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Deriving amplitude equations via evolutionary Γ-convergence

Dedicated to Jürgen Sprekels on the occasion of his sixty-fifth birthday

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Abstract. We discuss the justification of the Ginzburg–Landau equation with real coefficients as an amplitude equation for the weakly unstable one-dimensional Swift–Hohenberg equation. In contrast to classical justification approaches we employ the method of evolutionary Γ-convergence by reformulating both equations as gradient systems. Using a suitable linear transformation we show Γ-convergence of the associated energies in suitable function spaces.

The limit passage of the time-dependent problem relies on the recent theory of evolutionary variational inequalities for families of uniformly convex functionals as developed by Daneri and Savaré 2010. In the case of a cubic energy it suffices that the initial conditions converge strongly in L^2, while for the case of a quadratic nonlinearity we need to impose weak convergence in H^1. However, we do not need wellpreparedness of the initial conditions.

1 Introduction

We propose a new method for deriving amplitude equations in the case that the original model is a gradient system $(X, \mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$, i.e. the evolution is defined by the abstract balance between the viscous force and the potential restoring force:

$$0 = \mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u, \dot u) + \mathrm{D}\mathcal{F}_\varepsilon(u) \ \in X^*.$$

Here we will assume that the state space X is a Hilbert space and $\mathcal{F}_\varepsilon : X \to \mathbb{R}_\infty := \mathbb{R}\cup\{\infty\}$ denotes the energy functional. In general, the dissipation potential is such that $\mathcal{R}_\varepsilon(u, \cdot) : X \to [0, \infty]$ is a lower semicontinuous convex function satisfying $\mathcal{R}_\varepsilon(u, 0) = 0$. However, in this work we will restrict to the simplified setting that $\mathcal{R}_\varepsilon$ is independent of u and quadratic in $\dot u$, viz. $\mathcal{R}_\varepsilon(u, \dot u) = \frac{1}{2}\langle \mathbb{G}_\varepsilon \dot u, \dot u\rangle$.

Here ε is a small parameter characterizing the ratio between the microscopic and the macroscopic length scale. The main question in evolutionary Γ-convergence is to identify conditions for the convergence of the pair $(\mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$ to a limit $(\widehat{\mathcal{F}}, \widehat{\mathcal{R}})$ such that the solutions $u_\varepsilon : [0, T] \to X$ of $(X, \mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$ converge to the solutions $\widehat{u} : [0, T] \to X$ of $(X, \widehat{\mathcal{F}}, \widehat{\mathcal{R}})$. This work uses the theory of *evolutionary variational inequalities* and the Γ-convergence theory developed in [DaS10, Sav11]. For other approaches we refer to the survey [Mie14].

The abstract theory is devised to treat the Swift–Hohenberg equation (SHe)

$$v_\tau = -(1+\Delta)^2 v + \widehat{\mu} v + \gamma v^2 - v^3. \tag{1.1}$$

We will consider the one-dimensional case under the assumption that we are in the weakly unstable regime, i.e. $\widehat{\mu} = \mu\varepsilon^2$ for small positive ε. It was shown formally in [Eck65] that the typical solutions can be approximated by a modulated role pattern in the form $v(\tau, y) = \varepsilon \,\mathrm{Re}\big(A(\varepsilon^2\tau, \varepsilon y)\mathrm{e}^{\mathrm{i}y}\big)$ and that the amplitude function $A(t, x) \in \mathbb{C}$ satisfies the Ginzburg–Landau equation (GLe)

$$A_t = 4A_{xx} + \mu A - \rho|A|^2 A \quad \text{with } \rho = \tfrac{3}{4} - \tfrac{19}{18}\gamma^2, \tag{1.2}$$

see [MiS96, Eqn. (2.6)] where our ρ occurs as $c/4$ because of a differing factor 2 in the normalization of A. While τ and y denote the microscopic time and space scale, the variables $t = \varepsilon^2\tau$ and $x = \varepsilon y$ denote the macroscopic time and space scale. First mathematical justification of this approximation where given in [CoE90, vHa91, Eck93, Sch94]. We refer to [Mie02] for a survey and to [KSM92] for a 4-page proof of the result in the case of cubic nonlinearities, i.e. $\gamma = 0$. We also will see that the case $\gamma \neq 0$ is substantially different. The comparison of the global attractors and the inertial manifolds of (1.1) and (1.2) are done in [MiS96] and [MSZ00], respectively.

The traditional methodology for justifying the amplitude equations is most easily explained in [KSM92]: First, one considers a fixed and sufficiently smooth solution A of the amplitude equation. Then, using formal asymptotic expansions one constructs an approximate solution v_A^ε for the original system, where depending on the needed order of accuracy in ε one needs a suitable number of derivatives of A. Finally. one inserts the ansatz $v = v_A^\varepsilon + \varepsilon^\beta R$ into the original equation and derives an ε-independent bound for the scaled error R.

The method we propose in this work is quite different, because it uses the abstract method of evolutionary Γ-convergence as introduced in [SaS04], see also [Ser11, Mie14] for surveys. The main point of the present work is that we rely on the gradient structure of the SHe and study the convergence in the class of gradient systems. Of course, the theory of amplitude equation applies to general, non-gradient or non-Hamiltonian system, and the theories mentioned above apply to these more general classes. However, the restriction to gradient system is compensated for by much finer tools that allow us to reduce the assumption of the convergence theorem to an absolute minimum.

To be more specific, we consider (1.1) with $\widehat{\mu} = \varepsilon^2\mu$ on the real line with the periodicity in $y \in \mathbb{R}$ with period ℓ/ε, viz. $v(\tau, y{+}\ell/\varepsilon) = v(\tau, y)$. Upon the rescaling $u(t,x) = v(t/\varepsilon^2, x/\varepsilon)/\varepsilon$ we arrive at

$$\dot u = -\frac{1}{\varepsilon^2}(1{+}\varepsilon^2\partial_x^2)^2 u + \mu u + \frac{\gamma}{\varepsilon}u^2 - u^3 = -\mathrm{D}\mathcal{F}_\varepsilon^{\mathrm{SH}}(u), \quad u(t,x{+}\ell) = u(t,x), \tag{1.3}$$

where the energy functional reads $\mathcal{F}_\varepsilon^{\mathrm{SH}}(u) := \int_0^\ell \frac{1}{2\varepsilon^2}(u{+}\varepsilon^2 u'')^2 - \frac{\mu}{2}u^2 + \frac{\gamma}{3\varepsilon}u^3 + \frac14 u^4 \,\mathrm{d}x$. This form is not suitable for the limit passage due to the dominating harmonic oscillations which are expected because of the approximate solutions in the form $u(t,x) = \mathrm{Re}\big(A(t,x)\mathrm{e}^{\mathrm{i}x/\varepsilon}\big)$. In particular, we assume $\varepsilon = \ell/(2\pi N)$ with $N \gg 1$ such that A is a periodic function on $\mathbb{S}_\ell = \mathbb{R}/_{(\ell\mathbb{Z})}$. The case $\varepsilon = \ell(2\pi(N{+}\theta))$ with $N \in \mathbb{N}$ and $\theta \in \,]0,1[$ fixed can be handled as in [MSZ00] leading to a family of limit equations depending on $\theta \in \mathbb{S}_1$.

Using a suitable linear bijection $A = \Psi_\varepsilon u \in \mathrm{L}^2_{\mathbb{C}} := \mathrm{L}^2(\mathbb{S}_\ell;\mathbb{C})$ we obtain an equivalent evolutionary problem for A in the form

$$\dot A = L_\varepsilon A + F_\varepsilon(A) \quad \Longleftrightarrow \quad \mathbb{G}_\varepsilon \dot A = -\mathrm{D}\mathcal{F}_\varepsilon(A), \tag{1.4}$$

where the energy $\mathcal{F}_\varepsilon(A)$ and the dissipation potential $\mathcal{R}_\varepsilon(\dot A) = \frac12\langle \mathbb{G}_\varepsilon \dot A, \dot A\rangle$ are obtained by simply transforming the corresponding structures for the $\frac12\|\cdot\|^2_{\mathrm{L}^2}$ and $\mathcal{F}_\varepsilon^{\mathrm{SH}}$ via Ψ_ε. The aim is to show that solutions A_ε of (1.4) converge to solutions of the GLe (1.2).

The first main step is to show that the functionals $\mathcal{F}_\varepsilon$ converge to the Ginzburg–Landau functional $\mathcal{F}_{\mathrm{GL}}$ in a suitable sense of Γ-convergence. For the easy case $\gamma = 0$ we have Mosco convergence (=weak and strong Γ-convergence) of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{GL}}$ in $\mathrm{H}^1_{\mathbb{C}} := \mathrm{H}^1(\mathbb{S}_\ell;\mathbb{C})$. The case $\gamma \neq 0$ is considerably more difficult and needs, as for the classical way of justifying the amplitude equation (cf. [Sch94, MiS96]), a special normal-form theory to handle the quadratic term. This is reflected in a much more complicated proof of the Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{GL}}$ in the weak topology of $\mathrm{H}^1_{\mathbb{C}}$. Moreover, the dissipation potentials $\mathcal{R}_\varepsilon$ converge to $\mathcal{R}_{\mathrm{GL}} : V \mapsto \frac14\|V\|^2_{\mathrm{L}^2}$ in the sense that $V_\varepsilon \to V$ in $\mathrm{L}^2_{\mathbb{C}}$ implies $\mathcal{R}_\varepsilon(V_\varepsilon) \to \mathcal{R}_{\mathrm{GL}}(V)$.

The second main step is to derive uniform λ-convexity for the functionals, i.e. we need to find a $\lambda \in \mathbb{R}$ such that for all $\varepsilon \in \,]0,1]$ the functionals $\mathcal{E}_\varepsilon - \lambda\mathcal{R}_\varepsilon$ are convex. This again is trivial for $\gamma = 0$, but for $\gamma \neq 0$ it only holds in a weakened sense, where we have to restrict the analysis to large balls in $\mathrm{H}^1_{\mathbb{C}}$. The abstract theory of evolutionary Γ-convergence for uniformly λ-convex families was developed in [AGS05, DaS10, Sav11], and we present the main arguments in Section 3. The main tool is the reformulation of the gradient system in terms of the *Integrated Evolutionary Variational Inequality* $(\mathrm{IEVI})_\lambda$, i.e. A_ε is a solution of (1.4) if and only if

$$(\mathrm{IEVI})^\varepsilon_\lambda \quad \begin{cases} \mathrm{e}^{\lambda(t-s)}\mathcal{R}_\varepsilon(A_\varepsilon(t){-}B) - \mathcal{R}_\varepsilon(A_\varepsilon(s){-}B) \leq \int_0^{t-s} \mathrm{e}^{\lambda r}\,\mathrm{d}r\, \big(\mathcal{F}_\varepsilon(B) - \mathcal{F}_\varepsilon(A(t))\big) \\ \text{for all } s \text{ and } t \text{ with } 0 \leq s < t \text{ and all } B \in \mathrm{L}^2_{\mathrm{C}}. \end{cases}$$

The limit passage $\varepsilon \to 0$ in this formulation is surprisingly simply, see [DaS10, Thm. 2.17] or our Theorem 3.2.

For $\gamma = 0$ we obtain the following clear result: If A_ε denote the solutions of the SHe in the form (1.4) and A a solution of the GLe (1.2), then

$$\boxed{\gamma = 0}: \quad A_\varepsilon(0) \overset{\mathrm{L}^2}{\to} A(0) \quad \Longrightarrow \quad \forall\, t > 0: \; A_\varepsilon(t) \overset{\mathrm{H}^1}{\rightharpoonup} A(t) \text{ and } \mathcal{F}_\varepsilon(A_\varepsilon(t)) \to \mathcal{F}_{\mathrm{GL}}(A(t)).$$

Note that this result allows for solutions with arbitrary initial data $A_\varepsilon(0) \in \mathrm{L}^2_{\mathbb{C}}$. Thus, the energies $\mathcal{F}_\varepsilon(A_\varepsilon(t))$ may have singularities at $t = 0$. This improves the assumption $A(0) \in \mathrm{C}^4(\mathbb{S}_\ell; \mathbb{C})$ in [KSM92] considerably.

For general γ, we rely on a uniform equi-coercivity of the functionals $\mathcal{F}_\varepsilon$, which only can hold if $|\gamma|$ is not too large, since $\mathcal{F}_{\mathrm{GL}}$ is equi-coercive if and only if ρ in (1.2) is positive. The latter condition is equivalent to $\gamma^2 < \gamma_*^2 := 27/38$. In Proposition 2.2 we find a $\gamma_0 \in \,]0, \gamma_*]$ such that equi-coercivity of $\mathcal{F}_\varepsilon$ holds whenever $\gamma^2 < \gamma_0^2$. We conjecture that $\gamma_0 = \gamma_*$, but don't know how to prove this. The convergence result roughly reads

$$\boxed{|\gamma| < \gamma_0}: \quad A_\varepsilon(0) \overset{\mathrm{H}^1}{\rightharpoonup} A(0) \quad \Longrightarrow \quad \forall\, t > 0: \; A_\varepsilon(t) \overset{\mathrm{H}^1}{\rightharpoonup} A(t) \text{ and } \mathcal{F}_\varepsilon(A_\varepsilon(t)) \to \mathcal{F}_{\mathrm{GL}}(A(t)).$$

In Section 5 we discuss the obtained results and possible generalizations. In particular, we address the question of adding spatially inhomogeneous perturbations to the SHe and show that our approach of evolutionary Γ-convergence can easily handle such situation.

2 Gradient structures and convergence results

We consider first a general approach to derive generalized versions of the SHe by starting from general gradient systems. For a bounded domain $\Omega \subset \mathbb{R}^d$ we define the underlying Hilbert space $X := \mathrm{L}^2(\Omega)$. As a dissipation potential we choose the L^2-norm

$$\mathcal{R}(\dot{v}) := \frac{1}{2}\|\dot{v}\|^2_{\mathrm{L}^2} = \frac{1}{2}\int_\Omega (\dot{v}(y))^2\,\mathrm{d}y.$$

The energy functional is given in terms of a general nonlinear function $G(v, \nabla v)$ as

$$\mathcal{F}(v) = \int_\Omega \frac{1}{2}\big(v{+}\Delta v\big)^2 + G(v, \nabla v)\,\mathrm{d}x.$$

Now the generalized SHe is obtained as the gradient flow $\dot{v} = -\mathrm{D}\mathcal{F}(v)$, namely

$$\dot{v} = -(1{+}\Delta)^2 v - \partial_v G(v, \nabla v) + \mathrm{div}\,\big(\partial_{\nabla v} G(v, \nabla v)\big)$$

plus suitable boundary conditions, either periodic or natural ones. Choosing $G(v,\nabla v) = -\frac{\mu}{2}v^2 + \frac{\gamma}{3}v^3 + \frac14 v^4$ we obtain (1.1).

To study the amplitude equation we need to look at a microscopically very large domain, which still is large on the macroscopic scale. Thus, we consider $y \in [0,\ell/\varepsilon]$ for large, but fixed ℓ. By $\mathbb{S}_\ell$ we denote the macroscopic interval $[0,\ell]_{\mathrm{per}} = \mathbb{S}_\ell := \mathbb{R}_{/(\ell\mathbb{Z})}$ with periodic boundary conditions and use $X = \mathrm{L}^2_{\mathbb{R}} := \mathrm{L}^2(\mathbb{S}_\ell;\mathbb{R})$ as the basic Hilbert space.

Since we rescaled space and time we take the functional $\mathcal{F}^{\mathrm{SH}}_\varepsilon$ in the form

$$\mathcal{F}^{\mathrm{SH}}_\varepsilon(u) := \int_{\mathbb{S}_\ell} \frac{1}{2\varepsilon^2}\big(u(x){+}\varepsilon^2 u''(x)\big)^2 + \frac{1}{\varepsilon^4} G_\varepsilon(\varepsilon u(x), \varepsilon^2 u'(x))\,\mathrm{d}x.$$

For simplicity, we restrict our attention to $G_\varepsilon(a,b) = -\frac{\mu\varepsilon^2}{2}a^2 - \frac{\gamma}{3}a^3 + \frac14 a^4$ giving the rescaled SHe

$$\dot u = -\frac{1}{\varepsilon^2}(1{+}\varepsilon^2\partial_x^2)^2 u + \mu u + \frac{\gamma}{\varepsilon}u^2 - u^3. \tag{2.1}$$

The solutions u_ε of (2.1) behave to leading order like $\mathrm{Re}\,\big(A(t,x)\mathrm{e}^{\mathrm{i}x/\varepsilon}\big)$, so we cannot do the limiting procedure on u itself, but we need to define a variable converging to the limit A. For this we write $\boldsymbol{E}_\varepsilon(x) := \mathrm{e}^{\mathrm{i}x/\varepsilon}$ and assume

$$\varepsilon = \ell/(2\pi N) \text{ with } N \in \mathbb{N} \quad \Longrightarrow \quad \boldsymbol{E}_\varepsilon \in \mathrm{H}^k(\mathbb{S}_\ell;\mathbb{C}) \subset \mathrm{H}^1_{\mathbb{C}} := \mathrm{H}^1(\mathbb{S}_\ell;\mathbb{C}).$$

Subsequently we will use ε or N, whatever is more convenient, but we always assume the identity $2\pi\varepsilon N = \ell$.

We will often use Fourier series expansion which we will denote by $\mathfrak{f}_n(A) = a_n$ if $A = \sum_{n\in\mathbb{Z}} a_n \mathbb{E}^n$ where $\mathbb{E}(x) := \mathrm{e}^{\mathrm{i}x2\pi/\ell}$, and hence $\boldsymbol{E}_\varepsilon = \mathbb{E}^N$. We now set

$$\mathrm{X}_N := \{\, A \in \mathrm{L}^2(\mathbb{S}_\ell;\mathbb{C}) \mid \mathfrak{f}_n(A) = 0 \text{ for } n < -N \text{ and } \mathfrak{f}_{-N}(A) \in \mathbb{R}\,\} \subset \mathrm{L}^2_{\mathbb{C}} := \mathrm{L}^2(\mathbb{S}_\ell;\mathbb{C})$$

and define the bijection $\Psi_\varepsilon : \mathrm{L}^2_{\mathbb{R}} \to \mathrm{X}_N$ via

$$A = \Psi_\varepsilon u \quad \text{with} \quad a_n = \mathfrak{f}_n(A) = \begin{cases} 2u_{n+N} = 2\mathfrak{f}_{n+N}(u) & \text{for } n > -N, \\ u_0 = \mathfrak{f}_0(u) & \text{for } n = -N, \\ 0 & \text{for } n < -N \end{cases} \tag{2.2}$$

The important observation is that Ψ_ε is a right-inverse of the mapping $A \mapsto u = \mathrm{Re}\,\big(A\boldsymbol{E}_\varepsilon\big)$, viz. $u = \mathrm{Re}\,\big((\Psi_\varepsilon u)\boldsymbol{E}_\varepsilon\big)$ for all $u \in \mathrm{L}^2_{\mathbb{R}}$. Moreover, the mappings are almost norm-preserving, namely

$$\|\Psi_\varepsilon u\|_2^2 = 2\|u\|_2^2 - \ell|\mathfrak{f}_0(u)|^2, \quad 2\|\,\mathrm{Re}\,\big(A\boldsymbol{E}_\varepsilon\big)\|_2^2 = \|A\|_2^2 + \ell|\mathfrak{f}_{-N}(A)|^2. \tag{2.3}$$

We now define the transformed version of the SHe (2.1) by mapping the gradient system $(\mathrm{L}^2_{\mathbb{R}}, \mathcal{F}^{\mathrm{SH}}_\varepsilon, \frac12\|\cdot\|_2^2)$ via Ψ_ε to the gradient system $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$ with

$$\mathcal{F}_\varepsilon(A) := \begin{cases} \mathcal{F}^{\mathrm{SH}}_\varepsilon(\mathrm{Re}(A\boldsymbol{E}_\varepsilon)) & \text{for } A \in \mathrm{X}_N, \\ \infty & \text{otherwise.} \end{cases}$$

In particular, for $A \in \mathrm{X}_N$ we have the explicit form

$$\mathcal{F}_\varepsilon(A) = \int_{\mathbb{S}_\ell} \frac14|2\mathrm{i}A'{+}\varepsilon A''|^2 {-} \frac{\mu}{4}|A|^2 {-} \frac{\gamma}{3\varepsilon}\big(\mathrm{Re}(A\boldsymbol{E}_\varepsilon)\big)^3 {+} \frac14\big(\mathrm{Re}(A\boldsymbol{E}_\varepsilon)\big)^4\,\mathrm{d}x {+} \frac{\ell}{4}\big(\frac{1}{\varepsilon^2}{-}\mu\big)|\mathfrak{f}_{-N}(A)|^2.$$

The new dissipation potential $\mathcal{R}_\varepsilon$ has to be defined via the transformation Ψ_ε as well, i.e. we need $\mathcal{R}_\varepsilon(V) = \mathcal{R}_\varepsilon^{\rm SH}(\mathrm{Re}(V\boldsymbol{E}_\varepsilon))$. Outside of X_N we may define $\mathcal{R}_\varepsilon$ arbitrary. Hence, using $\mathcal{R}_\varepsilon^{\rm SH}(v) = \frac{1}{2}\|v\|_2^2$ and (2.3) we let

$$\mathcal{R}_\varepsilon(V) = \left\{ \begin{array}{ccc} \mathrm{L}^2_{\mathbb{C}} & \to & [0,\infty[, \\ V & \mapsto & \frac{1}{4}\|V\|_2^2 + \frac{\ell}{4}|\mathfrak{f}_{-N}(V)|^2. \end{array}\right. \tag{2.4}$$

The desired limiting gradient system $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}_{\rm GL}, \mathcal{R}_{\rm GL})$ is defined via

$$\mathcal{F}_{\rm GL}(A) = \int_{\mathbb{S}_\ell} |A'|^2 - \frac{\mu}{4}|A|^2 + \frac{27-38\gamma^2}{288}\,|A|^4\,\mathrm{d}x \quad \text{and} \quad \mathcal{R}_{\rm GL}(V) = \frac{1}{4}\|V\|_2^2,$$

because it generates the GLe (1.2) in the form $\frac{1}{2}\dot{A} = -\mathrm{D}\mathcal{F}_{\rm GL}(A)$. The first and elementary convergence result concerns the dissipation potentials.

Lemma 2.1 (Convergence of dissipation potentials) *Every sequence V_ε with $V_\varepsilon \to V$ in $\mathrm{L}^2_{\mathbb{C}}$ satisfies $\mathcal{R}_\varepsilon(V_\varepsilon) \to \mathcal{R}_{\rm GL}(V)$.*

Proof: Since $\|V_\varepsilon\|_2 \to \|V\|_2$, it remains to show that $f_{-N}(V_\varepsilon) \to 0$. However, $\boldsymbol{E}_\varepsilon = \mathbb{E}^N \rightharpoonup 0$ while $V_\varepsilon \to V$ in $\mathrm{L}^2_{\mathbb{C}}$. Hence $f_{-N}(V_\varepsilon) = \frac{2\pi}{\ell}\int_{\mathbb{S}_\ell} V_\varepsilon(x)\boldsymbol{E}_\varepsilon(x)\,\mathrm{d}x \to 0$ as desired. ■

The following equi-coercivity result will be crucial to derive the subsequent evolutionary Γ-convergence. We conjecture that $\gamma_0^2 = \frac{27}{38}$ but are unable to improve the crucial interpolation estimate in Theorem A.1.

Proposition 2.2 (Equi-coercivity of $\mathcal{F}_\varepsilon$) *There exists γ_0 with $\gamma_0^2 \in [\frac{2}{755}, \frac{27}{38}]$ such that for all γ with $|\gamma| < \gamma_0$ there exist constants $c, C > 0$ such that*

$$\forall\,\varepsilon \in\,]0,1]\ \forall\, A \in \mathrm{L}^2(\mathbb{S}_\ell): \quad \mathcal{F}_\varepsilon(A) \geq c\|A\|_{\mathrm{H}^1}^2 + c\|A\|_{\mathrm{L}^4}^4 - C. \tag{2.5}$$

Proof: We start the estimate in terms of the functional $\mathcal{F}_\varepsilon^{\rm SH}(u)$ where the formulas are simpler. By the bijection we obtain the corresponding estimate for $\mathcal{F}_\varepsilon(A)$. Using a rescaling we can restrict to the case $\ell = 2\pi$. The proof relies on the interpolation estimate (A.1) which allows us to estimate the cubic term $\gamma u^3/(3\varepsilon)$, namely

$$\left|\int_{\mathbb{S}_{2\pi}} u^3\,\mathrm{d}x\right| \leq \kappa_0\|u+\tfrac{1}{N^2}u''\|_2\|u\|_4^2 \leq \varepsilon\kappa_0\sqrt{2}\big(\tfrac{1}{2\varepsilon^2}\|u+\tfrac{1}{N^2}u''\|_2 + \tfrac{1}{4}\|u\|_4^4\big), \tag{2.6}$$

where we used $ab/(\sqrt{2}\varepsilon) \leq a^2/(2\varepsilon^2) + b^2/4$. This yields

$$\mathcal{F}_\varepsilon^{\rm SH}(u) \geq c_\gamma\left(\tfrac{1}{2\varepsilon^2}\|u+\tfrac{1}{N^2}u''\|_2^2 + \tfrac{1}{4}\|u\|_4^4\right) - \tfrac{\mu}{2}\|u\|_2^2 \quad \text{with } c_\gamma = 1 - \sqrt{2}|\gamma|\kappa_0/3. \tag{2.7}$$

Note that the constant c_γ is positive if and only if $\gamma^2 < \gamma_0^2 := 9/(2\kappa_0)$. The bounds for γ_0 follow from those for κ_0 at the end of the proof of Theorem A.1 and in Remark A.2.

We now return to $A = \Psi_\varepsilon u$ or $u = \mathrm{Re}(A\boldsymbol{E}_\varepsilon)$. According to (2.3) we have $\|u\|_2^2 \leq \|A\|_2^2 \leq 2\|u\|_2^2$, while $|u(x)| \leq |A(x)|$ and relation (A.2) yield $\|u\|_4 \leq \|A\|_4 \leq C_\Psi\|u\|_4$. Finally, a direct expansion in Fourier coefficients shows $\frac{1}{\varepsilon^2}\|u+\varepsilon^2u''\|_2^2 \geq \|A'\|_2^2$, see e.g. (A.3). Thus, we obtain

$$\mathcal{F}_\varepsilon(A) = \mathcal{F}_\varepsilon^{\rm SH}(\mathrm{Re}(A\boldsymbol{E}_\varepsilon)) \geq c_\gamma(\|A'\|_2^2 + \tfrac{1}{C_\Psi^4}\|A\|_4^4) - \tfrac{\mu}{2}\|A\|_2^2 \geq c\big(\|A'\|_2^2+\|A\|_2^2+\|A\|_4^4\big) - C,$$

where we used $\|A\|_4^4 \geq \frac{1}{2}\|A\|_4^4 + B\|A\|_2^2 - C_B$ for any $B > 0$. ■

We now present our two convergence results, the more elegant one is obtained in the case $\gamma = 0$, where we can use a uniform global λ-convexity, see below. It was already noticed in [KSM92] that the case $\gamma = 0$ is especially simple. Our present result simplifies the assumption of the convergence result to its absolute minimum. The results for the case $0 < |\gamma| < \gamma_0$ are weaker and need considerably more effort for the proof. We formulate both results for solutions $u(t, \cdot) \in \mathrm{L}^2_{\mathbb{R}}$ of the original Swift–Hohenberg equation (2.1) as gradient system $(\mathrm{L}^2_{\mathbb{R}}, \mathcal{F}^{\mathrm{SH}}_\varepsilon, \frac{1}{2}\|\cdot\|^2_2)$, whereas the proofs in Section 4.3 will be formulated in terms of the transformed equation $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$. These formulations are connected by the bijection Ψ_ε defined in (2.2).

Theorem 2.3 (Convergence for $\gamma = 0$) *For $\gamma = 0$ consider solutions $u_\varepsilon : [0, T] \to \mathrm{L}^2_{\mathbb{R}}$ for the Swift–Hohenberg equation* (2.1) *and a solution $A : [0, T] \to \mathrm{L}^2_{\mathbb{C}}$ for the associated Ginzburg–Landau equation* (1.2)*. Then, we have the evolutionary Γ-convergence:*

$$\begin{aligned} &\Psi_\varepsilon u_\varepsilon(0) \overset{\mathrm{L}^2}{\to} A_0(0) \\ \Longrightarrow \quad &\forall\, t > 0 : \ \Psi_\varepsilon u_\varepsilon(t) \overset{\mathrm{H}^1}{\rightharpoonup} A(t) \ \textit{and} \ \mathcal{F}^{\mathrm{SH},\gamma=0}_\varepsilon(u_\varepsilon(t)) \to \mathcal{F}^{\gamma=0}_{\mathrm{GL}}(A(t)). \end{aligned}$$

We emphasize that in the above result we do not make the assumption that the initial energies are bounded, i.e. we may assume $A(0) \in \mathrm{L}^2_{\mathbb{C}}$ is such a way that $\mathcal{F}^{\gamma=0}_{\mathrm{GL}}(A(0)) = \infty$. In particular, we do not assume any type of wellpreparedness of the initial conditions, see [Mie14] for a discussion on this. In contrast, we need for $\gamma \neq 0$ a uniform bound on the initial energies.

Theorem 2.4 (Evolutionary Γ-convergence for $\gamma \neq 0$) *Assume $0 < |\gamma| < \gamma_0$ and consider solutions $u_\varepsilon : [0, T] \to \mathrm{L}^2_{\mathbb{R}}$ for the Swift–Hohenberg equation* (2.1) *and a solution $A : [0, T] \to \mathrm{L}^2_{\mathbb{C}}$ for the associated Ginzburg–Landau equation* (1.2)*. Then, we have the evolutionary Γ-convergence:*

$$\begin{aligned} &\Psi_\varepsilon u_\varepsilon(0) \overset{\mathrm{L}^2}{\to} A_0(0) \quad \textit{and} \quad \sup_{0<\varepsilon<1} \mathcal{F}^{\mathrm{SH}}_\varepsilon(u_\varepsilon(0)) < \infty \\ \Longrightarrow \quad &\forall\, t > 0 : \ \Psi_\varepsilon u_\varepsilon(t) \overset{\mathrm{H}^1}{\rightharpoonup} A(t) \ \textit{and} \ \mathcal{F}^{\mathrm{SH}}_\varepsilon(u_\varepsilon(t)) \to \mathcal{F}_{\mathrm{GL}}(A(t)). \end{aligned}$$

In this result we need boundedness of the initial energies, which by Proposition 2.2 implies boundedness of $\Psi_\varepsilon u_\varepsilon(0)$ in $\mathrm{H}^1_{\mathbb{C}}$ and such turns the convergence of the initial data into weak convergence in $\mathrm{H}^1_{\mathbb{C}}$. However, it is still weaker than the wellpreparedness condition $\mathcal{F}^{\mathrm{SH}}_\varepsilon(u_\varepsilon(0)) \to \mathcal{F}_{\mathrm{GL}}(A(0))$ needed for the convergence results based on the energy-dissipation balance, see [SaS04, Ser11, Mie14].

The proofs of these two results are given via the proofs of Theorems 4.6 and 4.7.

3 Evolutionary Γ-convergence via uniform λ-convexity

Here we present the main abstract arguments for the limit passage which rely on the reformulation of the parabolic equations as gradient systems in the the form of an evolutionary variational inequality. We give a brief account of the much more general theory in [DaS10], by restricting ourselves to the case of Hilbert spaces which is relevant for our application in amplitude equations.

We start from a general separable Hilbert space $\boldsymbol{H}$, an energy functional $\mathcal{E}$ and a quadratic dissipation potential

$$\mathcal{R}(v) = \frac{1}{2}\langle \mathbb{G}v, v\rangle \text{ with } c_0\|v\|^2 \leq \mathcal{R}(v) \leq c_1\|v\|^2,$$

where $\mathbb{G}$ is a bounded, positive definite symmetric operator. Note that $v \mapsto \big(2\mathcal{R}(v)\big)^{1/2}$ generates an equivalent Hilbert space norm, but be prefer to keep $\mathcal{R}$ and $\|\cdot\|$ distinct, because later on $\mathcal{R}$ will depend on ε. The functional $\mathcal{E} : \boldsymbol{H} \to \mathbb{R}_\infty := \mathbb{R}\cup\{\infty\}$ is assumed to be lower semicontinuous (in the strong topology) and coercive, e.g.

$$\exists\, c, C > 0\ \forall\, u \in \boldsymbol{H} : \quad \mathcal{E}(u) \geq c\|u\| - C. \tag{3.1}$$

The important condition is the *geodesic λ-convexity*, which relates to classical convexity in Hilbert spaces, where the geodesic curves are unique and are given by straight lines:

$$\text{the mapping } u \mapsto \mathcal{E}(u) - \lambda\mathcal{R}(u) \text{ is convex.} \tag{3.2}$$

(For smooth functionals this is equivalent to the local condition $\mathrm{D}^2\mathcal{E}(u) \geq \mathbb{G}$ for all $u \in \boldsymbol{H}$.) For such functions the Fréchet subdifferential $\partial^{\mathrm{F}}\mathcal{E} : \boldsymbol{H} \rightrightarrows \boldsymbol{H}^*$ exists (cf. [MRS13]) and for all (u, ξ) with $\xi \in \partial^{\mathrm{F}}\mathcal{E}(u)$ we have

$$\mathcal{E}(w) \geq \mathcal{E}(u) + \langle \xi, w{-}u\rangle + \lambda\mathcal{R}(w{-}u) \text{ for all } w \in \boldsymbol{H}. \tag{3.3}$$

As a consequence of the last estimate we immediately see that for solutions $u : [0, T] \to \boldsymbol{H}$ of evolution equation in form of the subdifferential inclusion

$$0 \in \mathbb{G}\dot{u}(t) + \partial^{\mathrm{F}}\mathcal{E}(u(t)) \text{ for a.a. } t \in [0, T] \tag{3.4}$$

we obtain the differential form of *Evolutionary Variational Inequality* $(\mathrm{EVI})_\lambda$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(u(t){-}w) = \langle \mathbb{G}\dot{u}, u{-}w\rangle = \langle \xi, w{-}u\rangle \leq \mathcal{E}(w) - \mathcal{E}(u) - \lambda\mathcal{R}(u{-}w) \ \text{ for all } w,$$

(cf. [AGS05]) which corresponds to the Hilbert space version of Benilan's weak formulation [Bén72] in the case $\lambda = 0$. Note that existence and uniqueness of solution follows in the present Hilbert space setting from classical semigroup theory of maximal monotone operators with Lipschitz perturbation, see [Bré73, Ch. 3].

Multiplying with $\mathrm{e}^{\lambda t}$ and integrating over an interval $[s, t] \subset\ ]0, T]$ leads to an integrated form of $(\mathrm{EVI})_\lambda$. We call $u : [0, T] \to \boldsymbol{H}$ a solution of the *Integrated Evolutionary Variational Inequality* $(\mathrm{IEVI})_\lambda$, if

$$(\mathrm{IEVI})_\lambda \quad \begin{cases} \forall\, s, t \in [0, T] \text{ with } 0 \leq s < t \leq T \ \ \forall\, w \in \boldsymbol{H} : \\ \mathrm{e}^{\lambda(t-s)}\mathcal{R}(u(t){-}w) - \mathcal{R}(u(s){-}w) \leq M_\lambda(t{-}s)\big(\mathcal{E}(w) - \mathcal{E}(u(t))\big), \end{cases} \tag{3.5}$$

where $M_\lambda(r) := \int_0^r \mathrm{e}^{\lambda\rho}\,\mathrm{d}\rho$. The great advantage of this formulation is that it is absolutely derivative free, which makes it an ideal starting point for convergence theories involving functionals. Moreover, this formulation is sufficiently strong to return back to the differential equation, i.e. every solution of (3.5) is a solution of (3.4), see [DaS10, Thm. 2.5].

Our result on evolutionary Γ-convergence for $\varepsilon \to 0$ concerns a family of gradient systems $(\boldsymbol{H}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, $\varepsilon \in [0, 1]$, all of which are of the type as above. We collect all assumption to emphasize their uniformity for $\varepsilon \in [0, 1]$:

Energy functionals

$$\exists \text{ Banach space } \boldsymbol{Z}: \quad \boldsymbol{Z} \Subset \boldsymbol{H} \text{ (compact embedding)}, \tag{3.6a}$$

$$\exists\, c, C>0\ \forall \varepsilon\in[0,1]\ \forall u\in\boldsymbol{H}: \quad \mathcal{E}_\varepsilon(u) \geq c\|u\|_{\boldsymbol{Z}} - C, \tag{3.6b}$$

$$\exists\, \lambda\in\mathbb{R}\ \forall\varepsilon\in[0,1]: \quad \mathcal{E}_\varepsilon(\cdot) - \lambda\mathcal{R}_\varepsilon(\cdot) \text{ is convex.} \tag{3.6c}$$

Dissipation potentials

$$\mathcal{R}_\varepsilon(v) = \frac{1}{2}\langle \mathbb{G}_\varepsilon v, v\rangle \quad \text{and} \quad \exists\, c_0, c_1>0\ \forall v\in\boldsymbol{H}:\ c_0\|v\|^2_{\boldsymbol{H}} \leq \mathcal{R}_\varepsilon(v) \leq c_1\|v\|^2_{\boldsymbol{H}}. \tag{3.6d}$$

Convergence properties

$$\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0 \text{ in } \boldsymbol{H} \quad \text{(Mosco convergence)}, \tag{3.6e}$$

$$u_\varepsilon \to u_0 \text{ in } \boldsymbol{H} \implies \mathcal{R}_\varepsilon(u_\varepsilon) \to \mathcal{R}_0(u_0). \tag{3.6f}$$

In (3.6b) we set $\|u\|_{\boldsymbol{Z}} = \infty$ for $u\in\boldsymbol{H}\setminus\boldsymbol{Z}$. The Mosco convergence in (3.6e) means the two Γ-convergences $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0$ and $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in $\boldsymbol{H}$, see [Dal93]. The first means that (i) $u_\varepsilon \rightharpoonup u_0$ implies $\mathcal{E}_0(u_0) \leq \liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(u_\varepsilon)$ and (ii) that for each $\widehat{u}_0\in\boldsymbol{H}$ there exists $\widehat{u}_\varepsilon$ such that $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}_0$ and $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(\widehat{u}_0)$. The definition of $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ is similar, but all "$\rightharpoonup$" are replaced by "$\to$".

Remark 3.1 (Γ vs. Mosco convergence) *It is well-known (cf. [Dal93, Mie14]) that, given the compact embedding $\boldsymbol{Z}\Subset\boldsymbol{H}$ in* (3.6a) *and the equi-coercivity* (3.6b) *in $\boldsymbol{Z}$, the Mosco convergence $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ in* (3.6e) *is equivalent to the Γ-convergence $\mathcal{E}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{E}_0$ in $\boldsymbol{Z}$.*

The following convergence result is a variant of [DaS10, Thm. 2.17], since we allow the distance to depend mildly on ε but restrict to Hilbert spaces. To emphasize the strength of theory we remark that solutions $u:[0,T]\to\boldsymbol{H}$ of the $(\mathrm{IEVI})_\lambda$ do not need to have finite energy at $t=0$. Defining the domain of $\mathcal{E}$ by $\mathrm{dom}(\mathcal{E}) := \{\, u\in\boldsymbol{H} \mid \mathcal{E}(u)<\infty\,\}$ one has a unique solution for each $u(0)\in\overline{\mathrm{dom}(\mathcal{E})}$. This solution is still continuous, but may not be absolutely continuous. Hence, it is surprising that the mere convergence $u_\varepsilon(0)\to u_0(0)$ of the initial conditions is sufficient.

Theorem 3.2 (Evolutionary Γ-convergence for EVI$_\lambda$) *Assume that the gradient systems $(\boldsymbol{H},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$, $\varepsilon\in[0,1]$, satisfy* (3.6). *Then, consider any $T>0$ and solutions $u_\varepsilon:[0,T]\to\boldsymbol{H}$ for $(\boldsymbol{H},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$. Then, we have the evolutionary Γ-convergence*

$$u_\varepsilon(0)\in\mathrm{dom}(\mathcal{E}_\varepsilon) \text{ and } u_\varepsilon(0)\to u_0(0) \text{ in } \boldsymbol{H} \tag{3.7}$$

$$\implies \quad \forall\, t\in\,]0,T]: \quad u_\varepsilon(t)\rightharpoonup u_0(t) \text{ in } \boldsymbol{Z} \text{ and } \mathcal{E}_\varepsilon(u_\varepsilon(t))\to\mathcal{E}_0(u_0(t)). \tag{3.8}$$

Proof: We follow Section 2 of [DaS10]. The solutions u_ε satisfy $(\mathrm{IEVI})_\lambda$ in the form

$$(\mathrm{IEVI})^\varepsilon_\lambda \quad \begin{cases} \forall\, 0\leq s<t\leq T \quad \forall\, w\in\boldsymbol{H}: \\ \mathrm{e}^{\lambda(t-s)}\mathcal{R}_\varepsilon(u_\varepsilon(t){-}w) - \mathcal{R}_\varepsilon(u_\varepsilon(s){-}w) \leq M_\lambda(t{-}s)\big(\mathcal{E}_\varepsilon(w) - \mathcal{E}_\varepsilon(u_\varepsilon(t))\big). \end{cases} \tag{3.9}$$

We choose test states $w_\varepsilon\to w_0$ in $\boldsymbol{H}$ such that $\mathcal{E}_\varepsilon(w_\varepsilon)\leq 2\mathcal{E}_0(w_0)<\infty$, insert them into $(\mathrm{IEVI})^\varepsilon_\lambda$ with $s=0$, and obtain

$$\mathrm{e}^{\lambda t}\mathcal{R}_\varepsilon(u_\varepsilon(t){-}w_\varepsilon) + M_\lambda(t)\mathcal{E}_\varepsilon(u_\varepsilon) \leq \mathcal{R}_\varepsilon(u_\varepsilon(0){-}w_\varepsilon) + M_\lambda(t)\mathcal{E}_\varepsilon(w_\varepsilon) \leq C_1. \tag{3.10}$$

Using $M_\lambda(t) \geq m_0 > 00$ for $0 < t_0 \leq t \leq T0$ gives a uniform bound on the energy and

$$\|u_\varepsilon\|_{\mathrm{L}^\infty([t_0,T];\boldsymbol{Z})} \leq C_2(t_0) \quad \text{for all } \varepsilon \in [0,1],$$

because of the equi-coercivity (3.6b). Moreover, looking at a partition $t_k = t_0 + k\tau_N$ with $\tau_N = (T-t_0)/N$ we insert $t = t_k$, $s = t_{k-1}$, and $w = u_\varepsilon(s)$ into $(\mathrm{IEVI})^\varepsilon_\lambda$ and find

$$\mathrm{e}^{\lambda_N \tau}\mathcal{R}_\varepsilon(u_\varepsilon(t_k)-u_\varepsilon(t_{k-1})) \leq M_\lambda(\tau_N)\big(\mathcal{E}_\varepsilon(u_\varepsilon(t_{k-1})) - \mathcal{E}_\varepsilon(u_\varepsilon(t_k))\big).$$

Summing from $k = 1$ to N and taking the limit $N \to \infty$ we find

$$\int_{t_0}^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon(s))\,\mathrm{d}s \leq \mathcal{E}_\varepsilon(u_\varepsilon(t_0)) - \mathcal{E}_\varepsilon(u_\varepsilon(T)) \leq C_3. \tag{3.11}$$

Thus, for all $t_0 \in\,]0,T]$ we have uniform bound in $\mathrm{C}^{1/2}([t_0,T];\boldsymbol{H})$, such that by $\boldsymbol{Z} \Subset \boldsymbol{H}$ (cf. (3.6a)) we can find a subsequence (not relabeled) such that $u_\varepsilon(t) \rightharpoonup U(t)$ in $\boldsymbol{Z}$ for all $t > 0$. We set $U(0) = u_0(0)$ such that $u_\varepsilon(t) \to U(t)$ in $\boldsymbol{H}$ for all $t \in [0,T]$.

To pass to the limit in $(\mathrm{IEVI})^\varepsilon_\lambda$ we take any $\widehat{w}$ and insert $w = \widehat{w}_\varepsilon$, where $\widehat{w}_\varepsilon \to \widehat{w}$ is a recovery sequence with $\mathcal{E}_\varepsilon(\widehat{w}_\varepsilon) \to \mathcal{E}_0(\widehat{w})$. Using the convergences (3.6e) and (3.6f) yields

$$\mathrm{e}^{\lambda(t-s)}\mathcal{R}_0(U(t)-w) - \mathcal{R}_0(U(s)-w) \leq M_\lambda(t-s)\,\big(\mathcal{E}_0(w) - \mathcal{E}_0(U(t))\big). \tag{3.12}$$

Thus, $U : [0,T] \to \boldsymbol{H}$ is a solution of $(\mathrm{IEVI})^{\varepsilon=0}_\lambda$. To conclude that U is equal to the unique solution u_0 it suffices to show continuity of U at $t = 0$. Inserting $s = 0$ and any $w \in \mathrm{dom}(\mathcal{E}_0)$ we consider the limit $t \to 0^+$ to obtain

$$\lim_{t\to 0^+} \mathcal{R}_0(U(t)-w) - \mathcal{R}_0(u_0(0)-w) \leq \lim_{t\to 0^+} M_\lambda(t)\big(\mathcal{E}_0(w) - \inf \mathcal{E}_0\big) = 0,$$

because $M_\lambda(t) = O(t)$. Thus, we have $\lim_{t\to 0^+} \|U(t)-w\|_{\boldsymbol{H}} \leq \|u_0(0)-w\|_{\boldsymbol{H}}$ for all $w \in \mathrm{dom}(\mathcal{E}_0)$. Using $u_0(0) \in \overline{\mathrm{dom}(\mathcal{E}_0)}$ we conclude $U(t) \to u_0(0)$ for $t \to 0^+$ as desired.

It remains to show that the energies converge as well. This theory is more advanced (cf. [DaS10, Sav11]), and we give only the main idea, which relies on the notion of metric slope, which reads in our Hilbert-space setting as follows

$$|\partial\mathcal{E}|_\mathcal{R}(u) := \limsup_{w\to u} \frac{\max\{\mathcal{E}(u)-\mathcal{E}(w), 0\}}{\|u-w\|_\mathcal{R}}, \text{ where } \|v\|_\mathcal{R} := \big(2\mathcal{R}(v)\big)^{1/2}.$$

In the λ-convex Hilbert-space case the slope can be expressed via the Fréchet subdifferential $\partial^{\mathrm{F}}\mathcal{E}$ as $|\partial\mathcal{E}|_\mathcal{R}(u) = \inf\{\, \|\eta\|_{*,\mathcal{R}} \mid \eta \in \partial^{\mathrm{F}}\mathcal{E}(u)\,\}$, and (3.3) yields the lower bound

$$\mathcal{E}(w) \geq \mathcal{E}(u) - |\partial\mathcal{E}|_\mathcal{R}(u)\|w-u\|_\mathcal{R} + \lambda\mathcal{R}(w-u) \quad \text{for all } u, w \in \boldsymbol{H}. \tag{3.13}$$

The main observation (see [DaS10, eqn. (2.9)]) is that the a priori estimate (3.10) can be improved to an a priori bound including the slope as well, namely

$$\mathrm{e}^{\lambda t}\mathcal{R}_\varepsilon(u_\varepsilon(t)-w_\varepsilon) + M_\lambda(t)\mathcal{E}_\varepsilon(u_\varepsilon) + \tfrac{M_\lambda(t)^2}{2}|\partial\mathcal{E}_\varepsilon(u_\varepsilon(t)|^2_{\mathcal{R}_\varepsilon} \leq C_1. \tag{3.14}$$

Hence, as above the slopes are uniformly bounded by a constant $S(t_0)$ for all $t \in [t_0,T]$ and all $\varepsilon \in [0,1]$. Fixing $t \in [t_0,T]$ we choose a recovery sequence $\widehat{u}_\varepsilon \to u_0(t)$ with $\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \to \mathcal{E}_0(u_0(t))$, then the lower bound (3.13) gives the estimate

$$\mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \geq \mathcal{E}_\varepsilon(u_\varepsilon(t)) - S(t_1)\|\widehat{u}_\varepsilon-u_\varepsilon(t)\|_{\mathcal{R}_\varepsilon} + \lambda\mathcal{R}_\varepsilon(\widehat{u}_\varepsilon-u_\varepsilon(t)).$$

Since $u_\varepsilon(t) \to u_0(t)$ we have $\mathcal{R}_\varepsilon(\widehat{u}_\varepsilon - u_\varepsilon(t)) \to 0$ for $\varepsilon \to 0$ by (3.6d), and the limit $\varepsilon \to 0$ yields $\mathcal{E}_0(u_0(t)) = \lim \mathcal{E}_\varepsilon(\widehat{u}_\varepsilon) \geq \limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon(t))$. Since the opposite inequality follows from the Mosco convergence we conclude the desired energy convergence. ■

We will use the above result of the Swift–Hohenberg equation in the case $\gamma = 0$. However, for the case $\gamma \neq 0$ we need to modify the theory. For that purpose we restrict the λ-convexity to a closed convex set $\mathcal{B} \subset \boldsymbol{H}$, i.e. instead of (3.6c) we assume that $\mathcal{E}_\varepsilon - \lambda \mathcal{R}_\varepsilon$ are convex when restricted to $\mathcal{B}$.

Then, $(\text{IEVI})_\lambda$ can be derived from the subdifferential inclusion (3.4) in exactly the same manner as above, if the solution $u : [0,T] \to \boldsymbol{H}$ lies in $\mathcal{B}$ for all t and if we restrict the test states w to the set $\mathcal{B}$. For the derivation one only relies on the lower bounds (3.3) and (3.13), which are still valid for all $u, w \in \mathcal{B}$. Moreover, we can still go back from $(\text{IEVI})_\lambda$ to (3.4), if the solution u does not touch the boundary of $\mathcal{B}$.

The approach involving $\mathcal{B}$ can also be understood by changing the energy functionals and replacing $\mathcal{E}_\varepsilon$ by $\mathcal{E}_\varepsilon^{\mathcal{B}} = \mathcal{E}_\varepsilon + \chi_{\mathcal{B}}$, where $\chi_{\mathcal{B}}(u) = 0$ for $u \in \mathcal{B}$ and $+\infty$ otherwise. The problem is that $\mathcal{E}_\varepsilon \stackrel{\mathrm{M}}{\to} \mathcal{E}_0$ doesn't imply that $\mathcal{E}_\varepsilon^{\mathcal{B}} \stackrel{\mathrm{M}}{\to} \mathcal{E}_0^{\mathcal{B}}$. However, in (4.2b) we will be able to control the size of the recovery sequences in such a way that each $\widehat{u} \in \mathcal{B}_{\text{small}}$ admits recovery sequences $\widehat{u}_\varepsilon \in \mathcal{B}$, then the Γ-limsup and the Γ-liminf of $\mathcal{E}_\varepsilon^{\mathcal{B}}$ will coincide with $\mathcal{E}_0$ on $\mathcal{B}_{\text{small}}$. Thus, we can proceed as in the proof of Theorem 3.2 and conclude that limit solutions lying in $\mathcal{B}_{\text{small}}$ satisfy an $(\text{IEVI})_\lambda$ with test states $w \in \mathcal{B}_{\text{small}}$.

4 Evolutionary Γ-convergence for the SHe

We now employ the abstract theory from above for the justification of the GLe as an amplitude equation for the SHe. Relying on the equi-coercivity of Proposition 2.2 we still we have to establish Γ-convergence (see Section 4.1) and a suitable weakened version of the λ-convexity (see Section 4.2) that allows us to invoke the formulation based on $(\text{IEVI})_\lambda^\varepsilon$. The final results on evolutionary Γ-convergence are established in Section 4.3.

4.1 Γ-convergence of the energies $\mathcal{F}_\varepsilon$

We first consider the case $\gamma = 0$ and show the Mosco convergence of $\mathcal{F}_\varepsilon^{\gamma=0}$ to $\mathcal{F}_{\mathrm{GL}}^{\gamma=0}$ in $\mathrm{H}^1_{\mathbb{C}}$. The main effort concerns the case $\gamma \neq 0$ where we only obtain weak Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{GL}}$ in $\mathrm{H}^1_{\mathbb{C}}$. The essential difficulty is the treatment of the singular term $\int_{\mathbb{S}_\ell} \frac{\gamma}{3\varepsilon} u_\varepsilon(x)^3 \,\mathrm{d}x$, which relies on a careful treatment of the higher harmonics generated by the cubic term. Since $\mathrm{Re}(A\boldsymbol{E}_\varepsilon)$ is rapidly oscillating one can expect that its average is 0. In fact, we use an integration by parts based on $\varepsilon \boldsymbol{E}_\varepsilon' = \mathrm{i}\boldsymbol{E}_\varepsilon$, i.e. to eliminate the power $1/\varepsilon$ we have to replace one of the factors A by its derivative A'. Thus, this proof reflects the correction terms or the normal form transformations used in the classical approach to the derivation and justification of amplitude equations, see [Sch94, MiS96, Sch98].

To study the Γ-convergence of $\mathcal{F}_\varepsilon$ to $\mathcal{F}_{\mathrm{GL}}$ we decompose $\mathcal{F}_\varepsilon$ into the functionals

$$\mathcal{J}_\varepsilon^{\text{quadr}}(A) := \frac{\ell}{4\varepsilon^2}|\mathfrak{f}_{-N}(A)|^2 + \begin{cases} \int_{\mathbb{S}_\ell} \frac{1}{4}|2\mathrm{i}A' + \varepsilon A''|^2 \,\mathrm{d}x & \text{for } A \in \mathrm{X}_N, \\ \infty & \text{otherwise;} \end{cases}$$

$$\mathcal{J}_\varepsilon^{\text{cubic}}(A) := -\int_{\mathbb{S}_\ell} \frac{\gamma}{3\varepsilon} \mathrm{Re}(A\boldsymbol{E}_\varepsilon)^3 \,\mathrm{d}x, \qquad \mathcal{J}_\varepsilon^{\text{quart}}(A) := \int_{\mathbb{S}_\ell} \frac{1}{4} \mathrm{Re}(A\boldsymbol{E}_\varepsilon)^4 \,\mathrm{d}x,$$

$$\mathcal{J}_0^{\text{quadr}}(A) = \|A'\|_2^2, \quad \mathcal{J}_0^{\text{cubic}}(A) = -\frac{19\gamma^2}{144}\|A\|_4^4, \quad \mathcal{J}_0^{\text{quart}}(A) = \frac{3}{32}\|A\|_4^4.$$

Obviously, we have $\mathcal{F}_\varepsilon(A) = \mathcal{J}_\varepsilon^{\text{quadr}}(A) - \mu\mathcal{R}_\varepsilon(A) + \mathcal{J}_\varepsilon^{\text{cubic}}(A) + \mathcal{J}_\varepsilon^{\text{quart}}(A)$ and $\mathcal{F}_{\text{GL}}(A) = \mathcal{J}_0^{\text{quadr}}(A) - \mu\mathcal{R}_0(A) + \mathcal{J}_0^{\text{cubic}}(A) + \mathcal{J}_0^{\text{quart}}(A)$.

Our first proposition shows that the Γ-convergence of $\mathcal{J}_\varepsilon^{\text{quadr}}$ and $\mathcal{J}_\varepsilon^{\text{quadr}}$ is easily obtained and is in fact the better Mosco convergence. Thus, the full convergence result for the case $\gamma = 0$ is much simpler than the case $\gamma \neq 0$, which corresponds to the fact that the "cubic paper" [KSM92] is so much shorter that the "quadratic paper" [Sch94].

Proposition 4.1 (Case $\gamma = 0$) *We have the Mosco convergence* $\mathcal{F}_\varepsilon^{\gamma=0} \xrightarrow{\text{M}} \mathcal{F}_{\text{GL}}^{\gamma=0}$, *i.e.*

$$A_\varepsilon \rightharpoonup A \text{ in } \mathrm{H}^1_{\mathbb{C}} \quad \Longrightarrow \quad \mathcal{F}_{\text{GL}}^{\gamma=0}(A) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{\gamma=0}(A_\varepsilon) \tag{4.1a}$$

$$\forall\, A \in \mathrm{H}^1_{\mathbb{C}}\ \exists\, (A_\varepsilon)_{\varepsilon>0} : \quad A_\varepsilon \to A \ \text{ in } \mathrm{H}^1_{\mathbb{C}} \ \text{ and } \ \mathcal{F}_\varepsilon^{\gamma=0}(A_\varepsilon) \to \mathcal{F}_{\text{GL}}^{\gamma=0}(A). \tag{4.1b}$$

Proof: We write A_N instead of A_ε using the relation $2\pi\varepsilon N = \ell$.

Step 1: By the compact embedding of $\mathrm{H}^1_{\mathbb{C}}$ into $\mathrm{C}^0_{\mathbb{C}}$ we see that $A_N \rightharpoonup A$ in $\mathrm{H}^1_{\mathbb{C}}$ implies $A_N \to A$ in $\mathrm{C}^0_{\mathbb{C}}$, and thus in $\mathrm{L}^p_{\mathbb{C}}$ for $p = 2$ and $p = 4$. To study the convergence of $\mathcal{J}_\varepsilon^{\text{quart}}(A_N)$ use $16\,\mathrm{Re}(A\boldsymbol{E}_\varepsilon)^4 = A^4\boldsymbol{E}_\varepsilon^4 + 4|A|^2A^2\boldsymbol{E}_\varepsilon^2 + 6|A|^4 + 4|A|^2\overline{A}^2\overline{\boldsymbol{E}}_\varepsilon^2 + \overline{A}^4\overline{\boldsymbol{E}}_\varepsilon^4$ and $\boldsymbol{E}_\varepsilon^m \rightharpoonup 0$ in $\mathrm{L}^2_{\mathbb{C}}$ whenever $m \neq 0$. By the strong convergence of A_N in $\mathrm{L}^4_{\mathbb{C}}$ we have $16\,\mathrm{Re}(A_N\boldsymbol{E}_\varepsilon)^4 \rightharpoonup 6|A|^4$, which gives $\mathcal{J}_\varepsilon^{\text{quart}}(A_N) \to \mathcal{J}_0^{\text{quart}}(A)$.

Step 2: Similarly, we obtain $\mathcal{R}_\varepsilon(A_N) \to \mathcal{R}_0(A)$, see Lemma 2.1.

Step 3: The only remaining term is $\mathcal{J}_\varepsilon^{\text{quadr}}$. We first do the liminf estimate corresponding to (4.1a) using Fatou's lemma. Indeed, $A_N \rightharpoonup A$ implies $\mathfrak{f}_m(A_N) \to \mathfrak{f}_m(A)$ as $N \to \infty$ for all $m \in \mathbb{Z}$. Hence, dropping the term $\frac{\ell}{4\varepsilon^2}|\mathfrak{f}_{-N}(A_N)|^2$ we obtain

$$\begin{aligned}\liminf_{N\to\infty} \mathcal{J}_\varepsilon^{\text{quadr}}(A_N) &\ge \liminf_{N\to\infty} \ell \sum_{m=-N}^{\infty} \frac{1}{4}\big|(-2m - \varepsilon m^2)\mathfrak{f}_m(A_N)\big|^2 \\ &\ge \ \ell \sum_{m\in\mathbb{Z}} m^2|\mathfrak{f}_m(A)|^2 \ = \ \|A'\|_2^2 \ = \ \mathcal{J}_0^{\text{quadr}}(A).\end{aligned}$$

In summary, Step1 to 3 establish the liminf estimate (4.1a).

Step 4: The recovery sequence is constructed via $A_N = \sum_{|m|\le\sqrt{N}} \mathfrak{f}_m(A)\mathbb{E}^m$. Clearly, $A_N \to A$ in $\mathrm{H}^1_{\mathbb{C}}$ and the convergence of $\mathcal{J}_\varepsilon^{\text{quart}}(A_N)$ and $\mathcal{R}_\varepsilon(A_N)$ was shown in Steps 1 and 2, respectively. With $\alpha_m = |m\mathfrak{f}_m(A)|^2$ we have $\mathfrak{f}_{-N}(A_N) = 0$, $\sum_{m\in\mathbb{Z}} \alpha_m < \infty$, and obtain

$$\begin{aligned}&\Big|\mathcal{J}_\varepsilon^{\text{quadr}}(A_N) - \mathcal{J}_0^{\text{quadr}}(A)\Big| \le \ell \sum_{|m|\le\sqrt{N}} \Big|(1 + \frac{\varepsilon m}{2})^2 - 1\Big|\alpha_m + \ell \sum_{|m|>\sqrt{N}} \alpha_m \\ &\le \ell \sum_{|m|\le\sqrt{N}} 2\varepsilon|m|\alpha_m + \ell \sum_{|m|>\sqrt{N}} \alpha_m \ \le \ \ell \sum_{m\in\mathbb{Z}} \min\{\varepsilon^{1/2}|m|, 1\}\alpha_m \ \to \ 0 \text{ for } \varepsilon \to 0,\end{aligned}$$

by Lebesgue's dominated convergence theorem. This proves (4.1b). ∎

We now turn to the more difficult case involving the cubic term. We will see that the convergence will only be a weak Γ-convergence. The difficulty in treating the Γ-convergence for $\gamma \neq 0$ is the prefactor $1/\varepsilon$ in $\mathcal{J}_\varepsilon^{\text{cubic}}$ such that it is necessary to exploit that the average of $\big(\mathrm{Re}\, A\boldsymbol{E}_\varepsilon\big)^3$ is of order ε. The main point is that the quadratic part $\mathcal{J}_\varepsilon^{\text{quadr}}$ forces the functions $u_\varepsilon = \mathrm{Re}(A_N\boldsymbol{E}_\varepsilon)$ to be highly oscillatory. However, higher harmonics will contribute nontrivially in the sense of averaging.

Proposition 4.2 *Let $\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon^{quadr} + \mathcal{J}_\varepsilon^{cubic}$ for $\varepsilon \geq 0$. Then, $\mathcal{J}_\varepsilon \overset{\Gamma}{\rightharpoonup} \mathcal{J}_0$ in $\mathrm{H}^1_\mathbb{C}$, i.e.*

$$A_\varepsilon \rightharpoonup A \text{ in } \mathrm{H}^1_\mathbb{C} \quad \Longrightarrow \quad \mathcal{J}_0(A) \leq \liminf_{\varepsilon\to 0} \mathcal{J}_\varepsilon(A_\varepsilon) \tag{4.2a}$$

$$\begin{aligned} \forall\, A \in \mathrm{H}^1_\mathbb{C}\ \exists\, (A_\varepsilon)_{\varepsilon>0} :\quad & A_\varepsilon \rightharpoonup A, \quad \mathcal{J}_\varepsilon(A_\varepsilon) \to \mathcal{J}_0(A), \\ & \text{and } \|A_\varepsilon\|_{\mathrm{H}^1} \leq \|A\|_{\mathrm{H}^1} + \widehat{c}_\ell |\gamma| \|A\|^2_{\mathrm{H}^1}, \end{aligned} \tag{4.2b}$$

where $\widehat{c}_\ell$ depends only on $\ell > 0$. In the strong topology of $\mathrm{H}^1_\mathbb{C}$ we have $\mathcal{J}_\varepsilon \overset{\Gamma}{\to} \mathcal{J}_0^{quadr}$.

The following result is a simple consequence of the previous two propositions, see Remark 3.1 and the equi-coercivity in Proposition 2.2.

Corollary 4.3 (Γ-convergence of $\mathcal{F}_\varepsilon$ for $\gamma \neq 0$) *For $\gamma \neq 0$ we have $\mathcal{F}_\varepsilon \overset{\Gamma}{\rightharpoonup} \mathcal{F}_{\mathrm{GL}}$ in $\mathrm{H}^1_\mathbb{C}$, and for $|\gamma| < \gamma_0$ we have $\mathcal{F}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{F}_{\mathrm{GL}}$ in $\mathrm{L}^2_\mathbb{C}$.*

In the following proof we will use that the transformation operator Ψ_ε of (2.2) can be expressed by a general operator $\boldsymbol{P}_\theta$ defined via

$$\boldsymbol{P}_\theta : \left\{ \begin{array}{ccc} \mathrm{L}^2_\mathbb{C} & \to & \mathrm{L}^2_\mathbb{C}\,, \\ w & \mapsto & \theta \operatorname{Re} \mathfrak{f}_0(w) + \sum_{m>0} \mathfrak{f}_m(w) \mathbb{E}^m, \end{array} \right. \tag{4.3}$$

which maps $\mathrm{L}^2_\mathbb{C}$ into the space X_0 and gives $\Psi_\varepsilon u = (\boldsymbol{P}_{1/2} u) \overline{\boldsymbol{E}}_\varepsilon$. For general θ we obtain formulas similarly to (2.3), namely for $v \in \mathrm{L}^2_\mathbb{R}$ we have

$$\begin{aligned} &\boldsymbol{P}_1 v = \boldsymbol{P}_1\big(\boldsymbol{P}_1 v\big), && \int_{\mathbb{S}_\ell} v(\boldsymbol{P}_\theta v)\,\mathrm{d}x = \frac{1}{2}\|v\|_2^2 + \ell(\theta - \tfrac{1}{2})\mathfrak{f}_0(v)^2, \\ &v = \boldsymbol{P}_{1/2} v + \overline{\boldsymbol{P}_{1/2} v}, && \|v\|_2^2 = 2\|\boldsymbol{P}_\theta v\|_2^2 + \ell(1 - 2\theta^2)\mathfrak{f}_0(v)^2. \end{aligned} \tag{4.4}$$

Proof of Proposition 4.2: We first establish the liminf estimate (4.2a) in Steps 1 to 3. Step 4 provides the recovery sequence construction (4.2b), and the strong Γ-convergence is established in Step 5.

Step 1: For the arbitrary sequence A_N with $A_N \rightharpoonup A$ in $\mathrm{H}^1_\mathbb{C}$ we use the decomposition into its central part C_N and the remainder R_N as follows:

$$A_N = \sum_{n \geq -N} a_n \mathbb{E}^n = C_N + R_N \quad \text{with} \quad C_N := \sum_{|n| \leq \sqrt{N}} a_n \mathbb{E}^n, \quad R_N := \sum_{|n| > \sqrt{N}} a_n \mathbb{E}^n.$$

Because $(A_N)_N$ is bounded in $\mathrm{H}^1_\mathbb{C}$, we obtain $\|R_N\|_2 = O(N^{-1/2})$ and $\|R_N\|_\infty = O(N^{-1/4})$. With $\|C_N\|^2_{\mathrm{H}^1} + \|R_N\|^2_{\mathrm{H}^1} = \|A_N\|^2_{\mathrm{H}^1} \leq C$ we conclude $C_N \rightharpoonup A$ and $R_N \rightharpoonup 0$ in $\mathrm{H}^1_\mathbb{C}$.

Moreover, we have $\mathcal{J}_\varepsilon^{\mathrm{quadr}}(A_N) = \mathcal{J}_\varepsilon^{\mathrm{quadr}}(C_N) + \mathcal{J}_\varepsilon^{\mathrm{quadr}}(R_N)$ and $\mathfrak{f}_{-N}(C_N) = 0$. From $C_N \rightharpoonup A$ we obtain

$$\liminf_{N\to\infty} \mathcal{J}_\varepsilon^{\mathrm{quadr}}(C_N) \geq \mathcal{J}_0^{\mathrm{quadr}}(A) = \|A'\|_2^2$$

as in Step 1 of the previous proof.

Step 2: We transform the cubic term by an integration by parts, which gives for all $A \in \mathrm{X}_N$ the relation

$$\begin{aligned} \int_{\mathbb{S}_\ell} [\operatorname{Re}(A\boldsymbol{E}_\varepsilon)]^3\,\mathrm{d}x &= \frac{1}{8}\int_{\mathbb{S}_\ell} A^3 \boldsymbol{E}_\varepsilon^3 + 3|A|^2 A \boldsymbol{E}_\varepsilon + \text{c.c.}\,\mathrm{d}x \\ &= \frac{1}{8}\Big(\ell(\mathfrak{f}_{-N}(A))^3 + 3\mathrm{i}\varepsilon \int_{\mathbb{S}_\ell} \big(2|A|^2 A' + A^2 \overline{A'}\big)\boldsymbol{E}_\varepsilon\,\mathrm{d}x \ + \ \text{c.c.}\Big). \end{aligned}$$

Here $\int_{\mathbb{S}_\ell} A^3 \boldsymbol{E}_\varepsilon^3 \,\mathrm{d}x = \ell(\mathfrak{f}_{-N}(A))^3$ follows from $A \in \mathrm{X}_N$, which gives the representation $A\boldsymbol{E}_\varepsilon = \mathfrak{f}_{-N}(A) + \sum_{k>0} \mathfrak{f}_{k-N}(A)\mathbb{E}^k$. Hence also the cube $A^3\boldsymbol{E}_\varepsilon^3$ has only one term with $\mathbb{E}^0 = 1$. The other term is obtained by an integration by parts using $\varepsilon \boldsymbol{E}_\varepsilon' = \mathrm{i}\boldsymbol{E}_\varepsilon$.

The main point of the decomposition $A_N = C_N + R_N$ in Step 1 is that we can control the support of the Fourier series of powers $C_N^j \overline{C}_N^k$, i.e. we know $\mathfrak{f}_n(C_N^j\overline{C}_N^k) = 0$ if $|n| > (j{+}k)\sqrt{N}$. In particular, we have $\int_{\mathbb{S}_\ell} C_N^j\overline{C}_N^k \boldsymbol{E}_\varepsilon \,\mathrm{d}x = 0$ for sufficiently large N. The boundedness of $(A_N)_N$ in $\mathrm{H}^1(\mathbb{S}_\ell;\mathbb{C})$ also gives $\mathfrak{f}_{-N}(A_N) = O(\varepsilon)$, and we find

$$\mathcal{J}_\varepsilon^{\mathrm{cubic}}(A) = -\frac{\mathrm{i}}{8}\Big(\int_{\mathbb{S}_\ell} \big(2|C_N|^2 R_N' + C_N^2 \overline{R}_N'\big)\boldsymbol{E}_\varepsilon \,\mathrm{d}x \;+\; \text{c.c.}\Big) + O(\varepsilon^{1/2}).$$

Note that we also omitted the terms involving R_N without derivative as $\|R_N\|_2 = O(\varepsilon^{1/2})$.

Step 3: For the full functional $\mathcal{J}_\varepsilon$ we use the decomposition obtain

$$\begin{aligned}
&\mathcal{J}_\varepsilon(A_N) - \mathcal{J}_\varepsilon^{\mathrm{quadr}}(C_N) = \mathcal{J}_\varepsilon^{\mathrm{quadr}}(R_N) + \mathcal{J}_\varepsilon^{\mathrm{cubic}}(A_N)\\
&= \int_{\mathbb{S}_\ell} \frac{1}{4}|2\mathrm{i}R_N' {+} \varepsilon R_N'' + H_N|^2 - \frac{1}{4}|H_n|^2 \,\mathrm{d}x + \frac{\ell}{4\varepsilon^2}|\mathfrak{f}_{-N}(R_N){-}\alpha_N|^2 - \frac{\ell}{4\varepsilon^2}\alpha_N^2 - \widehat{J}_N\\
&\text{with } \widehat{J}_N := \frac{1}{4}\int_{\mathbb{S}_\ell} (2\mathrm{i}R_N' {+} \varepsilon R_N'')\overline{H}_N + \text{c.c.}\,\mathrm{d}x + \frac{\ell}{2\varepsilon^2}\alpha_N - \mathcal{J}_\varepsilon^{\mathrm{cubic}}(A_N),
\end{aligned}$$

where the function H_N and and the constant $\alpha_N \in \mathbb{R}$ are still arbitrary. Using the expansion for $\mathcal{J}_\varepsilon^{\mathrm{cubic}}$ derived in Step 2 we have

$$\begin{aligned}
\widehat{J}_N = \frac{\mathrm{i}}{8}\int_{\mathbb{S}_\ell} &\Big(2\gamma|C_N|^2\boldsymbol{E}_\varepsilon R_N' + \gamma \overline{C}_N^2 \overline{\boldsymbol{E}}_\varepsilon R_N' + 2(2R_N' {-} \mathrm{i}\varepsilon R_N'')\overline{H}_N \text{c.c.}\Big)\,\mathrm{d}x\\
&+ \frac{\ell}{2\varepsilon^2}\alpha_N \mathfrak{f}_{-N}(R_N) + O(\varepsilon^{1/2}),
\end{aligned}$$

In order to make $\widehat{J}_N$ small we set $H_N = H_N^{(1)} + H_N^{(2)}$ with

$$H_N^{(1)} := -\frac{\gamma}{2(2{+}1)} C_N^2 \boldsymbol{E}_\varepsilon \quad \text{ and } \quad H_N^{(2)} := -\frac{\gamma}{(2{-}1)} \overline{\boldsymbol{P}_\theta(|C_N|^2)}\,\overline{\boldsymbol{E}}_\varepsilon$$

with $\boldsymbol{P}_\theta$ from (4.3). We first find a cancellation induced by the choice of $H_N^{(1)}$, namely

$$\int_{\mathbb{S}_\ell} \gamma \overline{C}_N^2 \overline{\boldsymbol{E}}_\varepsilon R_N' + 2(2R_N' {-} \mathrm{i}\varepsilon R_N'')\overline{H}_N^{(1)} \,\mathrm{d}x = \frac{\gamma}{3}\int_{\mathbb{S}_\ell} \overline{C}_N^2 \overline{\boldsymbol{E}}_\varepsilon (\mathrm{i}\varepsilon R_N'' {-} R_N')\,\mathrm{d}x = O(\varepsilon^{1/2}),$$

because $C_N^2\boldsymbol{E}_\varepsilon$ has support in $[N{-}2\sqrt{N}, N{+}2\sqrt{N}]$ where the Fourier coefficients of $\mathrm{i}\varepsilon R_N''$ equal those of R_N' up to a multiplicative factor of order $O(N^{-1/2}) = O(\varepsilon^{1/2})$. Second, using $R_N \in \mathrm{X}_N$ we find

$$\begin{aligned}
&\int_{\mathbb{S}_\ell} 2\gamma |C_N|^2 \boldsymbol{E}_\varepsilon R_N' + 2(2R_N' {-} \mathrm{i}\varepsilon R_N'')\overline{H}_N^{(2)} \,\mathrm{d}x\\
&= 2\gamma \int_{\mathbb{S}_\ell} \overline{\boldsymbol{P}_1\big(|C_N|^2\big)}\,\boldsymbol{E}_\varepsilon R_N' - \overline{\boldsymbol{P}_\theta\big(|C_N|^2\big)}\,\boldsymbol{E}_\varepsilon(2R_N' {-} \mathrm{i}\varepsilon R_N'')\,\mathrm{d}x\\
&= 2\gamma(1{-}\theta)\mathfrak{f}_{-N}(R_N') \int_{\mathbb{S}_\ell} |C_N|^2\,\mathrm{d}x + 2\gamma \int_{\mathbb{S}_\ell} \overline{\boldsymbol{P}_0(|C_N|^2)}\,\boldsymbol{E}_\varepsilon(R_N' {+} \mathrm{i}\varepsilon R_N'')\,\mathrm{d}x\\
&= 2\gamma(1{-}\theta)\frac{-\mathrm{i}}{\varepsilon}\mathfrak{f}_{-N}(R_N)\|C_N\|_2^2 + O(\varepsilon^{1/2}),
\end{aligned}$$

because $R_N'+\mathrm{i}\varepsilon R_N''$ projected to the Fourier interval $[-2\sqrt{N},2\sqrt{N}]$ is $O(N^{-1/2})=O(\varepsilon^{1/2})$. Combining these relations we arrive at

$$\widehat{J}_N=\Big(\frac{\ell\alpha_N}{2\varepsilon^2}+\frac{\gamma(1{-}\theta)}{4\varepsilon}\|C_N\|_2^2\Big)\mathfrak{f}_{-N}(R_N)+O(\varepsilon^{1/2})$$

and choosing $\alpha_N=-\varepsilon\gamma(1{-}\theta)\|C_N\|_2^2/(2\ell)$ we conclude $\widehat{J}_N\to 0$. Inserting the above choices of H_N and α_N into the above decomposition of $\mathcal{J}_\varepsilon(A_N)$ and dropping the terms $|2\mathrm{i}R_N'{+}\varepsilon R_N''+H_N|^2\geq 0$ and $|\mathfrak{f}_{-N}(R_N){-}\alpha_N|^2\geq 0$ we obtain the lower estimate

$$\liminf_{\mathbb{N}\to\infty}\mathcal{J}_\varepsilon(A_N)\geq\mathcal{J}_0^{\mathrm{quadr}}(A)-\frac14\lim_{N\to\infty}\Big(\|H_N\|_2^2+\frac{\ell}{\varepsilon^2}\alpha_N^2\Big).$$

Since the Fourier supports of $H_N^{(1)}$ and $H_N^{(2)}$ are disjoint we obtain

$$\|H_N\|_2^2=\|H_N^{(1)}\|_2^2+\|H_N^{(2)}\|_2^2=\frac{\gamma^2}{36}\|C_N\|_4^4+\gamma^2\|\boldsymbol{P}_\theta(|C_N|^2)\|_2^2\to\frac{19\gamma^2}{36}\|A\|_4^4+\gamma^2\frac{2\theta^2{-}1}{2\ell}\|A\|_2^4,$$

due to $C_N\to A$ in $\mathrm{C}^0_{\mathbb{C}}$, (4.4), and $\ell\mathfrak{f}_0(|A|^2)=\|A\|_2^2$. Now $\alpha_N/\varepsilon\to\gamma(1{-}\theta)\|A\|_2^2/(2\ell)$ gives

$$\liminf_{N\to\infty}\mathcal{J}_\varepsilon(A_N)\geq\mathcal{J}_0^{\mathrm{quadr}}(A)-\frac{19\gamma^2}{144}\|A\|_4^4-\frac{\gamma^2}{4\ell}\widetilde{c}(\theta)\|A\|_2^2,$$

where $\widetilde{c}(\theta)=2\theta^2-1+2(1{-}\theta)^2$. We still have the option to choose θ to obtain the largest lower bound. Taking $\theta=1/2$ the lower bound (4.2a) is established.

Step 4: The construction of the recovery sequence A_N is guided by the constructions used for the liminf estimate. For a given $A\in\mathrm{H}^1_{\mathbb{C}}$ we set $A_N=C_N+\varepsilon G_N+\varepsilon F_n$ with

$$C_N=\sum\nolimits_{|m|\leq\sqrt{N}}\mathfrak{f}_m(A)\mathbb{E}^n,\quad G_N=c_0\overline{\boldsymbol{P}_{1/2}\big(|C_N|^2\big)}\,\overline{\boldsymbol{E}}_\varepsilon,\quad\text{and } F_N=-c_1C_N^2\boldsymbol{E}_\varepsilon,$$

where $c_0,\ c_1\in\mathbb{C}$ will be determined later. In particular, we have $A_N\in\mathrm{X}_N$, $\|A{-}C_N\|_{\mathrm{H}^1}\to 0$, $\|A_N-C_N\|_2=O(\varepsilon)$, and we will next show $\sup_{N\in\mathbb{N}}\|A_N\|_{\mathrm{H}^1}<\infty$. Hence, we conclude $A_N\rightharpoonup A$ in $\mathrm{H}^1_{\mathbb{C}}$.

For the bound of A_N in $\mathrm{H}^1_{\mathbb{C}}$ we observe $\|C_N\|_{\mathrm{H}^1}\leq\|A\|_{\mathrm{H}^1}$ and

$$\|\varepsilon G_N\|_2^2+\|\varepsilon G_N'\|_2^2\leq|c_0|^2\Big(\varepsilon^2\|C_N\|_4^4+4\varepsilon^2\|C_N'\|_2^2\|C_N\|_\infty^2+\|C_N\|_4^4\Big),$$

where we used $\varepsilon\boldsymbol{E}_\varepsilon'=\mathrm{i}\boldsymbol{E}_\varepsilon$. A similar estimate holds for F_N except for an additional factor $|c_1|^2$. Using $\varepsilon\in[0,1]$ and $\|C_N\|_4,\|C_N\|_\infty\leq C_\ell\|C_N\|_{\mathrm{H}^1}\leq C_\ell\|A\|_{\mathrm{H}^1}$ we obtain the bound

$$\|A_N\|_{\mathrm{H}^1}\leq\|C_N\|_{\mathrm{H}^1}+\|\varepsilon G_N\|_{\mathrm{H}^1}+\|\varepsilon F_N\|_{\mathrm{H}^1}\leq\|A\|_{\mathrm{H}^1}+\sqrt{6}\big(|c_0|{+}|c_1|\big)\|A\|_{\mathrm{H}^1}^2,$$

which establishes the bound for $A_\varepsilon=A_N$ stated in (4.2b) after c_0 and c_1 are chosen as at the end of Step 4.

Moreover, we can evaluate $\mathcal{J}_\varepsilon$ explicitly. For the quadratic term we can use that the three terms in A_N have disjoint support in Fourier space giving $\mathcal{J}_\varepsilon^{\mathrm{quad}}(A_N)=\mathcal{J}_\varepsilon^{\mathrm{quad}}(C_N)+\mathcal{J}_\varepsilon^{\mathrm{quad}}(\varepsilon G_N)+\mathcal{J}_\varepsilon^{\mathrm{quad}}(\varepsilon F_N)$. Since the support of C_N lies inside $[-\sqrt{N},\sqrt{N}]$ we have $\|\varepsilon C_N''\|_2\leq\varepsilon\sqrt{N}\|C_N'\|_2\leq\varepsilon\sqrt{N}\|A'\|_2=O(\varepsilon^{1/2})$ and conclude

$$\mathcal{J}_\varepsilon^{\mathrm{quad}}(C_N)=\frac14\|2\mathrm{i}C_N'+\varepsilon C_N''\|_2^2\quad\to\quad\|A'\|_2^2=\mathcal{J}_0^{\mathrm{quadr}}(A).$$

For the second term we use $\varepsilon(2\mathrm{i}G_N'+\varepsilon G_N'') = c_0\boldsymbol{P}_{1/2}(|C_N|^2)(2\mathrm{i}\varepsilon\overline{\boldsymbol{E}}_\varepsilon' + \varepsilon^2\overline{\boldsymbol{E}}_\varepsilon'') + O(\varepsilon)$ in $\mathrm{L}^2_\mathbb{C}$. Because of $2\mathrm{i}\varepsilon\overline{\boldsymbol{E}}_\varepsilon' + \varepsilon^2\overline{\boldsymbol{E}}_\varepsilon'' = (2{-}1)\overline{\boldsymbol{E}}_\varepsilon$ we arrive at

$$\mathcal{J}_\varepsilon^{\mathrm{quad}}(\varepsilon G_N) \quad\to\quad \frac{|c_0|^2}{4}(2{-}1)^2\|\boldsymbol{P}_{1/2}\big(|A|^2\big)\|_2^2 + \frac{|c_0|^2}{16\ell}\|A\|_2^2 \overset{(4.4)}{=} \frac{|c_0|^2}{8}\|A\|_4^4.$$

Similarly, the third term follows with $2\mathrm{i}\varepsilon\boldsymbol{E}_\varepsilon' + \varepsilon^2\boldsymbol{E}_\varepsilon'' = (-2{-}1)\boldsymbol{E}_\varepsilon$, namely

$$\mathcal{J}_\varepsilon^{\mathrm{quad}}(\varepsilon F_N) \quad\to\quad \frac{|c_1|^2}{4}(2{+}1)^2\|\,|A|^2\|_2^2 = \frac{9|c_1|^2}{4}\|A\|_4^4.$$

To evaluate the cubic term, we again use that the terms C_N, G_N, and F_N have their Fourier support localized near 0, $-N$, and $+N$, respectively. Hence, evaluating

$$J_N^3 := -\mathcal{J}_\varepsilon^{\mathrm{cubic}}(A_N) = \int_{\mathbb{S}_\ell} \frac{\gamma}{3\varepsilon}\Big(\operatorname{Re}\big(A_N\boldsymbol{E}_\varepsilon\big)\Big)^3\,\mathrm{d}x,$$

we see that the terms of order $1/\varepsilon$ arising from $\big(\operatorname{Re}(C_N\boldsymbol{E}_\varepsilon)\big)^3$ are identically 0 as the Fourier spectrum is bounded away from 0. Thus, the only contribution for the limit stems from the term of order ε^0, namely

$$\begin{aligned}
J_N^3 &= \gamma\int_{\mathbb{S}_\ell}\Big(\operatorname{Re}\big(C_N\boldsymbol{E}_\varepsilon\big)\Big)^2\Big(\operatorname{Re}\big((G_N{+}F_N)\boldsymbol{E}_\varepsilon\big)\Big)\,\mathrm{d}x + O(\varepsilon)\\
&= \frac{\gamma}{8}\int_{\mathbb{S}_\ell}\big(C_N^2\boldsymbol{E}_\varepsilon^2{+}2|C_N|^2{+}\overline{C}_N^2\overline{\boldsymbol{E}}_\varepsilon^2\big)\Big(c_0\boldsymbol{P}_{1/2}(|C_N|^2) + c_1C_N^2\boldsymbol{E}_\varepsilon^2 + \text{c.c.}\Big)\,\mathrm{d}x + O(\varepsilon)\\
&= \frac{\gamma}{8}\int_{\mathbb{S}_\ell}\Big(2|C_N|^2\;c_0\boldsymbol{P}_{1/2}(|C_N|^2) + \overline{C}_N^2c_1C_N^2 + \text{c.c.}\Big)\,\mathrm{d}x + O(\varepsilon)\\
&= \frac{\gamma}{8}\big(c_0{+}\overline{c}_0\big)\int_{\mathbb{S}_\ell}2|C_N|^2\boldsymbol{P}_{1/2}(|C_N|^2)\,\mathrm{d}x + \frac{\gamma}{8}\big(c_1{+}\overline{c}_1\big)\int_{\mathbb{S}_\ell}|C_N|^4\,\mathrm{d}x + O(\varepsilon),
\end{aligned}$$

Using $\|C_N\|_4^4 \to \|A\|_4^4$ and (4.4) we have derived the convergence

$$\begin{aligned}
\mathcal{J}_\varepsilon(A_N) &= \mathcal{J}_\varepsilon^{\mathrm{quadr}}(C_N) + \mathcal{J}_\varepsilon^{\mathrm{quadr}}(\varepsilon G_N) + \mathcal{J}_\varepsilon^{\mathrm{quadr}}(\varepsilon F_N) - J_N^3\\
&\to \|A'\|_2^2 + \big(\tfrac18|c_0|^2 + \tfrac94|c_1|^2 - \tfrac{\gamma}{4}\operatorname{Re}c_0 - \tfrac{\gamma}{4}\operatorname{Re}c_1\big)\|A\|_4^4.
\end{aligned}$$

We now determine c_0 and c_1 by minimization giving the minimizers $c_0 = \gamma$ and $c_1 = \gamma/18$ and the desired limit $\|A'\|_2^2 - \frac{19\gamma^2}{144}\|A\|_4^4 = \mathcal{J}_0(A)$.

Step 5: In the strong topology of $\mathrm{H}^1_\mathbb{C}$ we can argue as in Step 4 of Proposition 4.1 concerning the quadratic part. So it remains to show that the cubic term converges to 0 along all sequences $A_N \to A$ in $\mathrm{H}^1_\mathbb{C}$. For this we rewrite $\mathcal{J}_\varepsilon^{\mathrm{cubic}}(A_N)$ as in the beginning of Step 2, by eliminating the denominator via integration by parts. Passing to the limit $\varepsilon \to 0$ we have $A_N \to A$ uniformly in C^0 and $A_N' \to A'$ in $\mathrm{L}^2_\mathbb{C}$, while $\boldsymbol{E}_\varepsilon^k \rightharpoonup 0$ for $k \neq 0$. Hence, $\mathcal{J}_\varepsilon^{\mathrm{cubic}}(A_N) \to 0$ follows as desired. ∎

4.2 Geodesic λ-convexity for $\mathcal{F}_\varepsilon$ on balls

Again, the case $\gamma = 0$ is very simple.

Lemma 4.4 *For $\gamma = 0$ the functionals $\mathcal{F}_\varepsilon^{\gamma=0}$ are uniformly λ-convex with $\lambda = -\mu$.*

The proof is trivial, since by the definition of $\mathcal{F}_\varepsilon^{\gamma=0}$ all terms in $\mathcal{F}_\varepsilon^{\gamma=0}+\mu\mathcal{R}_\varepsilon$ are convex.

For $\gamma\neq 0$ the functionals $\mathcal{F}_\varepsilon$ are not uniformly λ-convex. Calculating the Hessian

$$\mathrm{D}^2\mathcal{F}_\varepsilon^{\mathrm{SH}}(u): w\mapsto +\frac{1}{\varepsilon^2}(1+\varepsilon^2\partial_x^2)^2 w-\mu w-\frac{2\gamma}{\varepsilon}\,u\,w+3u^2\,w,$$

we can insert the constant state $u_*\equiv\gamma/(3\varepsilon)$ and see that $\mathrm{D}^2\mathcal{F}_\varepsilon^{\mathrm{SH}}(u_*)$ has the smallest eigenvalue $-\mu-\gamma^2/(3\varepsilon)$, which tends to $-\infty$ for $\varepsilon\to 0$. Thus, global uniform λ-convexity does not hold. However, for the limit passage it is sufficient to have this property on sufficiently large balls. Unfortunately, we were not able to show that it is possible to use balls in $\mathrm{L}^2_{\mathbb{C}}$, and we have to use the stronger norm in $\mathrm{H}^1_{\mathbb{C}}$.

Proposition 4.5 *For each radius $R>0$ there exists $\Lambda_R\leq 0$ such that $\mathcal{F}_\varepsilon:\mathrm{L}^2_{\mathbb{C}}\to\mathbb{R}_\infty$ restricted to the ball $\mathcal{B}_R:=\{\,A\in\mathrm{H}^1_{\mathbb{C}}\mid \|A\|_{\mathrm{H}^1}\leq R\,\}$ is Λ_R-convex, i.e.*

$$\forall\, A_0,\, A_1\in\mathcal{B}_R\ \forall\,\varepsilon,\theta\in\,]0,1[\,:\quad \mathcal{F}_\varepsilon(A_\theta)\leq(1-\theta)\mathcal{F}_\varepsilon(A_0)+\theta\mathcal{F}_\varepsilon(A_1)-\frac{\Lambda_R}{2}\theta(1-\theta)\|A_1-A_0\|^2_{\mathrm{L}^2},$$

where $A_\theta=(1-\theta)A_0+\theta A_1$.

Proof: The functional $\mathcal{F}_\varepsilon$ decomposes into a nonnegative quadratic form $\mathcal{J}_\varepsilon^{\mathrm{quadr}}$ and a polynomial part $\mathcal{J}_\varepsilon^{\mathrm{cubic}}+\mathcal{J}_\varepsilon^{\mathrm{quart}}-\mu\mathcal{R}_\varepsilon$, which is smooth on $\mathcal{B}_R$.

Estimating $\mathcal{J}_\varepsilon^{\mathrm{quadr}}(A)\geq\|A'\|_2^2\geq 0$ (see Step 3 in the proof of Proposition 4.1) we certainly reduce the convexity properties, i.e. lowers the value of Λ_R. Similarly, the quartic contribution is convex, so we can drop it. Thus, it suffices to find a good lower bound $\widehat{\lambda}_{\mathbb{R}}$ for the λ-convexity of the functional

$$\mathcal{K}_\varepsilon:\ B\mapsto\|A'\|_2^2-\int_{\mathbb{S}_\ell}\tfrac{\gamma}{3\varepsilon}\operatorname{Re}\big(A\boldsymbol{E}_\varepsilon\big)^3\,\mathrm{d}x$$

on the ball $\mathcal{B}_R$. Then, the result is established with $\Lambda_R=\widehat{\lambda}_R-\mu$.

Hence, it remains to establish the desired bound

$$\forall\,A\in\mathcal{B}_R\ \forall\,B\in\mathrm{H}^1(\mathbb{S}_\ell):\quad\langle\mathrm{D}^2\mathcal{K}_\varepsilon(A)B,B\rangle\geq\widehat{\lambda}_R\|B\|_2^2,$$

where $\langle\mathrm{D}^2\mathcal{K}_\varepsilon(A)B,B\rangle=2\|B'\|_2^2-\frac{2\gamma}{\varepsilon}\int_{\mathbb{S}_\ell}\operatorname{Re}(A\boldsymbol{E}_\varepsilon)\big[\operatorname{Re}(B\boldsymbol{E}_\varepsilon)]^2\,\mathrm{d}x$. The major task is to get rid of the denominator ε, and this can be done by integration by parts. For this, we write $\operatorname{Re}z=\frac12(z+\overline{z})$ for each of the three terms in the integral over $\mathbb{S}_\ell$. Grouping into terms multiplying $\boldsymbol{E}_\varepsilon^k$ with $k\in\{-3,-1,1,3\}$, we can integrate $\boldsymbol{E}_\varepsilon^k$ and differentiate the corresponding factors. This yields the estimate

$$\frac{1}{\varepsilon}\Big|\int_{\mathbb{S}_\ell}\operatorname{Re}(A\boldsymbol{E}_\varepsilon)\big[\operatorname{Re}(B\boldsymbol{E}_\varepsilon)]^2\,\mathrm{d}x\Big|\leq\frac{14}{24}\int_{\mathbb{S}_\ell}|A'||B|^2+2|A||B||B'|\,\mathrm{d}x$$

Thus, using $\|B\|_4^2\leq C_\ell(\|B\|_2^2+\|B\|_2\|B'\|_2)$ we obtain $\widehat{\lambda}_R$ by estimating as follows:

$$\begin{aligned}\langle\mathrm{D}^2\mathcal{K}_\varepsilon(A)B,B\rangle&\geq 2\|B'\|_2^2-\frac54|\gamma|\|A'\|_2\|B\|_4^2-\frac52|\gamma|\|A\|_\infty\|B\|_2\|B'\|_2\\&\geq-\left(\frac{5|\gamma|}{4}C_\ell\|A'\|_2+\frac{\gamma^2}{5}\big(C_\ell\|A'\|_2{+}2\|A\|_\infty\big)^2\right)\|B\|_2^2.\end{aligned}$$

Due to $A\in\mathcal{B}_R$ we find a lower bound $\widehat{\lambda}_R=-C_*(|\gamma|R+\gamma^2R^2)$, and we are done. ■

4.3 Convergence result for SHe

The following two theorems provide the evolutionary Γ-convergence results in Theorem 2.3 for $\gamma = 0$ and Theorem 2.4 for $0 < |\gamma| < \gamma_0$, respectively. For $\gamma = 0$ we directly apply the theory of Section 3 to the gradient systems $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}^{\gamma=0}_\varepsilon, \mathcal{R}_\varepsilon)$ and obtain the following:

Theorem 4.6 (Evolutionary Γ-convergence for $\gamma = 0$) *Assume $\gamma = 0$ and consider solutions $A_\varepsilon : [0,T] \to \mathrm{L}^2_{\mathbb{C}}$ for the SHe given by $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}^{\gamma=0}_\varepsilon, \mathcal{R}_\varepsilon)$ and a solution A_0 for the GLe given by $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}^{\gamma=0}_{\mathrm{GL}}, \mathcal{R}_{\mathrm{GL}})$. Then, we have the evolutionary Γ-convergence*

$$\begin{aligned} &A_\varepsilon(0) \to A_0(0) \text{ in } \mathrm{L}^2_{\mathbb{C}} \\ &\implies \forall\, t > 0: \ A_\varepsilon(t) \to A_0(0) \text{ and } \mathcal{F}^{\gamma=0}_\varepsilon(A_\varepsilon(t)) \to \mathcal{F}_{\mathrm{GL}}(A_0(t)). \end{aligned}$$

We now concentrate on the case $0 < |\gamma| < \gamma_0$, where we need stronger assumptions on the convergence of the initial conditions, namely boundedness of the initial energies. This is still weaker than the wellpreparedness of the initial conditions as discussed in [Mie14].

Theorem 4.7 (Evolutionary Γ-convergence for $\gamma \neq 0$) *Assume $0 < |\gamma| < \gamma_0$ and consider solutions $A_\varepsilon : [0,T] \to \mathrm{L}^2_{\mathbb{C}}$ for the SHE given by $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}_\varepsilon, \mathcal{R}_\varepsilon)$ and a solution A_0 for the associated GLe given by $(\mathrm{L}^2_{\mathbb{C}}, \mathcal{F}_{\mathrm{GL}}, \mathcal{R}_{\mathrm{GL}})$. Then, we have*

$$\begin{aligned} &A_\varepsilon(0) \to A_0(0) \text{ in } \mathrm{L}^2_{\mathbb{C}} \text{ and } \sup_{0<\varepsilon<1} \mathcal{F}_\varepsilon(A_\varepsilon(0)) < \infty \\ &\implies \ \forall\, t > 0: \ A_\varepsilon(t) \rightharpoonup A(t) \text{ in } \mathrm{H}^1_{\mathbb{C}} \text{ and } \mathcal{F}_\varepsilon(A_\varepsilon(t)) \to \mathcal{F}_{\mathrm{GL}}(A(t)). \end{aligned}$$

By the equi-coercivity (2.5) of $\mathcal{F}_\varepsilon$ the initial data have to satisfy $A_\varepsilon(0) \rightharpoonup A(0)$ in $\mathrm{H}^1_{\mathbb{C}}$.

Proof: We denote by $F < \infty$ the supremum of $\mathcal{F}_\varepsilon(A_\varepsilon(0))$. Using the decay of the energy along solutions we find $\mathcal{F}_\varepsilon(A_\varepsilon(t)) \leq F$ for all t and $\varepsilon > 0$. By the equi-coercivity (2.5) we find an $R_1 > 0$ such that $\|A_\varepsilon(t)\|_{\mathrm{H}^1} \leq R_1$. For later purposes we set $R_2 = 2R_1 + \widehat{c}_\ell|\gamma|4R_1^2$, cf. the estimate in (4.2b). Setting $\Lambda_* := \Lambda_{R_2}$ with Λ_R from Proposition 4.5 and using the discussion at the end of Section 3, we obtain the IEVI

$$\begin{aligned} &\forall\, 0 \leq s < t \leq T \ \forall\, B \in \mathcal{B}_{R_2}: \\ &\mathrm{e}^{\Lambda_*(t-s)}\mathcal{R}_\varepsilon(A_\varepsilon(t){-}B) - \mathcal{R}_\varepsilon(A_\varepsilon(s){-}B) \leq M_{\Lambda_*}(t-s)\big(\mathcal{F}_\varepsilon(B) - \mathcal{F}_\varepsilon(A_\varepsilon(t))\big), \end{aligned} \tag{4.5}$$

where $\mathcal{B}_R$ is the ball of radius R in $\mathrm{H}^1_{\mathbb{C}}$. The solutions satisfy the a priori estimate

$$\|A_\varepsilon\|_{\mathrm{H}^1([0,T];\mathrm{L}^2_{\mathbb{C}})} \leq C \quad \text{and} \quad \|A_\varepsilon\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^1_{\mathbb{C}})} \leq R_1,$$

where in contrast to (3.11) we can use $t_0 = 0$, because of $\mathcal{F}_\varepsilon(A_\varepsilon(0)) \leq F$. Thus, for a subsequence (not relabeled) we have pointwise convergence $A_\varepsilon(t) \rightharpoonup A(t)$ in $\mathrm{H}^1_{\mathbb{C}}$ for all $t \in [0,T]$. To pass to the limit $\varepsilon \to 0$ in (4.5), we choose any test state $\widehat{B} \in \mathcal{B}_{2R_1}$ and use a recovery sequence $\widehat{B}_\varepsilon \rightharpoonup \widehat{B}$ in $\mathrm{H}^1_{\mathbb{C}}$ such that $\mathcal{F}_\varepsilon(\widehat{B}_\varepsilon) \to \mathcal{F}_{\mathrm{GL}}(\widehat{B})$ and $\widehat{B}_\varepsilon \in \mathcal{B}_{R_2}$, where we use the Γ-convergence established in Corollary 4.3 and the norm bound for the recovery sequences stated in (4.2b). Exploiting the continuous convergence of the dissipation potentials $\mathcal{R}_\varepsilon$, see Lemma 2.1, we obtain the limit expression

$$\begin{aligned} &\forall\, 0 \leq s < t \leq T \ \forall\, \widehat{B} \in \mathcal{B}_{2R_1}: \\ &\mathrm{e}^{\Lambda_*(t-s)}\mathcal{R}_{\mathrm{GL}}(A(t){-}\widehat{B}) - \mathcal{R}_{\mathrm{GL}}(A(s){-}\widehat{B}) \leq M_{\Lambda_*}(t-s)\big(\mathcal{F}_{\mathrm{GL}}(\widehat{B}) - \mathcal{F}_{\mathrm{GL}}(A(t))\big). \end{aligned} \tag{4.6}$$

By the arguments of Section 3 we know that this implies that A is a solution of the GLe. Since this solution is unique, we conclude that the whole family A_ε converges to the unique solution with the initial condition $A_0(0) = \lim_{\varepsilon\to 0} A_\varepsilon(0)$. ∎

5 Discussion

We believe that the results for the case $\gamma = 0$ are optimal in the sense that the conditions on the convergence $A_\varepsilon(0) \to A(0)$ in $\mathrm{L}^2_\mathbb{C}$ of the initial conditions are most natural.

The situation for $0 < |\gamma| < \gamma_0$ is less satisfactory. It is unclear to what respect the stronger conditions $A_\varepsilon(0) \rightharpoonup A(0)$ in $\mathrm{H}^1_\mathbb{C}$ and $\mathcal{F}_\varepsilon(A_\varepsilon(0)) \leq F < \infty$ are really necessary. To weaken these conditions it would be good to generalize the uniform λ-convexity result in Proposition 4.5 to balls in $\mathrm{L}^2_\mathbb{C}$ rather than balls in $\mathrm{H}^1_\mathbb{C}$.

Of course, for both cases it would be easily possible to improve the converge for $t > 0$ to higher Sobolev norms. For this one could use the explicit form of the parabolic equations and invoke higher regularity for SHe and GLe. Another aspect concerns quantitative error bounds as they are obtained in the classical justification results [vHa91, KSM92, Sch94]. Our present theory is not adapted for such results, since first we would need to turn the Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ to its limit $\mathcal{F}_{\mathrm{GL}}$ into quantitative statements. For this, one needs to introduce suitable recovery operators, see the folding and unfolding operators in [Han11, MRT13].

We kept the restriction $|\gamma| < \gamma_0$ throughout this work, but believe that it can be avoided without too much effort. In particular, this condition guarantees the global existence of solutions. For $\gamma^2 > 27/38$ the GLe has blowup and the associated justification results need to be restricted to time intervals on which the solutions stay smooth. Such a restriction can easily be achieved by modifying the SHe and the GLe outside of large balls by adding a suitable stabilizing term to the corresponding energies. The statement concerning the original equations is then restricted to time intervals on which the modifications are not yet seen.

A major advantage of our approach based on evolutionary Γ-convergence is that we can easily add perturbations to the problems and derive their effect on the limiting amplitude equation. For instance, we can study the perturbed SHe

$$\dot{u} = -\frac{1}{\varepsilon^2}(1+\varepsilon^2\partial_x^2)^2 u + \mu u + \frac{\gamma}{\varepsilon}u^2 - u^3 + h_\varepsilon(x) + a_\varepsilon(x)u, \quad u(t,x+\ell) = u(t,x), \tag{5.1}$$

where a_ε and h_ε are suitable coefficients depending on ε and x, e.g. in the form $a_\varepsilon(x) = \varepsilon^{-\alpha}\mathbb{A}(x, \frac{1}{\varepsilon}x)$. Thus, these terms may include localized or fast oscillating terms.

The perturbed SHe has still a gradient structure $(\mathrm{L}^2_\mathbb{R}, \mathcal{F}_\varepsilon^{\mathrm{SH,pert}}, \frac{1}{2}\|\cdot\|_2^2)$, i.e. it can be written in the form $\dot{u} = -\mathrm{D}\mathcal{F}_\varepsilon^{\mathrm{SH,pert}}(u)$. The transformation $A = \Psi_\varepsilon u$ or $u = \mathrm{Re}(A\boldsymbol{E}_\varepsilon)$ leads to the energy functional

$$\mathcal{F}_\varepsilon^{\mathrm{pert}}(A) = \mathcal{F}_\varepsilon(A) - \mathcal{P}_\varepsilon(A) \quad \text{with} \;\; \mathcal{P}_\varepsilon(A) = \int_{\mathbb{S}_\ell} h_\varepsilon \,\mathrm{Re}(A\boldsymbol{E}_\varepsilon) + \frac{a_\varepsilon}{2}\big(\mathrm{Re}(A\boldsymbol{E}_\varepsilon)\big)^2 \,\mathrm{d}x.$$

If the functionals $\mathcal{P}_\varepsilon$ continuously converge in the weak topology of $\mathrm{H}^1_\mathbb{C}$ to $\mathcal{P}_0$ (i.e. $A_\varepsilon \rightharpoonup A$ in $\mathrm{H}^1_\mathbb{C}$ implies $\mathcal{P}_\varepsilon(A_\varepsilon) \to \mathcal{P}_0(A)$), then $\mathcal{F}_\varepsilon^{\mathrm{pert}} \xrightharpoonup{\Gamma} \mathcal{F}_{\mathrm{GL}} - \mathcal{P}_0$ and the same convergence theory as above applies. Note that h_ε does not contribute to the λ-convexity, hence $a_\varepsilon \leq c_0$ is sufficient to make $\mathcal{P}_\varepsilon$ uniformly λ-convex. As an example we consider

$$h_\varepsilon(x) = \tfrac{1}{\varepsilon}\psi\big(\tfrac{1}{\varepsilon}x\big) + b_1(x)\cos\big(\tfrac{1}{\varepsilon}x\big) \;\; \text{and} \;\; a_\varepsilon(x) = b_2(x)\cos\big(\tfrac{2}{\varepsilon}x\big)$$

with $\psi \in \mathrm{C}^0_\mathrm{c}(\mathbb{R})$ and $b_j \in \mathrm{L}^2_\mathbb{R}$. Then, we find

$$\mathcal{P}_0(A) = zA(0) + \overline{z}\overline{A(0)} + \int_{\mathbb{S}_\ell} \frac{b_1}{4}(A+\overline{A}) + \frac{b_2}{16}\big(A^2+\overline{A}^2\big)\,\mathrm{d}x \text{ with } z = \int_\mathbb{R} \psi(y)\mathrm{e}^{\mathrm{i}y}\,\mathrm{d}y \in \mathbb{C},$$

and are led to a macroscopic perturbed GLe of the form

$$\dot{A} = 4A_{xx} + \mu A - \rho|A|^2 A + 4z\delta_0(x) + b_1(x) + \frac{b_2(x)}{2}\,\overline{A},$$

where δ_0 denotes the Dirac distribution located at $x = 0$.

A An interpolation inequality

The aim of this section is to prove the following interpolation inequality that is used in Proposition 2.2 to show the equi-coercivity of the energy $\mathcal{F}_\varepsilon^{\mathrm{SH}}$. Recall $\mathbb{S} := \mathbb{S}_{2\pi} = \mathbb{R}/_{2\pi\mathbb{Z}}$.

Theorem A.1 *There exists $\kappa_0 > 0$ such that*

$$\forall\, u \in \mathrm{H}^2(\mathbb{S})\ \forall\, N \in \mathbb{N}: \quad \Big(\int_{\mathbb{S}} u^3\,\mathrm{d}x\Big)^2 \leq \kappa_0^2 \int_{\mathbb{S}} \big(u{+}\tfrac{1}{N^2}u''\big)^2\,\mathrm{d}x \int_{\mathbb{S}} u^4\,\mathrm{d}x. \tag{A.1}$$

The inequality is not a classical interpolation inequality, because on the left-hand side we do not have the norm in L^3. It is really essential that we consider the power u^3 having negative and positive values. In fact, for the function $u(x) = \cos(Nx)$ we easily see that both sides vanish.

Proof: We use the linear operator $\Psi : \mathrm{L}^2(\mathbb{S};\mathbb{R}) \to \mathrm{L}^2(\mathbb{S};\mathbb{C})$ defined via

$$a = \Psi u = u_0 + 2\sum_{n=1}^{\infty} u_n \mathbb{E}^n \quad \text{where } u(x) = \sum_{k\in\mathbb{N}} u_k \mathbb{E}^k.$$

Note that Ψ can be written in terms of the Hilbert transform $\mathcal{H}u = \sum_{k\in\mathbb{N}} \mathrm{sign}(k) u_k \mathbb{E}^k$, namely $\Psi = I + \mathcal{H}$. It is well-known (cf. [McE88]) that $\mathcal{H} : \mathrm{L}^4(\mathbb{S};\mathbb{C}) \to \mathrm{L}^4(\mathbb{S},\mathbb{C})$ is bounded, namely $\|\mathcal{H}u\|_4 \leq (1{+}\sqrt{2})\|u\|_4$. Hence there exists $C_\Psi \leq 2{+}\sqrt{2} < 3.5$ such that

$$\forall\, u \in \mathrm{L}^4(\mathbb{S};\mathbb{R}): \quad \|\Psi u\|_4 \leq C_\Psi \|u\|_4. \tag{A.2}$$

We transform the desired interpolation inequality (A.1) via $A = \Psi_N u := (\Psi u)\overline{\mathsf{E}}_N$ where $\mathsf{E}_N(x) = \mathbb{E}^N(x) = \mathrm{e}^{\mathrm{i}Nx}$. The range of Ψ_N is given by

$$\mathrm{X}_N := \{\, A = \sum_{k\in\mathbb{Z}} a_k \mathbb{E}^k \mid a_k = 0 \text{ for } k < -N, \ \ a_{-N} \in \mathbb{R}\,\} \subset \mathrm{L}^2(\mathbb{S};\mathbb{C}).$$

The inverse mapping is $u = \mathrm{Re}(A\mathsf{E}_N)$ and for $u = \sum_{k\in\mathbb{N}} u_k \mathbb{E}^n$ we have

$$A = \Psi_N u = \sum_{n=-N}^{\infty} a_n \mathbb{E}^n \quad \text{with } a_{-N} = u_0 \text{ and } a_n = 2u_{n+N} \text{ for } n > -N.$$

Hence a direct computation using $u_{-k} = \overline{u}_k$ gives

$$\begin{aligned} \int_{\mathbb{S}} (u{+}\tfrac{1}{N^2}u'')^2\,\mathrm{d}x &= 2\pi \sum_{k\in\mathbb{Z}} \big(1{-}\tfrac{k^2}{N^2}\big)^2 |u_k|^2 = 2\pi|u_0|^2 + 4\pi \sum_{k\in\mathbb{N}} \big(1{-}\tfrac{k^2}{N^2}\big)^2 |u_k|^2 \\ &= 2\pi|a_{-N}|^2 + \pi \sum_{n>-N} \big(1{-}\tfrac{(n+N)^2}{N^2}\big)^2 |a_n|^2 \geq \pi \sum_{n\geq -N} \frac{n^2}{N^2}|a_n|^2 = \frac{1}{2N^2}\|A'\|_2^2. \end{aligned} \tag{A.3}$$

To deal with the cubic term we use $u^3 = (\mathrm{Re}(A\mathsf{E}_N)^3$ and obtain

$$\int_{\mathbb{S}} u^3 \,\mathrm{d}x = \frac{1}{8}\int_{\mathbb{S}} \big(A\mathsf{E}_N + \overline{A\mathsf{E}}_N\big)^3 \,\mathrm{d}x = \frac{1}{4}\,\mathrm{Re}\Big(\int_{\mathbb{S}} A^3\mathsf{E}_N^3 + 3A^2\overline{A}\mathsf{E}_N \,\mathrm{d}x\Big)$$

For the first term we can use the fact that $A = a_{-N}\overline{\mathsf{E}}_N + \sum_{n>-N} a_n\mathbb{E}^n$ and find

$$\int_{\mathbb{S}} A^3\mathsf{E}_N^3 \,\mathrm{d}x = 2\pi a_{-N}^3 \qquad \text{with } a_{-N} = \frac{1}{2\pi}\int_{\mathbb{S}} A(x)\mathsf{E}_N(x)\,\mathrm{d}x.$$

Doing either a simple estimate for $\mathbb{E}_N$ or integration by parts gives

$$|a_{-N}| \leq \tfrac{1}{2\pi}\|A\|_1 \leq \tfrac{1}{(2\pi)^{1/2}}\|A\|_4 \quad \text{and} \quad |a_{-N}| \leq \tfrac{1}{2N\pi}\|A'\|_1 \leq \tfrac{1}{N(2\pi)^{1/4}}\|A'\|_2.$$

The second term can be estimated by integration by parts as well:

$$\Big|\int_{\mathbb{S}} A^2\overline{A}\mathsf{E}_N \,\mathrm{d}x\Big| = \Big|\int_{\mathbb{S}} \big(2|A|^2A' + A^2\overline{A}'\big)\tfrac{1}{\mathrm{i}N}\mathsf{E}_N \,\mathrm{d}x\Big| \leq \frac{3}{N}\|A\|_4^2\|A'\|_2.$$

Combining the last two estimates we find

$$\Big|\int_{\mathbb{S}} u^3 \,\mathrm{d}x\Big| \leq \frac{5}{2N}\|A\|_4^2\|A'\|_2.$$

Combining this with (A.2) and (A.3) we obtain the desired estimate (A.1) with $\kappa_0 = 5C_\Psi^2/\sqrt{2} \leq 5(4{+}3\sqrt{2}) \approx 41,2132$. ■

Remark A.2 *We have* $\kappa_0^2 \geq 19/3$ *and conjecture equality. The estimate follows choosing* $u(x) = \cos(Nx) + 9\varepsilon + \varepsilon\cos(2Nx)$ *in the limit* $\varepsilon = 1/N \to 0$. *More precisely, we find*

$$\int_{\mathbb{S}} u^3 \,\mathrm{d}x = \tfrac{3}{2}\pi 19\varepsilon + O(\varepsilon^3), \quad \int_{\mathbb{S}} u^4 \,\mathrm{d}x = \tfrac{3}{4}\pi + O(\varepsilon^2), \quad \int_{\mathbb{S}} (u{+}\tfrac{1}{N^2}u'')^2 \,\mathrm{d}x = 9\pi 19\varepsilon^2.$$

Remark A.3 *Minimizing the right-hand side of* (A.1) *with respect to* N *we obtain the slightly stronger estimate*

$$\forall\, u \in \mathrm{H}^2(\mathbb{S}) : \quad \|u''\|_2^2\big(\textstyle\int_{\mathbb{S}} u^3 \,\mathrm{d}x\big)^2 \leq \kappa_1^2\|u\|_4^4\,\big(\|u\|_2^2\,\|u''\|_2^2 - \|u'\|_2^4\big).$$

This estimate implies (A.1) *with* $\kappa_0 \leq \kappa_1$. *We conjecture the optimal value* $\kappa_1^2 = 19/3$.

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