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On indecomposable polyhedra and the number of interior Steiner points

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Abstract

The existence of 3d *indecomposable polyhedra*, that is, the interior of every such polyhedron cannot be decomposed into a set of tetrahedra whose vertices are all of the given polyhedron, is well-known. While the geometry and combinatorial structure of such polyhedra are much less studied. In this article, we first investigate the geometry of some well-known examples, the so-called *Schönhardt polyhedron* [Schönhardt, 1928] and the Bagemihl's generalization of it [Bagemihl, 1948], which will be called *Bagemihl polyhedra*. We provide a construction of an interior point, so-called *Steiner point*, which can be used to tetrahedralize the Schönhardt and the Bagemihl polyhedra. We then provide a construction of a larger class of three-dimensional indecomposable polyhedra which often appear in grid generation problems. We show that such polyhedra have the same combinatorial structure as the Schönhardt and Bagemihl polyhedra, but they may need more than one interior Steiner point to be tetrahedralized. Given such a polyhedron with $n \geq 6$ vertices, we show that it can be tetrahedralized by adding at most $\left\lceil \frac{n-5}{2} \right\rceil$ interior Steiner points. We also show that this number is optimal in the worst case.

1 Introduction

The existence of three-dimensional (non-convex) polyhedra whose interior cannot be decomposed into a set of non-overlapping tetrahedra without new vertices has long been observed [Lennes, 1911]. In 1928, Schönhardt provided the simplest example, which is a twisted triangular prism with six vertices. It is now well-known as the *Schönhardt polyhedron* [Schönhardt, 1928]. Later, further such non-convex, non-tetrahedralizable polyhedron with an arbitrary number of vertices have been presented, see [Bagemihl, 1948, Chazelle, 1984, Rambau, 2005]. Among them, Bagemihl's construction [Bagemihl, 1948] is a direct generalization from the Schönhardt's construction.

The existence of indecomposable polyhedra is a major difficulty in many geometric and combinatorial problems. For example, Below shows that the complexity of finding minimal or maximal subdivisions of three-dimensional polyhedra are NP-hard [Below, 2002]. Rupert and Seidel show that to determine whether a given three-dimensional polyhedron can be decomposed or not is NP-complete [Ruppert and Seidel, 1992]. The Schönhardt polyhedron appears as an important example in the study of the flip-graph of all triangulations of a given point set [De Loera et al., 2010]. In 3d tetrahedral mesh generation, indecomposable polyhedra are the main obstacles in the design and proof of several key algorithms, such as the recovery of a non-existing edge and the deletion of an existing vertex, see e.g. [George et al., 1991, Weatherill and Hassan, 1994, George et al., 2003, Si, 2015].

When tetrahedralizing an indecomposable polyhedron, it is necessary to add additional points, so-called *Steiner points* [1]. It is easy to see that the Schönhardt polyhedron needs only one Steiner point. However, there are indecomposable polyhedra which may need many Steiner points, see an excellent exposition of this topic from Eppstein's "The Geometry Junkyard" [2], where two important examples, namely the Thurston polyhedron [Paterson and Yao, 1990] and Chazelle polyhedron [Chazelle, 1984], are demonstrated. In particular, Chazelle polyhedron needs as many as $\Omega(n^2)$ convex polyhedra to be decomposed. This implies that it needs many Steiner points. We can treat the Schönhardt polyhedron and Chazelle polyhedron as two extreme cases for the number of Steiner points. Surprisingly enough, very few work about the number of Steiner points is known. The **general question** is: *How many Steiner points are necessary to decompose a 3d indecomposable polyhedron?*

Note that in the general question, the locations of the Steiner points are allowed either on the boundary or in the interior of the given polyhedron. We are particularly interested in the latter type of Steiner points, i.e., they lie strictly in the interior of the polyhedron. This is mainly motivated by an important application from finite element mesh generation [George et al., 1991, Weatherill and Hassan, 1994, George et al., 2003, Si, 2015], in which the input (discretized) boundary of the mesh domain must be entirely preserved. This requirement comes from various reasons, e.g., to assign boundary conditions, to preserve the original geometry information, to mesh subdomains separately, to partition the mesh for parallel generation, etc. Therefore, the **main question** we are interested in is:

> *How many interior Steiner points are necessary to decompose a 3d indecomposable polyhedron?*

In order to answer this question, it is necessary to understand the geometry and the combinatorial structure of the given polyhedron.

[1] There exist several types of Steiner points, named after Jakob Steiner (1796 – 1863), a Swiss mathematician who worked primarily in geometry, in the literatures, like the Steiner points in the Steiner tree problem, see e.g. `http://en.wikipedia.org/wiki/Steiner_tree_problem`, and the Steiner point in a triangle, see e.g. `http://en.wikipedia.org/wiki/Steiner_point_(triangle)`. The Steiner points in this article are constructed for decomposing Schönhardt or other indecomposable polyhedra. They are different to the previous ones.

[2] David Eppstein, Three Untetrahedralizable Objects, `https://www.ics.uci.edu/~eppstein/junkyard/untetra/`

Comment. This question may be very different to the general question stated above. Note that Chazelle's $\Omega(n^2)$ lower bound is obtained assuming that we can add Steiner points on the boundary of a polyhedron. Therefore it does not apply in our main question.

1.1 Related Work

The general question has been studied in the context of convex decomposition. Most of work are inspired by the Chazelle polyhedron [Chazelle, 1984]. Erickson showed that a special class of polyhedra, so-called *locally polyhedra* [Erickson, 2005], can be decomposed into $O(n \log n)$ tetrahedra and this bound is tight. De Berg and Gray consider a different class of polyhedra, so-called *fat polyhedra* [de Berg and Gray, 2010], and showed that any *locally-fat* polyhedron with convex fat faces can be decomposed into $O(n)$ tetrahedra. Both locally polyhedra and fat polyhedra may be non-convex. However, it is not necessary that either a 3d locally polyhedron or locally-fat polyhedra is indecomposable, i.e., they may not need interior Steiner point to be tetrahedralized.

Another related work appears in the context of conforming Delaunay mesh generation [Edelsbrunner and Tan, 1993, **?**, **?**], where a Delaunay triangulation is constructed to decompose a space described by a piecewise linear complex (PLC) [**?**]. Edelsbrunner and Tan showed an upper bound $O(n^3)$ number of Steiner points for triangulating any 2d PLC [Edelsbrunner and Tan, 1993]. An upper bound for obtaining 3d conforming Delaunay tetrahedralization is still widely open.

The above work only consider the general question, i.e., the Steiner points can locate every where on the boundary. The problem of adding interior Steiner points is long addressed in the context of mesh generation for finite element applications [**?**, Frey and George, 2000], where a set of constraints (edges and faces) must be entirely preserved in the generated mesh. There are many work on how to add interior Steiner points, refer to the work and reference in [George et al., 1991, Weatherill and Hassan, 1994, George et al., 2003, Si, 2015]. However, the number of interior Steiner points is still an open question.

Finally, we comment that there is other interesting work on indecomposable polyhedra in the sense of Minkowsi sums of two convex polytopes, see e.g. [**?**, Yost, 2007]. But the meaning of this indecomposibility is very different to ours.

1.2 Outline

This paper is devoted to the main question stated above. It is organized as follows.

In Section 2, we first study this question for the class of polyhedra constructed by Bagemihl [Bagemihl, 1948], which is a generalisation of the Schönhardt polyhedron. Hereafter we will call them *Bagemihl polyhedra* (described in Section 2.1). A Bagemihl polyhedron can have an arbitrary number n of vertices, where $n \geq 6$. Due to its geometrical properties, we show that a Bagemihl polyhedron needs only one interior Steiner point to be decomposed. Our proof is based on a construction of a Steiner point in the interior of a given Bagemihl polyhedron, and we proof that all boundary faces of this polyhedron are visible by this Steiner point (in Section 2.2). Furthermore, we show that our constructed Steiner point is also valid for a class of *generalized Bagemihl polyhedra* which can be obtained by relaxing the symmetric and height requirements in Bagemihl's construction (in Section 2.3).

Next, in Section 3, we first extend Bagemihl's Theorem to show that there exists a larger class of 3d indecomposable polyhedra (in Section 3.1). We then give a general construction of such polyhedra with $n \geq 6$ vertices, denoted as σ_n. It is worth to mention, that the polyhedra from our construction are commonly encountered during the process of tetrahedral mesh generation. We show that our constructed polyhedra have the same combinatorial structure as the Schönhardt and Bagemihl polyhedra, but they may need more than one interior Steiner point to be decomposed (in Section 3.2). We then prove the following main result regarding the number of interior Steiner points for our constructed polyhedra (in Section 3.3):

> *Given a polyhedron σ_n that satisfies our construction, where n is the number of vertices of σ_n, it needs at most $\lceil \frac{n-5}{2} \rceil$ interior Steiner points to be decomposed.*

Our proof of this result is based on a construction of Steiner points in the interior of such polyhedra so that it can be decomposed into a set of tetrahedra. This construction also provides hints to design efficient algorithms to tetrahedralize such polyhedra.

Finally, some closing remarks and open questions are given in Section 4.

2 Bagemihl Polyhedra and a Construction of an Interior Steiner Point

In the Paper "On Indecomposable Polyhedra" by F. Bagemihl ([Bagemihl, 1948]) he proves the following theorem.

Theorem 1 ([Bagemihl, 1948]). *If n is an integer not less than 6, then there exists a polyhedron, π_n, with n vertices and the following properties:*

(I) π_n is simple and every one of its faces is a triangle.

(II) If τ is a tetrahedron, each of whose vertices is a vertex of π_n, then not every interior point of τ is an interior point of π_n.

(III) Every open segment whose endpoints are vertices of π_n, but which is not an edge of π_n, lies wholly exterior to π_n.

(IV) Every triangle whose sides are edges of π_n is a face of π_n.

Comments In the original version of Theorem 1 [Bagemihl, 1948], Bagemihl only stated the first three properties (I) - (III). However, he gave a construction of a class of polyhedra which also fulfills the property (IV). Note that by including the property (IV), we might decrease the size of the original class of polyhedra. But this is out of the question of this article. With the included property (IV), the property (II) becomes redundant. It is followed together from (I), (III) and (IV).

Note that the property (II) indicates that π_n is indecomposable, since no tetrahedron τ whose vertices are from π_n in the interior of π_n exists. The key fact is that τ must contain at least one open segment of π_n. Suppose the four edges of τ are all not open segments, then, by (IV), the boundary of π_n must form a tetrahedron. Since π_n is simple by (I), we can conclude that π_n is a tetrahedron. But this is contradict to the assumption $n \geq 6$. Then by (III), τ does not lie the interior of π_n.

Bagemihl provides a construction of a class of polyhedra that satisfy Theorem 1, which will be called *Bagemihl polyhedra*.

In this section, we will first review the construction of Bagemihl polyhedra. Our goal is to show that a Bagemihl polyhedron needs only one interior Steiner point to be decomposed. For this purpose, we first give a construction of a Steiner point in a given Bagemihl polyhedron. We then proof it is valid for the decomposition. We further proof that our construction of a Steiner point is also valid for a variation of Bagemihl polyhedra by relaxing the symmetry and edge length requirements in the original Bagemihl's construction.

2.1 Description of Bagemihl Polyhedra

Bagemihl's construction starts with the Schönhardt polyhedron π_6 which we will describe first. Take an equilateral triangle with edge lengths 1 and vertices A_1, B_1, C_1. Take a copy of it and lift it up orthogonally to the height $h = 1$ and rotate it around the axis connecting the centers of the top and bottom triangle by an angle of $\vartheta = 30°$. Call the so obtained vertices in the top triangle A_2, B_2, C_2, respectively. By connecting the vertices as shown in Figure 1 we obtain the polyhedron π_6.

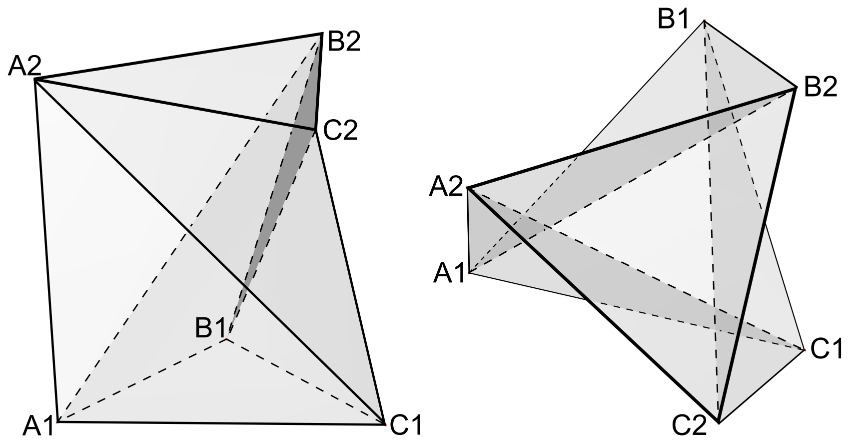

Figure 1: The Schönhardt polyhedron, which is the Bagemihl polyhedron π_6. A side view (left) and a top view (right) are shown.

In the case $n > 6$, Bagemihl adds an open circular arc $\widehat{A_1A_2}$ connecting A_1 and A_2 in the interior of π_6. The radius of this arc is chosen to be large enough such that every point of $\widehat{A_1A_2}$ is on the same side of the plane $C_1A_2C_2$ as A_1, and on the same side of the plane $B_1A_1B_2$ as A_2. On the arc $\widehat{A_1A_2}$ one can choose $k = n - 6$ distinct points, $D_1, D_2, \ldots, D_k$, in the order $A_1D_1D_2 \ldots D_{k-1}D_kA_2$

and add the edges $A_1D_1, D_1D_2, \ldots, D_{k-1}D_k, D_kA_2$ connecting the vertices. An example of a π_9 is shown in Figure 2.

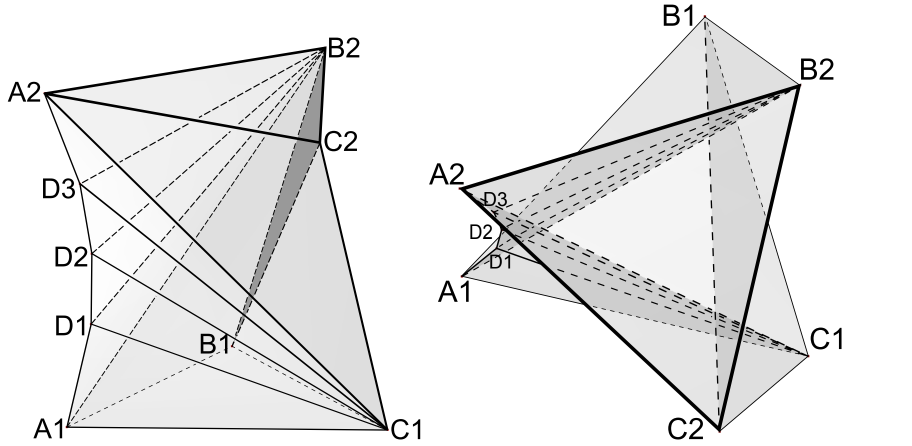

Figure 2: A Bagemihl polyhedron π_9. A side view (left) and a top view (right) are shown.

2.2 Construction of an Interior Steiner Point

Given a Bagemihl polyhedron, π_n with $n \geq 6$ vertices, it is clear that at least one Steiner point is needed. But it is not obvious how many Steiner points are necessary. We want to show that one interior Steiner point is already sufficient. In this section, we give a construction of such a Steiner point.

We first consider π_6, which is just the Schönhardt polyhedron. The region in which we can place a Steiner point is the intersection of the eight halfspaces defined by the boundary triangles of the Schönhardt polyhedron, see Figure 3.

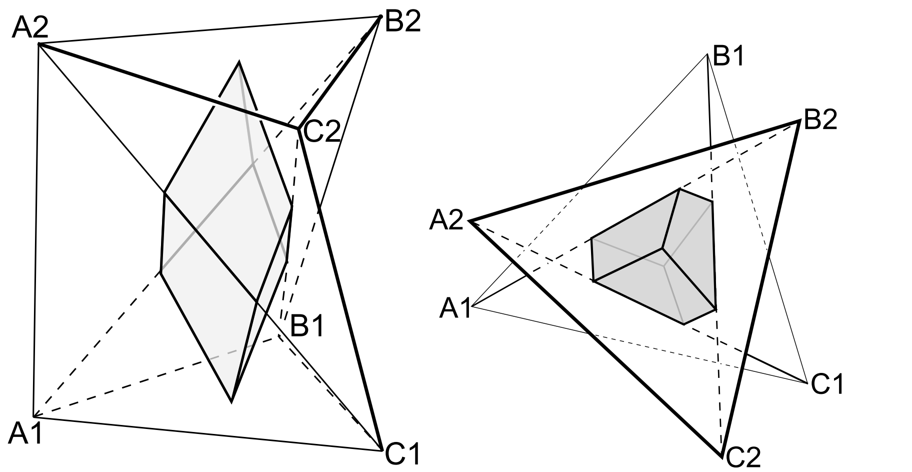

Figure 3: The (open) valid domain for placing Steiner points inside the Schönhardt polyhedron. A side view (left) and a top view (right) are shown.

Placing the points $D_1, \ldots, D_k$, where $k = n - 6$, in the interior of π_6 just as described by Bagemihl, the region for a Steiner point S in π_n is getting smaller. We want to show that it is not empty for all $k \in \mathbb{N}_{\geq 0}$ and all choices of additional points D_i. For this purpose we will show that it is always possible to place a Steiner point in the interior that has the required visibility properties.

We first determine the domain in which the valid arcs as described by Bagemihl can live. First, every arc has to lie inside π_6, so it is restricted by the halfspaces limited by the faces $A_1A_2C_1$ and $A_1A_2B_2$, respectively. Since every point of an open arc $\widehat{A_1A_2}$ has to be on the same side of the plane $C_1A_2C_2$ as A_1, and on the same side of the plane $B_1A_1B_2$ as A_2 [Bagemihl, 1948, p. 413], it is restricted by these two faces of π_6 as well. So, this domain is the intersection of four half spaces, which is a tetrahedron (see Figure 4), denoted as T, with vertices $A_1A_2G_1G_2$, where G_1 and G_2 are defined by

$$G_1 := \text{plane}_{C_1A_2C_2} \cap \text{line}_{A_1B_2}$$
$$G_2 := \text{plane}_{B_1A_1B_2} \cap \text{line}_{A_2C_1}.$$

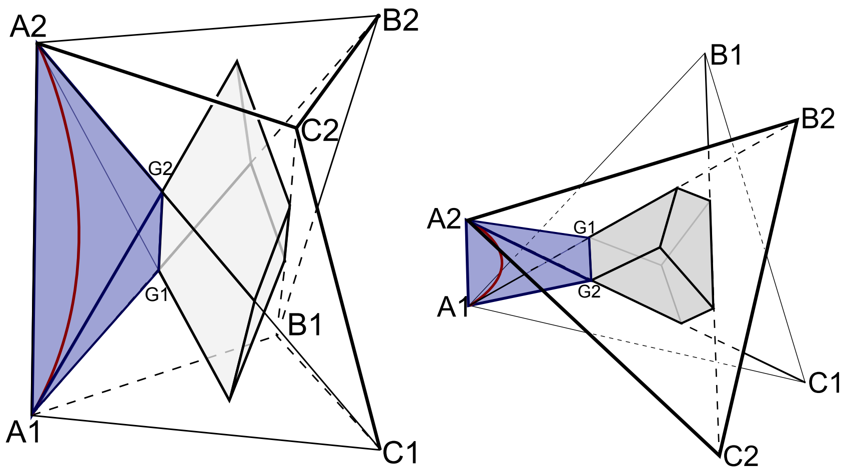

Figure 4: The tetrahedron $A_1A_2G_1G_2$ (blue) in which the arc in Bagemihl's construction can lie. An example of such an arc (red) is also shown. This arc cannot lie on a boundary face of the polyhedron.

Note that the points G_1 and G_2 are on the boundary of π_6: Since the polyhedron doesn't have interior edges, the points A_1 and B_2 will lie on different sides of the plane through $C_1A_2C_2$, so G_1 has to be on the edge A_1B_2 itself. Analogously G_2 has to be on the edge A_2C_1.

With the help of this tetrahedron T, we can determine the locations for the intersections of the tangent lines of all possible arcs at A_1 and A_2. Given a possible arc $\widehat{A_1A_2}$ (as defined by Bagemihl) in the tetrahedron T, define the intersection point of the tangent lines through A_1 and A_2 to the arc $\widehat{A_1A_2}$ as P (refer to Figure 5). P must lie in the closure of T. If the radius of the circle (containing the arc) is getting larger, the distance between P and the line through A_1 and A_2 will decrease.

It is enough to consider the extremal case, i.e. the case when the radius of the circle is the smallest possible.

Consider the points G_1 and G_2. By definition, they can see all vertices of π_n or are coplanar with a face containing them. The open segment G_1G_2 is obtained by first intersecting the two planes through $C_1A_2C_2$ and $B_1A_1B_2$, then by intersecting the interior of π_6. All points in the interior of the open segment G_1G_2 can see all interior points of π_n, but we cannot take a point on this segment as a Steiner point because of the coplanarity with some faces, however we are already close to it.

Now construct the Steiner point S as follows. Intersect the plane p_1 containing the arc $\widehat{A_1A_2}$ with the line passing through G_1 and G_2. We obtain a point $\tilde{S}$ in the interior of π_n. Now construct a plane p_2 which is orthogonal to the line passing through A_1 and A_2 and containing $\tilde{S}$. Intersect the planes p_1 and p_2. We obtain a line which we will call l. By construction $\tilde{S} \in l$. Now take the point $\tilde{S}$ and move it a little on the line l away from $\widehat{A_1A_2}$ but only so far, that it is still before the edge connecting B_1 with C_2. The so obtained point is our Steiner point S, refer to Figure 5.

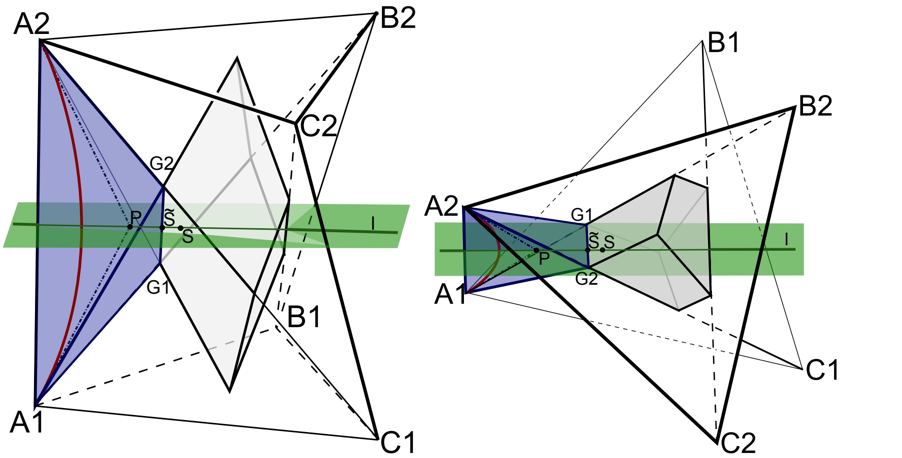

Figure 5: The construction of a Steiner point S in π_n. The plane p_1 (not shown) containing the arc $\widehat{A_1A_2}$ (red). The plane p_2 (green) is orthogonal to the line passing A_1A_2. The line $l := p_1 \cap p_2$ is shown as well as the points $P, \tilde{S}, S \in l$.

Proposition 2. *A Bagemihl polyhedron π_n with n vertices together with the so constructed Steiner point $S \in \pi_n$ from above can be tetrahedralized.*

Proof. By our construction, S lies beyond P. Recall that P is the intersection point of the tangent lines of the arc at A_1 and A_2, see Figure 6. By that, the vertices $A_1, D_1, \ldots, D_k, A_2$ are visible by S (from the interior of π_n).

The visibility of the remaining vertices C_1, C_2, B_1 and B_2 is given, since S is chosen by moving $\tilde{S}$ to the inside of the visible polytope of π_n. The point $\tilde{S}$ lies on the boundary of the visible domain of the polytope π_n (and not only the polytope π_6). This polytope is as well bounded by the edge B_1C_2, so we can be sure that there is space left in the interior. $\square$

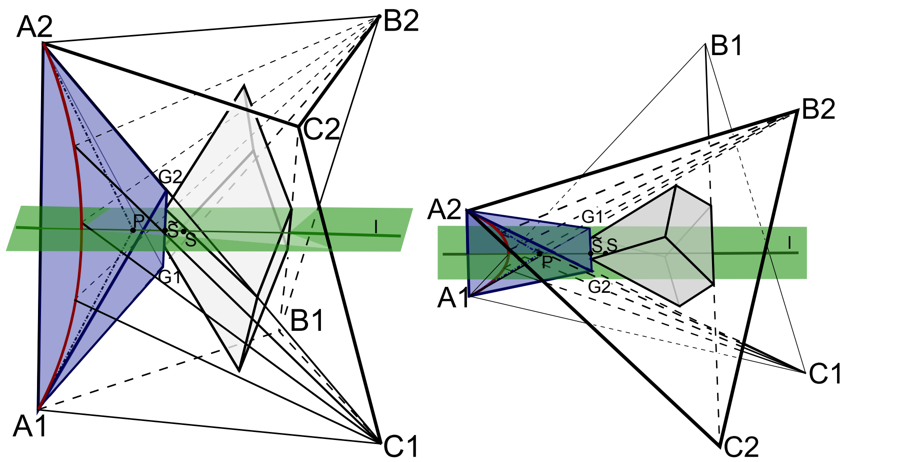

Figure 6: An example of the constructed Steiner point $S \in \pi_9$. The smaller polyhedron in the interior is the valid domain for π_9.

2.3 Generalized Bagemihl Polyhedra

Bagemihl gave the construction of a special class of polyhedra without leaving much space of freedom. The only choice one has is the one of the arc lying inside the original Schönhardt polyhedron which even has to fulfill some visibility constraints and the points one chooses on the arc. We can generalize these polyhedra without loosing their main properties, i.e. they will still fulfill the properties (I)-(IV) of Theorem 1.

A nearby way to generalize them is to let the rotation angle ϑ of the triangles be in $\vartheta \in (0°, 60°)$ instead of fixing it to a value of $\vartheta = 30°$. Another way is to let the height h of the polyhedron be arbitrary in $h \in \mathbb{R}_{>0}$ instead of fixing it to the value of $h = 1$.

One can even change the bottom and top triangle itself. It is not necessary that they are parallel, are equilateral or have the same size. One can construct a generalized form of Bagemihl polyhedra based on two triangles in space connecting the vertices like in the original case. As long as they fulfill the properties of his Theorem and the open circular arc has the same visibility properties as in the original case, we will call them *generalized Bagemihl polyhedra*. See Figure 7 for an example.

These polyhedra are still not decomposable, so at least one Steiner point is needed. But the construction of a Steiner point given above can be adapted to this class of polyhedra, so we can state the following Corollary. In Figure 8 the visible polytope of the example polyhedron from Figure 7 is shown.

Corollary 3. *For a tetrahedralization of $\tilde{\pi}_n$ with $n \in \mathbb{N}_{\geq 6}$ one needs exactly one Steiner point, where $\tilde{\pi}_n$ is a generalized Bagemihl polyhedron as described above.*

Proof. Since the property (II) mentioned in Theorem 1 is still fulfilled, at least one Steiner point is needed. On the other hand, one can use the construction as described in Section 2.2 to obtain a Steiner point S in $\tilde{\pi}_n$. Furthermore, the proof of Proposition 2 can be adapted to show that S is already sufficient to

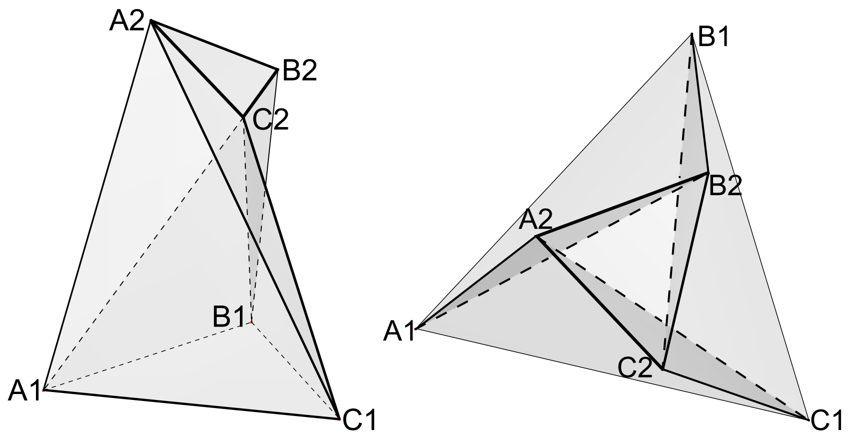

Figure 7: Example of a generalized Bagemihl polyhedron $\tilde{\pi}_6$. One still can choose a suitable arc to add more vertices.

tetrahedralize $\tilde{\pi}_n$. $\square$

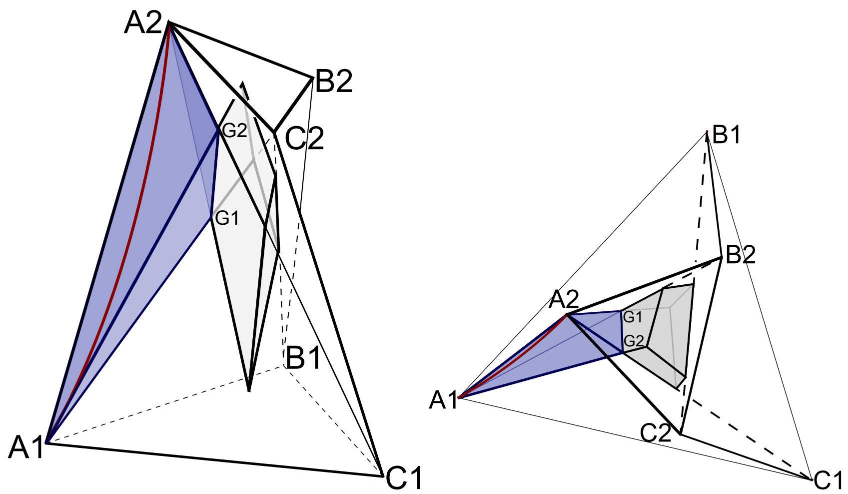

Figure 8: Visible polytope and the tetrahedron as the possible domain for arcs of a generalized Bagemihl polyhedron $\tilde{\pi}_6$.

2.4 About the Limiting Cases

In this section, we take a closer look on the polyhedra in the limiting cases of the constructions described above.

Let's begin with the cases of 6 vertices. As mentioned before, the rotation angle ϑ of the triangles has to be in $\vartheta \in (0°, 60°)$. At an angle of $\vartheta = 0°$ the faces are coplanar, which means that the polyhedron is a prism and so doesn't fulfill the properties (I) and (IV) of Theorem 1. However, with the chosen edges this prism is not decomposable, whereas it can be decomposed with a different choice of edges on the planar faces.

In the case $\vartheta = 60°$, the volume separates into two parts which are attached at a single point and the dihedral angles between the following pairs of faces are $0°$:

$$A_1A_2C_1 \text{ and } A_1A_2B_2,$$
$$B_1B_2A_1 \text{ and } B_1B_2C_2,$$
$$C_1C_2B_1 \text{ and } C_1C_2A_2.$$

So the polyhedron isn't simple any more and doesn't fulfill property (I) of Theorem 1 (see Figure 9). Furthermore it is not a 3d manifold and so it is not a valid polyhedron.

The question about decomposability doesn't have an obvious answer, since it is dependent on the definitions. By that it can happen that it is not decomposable, even though the volume already consists of two tetrahedra. Because the connecting point is not a part of the set of vertices we use to define the polyhedron with, we have to split the edges into two separate edges at this point.

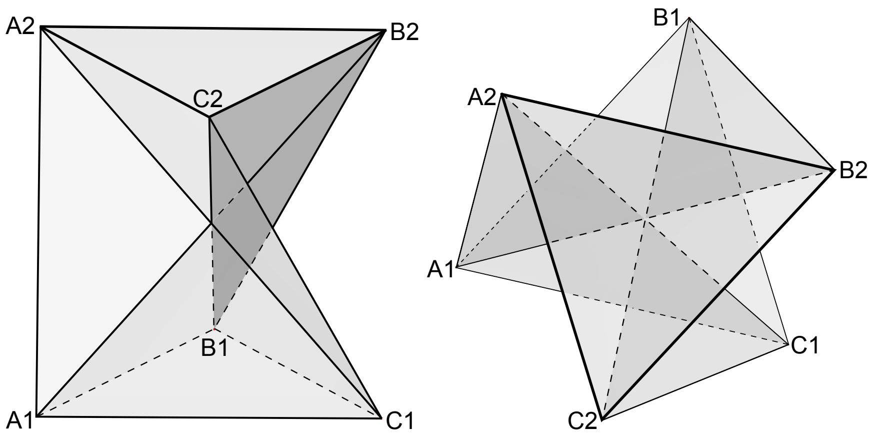

Figure 9: The Schönhardt polyhedron with a rotation angle of $\vartheta = 60°$. The edges A_1B_2 and A_2C_1, A_1B_2 and B_1C_2, and A_2C_1 and B_1C_2 are coplanar since the dihedral angles between some faces is $0°$. A side view (left) and a top view (right) are shown.

Now consider the limiting cases in Bagemihl's construction. These are the ones, in which the rotation angle is $\vartheta = 0°$ or $60°$ and the ones in which one chooses the additional points $D_1, \ldots, D_k$ on a straight line connecting A_1 and A_2 instead of lying on an arc or lying on a face of the tetrahedron $A_1A_2G_2G_1$ defined above.

First fix the angle to $\vartheta = 0°$, which means that the upper and lower triangle build a prism like in the case above. One has to add the addition points $D_1, \ldots, D_k$ on the line segment connecting A_1 and A_2. By that, the polyhedron stays indecomposable. However, this polyhedron becomes decomposable if we change the diagonal B_1C_2 to C_1B_2.

If the angle is $\vartheta = 60°$, the (non-valid) polyhedron separates like before into two parts whose volumes are connected only by a single point which is, or is not, a part of the vertex set. Let's call this point P. The arc containing the additional points has to lie in the triangle A_1PA_2, or in a limiting case, be the segment A_1A_2 itself. The polyhedron is only decomposable, when one splits all the edges crossing P and adds edges from P to all other vertices, as long as one allows tetrahedra with a volume of 0. Anyway, this case is not relevant for practical use.

Let $\vartheta \in (0°, 60°)$. When we choose the points $D_1, \ldots, D_k$ on a straight line, it is indecomposable even though properties (I) and (IV) from the Theorem are not satisfied. If the arc lies on a boundary facet of the tetrahedron $A_1A_2G_2G_1$ we are adding coplanar faces and it is still indecomposable.

3 A Larger Class of Indecomposable Polyhedra and The Number of Interior Steiner Points

In this section, we first extend Bagemihl's Theorem to show that there exists a larger class of indecomposable polyhedra. We then provide a different construction of one of such polyhedra. We show that our constructed polyhedra are combinatorially the same as the Schönhardt and Bagemihl polyhedra. But they may require more than one interior Steiner point to be decomposed. We then proof the maximum number of necessary Steiner points is $\lceil \frac{n-5}{2} \rceil$. Our proof is based on a construction of such Steiner points in the interior.

3.1 A Larger Class of Indecomposable Polyhedra

Note that Bagemihl polyhedra π_n all satisfy a crucial property, which is (III) in Theorem 1, i.e., *every open segment whose endpoints are vertices of π_n, but which is not an edge of σ_n, lies wholly exterior to π_n.* This property is sufficient but not necessary to guarantee that a polyhedron is indecomposable. By relaxing this property, we can obtain a larger class of indecomposable polyhedra. Their properties are given in the following Theorem. The only difference to Theorem 1 is the property (III), which is also highlighted.

Theorem 4. *If n is an integer not less than 6, then there exists a polyhedron, σ_n, with n vertices and the following properties:*

- *(I) σ_n is simple and every one of its faces is a triangle.*
- *(II) If τ is a tetrahedron, each of whose vertices is a vertex of σ_n, then not every interior point of τ is an interior point of σ_n.*
- *(III)* **Every open segment e, whose endpoints are vertices of σ_n, but which is not an edge of σ_n, satisfies interior$(e) \cap \sigma_n \not\subset$ interior(e).**
- *(IV) Every triangle whose sides are edges of σ_n is a face of σ_n.*

Comment The property (II) is redundant, since it can be derived from the other three properties. We keep it in order to keep the same form of Theorem 1.

The arguments that σ_n is an indecomposable polyhedron are exactly the same as those given by Bagemihl (see Section 2). The key fact is that every

(1)	$(\mathbf{a}, \mathbf{c}, \mathbf{d}), (\mathbf{b}, \mathbf{c}, \mathbf{d})$
(2)	$(\mathbf{a}, \mathbf{c}, \mathbf{g}_0), (\mathbf{b}, \mathbf{c}, \mathbf{g_0}), (\mathbf{a}, \mathbf{d}, \mathbf{g}_{k+1}), (\mathbf{b}, \mathbf{d}, \mathbf{g}_{k+1})$
(3)	$(\mathbf{a}, \mathbf{g}_i, \mathbf{g}_{i+1}), (\mathbf{b}, \mathbf{g}_i, \mathbf{g}_{i+1})$, where $i = 0, \ldots, k$

Table 1: The set $\{(1), (2), (3)\}$ of boundary faces of σ_n.

	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$	$\mathbf{d}$	$\mathbf{g}_0$	$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3$	$\mathbf{g}_4$
x	-5	5	0	0	1	-2.5	2	-2.5	1
y	0	0	-10	10	-8	-4	0	4	8
z	0	0	2	2	4	4	4	4	4

Table 2: The coordinates of the vertices of a σ_9.

tetrahedron τ whose vertices in σ_n must contain at least one open segment. By the relaxed property (III), it is sufficient to ensure that some points of τ do not lie in the interior of σ_n.

3.2 Construction of σ_n

Choose four non-coplanar points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$, and a (simple) curve γ starting at $\mathbf{c}$ and ending at $\mathbf{d}$, and γ lies in the intersection of the two open halfspaces bounded by the triangles $\mathbf{cda}$ and $\mathbf{dcb}$ (using the right-hand rule to orient the vertices of the triangles), refer to Figure 10 (a).

Now we will choose $k + 2$ ($k \geq 0$) distinct points, denoted as $\mathbf{g_0}, \ldots, \mathbf{g_{k+1}}$, on the curve γ from $\mathbf{c}$ to $\mathbf{d}$, so that they satisfy the following constraints (refer to Figure 10):

(c1) The line segment $\mathbf{cd}$ intersects all the triangles $\mathbf{abg}_i,\ i = 0, \ldots, k + 1$.

(c2) Given two adjacent points $\mathbf{g}_i$ and $\mathbf{g}_{i+1}$, for $i = 0, \ldots, k$, on the curve γ, the point $\mathbf{g}_{i+1}$ and $\mathbf{d}$ must lie in the same halfspace bounded by the plane containing $\mathbf{abg}_i$.

(c3) Let $\mathbf{g}_i$ and $\mathbf{g}_j$, for $i, j = -1, \ldots, k+2$ and $i \neq j$, be two non-adjacent points on the curve γ where $\mathbf{g}_{-1} := \mathbf{c}$ and $\mathbf{g}_{k+2} := \mathbf{d}$. Without loss of generality, assume $i < j$. Then the line segment $\mathbf{g}_i\mathbf{g}_j$ (except $\mathbf{g}_{-1}\mathbf{g}_{k+2} = \mathbf{cd}$) does not intersect all triangles $\mathbf{abg}_l$, where $i < l < j$.

(c4) Let $\mathbf{g}_i$, $\mathbf{g}_{i+1}$ and $\mathbf{g}_{i+2}$, for $i = -1, \ldots, k$, be three consecutive points on the curve γ. Then the three points are neither coplanar with $\mathbf{a}$ nor $\mathbf{b}$.

Now the polyhedron σ_n, $n = 6 + k$, where $k \geq 0$, is constructed by choosing the boundary faces listed in Table 1 (refer to Figure 10).

Comments The curve γ in our construction is only an assistant. One can construct σ_n by choosing k points that satisfy all constraints. However, it is easier to imagine the relations of the points $\mathbf{g}_i$ with a curve in mind. The condition (c4) ensures that all faces of σ_n are triangles.

Figure 10 gives an example of such a polyhedron σ_9. A particular choice of the coordinates of the 9 vertices is given in Table 2.

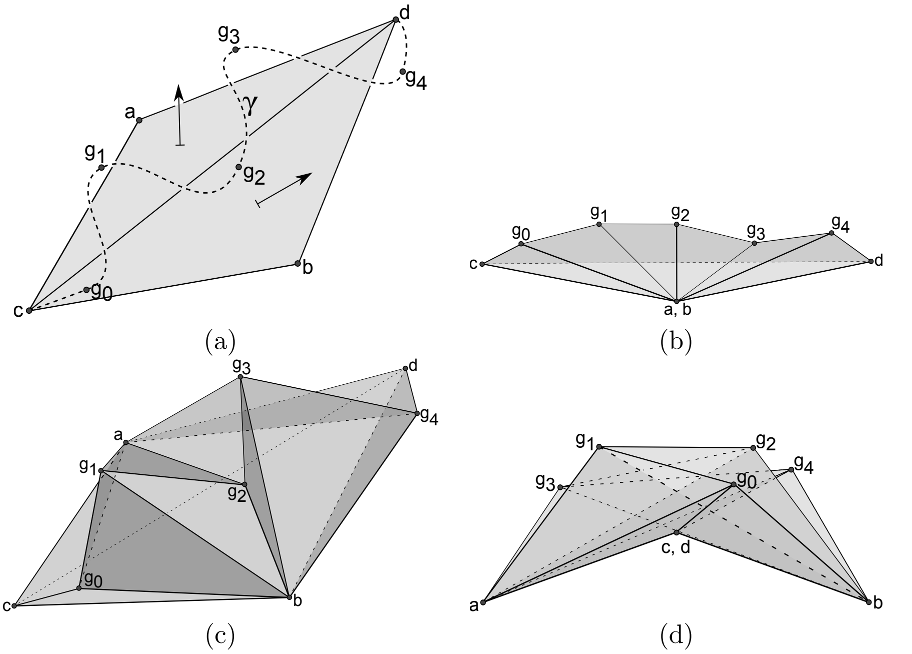

Figure 10: An example polyhedron σ_9 with 9 vertices. The four initial points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as well as the points $\mathbf{g}_i$, $i = 0, \ldots, 4$, chosen from a curve γ are shown in (a). Pictures (b), (c), and (d) are different views of the constructed polyhedron σ_9. A particular choice of the coordinates of the vertices is given in Table 2.

In the following we show that the so constructed polyhedron σ_n satisfies the properties (I) - (IV) in Theorem 4.

At first, we show that σ_n is a simple 3d polyhedron. Let $\mathcal{T}$ be the set of tetrahedra

$$\mathcal{T} = \{\mathbf{abcg}_0, \mathbf{abdg}_{k+1}\} \cup \{\mathbf{abg}_i\mathbf{g}_{i+1} \mid i = 0, \ldots, k\}.$$

The constraint (c2) ensures that every two tetrahedra in $\mathcal{T}$ must either share a common face or only share the common edge $\mathbf{ab}$. We see that the union of the set of tetrahedra of $\mathcal{T}$ is a 3d simple polyhedron $P := \cup\mathcal{T}$. By constraint (c1), we see that the open line segment $\mathbf{cd}$ lies wholly in the interior of P. Moreover, this constraint also ensures that the two open triangles $\mathbf{cda}$ and $\mathbf{dcb}$ lie wholly in the interior of P. Finally, by removing the tetrahedron $\mathbf{abcd}$ from P we obtain the polyhedron σ_n.

Next we show that it is combinatorially equivalent to the Bagemihl polyhedron π_n. The simplest case is when $n = 6$ ($k = 0$). The corresponding π_6 is just the well-known Schönhardt polyhedron (refer to Figure 1). The 6 vertices of σ_6 are: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{g}_0, \mathbf{g}_1$, respectively. We map them one-to-one to the vertices of the Schönhardt polyhedron as following (see Figure 11 Left):

- $\mathbf{g}_0 \to A_1$, $\mathbf{c} \to B_1$, $\mathbf{b} \to C_1$; and
- $\mathbf{g}_1 \to A_2$, $\mathbf{a} \to B_2$, $\mathbf{d} \to C_2$.

In general, when $n \geq 6$ ($k \geq 0$), the n vertices of σ_n are: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{k+1}$, respectively. We build a one-to-one map between the vertices of σ_n and the Bagemihl polyhedron π_n as following (refer to Figure 11 Right):

- $\mathbf{g}_0 \to A_1$, $\mathbf{c} \to B_1$, $\mathbf{b} \to C_1$;
- $\mathbf{g}_{k+1} \to A_2$, $\mathbf{a} \to B_2$, $\mathbf{d} \to C_2$; and
- $\mathbf{g}_i \to D_i$, for all $i = 1, \ldots, k$.

By this mapping, one can check that the faces of σ_n and π_n are also mapped one-to-one, so do the edges of them.

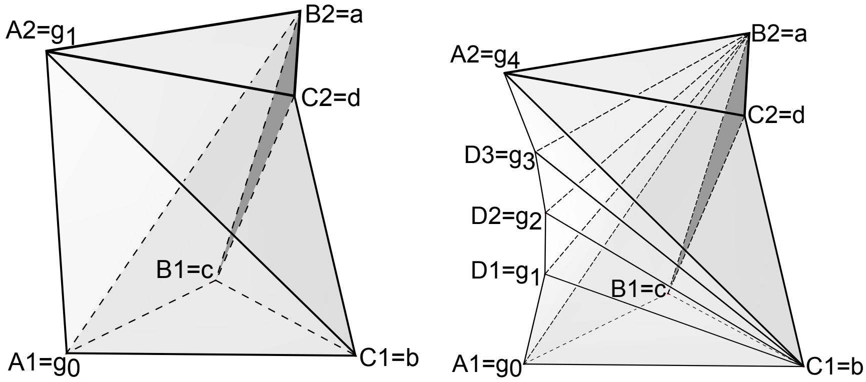

Figure 11: The mapping between the vertices of σ_n and the vertices of the Bagemihl polyhedron π_n.

Now we can show that σ_n satisfies the properties in Theorem 4 by borrowing Bagemihl's arguments in proving Theorem 1 [Bagemihl, 1948].

At first, σ_n satisfies the properties (I) and (IV) by a direct checking of the faces (listed above) and the edges of σ_n, using (c3) and (c4). Note that the property (IV) is fulfilled because of the constraint (c3) which prohibits to have three consecutive points $\mathbf{g}_i, \mathbf{g}_{i+1}, \mathbf{g}_{i+2}$ being collinear and constraint (c4) ensures the triangularity of the faces.

Remember that the main reason that causes σ_n being indecomposable is the property (II), since no open tetrahedron τ whose vertices of σ_n can lie in the interior of σ_n. The key fact is that τ must contain an open segment as stated in the property (III). This fact is true by the properties (I), (III) and (IV).

What remains is to show that σ_n satisfies the property (III). The open segments of σ_n are the line segment $\mathbf{ab}$, and all line segments with endpoints $\mathbf{g}_i\mathbf{g}_j$, where $\mathbf{g}_i$ and $\mathbf{g}_j$ are not adjacent vertices on the curve γ, where $i, j = -1, 0, \ldots, k{+}2$ and $i \neq j$, except the line segment $\mathbf{cd} = \mathbf{g}_{-1}\mathbf{g}_{k+2}$. The constraint (c3) in our construction ensures that such a line segment must not lie wholly in the interior of σ_n. This shows that the property (III) is satisfied.

3.3 The Number of Interior Steiner Points for σ_n

From now on, we study the question: "Given a σ_n, how many interior Steiner points are necessary to decompose it?"

At first, we show that a σ_n may need more than one interior Steiner point to be decomposed. We provide an explicit example of a σ_9. The geometry of this polyhedron is similar to the one shown in Figure 10. The coordinates of the 9 vertices are given in Table 2.

Given an arbitrary σ_n, we can associate every pair of its triangles, $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{a}$ and $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{b}$, to an *interval*, $\mathbf{t}_{a,i}\mathbf{t}_{b,i}$, on the line through $\mathbf{cd}$, for all $i = 0, 1, \ldots, k$, where:

$$\mathbf{t}_{a,i} := \text{plane}_{\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{a}} \cap \text{line}_{\mathbf{cd}}$$
$$\mathbf{t}_{b,i} := \text{plane}_{\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{b}} \cap \text{line}_{\mathbf{cd}}.$$

In general, such an interval is not necessarily inside the line segment $\mathbf{cd}$ for an arbitrary σ_n. In our particular example (in Table 2), the pair of planes containing the triangles $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{a}$ and $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{b}$ cut the line segment $\mathbf{cd}$ in its interior, for all $i = 0, 1, 2, 3$. Any point $\mathbf{p}$ in the interval $\mathbf{t}_{a,i}\mathbf{t}_{b,i}$ must see the two triangles, $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{a}$ and $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{b}$, from the interior of σ_9. However, if $\mathbf{p}$ does not lie in the interval defined by a pair of triangles, $\mathbf{p}$ can not see them. Furthermore, the two intervals $\mathbf{t}_{a,0}\mathbf{t}_{b,0}$ and $\mathbf{t}_{a,3}\mathbf{t}_{b,3}$ are disjoint. This implies that one cannot find a common interior Steiner point that is visible simultaneously by the four triangles, which are $\mathbf{g}_0\mathbf{g}_1\mathbf{a}$, $\mathbf{g}_0\mathbf{g}_1\mathbf{b}$, $\mathbf{g}_3\mathbf{g}_4\mathbf{a}$, and $\mathbf{g}_3\mathbf{g}_4\mathbf{b}$. Hence, more than one interior Steiner point is needed for decomposing this polyhedron.

For a general σ_n with n vertices, there are $k + 1$ intervals. It is easy to estimate that the required number of interior Steiner points for decomposing σ_n will not exceed the total number of such intervals, which is $k + 1 = n - 5$. However, this estimate is too rough. By an careful construction of interior Steiner points, one can get an explicit upper bound on the number of interior Steiner points for any σ_n. It is given by the following Theorem.

Theorem 5. *Given a 3d polyhedron σ_n as constructed in Section 3.2, where $n \geq 6$ is the number of vertices, it can be tetrahedralized by adding $\lceil \frac{n-5}{2} \rceil$ interior Steiner points.*

Proof. We prove it in two steps: at first, we will place this number of Steiner points in the interior of the σ_n, then we show how to tetrahedralize it with these Steiner points.

Step (1), placing interior Steiner points. From the previous analysis, we see that the requirement of multiple Steiner points comes from the fact that one may not find a common Steiner point that is simultaneously visible by all boundary faces. We will place a number of Steiner points in the interior of σ_n. We make sure that each Steiner point that we place will be visible by a certain number of boundary faces, and every boundary face will be visible by at least one of these Steiner points.

Consider the $k+2$ vertices, $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_{k+1}$ of σ_n. Each $\mathbf{g}_i$ ($i = 0, 1, \ldots, k+1$) is a vertex of four adjacent boundary faces of σ_n, i.e., $\mathbf{g}_{i-1}\mathbf{g}_i\mathbf{a}$, $\mathbf{g}_{i-1}\mathbf{g}_i\mathbf{b}$, $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{a}$, and $\mathbf{g}_i\mathbf{g}_{i+1}\mathbf{b}$ (recall that $\mathbf{g}_{-1} = \mathbf{c}$ and $\mathbf{g}_{k+2} = \mathbf{d}$). We will search a point inside σ_n and near to $\mathbf{g}_i$, hence it is visible by all these four faces. For this purpose, it is not necessary to use all $\mathbf{g}_i$. In particular, we choose the following subset of the set of vertices of σ_n,

$$\mathcal{G} := \{\mathbf{g}_1, \mathbf{g}_3, \mathbf{g}_5, \ldots, \mathbf{g}_m\},$$

where the last index is

$$m = k + ((k+1) \text{ modulo } 2),$$

which is the largest odd number of the indices. The cardinality is $|\mathcal{G}| = \lceil \frac{k+1}{2} \rceil$.

Let $\mathcal{G}^*$ be the set of points constructed from the points in $\mathcal{G}$, such that

$$\mathcal{G}^* := \{\mathbf{g}_j^* := \text{plane}_{\mathbf{g}_j\mathbf{ab}} \cap \text{line}_{\mathbf{cd}} \mid \mathbf{g}_j \in \mathcal{G}\},$$

see Figure 12. Note that the points in $\mathcal{G}^*$ have the following visibilities from the

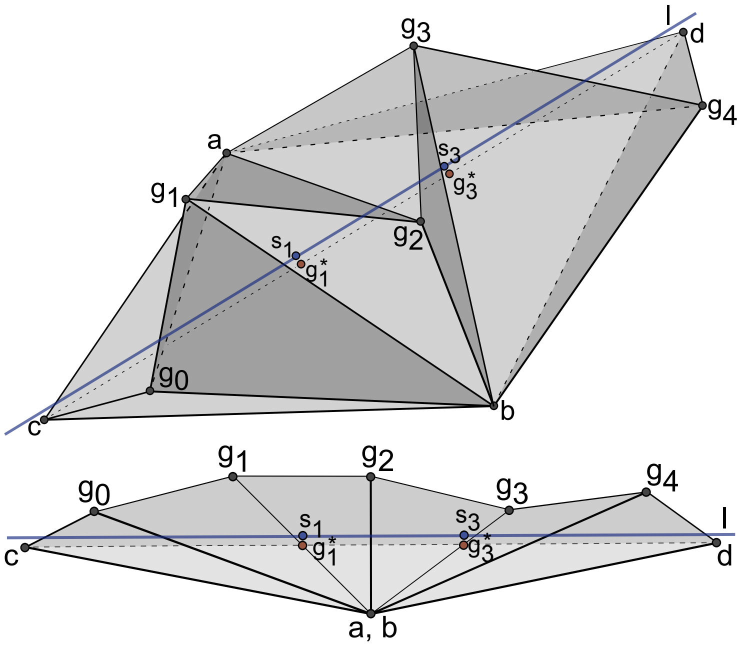

Figure 12: Two views of the example polyhedron σ_9 with the constructed points $\mathcal{G}^*$ and the Steiner points $\mathcal{S}$ on the common line l that is parallel to **cd**.

inside of σ_n:

(i) Each $\mathbf{g}_j^*$, where $j = 1, 3, \dots, m$, is visible by the faces $\mathbf{ag}_{j-1}\mathbf{g}_j$, $\mathbf{bg}_{j-1}\mathbf{g}_j$, $\mathbf{ag}_j\mathbf{g}_{j+1}$, and $\mathbf{bg}_j\mathbf{g}_{j+1}$;

(ii) Additionally, $\mathbf{g}_1^*$ is visible by the two faces $\mathbf{acg}_0$ and $\mathbf{bcg}_0$, and $\mathbf{g}_m^*$, is visible by the two faces $\mathbf{adg}_{k+1}$ and $\mathbf{bdg}_{k+1}$.

Since all $\mathbf{g}_j^*$ are on the line segment **cd**, i.e., they are coplanar with the two faces **cda** and **cdb**, they are not yet our wanted interior Steiner points.

Now we will place our interior Steiner points moving from the set of points $\mathcal{G}^*$. Choose the plane p along the middle axis of the two planes containing the two bottom triangles **cda** and **cdb**. The plane p must contain the line segment **cd**. Now we will move $\mathcal{G}^*$ within p and toward the interior of σ_n and define the so obtained set of points as $\mathcal{S}$. For our proof, we just move all points in such a way, that they all stay within a line, denoted as l, which is parallel to **cd**. And

(1)	$(\mathbf{a},\mathbf{c},\mathbf{d}),(\mathbf{b},\mathbf{c},\mathbf{d})$
(2)	$(\mathbf{a},\mathbf{c},\mathbf{s}_1),(\mathbf{b},\mathbf{c},\mathbf{s}_1),\ (\mathbf{a},\mathbf{d},\mathbf{s}_m),(\mathbf{b},\mathbf{d},\mathbf{s}_m)$
(3)	$(\mathbf{a},\mathbf{s}_p,\mathbf{s}_{p+2}),(\mathbf{b},\mathbf{s}_p,\mathbf{s}_{p+2})$, where $p=1,3,5,\ldots,m$

Table 3: The set $\{(1),(2),(3)\}$ of boundary faces of the remaining polyhedron.

we choose the distance between l and $\mathbf{cd}$ small enough such that all the points on l remain in the interior of σ_n and their visibilities by faces given in (i) and (ii) do not change. We then take the set of points on the line l as our interior Steiner points, denoted as

$$\mathcal{S} := \{\mathbf{s}_j \mid \mathbf{s}_j \text{ is moved from } \mathbf{g}_j^* \text{ onto } l,\ \mathbf{g}_j^* \in \mathcal{G}^*\},$$

then $|\mathcal{S}| = \lceil \frac{k+1}{2} \rceil$. See Figure 12 for an example.

Step (2), tetrahedralizing σ_n. With the created interior Steiner points, we are able to create a tetrahedralization of σ_n. The idea is first to create tetrahedra inside σ_n by using the set of interior Steiner points and the visibility properties of them. However, this set of tetrahedra still not tetrahedralizes the whole of σ_n. Then second to tetrahedralize the remaining part, which is a 3d polyhedron with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and the set of interior Steiner points.

By the visibility properties (i) and (ii), we can create the following sets of tetrahedra from the inside of σ_n:

$$\begin{aligned}
\mathcal{T}_1 &:= \{\mathbf{ag}_{j-1}\mathbf{g}_j\mathbf{s}_j, \mathbf{bg}_{j-1}\mathbf{g}_j\mathbf{s}_j, \mathbf{ag}_j\mathbf{g}_{j+1}\mathbf{s}_j, \mathbf{bg}_j\mathbf{g}_{j+1}\mathbf{s}_j \mid \mathbf{s}_j \in \mathcal{S}\},\\
\mathcal{T}_2 &:= \{\mathbf{acg}_0\mathbf{s}_1, \mathbf{bcg}_0\mathbf{s}_1, \mathbf{adg}_{k+1}\mathbf{s}_m, \mathbf{bdg}_{k+1}\mathbf{s}_m\},\\
\mathcal{T}_3 &:= \{\mathbf{ag}_{p+1}\mathbf{s}_p\mathbf{s}_{p+2}, \mathbf{bg}_{p+1}\mathbf{s}_p\mathbf{s}_{p+2} \mid p = 1,3,5,\ldots,m-2\}.
\end{aligned}$$

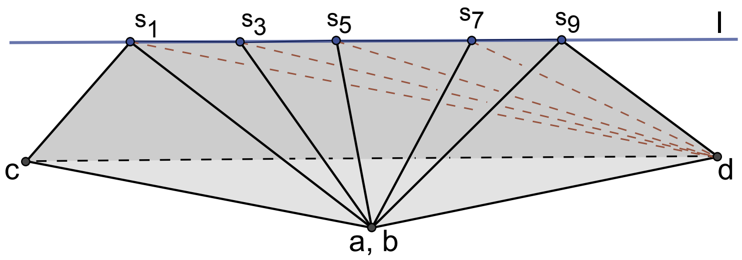

Figure 13: The polyhedron after removing the sets $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ of tetrahedra from σ_9. A decomposition of this polyhedron into the two sets $\mathcal{T}_4$ and $\mathcal{T}_5$ of tetrahedra is given. The internal edges are shown in brown.

The remaining region of σ_n after removing the above tetrahedra is a 3d polyhedron, see Figure 13 for an example. It has the vertices $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\} \cup \mathcal{S}$, and the boundary faces are given in Table 3. Since all interior Steiner points are collinear, this polyhedron can be decomposed into the following two sets of tetrahedra:

$$\begin{aligned}
\mathcal{T}_4 &:= \{\mathbf{acds}_1, \mathbf{bcds}_1\},\\
\mathcal{T}_5 &:= \{\mathbf{as}_p\mathbf{s}_{p+2}\mathbf{d}, \mathbf{bs}_p\mathbf{s}_{p+2}\mathbf{d} \mid p = 1,3,5,\ldots,m-2\}.
\end{aligned}$$

This concludes our proof. □

	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{c}$	$\mathbf{d}$	$\mathbf{g}_0$	$\mathbf{g}_1$	$\mathbf{g}_2$	$\mathbf{g}_3$
x	-1.294	4.830	4.830	-3.536	4.253	-0.301	3.117	-2.183
y	10	0	10	0	6.532	9.760	2.999	8.657
z	4.830	1.294	-1.294	3.536	-2.426	0	-2.571	0.646
	$\mathbf{g}_4$	$\mathbf{g}_5$	$\mathbf{g}_6$	$\mathbf{g}_7$				
x	1.874	-3.330	0.163	-4.051				
y	1.002	6.864	-0.105	3.184				
z	-1.808	1.350	-0.366	2.242				

Table 4: A choice of the coordinates of the vertices of a σ_{12}. The geometry of this polyhedron is shown in Figure 14. With these coordinates, this polyhedron needs at least 4 Steiner points to be decomposed.

There are indeed many possibilities to place interior Steiner points that may lead to a smaller number of interior Steiner points. However, we can show that this number $\lceil \frac{k+1}{2} \rceil = \lceil \frac{n-5}{2} \rceil$ of interior Steiner points is optimal in the worst case.

Theorem 6. *Given $n \in \mathbb{N}_{\geq 6}$, one can construct an a 3d polyhedron σ_n with n vertices which has the property that one needs exactly $\lceil \frac{n-5}{2} \rceil$ interior Steiner points to decompose it.*

Proof. We proof the Theorem by giving a general construction of a σ_n, so that it will always need at least this number of interior Steiner points. We then get the equality by Theorem 5. The basic idea is to control the overlap of the intervals $\mathbf{t}_{a,j}\mathbf{t}_{b,j}$, $j = 0, ..., n-6$, as defined at the beginning of Section 3.3. By the following construction we ensure that two consecutive intervals $\mathbf{t}_{a,j}\mathbf{t}_{b,j}$ and $\mathbf{t}_{a,j+1}\mathbf{t}_{b,j+1}$ overlap in their interior. Moreover, if two non-consecutive intervals don't overlap, we will obtain the desired number of interior Steiner points.

Fix $n \geq 6$ and start with non-coplanar points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3$ as described in Section 3.2. Then choose $n-4$ points $\mathbf{g}_i$, $i = 0, ..., n-5$ from the valid domain of a curve in a zig-zag shape, like in the polyhedron in Figure 10. By moving the points $\mathbf{g}_i$ lower, so that the segments $\mathbf{g}_i\mathbf{a}$ or $\mathbf{g}_i\mathbf{b}$ resp. are nearly crossing the line $\mathbf{cd}$, we obtain non overlapping intervals $\mathbf{t}_{a,i}\mathbf{t}_{b,i}$ and $\mathbf{t}_{a,i+2}\mathbf{t}_{b,i+2}$. So, we can achieve that except for the consecutive intervals $\mathbf{t}_{a,j}\mathbf{t}_{b,j}$ and $\mathbf{t}_{a,j+1}\mathbf{t}_{b,j+1}$ with $j = 0, ..., n-6$, no intervals overlap. Placing one interior Steiner point slightly above each overlap of the intervals gives the number of $\lceil \frac{n-5}{2} \rceil$ interior Steiner points. One can decompose the polyhedron as described in the proof of Theorem 5. □

Figure 14 shows an particular example of such a polyhedron with 12 vertices. The coordinates of the 12 vertices are given in Table 4. By our construction, this polyhedron satisfies the property that only two adjacent intervals are overlapping. A pair of such intervals is illustrated in Figure 14 (c). Therefore, this polyhedron needs at least 4 interior Steiner points to be decomposed, which is optimal for this case.

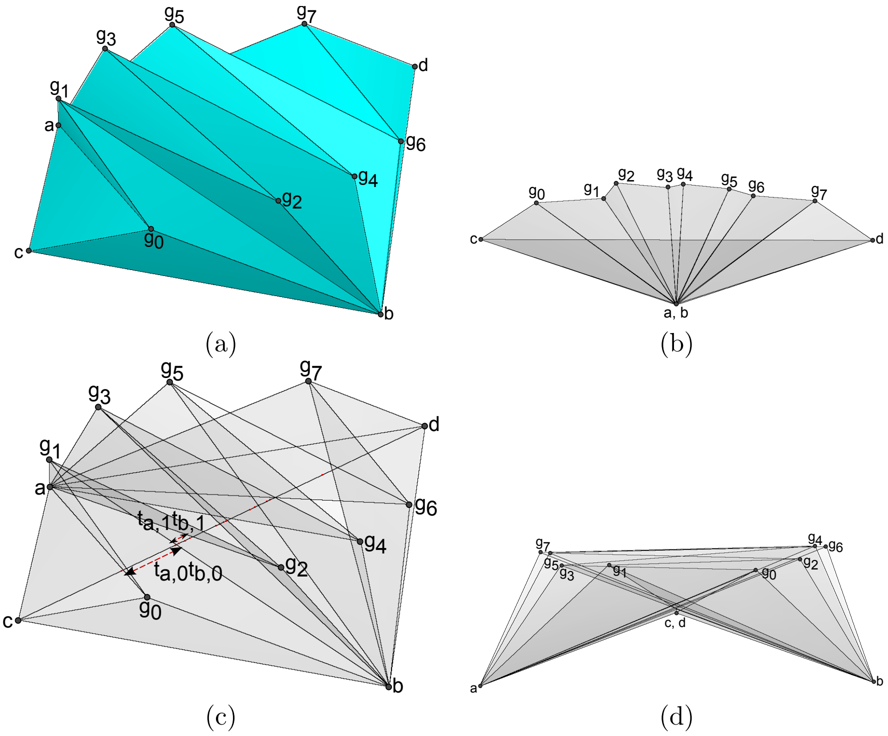

Figure 14: An example polyhedron, σ_{12}, with 12 vertices. The coordinates of the vertices are given in the Table 4. Different views of this polyhedron are shown in (a), (b), (c), and (d), respectively. In particular, two overlappings intervals are shown in (c).

4 Discussion

In this paper, we studied the question of how many interior Steiner points are needed for some special classes of 3d indecomposable polyhedra.

We comment that our construction of an interior Steiner point in Section 2.2 for Bagemihl polyhedra as well as it generalizations is not unique. Note that the line segment B_1C_2 in the Bagemihl polyhedron always touches the visible polyhedron. It is always possible to choose an interior Steiner point near this line segment such that it can decompose the polyhedron. The interior Steiner point can be chosen such that it is near the interval cut by the two planes containing the triangles $A_1A_2C_1$ and $A_1A_2B_2$ for π_6 (or $A_1A_2D_1$ and $D_kA_2B_2$ for π_n, where $n > 6$) and the line segment B_1C_2.

The original Theorem 1 of Bagemihl proves the existence of a class of polyhedra (that satisfy all these properties). The following two open questions are interesting:

(1) Except the generalized Bagemihl polyhedra $\tilde{\pi}_n$, is there an other construction that satisfies Theorem 1.

(2) Does every polyhedron satisfying the properties in Theorem 1 require only one interior Steiner point to be decomposed?

The result of Theorem 5 shows that any σ_7 needs only one interior Steiner points to be decomposed, regardless of its geometry. However, we do not know yet if this is true or false for an arbitrary 3d polyhedron with 7 vertices.

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