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On the representation of hysteresis operators acting on vector-valued, left-continuous and piecewise monotaffine and continuous functions

Olaf Klein[1]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Olaf.Klein@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In Brokate-Sprekels 1996, it is shown that hysteresis operators acting on scalar-valued, continuous, piecewise monotone input functions can be represented by functionals acting on alternating strings. In a number of recent papers, this representation result is extended to hysteresis operators dealing with input functions in a general topological vector space. The input functions have to be continuous and piecewise monotaffine, i.e., being piecewise the composition of two functions such that the output of a monotone increasing function is used as input for an affine function.

In the current paper, a representation result is formulated for hysteresis operators dealing with input functions being left-continuous and piecewise monotaffine and continuous. The operators are generated by functions acting on an admissible subset of the set of all strings of pairs of elements of the vector space.

1 Introduction

The representation of hysteresis operators acting on scalar-valued, continuous, piecewise monotone input functions by using functionals acting on alternating strings is introduced in [1, 3] and used, for example, in [2, 6, 7, 8, 9, 16, 18, 20, 22],

In a number of recent papers [10, 11, 12, 13] of the author, this representation result is extended to hysteresis operators dealing with inputs in a general topological vector space.

To generalize the notion of a monotone function that is only defined for scalar valued functions, a vector-valued function is denoted as *monotaffine function* if it is the composition of a monotone increasing function and an affine function, such that the monotone increasing function is applied first.

A string of real numbers is an alternating string as considered in [1, 3] if and only if no element in the string can be written as the convex combination of its predecessor and its successor, in other words, if no triple within the string forms a *convexity triple in* $\mathbb{R}^3$ according to the definition below.

In [10, 12], it has been proved that hysteresis operators acting on vector-valued continuous piecewise monotaffine input functions can be uniquely generated by considering functions acting on convexity triple free strings.

In [3], the hysteresis operator generated from alternating strings has been extend to an hysteresis operator dealing with scalar, piecewise monotone input functions.

In [12], it has been shown that the hysteresis operator acting on piecewise monotaffine and piecewise continuous input functions with values in a topological vector space can be uniquely generated by appropriate functions acting on appropriate quintuples. Each quintuple consists of three elements in the vector space combined with two boolean variables. In the current paper, a modification of this result is formulated for the special case that the input functions are in addition left-continuous, as it holds for the input function considered for example in [15]. It is shown that each hysteresis operator dealing with these input functions can be generated in a unique way from an appropriate function defined on an appropriate set of strings of pairs of elements of the vector space.

The paper is organized as the follows: In Section 2, some definitions will be presented, including an appropriate extension of piecewise monotaffinicity considered on half-open intervals, and an appropriate extension of the notion of convexity free strings to NCTC triple free strings. Moreover, the main result of the paper, the representation result will be presented in this section. In Section 3, the result will be proved, and an example for the representation of a concrete hysteresis operator is shown in Section 4.

2 Definitions and main result

2.1 Fundamental Definitions

Let X be a topological vector space, let Y be some nonempty set and let $T > 0$ be some final time. For functions $u : [0,T] \to X$ let $u(t+) := \lim_{\tau \searrow t} u(t)$ for all $t \in [0,T[$ and $u(T+) := u(T)$.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of all natural numbers and let $\mathrm{C}\left([0,T]; X\right)$ be the set of all continuous function from $[0,T]$ to X.

The following notations correspond to the ones in [3, Def. 2.2.2]:

Definition 2.1. Let a function $\beta : [0,T] \to \mathbb{R}$ and a nonempty interval $I \subseteq [0,T]$ be given. The function β is denoted as *(strictly) monotone increasing on* I if for all $s, t \in I$ with $s < t$ it holds that $\beta(s) \leq \beta(t)$ (resp. $\beta(s) < \beta(t)$).

2.2 Hysteresis operators

Following the monographs [3, 14, 21], it is defined, as in [10]:

Definition 2.2. Let $\mathcal{H} : D(\mathcal{H}) (\subseteq \mathrm{Map}\left([0,T], X\right)) \to \mathrm{Map}\left([0,T], Y\right)$ with $D(\mathcal{H}) \neq \emptyset$ be some operator.

a) The operator $\mathcal{H}$ is denoted as *hysteresis operator*, if it is causal and rate-independent according to the following definitions.

b) The operator $\mathcal{H}$ is said to be *causal* or to have the *Volterra property*, if for every $v, w \in D(\mathcal{H})$ and every $t \in [0,T]$ it holds: If $v(\tau) = w(\tau)$ is satisfied for all $\tau \in [0,t]$ then it follows that $\mathcal{H}[v](t) = \mathcal{H}[w](t)$.

c) The operator $\mathcal{H}$ is called *rate-independent*, if for every $v \in D(\mathcal{H})$ and every admissible time-transformation $\alpha : [0,T] \to [0,T]$ (see Def. 2.3 below) with $v \circ \alpha \in D(\mathcal{H})$ it holds that $\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t))$ for all $t \in [0,T]$.

Definition 2.3. A function $\alpha : [0,T] \to [0,T]$ is an admissible time transformation if and only if $\alpha(0) = 0$, $\alpha(T) = T$, α is continuous, and α is monotone increasing (not necessary strictly monotone increasing).

2.3 Convexity triple and strings

Definition 2.4. A triple $(v_1, v_2, v_3) \in X^3$ is a *convexity triple in* X^3 if

$$v_2 \in \operatorname{conv}(v_1, v_3) := \{(1-\lambda)v_1 + \lambda v_3 \,|\, \lambda \in [0,1]\}\,. \tag{1}$$

Definition 2.5. Let $(x_a, y_a), (x_b, y_b), (x_c, y_c) \in X^2$ be given

a) $\big((x_a, y_a), (x_b, y_b), (x_c, y_c)\big)$ is denoted as *convexity triple containing triple of elements of* X^2, *i.e.* as *CTC triple*, if $x_b = y_b$ and (y_a, x_b, x_c) is a convexity triple, in other words, if

$$x_b = y_b \in \operatorname{conv}(y_a, x_c). \tag{2}$$

b) $\big((x_a, y_a), (x_b, y_b), (x_c, y_c)\big)$ is denoted as *non-constant convexity triple containing triple of elements of* X^2, *i.e.* as *NCTC triple*, if it is a convexity triple containing triple of elements of X^2, such that $x_a \neq y_a = x_b = y_b \neq x_c$ does not hold.

Definition 2.6. a) We are considering the following subset of the set of all strings of elements of X^2:

$$S^2(X) := \Big\{\big((x_0, y_0), \ldots, (x_n, y_n)\big) \in \big(X^2\big)^{n+1} \,\Big|\, n \in \mathbb{N},\, x_n = y_n\Big\}. \tag{3}$$

b) $\big((x_0, y_0), \ldots, (x_n, y_n)\big) \in S^2(X)$ is denoted as NCTC triple free string if $n = 1$ or if $n > 1$ and it holds that:

$$\big((x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1})\big) \text{ is no NCTC triple,} \quad \forall\, i \in \{1, \ldots, n-1\}. \tag{4}$$

c) Let

$$S^2_F(X) := \Big\{\mathbf{V} \in S^2(X) \Big| \mathbf{V} \text{ is an NCTC triple free string}\Big\} \tag{5}$$

be the set of all NCTC triple free strings in $S^2_F(X)$.

2.4 Monotaffine functions

In order to generalize the notion of monotonicity from scalar-valued to vector-values functions, the composition of a **monot**one increasing and an **affine** function with the monotone increasing function being evaluated first is considered in [11, 10, 12, 13] and leads to **monotaffine** function considered on closed intervals. In [12], these monotaffine functions are also considered on open intervals. Adapting this definition to left-open, right-closed intervals, one ends up with the following definition.

Definition 2.7. Let some $t_a, t_b \in [0, T]$ with $t_a < t_b$ and some function $u : [0, T] \to X$ be given.

a) u is denoted as *affine on* $]t_a, t_b] := \{t \mid t_a < t \leq t_b\}$ if $u(t_a+) := \lim_{t \searrow t_a} u(t)$ exists and

$$u(t) = \frac{t_b - t}{t_b - t_a} u(t_a+) + \frac{t - t_a}{t_b - t_a} u(t_b), \quad \forall\, t \in]t_a, t_b]. \tag{6}$$

b) u is denoted as *monotaffine on* $]t_a, t_b]$ if $u(t_a+)$ exists and there exists a monotone increasing (not necessary strictly monotone increasing) function $\beta :]t_a, t_b] \to [0, 1]$ such that $\beta(t_a+) = 0$, $\beta(t_b) = 1$, and

$$u(t) = (1 - \beta(t))u(t_a+) + \beta(t)u(t_b), \quad \forall\, t \in]t_a, t_b]. \tag{7}$$

Remark 2.8. If in the definition of "monotaffine" in Def. 2.7 it were required that β is a function from $]t_a, t_b]$ to $[0, 1]$, a function $u : [0, T] \to X$ with u being monotaffine on $[0, T]$ such u is constant on $[T/2, T]$ and $u(0) \neq u(T/2)$ would not be monotaffine on $[T/2, T]$ according to this alternative definition. In contrast, for all $u : [0, T] \to X$ and all $t_a, t'_a t_b \in [0, T]$ with $t_a < t'_a < t_b$ such that u is monotaffine on $]t_a, t_b]$ according to Def. 2.7 if follows that u is monotaffine on $]t'_a, t_b]$according to Def. 2.7, see Lemma 3.2.

2.5 Piecewise left-open right-closed monotaffine-continuous function

Definition 2.9. a) A function $u : [0, T] \to X$ is denoted as *piecewise left-open, right-closed monotaffine-continuous (pw. lo. rc. monotaffine-continuous)* if there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that u is monotaffine and continuous on $]t_i, t_{i+1}]$ for all $i = 0, \ldots, n - 1$. In this case, the decomposition is denoted as *lo. rc. monotaffinicity continuity decomposition of* $[0, T]$ *for* u.

b) Let $\mathrm{Map}_{\mathrm{pw},*}\left([0, T], X\right)$ be the set of all pw. lo. rc. monotaffine-continuous functions from $[0, T]$ to X.

Remark 2.10. Every function in $\mathrm{Map}_{\mathrm{pw},*}\left([0, T], X\right)$ is left-continuous.

Definition 2.11. For $u \in \mathrm{Map}_{\mathrm{pw},*}\left([0, T], X\right)$ the *standard lo. rc. monotaffinicity continuity decomposition of* $[0, T]$ *for* u is the lo. rc. monotaffinicity continuity decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ for u, such that for all for all $i = 0, \ldots, n - 1$, it holds

$$t_{i+1} = \max \left\{ t \in]t_i, T] \mid u \text{ is monotaffine and continuous on }]t_i, t] \right\}. \tag{8}$$

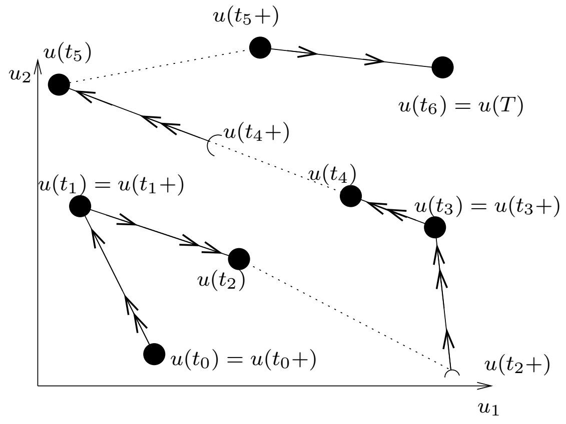

Figure 1: An example for a piecewise left-open, right-closed monotaffine-continuous function $u : [0,T] \to \mathbb{R}^2$ and its evaluation at the times t_i of the lo. rc. monotaffinicity continuity decomposition $0 = t_0 < t_1 < \cdots < t_6 = T$ of $[0,T]$ for u. The arrow density indicates the passing velocity, the dotted lines the jumps at the discontinuities, i.e. the jump between $u(t_2)$ and $u(t_2+)$, the jump between $u(t_4)$ and $u(t_4+)$, and the jump between $u(t_5)$ and $u(t_5+)$. Here, it holds that $u(t_2+) \notin u(]t_2,t_3])$ and $u(t_4+) \notin u(]t_4,t_5])$ but $u(t_5+) \in u(]t_5,t_6])$.

Remark 2.12. a) For the *set* $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ *of all continuous, piecewise monotaffine functions form* $[0,T]$ *to* X considered in [10, 11, 12, 13] it holds that $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X) = \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \cap \mathrm{C}([0,T];X)$.

b) It holds for $u \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ that the standard lo. rc. monotaffinicity continuity decomposition $0 = t_0 < \cdots < t_n = T$ of $[0,T]$ for u coincides with the *standard monotaffinicity partition of* $[0,T]$ *for* u considered in [10, 11, 12, 13].

2.6 Memory operator

Definition 2.13. The *memory operator* $\rho : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],S_F^2(X))$ is defined by mapping $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ to $\rho[u] : [0,T] \to S_F^2(X)$ according to:

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u.

I) Let $r_0,\ldots,r_{n-1} \in \{0,1\}$ and $m_0,\ldots,m_{n-1} \in \{0,\ldots,2n\}$ be defined by

$$r_i := \begin{cases} 1, & \text{if} \quad i = 0 \quad \text{and} \quad u(t_0+) \in u(]t_0,t_1[) \quad \text{and} \quad u(t_0) \neq u(t_0+) \neq u(t_1), \\ 1, & \text{if} \quad i > 0 \quad \text{and} \quad u(t_i+) \in u(]t_i,t_{i+1}[) \quad \text{and} \quad u(t_i+) \neq u(t_{i+1}), \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

$$m_i := i + \sum_{j=0}^{i} r_j, \tag{10}$$

for all $i \in \{0, \ldots, n-1\}$.

II) Let $V_0, \ldots, V_{m_{n-1}} \in X^2$ be defined by

$$\begin{cases} V_{m_i} = (u(t_i), u(t_i+)) \,, & \text{if} \quad r_i = 0, \\ V_{m_i - 1} = (u(t_i), u(t_i+)) \,, \quad V_{m_i} = (u(t_i+), u(t_i+)) \,, & \text{if} \quad r_i = 1, \end{cases} \tag{11}$$

for all $i \in \{0, \ldots, n-1\}$.

III) Let $\rho[u] : [0,T] \to S_F^2(X)$ be defined by

$$\rho[u](0) := \big((u(0), u(0)),\, (u(0), u(0))\big), \tag{12a}$$

$$\rho[u](t) := \begin{cases} \big(V_0, (u(t), u(t))\big), & \text{if} \quad r_0 = 0, \\ \big(V_0, (u(t), u(t))\big), & \text{if} \quad r_0 = 1 \quad \text{and} \quad u(0+) = u(t), \\ \big(V_0, V_1, (u(t), u(t))\big), & \text{if} \quad r_0 = 1 \quad \text{and} \quad u(0+) \neq u(t), \end{cases}$$
$$\forall\, t \in]t_0, t_1], \tag{12b}$$

$$\rho[u](t) := \begin{cases} \big(V_0, \ldots, V_{m_i}, (u(t), u(t))\big), & \text{if} \quad r_i = 0, \\ \big(V_0, \ldots, V_{m_i - 1}, (u(t), u(t))\big), & \text{if} \quad r_i = 1 \quad \text{and} \quad u(t_i+) = u(t), \\ \big(V_0, \ldots, V_{m_i}, (u(t), u(t))\big), & \text{if} \quad r_i = 1 \quad \text{and} \quad u(t_i+) \neq u(t), \end{cases}$$
$$\forall\, t \in]t_i, t_{i+1}],\ i \in \{1, \ldots, n-1\}. \tag{12c}$$

Remark 2.14. For the memory operator considered in [12] it holds that for u, t_i, t_{i+1} as in Def. 2.13 the corresponding memory string determined for u on $]t_i, t_{i+1}]$ would have exactly $i+1$ components.

If one would prefer to deal with such an memory operator instead of the one defined in Def. 2.13, one would need to replace the strings of elements of X^2 considered here by strings such that each componend is the combination of an element of X^2 with a boolean variable, keeping track of the fact, if $u(t_i+) \in u\,(]t_i, t_{i+1}])$ holds or does not hold. In other words, each string component would consits of the last three components of the quintuples considered as string components in [12, Sec. 17 –20].

2.7 Generated hysteresis operators and main result

Using the memory operator introduced in the last section, we can define:

Definition 2.15. a) Let $\mathbf{U}, \mathbf{V} \in S_F^2(X)$ be given. Then we denote that $\mathbf{V}$ *is generated from* $\mathbf{U}$ *by stretching the first component* if for $((x_0, y_0), \ldots, (x_n, y_n)) := \mathbf{U}$ it holds that $\mathbf{V} = ((x_0, x_0), (x_0, y_0), \ldots, (x_n, y_n))$. [1]

b) A function $F : S_F^2(X) \to Y$ is denoted as *invariant with respect to stretching the first component*, if for all $\mathbf{U}, \mathbf{V} \in S_F^2(X)$ such that $\mathbf{V}$ is generated from $\mathbf{U}$ by stretching the first component it holds that $F(\mathbf{U}) = F(\mathbf{V})$

[1] This implies that $x_0 \neq y_0$.

c) For $F : S_F^2(X) \to Y$ being invariant with respect to stretching the first component the *hysteresis operator* $\mathcal{G}_F : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ *generated by* F is defined by $\mathcal{G}_F[u](t) := F(\rho[u](t))$ for all $t \in [0,T]$ and for all $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$.

d) Let $\mathcal{H} : D(\mathcal{H}) \to \mathrm{Map}([0,T],Y)$ be a hysteresis operator with $\mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ being a subset of $D(\mathcal{H})$. The *function* $G_{\mathcal{H}} : S_F^2(X) \to Y$ *generated by* $\mathcal{H}$ is defined by

$$G_{\mathcal{H}}(\mathbf{U}) = \mathcal{H}[\pi[\mathbf{U}]](T), \quad \forall\, \mathbf{U} \in S_F^2(X). \tag{13}$$

The following Main Theorem yields that the above notion is well defined and that for all sets Y the mapping $F \mapsto \mathcal{G}_F$ is a bijective mapping between all maps $S_F^2(X) \to Y$ being invariant with respect to stretching the first component and the set of all hysteresis operators from $\mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ to $\mathrm{Map}([0,T],Y)$.

Main Theorem 1. *a) Let $F : S_F^2(X) \to Y$ be some function that is invariant with respect to stretching the first component. Then it follows that the operator $\mathcal{G}_F : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ defined in Def. 2.15 is a hysteresis operator.*

b) For every hysteresis operator $\mathcal{H} : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ there exists a unique function $F : S_F^2(X) \to Y$ that is invariant with respect to stretching the first component, such that $\mathcal{H}$ is the hysteresis operator generated by G, i.e. such that $\mathcal{H} = \mathcal{G}_F$. It holds $F = G_{\mathcal{H}}$, i.e., F is the function generated by the hysteresis operator.

2.8 Connection to former results for hysteresis operators dealing with continuous inputs

The following strings are considered in in [11, 10, 12, 13]:

Definition 2.16. a) $\big(v_0, \ldots, v_n)\big) \in X^{n+1}$ with $n \in \mathbb{N}$ is denoted as convexity triple free string of elements of X if $n = 1$ or if $n > 1$ and it holds that:

$$\big(v_{i-1}, v_i, v_{i+1}\big) \text{ is no convexity triple}, \quad \forall\, i \in \{1, \ldots, n-1\}. \tag{14}$$

b) Let $S_F(X)$ be the set of all convexity triple free strings of elements of X.

Remark 2.17. Let $F : S_F^2(X) \to Y$ be some function that is invariant with respect to stretching the first component. Let $\mathcal{H}_0$ be the restrition of $\mathcal{G}_F : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],Y)$ to $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$. Let $F_0 : S_F(X) \to Y$ be defined by

$$F_0(v_0, \ldots, v_n) = F\big((v_0, v_0), \ldots, (v_n, v_n)\big), \quad \forall\, (v_0, \ldots, v_n) \in S_F(X). \tag{15}$$

Then it holds for all $u \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ and the standard monotaffinicity partition $0 = t_0 < \cdots < t_n = T$ of $[0,T]$ for u that

$$\mathcal{H}_0 := G_0(u(t_0), u(t)), \quad \forall\, t \in [t_0, t_1], \tag{16a}$$

$$\mathcal{H}_0[u](t) := G_0(u(t_0), \ldots, u(t_{i-1}), u(t)), \qquad \forall\, t \in]t_{i-1}, t_i],\ i = 2, \ldots, n. \tag{16b}$$

This yield that the *hysteresis operator on* $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ *generated by* $F_0 : S_F(X) \to Y$ considered in [10, 11, 12, 13] conicides with $\mathcal{H}_0$.

The following representation result is presented in [10, 11, 12, 13] and proved in [10, 12]. It extends the representation result derived in [1, 3] for hysteresis operators defined on scalar valued, continuous and piecewise monotone inputs functions.

Theorem 2.18. *For every hysteresis operator $\mathcal{H}_0 : \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\,([0,T],Y)$ there exists a unique function $F_0 : S_F(X) \to Y$ such that $\mathcal{H}_0$ is the hysteresis operator on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by F_0.*

2.9 Extension of hysteresis operators dealing with continuous inputs

Remark 2.19. Let a hysteresis operator $\mathcal{H}_0 : \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\,([0,T],Y)$ and the a unique function $F_0 : S_F(X) \to Y$ such that $\mathcal{H}_0$ is the hysteresis operator on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by F_0 be given.

a) It holds for every hysteresis operator $\mathcal{H} : \mathrm{Map}_{\mathrm{pw},*}\,([0,T],X) \to \mathrm{Map}\,([0,T],Y)$ that $\mathcal{H}$ is an extension of $\mathcal{H}_0$ if and only if 15 holds for $F := G_{\mathcal{H}}$.

b) If X is a Hilbert-space, one can compute the *"arclen" extension $\mathcal{H}_{arclen}$ of $\mathcal{H}_0$* to the space of all function of bounded variation with values in X introduced in [19]. This extension can be described in the following way: For determing $\mathcal{H}_{\mathrm{arclen}}[u]$ for a function $u \in \mathrm{Map}_{\mathrm{pw},*}\,([0,T],X)$ the jumps between $u(t)$ and $u(t+)$ are filled with straight lines, i.e., affine functions are inserted.

   Hence, one can get from [12, Sec. 20.3.1] that for the function $F_{\mathrm{arclen}} : S_F^2(X) \to Y$ generated by $\mathcal{H}_{\mathrm{arclen}}$ it holds for all $\big((x_0,y_0),\ldots,(x_n,y_n)\big) \in S_F^2(X)$ that

$$F_{\mathrm{arclen}}\big((x_0,y_0),\ldots,(x_n,y_n)\big) = G\left((x_0,y_0,x_1,y_1,\ldots,x_{n-1},y_{n-1}x_n,y_n)\right). \tag{17}$$

3 Proof of the Main Theorem

3.1 Tools for preparing the proof

The following Lemma corresponds to [12, Lem. 7.2.3.c]:

Lemma 3.1. *Let $v_0, v_1, v_2, v_3 \in X^4$ be given such that (v_0,v_1,v_2) and (v_1,v_2,v_3) are convexity triples and $v_1 \neq v_2$. Then it follows that (v_0,v_1,v_3) is a convexity triple.*

Proof. Since (v_0,v_1,v_2) and (v_1,v_2,v_3) are convexity triples, we have $\lambda_1,\lambda_2 \in [0,1]$ such that

$$v_1 = (1-\lambda_1)v_0 + \lambda_1 v_2, \quad v_2 = (1-\lambda_2)v_1 + \lambda_2 v_3. \tag{18}$$

- If $\lambda_1 = 0$ then it follows by using 18 that (v_0,v_1,v_3) is a convexity triple.
- If $\lambda_2(1-\lambda_1) = 0$ then 18 yields that $v_2 = v_1$ in contradiction to the assumption.

- If $1 > \lambda_1 > 0$ and $\lambda_2 > 0$ then one can consider the first equation in 18 and insert it into the second. This yields that

$$\begin{aligned} v_2 &= (1-\lambda_2)\left((1-\lambda_1)v_0 + \lambda_1 v_2\right) + \lambda_2 v_3 \\ &= (1-\lambda_2)(1-\lambda_1)v_0 + (\lambda_1 - \lambda_1\lambda_2)\, v_2 + \lambda_2 v_3. \end{aligned} \tag{19}$$

Hence, it holds that

$$(1-\lambda_1+\lambda_1\lambda_2)v_2 = (1-\lambda_2)(1-\lambda_1)v_0 + \lambda_2 v_3. \tag{20}$$

We have

$$1-\lambda_1+\lambda_2\lambda_1 = (1-\lambda_1)+\lambda_2\lambda_1 > \lambda_2(1-\lambda_1)+\lambda_2\lambda_1 \geq \lambda_2 > 0. \tag{21}$$

Therefore, the following definition creates a well defined number in $[0,1]$:

$$\lambda' := \frac{\lambda_2}{1-\lambda_1+\lambda_1\lambda_2}. \tag{22}$$

It holds that

$$1-\lambda' = \frac{1-\lambda_1+\lambda_1\lambda_2-\lambda_2}{1-\lambda_1+\lambda_1\lambda_2} = \frac{(1-\lambda_1)(1-\lambda_2)}{1-\lambda_1+\lambda_1\lambda_2}. \tag{23}$$

Recalling 20, we deduce that $v_2 = (1-\lambda')v_0 + \lambda' v_3 \in \mathrm{conv}(v_0, v_3)$.

This yields that (v_0, v_2, v_3) is a convexity triple. □

Lemma 3.2. *Let some $t_a, t_b, t_c \in [0,T]$ with $t_a < t_b < t_c$ and some function $u : [0,T] \to X$ be given. Then it holds: u is monotaffine and continuous on $]t_a, t_c]$ if and only if $u(t_b) = u(t_b+)$, $(u(t_a+), u(t_b), u(t_c))$ is a convexity triple and u is monotaffine and continuous on $]t_a, t_b]$ and on $]t_b, t_c]$.*

Proof. $\Longrightarrow$ Assume that u is monotaffine and continuous on $]t_a, t_c]$. Then it follows that $u(t_a+)$ and $u(t_b+)$ are well defined, that $u(t_b+) = u(t_b)$, and that u is continuous on $]t_a, t_b]$ and on $]t_b, t_c]$. Moreover, there exists an monotone increasing function $\beta :]t_a, t_c] \to [0,1]$ such that $\beta(t_a+) = 0$, $\beta(t_c) = 1$ and 7 holds with t_b replaced by t_c.

To prove that u is monotaffine on $]t_a, t_b]$, we have do distinguish two situations:

- If $\beta(t_b) = 0$ then it follows, since β is monotone increasing and $\beta(t_a+) = 0$ that $\beta(t) = 0$ for all $t \in]t_a, t_b]$. Using now 7 with t_b replaced by t_c, we deduce that $u(t) = u(t_a+)$ for all $t \in]t_a, t_b]$. This yields that u is monotaffine and continuous on $]t_a, t_b]$.
- If $\beta(t_b) > 0$ then let $\beta_1 :]t_a, t_b] \to [0,1]$ be defined by

$$\beta_1(t) := \frac{\beta(t)}{\beta(t_b)}, \quad \forall\, t \in]t_a, t_b]. \tag{24}$$

Then it follows that β_1 is monotone increasing and that

$$\beta_1(t_a+) = \frac{\beta(t_a+)}{\beta(t_b)} = 0 \quad \text{and} \quad \beta_1(t_b) = \frac{\beta(t_b)}{\beta(t_b)} = 1. \tag{25}$$

Using 7 with t_b replaced by t_c, we deduce that for all $t \in]t_a, t_b]$ it holds that

$$\begin{aligned} &(1-\beta_1(t))u(t_a+) + \beta_1(t)u(t_b) \\ &= (1-\beta_1(t))u(t_a+) + \beta_1(t)\big((1-\beta(t_b))u(t_a+) + \beta(t_b)u(t_c)\big) \\ &= (1-\beta(t))u(t_a+) + \beta(t)u(t_c) = u(t). \end{aligned} \tag{26}$$

This yields that u is monotaffine on $]t_a, t_b]$.

For proving that u is monotaffine on $]t_b, t_c]$, we have to deal with three situations:

- If $\beta(t_b) = 1$ then it follows that $\beta(t) = 1$ for all $t \in]t_b, t_c]$, since β is monotone increasing and $\beta(t_b) = 1$. Using now 7 with t_b replaced by t_c, we deduce that $u(t) = u(t_c)$ for all $t \in]t_b, t_c]$. This yields that u is monotaffine and continuous on $]t_b, t_c]$.
- If $u(t_a+) = u(t_c)$ then we observe by recalling 7 with t_b replaced by t_c that $u(t) = u(t_a+)$ for all $t \in]t_b, t_c]$. This yields that u is monotaffine and continuous on $]t_b, t_c]$.
- If $\beta(t_b) < 1$ and $u(t_a+) \neq u(t_c)$ then let $\beta_2 :]t_b, t_c] \to [0, 1]$ be defined by

$$\beta_2(t) := \frac{\beta(t) - \beta(t_b)}{1 - \beta(t_b)}, \quad \forall\, t \in]t_a, t_b]. \tag{27}$$

This is a monotone increasing function. Applying 7 with t_b replaced by t_c and using the continuity of u at t_b, we deduce that

$$\begin{aligned} &u(t_a+) + \beta(t_b)\left(u(t_c) - u(t_a+)\right) = u(t_b) = u(t_b+) \\ &= u(t_a+) + \beta(t_b+)\left(u(t_c) - u(t_a+)\right). \end{aligned} \tag{28}$$

This yields that $\beta(t_b) = \beta(t_b+)$. Hence, we have

$$\beta_2(t_b+) = \frac{\beta(t_b+) - \beta(t_b)}{1 - \beta(t_b)} = 0 \quad \text{and} \quad \beta_2(t_c) = \frac{\beta(t_c) - \beta(t_b)}{1 - \beta(t_b)} = 1. \tag{29}$$

Using 28 and 7 with t_a replaced by t_b, we deduce that for all $t \in]t_b, t_c]$ it holds that

$$\begin{aligned} &(1-\beta_2(t))u(t_b+) + \beta_2(t)u(t_c) \\ &= \frac{1-\beta(t)}{1-\beta(t_b)}\big((1-\beta(t_b))u(t_a+) + \beta(t_b)u(t_c)\big) + \frac{\beta(t)-\beta(t_b)}{1-\beta(t_b)}u(t_c) \\ &= (1-\beta(t))u(t_a+) + \beta(t)u(t_c) = u(t). \end{aligned} \tag{30}$$

This yields that u is monotaffine on $]t_a, t_c]$.

Hence, it is proved that u is monotaffine and continuous on $]t_a, t_b]$ and also on $]t_b, t_c]$.

$\Longleftarrow$ Assume that $u(t_b) = u(t_b+)$, that $(u(t_a+), u(t_b), u(t_c))$ is a convexity triple and that u is monotaffine and continuous on $]t_a, t_b]$ and on $]t_b, t_c]$. Hence, it follows that $u(t_a+)$ is well defined, and that there exists some $\lambda \in [0,1]$ and monotone increasing functions $\beta_1 :]t_a.t_b] \to [0,1]$ and $\beta_2 :]t_b.t_c] \to [0,1]$ such that

$$u(t_b+) = u(t_b) = (1-\lambda)u(t_a+) + \lambda u(t_c), \tag{31}$$
$$\beta_1(t_a+) = 0, \quad \beta_1(t_b) = 1, \quad \beta_2(t_b+) = 0, \quad \beta_2(t_c) = 1, \tag{32}$$
$$u(t) = (1-\beta_1(t))u(t_a+) + \beta_1(t)u(t_b), \quad \forall\, t \in]t_a, t_b], \tag{33}$$
$$u(t) = (1-\beta_2(t))u(t_b+) + \beta_2(t)u(t_c), \quad \forall\, t \in]t_b, t_c]. \tag{34}$$

Defining now $\beta :]t_a, t_c] \to [0,1]$ by

$$\beta(t) := \begin{cases} \beta_1(t)\lambda, & \text{if} \quad t \in]t_a, t_b] \\ \lambda + \beta_2(t)(1-\lambda), & \text{if} \quad t \in]t_a, t_b] \end{cases} \tag{35}$$

we get an monotone increasing function with

$$\beta(t_a+) = \lambda\beta(t_a+) = 0, \quad \beta(t_c) = \lambda + (1-\lambda)\beta(t_c) = 1, \tag{36}$$

such that 7 with t_b replaced by t_c holds.

Therefore, we can deduce that u is monotaffine on $]t_a, t_c]$.

Combining $u(t_b) = u(t_b+)$ and the continuity of u on $]t_a, t_b]$ and on $]t_b, t_c]$, we conclude that u is continuous von $]t_a, t_c]$.

$\square$

Lemma 3.3. *Let $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T], X)$ be given. Then it holds: The memory operator $\rho[u] : [0,T] \to S^2_F(X)$ as in Def. 2.13 is well defined.*

Proof. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u. Let $r_0, \ldots, r_{n-1} \in \{0,1\}$, $m_0, \ldots, m_{n-1} \in \mathbb{N}_0$ and $V_0, \ldots, V_{m_{n-1}} \in X^2$ be defined as in Def. 2.13. Recalling 12, we see that $\rho[u]$ is a well defined function from $[0,T]$ to $S^2(X)$. Hence, it remains to show that for all $t \in [0,T]$ it holds that $\rho[u]$ is a NCTC triple free strings.

- All strings consisting of only two elements of X^2 as components are NCTC triple free string. Using 12a, we see therefore that $\rho[u](0)$ is a NCTC triple free string.
- Recalling 12b, we deduce for all $t \in]0, t_1]$ it holds:
  - $\rho[u](t)$ has two components and is therefore a NCTC triple free string.
  - $\rho[u](t)$ has three components and it holds therefore $r_0 = 1$, $u(0) \neq u(0+) \neq u(t)$ and
  $$\rho[u](t) = \big(V_0, V_1, (u(t), u(t))\big). \tag{37}$$
  Invoking 9 and 11, we see that $V_0 = (u(0), u(0+))$ and $V_1 = (u(0+), u(0+))$. Thus, Def. 2.5 yields that $\rho[u](t)$ considered as triple is no NCTC triple, and that $\rho[u](t)$ considered as string is a NCTC triple free string.

Therefore, it is proved that $\rho[u](t) \in S_F^2(X)$ for all $t \in]0, t_1]$.

- Using the above result, we see that for an induction proof of the assertion $\rho[u](t) \in S_F^2(X)$ for all $t \in]0, t_i]$ for all $i \in \{1, \dots, n\}$ the base case for $i = 1$ is done.

  In the induction step, we assume that $i \in \{1, \dots, n-1\}$ s given such that $\rho[u](t) \in S_F^2(X)$ for all $t \in]0, t_i]$. Now, it is to prove that $\rho[u](t) \in S_F^2(X)$ for all $t \in]t_i, t_{i+1}]$. Let $t \in]t_i, t_{i+1}]$ be arbitrary.

  Recalling 12, we see that

  $$S_F^2(X) \ni \rho[u](t_i) = \big(V_0, \dots, V_{m_{i-1}}, (u(t_i), u(t_i))\big). \tag{38}$$

  Using also that 9 and 11 yield that $m_{i-1} + 1 = m_i - r_i$ and $V_{m_i - r_i} = (u(t_i), u(t_i+))$, we observe that:

  $$(V_{j-1}, V_j, V_{j+1}) \text{ is no NCTC triple} \quad \forall\, j \in \mathbb{N} \text{ with } j < m_i - r_i. \tag{39}$$

  Since u is monotaffine and continuous on $]t_{i-1}, t_i]$ and on $]t_i, t]$ but not monotaffine on $]t_{i-1}, t]$, we deduce from Lemma 3.2 that at least one of the two conditions $(u(t_{i-1}+), u(t_i), u(t))$ is a convexity triple and $u(t_i) = u(t_i+)$ is not satisfied.

  Using that 11 yields that there is some $x^* \in X$ such that

  $$\big(V_{m_i - r_i - 1}, V_{m_i - r_i}, (u(t), u(t)\big) = \big((x^*, u(t_{i-1}+)), (u(t_i), u(t_i+)), (u(t), u(t)\big), \tag{40}$$

  we observe that

  $$\big(V_{m_i - r_i - 1}, V_{m_i - r_i}, (u(t), u(t))\big) \text{ is no CTC triple.} \tag{41}$$

  - If $r_i = 0$ then it follows from 12c that

    $$\rho[u](t) = \big(V_0, \dots, V_{m_i}, (u(t), u(t))\big). \tag{42}$$

    Combining this with 39 and 41, we see that $\rho[u](t) \in S_F^2(X)$.

  - If $r_i = 1$ and $u(t_i+) = u(t)$ then if follows from 12c that

    $$\rho[u](t) = \big(V_0, \dots, V_{m_i - r_i}, (u(t), u(t))\big). \tag{43}$$

    Invoking 39 and 41, we deduce that $\rho[u](t) \in S_F^2(X)$.

  - If $r_i = 1$ and $u(t_i+) \neq u(t)$ then if follows from 9 and the monotaffinicity of u on $]t_{i-1}, t_i]$ that there exists some $\tau \in]t_i, t[$ such that $u(t_i+) = u(\tau)$, and therefore, by 11, $V_{m_i} = (u(\tau), u(\tau))$. Hence, the above discussion holds with t replaced by τ such that the corresponding modified version of 41 yields that $\big(V_{m_i-2}, V_{m_i-1}, V_{m_i}\big) = \big(V_{m_i - r_i - 1}, V_{m_i - r_i}, (u(\tau), u(\tau))\big)$ is no CTC triple.

    Since $u(t_i+) \neq u(t)$ and $u(t_i) \neq u(t_i+)$, because of $r_i = 1$ and 9, we see by recalling Def. 2.5 that

    $$\big(V_{m_i-1}, V_{m_i}, (u(t), u(t))\big) = \big((u(t_i), u(t_i+)), (u(t_i+), u(t_i+)), (u(t), u(t))\big)$$

    is an CTC triple but no NCTC triple.

    Combining these results with 39 and 12c implies that

    $$S_F^2(X) \ni \big(V_0, \dots, V_{m_i}, (u(t), u(t))\big) = \rho[u](t). \tag{44}$$

$\square$

Lemma 3.4. *The memory operator* $\rho : \mathrm{Map}_{\mathrm{pw},*}([0,T],X) \to \mathrm{Map}([0,T],S_F^2(X))$ *defined in Def. 2.13 is causal.*

Proof. Let $u, u' \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ and $t \in [0,T]$ be given such that $u(\tau) = u'(\tau)$ for all $\tau \in [0,t]$.

Let $0 = t_0 < t_1 < \cdots < t_n = T$ and $0 = t'_0 < t'_1 < \cdots < t'_{n'} = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u and for u', respectively.

Let $r_0, \ldots, r_{n-1}, m_0, \ldots, m_{n-1}$ and $V_0, \ldots, V_{m_{n-1}}$ be as in Def. 2.13 when $\rho[u]$ is determined and let $r'_0, \ldots, r'_n, m'_0, \ldots, m'_{n'}$ and $V'_0, \ldots, V'_{m'_{n'}}$ be as in Def. 2.13 when $\rho[u']$ is determined.

I) If $t = 0$ then we can use 12a to deduce that
$$\rho[u](0) = \big((u(0),u(0)),(u(0),u(0))\big) = \big((u'(0),u'(0)),(u'(0),u'(0))\big) = \rho[u'](0). \tag{45}$$

II) if $t \in]0,t_1]$ then it follows that $t \in]0,t'_1]$ and that $V_0 = (u(0),u(0+)) = (u'(0),u'(0+)) = V'_0$ and $(u(0+),u(0+)) = (u'(0+),u'(0+))$.

- If $u(t) = u(0+)$ then it follows that also $u'(t) = u'(0+)$. Invoking now 12b, we deduce that
$$\rho[u](t) = \big(V_0,(u(t),u(t)\big) = \big(V'_0,(u'(t),u'(t)\big) = \rho[u'](t). \tag{46}$$
- If $u(t) \neq u(0+)$ then it follows that also $u'(t) \neq u'(0+)$. Since u is monotaffine on $]0,t_1]$ and u' is monotaffine on $]0,t'_1]$, we obtain that $u(t_1) \neq u(0+)$ and $u'(t'_1) \neq u'(0+)$.
  - If $r_0 = 0$ then it holds $u(0+) \notin u(]0,t_1[)$ and therefore $u'(0+) = u(0+) \notin u(]0,t[) = u'(]0,t])$. Using that u' is monotaffine on $]0,t'_1]$, we conclude that $u'(0+) \notin u'(]0,t'_1])$. Hence, it holds that $r'_0 = 0$. such that 12b yields that 46 is valid.
  - If $r_0 = 1$ then it holds $u(0+) \in u(]0,t_1[)$ and therefore, thanks to the monotaffinicity of u, that $u'(0+) = u(0+) \in u(]0,t[) = u'(]0,t]) \subseteq u'(]0,t'_1]$. Therefore, we have $r'_0 = 1$ and $V_1 = (u(0+),u(0+)) = (u'(0+),u'(0+)) = V'_1$ such that 12b yields that
$$\rho[u](t) = \big(V_0,V_1,(u(t),u(t)\big) = \big(V'_0,V'_1,(u'(t),u'(t))\big) = \rho[u'](t). \tag{47}$$

III) If $t \in]t_i,t_{i+1}]$ for some $i \in \{1,\ldots,n\}$ then it follows that $t \in]t'_i,t'_{i+1}]$, that $r_k = r'_k$ and $m_k = m'_k$ for all $k \in \{0,\ldots,i-1\}$, $V_j = V_j$ for all $j \in \{0,\ldots,m_i-1\}$, that $m_i - r_i = m'_i - r'_i$ and that $V_{m_i-r_i} = (u(t_i),u(t_i+)) = (u'(t_i),u'(t_i+)) = V_{m'_i-r'_i}$

- If $u(t) = u(t_i+)$ then it follows that also $u'(t) = u'(t_i+)$. Invoking now 12c, we deduce that
$$\begin{aligned}\rho[u](t) &= \big(V_0,\ldots,V_{m_i-r_i},(u(t),u(t))\big)\\ &= \big(V'_0,\ldots,V'_{m'_i-r'_i},(u'(t),u'(t))\big) = \rho[u'](t).\end{aligned} \tag{48}$$

- If $u(t) \neq u(t_i+)$ then it follows that also $u'(t) \neq u'(t_i+)$. Since u is monotaffine on $]t_i, t_{i+1}]$ and u' is monotaffine on $]t_i, t'_{i+1}]$ it follows that $u(t_{i+1}) \neq u(t_i+)$ and $u'(t'_{i+1}) \neq u'(t_i+)$.
  - If $r_i = 0$ then it follows that $u(t_i+) \notin u(]t_i, t_{i+1}[)$ and therefore $u'(t_i+) = u(t_i+) \notin u(]t_i, t[) = u'(]t_i, t])$. Using that u' is monotaffine on $]t_i, t'_{i+1}]$, we conclude that $u'(t_i+) \notin u'(]t_i, t'_{i+1}])$. Hence, $r'_i = 0$ holds such that 12c yields that 48 is valid.
  - If $r_i = 1$ then it holds that $u(t_i+) \in u(]t_i, t_{i+1}[)$ and therefore, thanks to the monotaffinicity of u:

$$u'(t_i+) = u(t_i+) \in u\left(]t_i, t[\right) = u'\left(]t_i, t[\right) \subseteq u'\left(]t_i, t'_{i+1}]\right). \tag{49}$$

Therefore, we have $r'_i = 1$, $m_i = m'_i$, and

$$V_{m_i} = (u(t_i+), u(t_i+)) = (u'(t_i+), u'(t_i+)) = V'_{m_i}, \tag{50}$$

such that 12c yields that

$$\begin{aligned} \rho[u](t) &= \big(V_0, \dots, V_{m_i}, (u(t), u(t))\big) \\ &= \big(V'_0, \dots, V'_{m_i}, (u(t), u(t))\big) = \rho[u'](t). \end{aligned} \tag{51}$$

$\square$

Lemma 3.5. *Let $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T], X)$ be given. Let $0 = t_0 < t_1 < \dots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u. Let $\alpha : [0,T] \to [0,T]$ be an admissible time transformation of $[0,T]$.*

a) If $u(0+) = u(0)$ and/or $\alpha(t) > 0$ for all $t > 0$ then we have $\rho[u](\alpha(s)) = \rho[u \circ \alpha](s)$ for all $s \in [0,T]$.

b) If $u(0+) \neq u(0)$ and there is some $t \in]0,T]$ with $\alpha(t) = 0$ then we have:

- *For all $s \in [0,T]$ with $\alpha(s) = 0$ it holds $\rho[u](\alpha(s)) = \rho[u \circ \alpha](s)$.*
- *For all $s \in]0,T]$ with $\alpha(s) > 0$ it holds that $\rho[u \circ \alpha](s)$ is generated from $\rho[u](\alpha(s))$ by stretching the first component.*

Proof. Let $0 = t_0 < t_1 < \dots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u. Let $\sigma_0 < \sigma_1 < \dots < \sigma_n$ be defined by

$$\sigma_i := \max\{s \in [0,T] \,|\, \alpha(s) = t_i\}, \quad \forall\, i = 0, \dots, n. \tag{52}$$

Since α is monotone increasing and continuous, we see that for all $s, s' \in [0,T]$ with $s < s'$ it holds that $u \circ \alpha$ is monotaffine and continuous on $]s, s']$ if and only if u is monotaffine and continuous on $]\alpha(s), \alpha(s')]$.

Therefore, using the above settings and recalling Def. 2.11, we deduce that

$$\sigma_i = \max\left\{s \in]\sigma_{i-1}, T] \mid u \circ \alpha \text{ is monotaffine and continuous on }]\sigma_{i-1}, s]\right\}, \tag{53}$$

for all $i \in \{1, \ldots, n\}$. Moreover, the definition of σ_i in 52 yields that

$$u \circ \alpha(\sigma_i) = u(t_i), \quad u \circ \alpha(\sigma_i+) = u(t_i+), \quad \forall\, i \in \{0, \ldots, n\}. \tag{54}$$

I) If for all $s \in]0, T]$ it holds $\alpha(s) > 0$, then we have $\sigma_0 = 0$ and it follows that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = T$ is the standard lo. rc. monotaffinicity continuity decomposition of $[0, T]$ for $u \circ \alpha$. Recalling Def. 2.13 and 54, we see that $\rho[u \circ \alpha](t) = \rho[u](\alpha(t))$ for all $t \in [0, T]$.

II) If $u(0) = u(0+)$ and for some $s \in]0, T]$ it holds $\alpha(s) = 0$, then we have $\sigma_0 > 0$. Thanks to 54, we have $u(0) = u \circ \alpha(\sigma_0) = u \circ \alpha(\sigma_0+)$. Since $u \circ \alpha$ is constant on $[0, \sigma_0]$ and monotaffine and continuous on $]\sigma_0, \sigma_1]$, we can now apply Lemma 3.2 and deduce that $u \circ \alpha$ is monotaffine and continuous on $]0, \sigma_1]$. Defining $\sigma'_0, \ldots, \sigma'_n$ by $\sigma'_0 := 0$ and $\sigma'_i = \sigma_i$ for $i = 1, \ldots, n$, we conclude that $0 = \sigma'_0 < \sigma'_1 < \cdots < \sigma'_n = T$ is the standard lo. rc. monotaffinicity continuity decomposition of $[0, T]$ for $u \circ \alpha$. Recalling Def. 2.13 and 54, we see that $\rho[u \circ \alpha](t) = \rho[u](\alpha(t))$ for all $t \in [0, T]$.

III) If $u(0) \neq u(0+)$ and for some $s \in]0, T]$ it holds $\alpha(s) = 0$, then we have $\sigma_0 > 0$. Thanks to 54, we have $u \circ \alpha(\sigma_0) = u(0) \neq u(0+) = u \circ \alpha(\sigma_0+)$. Since $u \circ \alpha$ is constant on $[0, \sigma_0]$, we deduce that

$$\sigma_0 = \max\left\{s \in]0, T] \mid u \circ \alpha \text{ is monotaffine and continuous on }]0, s]\right\}. \tag{55}$$

Defining $\sigma''_0, \ldots, \sigma''_{n+1}$ by $\sigma''_0 := 0$ and $\sigma'_i = \sigma_{i-1}$ for $i = 1, \ldots, n+1$, we conclude that $0 = \sigma''_0 < \sigma''_1 < \cdots < \sigma''_{n+1} = T$ is the standard lo. rc. monotaffinicity continuity decomposition of $[0, T]$ for $u \circ \alpha$. Let $r_0, \ldots, r_{n-1}, m_0, \ldots, m_{n-1}$ and $V_0, \ldots, V_{m_{n-1}}$ be as in Def. 2.13 when u is determined and let $r''_0, \ldots, r''_n$, $m''_0, \ldots, m''_n$ and $V''_0, \ldots, V''_{m''_n}$ be as in this Definition when $\rho[u \circ \alpha]$ is determined. Recalling 54, we see that

$$r''_0 = 0, \quad m''_0 = 0, \quad r''_i = r_{i-1}, \quad m''_i = m_{i-1} + 1, \quad \forall\, i \in \{1, \ldots, n+1\}. \tag{56}$$

and

$$V_0 = (u(0), u(0+)), \quad V''_0 = (u \circ \alpha(0), u \circ \alpha(0+)) = (u(0), u(0)), \tag{57}$$

$$V''_1 = (u \circ \alpha(\sigma_1), u \circ \alpha(\sigma_1+)) = (u(0), u(0+)), \tag{58}$$

$$V''_j = V_{j-1}, \quad \forall\, j \in \{1, \ldots, m''_n\} \tag{59}$$

are satisfied.

- For $s \in]0, T]$ such that $\alpha(s) = 0$, we deduce by using 12

$$\begin{aligned}\rho[u](\alpha(s)) &= \rho[u](0) = \big((u(0), u(0)), (u(0), u(0))\big) \\ &= \big((u \circ \alpha(0), u \circ \alpha(0)), (u \circ \alpha(s), u \circ \alpha(s))\big) = \rho[u \circ \alpha](s).\end{aligned} \tag{60}$$

- For $s \in]0,T]$ such that $\alpha(s) \in]t_0,t_1]$ it holds $s \in]\sigma_0,\sigma_1] =]\sigma''_1,\sigma''_2]$. Using now 12 $m''_1 = m_0 + 1 = r_0 + 1$, $r''_1 = r_0$ and $u \circ \alpha(\sigma''_1+) = u(0+)$, we conclude that

$$
\begin{aligned}
&\rho[u](\alpha(s))\\
&= \begin{cases}
\big((u(0),u(0+)),(u(\alpha(s)),u(\alpha(s)))\big), & \text{if} \quad r_0 = 0,\\
\big((u(0),u(0+)),(u(\alpha(s)),u(\alpha(s)))\big), & \text{if} \quad r_0 = 1\\
\qquad\qquad \text{and} \quad u(0+) = u(\alpha(s)),\\
\big((u(0),u(0+)),V_1,(u(\alpha(s)),u(\alpha(s)))\big), & \text{if} \quad r_0 = 1\\
\qquad\qquad \text{and} \quad u(0+) \neq u(\alpha(s)),
\end{cases}
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&\rho[u\circ\alpha](s)\\
&= \begin{cases}
\big(V''_0,V''_1,(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r''_1 = 0,\\
\big(V''_0,V''_1,(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r''_i = 1 \quad \text{and}\\
\qquad u\circ\alpha(\sigma''_1+) = u\circ\alpha(t),\\
\big(V''_0,V''_1,V''_2,(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r''_1 = 1 \quad \text{and}\\
\qquad u\circ\alpha(\sigma''_1+) \neq u\circ\alpha(t),
\end{cases}\\
&= \begin{cases}
\big((u(0),u(0)),(u(0),u(0+)),(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r_0 = 0,\\
\big((u(0),u(0)),(u(0),u(0+)),(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r_0 = 1 \quad \text{and}\\
\qquad u(0+) = u\circ\alpha(s),\\
\big((u(0),u(0)),(u(0),u(0+)),V_1,(u\circ\alpha(s),u\circ\alpha(s))\big), & \text{if} \quad r_0 = 1 \quad \text{and}\\
\qquad u(0+) \neq u\circ\alpha(s).
\end{cases}
\end{aligned}
\tag{62}
$$

Therefore, we observe that $\rho[u\circ\alpha](t)$ is generated from $\rho[u](\alpha(t))$ by stretching the first component.

- For $s \in]0,T]$ and $i \in \{1,\dots,n-1\}$ such that $\alpha(s) \in]t_i,t_{i+1}]$ it holds $s \in]\sigma_i,\sigma_{i+1}] =]\sigma''_{i+1},\sigma''_{i+2}]$. Using now Def. 2.13, $m''_i = m_{i-1}+1$, $r''_{i-1} = r_i$ and $u\circ\alpha(\sigma''_{i+1}+) = u(t_i+)$, we see that

$$
\begin{aligned}
&\rho[u](\alpha(s)) \\
&= \begin{cases}
\big(V_0, \dots, V_{m_i}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 0, \\
\big(V_0, \dots, V_{m_i - 1}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 1 \quad \text{and} \\
\qquad u(t_i+) = u(\alpha(s)), & \\
\big(V_0, \dots, V_{m_i}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 1 \quad \text{and} \\
\qquad u(t_i+) \neq u(\alpha(s)), &
\end{cases} \\
&= \begin{cases}
\big((u(0), u(0)), V_1, \dots, V_{m_i}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 0, \\
\big((u(0), u(0)), V_1, \dots, V_{m_i - 1}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 1 \quad \text{and} \\
\qquad u(t_i+) = u(\alpha(s)), & \\
\big((u(0), u(0)), V_1, \dots, V_{m_i}, (u(\alpha(s)), u(\alpha(s)))\big), & \text{if} \quad r_i = 1 \quad \text{and} \\
\qquad u(t_i+) \neq u(\alpha(s)). &
\end{cases}
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
&\rho[u \circ \alpha](s) \\
&= \begin{cases}
\big(V_0'', \dots, V''_{m''_{i+1}}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r''_{i+1} = 0, \\
\big(V_0'', \dots, V''_{m''_{i+1} - 1}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r''_{i+1} = 1 \quad \text{and} \\
\qquad u \circ \alpha(\sigma''_{i+1}+) = u \circ \alpha(s), & \\
\big(V_0', \dots, V''_{m''_{i+1}}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r''_{i+1} = 1 \quad \text{and} \\
\qquad u \circ \alpha(\sigma''_{i+1}+) \neq u \circ \alpha(s), &
\end{cases} \\
&= \begin{cases}
\big((u(0), u(0)), (u(0), u(0+)), V_1, \dots, V_{m_i}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r_i = 0, \\
\big((u(0), u(0)), (u(0), u(0+)), V_1, \dots, V_{m_i - 1}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r_i = 1 \quad \text{and} \\
\qquad u(t_i+) = u \circ \alpha(s), & \\
\big((u(0), u(0)), (u(0), u(0+)), V_1, \dots, V''_{m''_{i+1}}, (u \circ \alpha(s), u \circ \alpha(s))\big), & \text{if} \quad r''_{i+1} = 1 \quad \text{and} \\
\qquad u(k_i+) \neq u \circ \alpha(s), &
\end{cases}
\end{aligned}
\tag{64}
$$

This proves that $\rho[u \circ \alpha](t)$ is generated from $\rho[u](\alpha(t))$ by stretching the first component.

□

Definition 3.6. a) Let $\mathbf{U} = \big((x_0, y_0), \dots, (x_m, y_m)\big)$ in $S_F^2(X)$ be given. The *interpolation of* $\mathbf{U}$ *in* $\mathrm{Map}_{\mathrm{pw},*}([0,T], X)$ is the function $\pi[\mathbf{U}] \in \mathrm{Map}_{\mathrm{pw},*}([0,T], X)$ defined by $\pi[\mathbf{U}](0) = x_0$ and

$$
\pi[\mathbf{U}](t) = \frac{s_{k+1} - t}{s_{k+1} - s_k} y_k + \frac{t - s_k}{s_{k+1} - s_k} x_{k+1}, \quad \forall\, t \in]s_k, s_{k+1}],\ k = 1, \dots, m \tag{65}
$$

with $s_k := \frac{k}{m} T$ for all $k \in \{0, \dots, m\}$.

b) The *interpolation operator* $\pi : S_F^2(X) \to \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ maps $\mathbf{U} \in S_F^2(X)$ to $\pi[\mathbf{U}] \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ defined in a).

Lemma 3.7. *For every $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ there exists some admissible time-transformation $\alpha_u : [0,T] \to [0,T]$, such that*

$$u = \pi\left[\rho[u](T)\right] \circ \alpha_u. \tag{66}$$

Proof. Let $u \in \mathrm{Map}_{\mathrm{pw},*}([0,T],X)$ be given. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard lo. rc. monotaffinicity continuity decomposition of $[0,T]$ for u. Let $r_0,\ldots,r_{n-1} \in \{0,1\}$ and $m_0,\ldots,m_{n-1} \in \mathbb{N}_0$ be as in Def. 2.13. Then it holds for $m := m_{n-1}+1$ that $\rho[u](T) \in X^{m+1}$. Let

$$\big((x_0,y_0),\ldots,(x_m,y_m)\big) := \rho[u](T). \tag{67}$$

Then it follows that

$$\begin{cases} x_{m_i} = u(t_i), \quad y_{m_i} = u(t_i+), \quad \text{if} \quad r_i = 0, \\ x_{m_i-1} = u(t_i), \quad y_{m_i-1} = x_{m_i} = y_{m_i} = u(t_i+), \quad \text{if} \quad r_i = 1, \end{cases} \tag{68}$$

for all $i \in \{0,\ldots,n-1\}$ and that $x_m = y_m = u(T)$.

Let $s_0,\ldots,s_m \in [0,1]$ be defined by $s_k := \frac{k}{m}T$ for all $k \in \{0,\ldots,m\}$.

I) Let $\alpha_u(0) := 0$. Then it follows that

$$u(0) = x_0 = \pi[\rho[u](T)](0) = \pi[\rho[u](T)] \circ \alpha_u(0). \tag{69}$$

II) For $i \in \{0,\ldots,n-1\}$ such that $u(t_i+) = u(t_{i+1})$ it holds that $r_i = 0$, $m_i = m_{i-1}+1$ and $y_{m_i} = u(t_i+) = u(t_{i+1}) = x_{m_i+1}$. Hence, thanks to the monotaffinicity of u and Def. 3.6 it holds that u is equal to $u(t_i+)$ on $]t_i,t_{i+1}]$ and that $\pi\left[\rho[u](T)\right]$ is also equal to $u(t_i+)$ on $]s_{m_i},s_{m_i+1}]$. Defining α_u on $]t_i,t_{i+1}]$ by

$$\alpha_u(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i}s_{m_i} + \frac{t-t_i}{t_{i+1}-t_i}s_{m_i+1}, \quad \forall\, t \in]t_i,t_{i+1}], \tag{70}$$

we conclude that α is continuous and monotone increasing and that

$$\alpha_u(]t_i,t_{i+1}]) =]s_{m_i},s_{m_{i+1}}], \tag{71}$$

$$u(t) = \pi\left[\rho[u](T)\right] \circ \alpha_u(t), \quad \forall t \in]t_i,t_{i+1}]. \tag{72}$$

III) For $i \in \{0,\ldots,n-1\}$ such that $u(t_i+) \neq u(t_{i+1})$, it holds that:

Since u it monotaffine and continuous on $]t_i,t_{i+1}]$, $u(t_i+) = y_{m_i-r_i}$, and $u(t_{i+1}) = x_{m_i+1}$, it follows that there is some continuous, monotone increasing function $\beta :]t_i,t_{i+1}] \to [0,1]$ such that $\beta(t_i+) = 0$, $\beta(t_{i+1}) = 1$, and

$$u(t) = (1-\beta(t))y_{m_i-r_i} + \beta(t)x_{m_i+1}, \quad \forall\, t \in]t_i,t_{i+1}]. \tag{73}$$

- If $r_i = 0$ then it follows that $m_i = m_{i-1} + 1$.
  Defining now

$$\alpha_u(t) = (1 - \beta(t))s_{m_i} + \beta(t)s_{m_i+1}, \quad \forall\, t \in]t_i, t_{i+1}], \tag{74}$$

  we deduce that α_u is continuous and monotone increasing.
  - If $u(t_i+) \notin u(]t_i, t_{i+1}])$ then it follows from 73 that we have therefore $\beta(t) > 0$ for all $t \in]t_i, t_{i+1}]$. Hence, we see that 71 holds and that 73 yields that 72 is valid.
  - If $u(t_i+) \in u\,(]t_i, t_{i+1}])$ then if follows from 9 that $x_{m_i} = u(t_i) = u(t_i+) = y_{m_i}$ and by applying 73 we observe that is some $\beta(t) \in]t_i, t_{i+1}]$ such that $\beta(t) = 0$.
    Hence, we deduce by recalling Def. 3.6 that $\phi[\rho[u](T)](t_i) = y_{m_i - 0} = y_{m_i - r_i}$, such that 73 yields that 72 is valid, and we see that $\alpha_u(]t_i, t_{i+1}]) = [s_{m_i}, s_{m_{i+1}}]$.
- If $r_i = 1$ then it follows that $m_i = m_{i-1} + 2$ and that $u(t_i+) \in u\,(]t_i, t_{i+1}])$. Therefore, it holds that

$$t_* := \max\{t \in]t_i, t_{i+1}[\,|\,\beta(t) = 0\} \tag{75}$$

  is a well-defined number. Defining now $\alpha_u :]t_i, t_{i+1}] \to]s_{m_i-1}, s_{m_i+1}]$ by

$$\alpha_u(t) = \begin{cases} \frac{t_*-t}{t_*-t_i} s_{m_i-1} + \frac{t-t_i}{t_*-t_i} s_{m_i}, & \text{if} \quad t \le t_*, \\ (1-\beta(t))s_{m_i} + \beta(t)s_{m_i+1}, & \text{if} \quad t > t_*, \end{cases} \tag{76}$$

  we see that α is continuous and monotone increasing. Invoking 73 and that 71 and 72 hold.

Hence, we see that $\alpha_u : [0, T] \to [0, T]$ generated by this method is an admissible time-transformation of $[0, T]$, such that 66 holds. □

Lemma 3.8. *For all $\mathbf{U} \in S_F^2(X)$ it holds that*

$$\rho\,[\pi[\mathbf{U}]]\,(T) = \mathbf{U}. \tag{77}$$

Proof. Let $((x_0, y_0), \ldots, (x_m, y_m)) := \mathbf{U}$. Considering $s_0, \ldots, s_m$ as in Def. 3.6, we see that $\pi[\mathbf{U}]$ is continuous and monotaffine on $]s_{k-1}, s_k]$ for all $k = 1, \ldots, m$.

Let $r'_0, \ldots, r'_{m-1} \in \{0, 1\}$ be defined by

$$r'_i := \begin{cases} 1, & \text{if} \quad y_i = x_{i+1} = y_{i+1} \quad \text{and} \quad i < m-1 \\ 0, & \text{otherwise,} \end{cases} \qquad \forall\, i \in \{0, \ldots, m-1\}. \tag{78}$$

Let $n := m - \sum_{j=0}^{m-1} r'_j$ and $m'_0, \ldots, m'_n \in \{0, \ldots, m\}$ be defined by

$$m'_0 := 0, \quad m'_{j+1} := m'_j + 1 + r_{m'_j}, \quad \forall\, j \in \{0, \ldots, n-1\}. \tag{79}$$

Let $t_0, \ldots, t_n \in [0, T]$ be defined by

$$t_j := s_{m'_j} \quad \forall\, j \in \{0, \ldots, n\}. \tag{80}$$

I) Now, it will be proved that $0 = t_0 < \cdots < t_n = T$ is the standard lo. rc. monotaffinicity continuity decomposition of $[0, T]$ for $\pi[\mathbf{U}]$.

- Let $j \in \{0, \ldots, n-1\}$ be arbitrary. We have to prove that 8 holds for $u := \pi[\mathbf{U}]$ and $i := j$.
  - It follows from Def. 3.6, that $\pi[\mathbf{U}]$ is monotaffine and continuous on $]s_{m_j}, s_{m_j+1}]$ and on $]s_{m_j+r_{m_j}}, s_{m_j+1+r_{m_j}}]$.
    - If $r_j = 0$, we have $m_{j+1} = m_j + 1$ and it follows immediately that $\pi[\mathbf{U}]$ is monotaffine and continuous on $]s_{m_j}, s_{m_{j+1}}]$.
    - If $r_j = 1$, then it holds $y_{m_j} = x_{m_j+1} = y_{m_j+1}$. Hence, we deduce by Def. 3.6 that $\pi[\mathbf{U}]$ is on $]s_{m_j}, s_{m_j+1}]$ constant equal to y_{m_j+1} and that $y_{m_j+1} = \pi[\mathbf{U}](s_{m_j+1}+)$. Since $\pi[\mathbf{U}]$ is monotaffine and continuous on $]s_{m_j+1}, s_{m_j+2}]$, we conclude that $\pi[\mathbf{U}]$ is monotaffine and continuous on $]s_{m_j}, s_{m_j+2}] =]s_{m_j}, s_{m_{j+1}}]$.

    Hence, it is shown that $\pi[\mathbf{U}]$ is monotaffine and continuous on

$$]s_{m_j}, s_{m_{j+1}}] =]t_j, t_{j+1}].$$

  - Now, it will be proved by a contradiction argument that for all $s \in [0, T]$ with $s > t_{j+1} = s_{m_{j+1}}$ it holds that $\pi[\mathbf{U}]$ is not monotaffine and continuous on $]t_j, s] =]s_{m_j}, s]$: Assume that this assertion does not hold. Then there exits some $s \in]s_{m_{j+1}}, s_{m_{j+1}+1}]$ such that $\pi[\mathbf{U}]$ is monotaffine and continuous on $]s_{m_j}, s]$. Thus, we can apply Lemma 3.2 to conclude that

$$x_{m_{j+1}} = \pi[\mathbf{U}](s_{m_{j+1}}) = \pi[\mathbf{U}](s_{m_{j+1}}+) = y_{m_{j+1}} \tag{81}$$

    and that

$$\big(\pi[\mathbf{U}](s_{m_j}+),\, \pi[\mathbf{U}](s_{m_{j+1}}),\, \pi[\mathbf{U}](s)\big) = \big(y_{m_j}, x_{m_{j+1}}, \pi[\mathbf{U}](s)\big)$$

    is a convexity triple.
    - If $\pi[\mathbf{U}](s_{m_{j+1}}+) = \pi[\mathbf{U}](s)$ then it follows by Def. 3.6 that $y_{m_{j+1}} = x_{m_{j+1}+1}$. Since $x_{m_{j+1}} = y_{m_{j+1}}$, we deduce that $\big(y_{m_j}, x_{m_{j+1}}, x_{m_{j+1}+1}\big)$. is a convexity triple.
    - If $\pi[\mathbf{U}](s_{m_j+1}+) \neq \pi[\mathbf{U}](s)$ then we can use Lemma 3.2 to deduce that $\pi[\mathbf{U}](s) = \pi[\mathbf{U}](s+)$ and that

$$\big(\pi[\mathbf{U}](s_{m_j+1}+),\, \pi[\mathbf{U}](s),\, \pi[\mathbf{U}](s_{m_{j+1}+1})\big) = \big(x_{m_{j+1}},\, \pi[\mathbf{U}](s),\, x_{m_{j+1}+1}\big)$$

      is a convexity triple. Applying Lemma 3.1 for $\big(y_{m_j}, x_{m_{j+1}},\, \pi[\mathbf{U}](s)\big)$ and $\big(x_{m_{j+1}},\, \pi[\mathbf{U}](s),\, x_{m_{j+1}+1}\big)$ yields that $\big(y_{m_j}, x_{m_{j+1}},\, x_{m_{j+1}+1}\big)$ is a convexity triple.

    Using that $y_{m_j} = y_{m_j+r_j} = y_{m_{j+1}-1}$, we deduce that $\big(y_{m_{j+1}-1}, x_{m_{j+1}},\, x_{m_{j+1}+1}\big)$ is a convexity triple. Recalling now 81, it follows that

$$\Big(\big(x_{m_{j+1}-1}, y_{m_{j+1}-1}\big)\,,\, \big(x_{m_{j+1}},\, y_{m_{j+1}}\big)\,,\, \big(x_{m_{j+1}+1},\, y_{m_{j+1}+1}\big)\Big)$$

is a CTC triple. This is no NCTC triple since it is a triple in the string $\rho[\pi[[U]]$ being a NCTC triple free string. Combining these properties of the triple, we deduce by Def. 2.5 that

$$x_{m_{j+1}-1} \neq y_{m_{j+1}-1} = x_{m_{j+1}} = y_{m_{j+1}} \neq x_{m_{j+1}+1}. \tag{82}$$

- If $r_{m_j} = 0$ then it follows that $m_{j+1} = m_j + 1$ such that 82 yields that $y_{m_j} = x_{m_j+1} = y_{m_j+1}$. Hence, we see that 78 yields that $r_{m_j} = 1$ which is a contradiction to the considered situation.
- If $r_{m_j} = 1$ then it follows that $m_{j+1} = m_j + 2$ and that $y_{m_j} = x_{m_j+1} = y_{m_j+1}$. Hence, we have a contradiction to 82.

Since a contradiction could be derived in both possible situations, it follows that the considered assertion is wrong. Therefore, is proved that for all $s \in [0, T]$ with $s > t_{j+1} = s_{m_{j+1}}$ it holds that $\pi[\mathbf{U}]$ is not monotaffine and continuous on $]t_j, s] =]s_{m_j}, s]$.

Combining both results, we see that 8 holds for $u := \pi[\mathbf{U}]$ and $i := j$.

Thus, it is derived that $0 = t_0 < \cdots < t_n = T$ is the standard lo. rc. monotaffinicity continuity decomposition of $[0, T]$ for $\pi[\mathbf{U}]$.

II) Let $r_0, \ldots, r_{n-1}$, $m_0, \ldots, m_{n-1}$ and $V_0, \ldots, V_{m_{n-1}}$ be as in Def. 2.13 when $\rho[\pi[\mathbf{U}]]$ is determined. Hence, we deduce by Def. 3.6, that $r_i = r_{m'_i}$ and $m_i = m'_i$ for all $i \in \{0, \ldots, n-1\}$ and that $V_j = (x_j, y_j)$ for all $i = 1, \ldots m_{n-1}$. Using 12, we see that

$$\rho[\pi[\mathbf{U}]](T) = \big(V_0, \ldots, V_{m_{n-1}}, (u(T), u(T))\big). \tag{83}$$

Using that $u(T) = x_m = y_m$ and $m_{n-1} = m - 1$, we conclude that $\rho[\pi[\mathbf{U}]](T) = \mathbf{U}$.

□

3.2 Proof of Assertion a) in Thm. 1

Let $F : S^2_F(X) \to Y$ be some function that is invariant with respect to stretching the first component. Let $\mathcal{G}_F : \mathrm{Map}_{\mathrm{pw},*}([0, T], X) \to \mathrm{Map}([0, T], Y)$ be defined as in Def. 2.15, i.e. it holds $\mathcal{G}_F[u](t) = F(\rho[u](t))$ for all $t \in [0, T]$ and for all $u \in \mathrm{Map}_{\mathrm{pw},*}([0, T], X)$.

- For all $u, v \in \mathrm{Map}_{\mathrm{pw},*}([0, T], X)$ and $t \in [0, T]$ such that $u(\tau) = v(\tau)$ for all $\tau \in [0, t]$, we observe by recalling Lemma 3.4 that $\rho[u](t) = \rho[v](t)$ and therefore

$$\mathcal{G}_F[u](t) = F(\rho[u](t)) = F(\rho[v](t)) = \mathcal{G}_F[v](t). \tag{84}$$

  Hence, it is proved that $\mathcal{G}_F$ is causal.

- Let $u \in \mathrm{Map}_{\mathrm{pw},*}([0, T], X)$ and $t \in [0, T]$ be given. Let $\alpha : [0, T] \to [0, T]$ be an admissible time transformation of $[0, T]$.

- If $\rho[u](\alpha(t)) = \rho[u \circ \alpha](t)$ then it follows that

$$\mathcal{G}_F[u](\alpha(t)) = F\left(\rho[u](\alpha(t))\right) = F\left(\rho[u \circ \alpha](t)\right) = \mathcal{G}_F[u \circ \alpha](t). \tag{85}$$

- If $\rho[u](\alpha(t)) \neq \rho[u \circ \alpha](t)$ then we conclude by recalling Lemma 3.5 that $\rho[u \circ \alpha](t)$ is generated from $\rho[u](\alpha(t))$ by stretching the first component. Since F is invariant with respect to stretching the first component, it follows also that 85 holds.

Since 85 is proved for general u, α, and t, it is proved that $\mathcal{G}_F$ is rate-independent.

Using that $\mathcal{G}_F$ is causal and rate-independent, we deduce that $\mathcal{G}_G$ is an hysteresis-operator, i.e. Ass. a) in Thm. 1 is proved. □

3.3 Proof of Assertion b) in Thm. 1

Let $\mathcal{H} : \mathrm{Map}_{\mathrm{pw},*}\left([0,T], X\right) \to \mathrm{Map}\left([0,T], Y\right)$ be some hysteresis operator.

Let $G_{\mathcal{H}} : S_F^2(X) \to Y$ be defined as in Def. 2.15.

- To prove that $G_{\mathcal{H}}$ is invariant with respect to stretching the first component, let $\mathbf{U} := ((x_0, y_0), \ldots, (x_n, y_n)) \in S_F^2(X)$ with be given such that $\mathbf{V} := \big((x_0, x_0), (x_0, y_0), \ldots, (x_n, y_n)\big) \in S_F^2(X)$. The function $\alpha : [0,T] \to [0,T]$ that is constant equal to 0 on $\frac{1}{n}T$ and increases linearly from 0 to T on $\left[\frac{1}{n}T, T\right]$ is an admissible time transformation of $[0,T]$. Recalling Def. 3.6, one observes that $\pi[\mathbf{V}] = \pi[\mathbf{U}] \circ \alpha$. Since $\mathcal{H}$ is rate-independent. we can deduce that

$$G_{\mathcal{H}}[\mathbf{V}] = \mathcal{H}[\pi[\mathbf{V}]](T) = \mathcal{H}[\pi[\mathbf{U}] \circ \alpha](T) = \mathcal{H}[\pi[\mathbf{U}]](\alpha(T)) = \mathcal{H}[\pi[\mathbf{U}]](T) = G_{\mathcal{H}}[\mathbf{U}]. \tag{86}$$

Hence, it is shown that $G_{\mathcal{H}}$ is invariant with respect to stretching the first component.

- Now, we consider $u \in \mathrm{Map}_{\mathrm{pw},*}\left([0,T], X\right)$ and prove that $\mathcal{H}[u](T) = \mathcal{G}_{G_{\mathcal{H}}}[u](T)$.

Thanks to Lemma 3.7 there exists an admissible time-transformation $\alpha_u : [0,T] \to [0,T]$ such that $u = \pi\left[\rho[u](T)\right] \circ \alpha_u$.

Using that $\mathcal{H}$ is rate-independent, we deduce that

$$\begin{aligned}\mathcal{H}[u](T) &= \mathcal{H}\left[\pi\left[\rho[u](T)\right] \circ \alpha_v\right](T) = \mathcal{H}\left[\pi\left[\rho[u](T)\right]\right](\alpha_u(T)) \\ &= \mathcal{H}\left[\pi\left[\rho[u](T)\right]\right](T) = G_{\mathcal{H}}\left(\rho[u](T)\right) = \mathcal{G}_{G_{\mathcal{H}}}[u](T).\end{aligned} \tag{87}$$

- Now, let $u \in \mathrm{Map}_{\mathrm{pw},*}\left([0,T], X\right)$ and $t \in [0,T[$ be arbitrary. Let $v_t : [0,T] \to X$ be defined by

$$v_t(t) := \begin{cases} u(t), & \text{if} \quad t \leq s \\ u(s), & \text{if} \quad t > s \end{cases} \tag{88}$$

Since $\mathcal{H}$ and $\mathcal{G}_{G_{\mathcal{H}}}$ are causal, it holds that $\mathcal{H}[u](t) = \mathcal{H}[v_t](t)$ and $\mathcal{G}_{G_{\mathcal{H}}}[u](t) = \mathcal{G}_{G_{\mathcal{H}}}[v_t](t)$. Moreover, taking into account that $\mathcal{H}$ and $\mathcal{G}_{G_{\mathcal{H}}}$ are rate-independent and that v_t is constant on $[t,T]$, we deduce that $\mathcal{H}[v_t]$ and $\mathcal{G}_{G_\mathcal{H}}[v_t]$ are also constant on $[t,T]$, and therefore $\mathcal{H}[u](t) = \mathcal{H}[v_t](T)$ and $\mathcal{G}_{G_\mathcal{H}}[u](t) = \mathcal{G}_{G_\mathcal{H}}[v_t](T)$. Applying now 87 with u replaced by v_t, we observe that

$$\mathcal{H}[u](t) = \mathcal{H}[v_t](T) = \mathcal{G}_{G_\mathcal{H}}[v_t](T) = \mathcal{G}_{G_\mathcal{H}}[u](t).$$

- Let $F : S_F^2(X) \to Y$ be some function that is invariant with respect to stretching the first component such that $\mathcal{G}_F = \mathcal{H}$. Then it holds by Lem. 3.8 for all $\mathbf{U} \in S_F^2(X)$ that

$$\begin{aligned} F(\mathbf{U}) = F\left(\rho[\pi[\mathbf{U}]](T)\right) = \mathcal{G}_F[\pi[\mathbf{U}]](T) = \mathcal{H}[\pi[\mathbf{U}]](T) \\ = \mathcal{G}_{G_\mathcal{H}}[\pi[\mathbf{U}]](T) = G_\mathcal{H}\left(\rho[\pi[\mathbf{U}]](T)\right) = G_\mathcal{H}(\mathbf{U}). \end{aligned} \tag{89}$$

  Hence, it is shown that $G_\mathcal{H} = F$.

Collecting all results, we see that $G_\mathcal{H}$ is the unique function $F : S_F^2(X) \to Y$ being invariant with respect to stretching the first component such that $\mathcal{G}_F = \mathcal{H}$.

□

4 Example: Generalized vectorial relay

4.1 Definition of the generalized vectorial relay

Many vectorial relays considered in the literature (see, e.g. [4, 5, 17, 21]) can be rewritten as a hysteresis operator of the following form (see [10, 12]), acting on the set $\mathrm{C}([0,T];X)$ of all continuous function from $[0,T]$ to X.

Let a nonempty, open subset Ω of X, a function $\zeta : X \setminus \Omega \to Y$ and $\eta^0 \in Y$ be given.

Definition 4.1. The *generalized vectorial relay operator* $\mathcal{R}$ is defined by

$$\mathcal{R} : \mathrm{C}([0,T];X) \to \mathrm{Map}([0,T],Y), \tag{90}$$

$$\mathcal{R}[u](t) := \begin{cases} \zeta(u(t)), & \text{if} \quad u(t) \notin \Omega, \\ \eta^0, & \text{if} \quad u([0,t]) \subset \Omega, \\ \zeta\left(u\left(\max\left\{s \in [0,t] \,\middle|\, u(s) \notin \Omega\right\}\right)\right), & \text{otherwise}, \end{cases} \tag{91}$$

for all $u \in \mathrm{C}([0,T];X)$.

4.2 The "Definition reuse" extension of the relay and the generated function

The generalized vectorial relay operator can be extended to a hysteresis operator $\mathcal{R}_{\text{reuse}}$ on the space $C_l([0,T];X)$ of all left-continuous function from $[0,T]$ to X by requesting that 91 holds with $\mathcal{R}$ replaced by $\mathcal{R}_{\text{reuse}}$ for all $u \in C_l([0,T];X)$. For the function $G_{\mathcal{R}_1}$ generated by $\mathcal{R}_{\text{reuse}}$ is holds for all $\big((x_0,y_0),\dots,(x_n,y_n)\big) \in S_F^2(X)$ that

$$
\begin{aligned}
&G_{\mathcal{R}_{\text{reuse}}}\big((x_0,y_0),\dots,(x_n,y_n)\big)\\
&= \begin{cases}
\zeta(x_n), \quad \text{if} \quad x_n \notin \Omega,\\
\zeta(x_0), \quad \text{if} \quad x_0 \notin \Omega \text{ and } \operatorname{conv}_{>0}(y_i,x_{i+1}) \subseteq \Omega, \quad \forall\, i \in \{0,\dots,n-1\},\\
\eta^0, \quad \text{if} \quad x_0 \in \Omega \text{ and } \operatorname{conv}_{>0}(y_i,x_{i+1}) \subseteq \Omega, \quad \forall\, i \in \{0,\dots,n-1\},\\
\zeta\left((1-s)y_k + s x_{k+1}\right) \quad \text{with}\\
\qquad s \in]0,1] \text{ maximal such that } \ (1-s)y_k + s x_{k+1} \in \partial\Omega,\\
\qquad k := \max\big\{ i \in \{0,\dots,n-1\} \,\big|\, \operatorname{conv}_{>0}(y_i,x_{i+1}) \setminus \Omega \neq \emptyset \big\}, \quad \text{otherwise},
\end{cases}
\end{aligned}
\tag{92}
$$

with

$$
\operatorname{conv}_{>0}(v,w) := \big\{(1-\lambda)v + \lambda w \,\big|\, \lambda \in]0,1]\big\} = \begin{cases} \operatorname{conv}(v,w) \setminus \{v\}, & \text{if} \quad v \neq w,\\ \{v\}, & \text{if} \quad v = w, \end{cases}
\tag{93}
$$

for all $v, w \in X$,

4.3 The “arclen” extension of the relay and the generated function

If X is a Hilbert-space, we can extend $\mathcal{R}$ the “arclen” extension $\mathcal{R}_{\text{arclen}}$ introduced in [19], see Rem. 2.19.

Hence, starting from 17, we deduce for all $\big((x_0,y_0),\dots,(x_n,y_n)\big) \in S_F^2(X)$ that:

$$
\begin{aligned}
&G_{\mathcal{R}_{\text{arclen}}}\big((x_0,y_0),\dots,(x_n,y_n)\big)\\
&= \begin{cases}
\zeta(x_n), \quad \text{if} \quad x_n \notin \Omega,\\
\eta^0, \quad \text{if} \quad (\operatorname{conv}(x_i,y_i) \cup \operatorname{conv}(y_i,x_{i+1})) \subseteq \Omega, \quad \forall\, i \in \{0,\dots,n-1\},\\
\begin{cases}
\zeta\left((1-s)y_k + s x_{k+1}\right) \quad \text{with}\\
\quad s \in]0,1] \text{ maximal such that } \ (1-s)y_k + s x_{k+1} \in \partial\Omega,\\
\quad \text{if} \quad \operatorname{conv}(y_k,x_{k+1}) \setminus \Omega \neq \emptyset,\\
\zeta\left((1-s)x_k + s y_k\right) \quad \text{with}\\
\quad s \in]0,1] \text{ maximal such that } \ (1-s)x_k + s y_k \in \partial\Omega,\\
\quad \text{if} \quad \operatorname{conv}(y_k,x_{k+1}) \setminus \Omega = \emptyset,
\end{cases}\\
\quad \text{with} \quad k := \max\big\{ i \in \{0,\dots,n-1\} \,\big|\, (\operatorname{conv}(x_i,y_i) \cup \operatorname{conv}(y_i,x_{i+1})) \setminus \Omega \neq \emptyset \big\},\\
\quad \text{otherwise.}
\end{cases}
\end{aligned}
\tag{94}
$$

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