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A curvature estimate for open surfaces subject to a general mean curvature operator and natural contact conditions at their boundary

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Abstract

In the seventies, L. Simon showed that the main curvatures of two-dimensional hypersurfaces obeying a general equation of mean curvature type are *a priori* bounded by the Hölder norm of the coefficients of the surface differential operator. This was an essentially interior estimate. In this paper, we provide a complement to the theory, proving a global curvature estimate for open surfaces that satisfy natural contact conditions at the intersection with a given boundary.

1 Introduction

In this paper, we consider a two-dimensional hypersurface $S \subset \Omega \subset \mathbb{R}^3$ of class $\mathcal{C}^2$ obeying the geometric equation

$$\operatorname{div}_S \sigma_q(x,\, \nu) + \sigma_x(x,\, \nu) \cdot \nu = \Phi(x,\, \nu) \text{ on } S\,. \tag{1}$$

The domain $\Omega \subset \mathbb{R}^3$ is assumed bounded and has a boundary of class $\mathcal{C}^2$. The functions σ and Φ map from $\Omega \times \mathbb{R}^3$ into $\mathbb{R}$, $(x,\, q) \mapsto \sigma(x,\, q),\, \Phi(x,\, q)$. The function $q \mapsto \sigma(x,\, q)$ is moreover assumed positively homogeneous of degree one and convex. Throughout the paper, ν denotes a unit normal to S. In the case that Φ does not depend on ν, the equation (1) is often originating in the first variation of a parametric convex functional.

The paper is dealing with surfaces satisfying contact conditions at their boundary. It is here assumed that the (relative) boundary of the given surface S is a closed simple curve contained in $\partial\Omega$, and that

$$\sigma_q(x,\, \nu) \cdot n(x) = \kappa(x) \text{ on } \partial S \tag{2}$$

where n denotes the outward unit normal to $\partial\Omega$ and κ is a given function on $\partial\Omega$. Our main result is an *a priori* estimate *up to the boundary* for the main curvatures of the surface S (note $S^1 =$ unit sphere):

$$\|\, \delta\, \nu\|_{L^\infty(S)} \leq c\, \left(\|\Phi\|_{L^\infty(S^1;\, C^\alpha(\overline{\Omega}))} + \|\sigma_x\|_{C^{1,\alpha}(S^1;\, C^\alpha(\overline{\Omega}))} + \|\kappa\|_{C^{1,\alpha}(\partial\Omega)}\right). \tag{3}$$

An interior version (for $S' \subset S$ not intersecting ∂S) of the *a priori* estimate (3) was first proved in the work of L. Simon and N. Trudinger ([Sim77a, Sim77b] and [GT01], Chapter 16). The validity of their approach based on elementary surface differential calculus is essentially restricted to two-dimensional hypersurfaces. However, it is not outdated, even in the light of newer theorems of geometric measure theory, for at least two reasons: The starting point is a *surface differential equation* instead of a geometric minimisation property and, moreover, most general operators of mean curvature type can be treated with the method.

In this paper we prove a complement of the theory exposed in the Chapter 16 of [GT01]: We formulate sufficient conditions on σ, κ and Ω that allow to prove the validity of (3) up to ∂S under the condition (2).

In the context of the mean curvature equation with contact boundary conditions, it is usual to assume that the functions σ, κ and the domain Ω satisfy a compatibility condition

$$0 < \gamma_1 := \inf_{x\in\partial S} \{\sigma(x,\, n(x)) - |\kappa(x)|\} \;. \tag{4}$$

This condition in connexion to (2) ensures that the surfaces S and $\partial\Omega$ do not meet tangentially (cf. [Ura73, Ger79, SS76, Lie83, Dru12] among others). In particular the angle of contact α between the two surfaces is such that

$$|\sin\alpha(x)| := \sqrt{1-(\nu(x)\cdot n(x))^2} \geq \frac{\gamma_1}{\|\sigma_q\|_{L^\infty(\Omega\times\mathbb{R}^3)}} \text{ for all } x \in \partial S\,. \tag{5}$$

For our boundary estimates, we shall need instead of (4) the assumption

$$0 < \gamma_2 := \inf_{x\in\partial S} \left\{ \frac{1}{\sigma^*(x,\, n(x))} - |\kappa(x)| \right\} \;. \tag{6}$$

where $q \mapsto \sigma^*(x,\, q)$ is the dual convex function to $q \mapsto \sigma(x,\, q)$ (Appendix, Section A or among others [Roc70] Section 15, [Kra11] Section 2.2 for detailed discussions on σ^*). The definition of the dual convex function σ^* directly implies that $\sigma^*(x,\, n(x))\, \sigma(x,\, n(x)) \geq 1$. Therefore $\gamma_2 \leq \gamma_1$ and the condition (6) turns out slightly stronger than (4). A second assumption is a compatibility condition between the domain Ω and *the topology* of S. As in the theory by L. Simon, we require that S possesses a representation as a graph, but it must in addition be possible to choose the $Z-$axis of the coordinate system in which this graph-representation is valid tangent to $\partial\Omega$. We assume that there is a unit vector $\vec{g}$ such that

$$\nu(x)\cdot\vec{g} > 0 \text{ for all } x \in S \text{ and } n(x)\cdot\vec{g} = 0 \text{ for all } x \in \partial S\,. \tag{7}$$

The assumption (7) implies that the surface boundary ∂S can oscillate only vertically with respect to some local coordinates. The assumption is satisfied in the setting of the mean-curvature equation : the domain Ω is a cylinder $G\times]-L,\, L[$, with $G \subset \mathbb{R}^2$ and $L > 0$, and the surface $S \subset \Omega$ is a graph in the standard coordinates. Thus $\nu_3 > 0$ on S and $n_3 = 0$ on $\partial G\times]-L,\, L[$, and (7) is valid with the constant unit vector $\vec{g} = e^3$.

We then show that the constant c in the estimate (3) will only depend on the coefficients of the differential operator (1) and the boundary operator (2), and on γ_2. The structure of the paper is as follows. The Section 2 is devoted to formulating and interpreting more exhaustively the assumptions on the data needed for the proof of (3). The main idea used in the Section 3 is that under the condition (2), the mapping $\varphi(x) := \sigma_q(x,\, \nu(x))$, $x \in \overline{S}$, solves a system of nonlinear equation

$$\sigma^*(x,\, \varphi(x)) = 1, \quad \varphi(x)\cdot n(x) = \kappa(x) \text{ for all } x \in \partial S\,. \tag{8}$$

This fact allows under the assumption (6) to express the vector field $\varphi(x)$, $x \in \partial S$ as a smooth function of *only one* of its components. The next sections are devoted to prove the estimate (3) using [GT01], Chapter 16 as a road map: With the boundary reduction of Section 3 at hand, we can apply a variant of the theory of surfaces with $K - K'$ quasi-conformal Gaussian map to φ.

2 Notations and statement of the main result

We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain of class $\mathcal{C}^2$. Throughout the paper, the function σ is assumed to satisfy

$$\sigma \in C^{1,\alpha}(\overline{\Omega};\, C^{1,\alpha}(\mathbb{R}^3 \setminus \{0\})) \cap C^1(\overline{\Omega};\, C^{2,\alpha}(\mathbb{R}^3 \setminus \{0\})) \quad \alpha > 0\,. \tag{9}$$

We assume that $\sigma = \sigma(x,\, q)$ is convex and positively one-homogeneous in the $q-$variable. In particular, there are positive constants λ_j ($j = 0, 1$) and μ_i ($i = 0, \ldots, 2$) such that for all $(x,\, q) \in \Omega \times \mathbb{R}^3 \setminus \{0\}$

$$\lambda_0\, |q| \leq \sigma(x,\, q) \leq \mu_0\, |q| \tag{10a}$$

$$\frac{\lambda_1}{|q|}\, |\xi|^2 \leq \sum_{i,j=1}^3 \sigma_{q_i,q_j}(x,\, q)\, \xi_i\, \xi_j \leq \frac{\mu_1}{|q|}\, |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3\ :\ \xi \cdot q = 0 \tag{10b}$$

$$\sum_{j=1}^3 \sigma_{q_i,q_j}(x,\, q)\, q_j = 0 \quad \text{for } i = 1, \ldots, 3 \tag{10c}$$

$$|\sigma_{q,x}(x,\, q)| \leq \mu_2\,. \tag{10d}$$

The relations (10) are well-known consequences of the smoothness, the convexity and of the positive one-homogeneity of σ (cf. [LU70] for a proof). Note moreover the consequences of (10)

$$\sigma_q(x,\, q) \cdot q = \sigma(x,\, q), \quad |\sigma_q(x,\, q)| \leq \mu_0, \quad |\sigma_x(x,\, q)| \leq \mu_2\, |q|\,. \tag{11}$$

For the right-hand side Φ of (1), we assume that there is $\alpha > 0$ such that

$$\begin{aligned} x \mapsto \Phi(x,\, q) \in C^{\alpha}(\overline{\Omega}) \text{ for all } q \in \mathbb{R}^3\,, \\ q \mapsto \Phi(x,\, q) \in C^{0,1}(\mathbb{R}^3) \text{ for all } x \in \overline{\Omega}\,, \end{aligned} \tag{12}$$

and moreover that

$$\begin{aligned} \|\Phi\|_{L^\infty(\mathbb{R}^3;\, C^\alpha(\overline{\Omega}))} &:= \sup_{q \in \mathbb{R}^3} \|\Phi(\cdot, q)\|_{C^\alpha(\overline{\Omega})} < \infty\,, \\ \|\Phi\|_{L^\infty(\Omega;\, C^{0,1}(\mathbb{R}^3))} &:= \sup_{x \in \overline{\Omega}} \|\Phi(x, \cdot)\|_{C^{0,1}(\mathbb{R}^3)} < \infty\,. \end{aligned} \tag{13}$$

For the right-hand side κ of (2), we assume the regularity

$$\kappa \in C^{1,\alpha}(\partial\Omega) \quad \alpha > 0\,. \tag{14}$$

Theorem 2.1. *Assume that $S \subset \Omega$ satisfies* (1) *and* (2)*. Assume that σ satisfies* (9) *and is convex and positively one-homogeneous in the variable q. Assume that Φ satisfies* (12) *and* (13) *and that κ satisfies* (14)*. If the compatibility conditions* (6) *and* (7) *are valid, there is a constant $c > 0$ that depends only on Ω, on the constants appearing in the conditions* (10)*, on $\|\kappa\|_{C^1(\partial\Omega)}$, on $\|\Phi\|_{L^\infty(\Omega;\, C^{0,1}(S^1))}$ and on the number γ_2 such that* (3) *holds true.*

3 The preliminary boundary reduction

Throughout the section, we assume that $S \in \mathcal{C}^2$ satisfies (2). It is not necessary that S be a graph-surface. For $x \in \overline{S}$, define $\varphi(x) := \sigma_q(x,\, \nu(x))$. Then $\varphi \in C^1(\overline{S};\, \mathbb{R}^3)$. For $x \in \Gamma := \partial S$, the point $\varphi(x)$ belongs to the solution manifold for the system (8) of two algebraic equations. For $x \in \Gamma$ treated as parameter, we accordingly consider the following system of nonlinear equations for the variable $q \in \mathbb{R}^3$:

$$\mathcal{F}(x,\, q) = 0 \qquad \text{where } \begin{cases} \mathcal{F}_1(x,\, q) := q \cdot n(x) - \kappa(x) \\ \mathcal{F}_2(x,\, q) := \sigma^*(x,\, q) - 1 \end{cases} . \tag{15}$$

Lemma 3.1. *For $x \in \Gamma$, denote $K(x) := \{q \in \mathbb{R}^3 \,:\, \mathcal{F}(x,\, q) = 0\}$. Under the assumptions of Theorem 2.1, $K(x)$ is a nonempty closed planar curve of class $\mathcal{C}^2$ enclosing a convex domain. The point $q_0(x) := \kappa(x)\, n(x)$ belongs to the interior of this domain and $\operatorname{dist}(q_0(x),\, K(x)) \geq \gamma_2\, \frac{\lambda_0}{\mu_0}$.*

Proof. At first, we compute $\mathcal{F}'(x,\, q) = (\sigma^*_q(x,\, q),\, n(x))$. For all $x \in \Gamma$, the range of $\mathcal{F}'(x,\, q)$ is equal to 2 on solutions to (15). Otherwise, there would exist a $\lambda = \lambda(x,\, q) \neq 0$ such that $\sigma^*_q(x,\, q) = \lambda\, n(x)$. Thus, using the properties of dual convex functions

$$\frac{q}{\sigma^*(x, q)} = \operatorname{sign}(\lambda)\, \sigma_q(x,\, n(x)) \Longrightarrow \kappa(x) = q \cdot n(x) = \pm\sigma(x,\, n(x))\,,$$

which contradicts (4). Thus, we see that if q is a solution to $\mathcal{F}(x,\, q) = 0$, there is a neighbourhood $B_\rho(q)$ in $\mathbb{R}^3$ such that the range of $\mathcal{F}'(x)$ is equal to 2 therein, and the solution manifold for (15) has to be a curve.

To proceed, we introduce unit tangent vectors τ^j, $j = 1, 2$ at $\partial\Omega$ such that the system $\{n(x),\, \tau^1(x),\, \tau^2(x)\}$ is orthonormal for all $x \in \partial\Omega$. The set $K(x)$ can alternatively be described as

$$K(x) := \{q \in \mathbb{R}^3 \,:\, q = \kappa(x)\, n(x) + \sum_{j=1}^2 p_j\, \tau^j(x),\, p \in \mathbb{R}^2,\, \sigma^*(x,\, q) = 1\}\,. \tag{16}$$

We compute the second partial derivatives

$$\partial^2_{p_i,p_j} \sigma^*(x,\, \kappa(x)\, n(x) + p_j\, \tau^j(x)) = \sigma_{q,q}(x,\, \kappa(x)\, n(x) + p_j\, \tau^j(x))\tau^i(x) \cdot \tau^j(x)\,,$$

and using (10b), we easily show that if $K(x)$ is not empty, it is a convex curve in the plane $P(x) := \{q \,:\, q \cdot n(x) = \kappa(x)\}$. The assumption (6) yields $\sigma^*(x,\, n(x))\, |\kappa(x)| < 1$, and it implies that the point $q_0(x) := \kappa(x)\, n(x)$ belongs to the interior of the unit ball $W^1_{\sigma^*(x)}$ of the function $\sigma^*(x)$ (Appendix, Section A). As an obvious by-product $K(x)$ cannot be empty.

Calling $G(x)$ the planar convex domain enclosed by $K(x)$, we see that $q_0(x)$ is an interior point $G(x)$. Moreover, writing $\sigma^*(x,\, q_0(x)) = 1 - \delta$, and using also (10a) it follows that

$$\delta = 1 - |\kappa(x)|\, \sigma^*(x,\, n(x)) \geq \gamma_2\, \sigma^*(x,\, n(x)) \geq \frac{\gamma_2}{\mu_0}\,. \tag{17}$$

For arbitrary $q \in \partial W^1_{\sigma^*(x)}$, it follows that $\delta \leq \sigma^*(x,\, q) - \sigma^*(x,\, q_0) \leq \lambda_0^{-1}\, |q - q_0|$. This shows that $\operatorname{dist}(q_0(x),\, K(x)) \geq \operatorname{dist}(q_0(x),\, \partial W^1_{\sigma^*(x)}) \geq \gamma_2\, \frac{\lambda_0}{\mu_0}$. $\square$

Next, we construct a parametrisation of the manifold $K(x)$.

Lemma 3.2. *There is $\psi \in C^1(\Gamma;\, C^1([0,\, 2\pi])) \cap C(\Gamma;\, C^2([0,\, 2\pi]))$ such that*

$$K(x) = \{q \in \mathbb{R}^3 \,:\, q = \psi(x,\, \theta), \quad \theta \in [0,\, 2\pi[\}\,.$$

Moreover, there is c depending on Ω, on the constants in the conditions (10) *and on γ_2 such that*

$$\|\psi\|_{C^1(\Gamma;\, C^1([0,\, 2\pi]))} + \|\psi\|_{C(\Gamma;\, C^2([0,\, 2\pi]))} \leq c(\|\sigma\|_{C^1(\overline{\Omega};\, C^2(S^1))} + \|\kappa\|_{C^1(\partial\Omega)})\,.$$

The function $\theta \mapsto \psi(x,\, \theta)$ is invertible with inequality

$$|\theta_1 - \theta_2| \leq \frac{\lambda_0\, \gamma_2}{\mu_0}\, |\psi(x,\, \theta_1) - \psi(x,\, \theta_2)| \text{ for all } x \in \Gamma \text{ and } \theta_1,\, \theta_2 \in [0, 2\pi[\,. \tag{18}$$

Proof. In Lemma 3.1, we proved that the curve $K(x)$ is closed and contained in the plane $P(x) = \{q \,:\, q \cdot n(x) = \kappa(x)\}$ where it encloses a convex domain. Due to Lemma 3.1, the point $q_0(x) = \kappa(x)\, n(x) \in P(x)$ must belong to the interior of this domain. It follows that for each $x \in \Gamma$ and $\theta \in [0,\, 2\pi[$, the equation

$$1 = f(x,\, \theta,\, r) := \sigma^*(x,\, q_0(x) + r\, \cos\theta\, \tau^1(x) + r\, \sin\theta\, \tau^2(x)) \tag{19}$$

possesses a solution $r \in]0,\, +\infty[$. The inequality $\operatorname{dist}(q_0,\, K) \geq \gamma_2\, \lambda_0/\mu_0$ even implies that every solution to (19) satisfies

$$\underline{r} := \frac{\lambda_0}{\mu_0}\, \gamma_2 \leq \operatorname{dist}(q_0(x),\, K(x)) \leq r \leq \bar{r} := \sup_{x\in\Gamma} \operatorname{diam}(W^1_{\sigma^*(x)})\,. \tag{20}$$

The solution to (19) is also unique. To see this, we abbreviate $e = e(x,\, \theta) = \cos\theta\, \tau^1(x) + \sin\theta\, \tau^2(x)$, and we compute

$$f_r(x,\, \theta,\, r) = \sigma^*_q(x,\, q_0 + r\, e) \cdot e\,, \tag{21}$$

$$f_{r,r}(x,\, \theta,\, r) = \sigma^*_{q,q}(x,\, q_0 + r\, e)\, e \cdot e \overset{(10b)}{\geq} \frac{\lambda_1}{(1 + r^2)^{3/2}}\,. \tag{22}$$

We note that $\lim_{r\to\pm\infty} f_r(x,\, \theta,\, r) = \pm\sigma^*(x,\, e)$ (cp. (11)). Due to (22), there thus exists for x and θ fixed a unique $r_0 \in \mathbb{R}$ such that $f_r(x,\, \theta,\, r_0) = 0$, and the function $f(x,\, \theta,\, r)$ attains its global minimum at r_0. In particular, $\sigma^*(x,\, q_0) = f(0) \geq f(r_0)$. If r is a solution to (19), it follows that (cp. (17))

$$\frac{\gamma_2}{\mu_0} \leq 1 - \sigma^*(x,\, q_0) \leq f(x,\, \theta,\, r) - f(x,\, \theta,\, r_0) \leq \frac{r - r_0}{\lambda_0}\,.$$

Thus, on a solution to (19), we also obtain for some $t \in [0,\, 1]$ that

$$\begin{aligned} f_r(x,\, \theta,\, r) &= f_r(x,\, \theta,\, r_0) + f_{r,r}(x,\, \theta,\, t\, r_0 + (1-t)\, r)\, (r - r_0) \\ &\geq \frac{\lambda_1}{(1 + (t\, r_0 + (1-t)\, r)^2)^{3/2}}\, \frac{\lambda_0\, \gamma_2}{\mu_0} \\ &\geq \frac{\lambda_1}{(1 + \bar{r}^2)^{3/2}}\, \frac{\lambda_0\, \gamma_2}{\mu_0}\,. \end{aligned} \tag{23}$$

This shows that (19) possesses a unique solution, and the implicit function theorem allows to introduce a function $r\,:\,\Gamma\times[0,\,2\pi]\,\to\,[\underline{r},\,\bar{r}]$ such that $f(x,\,\theta,\,r(x,\,\theta))\,=\,0$ for all $(x,\,\theta)\in\Gamma\times[0,\,2\pi]$. Using the equation (19) and the regularity of the function r implied by the implicit function theorem, we obtain the identities (subscript s means tangent derivative along Γ)

$$
\begin{aligned}
f_r(x,\,\theta,\,r)\,\partial_s r + f_s(x,\,\theta,\,r) = 0, \quad f_r(x,\,\theta,\,r)\,\partial_\theta r + f_\theta(x,\,\theta,\,r) = 0\\
f_r(x,\,\theta,\,r)\,\partial^2_{s,\theta} r + f_{r,\theta}(x,\,\theta,\,r)\,\partial_s r + f_{s,\theta}(x,\,\theta,\,r) = 0,\\
f_r(x,\,\theta,\,r)\,\partial^2_{\theta^2} r + f_{r,\theta}(x,\,\theta,\,r)\,\partial_\theta r + f_{\theta,\theta}(x,\,\theta,\,r) = 0\,.
\end{aligned}
$$

From the two latter relations and owing to (23), we can easily obtain the announced bounds for the derivatives of r. We define

$$
\psi(x,\,\theta) := \kappa(x)\,n(x) + r(x,\,\theta)\,\cos\theta\,\tau^1(x) + r(x,\,\theta)\,\sin\theta\,\tau^2(x)\,. \tag{24}
$$

Using (20), we compute $|\partial_\theta\psi| = (r_\theta^2 + r^2)^{1/2} \geq \underline{r}$, which shows that the function $\psi(x)$ possesses a Lipschitz continuous inverse. $\square$

Next, we observe that $\varphi(x)\in K(x)$ for all $x\in\Gamma$ (cf. (8)). Thus, the Lemma 3.2 implies that there is a unique $\theta(x)\in[0,\,2\pi]$ such that

$$
\varphi(x) = \psi(x,\,\theta(x)) \text{ for } x\in\Gamma\,. \tag{25}
$$

The differentiability of ψ just proved yields the identity

$$
\partial_s\varphi(x) = \psi_s(x,\,\theta(x)) + \psi_\theta(x,\,\theta(x))\,\partial_s\theta(x), \quad x\in\Gamma\,. \tag{26}
$$

Lemma 3.3. *For a mapping $f\in C^1(\Gamma\times\mathbb{R}^3;\,\mathbb{R}^3)$, $(x,\,q)\mapsto f(x,\,q)$, we introduce*

$$
F(x,\,\theta) := \int_0^\theta f(x,\,\psi(x,\,z))\cdot\psi_\theta(x,\,z)\,dz, \quad x\in\Gamma,\,\theta\in[0,\,2\pi]\,.
$$

For $x\in\Gamma$, denote $\theta(x) := \psi(x)^{-1}(\varphi(x))$, with ψ according to Lemma 3.2. Then, there is a function $a\in L^\infty(\Gamma)$ such that for all $x\in\Gamma$

$$
\begin{aligned}
f(x,\,\varphi(x))\cdot\partial_s\varphi(x) = \frac{d}{ds}F(x,\,\theta(x)) + a(x)\\
\|a\|_{L^\infty(\Gamma)} \leq 2\pi\,\|\psi\|_{C^1(\Gamma\times C^1([0,\,2\pi]))}\,\|f\|_{C^1(\Gamma\times\mathbb{R}^3;\,\mathbb{R}^3)}
\end{aligned}
$$

Proof. Due to (26), we obtain that

$$
\begin{aligned}
f(x,\,\varphi(x))\cdot\partial_s\varphi(x) &= f(x,\,\psi(x,\,\theta(x)))\cdot(\psi_s(x,\,\theta(x)) + \psi_\theta(x,\,\theta(x))\,\partial_s\theta(x))\\
&= f(x,\,\psi(x,\,\theta(x)))\cdot\psi_s(x,\,\theta(x)) + \frac{d}{ds}\int_0^{\theta(x)} f(x,\,\psi(x,\,z))\cdot\psi_\theta(x,\,z)\,dz\\
&\quad - \int_0^{\theta(x)} \{[f_s(x,\,\psi(x,\,z)) + f_q(x,\,\psi(x,\,z))\cdot\psi_s(x,\,z)]\cdot\psi_\theta(x,\,z)\\
&\qquad + f(x,\,\psi(x,\,z))\cdot\psi_{s,\theta}(x,\,z)\}\,dz\,.
\end{aligned}
$$

We introduce the abbreviation

$$
\begin{aligned}
a(x) :=& f(x,\, \psi(x,\, \theta(x))) \cdot \psi_s(x,\, \theta(x)) - \int_0^{\theta(x)} \{[f_s(x,\, \psi(x,\, z)) \\
&+ f_q(x,\, \psi(x,\, z)) \cdot \psi_s(z,\, z)] \cdot \psi_\theta(x,\, z) + f(x,\, \psi(x,\, z)) \cdot \psi_{s,\theta}(x,\, z)\}\, dz\,.
\end{aligned}
$$

The inequality for a follows from Lemma 3.2. □

4 A Morrey-type estimate for $\delta\,\nu$

For $x \in \overline{S}$ and $\rho > 0$ denote $S_\rho(x) := S \cap B_\rho(x)$ where $B_\rho(x)$ is the three-dimensional ball of radius ρ centred at x. We also introduce a decomposition of ∂S_ρ according to

$$
\partial S_\rho(x) = \Gamma_\rho(x) \cup \Sigma_\rho(x), \quad \Gamma_\rho(x) := \partial S \cap \overline{B_\rho(x)},\, \Sigma_\rho(x) := \partial B_\rho(x) \cap S\,. \tag{27}
$$

For $\mu > 0$ we introduce the notation $[u]_{\mu,S} := \sup_{x\in\overline{S},\,\rho>0} \rho^{-\mu} \int_{S_\rho(x)} |u|\, dS$. Throughout the section, we denote $c_0 = c_0(S) > 0$ the smallest constant such that

$$
\text{meas}(S_\rho(x)) \leq c_0\, \rho^2, \quad \forall\, x \in \overline{S},\, \rho > 0\,. \tag{28}
$$

If S satisfies (7), it is locally a graph subject to (1) and (2), and there are well known estimates for the constant c_0 (Appendix, Lemma A.6). The following theorem states a Morrey-type estimate for $\delta\,\nu$. The original interior version (for $S_\rho(x)$ not intersecting ∂S) was proved in [GT01, Sim77b].

Theorem 4.1. *Assumptions of the theorem 2.1. Then, there are numbers $c > 0$ and $1 > \beta > 0$ depending only on the constants in the conditions* (6), (10)*, on the constant of* (28)*, on Ω, on $\|\Phi\|_{L^\infty(S\times S^1)}$ and on $\|\kappa\|_{C^1(\partial\Omega)}$ such that $[\delta\,\nu]_{1+\beta,S} \leq c$.*

To prove the Theorem 4.1, we use on the one hand a variation of the theory of surfaces with $K - K'$quasi-conformal Gaussian map (see [GT01], Ch. 16). On the other hand we rely on the boundary estimates proved independently in the Section 3. For $i, j = 1, 2, 3$, we compute

$$
\delta_i\, \varphi_j(x) = \sigma_{q_j,\delta_i}(x,\, \nu(x)) + \sigma_{q_j,q}(x,\, \nu(x)) \cdot \delta_i\, \nu(x)\,. \tag{29}
$$

Here and in the following, $u_{\delta_i} := u_x \cdot (e^i - \nu_i\, \nu)$. According to the Appendix Lemma A.1 and Remark A.2, we can choose a vector field $\omega \in C^1(\overline{\Omega} \times \mathbb{R}^3;\, \mathbb{R}^3)$ satisfying

$$
\text{curl}_q\, \omega(x,\, \varphi(x)) \cdot \nu(x) = 1 \text{ for all } x \in \overline{S}\,. \tag{30}
$$

Lemma 4.2. *Assume that S is a graph solution to* (1)*. For $i, j = 1, 2, 3$, denote $(M_\varphi)_{i,j}(x) := \sigma_{q_j,q}(x,\, \nu(x)) \cdot \delta_i\, \nu(x)$. Then, zero is an eigenvalue of the matrix $M_\varphi(x)$, associated with the eigenvector $\nu(x)$. Denoting $m_i(x)$, $i = 1, 2$ the two remaining eigenvalues of M_φ*

$$
m_1(x)\, m_2(x) = \nu(x) \cdot (\delta\, \omega_i(x,\, \phi(x)) \times \delta\, \phi_i(x)) + a(x)\,, \tag{31}
$$

with a function $a \in L^\infty(S)$ such that $|a(x)| \leq c\, |\, \delta\, \varphi(x)|$, where c depends only on the constants in the conditions (10)*.*

Proof. Since (10b) implies that $M_\varphi \nu(x) = 0 = M_\varphi^T \nu(x)$, zero is an eigenvalue of M_φ. By assumption, the surface S is globally a graph. To avoid technicalities, we assume that S is already a graph in the standard coordinates: There is $G \in \mathbb{R}^2$ a bounded domain, and $\psi \in C^2(\overline{G})$ such that $S = \mathrm{graph}(\psi;\, G)$.

Every $x \in S$ can be represented $x = (\bar{x},\, \psi(\bar{x}))$ with $\bar{x} \in G$ and $\nu(x) = \frac{(-\nabla\psi(\bar{x}),\,1)}{\sqrt{1+|\nabla\psi(\bar{x})|^2}}$. Due to this representation, the unit normal possesses a natural extension in the whole of $G \times \mathbb{R}$. This can be used to simplify the computations, since $D_3\nu = 0$ (We denote D_i, $i = 1, 2, 3$ the derivative according to the standard coordinates). Exploiting the symmetry of $\delta\,\nu$ and (10b), we see that $(M_\varphi)_{i,j} := \sigma_{q_j,q} \cdot \delta\,\nu_i = \sigma_{q_j,q} \cdot D\nu_i$. We use that there is a zero eigenvalue, so that the product $m_1\, m_2$ is the sum of the three co-factors associated with the diagonal of M_φ. From direct computation, we obtain that

$$
\begin{aligned}
m_1\, m_2 =& (\sigma_{q_1,q_2}\, \sigma_{q_2,q_3} - \sigma_{q_2,q_2}\, \sigma_{q_1,q_3})\, (D_1\nu_2\, D_2\nu_3 - D_2\nu_2\, D_1\nu_3)\\
&+ (\sigma_{q_2,q_1}\, \sigma_{q_1,q_3} - \sigma_{q_1,q_1}\, \sigma_{q_2,q_3})\, (D_2\nu_1\, D_1\nu_3 - D_1\nu_1\, D_2\nu_3)\\
&+ (\sigma_{q_1,q_1}\, \sigma_{q_2,q_2} - \sigma_{q_1,q_2}\sigma_{q_2,q_1})\, (D_1\nu_1\, D_2\nu_2 - D_1\nu_2\, D_2\nu_1)\,,
\end{aligned}
$$

where $\sigma_{q_i,q_j} = \sigma_{q_i,q_j}(x,\, \nu(x))$. Recall that $\det(D^2\psi)(\bar{x})/(1 + |\nabla\psi(\bar{x})|^2)^2 = K_G(x) =$ Gaussian curvature of S at x. Direct computations further show that

$$
\begin{aligned}
&D_1\nu_2\, D_2\nu_3 - D_2\nu_2\, D_1\nu_3 = -\psi_{x_1}\, K_G,\;\; D_2\nu_1\, D_1\nu_3 - D_1\nu_1\, D_2\nu_3 = -\psi_{x_2}\, K_G\,,\\
&D_1\nu_1\, D_2\nu_2 - D_1\nu_2\, D_2\nu_1 = K_G\,.
\end{aligned}
\tag{32}
$$

The latter relations yield

$$
\begin{aligned}
m_1\, m_2 =& K_G\, [-(\sigma_{q_1,q_2}\, \sigma_{q_2,q_3} - \sigma_{q_2,q_2}\, \sigma_{q_1,q_3})\, \psi_{x_1}\\
&- (\sigma_{q_2,q_1}\, \sigma_{q_1,q_3} - \sigma_{q_1,q_1}\, \sigma_{q_2,q_3})\, \psi_{x_2} + \sigma_{q_1,q_1}\, \sigma_{q_2,q_2} - \sigma_{q_1,q_2}\sigma_{q_2,q_1}]\\
=& K_G\, \nu_3^{-1}\, \sigma^{(C)}_{q,q}(x,\, \nu)\nu \cdot e^3\,,
\end{aligned}
$$

with $\sigma^{(C)}_{q,q} =$ co-factor matrix of $\sigma_{q,q}$ and $e^3 =$ third standard basis vector. Further, the property (10c) ensures that there is an orthonormal system $\{\tau^1(x),\, \tau^2(x)\}$ of unit tangent eigenvectors for the matrix $\sigma_{q,q}(x,\, \nu(x))$. Thus, we can use the formula

$$
\begin{aligned}
\sigma^{(C)}_{q,q}\nu(x) &= \sigma^{(C)}_{q,q}(\tau^1(x) \times \tau^2(x)) = \sigma_{q,q}\tau^1(x) \times \sigma_{q,q}\tau^2(x)\\
&= \lambda_1(x)\, \lambda_2(x)\, \tau^1 \times \tau^2 = \lambda_1(x)\, \lambda_2(x)\, \nu(x)\,,
\end{aligned}
\tag{33}
$$

where $\lambda_i(x)$, $i = 1, 2$ are the (positive) eigenvalues of $\sigma_{q,q}(x,\, \nu(x))$ (cp. (10b)). Thus

$$
m_1(x)\, m_2(x) = K_G(x)\, \lambda_1(x)\, \lambda_2(x)\,. \tag{34}
$$

To prove the validity of (31), we now reexpress $\nu(x) \cdot (\delta\,\omega_i(x,\, \phi(x)) \times \delta\,\phi_i(x))$. Owing to the chain rule

$$
\begin{aligned}
&\nu \cdot (\delta\,\omega_i(x,\, \phi) \times \delta\,\phi_i) = \omega_{i,x_k}(x,\, \phi)\, \nu \cdot (\delta\,\phi_k \times \delta\,\phi_i) + \nu \cdot (\omega_{i,\delta}(x,\, \phi) \times \delta\,\phi_i)\\
&\quad = \omega_{i,x_k}(x,\, \phi)\, \sigma_{q_k,q_l}\, \sigma_{q_i,q_j}\, \nu \cdot (\delta\,\nu_l \times \delta\,\nu_j) + \nu \cdot (\omega_{i,\delta}(x,\, \phi) \times \delta\,\phi_i)\\
&\qquad + \omega_{i,x_k}(x,\, \phi)\, \nu \cdot [(\sigma_{\delta,q_k} \times \delta\,\phi_i) + (\delta\,\phi_k \times \sigma_{\delta,\,q_i})]\,.
\end{aligned}
$$

We define

$$a := \nu \cdot (\omega_{i,\delta}(x,\, \phi) \times \delta\, \phi_i) + \omega_{i,x_k}(x,\, \phi)\, \nu \cdot [(\sigma_{\delta,q_k} \times \delta\, \phi_i) + (\delta\, \phi_k \times \sigma_{\delta,\, q_i})\,.$$

The estimate $|a| \leq c\,|\,\delta\,\phi|$ with $c = c(\sigma, \omega)$ follows directly. By construction, the norm $\|\omega\|_{C^1(\overline{\Omega}\times\mathbb{R}^3;\,\mathbb{R}^3)}$ depends only on the dual convex function σ^*, and therefore this quantity is also estimated by the constants in the conditions (10).

Abbreviating also for $j, l = 1, 2, 3$ that $\zeta_{j,l} = \omega_{i,x_k}(x,\, \phi)\, \sigma_{q_k,q_l}\, \sigma_{q_i,q_j}$, and using the graph representation of S, we see that

$$\begin{aligned}\nu \cdot (\delta\, \omega_i(x,\, \phi) \times \delta\, \phi_i) - a = \nu_3\, [&(\zeta_{1,2} - \zeta_{2,1})\, (D_1\nu_1\, D_2\nu_2 - D_1\nu_2\, D_2\nu_1) \\ &+ (\zeta_{3,1} - \zeta_{1,3})\, (D_2\nu_1\, D_1\nu_3 - D_1\nu_1\, D_2\nu_3) \\ &+ (\zeta_{2,3} - \zeta_{3,2})\, (D_1\nu_2,\, D_2\nu_3 - D_2\nu_2\, D_1\nu_3)]\,.\end{aligned}$$

Using the identities (32) again

$$\nu \cdot (\delta\, \omega_i \times \delta\, \phi_i) - a = K_G\, [(\zeta_{1,2} - \zeta_{2,1})\, \nu_3 + (\zeta_{3,1} - \zeta_{1,3})\, \nu_2 + (\zeta_{2,3} - \zeta_{3,2})\, \nu_1]\,.$$

It remains to observe that (cf. (33))

$$\begin{aligned}&[(\zeta_{1,2} - \zeta_{2,1})\, \nu_3 + (\zeta_{3,1} - \zeta_{1,3})\, \nu_2 + (\zeta_{2,3} - \zeta_{3,2})\, \nu_1] = \sigma^{(C)}_{q,q}\, \mathrm{curl}_q\, \omega(x,\, \phi) \cdot \nu \\ &= \sigma^{(C)}_{q,q} \nu \cdot \nu\;\; \mathrm{curl}_q\, \omega(x,\, \phi) \cdot \nu = \lambda_1\, \lambda_2\;\, \mathrm{curl}_q\, \omega(x,\, \phi) \cdot \nu\;.\end{aligned}$$

Therefore, we finally obtain that

$$\nu \cdot (\delta\, \omega_i(x,\, \phi) \times \delta\, \phi_i) - a = \lambda_1\, \lambda_2\, K_G\;\, \mathrm{curl}_q\, \omega(x,\, \sigma_q(x,\, \nu)) \cdot \nu\;.$$

Due to lemma A.1 and to the choice of ω, one sees that $\mathrm{curl}_q\, \omega(x,\, \sigma_q(x,\, \nu)) \cdot \nu = 1$, and the claim follows comparing to (34). □

Corollary 4.3. *Assumptions of Lemma 4.2. Then there is a number $c > 0$ depending only on the constants in the conditions* (10) *such that*

$$|\,\delta\, \varphi|^2 \leq c\, (-\nu \cdot \mathrm{curl}(\omega_i(x,\, \varphi)\;\, \delta\, \varphi_i) + |\Phi(x,\, \nu)|^2 + 1) \qquad \text{on } S\,.$$

Proof. We can perform the differentiation in the equation (1) to see that

$$\sigma_{q_i,q_j}(x,\, \nu)\;\, \delta_i\, \nu_j = \Phi(x,\, \nu) - \sigma_x(x,\, \nu) \cdot \nu - \sigma_{\delta_i,q_i}(x,\, \nu)\,. \tag{35}$$

Define $\widetilde{\Phi}(x,\, \nu)$ the right-hand of (35). This is thus equivalent to $m_1(x) + m_2(x) = \widetilde{\Phi}(x,\, \nu)$, where m_i are the eigenvalues of the matrix M_φ of Lemma 4.2. We square this identity to obtain that $m_1^2 + m_2^2 = \widetilde{\Phi}^2 - 2\, m_1\, m_2$. We estimate the $2-$norm of the matrix M_φ from above with its spectral norm, and obtain that

$$\frac{1}{2}\, |M_\varphi|^2 \leq \frac{c}{2}\, (m_1^2 + m_2^2) = c\, (\widetilde{\Phi}^2 - a - \nu \cdot (\delta\, \omega_i(x,\, \varphi) \times \delta\, \varphi_i))\,.$$

Recall that $\delta\, \varphi = M_\varphi + \sigma_{q,\delta}$ (cf. (29)). It follows that

$$\begin{aligned}\frac{1}{2}\, |\,\delta\, \varphi|^2 &\leq -c\, \nu \cdot (\delta\, \omega_i(x,\, \varphi) \times \delta\, \varphi_i) + c\, (\mu_2 + \widetilde{\Phi}^2 + |a|) \\ &\leq -c\, \nu \cdot (\delta\, \omega_i(x,\, \varphi) \times \delta\, \varphi_i) + c\, (\mu_2 + \widetilde{\Phi}^2 + |\,\delta\, \varphi|)\,.\end{aligned}$$

□

We now have to introduce another variation to the proof in [GT01]. Again this is owing to the presence of a surface boundary in the case here under study. Observe at first that since we assume S being of class $\mathcal{C}^2$, the sets $S_\rho(x) = S \cap B_\rho(x)$ consists of finitely many connected, relatively open surfaces $S^1_\rho(x), \ldots, S^m_\rho(x)$, $m = m(\rho, x) \in \mathbb{N}$. For each of these surfaces and almost all $\rho > 0$, the Theorem of Sard implies that the boundary $\partial S^i_\rho(x)$ is a closed simple curve (cp. [GT01], formula (16.78)) such that $|\,\delta\, r| > 0$ on $\partial S^i_\rho(x)$. However, the curves $\Sigma^i_\rho(x) := \partial S^i_\rho(x) \cap S$ and $\Gamma^i_\rho(x) := \partial S^i_\rho(x) \cap \partial S$ need not being themselves connected. This fact counteracts our proof idea based on Section 3, and we have to introduce a modification of the sets $S_\rho(x)$. In the following Lemma, we want to construct for each $i = 1, \ldots, m$ a set $K^i_\rho(x)$ as the smallest sup-set of $S^i_\rho(x)$ such that the boundary $\partial K^i_\rho(x) \cap \partial S$ is connected.

Lemma 4.4. *Let $S \subset \Omega$ be of class $\mathcal{C}^2$ with boundary $\Gamma \subset \partial\Omega$ a simple curve. For $x \in \overline{S}$ and $\rho > 0$, define $S_\rho(x) = S \cap B_\rho(x)$. Then, there exists a set $S_\rho(x) \subseteq K_\rho(x) \subset S$ with the following properties:*

1 *For all $x \in \overline{S}$ and almost all $\rho > 0$, the boundary $\partial K_\rho(x)$ consists of finitely many Lipschitz closed simple curves $\partial K^i_\rho(x)$, $i = 1, \ldots, m$ with $m = m(\rho,\, x) \in \mathbb{N}$;*

2 *For each $i \in \{1, \ldots, m\}$, the curve $\partial K^i_\rho(x) \cap S$ and the curve $\partial K^i_\rho(x) \cap \partial S$ are connected;*

3 *The set $\partial K_\rho(x) \cap S$ is a subset of the set $\Sigma_\rho(x) := \partial S_\rho(x) \cap S$;*

There is $c > 0$ depending only on the constants of the conditions (10) *such that*

$$\operatorname{meas}(\partial K_\rho(x) \cap \Gamma) \leq c\, \gamma_1^{-1} \left(\operatorname{meas}(\Sigma_\rho(x)) + (1 + \|\Phi\|_{L^\infty(\Omega \times S^1)}) \operatorname{meas}(K_\rho(x)) \right) \quad (36)$$

Assume that S moreover satisfies (7)*. Then, there is a $R_0 = R_0(\Omega) > 0$ and a constant $c > 0$ (see Lemma A.5 below for details) such that*

$$\operatorname{meas}(K_\rho(x)) \leq c\, \rho, \quad \text{for all } x \in \overline{S},\ 0 < \rho \leq R_0\,. \quad (37)$$

Proof. As we already observed, the surface $S_\rho(x)$ consists of finitely many connected, relatively open surfaces $S^1_\rho(x), \ldots, S^m_\rho(x)$, $m = m(\rho, x) \in \mathbb{N}$. For each i, the boundary $\partial S^i_\rho(x)$ is a closed simple curve such that $|\,\delta\, r| > 0$ therein.

Consider arbitrary $i \in \{1, \ldots, m\}$. If the intersection $\partial S^i_\rho(x) \cap \partial S$ has vanishing arclength measure, we define $K^i_\rho(x) := S^i_\rho(x)$.

Otherwise, the simple curve $\partial S^i_\rho(x)$ possesses an intersection with ∂S of positive arclength measure. We define $x^i_0 = x^i_0(x, \rho) \in \partial S$ and $x^i_1 = x^i_1(x, \rho) \in \partial S$ as the first and the last (according to some fixed orientation of $\partial S^i_\rho(x)$) intersection point between the two curves $\partial S^i_\rho(x)$ and Γ. We call $\Gamma(x^i_0,\, x^i_1)$ the connected segment of Γ joining x^i_0 and x^i_1. By definition, x^i_0 and x^i_1 belong to the intersection of $\Sigma^i_\rho(x) = \overline{\partial S^i_\rho(x) \cap S}$ and Γ; Moreover, there is a connected segment $\Sigma(x^i_0,\, x^i_1)$ of $\Sigma^i_\rho(x)$ joining these two points. The curve $\Sigma(x^i_0,\, x^i_1) \cup \Gamma(x^i_0,\, x^i_1)$ is a Lipschitz continuous closed and simple curve. We define $K^i_\rho(x)$ as the piece of the surface S enclosed by this curve. Finally, we define $K_\rho(x) := \bigcup_{i=1}^m K^i_\rho(x)$.

In order to prove the inequality (36), we start from the following identity valid for all $k = 1, 2, 3$, all surfaces $S' \subseteq S$ bounded by a simple curve, and all $v \in C^1(\overline{S'})$:

$$\int_{S'} (\sigma\,\delta_j^k - \sigma_{q_j}\,\nu_k)\;\delta_j\,v = \int_{S'} (\Phi\,\nu_k - \sigma_{x_k})\,v + \int_{\partial S'} (\sigma\,\delta_j^k - \sigma_{q_j}\,\nu_k)\,\nu'_j\,v\,ds\,, \tag{38}$$

where δ_j^k is the Kronecker symbol ((38) is a direct consequence of the Gauss integral theorem and (1)). We apply this identity to $S' = K^i_\rho(x)$, and we choose $v = n_k$. Observe that on $\partial K^i_\rho(x) \cap \Gamma$, the unit co-normal is given by $\nu' = |\sin\alpha|^{-1}\,(n - \nu\cdot n\,\nu)$, which implies that

$$(\sigma\,\delta_j^k - \sigma_{q_j}\,\nu_k)\,\nu'_j\,n_k = |\sin\alpha|^{-1}\,(\sigma - \kappa\,(\nu\cdot n)) \geq \gamma_1/|\sin\alpha| \geq \gamma_1\,. \tag{39}$$

Therefore, as the unit co-normal on $\Sigma(x^i_0,\, x^i_1) \subset \Sigma_\rho(x)$ is given by $\delta\,r/|\,\delta\,r|$

$$\gamma_1\ \mathrm{meas}(\Gamma(x^i_0,\, x^i_1)) \leq c\,(\mathrm{meas}(\Sigma(x^i_0,\, x^i_1)) + (1 + \|\Phi\|_{L^\infty(\Omega\times S^1)})\ \mathrm{meas}(K^i_\rho(x)))\,,$$

where c depends only on the constants of the conditions (10). Summing up these inequalities for $i = 1, \ldots, m$, (36) follows.

We now prove (37). For simplicity, we assume that S is globally the graph of a function in the standard coordinates, and that the vector $\vec{g} := e^3$ is tangent on $\partial\Omega$ at ∂S. Thus, $S = \{(\bar{x},\, \psi(\bar{x}))\ :\ \bar{x} \in G \subset \mathbb{R}^2\}$ where $G = \pi(S)$ is the projection of S onto $\mathbb{R}^2 \times \{0\}$. Since $n\cdot e^3 = 0$ on ∂S, the unit normal $n(\bar{x})$ for $\bar{x} \in \partial G$ is nothing else but $n(\bar{x},\, \psi(\bar{x}))$, and we see that the curvature of ∂G depends only on Ω. Thus, there is a certain $R_0 > 0$ depending only on Ω, such that for all $\bar{x} \in G$ and $0 \leq \rho \leq R_0$, the intersection of the ball $B_\rho(\bar{x})$ with ∂G is a connected curve.

For $i = 1, \ldots, m$ and $\rho \leq R_0$, consider the set $K^i_\rho(x)$ as above. The boundary $\partial K^i_\rho(x)$ consists of two simple curves $\Sigma(x^i_0,\, x^i_1)$ and $\Gamma(x^i_0,\, x^i_1)$. Denote $\bar{x}^i_0,\, \bar{x}^i_1 \in \partial G$ the projections of the extremal points $x^i_0,\, x^i_1$. The projection of the connected curve segment $\pi(\Gamma(x^i_0,\, x^i_1))$ can be nothing else but the entire piece of curve $\partial G(\bar{x}^i_0,\, \bar{x}^i_1)$. Since $x^i_0,\, x^i_1$ belong to $\Sigma_\rho(x)$, their projections $\bar{x}^i_0,\, \bar{x}^i_1 \in G$ satisfy $|\bar{x}^i_k - \bar{x}| \leq \rho$ for $k = 0, 1$ with $\bar{x} := \pi(x)$. Owing to the choice of ρ, the intersection $B_\rho(\bar{x}) \cap \partial G$ is connected, so that the entire $\pi(\Gamma(x^i_0,\, x^i_1))$ must be contained in $B_\rho(\bar{x})$. Thus, $K^i_\rho(x)$ is contained in the cylinder $Z_\rho(x) := B_\rho(\bar{x}) \times \mathbb{R}$. Using Lemma A.5, we obtain the inequality $\mathrm{meas}(K_\rho(x)) \leq \mathrm{meas}(Z_\rho(x)) \leq c\,\rho$. □

Corollary 4.5. *Assume that S is a graph solution to* (1)*. Then there is a constant c that depends only on σ such that*

$$\int_{K_\rho(x)} |\,\delta\,\varphi|^2\,dS \leq c\int_{K_\rho(x)} (1 + \Phi^2(x,\, \nu))\,dS - c\int_{\partial K_\rho(x)} \omega_i(x,\, \varphi)\cdot\partial_s\varphi_i\,ds\,.$$

Proof. We integrate the inequality of the Corollary 4.3 over K_ρ and use the structure of the boundary ∂K_ρ to apply the Stokes integral theorem. □

The most consequent step in the proof of the theorem 4.1 is the following statement.

Proposition 4.6. *Assume that S is a graph solution to* (1), (2)*. Let $x \in \overline{S}$ and $0 \leq \rho \leq R_0$ with $R_0 = R_0(\Omega)$ according to Lemma 4.4. Then there is a constant $c > 0$ depending on all the quantities mentioned in the statement of Theorem 2.1 such that*

$$\left| \int_{\partial K_\rho(x)} \omega_i(x,\, \varphi)\, \partial_s \varphi_i \, ds \right| \leq c \left[\rho + \mathrm{meas}(\Sigma_\rho(x)) + \left(\int_{\Sigma_\rho(x)} |\partial_s \varphi| \, ds \right)^2 \right] .$$

Proof. For all $x \in \overline{S}$ and $0 \leq \rho \leq R_0$, define $\int_{\partial K_\rho(x)} \omega(x,\, \varphi) \cdot \partial_s \varphi \, ds =: I_S$. For arbitrary constant vectors $\omega_0^1, \ldots,\, \omega_0^m$ the identity

$$I_S = \sum_{i=1}^m \int_{\partial K^i_\rho(x)} (\omega(x,\, \varphi) - \omega_0^i) \cdot \partial_s \varphi \, ds =: \int_{\partial K_\rho(x)} (\omega(x,\, \varphi) - \omega_0) \cdot \partial_s \phi \, ds$$

is a consequence of the definition of I_S and the fact that the curves $\partial K^i_\rho(x)$ are closed. For $i = 1, \ldots, m$, we can choose points $x_0^i,\, x_1^i \in \Sigma^i_\rho(x)$ such that $\partial K^i_\rho(x) \cap \partial S$ is the connected segment $\Gamma(x_0^i,\, x_1^i)$ of the curve Γ, and such that $\partial K^i_\rho(x) \cap S$ is the connected segment $\Sigma(x_0^i,\, x_i^1)$ of the curve $\Sigma_\rho(x)$ (Lemma 4.4). We then choose

$$\omega_0^i := \omega(x_0^i,\, \varphi(x_0^i)) \,. \tag{40}$$

The identity $I_S =: I_\Gamma + I_\Sigma$ is valid, where $I_\Sigma := \int_{\partial K_\rho(x) \cap S} (\omega(x,\, \varphi) - \omega_0) \cdot \partial_s \varphi \, ds$, and $I_\Gamma := \int_{\partial K_\rho(x) \cap \partial S} (\omega(x,\, \varphi) - \omega_0) \cdot \partial_s \varphi \, ds$. For $y \in \Sigma(x_0^i,\, x_1^i)$, we can use the smoothness of ω and the fact that $\Sigma(x_0^i,\, x_1^i) \subseteq \Sigma^i_\rho(x)$ to show that

$$|\omega(y,\, \varphi(y)) - \omega_0| \leq c_\omega \left(|y - x_0^i| + |\varphi(y) - \varphi(x_0^i)| \right) \leq c \left(\rho + \int_{\Sigma^i_\rho(x)} |\partial_s \varphi| \, ds \right) ,$$

implying that $\max_{y \in \Sigma(x_0^i,\, x_1^i)} |\omega(y,\, \varphi(y)) - \omega_0| \leq c\, (\rho + \int_{\Sigma^i_\rho} |\partial_s \varphi| \, ds)$. Thus

$$\begin{aligned} |I_\Sigma| &\leq \sum_{i=1}^m \int_{\Sigma(x_0^i,\, x_1^i)} |\partial_s \varphi| \, ds \; (\max_{y \in (x_0^i,\, x_1^i)} |\omega(y,\, \varphi(y)) - \omega_0|) \\ &\leq c \left(\rho + \int_{\Sigma_\rho} |\partial_s \varphi| \, ds \right) \int_{\Sigma_\rho} |\partial_s \varphi| \, ds \,. \end{aligned}$$

In order to estimate I_Γ, we apply the Lemma 3.3 with $f = \omega - \omega_0$. We obtain that $(\omega(x,\, \varphi) - \omega_0) \cdot \partial_s \varphi = \frac{d}{ds} F(x,\, \theta(x)) + a(x)$, with $a \in L^\infty(\Gamma)$ satisfying the estimate of Lemma 3.3. The function F is given on $\Gamma \times [0,\, 2\,\pi]$ by the expression

$$F(x,\, \theta) := \int_0^\theta (\omega - \omega_0)(x,\, \psi(x,\, z)) \cdot \psi_\theta(x,\, z) \, dz \,.$$

In particular, recall that the functions ψ and θ are such that $\psi(x,\,\theta(x)) = \varphi(x)$ for all $x \in \Gamma$ (Lemma 3.2). We obtain that

$$\begin{aligned}\int_{\Gamma(x_0^i,\,x_1^i)} (\omega(x,\,\varphi) - \omega_0)\cdot\partial_s\varphi\,ds &= \int_{\Gamma(x_0^i,\,x_1^i)} \left\{a(x) + \frac{d}{ds}F(x,\,\theta(x))\right\}\,ds(x)\\ &= \int_{\Gamma(x_0^i,\,x_1^i)} a(x)\,ds(x) + F(x_1^i,\,\theta(x_1^i)) - F(x_0^i,\,\theta(x_0^i))\\ &\le \|a\|_{L^\infty(\Gamma)}\ \mathrm{meas}(\Gamma[x_0^i,\,x_1^i]) + |F(x_1^i,\,\theta(x_1^i)) - F(x_0^i,\,\theta(x_0^i))|\,.\end{aligned}$$

The choice (40) of the constant ω_0 implies that

$$\begin{aligned}F_\theta(x_0^i,\,\theta(x_0^i)) &= (\omega-\omega_0)(x_0^i,\,\psi(x_0^i,\,\theta(x_0^i)))\cdot\psi_\theta(x_0^i,\,\theta(x_0^i))\\ &= (\omega-\omega_0)(x_0^i,\,\varphi(x_0^i))\cdot\psi_\theta(x_0^i,\,\theta(x_0^i)) = 0\,.\end{aligned}$$

Thus, owing also to (18), and Lemma 3.2

$$\begin{aligned}&|F(x_1^i,\,\theta(x_1^i)) - F(x_0^i,\,\theta(x_0^i))|\\ &\quad\le |F(x_1^i,\,\theta(x_1^i)) - F(x_0^i,\,\theta(x_1^i))| + |F(x_0^i,\,\theta(x_1^i)) - F(x_0^i,\,\theta(x_0^i))|\\ &\quad\le \sup_{\theta\in[0,2\pi]}\|F_s(\theta)\|_{C(\Gamma)}\ \mathrm{meas}(\Gamma[x_0^i,\,x_1^i]) + \sup_{x\in\Gamma}\|F_{\theta,\theta}(x)\|_{C([0,2\pi])}\,|\theta(x_1^i) - \theta(x^i)|^2\\ &\quad\le c\,(\mathrm{meas}(\Gamma[x_0^i,\,x_1^i]) + |\varphi(x_1^i) - \varphi(x^i)|^2)\\ &\quad\le c\,\left(\mathrm{meas}(\Gamma[x_0^i,\,x_1^i]) + \left(\int_{\Sigma[x^i,\,x_1^i]}|\partial_s\varphi|\right)^2\right)\end{aligned}$$

We now estimate with the help of Lemma 4.4

$$\sum_{i=1}^{m}\mathrm{meas}(\Gamma[x_0^i,\,x_1^i]) \le c\,(\mathrm{meas}(K_\rho) + \mathrm{meas}(\Sigma_\rho)) \le c\,(\rho + \mathrm{meas}(\Sigma_\rho))\,.$$

The claim follows. □

For the proof of Theorem 4.1, we need a few more elementary inequalities.

Lemma 4.7. *Assume that S is a surface of class $\mathcal{C}^2$ that satisfies* (1), (2)*. Then, there are constants $c_i > 0$ ($i = 1,\ldots,5$) such that the following inequalities are valid:*

(1) $\mathrm{meas}(\partial S) \le c_1\ \mathrm{meas}(S)$*;*

(2) $\int_S |\,\delta\,\varphi|^2\,dS \le c_2\ \mathrm{meas}(S)$.

(3) $\int_S |\,\delta\,\nu|^2\,dS \le c_3\ \mathrm{meas}(S)$.

(4) For all $\rho > 0,\, x \in \overline{S},\, \mathrm{meas}(\Gamma_\rho(x)) \le c_4\,\rho$.

(5) For all $\rho > 0,\, x \in \overline{S}$, $\int_{\Sigma_\rho(x)} |\,\delta\,r|\,ds \le c_5\,\rho$, *where* $r(y) = r(y;\,x) := |y-x|$.

The constants c_i depend for $i = 1, 2, 3$ only on Ω, on the constants in the conditions (10) *and* (4) *and on $\|\Phi\|_{L^\infty(S\times S^1)}$. The constants c_4, c_5 additionally depend on $\operatorname{meas}(S)$ and on the constant of* (28).

Proof. To prove (1), we choose $v = n_k$ in formula (38) for $S' = S$. Due to (39), it follows that $\gamma_1 \operatorname{meas}(\partial S) \leq \int_S \{c_\sigma \,|\,\delta\, n| + |\Phi| + |\sigma_x|\}\, dS \leq c \operatorname{meas}(S)$.

In order to prove (2), we first apply the inequality of lemma 4.5 and integrate over S, to obtain that

$$\int_S |\,\delta\,\varphi|^2\, dS \leq c \int_S (1 + |\Phi|^2)\, dS + c \left| \int_{\partial S} \omega(x,\, \varphi) \cdot \partial_s \varphi \right| . \tag{41}$$

Due to the Lemma 3.3 with $f = \omega$, $\omega(x,\, \varphi) \cdot \partial_s \varphi = \frac{d}{ds} F(x,\, \theta(x)) + a(x)$. Since Γ is a closed curve, it follows that $|\int_{\partial S} \omega(x,\, \varphi) \cdot \partial_s \varphi\, ds| \leq \|a\|_{L^\infty} \operatorname{meas}(\partial S)$. Using also point (1) in this Lemma, the claim (2) follows.

The point (3) is a direct corollary. The properties of the dual convex function σ^* (Appendix, (66)) imply that $\nu = \frac{\sigma^*_q(x,\varphi)}{|\sigma^*_q(x,\varphi)|}$. Also $|\sigma^*_q(x,\, \varphi)| = \frac{|\nu|}{\sigma(x,\nu)} \geq \mu_0^{-1}$. Thus,

$$\delta_i\, \nu_j = \frac{\sigma^*_{q_j,q_k}(x,\, \varphi)}{\sigma(x,\, \nu)}\ \delta_i\, \varphi_l\, (\delta_{k,l} - \frac{\sigma^*_{q_k}(x,\, \varphi)\, \sigma^*_{q_l}(x,\, \varphi)}{\sigma(x,\, \nu)^2}) . \tag{42}$$

Thus, $|\,\delta\,\nu| \leq c\,|\,\delta\,\varphi|$ and (2) proves (3).

We prove (4) in the same fashion as (1). Consider $x \in \overline{S}$ and $\rho > 0$ arbitrary. Let $\zeta \in C^\infty(\mathbb{R}^3)$ satisfy $\zeta = 1$ on $B_\rho(x)$, $0 \leq \zeta \leq 1$ in $\mathbb{R}^3$, $\operatorname{supp}(\zeta) \subseteq B_{2\,\rho}(x)$ and $|\nabla\zeta| \leq \rho^{-1}$. We choose $v = \zeta\, n_k$ in the formula (38), and it follows that

$$\begin{aligned}\gamma_1 \operatorname{meas}(\Gamma_\rho) &\leq c_\sigma \int_S |\,\delta\,\zeta|\, dS + \int_S \{c_\sigma\,|\,\delta\,n| + |\Phi| + |\sigma_x|\}\,\zeta\, dS \\ &\leq c\,(\rho^{-1} + 1) \operatorname{meas}(S_{2\rho}) .\end{aligned}$$

Thus using the definition (28), the estimate (4) follows.

We at last prove (5). Since $\Sigma_\rho = \partial B_\rho(x) \cap S$, the co-normal unit vector n' on Σ_ρ is given by $\delta\, r/|\,\delta\, r|$. We denote ν' = co-normal on ∂S. Using the Gauss theorem $\int_{\Sigma_\rho} |\,\delta\, r|\, ds = \int_{\Sigma_\rho} \delta\, r \cdot n'\, ds = \int_{S_\rho} \triangle_S r\, dS - \int_{\Gamma_\rho} \delta\, r \cdot \nu'\, ds$. Since $\triangle_S r = r^{-1}\,(1 + (\nu\cdot\nabla r)^2) - \operatorname{div} \nu\,(\nu\cdot\nabla r)$, the estimate $|\triangle_S r| =\leq 2\, r^{-1} + |\,\delta\,\nu|$ is valid. We also observe that

$$\begin{aligned}&\int_{S_\rho} r^{-1}\, dS = \int_0^\rho t^{-2} \operatorname{meas}(S_t)\, dt \leq c_0\, \rho\, , \\ &\int_{S_\rho} |\,\delta\,\nu|\, dS \leq \|\,\delta\,\nu\|_{L^2(S)} \operatorname{meas}(S_\rho)^{1/2} \leq \|\,\delta\,\nu\|_{L^2(S)}\, \sqrt{c_0}\, \rho\, .\end{aligned}$$

Thus, due also to the estimate (3) in this Lemma, $\int_{S_\rho} |\triangle_S r|\, dS \leq c\,\rho$. On the other hand, $|\,\delta\, r \cdot \nu'| \leq 1$ on Γ_ρ, and the estimate (4) yields $|\int_{\Gamma_\rho} \delta\, r \cdot \nu'\, ds| \leq c_4\, \rho$, achieving to prove the claim. □

We are now able to give the final argument and finish the proof of Theorem 4.1. We denote c a generic constant that depends on the constants of the conditions (10), (6), (28) and moreover on $\|\kappa\|_{C^1(\partial\Omega)}$, $\|\Phi\|_{L^\infty(S\times S^1)}$, on $\operatorname{meas}(S)$ and on the domain Ω.

For $x \in \overline{S}$, define $f(\rho) := \int_{S_\rho(x)} |\,\delta\,\varphi|^2\,dS$. Due to the statements 4.5 and 4.6, we obtain the inequality

$$f(\rho) \leq c \left[\rho + \operatorname{meas}(\Sigma_\rho(x)) + \left(\int_{\Sigma_\rho(x)} |\,\delta\,\varphi|\,ds\right)^2\right]. \tag{43}$$

The Sard Theorem implies that for $x \in S$ and almost all $\rho > 0$, $|\,\delta\,r|$ vanishes at no point of $\partial S_\rho(x)$ (see also the proof of Corollary 4.5). Thus, we can follow the ideas of Theorem 16.4 in [GT01] to estimate

$$\begin{aligned}\left(\int_{\Sigma_\rho(x)} |\,\delta\,\varphi|\,dS\right)^2 &= \left(\int_{\Sigma_\rho(x)} |\,\delta\,r|^{1/2}\,\frac{|\,\delta\,\varphi|}{|\,\delta\,r|^{1/2}}\,ds\right)^2\\ &\leq \left(\int_{\Sigma_\rho(x)} |\,\delta\,r|\,ds\right)\left(\int_{\Sigma_\rho(x)} \frac{|\,\delta\,\varphi|^2}{|\,\delta\,r|}\,ds\right).\end{aligned}$$

The Lemma A.4 below shows that $\int_{\Sigma_\rho(x)} \frac{|\,\delta\,\varphi|^2}{|\,\delta\,r|}\,ds = f'(\rho)$. Moreover, $\operatorname{meas}(\Sigma_\rho(x)) \leq g'(\rho)$, where $g(\rho) := \operatorname{meas}(S_\rho(x))$. Exploiting moreover the inequality of Lemma 4.7, (5), it follows that $(\int_{\Sigma_\rho(x)} |\,\delta\,\varphi|\,ds)^2 \leq c_5\,\rho\,f'(\rho)$.

Using this latter inequality in (43) yields the differential inequality

$$f(\rho) \leq c\,[\rho + g'(\rho) + \rho\,f'(\rho)]\,. \tag{44}$$

For $\beta := 1/c$, and $h(\rho) := \rho^{-\beta} f(\rho)$, it follows that $0 \leq \rho^{-\beta} + g'(\rho)\,\rho^{-1-\beta} + h'(\rho)$, and therefore, after intergation on the interval $(R,\,R_0)$

$$h(R) \leq h(R_0) + \frac{R_0^{1-\beta}}{1-\beta} + \int_R^{R_0} g'(\rho)\,\rho^{-1-\beta}\,d\rho\,.$$

Using integration by parts

$$\int_R^{R_0} g'(\rho)\,\rho^{-1-\beta} = \frac{g(R_0)}{R_0^{1+\beta}} - \frac{g(R)}{R^{1+\beta}} - (1+\beta)\int_R^{R_0} g(\rho)\,\rho^{-2-\beta}\,d\rho\,.$$

Owing to the definition of g, the condition (28) implies that $g \leq c_0\,\rho^2$, and therefore $|\int_R^{R_0} g'(\rho)\,\rho^{-1-\beta}| \leq c\,R_0^\beta$. It follows that

$$f(R) \leq \left(\frac{R}{R_0}\right)^\beta \left(\int_S |\,\delta\,\varphi|^2 + c(R_0)\right) \qquad 0 < R \leq R_0\,.$$

Using finally that $|\,\delta\,\nu| \leq c\,|\,\delta\,\varphi|$ (cf. (42)) yields $\int_{S_R(x)} |\,\delta\,\nu|^2\,dS \leq c\,(R/R_0)^{2\beta}$. Thus, by Hölder's inequality

$$\int_{S_R} |\,\delta\,\nu|\,dS \leq \operatorname{meas}^{1/2}(S_R)\left(\int_{S_R} |\,\delta\,\nu|^2\,dS\right)^{1/2} \leq \sqrt{c_0}\,c\,R^{1+\beta}\,,$$

and the claim of Theorem 4.1 follows.

5 The Hölder estimate and the curvature estimates

The inequality of Theorem 4.1 in principle implies a Hölder estimate for the vector ν, because the Campanato space $L^{(C)}_{1,2+\beta}(S)$ is continuously embedded in $C^{\beta}(\overline{S})$. But the challenge is to prove that the embedding constant can be controlled by the the coefficients of the operators in (1), (2). The interior Hölder estimate was shown in [GT01], Theorem 16.15. But the Hölder estimate on a curved manifold *with contact boundary conditions* must now be worked out. Unfortunately, we find no way to directly apply representation theorems on on the surface S like in the interior case: We obtain in the first subsection 5.1 a result at once more technical and weaker, though it is still sufficient for our purposes. In the second subsection 5.2 we finish the proof of the main result, the curvatures estimate.

The results of both sections are based on local transformations of the surface that allows to pass to a flat configuration. This flattening happens to be more concisely handled if we consider a piece of surface $S_0 \subseteq S$ near a portion of $\partial\Omega$ itself assumed to be planar. By this we mean that $S_0 \subseteq S \cap U$, with an open smooth bounded domain $U \subset \mathbb{R}^3$ such that $U \cap \Omega \subset \{x \in \mathbb{R}^3 \, : \, x_1 < 0\}$, and $U \cap \partial\Omega$ is contained in the plane $\{x \in \mathbb{R}^3 \, : \, x_1 = 0\}$ (so that $n(x) = e_1$ for all $x \in U \cap \partial\Omega$). There is not enough room here to describe the technical step how to locally flatten the boundary of Ω, but we claim it to be a standard procedure that does not affect the generality of the result.

In the next lemma, we want to show how to locally flatten S. At first we need a notation.

Notation 5.1. *For $i = 1, 2, 3$, we denote $\bar{x}^i := \sum_{j=1,\, j\neq i}^{3} x \cdot e^j$ the projection of x on the plane $\{x_i = 0\}$. Here, e^j ($j = 1, 2, 3$) are the standard basis vectors in $\mathbb{R}^3$. For $i = 1, 2, 3$ we employ the reordering $(\bar{x}^i,\, x_i) := x$ of the coordinates.*

Lemma 5.2. *Let $\mathcal{F} \subseteq S$ be a relatively open two-dimensional hypersurface such that $|\nu_i| \geq c_0 > 0$ on $\mathcal{F}$. Then there are: An open neighbourhood $B \subset \mathbb{R}^3$ of $\mathcal{F}$ and a Lipschitz continuous diffeomorphism T in B with Lipschitz constant equal 1, such that the set $G := T(\mathcal{F})$ is contained in the plane $\{x \in \mathbb{R}^3 \, : \, x_i = 0\}$ and such that $\sup_{y\in G} |(T^{-1})'(y)| \leq c_0^{-1}$. Moreover, if $\partial\mathcal{F} \cap \Gamma$ is contained in a flat portion of $\partial\Omega$, the curve $\gamma := T(\partial\mathcal{F} \cap \Gamma)$ is contained in the line $\{x_1 = 0,\, x_i = 0\}$.*

Proof. Observe that the surface S is the zero level-set of $d(x) = \operatorname{dist}(x,\, S)$, and that $\partial_{x_i} d(x) = \pm\nu_i(x)$ for $x \in S$, $i = 1, 2, 3$. For $x \in \mathcal{F}$ arbitrary, the implicit function theorem implies that there are an open neighbourhood B of the point $\bar{x}^i$ in $\mathbb{R}^2$ (cf. the Notation 5.1), and a function $\psi \in C^1(\overline{B})$ such that

$$d(\bar{y},\, \psi(\bar{y})) = 0 \text{ for } \bar{y} \in B(\bar{x}^i), \quad \psi(\bar{x}^i) = x_i\,.$$

Since $|\partial_{x_i} d| \geq c_0$ on $\mathcal{F}$, it also follows that $\psi_{\bar{y}_j}(\bar{y}) = -\partial_{\bar{x}_j} d/\nu_i(\bar{y},\, \psi(\bar{y}))$. Thus, $\sup_{\bar{y}\in B} |\psi_{\bar{y}}| \leq c_0^{-1}$. The latest bound being uniform for $x \in \mathcal{F}$, the construction can be extended to the entire $\mathcal{F}$. $\square$

5.1 The Hölder estimate

Throughout this section, $S_0 \subset S$ is a connected surface such that the curve $\Gamma_0 := \partial S_0 \cap \partial S$ is connected and such that

$$S_0 \subset \{x \in \mathbb{R}^3 \,:\, x_1 < 0\}, \quad \Gamma_0 \subset \{x \in \mathbb{R}^3 \,:\, x_1 = 0\}\,. \tag{45}$$

Since $n = e^1$ on Γ_0, the inequality (5) implies that

$$(\nu_2^2(x) + \nu_3^2(x))^{1/2} = |\sin\alpha(x)| \geq \frac{\gamma_1}{\mu_0} =: \gamma_0, \quad x \in \Gamma_0\,. \tag{46}$$

We define a function $f_{\gamma_0} :\, [0,1] \to [0,1]$ via

$$f(t) := \begin{cases} 1 & \text{for } t \geq \gamma_0/2 \\ \frac{4}{\gamma_0}\,(t - \frac{\gamma_0}{4}) & \text{for } t \in [\gamma_0/4,\, \gamma_0/2[\\ 0 & \text{otherwise} \end{cases}\,.$$

For $x \in S$, we introduce functions

$$\begin{aligned} &\zeta_3(x) := f_{\gamma_0}(|\nu_3(x)|), \quad \zeta_2(x) := f_{\gamma_0}(|\nu_2(x)|)\,(1 - \zeta_3(x)) \\ &\zeta_1(x) := 1 - \zeta_3(x) - \zeta_2(x)\,. \end{aligned} \tag{47}$$

Lemma 5.3. *For $i = 1,2,3$, let $\zeta_i \in C^{0,1}(\overline{S})$ be given by* (47). *Then $0 \leq \zeta_i(x) \leq 1$, and $\sum_{i=1}^3 \zeta_i(x) = 1$ for all $x \in \overline{S}$. Moreover*

$$|\,\delta\,\zeta_i(x)| \leq \frac{4}{\gamma_0}\,|\,\delta\,\nu(x)| \quad x \in \overline{S}\,.$$

Denote $\operatorname{supp}(\zeta_i) := \overline{\{x \in \overline{S} \,:\, \zeta_i > 0\}}$. *Then, $|\nu_i| \geq \gamma_0/4$ on $\operatorname{supp}(\zeta_i)$ for $i = 1,2,3$. Let Γ_0 be defined by* (45). *Then* $\operatorname{dist}(\operatorname{supp}(\zeta_1),\, \Gamma_0) > 0$.

Proof. Looking at the definition of ζ_1 it immediately follows that $\sum_{i=1}^3 \zeta_i(x) = 1$ on $\overline{S}$. Moreover

$$\zeta_1 = 1 - \zeta_3 - (1 - \zeta_3)\,f_{\gamma_0}(|\nu_2|) = (1 - f_{\gamma_0}(|\nu_3|))\,(1 - f_{\gamma_0}(|\nu_2|))\,.$$

Thus, $0 \leq \zeta_i \leq 1$. For $x \in \operatorname{supp}(\zeta_3)$, the choice of f ensures that $|\nu_3(x)| \geq \gamma_0/4$, and for $x \in \operatorname{supp}(\zeta_2)$, it implies that $|\nu_2(x)| \geq \gamma_0/4$. If $x \in \operatorname{supp}(\zeta_1)$, then $\nu_3(x),\, \nu_2(x) \leq \gamma_0/2$, and since we can assume $\gamma_0 < 1$,

$$|\nu_1(x)| \geq \sqrt{1 - 2\,(\gamma_0/2)^2} \geq \gamma_0/4\,.$$

Finally, due to the property (46), $\sup\{|\nu_3(x)|,\, |\nu_2(x)|\} \geq \gamma_0/\sqrt{2}$ for $x \in \Gamma_0$. Therefore, $\operatorname{dist}(\operatorname{supp}(\zeta_1),\, \Gamma_0) > 0$. □

For $x \in S$ and $i \in \{1,2,3\}$, either $\zeta_i(x) = 0$, or we can introduce the largest relatively open connected surface $\mathcal{F}^{i,x} \subseteq \operatorname{supp}(\zeta_i)$ such that $x \in \mathcal{F}^{i,x}$. This definition implies that $\partial\mathcal{F}^{i,x} \subseteq \{\zeta_i = 0\} \cup \partial S$. For $i = 1$ and $x \in S$ such that $\zeta_1(x) > 0$, the Lemma 5.3 implies that $\operatorname{dist}(\mathcal{F}^{1,x},\, \partial S) > 0$. Thus $\partial\mathcal{F}^{1,x} \subseteq \{\zeta_1 = 0\}$.

We now prove a kind of mean value inequality in the spirit of [GT01], (16.27) and (16.29). As already mentioned, it is weaker but allows to consider points $x \in S$ arbitrary near to ∂S.

Lemma 5.4. *Let S_0 satisfy* (45)*. Let $u \in C^1(\overline{S})$ be nonnegative and satisfy*

$$u(x) = 0 \text{ for all } x \in S \setminus S_0 \,. \tag{48}$$

Then there is $c = c(\gamma_0) > 0$ such that for all $x \in S$

$$u(x) \le c \int_S \frac{\{|\,\delta\, u(y)| + |u(y)|\,|\,\delta\,\nu(y)|\}}{|x-y|}\, dS(y) + \max_{y\in\partial S} u(y) \text{ for all } x \in S \tag{49}$$

$$u(x) \le c \int_S \frac{\{|\,\delta\, u(y)| + |u(y)|\,|\,\delta\,\nu(y)|\}}{|x-y|}\, dS(y) \qquad \text{for all } x \in \partial S\,. \tag{50}$$

Proof. For $i = 1, 2, 3$, set $u_i(x) := \zeta_i(x)\, u(x)$ for $x \in \overline{S}$. Then $\sum_{i=1}^3 u_i = u$ on $\overline{S}$ (Lemma 5.3). Let $i \in \{1, 2, 3\}$. If $\zeta_i(x) = 0$, then

$$u_i(x) = 0\,. \tag{51}$$

Otherwise, $\zeta_i(x) > 0$, and we consider the set $\mathcal{F} = \mathcal{F}^{i,x}$. Owing to Lemma 5.2, there is an open neighbourhood $B \subset \mathbb{R}^3$ of $\mathcal{F}^{i,x}$ and a Lipschitz continuous diffeomorphism T in B, such that the set $G := T(\mathcal{F})$ is contained in the plane $\{x \in \mathbb{R}^3 \,:\, x_i = 0\}$ and such that $\sup_{y\in G} |(T^{-1})'(y)| \le 4/\gamma_0$. Moreover, for $i = 2, 3$, the set G is contained in the half plane $\{x_i = 0,\, x_1 < 0\}$ and the curve $\gamma := T(\partial\mathcal{F}^{i,x} \cap \Gamma)$ is contained in the line $\{x_1 = 0,\, x_i = 0\}$. We identify G with a domain in $\mathbb{R}^2$ and denote $\bar{y} \in G$ its elements. For $\bar{y} \in G$, we furthermore define $\widetilde{u}_i(\bar{y}) := u_i(T^{-1}(\bar{y}))$.

Let $\bar{x} := T(x) \in G$. For $\epsilon > 0$, the vector field

$$V^\epsilon(\bar{y}) := \begin{cases} \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} & \text{if } |\bar{y}-\bar{x}| \ge \epsilon \\ \frac{\bar{y}-\bar{x}}{\epsilon^2} & \text{otherwise} \end{cases} \quad \text{for } \bar{y} \in G$$

is Lipschitz continuous in G, and it satisfies $\operatorname{div} V^\epsilon = 2\,\epsilon^{-2}\,\chi_{B_\epsilon(\bar{x})}$. Thus

$$\int_G (\operatorname{div} V^\epsilon)(\bar{y})\, \widetilde{u}_i(\bar{y})\, d\bar{y} = \frac{2}{\epsilon^2} \int_{B_\epsilon(\bar{x})\cap G} \widetilde{u}_i(\bar{y})\, d\bar{y}\,. \tag{52}$$

Since $\partial\mathcal{F} \subseteq \partial S \cup \{\zeta_i = 0\}$, it follows that $\widetilde{u}_i = 0$ on $\partial G \setminus \gamma$. Moreover, γ is either empty ($i = 1$), or it is contained in a straight line and the outward normal relatively to G satisfies $n_\gamma = e^1$ ($i = 2, 3$). Thus, integration by parts in (52) yields

$$\frac{1}{\pi\,\epsilon^2} \int_{B_\epsilon(\bar{x})\cap G} \widetilde{u}_i\, d\lambda_2 = -\frac{1}{2\pi} \int_G V^\epsilon \cdot \nabla\widetilde{u}_i\, d\lambda_2 + \frac{1}{2\pi} \int_\gamma V^\epsilon \cdot e^1\, \widetilde{u}_i\, d\lambda_1\,. \tag{53}$$

Note that $x \in \mathcal{F}$ being an interior point, $\bar{x} \in G$ is also an interior point. For ϵ sufficiently small, the ball $B_\epsilon(\bar{x})$ is entirely contained in G. Moreover V^ϵ is bounded on γ. Since the singularity of V^ϵ is integrable over G, we can let ϵ tend to zero in the latest relation to obtain that

$$\widetilde{u}_i(\bar{x}) = -\frac{1}{2\pi} \int_G \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla\widetilde{u}_i\, d\lambda_2 + \frac{1}{2\pi} \int_\gamma \frac{\bar{y}_1 - \bar{x}_1}{|\bar{y}-\bar{x}|^2}\, \widetilde{u}_i\, d\lambda_1\,. \tag{54}$$

Denote Φ the double-layer potential of the density $\widetilde{u}_i$ associated with the curve γ, that is $\pi^{-1} \int_{\gamma} \frac{\bar{y}_1-\bar{x}_1}{|\bar{y}-\bar{x}|^2} \widetilde{u}_i \, d\lambda_1 = -\Phi(\bar{x})$. Consider $\bar{x}^0 \in \gamma$ arbitrary. Since γ is contained in a line, it is well known that $\lim_{\bar{x}\to\bar{x}^0} \Phi(\bar{x}) = -\widetilde{u}_i(\bar{x}^0)$. For a proof of this fundamental fact, the reader may consult for instance [Hac95], Section 8.2. It is also well known that the double-layer potential Φ is an harmonic function in the half plane $\{x_1 \leq 0\}$, and for this reason it must attain its extrema on γ. Thus, the relation (54) also implies for $\bar{x} \in G$ that

$$\widetilde{u}_i(\bar{x}) \leq -\frac{1}{2\pi} \int_G \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla \widetilde{u}_i \, d\lambda_2 + \frac{1}{2} \max_{\bar{y}\in\gamma} \widetilde{u}_i(\bar{y}) \,, \tag{55}$$

For $i = 1$, the curve γ being empty, we obtain that

$$\widetilde{u}_1(\bar{x}) \leq -\frac{1}{2\pi} \int_G \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla \widetilde{u}_1 \, d\lambda_2 \,. \tag{56}$$

For $j = 1, 2$, $\bar{y} \in G$, we compute $\partial_{\bar{y}} \widetilde{u}_i(\bar{y}) = \delta_k \, u_i(T^{-1}(\bar{y})) \, T^{-1}_{\bar{y},k}(\bar{y})$. Thus

$$\left| \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla \widetilde{u}_i \right| \leq \frac{4}{\gamma_0} \frac{|\, \delta \, u_i(T^{-1}(\bar{y}))|}{|\bar{y}-\bar{x}|} \leq \frac{4}{\gamma_0} \left(1 + \left(\frac{4}{\gamma_0}\right)^2\right)^{1/2} \frac{|\, \delta \, u_i(T^{-1}(\bar{y}))|}{|T^{-1}(\bar{y}) - T^{-1}(\bar{x})|} \,.$$

Using the transformation formula

$$\left| \int_G \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla \widetilde{u}_i \, d\lambda_2 \right| \leq \frac{4}{\gamma_0} \left(1 + \left(\frac{4}{\gamma_0}\right)^2\right)^{1/2} \int_{\mathcal{F}} |x-y|^{-1} \, |\, \delta \, u_i| \, dS \,. \tag{57}$$

The relations (55), (56) and (57) altogether imply that

$$\begin{aligned} u_i(x) &\leq \frac{c(\gamma_0)}{\pi} \int_{\mathcal{F}^{i,x}} |x-y|^{-1} |\, \delta \, u_i| \, dS + \frac{1}{2} \max_{y\in\partial\mathcal{F}^{i,x}\cap\partial S} u_i(y), \quad \text{for } i = 2, 3 \\ u_1(x) &\leq \frac{c(\gamma_0)}{\pi} \int_{\mathcal{F}^{1,x}} |x-y|^{-1} |\, \delta \, u_1| \, dS \,. \end{aligned}$$

We sum up these inequalities. Observe that if $\mathcal{F}^{i,x}$ is empty, then the case (51) applies. Thus, considering Lemma 5.3, we obtain that

$$u(x) \leq \frac{c(\gamma_0)}{\pi} \int_{S^x} |x-y|^{-1} \sum_{i=1}^{3} |\, \delta \, u_i| \, dS(y) + \max_{y\in\partial S^x\cap\partial S} u(y) \,, \quad S^x := \bigcup_{i=1}^{3} \mathcal{F}^{i,x}$$

It remains to observe that

$$|\, \delta \, u_i| \leq \zeta_i \, |\, \delta \, u| + |\, \delta \, \zeta_i| \, u \leq \zeta_i \, |\, \delta \, u| + \frac{4}{\gamma_0} \, |\, \delta \, \nu| \, u \,, \tag{58}$$

and the inequality (49) follows. For $x \in \partial S$, we choose a sequence $\{x^k\} \subset S$ such that $x^k \to x$. If $x \in \mathcal{F}^{i,x}$, then due to the fact that $\mathcal{F}^{i,x}$ is open, also $x^k \in \mathcal{F}^{i,x}$ for k sufficiently large. Thus, we can choose for $i = 2$ or $i = 3$ a fixed transformation T of the set $\mathcal{F}^{i,x_k}$. We obtain the relation (54) with $\bar{x} = \bar{x}^k$. For $k \to \infty$, $\bar{x}^k \to \bar{x}$, we use the properties of the double-layer potential, to obtain that

$$\frac{\widetilde{u}_i(\bar{x})}{2} = -\frac{1}{2\pi} \int_G \frac{\bar{y}-\bar{x}}{|\bar{y}-\bar{x}|^2} \cdot \nabla \widetilde{u}_i \, d\lambda_2 \,. \tag{59}$$

For $x \in \partial S$ and $i = 2, 3$, the relations (59) and (57) yield

$$u_i(x) \leq \frac{c(\gamma_0)}{\pi} \int_{\mathcal{F}^{i,x}} |x-y|^{-1} \, |\, \delta \, u_i| \, dS \,.$$

It follows that $u(x) \leq \frac{c(\gamma_0)}{\pi} \int_{S^x} |x-y|^{-1} \sum_{i=1}^3 |\, \delta \, u_i| \, dS(y)$, and the claim follows from (58).
□

For $x \in S$ fixed, denote $r_x(y) := |x-y|$, $y \in \mathbb{R}^3$. For $R > 0$ fixed, let ϕ be the cutoff function

$$\phi(t) := \begin{cases} 1 - t^2/R^2 & \text{for } 0 \leq t \leq R \\ 0 & \text{for } t > R \,. \end{cases} \tag{60}$$

Corollary 5.5. *Same assumptions as Lemma 5.4 for the function* u. *There is* $c = c(\gamma_0) > 0$ *such that for all* $R > 0$ *and* $x \in \overline{S}$, *we can find* $z = z(R, x) \in S_R(x)$ *such that*

$$u(x) \leq c \, \frac{1}{R^2} \int_{S_R(z)} u \, dS + c \int_{S_R(z)} r_z^{-1} \, \phi(r_z) \, (|\, \delta \, u| + |\, \delta \, \nu| \, u) \, dS \,.$$

Proof. Consider first $x \in \partial S$. In Lemma 5.4, we can choose the function $u \, \phi(r_x)$ instead of u. Straightforward calculations yield

$$u(x) \leq c \, \frac{1}{R^2} \int_{S_R(x)} u \, dS + c \int_{S_R(x)} r_x^{-1} \, \phi(r_x) \, (|\, \delta \, u| + |\, \delta \, \nu| \, u) \, dS \,, \tag{61}$$

which proves the claim with $z = x$. If $x \in S$, we also choose choose the function $u \, \phi(r_x)$ in Lemma 5.4 to obtain that

$$\begin{aligned} u(x) \leq & c \, \frac{1}{R^2} \int_{S_R(x)} u \, dS + c \int_{S_R(x)} r_x^{-1} \, \phi(r_x) \, (|\, \delta \, u| + |\, \delta \, \nu| \, u) \, dS \\ & + \max_{y \in \partial S \cap B_R(x)} \{u(y) \, \phi(r_x(y))\} \,. \end{aligned} \tag{62}$$

Considering (61), and choosing $y \in \partial S \cap B_R(x)$ such that $u(y) = \max_{\partial S \cap B_R(x)} u$, we obtain that

$$\begin{aligned} \max_{\partial S \cap B_R(x)} & \{u \, \phi(r_x)\} \leq u(y) \\ & \leq c \, \frac{1}{R^2} \int_{S_R(y)} u \, dS + c \int_{S_R(y)} r_y^{-1} \, \phi(r_y) \, (|\, \delta \, u| + |\, \delta \, \nu| \, u) \, dS \,. \end{aligned} \tag{63}$$

Thus, taking (62), (63) into account

$$u(x) \leq 2c \, \frac{1}{R^2} \int_{S_R(z)} u \, dS + c \int_{S_R(z)} r_z^{-1} \, \phi(r_z) \, (|\, \delta \, u| + |\, \delta \, \nu| \, u) \, dS \,,$$

where $z = x$ or $z = y$. □

For the proof following main result of this section, we can in view of our previous preparations at last follow the lines of Lemma 16.4 in [GT01].

Theorem 5.6. *Assumptions of Lemma 5.4. Let* $c = c(\gamma_0)$ *denote the constant of Lemma 5.5 and let* $\beta \in]0,1]$. *Define* $R_0 := \left(c \frac{\beta}{4[\delta\,\nu]_{1+\beta,S}}\right)^{1/\beta}$. *Then, there is* $\tilde{c} = \tilde{c}(\gamma_0) > 0$ *such that for all* $u \in C^1(\overline{S})$ *satisfying* (48)*, and for all* $x \in \overline{S}$ *and* $0 < R \leq R_0$

$$\mathrm{osc}_{S^*_R(x)}\, u \leq \tilde{c}\, \beta^{-1}\, [\delta\, u]_{1+\beta,S} \left(1 + \frac{\mathrm{meas}(S_{2R})}{4\pi\, R^2}\right) R^\beta\,.$$

Here, $S^*_R(x) \subseteq S_R(x)$ *denotes the connected part of* $S_R(x)$ *that contains* x.

Proof. Denote $u_1 := \sup_{S^*_R(x)} u$, $u_0 = \inf_{S^*_R(x)} u$. Set $c_0 = 2\, c\, [\delta\, u]_{1+\beta,S}/\beta$. If $u_1 - u_0 \leq c_0\, R^\beta$, we are already done. Otherwise, denote N the largest integer such that $N \leq (u_1 - u_0)/c_0\, R^\beta$. The interval $[u_0,\, u_1]$ can be subdivided into N subintervals $I_1,\, \ldots, I_N$ of length larger than (or equal to) $c_0\, R^\beta$. For each $j = 1, \ldots, N$, let ψ_j be a Lipschitz continuous function such that $\mathrm{supp}(\psi_j) \subseteq I_j$, $0 \leq \psi_j \leq 1$ in I_j, $\max_{I_j} \psi_j = 1$, and $|\psi_j'| \leq 1/(2c_0\, R^\beta)$.

Exploiting that $S^*_R(x)$ is connected, there is for each $j = 1, \ldots, N$ a $x_j \in S^*_R(x)$ such that $\psi_j(u(x_j)) = 1$. We apply Corollary the 5.5 with $x = x_j$ and $u = \psi_j(u)$, to find a $z_j \in S_R(x)$ such that

$$\begin{aligned} 1 = \psi_j(u(x_j)) \leq & c\, \frac{1}{R^2} \int_{S_R(z_j)} \psi_j(u)\, dS \\ & + c \int_{S_R(z_j)} r_{z_j}^{-1}\, \phi(r_{z_j})\, (\psi_j'(u)\, |\, \delta\, u| + |\, \delta\, \nu|\, \psi_j(u))\, dS\,. \end{aligned} \tag{64}$$

We now use the formula $\int_{S_R} r^{-1}\, \phi(r)\, g\, dS \leq \int_0^R \rho^{-2} \int_{S_\rho} g\, dS\, d\rho$, and the fact that $\psi_j \leq 1$ in order to prove that

$$\begin{aligned} \int_{S_R(z_j)} r_{z_j}^{-1}\, \phi(r_{z_j})\, |\, \delta\, \nu|\, \psi_j(u)\, dS \leq & \int_0^R \rho^{-2} \int_{S_\rho} |\, \delta\, \nu|\, dS\, d\rho \\ & \leq [\delta\, \nu]_{1+\beta,S} \int_0^R \rho^{-1+\beta}\, d\rho = \beta^{-1}\, R^\beta\, [\delta\, \nu]_{1+\beta,S}\,. \end{aligned}$$

Thus, choosing $R \leq R_0$, we obtain that $c \int_{S_R(z_j)} r_{z_j}^{-1}\, \phi(r_{z_j})\, |\, \delta\, \nu|\, \psi_j(u)\, dS \leq 1/4$. Analogously, we estimate

$$\int_{S_R(z_j)} r_{z_j}^{-1}\, \phi(r_{z_j})\, \psi_j'(u)\, |\, \delta\, u|\, dS \leq \frac{1}{2\, c_0\, R^\beta} \int_0^R \rho^{-2} \int_{S_\rho} |\, \delta\, u|\, dS\, d\rho \leq \frac{[\delta\, u]_{1+\beta,S}}{2c_0\beta}\,.$$

Thus, by the choice of c_0 we see that $c \int_{S_R(z_j)} r_{z_j}^{-1}\, \phi(r_{z_j})\, \psi_j'(u)\, |\, \delta\, u|\, dS \leq 1/4$. Tor all $j = 1, \ldots, N$, (64) now implies that

$$\frac{1}{2} \leq c\, \frac{1}{R^2} \int_{S_R(z_j)} \psi_j(u)\, dS\,.$$

Since $z_j \in S_R(x)$, it follows that $S_R(z_j) \subset S_{2R}(x)$. Thus, since by construction $\sum_{j=1}^N \psi_j \leq 1$, we obtain that

$$\frac{N}{2} \leq \frac{c}{R^2} \int_{S_{2R}(x)} \sum_{j=1}^N \psi_j(u)\, dS \leq c\, R^{-2}\, \text{meas}(S_{2R}(x))\, .$$

Due to the choice of N, we finally obtain that

$$u_1 - u_0 \leq (N+1)\, c_0\, R^\beta \leq c\, \beta^{-1}\, [\delta\, u]_{1+\beta,S}\, (1 + \frac{\text{meas}(S_{2R}(x))}{4\pi\, R^2})\, R^\beta\, .$$

□

5.2 Final proof of the curvature estimates

Let us note the following essential consequence of the Theorems 4.1 and 5.6.

Corollary 5.7. *Assumptions of the Theorem 4.1. Define $\beta \in]0, 1[$ as in this Theorem. Then, there are $c > 0$ and $R_0 > 0$ such that*

$$\text{osc}_{S^*_R(x)}\, \nu \leq c\, R^\beta, \quad \forall x \in \overline{S},\; R \leq R_0\, .$$

Here, the number c and R_0 depended on all quantities mentioned in the statement of Theorem 2.1.

With the Corollary 5.7 at hand, the proof of the main Theorem 2.1 is standard. The idea is that for all $x \in \overline{S}$, a certain neighbourhood $S^*_{R_1}(x)$ can be flattened in such a way that the problem (1), (2) is equivalent to a second order, elliptic boundary value problem in two space dimensions. For the proof of the following Lemma, we use the same flattening technique as in the Lemma 5.2, since we can ensure that there is a fixed vector $\xi = \nu(x)$ such that $\text{osc}_{S^*_R(x)}\, |\nu \cdot \xi - 1| \leq c\, R^\beta$. Thus, up to a rotation $S^*_R(x)$ is the graph of a function $\psi \in C^2(\mathbb{R}^2)$ in the standard coordinates.

Moreover, under the simplifying assumption that $\partial\Omega$ is locally flat, the curve $\Gamma^*_R(x)$ is contained in the plane $\{x_1 = 0\}$. Define

$$G := \{(\bar{x},\, 0)\, :\, (\bar{x},\, \psi(\bar{x})) \in S^*_R(x)\}\, , \quad \gamma := \{(\bar{x},\, 0)\, :\, (\bar{x},\, \psi(\bar{x})) \in \Gamma^*_R(x)\}\, .$$

The problem (1), (2) is then equivalent to

$$-\frac{d}{dx_i} \bar{\sigma}_{q_i}(\bar{x},\, \psi,\, \nabla\psi) + \bar{\sigma}_{x_3}(\bar{x},\, \psi,\, \nabla\psi) = \bar{\Phi}(\bar{x},\, \psi,\, \nabla\psi) \text{ in } G$$
$$-\bar{\sigma}_{q_i}(\bar{x},\, \psi,\, \nabla\psi)\, n_i(\bar{x}) = \kappa(\bar{x},\, \psi) \qquad \text{on } \partial G\, .$$

Here, $\bar{\sigma}(\bar{x},\, z,\, p) = \sigma(\bar{x},\, z,\, -p,\, 1)$ for $p \in \mathbb{R}^2$, and $\bar{\Phi}(\bar{x},\, z,\, p) := \Phi(\bar{x},\, z,\, \nu(p))$ with $\nu(p) := (-p_i,\, 1)\, (1 + p^2)^{-1/2}$. This is a quasilinear elliptic equation with a singularity of mean curvature type. Moreover, $\sqrt{1 + |\nabla\psi|^2} = \nu_3^{-1} \leq \underline{c}$ in G. Thus, the equation is even uniformly elliptic. The arguments to obtain the higher order estimates are well-known. First it is possible to obtain a Hölder estimate $\|\nabla\psi\|_{C^\alpha(G\cup\gamma)} \leq c\, (\|\Phi\|_{L^\infty(S\times S^1)} + \|\sigma_x\|_{L^\infty(S\times S^1)} + \|\kappa\|_{C^\alpha(\partial S)})$. Due to a bootstrapping argument well known in this context one obtains that $\|\nabla\psi\|_{C^{1,\alpha}(G\cup\gamma)} \leq c\, (\|\Phi\|_{L^\infty(S^1;\, C^\alpha(\overline{S}))} + \|\sigma_x\|_{C^{1,\alpha}(S^1;\, C^\alpha(\overline{S}))} + \|\kappa\|_{C^{1,\alpha}(\partial S)})$. For details, see the classical literature [LU70, Ura73, SS76, Ger79, Lie83] and further references therein.

A Auxiliary results

Let $\sigma \in C^1(\overline{\Omega};\, C^{2,\alpha}(\mathbb{R}^3 \setminus \{0\}))$ be a convex, one-homogeneous function satisfying the assumptions (10). Define

$$\sigma^*(x,\, q) := \sup_{p\in\mathbb{R}^3} q \cdot \frac{p}{\sigma(x,\, p)}\,. \tag{65}$$

In can be shown that σ^* is as smooth as σ, convex and one-homogeneous in the $q-$variable. It moreover satisfies (10) with constants $\lambda_i^*,\, \mu_j^*$. In particular, $\mu_0^* = \lambda_0^{-1}$, and $\lambda_0^* = \mu_0^{-1}$. Moreover, the identities

$$\sigma^*(x,\, \sigma_q(x,\, q)) = 1, \quad \sigma_q^*(x,\, \sigma_q(x,\, q)) = \frac{q}{\sigma(x,\, q)}\,, \tag{66}$$

are valid for all $(x,\, q) \in \overline{\Omega} \times \mathbb{R}^3 \setminus \{0\}$.

For all $x \in \Omega$, we can introduce the unit ball/sphere of the function $\sigma^*(x,\, \cdot)$

$$W^1_{\sigma^*(x)} = \{q \in \mathbb{R}^3 \,:\, \sigma^*(x,\, q) < 1\}, \;\; S^1_{\sigma^*(x)} = \partial W^1_{\sigma^*(x)}\,, \tag{67}$$

which is a convex domain of class $\mathcal{C}^2$ containing the origin. For $x \in \overline{\Omega}$, the properties (66) show that $q \mapsto \sigma_q(x,\, q)$ is a mapping from $\mathbb{R}^3$ onto $S^1_{\sigma^*(x)}$, and in particular from the standard unit sphere S^1 onto $S^1_{\sigma^*(x)}$.

Consider an arbitrary surface $S \subset \Omega$ class $\mathcal{C}^2$. Then, $\nu(x)$, $x \in S$ maps from S into S^1, and $\varphi(x) := \sigma_q(x,\, \nu(x))$ maps from S into $S^1_{\sigma^*(x)}$. The following proposition states sufficient conditions for the existence of a C^1-vector field ω satisfying the property (30).

Lemma A.1. *Let $S \subset \Omega$ be a surface of class $\mathcal{C}^2$. Assume that there is a connected free surface*[1] *$\mathcal{E} \subset S^1$ of class $\mathcal{C}^1$ such that $\nu(x) \in \mathcal{E}$ for all $x \in S$.*

Then, there exists a vector fields $\omega \in C^1(\overline{\Omega} \times \mathbb{R}^3;\, \mathbb{R}^3)$ such that

$$\mathrm{curl}_q\, \omega(x,\, \sigma_q(x,\, \nu(x))) \cdot \nu(x) = 1 \text{ for all } x \in S\,. \tag{68}$$

Moreover, $\|\omega\|_{C^1(\overline{\Omega}\times\mathbb{R}^3;\mathbb{R}^3)}$ depends on σ and $\mathcal{E}$, but not on S.

Proof. For $x \in \overline{\Omega}$, $r \in \mathbb{R}^3 \setminus \{0\}$ consider the change of coordinates $T_i(x,\, r) := |r|\, \sigma_{q_i}(x,\, r)$. The Jacobian $(dT)_{i,j} = \partial_{r_j} T_i$, $i, j = 1, 2, 3$ is given by

$$(dT(x,\, r))_{i,j} = |r|\, \sigma_{q_i,q_j}(x,\, r) + \sigma_{q_i}(x,\, r)\, \frac{r_j}{|r|}\,.$$

Using (10c), observe that $(dT(x,\, r))^t\, r = \sigma(x,\, r)\, \frac{r}{|r|}$. Therefore $(dT(x,\, r))^{-t}\, r = \frac{|r|}{\sigma(x,\, r)}\, r$, and we see that

$$|(dT(x,\, r))^{-t}\, r| = \frac{|r|^2}{|\sigma(x,\, r)|} \geq \frac{|r|}{\mu_0} \quad \text{for all } r \in \mathbb{R}^3 \setminus \{0\}\,. \tag{69}$$

[1] A free surface is a nonclosed, connected surface

Moreover, we can compute that

$$(dT(x,\, r))\, r \cdot r = |r|\, \sigma(x,\, r) \geq \lambda_0\, |r|^2\,,$$
$$(dT(x,\, r))\, \eta \cdot \eta = |r|\, \sigma_{q,q}(x,\, r)\eta \cdot \eta \geq \lambda_1\, |\eta|^2 \text{ for all } \eta \cdot r = 0\,.$$

Thus, $dT(x,\, r)$ is strictly positive definite, and its smallest eigenvalue is larger than $\min\{\lambda_1,\, \lambda_0\}$. It follows that

$$\det(dT(x,\, r)) \geq \lambda_1^2\, \lambda_0\,. \tag{70}$$

Consider now the function $u(x,\, r) := \det(dT(x,\, r))\, |(dT(x,\, r))^{-t}\, r|$. Thanks to the assumptions on σ, $u \in C^1(\overline{\Omega};\, C^\alpha(\mathbb{R}^3 \setminus \{0\}))$. Applying Lemma A.3 below, we find $\hat{\omega} \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R}^3;\, \mathbb{R}^3)$ such that $\operatorname{curl}_r \hat{\omega}(x,\, r) \cdot r = u(x,\, r)$ for all $r \in \mathcal{E} \subset S^1$. Define for the new coordinates $q := T(x,\, r)$

$$\omega(x,\, q) := dT(x,\, r)\, \hat{\omega}(x,\, r)\,.$$

Then, $\|\omega\|_{C^{1,\alpha}(\overline{\Omega}) \times \mathbb{R}^3)} \leq c(\sigma,\, \mathcal{E})$, and the transformation formula of the curl operator under coordinate changes yields

$$\operatorname{curl}_q \omega(x,\, q) = \frac{1}{\det(dT)(x,\, r)}\, dT(x,\, r) \operatorname{curl}_r \hat{\omega}(x,\, r)\,,$$

where we also used (70). Note now that the transformation $T(x,\, \cdot)$ maps S^1 into $S^1_{\sigma^*(x)}$. Thus, taking also (69) into account, the unit normal $n(r) = r$ on S^1 transforms according to the formula $n^{\sigma^*(x)}(q) = \frac{(dT(x,r))^{-t}\, r}{|(dT(x,r))^{-t}\, r|}$. Thus, for $q = T(x,\, r)$ with $r \in \mathcal{E}$

$$\begin{aligned}\operatorname{curl}_q \omega(x,\, q) \cdot n^{\sigma^*(x)}(q) &= \frac{[dT(x,\, r)\, \operatorname{curl}_r \hat{\omega}(x,\, r)] \cdot [(dT(x,r))^{-t}\, r]}{\det(dT)(x,\, r)\, |(dT(x,\, r))^{-t}\, r|} \\ &= \frac{u(x,\, r)}{\det(dT)(x,\, r)\, |(dT(x,r))^{-t}\, r|} = 1\,.\end{aligned}$$

Since $\nu(x) \in \mathcal{E}$, we can choose $q = \sigma_q(x,\, \nu(x))$, and using that $n^{\sigma^*(x)}(q) = \nu(x)$ (cf. (66)), the claim follows. ☐

Remark A.2. *If the surface S is a graph, there is a vector $\vec{g}$ such that $\vec{g} \cdot \nu(x) > 0$ for all $x \in \overline{S}$. Thus, the assumptions of Lemma A.1 are satisfied with $\mathcal{E} := \{q \in S^1 \;:\; \vec{g} \cdot q > 0\}$.*

Lemma A.3. *Let $\mathcal{E} \subset S^1$ be a connected free surface of class $\mathcal{C}^1$, and $u \in C^1(\overline{\Omega};\, C^\alpha(S^1))$, $\alpha > 0$. Then, there is $\hat{\omega} \in C^1(\overline{\Omega};\, \mathbb{R}^3;\, \mathbb{R}^3)$ such that*

$$\operatorname{curl} \hat{\omega}(x,\, r) \cdot r = u(x,\, r) \text{ for all } (x,\, r) \in \overline{\Omega} \times \mathcal{E}\,, \tag{71}$$

and the estimate $\|\omega\|_{C^{1,\alpha}(\overline{\Omega} \times \mathbb{R}^3;\, \mathbb{R}^3)} \leq c_{\mathcal{E}}\, \|u\|_{C^1(\overline{\Omega};\, C^\alpha(S^1))}$ is valid.

Proof. In a first step we show that for arbitrary $u \in C^\alpha(S^1)$, there is $\hat{\omega} \in C^1(\mathbb{R}^3;\, \mathbb{R}^3)$ such that $\operatorname{curl} \hat{\omega} \cdot n = u$ on $\mathcal{E}$ and $\|\hat{\omega}\|_{C^{1,\alpha}(\mathbb{R}^3)]^3} \leq c_{\mathcal{E}}\, \|u\|_{C^\alpha(S^1)}$. First we choose $T^1 \,:\, C^\alpha(S^1) \to C^\alpha_M(S^1)$ (sub-script M means vanishing mean value) a linear extension operator such that

$$T^1(u) = u \text{ on } \mathcal{E}, \quad \int_{S^1} T^1(u)\, dS = 0, \quad \|T^1(u)\|_{C^\alpha(S^1)} \leq c_{\mathcal{E}}\, \|u\|_{C^\alpha(S^1)}\,.$$

Since $\partial\mathcal{E}$ is a $\mathcal{C}^{1,\alpha}$ curve, $T^1(u)$ can be constructed by standard techniques. We then find a solution $p \in C^{2,\alpha}(S^1)$ to the problem $-\triangle_{S^1} p = T^1(u)$, and we extend p outside of S^1 so that $\|p\|_{C^{2,\alpha}(\mathbb{R}^3)} \leq c\,\|p\|_{C^{2,\alpha}(S^1)}$. We then set $\hat{w}(r) := \nabla p \times r$. Suppose further that u depends on the parameter $x \in \overline{\Omega}$ and that the mapping $x \mapsto u(x,\cdot) \in C^{\alpha}(S^1)$ is continuously differentiable. Then, the vector field $\hat{\omega}(x,\,\cdot)$ constructed as above belongs to $C^1(\overline{\Omega};\, C^{1,\alpha}(\mathbb{R}^3;\, \mathbb{R}^3))$. □

The next property extends the validity of formula (16.77) in [GT01] for $S_\rho(x_0) = S \cap B_\rho(x_0)$ that might intersect ∂S. The proof being completely similar, we shall omit it.

Lemma A.4. *There is a set $\mathcal{N} \subset [0,\, R_0]$ of measure zero, such that for all $\rho \in [0,\, R_0] \setminus \mathcal{N}$ and all $x_0 \in \overline{S}$, the quantity $|\,\delta\, r|$ is strictly positive on $\Sigma_\rho(x_0)$. Moreover, for all $g \in C(\overline{S})$, the identity $\int_{\Sigma_\rho(x_0)} g\,|\,\delta\, r|^{-1}\, ds = \frac{d}{d\rho}\, \int_{S_\rho(x_0)} g\, dS$ is valid.*

Lemma A.5. *Let S be a solution to the problem* (1) *and globally the graph of a function (denote e^3 the $Z-$axis of the coordinate system in which S is a graph). Assume that the following conditions are satisfied:*

(1) There is a constant μ_3 such that $\sigma_x(x,\, q)\cdot e^3 \leq \mu_3\,|q\cdot e^3|$ for $q \in \mathbb{R}^3$;

(2) The vector e^3 is tangent to $\partial\Omega$ for all $x \in \partial S$.

Then, there is a constant c depending on the constants in the conditions (10) *and on μ_3, on $\|\Phi\|_{L^\infty(S\times S_1)}$, on γ_1 and on Ω, such that for all $\bar{x} \in G := \pi(S)$ and $\rho > 0$, the intersection of S with the cylinder $Z_\rho := B_\rho(\bar{x}) \times \mathbb{R}$ satisfies $\operatorname{meas}(S \cap Z_\rho) \leq c\,\rho$.*

Proof. The formula (38) on the whole of S for $k = 3$ implies that

$$\int_S (\sigma\;\delta_3\, v - \sigma_q \cdot \delta\, v\, \nu_3) = \int_S (\Phi\, \nu_3 - \sigma_{x_3})\, v + \int_{\partial S} (\sigma\, \nu_3' - \sigma_q \cdot \nu'\, \nu_3)\, v\, ds\,.$$

Let $\rho > 0$, and $\bar{x} \in \mathbb{R}^2$ such that the point $x = (\bar{x},\, \psi(\bar{x}))$ belongs to S. We fix $\zeta \in C^1_c(B_{2\rho}(\bar{x}))$ equal to one on $B_\rho(\bar{x})$ and nonnegative, and satisfying $|\overline{\nabla}\zeta| \leq \rho^{-1}$. Here, $\overline{\nabla}$ means differentiation in the $\bar{x}$ coordinates. We choose the test function $v(\bar{x},\, z) = \zeta(\bar{x})\, z$. Note that $|\,\delta_3\, \zeta| \leq \rho^{-1}\, \nu_3$ and that $\delta_3\, z = 1 - \nu_3^2$ on S. Moreover, the co-normal on the outer boundary satisfies by assumption $\nu_3' = |\sin\alpha|^{-1}\,(n_3 - \cos\alpha\;\nu_3 = |\sin\alpha|^{-1}\,\cos\alpha\;\nu_3$. It follows that

$$\begin{aligned}\int_S \sigma\,\zeta \leq & \int_S \nu_3\,\{(\sigma + |\sigma_q|)\,|\overline{\nabla}\zeta| + (|\,\delta\, z|\,|\sigma_q| + |\Phi|\,|z| + |\sigma_{x_3}|\,\nu_3^{-1})\,\zeta\} \\ & + \int_{\partial S} \nu_3\,\{\sigma\,|\cot\alpha| + |\sigma_q|\}\,|z|\,|\zeta|\,.\end{aligned}$$

We denote $\gamma = \pi(\partial S)$ the projection of ∂S on $\mathbb{R}^2$. Owing to the assumption $n_3 = 0$ on ∂S, the curvature of γ depends only on $\partial\Omega$. Since $|\sigma_{\delta_3}| \leq (\mu_3 + \mu_0)\,\nu_3$, and since $|\cot\alpha| \leq \mu_0/\gamma_1$, we thus obtain the inequality

$$\int_S \sigma\,\zeta \leq c\,((\rho^{-1} + 1)\;\operatorname{meas}(B_{2\rho}(\bar{x})) + \operatorname{meas}(\gamma \cap B_{2\rho}(\bar{x}))) \leq c\,\rho\,.$$

The constant c depends on the constants in (10) and μ_3, on γ_1, $\|\Phi\|_{L^\infty}$ and $\|z\|_{L^\infty(S)}$. □

Applying similar ideas, we can choose $v = (z - z_0)_\rho\, \zeta$ with ζ like in the proof of Lemma A.5 and $(\cdot)_\rho$ the truncation at levels $\pm\rho$. We obtain the following result, achieving to show that the constant c of the main estimate (3) is independent of the surface S.

Lemma A.6. *Assumptions of Lemma A.5. Then the constant c_0 of the condition* (28) *depends only on the constants in the condition* (10) *and μ_3, on $\|\Phi\|_{L^\infty(S\times S^1)}$ and on γ_1.*

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