuration $v^{(n)}$ by setting

$$v_i^{(n)} := \begin{cases} 1 & \text{if } Q_{\varepsilon/2}(\varepsilon i) \in \mathcal{G}_n, \\ -1 & \text{if } Q_{\varepsilon/2}(\varepsilon i) \in \mathcal{Q}_n \setminus \mathcal{G}_n, \end{cases}$$

and denote by $\bar{v}^{(n)} = \bar{v}^{(n)}_{\varepsilon_n}$ its extension to $\mathbb{R}^d$, as in (1.24). Note that $\bar{v}^{(n)} = \chi_{G_n} - \chi_{\mathbb{R}^d \setminus G_n}$. We claim that

$$\bar{v}^{(n)} \longrightarrow \chi_E - \chi_{\mathbb{R}^d \setminus E} \quad \text{a.e. in } \mathbb{R}^d, \text{ as } n \to +\infty. \tag{7.2}$$

Indeed, since $G_n \subset E$ for any $n \in \mathbb{N}$ and E is open by hypothesis, we have that $\chi_{E \Delta G_n} \to 0$ a.e. in $\mathbb{R}^d$, as $n \to +\infty$. Hence, (7.2) follows.

Let now

$$\ell_n := \left\lceil \frac{R}{\varepsilon_n} \right\rceil,$$

and set

$$R_n := \left(\ell_n + \frac{1}{2} \right) \varepsilon_n.$$

Clearly, $R \leqslant R_n \leqslant R + 2\varepsilon_n$, so that $R_n \to R$, as $n \to +\infty$.

We consider the minimizer $u^{(n)}$ for $H^{(n)}$ in Q_{ℓ_n}, with datum $v^{(n)}$ outside of Q_{ℓ_n}, that is a configuration $u^{(n)}$ for which

$$H^{(n)}_{Q_{\ell_n}}(u^{(n)}) \leqslant H^{(n)}_{Q_{\ell_n}}(w) \quad \text{for any configuration } w \text{ such that } w_i = v_i^{(n)} \text{ for any } i \in \mathbb{Z}^d \setminus Q_{\ell_n}.$$

As in (5.3), we associate to each $u^{(n)}$ the set

$$E_n := \bigcup_{i\in\{u_i^{(n)}=1\}} Q_{\varepsilon_n/2}(\varepsilon_n i).$$

By arguing as for Lemma 5.2, we use the uniform Hamiltonian estimate given by Proposition 2.9 (in its refined form (3.10)) and the identity (5.4) to obtain that

$$\mathrm{Per}_K(E_n; Q_R) \leqslant \mathrm{Per}_K(E_n; Q_{R_n}) \leqslant C_1 R_n^{d-s} \leqslant C_2 R^{d-s},$$

for some constants $C_2 \geqslant C_1 \geqslant 1$ independent of n (and R). By this, we may then extract a subsequence $\{n_k\}$ in such a way that $\chi_{E_{n_k}}$ converges a.e. in Q_R to $\chi_{\widetilde{E}}$, for some measurable set $\widetilde{E} \subseteq Q_R$, as $k \to +infty$.

Set now

$$\widehat{E} := \widetilde{E} \cup (E \setminus Q_R)\,.$$

By (7.2) and the definition of $\widetilde{E}$, we see that

$$\bar{u}^{(n_k)} = \chi_{E_{n_k}} - \chi_{\mathbb{R}^d\setminus E_{n_k}} \longrightarrow \chi_{\widehat{E}} - \chi_{\mathbb{R}^d\setminus \widehat{E}} \quad \text{a.e. in } \mathbb{R}^d, \text{ as } k \to +\infty,$$

where $\bar{u}^{(n)} = \bar{u}^{(n)}_{\varepsilon_n}$ denotes as usual the extension of $u^{(n)}$ to $\mathbb{R}^d$ as of definition (1.24). Moreover, by arguing as in Section 5, one checks that the set $\widehat{E}$ is a minimizer for Per_K in Q_R. But then, since E is a strict minimizer and $\widehat{E} \setminus Q_R = E \setminus Q_R$, we conclude that $\widehat{E} = E$, and so Theorem 1.8 follows.

8. The Γ-convergence result. Proof of Theorem 1.9

In this section we show Theorem 1.9. For this, notice that the Γ-$\liminf$ inequality is a trivial consequence of Fatou's Lemma.

We can also easily check the validity of the third statement by applying the compact fractional Sobolev embedding (see e.g. [DNPV12, Corollary 7.2]) and recalling definition (1.27).

The proof of the Γ-$\limsup$ inequality is slightly more involved. To begin with, observe that we may restrict ourselves to assuming that $\mathscr{G}_K(u;\Omega) < +\infty$ and thus that $u = \chi_E - \chi_{R^d\setminus E}$ in Ω, for some measurable set $E \subseteq \mathbb{R}^d$ with finite K-perimeter in Ω.

We first prove the statement under the additional hypothesis that

$$\begin{cases} u = \chi_E - \chi_{\mathbb{R}^d\setminus E} \text{ in } \Omega' \\ \partial E \cap \Omega' \text{ is smooth} \\ u \in C^0(\mathbb{R}^d \setminus \Omega') \end{cases} \qquad \text{for some open bounded Lipschitz set } \Omega' \supset\supset \Omega. \tag{8.1}$$

We fix $\varepsilon > 0$ and, as in (7.1), we consider the (almost) partition of $\mathbb{R}^d$ given by the family

$$\mathcal{Q}_\varepsilon := \left\{ \mathring{Q}_{\varepsilon/2}(\varepsilon i) : i \in \mathbb{Z}^d \right\}.$$

We define the set

$$\Omega_\varepsilon := \bigcup_{Q\in\mathcal{Q}_\varepsilon : Q\cap\Omega\neq\varnothing} Q,$$

and, recalling (1.26), the function $u_\varepsilon \in \mathcal{X}_\varepsilon$, by setting for a.e. $x \in \mathbb{R}^d$

$$u_\varepsilon(x) := \inf_{Q_{\varepsilon/2}(\varepsilon i)} u, \quad \text{where } i \in \mathbb{Z}^d \text{ is the only site for which } x \in \mathring{Q}_{\varepsilon/2}(\varepsilon i).$$

Note that $\Omega \subseteq \Omega_\varepsilon \subset \Omega'$ for any ε sufficiently small and, consequently, that $u_\varepsilon = \chi_{E_\varepsilon} - \chi_{\mathbb{R}^d\setminus E_\varepsilon}$ in Ω_ε, for some measurable set E_ε.

Let now $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be any infinitesimal sequence for which

(8.2)
$$\limsup_{\varepsilon\to 0^+} \mathscr{G}_K^{(\varepsilon)}(u_\varepsilon;\Omega) = \lim_{n\to+\infty} \mathscr{G}_K^{(\varepsilon_n)}(u_{\varepsilon_n};\Omega).$$

Thanks to the regularity assumptions on E and u, we see that $u_{\varepsilon_n} \to u$ a.e. in $\mathbb{R}^d$ and thus in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, as $n \to +\infty$. Furthermore, by arguing as for (5.15) we can strengthen such convergence inside Ω and obtain that

$$|(E_{\varepsilon_n}\Delta E)\cap\Omega| \leqslant C\varepsilon_n^s,$$

for some constant $C > 0$ independent of n. As in the proof of Lemma 5.4, from this we then easily deduce

(8.3)
$$\lim_{n\to+\infty} \mathscr{K}_K(u_{\varepsilon_n};\Omega,\Omega) = \mathscr{K}_K(u;\Omega,\Omega).$$

On the other hand, by Lemma 4.3 we may use the Lebesgue's Dominated Convergence Theorem to get that

$$\lim_{n\to+\infty} \mathscr{K}_K(u_{\varepsilon_n};\Omega,\mathbb{R}^d\setminus\Omega) = \mathscr{K}_K(u;\Omega,\mathbb{R}^d\setminus\Omega).$$

By combining this with (8.3) and (8.2), we conclude that the Γ-$\limsup$ inequality holds true under hypothesis (8.1).

To finish the proof, we show that the Γ-$\limsup$ inequality may be proved without assuming (8.1). Recall that $u \in \mathcal{X}$ is such that $\mathscr{G}_K(u;\Omega) < +\infty$ and $u = \chi_E - \chi_{\mathbb{R}^d\setminus E}$ in Ω, for some measurable $E \subset \mathbb{R}^d$.

We first apply Lemma 5.3 to obtain[8] a sequence of measurable sets $\{E_k\}_{k\in\mathbb{N}}$ that satisfy

(8.4)
$$\partial E_k \cap \Omega_{1/k} \text{ is smooth}, \quad E_k\setminus\Omega_{1/k} = E\setminus\Omega_{1/k}, \quad \lim_{k\to+\infty} |E_k\Delta E| = 0,$$

where, for any $t \geqslant 0$, we set $\Omega_t := \{x\in\mathbb{R}^d : \operatorname{dist}(x,\Omega)\leqslant t\}$, and

(8.5)
$$\lim_{k\to+\infty} \operatorname{Per}_K(E_k;\Omega) = \operatorname{Per}_K(E;\Omega).$$

Next, we consider a sequence $\{\varphi_k\}_{k\in\mathbb{R}^d} \subset C^0(\mathbb{R}^d\setminus\Omega)$ such that $\varphi_k \to u$ a.e. in $\mathbb{R}^d\setminus\Omega$, as $k \to +\infty$. Note that, to obtain such approximating sequence, one may argue as follows. Fix $N \in \mathbb{N}$ in such a way that $\Omega_1 \subset B_N$. Set $F_0 := B_N\setminus\Omega$ and $F_j := B_{N+j}\setminus B_{N+j-1}$, if $j \in \mathbb{N}$. For any fixed $j \in \mathbb{N}\cup\{0\}$, we can find a sequence of functions $\{\varphi_k^{(j)}\}_{k\in\mathbb{N}} \subset C_0^\infty(F_j)$ such that $\varphi_k^{(j)} \to u$ in $L^1(F_j)$, as $k \to +\infty$. We then define

$$\varphi_k(x) := \sum_{j=0}^{+\infty} \chi_{F_j}(x)\varphi_k^{(j)}(x) \quad \text{for any } x\in\mathbb{R}^d\setminus\Omega.$$

Up to a subsequence, the sequence $\{\varphi_k\}$ has the desired convergence properties.

For any $x\in\mathbb{R}^d$, we define

$$u^{(k)}(x) := \begin{cases} \chi_{E_k}(x) - \chi_{\mathbb{R}^d\setminus E_k}(x) & \text{if } x\in\Omega_{1/k}, \\ \varphi_k & \text{if } x\in\mathbb{R}^d\setminus\Omega_{1/k}. \end{cases}$$

Observe that

$$u^{(k)} \to u \text{ a.e. in } \mathbb{R}^d \quad\text{and}\quad \mathscr{K}_K(u^{(k)};\Omega) \to \mathscr{K}_K(u;\Omega), \quad \text{as } k\to+\infty.$$

[8] To be extremely precise, Lemma 5.3 gives a sequence of sets $\{\widetilde{E}_k\}_{k\in\mathbb{N}}$ with smooth boundaries such that

$$\left|[(\widetilde{E}_k\cap\Omega)\cup(E\setminus\Omega)]\Delta E\right| \to 0 \quad\text{and}\quad \operatorname{Per}_K\Big((\widetilde{E}_k\cap\Omega)\cup(E\setminus\Omega);\Omega\Big) \to \operatorname{Per}_K(E;\Omega), \quad \text{as } k\to+\infty.$$

Then, it is not hard to check that the sets $E_k := (\widetilde{E}_k\cap\Omega_{1/k})\cup(E\setminus\Omega_{1/k})$ fulfill (8.4) and (8.5).

These facts are true thanks to (8.4), (8.5), the definition of $u^{(k)}$ and an application of the Lebesgue's Dominated Convergence Theorem together with Lemma 4.3.

Moreover, each $u^{(k)}$ satisfies assumption (8.1). Hence, for any $\varepsilon > 0$ we deduce the existence of $u_\varepsilon^{(k)} \in \mathcal{X}_\varepsilon$ such that $u_\varepsilon^{(k)} \to u^{(k)}$ a.e. in $\mathbb{R}^d$ and $\mathscr{K}_K(u_\varepsilon^{(k)};\Omega) \to \mathscr{K}_K(u^{(k)};\Omega)$, as $\varepsilon \to 0^+$. More precisely, we can find a strictly decreasing, infinitesimal sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive numbers such that

$$d_{L^1_{\rm loc}}(u_\varepsilon^{(k)}, u^{(k)}) + \left|\mathscr{K}_K(u_\varepsilon^{(k)};\Omega) - \mathscr{K}_K(u^{(k)};\Omega)\right| < \frac{1}{k} \quad \text{for any } \varepsilon \in (0,\varepsilon_k],\ k \in \mathbb{N}, \tag{8.6}$$

where $d_{L^1_{\rm loc}}$ is some metric on $L^1_{\rm loc}(\mathbb{R}^d)$ inducing the standard $L^1_{\rm loc}$ topology, e.g.

$$d_{L^1_{\rm loc}}(v,w) := \sum_{j=1}^{+\infty} \frac{1}{2^j} \frac{\|v-w\|_{L^1(B_j)}}{1+\|v-w\|_{L^1(B_j)}} \quad \text{for any } v,w \in L^1_{\rm loc}(\mathbb{R}^d).$$

For $\varepsilon \in (0,\varepsilon_1]$, we set

$$u_\varepsilon := u_\varepsilon^{(k)} \quad \text{where } k \in \mathbb{N} \text{ is the only integer for which } \varepsilon_{k+1} < \varepsilon \leqslant \varepsilon_k.$$

Clearly, $u_\varepsilon \in \mathcal{X}_\varepsilon$. Moreover, $u_\varepsilon \to u$ in $L^1_{\rm loc}(\mathbb{R}^d)$ and $\mathscr{K}_K(u_\varepsilon;\Omega) \to \mathscr{K}_K(u;\Omega)$, as $\varepsilon \to 0^+$. Indeed, given any $\delta > 0$, we may select $k = k_\delta \in \mathbb{N}$ large enough to have

$$d_{L^1_{\rm loc}}(u^{(j)}, u) < \frac{\delta}{2}, \quad |\mathscr{K}_K(u^{(j)};\Omega) - \mathscr{K}_K(u;\Omega)| < \frac{\delta}{2} \quad \text{and} \quad \frac{1}{j} < \frac{\delta}{2} \quad \text{for any } j \geqslant k. \tag{8.7}$$

Let now $\varepsilon \leqslant \varepsilon_k$ and select the only integer $j \geqslant k$ for which $\varepsilon \in (\varepsilon_{j+1}, \varepsilon_j]$. By combining (8.7) with (8.6), we conclude that

$$d_{L^1_{\rm loc}}(u_\varepsilon, u) = d_{L^1_{\rm loc}}(u_\varepsilon^{(j)}, u) \leqslant d_{L^1_{\rm loc}}(u_\varepsilon^{(j)}, u^{(j)}) + d_{L^1_{\rm loc}}(u^{(j)}, u) < \frac{1}{j} + \frac{\delta}{2} < \delta,$$

and, analogously,

$$|\mathscr{K}_K(u_\varepsilon;\Omega) - \mathscr{K}_K(u;\Omega)| \leqslant |\mathscr{K}_K(u_\varepsilon^{(j)};\Omega) - \mathscr{K}_K(u^{(j)};\Omega)| + |\mathscr{K}_K(u^{(j)};\Omega) - \mathscr{K}_K(u;\Omega)| < \delta.$$

This concludes the proof of the Γ-$\limsup$ inequality and, hence, of Theorem 1.9.

Appendix A. Proof of Lemma 5.3

In the present appendix, we provide a proof of Lemma 5.3 in full details. As mentioned right after its statement in Section 5, our argument is based on the strategies already followed in e.g. [G84, CSV16, L16].

Throughout the section, we implicitly suppose conditions (1.15) and (1.16) to be in force. Although the result may in fact hold under weaker hypotheses, we always suppose for simplicity that K satisfies both these assumptions. However, we stress that none of the steps of the proof require the periodicity hypothesis (1.17) to be valid, that we therefore do not suppose to hold.

After these introductory remarks, we may now head to the proof of Lemma 5.3.

Proof of Lemma 5.3. First, notice that, by (1.16) and the fact that F has finite K-perimeter, the characteristic function χ_F belongs to the fractional Sobolev space $W^{s,1}(\Omega)$. Hence, by standard density results (see e.g. [G85, Theorem 1.4.2.1]), there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset W^{s,1}(\Omega) \cap C^\infty(\overline{\Omega})$ such that

$$\varphi_n \to \chi_F \text{ in } W^{s,1}(\Omega), \text{ as } n \to +\infty. \tag{A.1}$$

By using again (1.16), this ensures that

(A.2) $$\lim_{n\to+\infty} \mathscr{K}_K(\varphi_n;\Omega,\Omega) = \mathscr{K}_K(\chi_F;\Omega,\Omega).$$

For $t \in (0,1)$, we let

$$F_n := \big(\{\varphi_n > t\} \cap \overline{\Omega}\big) \cup (F \setminus \overline{\Omega}).$$

Clearly, $F_n \setminus \overline{\Omega} = F \setminus \overline{\Omega}$, which proves (5.9).

Also, Morse-Sard's Theorem tells that, for a.e. $t \in (0,1)$, the boundary of the level set $\{\varphi_n > t\}$ is a smooth hypersurface. Hence ∂F_n is smooth inside $\overline{\Omega}$, which gives (5.8).

We now claim that for a.e. $t \in (0,1)$ fixed,

(A.3) $$\lim_{n\to+\infty} |F_n \Delta F| = 0,$$

and

(A.4) $$\lim_{n\to+\infty} \mathrm{Per}_K(F_n;\Omega) = \mathrm{Per}_K(F;\Omega),$$

up to a subsequence, that is (5.10) and (5.11), respectively.

We begin by checking (A.3). Let $\tau \in (0,1)$ and notice that

$$\begin{aligned} &\varphi_n - \chi_F > \tau && \text{in } (\{\varphi_n > \tau\} \setminus F) \cap \Omega \\ \text{and} \quad &\chi_F - \varphi_n \geqslant 1 - \tau && \text{in } (F \setminus \{\varphi_n > \tau\}) \cap \Omega. \end{aligned}$$

From this, we deduce that

$$\begin{aligned} \|\varphi_n - \chi_F\|_{L^1(\Omega)} &\geqslant \int_{(\{\varphi_n>\tau\}\setminus F)\cap\Omega} (\varphi_n(x) - \chi_F(x))\,dx + \int_{(F\setminus\{\varphi_n>\tau\})\cap\Omega} (\chi_F(x) - \varphi_n(x))\,dx \\ &\geqslant \tau|(\{\varphi_n > \tau\} \setminus F) \cap \Omega| + (1-\tau)|(F \setminus \{\varphi_n > \tau\}) \cap \Omega| \\ &\geqslant \min\{\tau, 1-\tau\}\, |(\{\varphi_n > \tau\}\Delta F) \cap \Omega|. \end{aligned}$$

Therefore, using this and (A.1),

(A.5) $$\{\varphi_n > \tau\} \longrightarrow F \quad \text{in } L^1(\Omega), \text{ for a.e. } \tau \in (0,1).$$

Claim (A.3) follows as a particular case by taking $\tau = t$ in formula (A.5) above and recalling that $F_n \setminus \overline{\Omega} = F \setminus \overline{\Omega}$.

Next, we address the convergence of the perimeters stated in (A.4). Thanks to (A.5) and Lemma 4.1, we have

$$\mathcal{L}_K(F \cap \Omega, \Omega \setminus F) \leqslant \liminf_{n\to+\infty} \mathcal{L}_K(\{\varphi_n > \tau\} \cap \Omega, \Omega \setminus \{\varphi_n > \tau\}) \quad \text{for a.e. } \tau \in (0,1),$$

or, equivalently,

(A.6) $$\mathscr{K}_K(\chi_F;\Omega,\Omega) \leqslant \liminf_{n\to+\infty} \mathscr{K}_K(\chi_{\{\varphi_n>\tau\}};\Omega,\Omega) \quad \text{for a.e. } \tau \in (0,1).$$

By applying, in sequence, (A.2), the generalized Coarea Formula of Lemma 4.2, Fatou's Lemma and (A.6), we compute

$$\begin{aligned} \mathscr{K}_K(\chi_F;\Omega,\Omega) &= \lim_{n\to+\infty} \mathscr{K}_K(\varphi_n;\Omega,\Omega) = \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \mathscr{K}_K(\chi_{\{\varphi_n>\tau\}};\Omega,\Omega)\,d\tau \\ &\geqslant \int_0^1 \liminf_{n\to+\infty} \mathscr{K}_K(\chi_{\{\varphi_n>\tau\}};\Omega,\Omega)\,d\tau \geqslant \int_0^1 \mathscr{K}_K(\chi_F;\Omega,\Omega)\,d\tau = \mathscr{K}_K(\chi_F;\Omega,\Omega). \end{aligned}$$

By this and, again, (A.6) we conclude that

$$\liminf_{n\to+\infty} \mathscr{K}_K(\chi_{\{\varphi_n>\tau\}};\Omega,\Omega) = \mathscr{K}_K(\chi_F;\Omega,\Omega) \quad \text{for a.e. } \tau \in (0,1),$$

and thence

$$\lim_{n\to+\infty} \mathcal{L}_K(F_n\cap\Omega,\Omega\setminus F_n)=\mathcal{L}_K(F\cap\Omega,\Omega\setminus F). \tag{A.7}$$

On the other hand, we claim that

$$\begin{aligned} &\lim_{n\to+\infty} \mathcal{L}_K(F_n\setminus\Omega,\Omega\setminus F_n)=\mathcal{L}_K(F\setminus\Omega,\Omega\setminus F)\\ \text{and}\quad &\lim_{n\to+\infty} \mathcal{L}_K(F_n\cap\Omega,\mathbb{R}^d\setminus(F_n\cup\Omega))=\mathcal{L}_K(F\cap\Omega,\mathbb{R}^d\setminus(F\cup\Omega)), \end{aligned} \tag{A.8}$$

up to subsequences. To check the validity of (A.8), we first notice that, by (A.3), $\chi_{F_n}\longrightarrow\chi_F$ a.e. in $\mathbb{R}^d$ (up to extracting a subsequence), as $n\longrightarrow+\infty$. Therefore, in view of Lemma 4.3 we may apply the Lebesgue's Dominated Convergence Theorem to get

$$\begin{aligned} \lim_{n\to+\infty} \mathcal{L}_K(F_n\setminus\Omega,\Omega\setminus F_n) &= \lim_{n\to+\infty}\int_\Omega \chi_{\mathbb{R}^d\setminus F_n}(x)\left(\int_{\mathbb{R}^d\setminus\Omega}\chi_{F_n}(y)K(x,y)\,dy\right)dx\\ &=\int_\Omega \chi_{\mathbb{R}^d\setminus F}(x)\left(\int_{\mathbb{R}^d\setminus\Omega}\chi_F(y)K(x,y)\,dy\right)dx\\ &=\mathcal{L}_K(F\setminus\Omega,\Omega\setminus F), \end{aligned}$$

and similarly for the limit on the second line of (A.8). The combination of (A.7) and (A.8) yields the convergence of the K-perimeters claimed in (A.4).

The proof of Lemma 5.3 is thus finished. □

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