Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Strongly nonlocal dislocation dynamics in crystals

Serena Dipierro[1], Alessio Figalli[2], Enrico Valdinoci[3]

submitted: December 11, 2013

[1] School of Mathematics
University of Edinburgh
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
Scotland
UK
E-Mail: serena.dipierro@ed.ac.uk

[2] Mathematics Department
University of Texas at Austin
RLM 8.100
2515 Speedway Stop C1200
Austin TX 78712-1202
USA
E-Mail: figalli@math.utexas.edu

[3] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: enrico.valdinoci@wias-berlin.de

No. 1894
Berlin 2013

2010 *Mathematics Subject Classification.* 49N60, 35B05, 35Q99, 35B40, 35J25, 35D30, 35G25.

Key words and phrases. Nonlocal Peierls–Nabarro model, dislocation dynamics, fractional Laplacian, oscillation and regularity results.

The first author has been supported by EPSRC grant EP/K024566/1 "Monotonicity formula methods for nonlinear PDEs". The second author has been supported by NSF grant DMS-1262411 "Regularity and stability results in variational problems". The third author has been supported by ERC grant 277749 "EPSILON Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. We consider the equation

$$v_t = L_s v - W'(v) + \sigma_\varepsilon(t,x) \quad \text{in } (0,+\infty) \times \mathbb{R},$$

where L_s is an integro-differential operator of order $2s$, with $s \in (0,1)$, W is a periodic potential, and σ_ε is a small external stress. The solution v represents the atomic dislocation in the Peierls–Nabarro model for crystals, and we specifically consider the case $s \in (0,1/2)$, which takes into account a strongly nonlocal elastic term.

We study the evolution of such dislocation function for macroscopic space and time scales, namely we introduce the function

$$v_\varepsilon(t,x) := v\left(\frac{t}{\varepsilon^{1+2s}}, \frac{x}{\varepsilon}\right).$$

We show that, for small ε, the function v_ε approaches the sum of step functions. From the physical point of view, this shows that the dislocations have the tendency to concentrate at single points of the crystal, where the size of the slip coincides with the natural periodicity of the medium. We also show that the motion of these dislocation points is governed by an interior repulsive potential that is superposed to an elastic reaction to the external stress.

CONTENTS

1. INTRODUCTION

In this paper we deal with an integro-differential equation of fractional order derived from the classical Peierls–Nabarro model for crystal dislocations. Specifically we will focus on the case in which the fractional order of the equation is low, which corresponds to a situation in which the long-range elastic interactions give the highest contribute to the energy. In this framework, we will describe the evolution of the atom dislocation function by showing that, for sufficiently long times and at a macroscopic scale, the dislocation function approaches the superposition of a finite number of dislocations. These individual dislocations have size equal to the characteristic period of the crystal and they occur at some specific points, which in turn evolve according to a repulsive potential and reacting elastically to the external stress.

More precisely, we consider the problem

$$v_t = L_s v - W'(v) + \sigma_\varepsilon(t,x) \quad \text{in } (0,+\infty)\times\mathbb{R}, \tag{1.1}$$

where $s \in (0,1)$, L_s is the so-called fractional Laplacian, and W is a 1-periodic potential. More explicitly, given $\varphi \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$, we define

$$L_s\varphi(x) := \frac{1}{2}\int_{\mathbb{R}} \frac{\varphi(x+y)+\varphi(x-y)-2\varphi(x)}{|y|^{1+2s}}\,dy.$$

We refer to [10, 3] for a basic introduction to the fractional Laplace operator. As for the potential, we assume that

$$\begin{cases} W \in C^{3,\alpha}(\mathbb{R}), \quad \text{for some } 0<\alpha<1,\\ W(x+1)=W(x) \quad \text{for any } x\in\mathbb{R},\\ W(k)=0 \quad \text{for any } k\in\mathbb{Z},\\ W>0 \quad \text{in } \mathbb{R}\setminus\mathbb{Z},\\ W''(0)>0. \end{cases} \tag{1.2}$$

As customary, $\varepsilon>0$ is a small scale parameter, and σ_ε plays the role of an exterior stress acting on the material. We suppose that

$$\sigma_\varepsilon(t,x) := \varepsilon^{2s}\sigma(\varepsilon^{1+2s}t, \varepsilon x),$$

where σ is a bounded uniformly continuous function such that, for some $\alpha \in (s,1)$ and $M>0$, it holds

$$\begin{aligned} &\|\sigma_x\|_{L^\infty([0,+\infty)\times\mathbb{R})} + \|\sigma_t\|_{L^\infty([0,+\infty)\times\mathbb{R})} \leqslant M,\\ &|\sigma_x(t,x+h)-\sigma_x(t,x)| \leqslant M|h|^\alpha, \quad \text{for every } x,h\in\mathbb{R} \text{ and } t\in[0,+\infty). \end{aligned} \tag{1.3}$$

The problem in (1.1) arises in the classical Peierls–Nabarro model for atomic dislocation in crystals, see e.g. [7] and references therein. In this paper, our main focus is on the fractional parameter range $s\in(0,1/2)$, which corresponds to a strongly nonlocal elastic term, in which the energy contributions coming from far cannot be neglected and, in fact, may become predominant. We refer to [6] for the case $s=1/2$ and to [4] for the case $s\in(1/2,1)$.

We define

$$v_\varepsilon(t,x) := v\left(\frac{t}{\varepsilon^{1+2s}}, \frac{x}{\varepsilon}\right)$$

and we look at the equation satisfied by the rescaled function v_ε, that is, recalling (1.1),

$$\begin{cases} (v_\varepsilon)_t = \dfrac{1}{\varepsilon}\left(L_s v_\varepsilon - \dfrac{1}{\varepsilon^{2s}}W'(v_\varepsilon) + \sigma(t,x)\right) \quad \text{in } (0,+\infty)\times\mathbb{R},\\ v_\varepsilon(0,\cdot) = v_\varepsilon^0 \quad \text{in } \mathbb{R}. \end{cases} \tag{1.4}$$

Following [8, 1], we introduce the basic layer solution $u \in C^{2,\alpha}(\mathbb{R})$ (here $\alpha = \alpha(s) \in (0,1)$), that is, the solution of the problem

$$\begin{cases} L_s u - W'(u) = 0 \quad \text{in } \mathbb{R}, \\ u' > 0, \quad u(-\infty) = 0, \quad u(0) = 1/2, \quad u(+\infty) = 1. \end{cases} \tag{1.5}$$

The name of layer solution is motivated by the fact that u approaches the limits 0 and 1 at $\pm\infty$. More quantitatively, there exists a constant $C \geqslant 1$ such that

$$|u(x) - H(x)| \leqslant C|x|^{-2s} \quad \text{and} \quad |u'(x)| \leqslant C|x|^{-(1+2s)}, \tag{1.6}$$

where H is the Heaviside function, see Theorem 2 in [8].

As a preliminary result, we will prove a finer asymptotic estimate on the decay of the layer solution:

Theorem 1.1. *Let $s \in (0, 1/2)$. There exist constants $C > 0$ and $\vartheta > 2s$ such that*

$$\left| u(x) - H(x) + \frac{1}{2s\, W''(0)} \frac{x}{|x|^{1+2s}} \right| \leqslant \frac{C}{|x|^{\vartheta}} \quad \text{for any } x \in \mathbb{R},$$

with ϑ depending only on s.

To state our next result, we recall that the semi-continuous envelopes of u are defined as

$$u^*(t,x) := \limsup_{(t',x') \to (t,x)} u(t',x')$$

and

$$u_*(t,x) := \liminf_{(t',x') \to (t,x)} u(t',x').$$

Moreover, given $x_1^0 < x_2^0 < \ldots < x_N^0$, we consider the solution $\big(x_i(t)\big)_{i=1,\ldots,N}$ to the system

$$\begin{cases} \dot{x}_i = \gamma \left(-\sigma(t, x_i) + \displaystyle\sum_{j \neq i} \frac{x_i - x_j}{2s\, |x_i - x_j|^{2s+1}} \right) \text{ in } (0, +\infty), \\ x_i(0) = x_i^0, \end{cases} \tag{1.7}$$

where

$$\gamma = \left(\int_{\mathbb{R}} (u')^2 \right)^{-1}. \tag{1.8}$$

For the existence and uniqueness of such solution see Section 8 in [5]. We consider as initial condition in (1.4) the state obtained by superposing N copies of the transition layers, centered at $x_1^0, \ldots, x_N^0$, that is

$$v_\varepsilon^0(x) = \frac{\varepsilon^{2s}}{\beta} \sigma(0,x) + \sum_{i=1}^N u\left(\frac{x - x_i^0}{\varepsilon} \right), \tag{1.9}$$

where

$$\beta := W''(0) > 0. \tag{1.10}$$

The main result obtained in this framework is the following:

Theorem 1.2. *Let $s \in (0, 1/2)$, assume that* (1.2), (1.3) *and* (1.9) *hold, and let*

$$v_0(t,x) = \sum_{i=1}^N H(x - x_i(t)),$$

where H is the Heaviside function and $(x_i(t))_{i=1,\ldots,N}$ is the solution to (1.7).

Then, for every $\varepsilon > 0$ there exists a unique viscosity solution v_ε to (1.4)*. Furthermore, as $\varepsilon \to 0$, the solution v_ε exhibits the following asymptotic behavior:*

$$\limsup_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} v_\varepsilon(t',x') \leqslant (v_0)^*(t,x)$$

and

$$\liminf_{\substack{(t',x')\to(t,x)\\ \varepsilon\to 0}} v_\varepsilon(t',x') \geqslant (v_0)_*(t,x)$$

for any $t \in [0,+\infty)$ and $x \in \mathbb{R}$.

When $s = 1/2$ the result above was proved in [6], where it was also raised the question about what happens for other values of the parameter s.

In [4], the result was extended to the case $s \in (1/2, 1)$. So the main purpose of this paper was to obtain the result for the remaining range of $s \in (0, 1/2)$. From the physical point of view, this range of parameters is important since it corresponds to the case of a strong nonlocal elastic effect: notice indeed that the lower the value of s the stronger become the energy contributions coming from far. We refer to [6, 4] for a more exhaustive set of physical motivations and heuristic asymptotics of the model we study.

We also remark that, differently from [6], we do not make use of any harmonic extension results, that are specific for the fractional powers of the Laplacian, and so our proof is feasible for more general types of integro-differential equations.

The cornerstone to prove Theorem 1.1 (and hence Theorem 1.2) is given by the following decay estimate at infinity, which we think has also independent interest:

Theorem 1.3. *Let $s \in (0, 1/2)$, and let $v \in L^\infty(\mathbb{R}) \cap C^2(\mathbb{R})$ such that*

$$\lim_{x\to\pm\infty} v(x) = 0. \tag{1.11}$$

Suppose that there exists a function $c \in L^\infty(\mathbb{R})$ such that $c(x) \geqslant \delta > 0$ for any $x \in \mathbb{R}$ and for some $\delta > 0$, and

$$-L_s v + cv = g, \tag{1.12}$$

where g is a function that satisfies the following estimate

$$|g(x)| \leqslant \frac{C}{1+|x|^{4s}} \quad \text{for any } x \in \mathbb{R}, \tag{1.13}$$

for some constant $C \geqslant 0$.

Then, there exist $\vartheta \in (2s, 1+2s]$ depending only on s, and a constant $\overline{C} \geqslant 0$ depending on C, δ, $\|c\|_{L^\infty(\mathbb{R})}$, and s, such that

$$|v(x)| \leqslant \frac{\overline{C}}{1+|x|^{\vartheta}} \quad \text{for any } x \in \mathbb{R}.$$

In our setting, we will use Theorem 1.3 in the proof of Theorem 1.1 (there, the function v in the statement of Theorem 1.3 will be embodied by the difference between the solution u of problem (1.5) and a suitable heteroclinic solution of a model problem, so that in this case condition (1.11) is automatically satisfied).

The explicit value of the exponent ϑ that appears in the statement of Theorem 1.3 will be given in formula (5.4), but such explicit value will not play any role in this paper (the only relevant feature for us is that $\vartheta > 2s$). We think that it is an interesting open problem to determine the optimal value of the exponent ϑ in a general setting.

Theorem 1.3 may be seen as the strongly nonlocal version of Corollary 5.13 in [6] and Corollary 7.1 in [4], where similar decay estimates (with different exponents) where obtained when $s = 1/2$ and $s \in (1/2, 1)$, respectively. However, the techniques in [6, 4] are not sufficient to obtain the desired decay estimates when $s \in (0, 1/2)$, so the proof of Theorem 1.3 here will rely on completely different methods. Roughly speaking, we use suitable test functions in order to obtain an integral decay estimates (this will be accomplished in Proposition 5.1) and then we use barriers and sliding arguments to infer from it a pointwise estimate. Remarkably, differently from the classical case where pointwise estimates follow from integral ones using a suitable version of the weak Harnack inequality (see e.g. Theorem 4.8(2) in [2]), in our case, to the best of our knowledge, the fractional analog of this weak Harnack inequality is not known. To overcome this difficulty, some careful estimates on the fractional Laplacian of a function below a barrier are employed (these estimates will be obtained in Corollary 4.2).

The rest of the paper is organized as follows. The proof of Theorem 1.3 is contained in Sections 2–6. More precisely, we collect some preliminary elementary estimates in Section 2. Then, in Sections 3 and 4, we estimate the fractional Laplacian of a function below a barrier by taking into account the contribution in a neighborhood of a given point and the contribution coming from infinity. An integral decay estimate is given in Section 5 and the proof of Theorem 1.3 is completed in Section 6.

With this we have the basic technical tools to prove Theorem 1.1 in Section 7. Then, Sections 8–10 are devoted to the proof of Theorem 1.2. Namely, Section 8 collects some uniform bounds that are used in Section 9 to construct the solution of a corrector equation and prove its regularity. With this, the proof of Theorem 1.2 is completed in Section 10.

2. An auxiliary summation lemma

Here we present some technical summation estimates, to be used in the forthcoming Section 4. For the sake of generality, we prove the results in Sections 2-5 in $\mathbb{R}^n$, for any $s \in (0, 1)$ and $n \geqslant 1$.

Lemma 2.1. *Let $s \in (0, 1)$, $x_0 \in \mathbb{R}^n$ such that $|x_0| \geqslant 3$, and $\vartheta \in (0, n + 2s]$. Then*

$$\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\leqslant|x_0|/2}} \frac{1}{|k|^{n+2s}\,(1 + |x_0 + k|)^{\vartheta}} \leqslant \frac{C}{(1 + |x_0|)^{\vartheta}},$$

for some $C > 0$ depending on n, s and ϑ.

Proof. If $|x_0 + k| \leqslant |x_0|/2$ then $|k| \geqslant |x_0| - |x_0 + k| \geqslant |x_0|/2$, therefore

$$\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\leqslant|x_0|/2}} \frac{1}{|k|^{n+2s}\,(1 + |x_0 + k|)^{\vartheta}} \leqslant \frac{2^{n+2s}}{|x_0|^{n+2s}} \sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\leqslant|x_0|/2}} \frac{1}{(1 + |x_0 + k|)^{\vartheta}}. \tag{2.1}$$

Moreover,

$$\int_1^{|x_0|} \frac{\rho^{n-1}\,d\rho}{\rho^{\vartheta}} = Z(n, \vartheta, x_0),$$

where

$$Z(n, \vartheta, x_0) := \begin{cases} (n-\vartheta)^{-1}(|x_0|^{n-\vartheta} - 1) & \text{if } n > \vartheta, \\ \log|x_0| & \text{if } n = \vartheta, \\ (\vartheta - n)^{-1}(1 - |x_0|^{n-\vartheta}) & \text{if } n < \vartheta. \end{cases}$$

In any case

$$\frac{Z(n, \vartheta, x_0)}{|x_0|^{n+2s}} \leqslant \frac{c_{n,\vartheta}}{|x_0|^{\vartheta}}, \tag{2.2}$$

for some constant $c_{n,\vartheta} > 0$ only depending on n and ϑ. Therefore

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\leqslant|x_0|/2}} \frac{1}{(1+|x_0+k|)^\vartheta} &\leqslant \int_{B_{|x_0|}(-x_0)} \frac{dx}{(1+|x+x_0|)^\vartheta}\\ &= \omega_{n-1}\int_0^{|x_0|} \frac{\rho^{n-1}\,d\rho}{(1+\rho)^\vartheta}\\ &\leqslant \omega_{n-1}\left[\int_0^1 \rho^{n-1}\,d\rho + \int_1^{|x_0|}\frac{\rho^{n-1}\,d\rho}{\rho^\vartheta}\right]\\ &= \omega_{n-1}\left[\frac{1}{n}+Z(n,\vartheta,x_0)\right].\end{aligned}$$

This and (2.1) give that

$$\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\leqslant|x_0|/2}} \frac{1}{|k|^{n+2s}\,(1+|x_0+k|)^\vartheta} \leqslant \frac{C_1\,(1+Z(n,\vartheta,x_0))}{|x_0|^\vartheta},$$

for some $C_1 > 0$. Then, the desired result follows from (2.2). $\square$

Corollary 2.2. *Let $s \in (0,1)$, $x_0 \in \mathbb{R}^n$ such that $|x_0| \geqslant 3$, and $\vartheta \in (0, n+2s]$. Then*

$$\sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{1}{|k|^{n+2s}\,(1+|x_0+k|)^\vartheta} \leqslant \frac{C}{(1+|x_0|)^\vartheta},$$

for some $C > 0$ depending on n, s and ϑ.

Proof. Notice that

$$\sum_{\substack{k\in\mathbb{Z}^n\setminus\{0\}\\ |x_0+k|\geqslant|x_0|/2}} \frac{1}{|k|^{n+2s}\,(1+|x_0+k|)^\vartheta} \leqslant \frac{1}{(1+|x_0|/2)^\vartheta}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\frac{1}{|k|^{n+2s}} \leqslant \frac{C_0}{(1+|x_0|)^\vartheta},$$

for some $C_0 > 0$, and so the result follows from Lemma 2.1. $\square$

3. Fractional Laplace computations I – Integral estimates at a point

Here we estimate the local contribution of the fractional Laplacian of a function touched by above by a polynomial barrier. By local, we mean here the contribution coming from a neighborhood of a given point. The contribution coming from far will then be studied in Section 4.

Though the main focus of this paper is the fractional parameter range $s \in (0,1/2)$ the results presented hold true for any $s \in (0,1)$. For this, it is convenient to recall the notation on singular integrals in the principal value sense, that is

$$\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+2s}}\,dy := \lim_{\rho\searrow 0}\int_{\mathbb{R}^n\setminus B_\rho} \frac{u(x+y)-u(x)}{|y|^{n+2s}}\,dy.$$

As a matter of fact, when $s \in (0,1/2)$ the above notation may be dropped since the integrand is indeed Lebesgue summable and no cancellations are needed to make the integral convergent near the origin.

With this notation, we can estimate the contribution in a given ball according to the following result:

Lemma 3.1. *Let $s \in (0,1)$, $\vartheta > 0$, $\varepsilon \in (0,1)$, and*

$$F_1(x) := \frac{1}{(1+|x|)^\vartheta}.$$

For any fixed $M>0$ let $F_M(x):=MF_1(x)$. Suppose that $u\in L^\infty(\mathbb{R}^n)\cap C^2(\mathbb{R}^n)$ satisfies

$$F_M(x_0)+\varepsilon = u(x_0) \text{ for some point } x_0\in\mathbb{R}^n, \tag{3.1}$$

$$F_M(x)+\varepsilon \geqslant u(x) \text{ for every } x\in\mathbb{R}^n, \tag{3.2}$$

$$\int_{B_1(x_0)}|u(\zeta)|\,d\zeta\leqslant\frac{C_0}{(1+|x_0|)^\vartheta} \tag{3.3}$$

for some $C_0>0$.

Then there exists $M_0>0$, depending only on n, s, $\|u\|_{L^\infty(\mathbb{R}^n)}$, ϑ, and C_0, such that if $M\geqslant M_0$ then

$$\text{P.V.}\int_{B_1}\frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy\leqslant-\frac{M\,|B_1|}{10\,(1+|x_0|)^\vartheta}.$$

Proof. First of all we observe that, without loss of generality, we can suppose that

$$|x_0|>3. \tag{3.4}$$

Indeed, if $|x_0|\leqslant3$ we deduce from (3.1) that

$$\frac{M}{4^\vartheta}\leqslant\frac{M}{(1+|x_0|)^\vartheta}=F_M(x_0)=u(x_0)-\varepsilon\leqslant\|u\|_{L^\infty(\mathbb{R}^n)}$$

that gives an upper bound on M which would be violated by choosing M_0 large enough.

From (3.4), we have that

$$\text{for any } y\in B_1,\ |x_0+y|\geqslant|x_0|-|y|\geqslant|x_0|/2. \tag{3.5}$$

Now we define

$$\begin{aligned} D_1 &:= \left\{y\in B_1 \text{ s.t. } |u(x_0+y)|\geqslant\frac{M}{2\,(1+|x_0|)^\vartheta}\right\},\\ D_2 &:= \left\{y\in B_1 \text{ s.t. } |u(x_0+y)|<\frac{M}{2\,(1+|x_0|)^\vartheta}\right\}.\end{aligned}$$

Then, by (3.3),

$$\frac{C_0}{(1+|x_0|)^\vartheta}\geqslant\int_{D_1}|u(x_0+y)|\,dy\geqslant\frac{M\,|D_1|}{2\,(1+|x_0|)^\vartheta}.$$

Hence

$$|D_1|\leqslant\frac{2C_0}{M} \tag{3.6}$$

and, as a consequence, if M is large enough,

$$|D_2|\geqslant|B_1|-|D_1|\geqslant\frac{9\,|B_1|}{10}. \tag{3.7}$$

Now we define

$$\begin{aligned} r_0 &:= \left(\frac{(1+|x_0|)^2}{M}\right)^{1/(n+2)},\\ D_3 &:= D_1\cap B_{r_0},\\ D_4 &:= D_1\setminus B_{r_0}.\end{aligned}$$

If $y\in D_3$ we use (3.1), (3.2) and a Taylor expansion of F_1 to obtain that

$$\begin{aligned} u(x_0+y)-u(x_0) &\leqslant M\Big(F_1(x_0+y)-F_1(x_0)\Big)\\ &\leqslant M\nabla F_1(x_0)\cdot y+M\sup_{\xi\in B_1}|D^2F_1(x_0+\xi)|\,|y|^2.\end{aligned}$$

Notice that

$$|\partial^2_{x_i,x_j} F_1(x)| \leqslant \frac{2\vartheta}{(1+|x|)^{\vartheta+1}\,|x|} + \frac{\vartheta\,(\vartheta+1)}{(1+|x|)^{\vartheta+2}}$$

and so, by (3.4) and (3.5),

$$\sup_{\xi\in B_1} |D^2 F_1(x_0+\xi)| \leqslant \frac{C_1}{(1+|x_0|)^{\vartheta+2}},$$

for some $C_1 > 0$. Therefore, for any $y \in D_3$,

$$u(x_0+y) - u(x_0) \leqslant M\nabla F_1(x_0)\cdot y + \frac{C_1\,M\,|y|^2}{(1+|x_0|)^{\vartheta+2}}$$

and so, since the odd term vanishes in the principal value integral,

$$\begin{aligned} \mathrm{P.V.}\int_{D_3} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy &\leqslant \frac{C_1\,M}{(1+|x_0|)^{\vartheta+2}} \int_{D_3} |y|^{2-n-2s}\,dy \\ &\leqslant \frac{C_1\,M}{(1+|x_0|)^{\vartheta+2}} \int_{B_{r_0}} |y|^{2-n-2s}\,dy \\ &= \frac{C_2\,M\,r_0^{2-2s}}{(1+|x_0|)^{\vartheta+2}}. \end{aligned} \tag{3.8}$$

Moreover, by (3.1), (3.2), and (3.5), we have that, if $y \in D_4$,

$$\begin{aligned} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}} &\leqslant \frac{F_M(x_0+y)-F_M(x_0)}{|y|^{n+2s}} \\ &\leqslant \frac{F_M(x_0+y)}{|y|^{n+2s}} \\ &\leqslant \frac{M}{r_0^{n+2s}(1+|x_0+y|)^{\vartheta}} \\ &\leqslant \frac{2^{\vartheta}\,M}{r_0^{n+2s}(1+|x_0|)^{\vartheta}}.. \end{aligned}$$

Accordingly, making use of (3.6), we conclude that

$$\begin{aligned} \mathrm{P.V.}\int_{D_4} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy &\leqslant \frac{2^{\vartheta}\,M\,|D_4|}{r_0^{n+2s}(1+|x_0|)^{\vartheta}} \\ &\leqslant \frac{2^{\vartheta}\,M\,|D_1|}{r_0^{n+2s}(1+|x_0|)^{\vartheta}} \\ &\leqslant \frac{C_3}{r_0^{n+2s}(1+|x_0|)^{\vartheta}}. \end{aligned} \tag{3.9}$$

for some $C_3 > 0$. Thus, by (3.8) and (3.9), we obtain

$$\begin{aligned} \mathrm{P.V.}\int_{D_1} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy &\leqslant \frac{C_2\,M\,r_0^{2-2s}}{(1+|x_0|)^{\vartheta+2}} + \frac{C_3}{r_0^{n+2s}(1+|x_0|)^{\vartheta}} \\ &\leqslant \frac{C_4\,M^{\beta}}{(1+|x_0|)^{\vartheta+2\beta}} \end{aligned} \tag{3.10}$$

for a suitable $C_4 > 0$, where

$$\beta := \frac{n+2s}{n+2} \in (0,1). \tag{3.11}$$

This completes the estimate of the contribution in D_1. Now we estimate the contribution in D_2. For this, we notice that, if $y \in D_2$, then

$$u(x_0 + y) - u(x_0) = u(x_0 + y) - \frac{M}{(1 + |x_0|)^\vartheta} - \varepsilon \leqslant -\frac{M}{2\,(1 + |x_0|)^\vartheta}$$

and therefore

$$\begin{aligned} \text{P.V.} \int_{D_2} \frac{u(x_0 + y) - u(x_0)}{|y|^{n+2s}}\, dy &\leqslant -\frac{M}{2\,(1 + |x_0|)^\vartheta} \int_{D_2} \frac{dy}{|y|^{n+2s}} \\ &\leqslant -\frac{M}{2\,(1 + |x_0|)^\vartheta} \int_{D_2} dy \\ &\leqslant -\frac{9M\,|B_1|}{20\,(1 + |x_0|)^\vartheta}, \end{aligned} \tag{3.12}$$

thanks to (3.7). By collecting the estimates in (3.10) and (3.12), we obtain that

$$\begin{aligned} \text{P.V.} \int_{B_1} \frac{u(x_0 + y) - u(x_0)}{|y|^{n+2s}}\, dy &\leqslant \frac{C_4\, M^\beta}{(1 + |x_0|)^{\vartheta + 2\beta}} - \frac{9M\,|B_1|}{20\,(1 + |x_0|)^\vartheta} \\ &= -\frac{9M\,|B_1|}{20\,(1 + |x_0|)^\vartheta} \left(1 - \frac{C_5}{M^{1-\beta}\,(1 + |x_0|)^{2\beta}}\right) \\ &\leqslant -\frac{9M\,|B_1|}{20\,(1 + |x_0|)^\vartheta} \left(1 - \frac{C_5}{M^{1-\beta}}\right) \end{aligned}$$

for some $C_5 > 0$. So, since $\beta \in (0, 1)$ due to (3.11), for M large we obtain the desired result. □

4. Fractional Laplace computations II – Integral estimates at infinity

This is the counterpart of Section 3, since here we study the contribution coming from infinity of the fractional Laplacian of a function touched by above by a polynomial barrier (since the singularity of the integral only occur at the origin, we do not need to use the principal value notation for such contribution).

Lemma 4.1. *Let $s \in (0, 1)$, $\vartheta \in (0, n + 2s]$, $\varepsilon \in (0, 1)$, and*

$$F_1(x) := \frac{1}{(1 + |x|)^\vartheta}.$$

For any fixed $M > 0$ let $F_M(x) := M F_1(x)$. Suppose that $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfies

$$F_M(x_0) + \varepsilon = u(x_0) \quad \text{for some point } x_0 \in \mathbb{R}^n, \tag{4.1}$$

$$F_M(x) + \varepsilon \geqslant u(x) \quad \text{for every } x \in \mathbb{R}^n \tag{4.2}$$

$$\int_{B_1(x)} |u(\zeta)|\, d\zeta \leqslant \frac{C_0}{(1 + |x|)^\vartheta} \quad \text{for every } x \in \mathbb{R}^n \tag{4.3}$$

for some $C_0 > 0$.

Then there exists $M_0 > 0$, depending only on n, s, $\|u\|_{L^\infty(\mathbb{R}^n)}$, ϑ, and C_0, such that if $M \geqslant M_0$ then

$$\int_{\mathbb{R}^n \setminus B_1} \frac{u(x_0 + y) - u(x_0)}{|y|^{n+2s}}\, dy \leqslant \frac{M\,|B_1|}{20\,(1 + |x_0|)^\vartheta}.$$

Proof. We notice that

$$u(x_0 + y) - u(x_0) = u(x_0 + y) - F_M(x_0) - \varepsilon \leqslant u(x_0 + y) - \varepsilon \leqslant \big(u(x_0 + y) - \varepsilon\big)^+.$$

Also, the cube centered at zero with side $1/\sqrt{n}$ lies inside the unit ball, namely $Q_{1/\sqrt{n}} \subset B_1$. Therefore

$$\int_{\mathbb{R}^n\setminus B_1} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy \leqslant \int_{\mathbb{R}^n\setminus Q_{1/\sqrt{n}}} \frac{\big(u(x_0+y)-\varepsilon\big)^+}{|y|^{n+2s}}\,dy. \tag{4.4}$$

Now we cover $\mathbb{R}^n \setminus Q_{1/\sqrt{n}}$ with cubes of side $1/(8n\sqrt{n})$ centered at points of a sublattice $\mathcal{Z}$ (roughly speaking, this sublattice is just a scaling of $\mathbb{Z}^n$ by a factor $1/(8n\sqrt{n})$, outside $Q_{1/\sqrt{n}}$). In this way,

$$\text{if } k\in\mathcal{Z}, \text{ then } |k| \geqslant \frac{1}{2\sqrt{n}}. \tag{4.5}$$

Therefore

$$\text{if } k\in\mathcal{Z} \text{ and } y\in Q_{1/(8n\sqrt{n})}(k) \text{ then } |y|\geqslant |k|-|y-k| \geqslant \frac{|k|}{2}+\frac{1}{4\sqrt{n}}-\frac{1}{8n}\geqslant\frac{|k|}{2}. \tag{4.6}$$

Moreover,

$$\begin{aligned}&\text{if } k\in\mathcal{Z} \text{ and } y\in Q_{1/(8n\sqrt{n})}(k) \text{ then}\\ &1+|x_0+y|\geqslant 1+|x_0+k|-|y-k|\geqslant 1+|x_0+k|-\frac{1}{8n}\geqslant\frac{1}{2}\big(1+|x_0+k|\big).\end{aligned} \tag{4.7}$$

Now we observe that, from (4.4),

$$\int_{\mathbb{R}^n\setminus B_1} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy \leqslant \sum_{k\in\mathcal{Z}}\int_{Q_{1/(8n\sqrt{n})}(k)} \frac{\big(u(x_0+y)-\varepsilon\big)^+}{|y|^{n+2s}}\,dy. \tag{4.8}$$

We define

$$\begin{aligned} D_1(k) &:= \left\{ y\in Q_{1/(8n\sqrt{n})}(k) \text{ s.t. } |u(x_0+y)|\geqslant \frac{\sqrt{M}}{(1+|x_0+k|)^{\vartheta}}\right\},\\ D_2(k) &:= \left\{ y\in Q_{1/(8n\sqrt{n})}(k) \text{ s.t. } |u(x_0+y)|< \frac{\sqrt{M}}{(1+|x_0+k|)^{\vartheta}}\right\}.\end{aligned}$$

Then, from (4.3),

$$\begin{aligned}\frac{C_0}{(1+|x_0+k|)^{\vartheta}} &\geqslant \int_{B_1(x_0+k)} |u(\zeta)|\,d\zeta\\ &\geqslant \int_{Q_{1/(8n\sqrt{n})}(x_0+k)} |u(\zeta)|\,d\zeta\\ &\geqslant \int_{D_1(k)} |u(x_0+y)|\,dy\\ &\geqslant \frac{\sqrt{M}\,|D_1(k)|}{(1+|x_0+k|)^{\vartheta}}\end{aligned}$$

and so

$$|D_1(k)|\leqslant\frac{C_0}{\sqrt{M}}.$$

Consequently, using (4.2), (4.6) and (4.7), we see that

$$\begin{aligned}\int_{D_1(k)}\frac{\big(u(x_0+y)-\varepsilon\big)^+}{|y|^{n+2s}}\,dy &\leqslant \int_{D_1(k)}\frac{F_M(x_0+y)}{|y|^{n+2s}}\,dy\\ &\leqslant \int_{D_1(k)}\frac{C_1\,M}{(1+|x_0+k|)^{\vartheta}\,|k|^{n+2s}}\,dy\\ &= \frac{C_1\,M\,|D_1(k)|}{(1+|x_0+k|)^{\vartheta}\,|k|^{n+2s}}\\ &\leqslant \frac{C_0\,C_1\,\sqrt{M}}{(1+|x_0+k|)^{\vartheta}\,|k|^{n+2s}},\end{aligned} \tag{4.9}$$

for a suitable $C_1>0$. Now we use again (4.6) to estimate the contribution in $D_2(k)$ in the following computation:

$$\begin{aligned}\int_{D_2(k)}\frac{u^+(x_0+y)}{|y|^{n+2s}}\,dy &\leqslant \int_{D_2(k)}\frac{\sqrt{M}}{(|k|/2)^{n+2s}\,(1+|x_0+k|)^{\vartheta}}\,dy\\ &\leqslant \frac{2^{n+2s}\,|Q_{1/(8n\sqrt{n})}|\,\sqrt{M}}{|k|^{n+2s}\,(1+|x_0+k|)^{\vartheta}}.\end{aligned} \tag{4.10}$$

Using (4.9) and (4.10), and the fact that

$$\big(u(x_0+y)-\varepsilon\big)^+\leqslant u^+(x_0+y),$$

we conclude that

$$\int_{Q_{1/(8n\sqrt{n})}(k)}\frac{\big(u(x_0+y)-\varepsilon\big)^+}{|y|^{n+2s}}\,dy\leqslant\frac{C_2\,\sqrt{M}}{(1+|x_0+k|)^{\vartheta}\,|k|^{n+2s}},$$

for a suitable $C_2>0$. So we plug this estimate into (4.8) and we deduce that

$$\int_{\mathbb{R}^n\setminus B_1}\frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy\leqslant C_2\,\sqrt{M}\sum_{k\in\mathcal{Z}}\frac{1}{(1+|x_0+k|)^{\vartheta}\,|k|^{n+2s}}.$$

Thus we estimate the latter series using Corollary 2.2 (notice that $\mathcal{Z}$ may be seen as a scaled version of $\mathbb{Z}^n\setminus\{0\}$, due to (4.5), and x_0 stays away from 0, as pointed out in (3.4), so the assumptions of Corollary 2.2 are satisfied, up to scaling): we obtain that

$$\int_{\mathbb{R}^n\setminus B_1}\frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy\leqslant\frac{C_3\,\sqrt{M}}{(1+|x_0|)^{\vartheta}},$$

for a suitable $C_3>0$, hence the claim plainly follows if M is large enough. □

Combining the estimates of Lemmata 3.1 and 4.1 we obtain that the negative local contribution cannot be compensated by the contribution at infinity. More explicitly, we have:

Corollary 4.2. *Let $s\in(0,1)$, $\vartheta\in(0,n+2s]$, $\varepsilon\in(0,1)$, and*

$$F_1(x):=\frac{1}{(1+|x|)^{\vartheta}}.$$

For any fixed $M>0$ let $F_M(x):=MF_1(x)$. Suppose that $u\in L^\infty(\mathbb{R}^n)\cap C^2(\mathbb{R}^n)$ satisfies

$$\begin{aligned}&F_M(x_0)+\varepsilon=u(x_0)\ \text{ for some point } x_0\in\mathbb{R}^n,\\ &F_M(x)+\varepsilon\geqslant u(x)\ \text{ for every } x\in\mathbb{R}^n\\ &\int_{B_1(x)}|u(\zeta)|\,d\zeta\leqslant\frac{C_0}{(1+|x|)^{\vartheta}}\ \text{ for every } x\in\mathbb{R}^n\end{aligned}$$

for some $C_0 > 0$.

Then there exists $M_0 > 0$, depending only on n, s, $\|u\|_{L^\infty(\mathbb{R}^n)}$, ϑ, and C_0, such that if $M \geqslant M_0$ then

$$L_s u(x_0) = \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x_0+y)-u(x_0)}{|y|^{n+2s}}\,dy \leqslant -\frac{M\,|B_1|}{20\,(1+|x_0|)^\vartheta}. \tag{4.11}$$

5. Decay estimates in average

Here we obtain some precise information on the decay at infinity of the solution of a nonlocal equation with decaying nonlinearity:

Proposition 5.1. *Let $s \in (0,1)$, $u \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ satisfy*

$$-L_s u + cu = g \quad \textit{in } \mathbb{R}^n, \tag{5.1}$$

where $c(x) \in (c_0,\, c_0^{-1})$, for some $c_0 \in (0,1)$ and

$$|g(x)| \leqslant \frac{C}{(1+|x|)^\alpha} \tag{5.2}$$

for some $C > 0$ and $\alpha > 0$.

Then, for any $x \in \mathbb{R}^n$,

$$\int_{B_1(x)} |u(y)|\,dy \leqslant \frac{C_*}{|x|^\vartheta} \tag{5.3}$$

where $C_* > 0$ is a suitable constant and

$$\vartheta := \frac{\min\{n+2s-(n-2\alpha)^+,\, 2\alpha\}}{2}. \tag{5.4}$$

Proof. We use that u satisfies (5.1) in the weak sense, that is, for any test function ψ,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} c\,u\psi\,dx = \int_{\mathbb{R}^n} g\psi\,dx.$$

Choosing $\psi = u\varphi^2$ we get

(5.5)

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\Big(u(x)\varphi^2(x)-u(y)\varphi^2(y)\Big)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} c\,u^2\varphi^2\,dx = \int_{\mathbb{R}^n} gu\varphi^2\,dx.$$

Notice that we can write

$$\begin{aligned}
& \big(u(x)-u(y)\big)\Big(u(x)\varphi^2(x)-u(y)\varphi^2(y)\Big) \\
= \;& \big(u(x)-u(y)\big)\Big(u(x)\varphi^2(x)-u(y)\varphi^2(x)+u(y)\varphi^2(x)-u(y)\varphi^2(y)\Big) \\
= \;& \big(u(x)-u(y)\big)\Big[\big(u(x)-u(y)\big)\varphi^2(x)+u(y)\big(\varphi^2(x)-\varphi^2(y)\big)\Big] \\
= \;& \big(u(x)-u(y)\big)^2\varphi^2(x)+u(y)\big(u(x)-u(y)\big)\big(\varphi(x)+\varphi(y)\big)\big(\varphi(x)-\varphi(y)\big).
\end{aligned}$$

Hence (5.5) becomes

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2\varphi^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y)\big(u(x)-u(y)\big)\,\big(\varphi(x)+\varphi(y)\big)\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy \\
&+\int_{\mathbb{R}^n} c\,u^2\varphi^2\,dx = \int_{\mathbb{R}^n} gu\varphi^2\,dx.
\end{aligned}
\tag{5.6}
$$

Now we estimate the second term in (5.6) in the following way

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y)\big(u(x)-u(y)\big)\,\big(\varphi(x)+\varphi(y)\big)\,\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\right| \\
\leqslant\;& \frac{1}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2(\varphi(x)+\varphi(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(y)(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\
\leqslant\;& \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2(\varphi^2(x)+\varphi^2(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(y)(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx\,dy \\
\leqslant\;& \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2\varphi^2(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(y)(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx\,dy.
\end{aligned}
$$

Using this and (5.6) we obtain

(5.7)

$$
\begin{aligned}
c_0\int_{\mathbb{R}^n} u^2\varphi^2\,dx \quad &\leqslant \int_{\mathbb{R}^n} c\,u^2\varphi^2\,dx \\
&= \int_{\mathbb{R}^n} gu\varphi^2\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2\varphi^2(x)}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y)(u(x)-u(y))(\varphi(x)+\varphi(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\leqslant \int_{\mathbb{R}^n} gu\varphi^2\,dx + I,
\end{aligned}
$$

where

$$
I := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u^2(y)(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx\,dy.
$$

On the other hand

$$
\begin{aligned}
\int_{\mathbb{R}^n} gu\varphi^2\,dx \quad &= \quad \int_{\mathbb{R}^n} 2\,(\sqrt{1/(2c_0)}\,g\varphi)(\sqrt{c_0/2}\,u\varphi)\,dx \\
&\leqslant \quad \frac{1}{2c_0}\int_{\mathbb{R}^n} g^2\varphi^2 + \frac{c_0}{2}\int_{\mathbb{R}^n} u^2\varphi^2\,dx.
\end{aligned}
$$

By plugging this into (5.7) and reabsorbing one term on the left hand side we obtain

$$
\frac{c_0}{2}\int_{\mathbb{R}^n} u^2\varphi^2\,dx \leqslant \frac{1}{2c_0}\int_{\mathbb{R}^n} g^2\varphi^2\,dx + I.
\tag{5.8}
$$

Our goal is now twofold: to estimate $\int_{\mathbb{R}^n} g^2\varphi^2\,dx$ and to reabsorb I on the left hand side. For this, we choose

$$
\varphi(x) := \frac{1}{(1+\varepsilon^2|x-x_0|^2)^N},
$$

where $x_0 \in \mathbb{R}^n$ is fixed,

$$
N := \frac{n+2s}{4},
\tag{5.9}
$$

and $0 < \varepsilon \ll 1/N$. Notice that $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We set

$$R := |x_0|/2 > 10, \tag{5.10}$$

and we claim that

$$\int_{\mathbb{R}^n} g^2 \varphi^2 \, dx \leqslant C_\varepsilon R^{-\gamma}, \tag{5.11}$$

for some $C_\varepsilon > 0$ and

$$\gamma := \min\{n + 2s - (n - 2\alpha)^+, \, 2\alpha\}.$$

Notice that

$$\vartheta = \gamma/2, \tag{5.12}$$

see (5.4).

To prove the claim, we first observe that if $x \in B_R$ then

$$|x - x_0| \geqslant |x_0| - |x| \geqslant 2R - R = R,$$

so

$$\varphi(x) \leqslant \frac{1}{(1 + \varepsilon^2 R^2)^N} \leqslant \frac{1}{\varepsilon^{2N} R^{2N}}.$$

Accordingly, using also (5.2) and (5.9), we obtain

$$\begin{aligned} \int_{B_R} g^2 \varphi^2 \, dx &\leqslant \frac{1}{\varepsilon^{4N} R^{4N}} \int_{B_R} g^2 \, dx \\ &\leqslant \frac{1}{\varepsilon^{4N} R^{4N}} \int_{B_R} \frac{C}{(1 + |x|)^{2\alpha}} \, dx \\ &\leqslant \frac{C}{\varepsilon^{4N} R^{4N}} \left[\int_{B_1} 1 \, dx + \int_{B_R \setminus B_1} \frac{C}{|x|^{2\alpha}} \, dx \right] \\ &\leqslant C_\varepsilon R^{-4N} \Big(1 + \ell(R) \, R^{(n-2\alpha)^+} \Big) \\ &\leqslant 2 C_\varepsilon \ell(R) \, R^{-n-2s+(n-2\alpha)^+}, \end{aligned} \tag{5.13}$$

for some $C_\varepsilon > 0$, where

$$\ell(R) := \begin{cases} \log R & \text{if } 2\alpha = n, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, if $x \in B_R(x_0)$ then

$$|x| \geqslant |x_0| - |x - x_0| \geqslant 2R - R = R$$

and so, from (5.2), we have

$$|g(x)| \leqslant \frac{C}{(1 + R)^\alpha} \leqslant \frac{C}{R^\alpha}.$$

As a consequence

$$\begin{aligned} \int_{B_R(x_0)} g^2 \varphi^2 \, dx &\leqslant \frac{C^2}{R^{2\alpha}} \int_{B_R(x_0)} \varphi^2 \, dx \\ &\leqslant \frac{C^2}{R^{2\alpha}} \int_{\mathbb{R}^n} \varphi^2 \, dx \\ &\leqslant C_\varepsilon R^{-2\alpha}, \end{aligned} \tag{5.14}$$

for some $C_\varepsilon > 0$ (up to renaming it). Now, if $x \in \mathbb{R}^n \setminus (B_R(x_0) \cup B_R)$ then $|x| \geqslant R$ and so, from (5.2) and (5.9),

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus(B_R(x_0)\cup B_R)} g^2\varphi^2\,dx &\leqslant \frac{C^2}{R^{2\alpha}}\int_{\mathbb{R}^n\setminus(B_R(x_0)\cup B_R)} \varphi^2\,dx \\
&\leqslant \frac{C^2}{R^{2\alpha}}\int_{\mathbb{R}^n\setminus B_R(x_0)} \frac{1}{\varepsilon^{4N}|x-x_0|^{4N}}\,dx \\
&\leqslant C_\varepsilon R^{n-2\alpha-4N} \\
&= C_\varepsilon R^{-2\alpha-2s}.
\end{aligned} \tag{5.15}$$

Then (5.11) follows from (5.13), (5.14) and (5.15).

Now we claim that, for any $\varepsilon' > 0$, we can choose ε sufficiently small (in the definition of φ) so that

$$\int_{\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx \leqslant \varepsilon'\varphi^2(y), \tag{5.16}$$

holds.

To prove this, we first observe that

$$|\nabla\varphi(x)| = \frac{2\varepsilon^2 N|x-x_0|}{(1+\varepsilon^2|x-x_0|^2)^{N+1}} \leqslant 2\varepsilon N\varphi(x). \tag{5.17}$$

In particular we have that $|\nabla\varphi| \leqslant 2\varepsilon N$ and therefore, for any $r > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx &\leqslant \int_{B_r(y)} \frac{4\varepsilon^2N^2|x-y|^2}{|x-y|^{n+2s}}\,dx + \int_{\mathbb{R}^n\setminus B_r(y)} \frac{4}{|x-y|^{n+2s}}\,dx \\
&\leqslant C(\varepsilon^2 r^{2-2s} + r^{-2s}),
\end{aligned}$$

for some $C > 0$. Accordingly, if we choose $r := 1/\sqrt{\varepsilon}$, we obtain

$$\int_{\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx \leqslant 2C\varepsilon^s.$$

Hence if y is such that $\varepsilon|y-x_0| \leqslant \varepsilon^{-s/(4N)}/|\log\varepsilon|$ then we have that

$$\begin{aligned}
|\log\varepsilon|^{-N}\varphi^2(y) &= \frac{|\log\varepsilon|^{-N}}{(1+\varepsilon^2|y-x_0|^2)^{2N}} \\
&\geqslant \frac{|\log\varepsilon|^{-N}}{\big(1+(\varepsilon^{-s/(4N)}/|\log\varepsilon|)^2\big)^{2N}} \\
&\geqslant \frac{|\log\varepsilon|^{-N}}{\big(2(\varepsilon^{-s/(4N)}/|\log\varepsilon|)^2\big)^{2N}} \\
&= 2^{-2N}\varepsilon^s|\log\varepsilon|^{3N} \\
&\geqslant 2C\varepsilon^s \\
&\geqslant \int_{\mathbb{R}^n} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx,
\end{aligned}$$

provided that ε is small enough, and this shows that (5.16) holds true if $\varepsilon|y-x_0| \leqslant \varepsilon^{-s/(4N)}/|\log\varepsilon|$. So we may and do suppose that

$$\varepsilon|y-x_0| \geqslant \varepsilon^{-s/(4N)}/|\log\varepsilon|. \tag{5.18}$$

Notice that, in this case, $\varepsilon|y-x_0|\geqslant 1$ if ε is small enough and so

$$\varphi^2(y)=\frac{1}{(1+\varepsilon^2|y-x_0|^2)^{2N}}\geqslant\frac{1}{(2\varepsilon^2|y-x_0|^2)^{2N}}=\frac{1}{4^N\varepsilon^{n+2s}|y-x_0|^{n+2s}},\tag{5.19}$$

thanks to (5.9). Now we set

$$r_\varepsilon:=\frac{\varepsilon^{-(n+3s)/(n+2s)}}{2|\log\varepsilon|}$$

and we study the contributions in $B_{r_\varepsilon}(x_0)$ and in $B_{r_\varepsilon}(y)$.

For this, we point out that, by (5.9) and (5.18),

$$|y-x_0|\geqslant\frac{\varepsilon^{-(4N+s)/(4N)}}{|\log\varepsilon|}=\frac{\varepsilon^{-(n+3s)/(n+2s)}}{|\log\varepsilon|}=2r_\varepsilon.\tag{5.20}$$

Therefore, if $x\in B_{r_\varepsilon}(x_0)$ we have that

$$|x-y|\geqslant|x_0-y|-|x-x_0|\geqslant|x_0-y|-r_\varepsilon\geqslant\frac{|x_0-y|}{2}$$

hence, using (5.19), we see that

$$\begin{aligned}\int_{B_{r_\varepsilon}(x_0)}\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx&\leqslant\int_{B_{r_\varepsilon}(x_0)}\frac{4^{n+1+2s}}{|x_0-y|^{n+2s}}\,dx\\&\leqslant C\frac{r_\varepsilon^n}{|x_0-y|^{n+2s}}\\&\leqslant 4^NC\frac{\varepsilon^{-n(n+3s)/(n+2s)}}{2|\log\varepsilon|^n}\,\varepsilon^{n+2s}\varphi^2(y)\\&=4^NC\frac{\varepsilon^{s(n+4s)/(n+2s)}}{2|\log\varepsilon|^n}\,\varphi^2(y).\end{aligned}\tag{5.21}$$

Now we estimate the contribution in $B_{r_\varepsilon}(y)$. For this, we take $x\in B_{r_\varepsilon}(y)$ and $\xi=tx+(1-t)y$ with $t\in[0,1]$ such that

$$|\varphi(x)-\varphi(y)|\leqslant|\nabla\varphi(\xi)|\,|x-y|.$$

Notice that, in this case,

$$|\xi-y|=t|x-y|\leqslant r_\varepsilon\leqslant\frac{|y-x_0|}{2}$$

thanks to (5.20), and therefore

$$|\xi-x_0|\geqslant|y-x_0|-|\xi-y|\geqslant\frac{|y-x_0|}{2}.$$

Using this and (5.17) we obtain that

$$\begin{aligned}|\nabla\varphi(\xi)|&\leqslant 2\varepsilon N\varphi(\xi)\\&=\frac{2\varepsilon N}{(1+\varepsilon^2|\xi-x_0|^2)^N}\\&\leqslant\frac{2^{2N+1}\varepsilon N}{(1+2^2\,\varepsilon^2|\xi-x_0|^2)^N}\\&\leqslant\frac{2^{2N+1}\varepsilon N}{(1+\varepsilon^2|y-x_0|^2)^N}\\&=2^{2N+1}\varepsilon N\varphi(y).\end{aligned}$$

As a consequence

$$\begin{aligned}\int_{B_{r_\varepsilon}(y)} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx &\leqslant \int_{B_{r_\varepsilon}(y)} \frac{4^{2N+2}\varepsilon^2 N^2 \varphi^2(y)}{|x-y|^{n+2s-2}}\,dx \\ &= C\varepsilon^2 r_\varepsilon^{2-2s}\varphi^2(y) \\ &= \frac{C\,\varepsilon^{2s(n-1+3s)/(n+2s)}}{2^{2-2s}\,|\log\varepsilon|^{2-2s}}\varphi^2(y).\end{aligned} \tag{5.22}$$

It remains to estimate the contribution in $\mathbb{R}^n \setminus \big(B_{r_\varepsilon}(x_0) \cup B_{r_\varepsilon}(y)\big)$. For this we will use the following estimate: fixed $p \in \mathbb{R}^n$ we have that

$$\int_{\mathbb{R}^n\setminus B_{r_\varepsilon}(p)} \frac{dx}{|x-p|^{n+2s}} = \frac{C}{r_\varepsilon^{2s}} = 2^{2s} C\,\varepsilon^{2s(n+3s)/(n+2s)}\,|\log\varepsilon|^{2s}.. \tag{5.23}$$

Moreover

$$\frac{|y-x_0|}{|x-x_0|\,|x-y|} \leqslant \frac{|y-x|+|x-x_0|}{|x-x_0|\,|x-y|} = \frac{1}{|x-x_0|} + \frac{1}{|x-y|}$$

and therefore

$$\frac{|y-x_0|^{n+2s}}{|x-x_0|^{n+2s}\,|x-y|^{n+2s}} \leqslant 2^{n+2s}\left(\frac{1}{|x-x_0|^{n+2s}} + \frac{1}{|x-y|^{n+2s}}\right).$$

Hence, if we integrate over $\mathbb{R}^n \setminus \big(B_{r_\varepsilon}(x_0) \cup B_{r_\varepsilon}(y)\big)$ and we use (5.23) we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\big(B_{r_\varepsilon}(x_0)\cup B_{r_\varepsilon}(y)\big)} \frac{|y-x_0|^{n+2s}}{|x-x_0|^{n+2s}\,|x-y|^{n+2s}}\,dx \\ &\leqslant 2^{n+2s}\left(\int_{\mathbb{R}^n\setminus B_{r_\varepsilon}(x_0)} \frac{dx}{|x-x_0|^{n+2s}} + \int_{\mathbb{R}^n\setminus B_{r_\varepsilon}(y)} \frac{dx}{|x-y|^{n+2s}}\right) \\ &\leqslant C\,\varepsilon^{2s(n+3s)/(n+2s)}\,|\log\varepsilon|^{2s},\end{aligned} \tag{5.24}$$

up to renaming constants. Moreover, exploiting (5.9) and (5.19) we see that

$$\varphi^2(x) = \frac{1}{(1+\varepsilon^2|x-x_0|^2)^{(n+2s)/2}} \leqslant \frac{1}{\varepsilon^{n+2s}|x-x_0|^{n+2s}} \leqslant \frac{4^N\,|y-x_0|^{n+2s}}{|x-x_0|^{n+2s}}\,\varphi^2(y).$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\big(B_{r_\varepsilon}(x_0)\cup B_{r_\varepsilon}(y)\big)} \frac{\varphi^2(x)}{|x-y|^{n+2s}}\,dx \\ &\leqslant 4^N\varphi^2(y)\int_{\mathbb{R}^n\setminus\big(B_{r_\varepsilon}(x_0)\cup B_{r_\varepsilon}(y)\big)} \frac{|y-x_0|^{n+2s}}{|x-x_0|^{n+2s}\,|x-y|^{n+2s}}\,dx \\ &\leqslant 4^N C\,\varepsilon^{2s(n+3s)/(n+2s)}\,|\log\varepsilon|^{2s}\,\varphi^2(y),\end{aligned} \tag{5.25}$$

thanks to (5.24). Furthermore, by (5.23) we have that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\big(B_{r_\varepsilon}(x_0)\cup B_{r_\varepsilon}(y)\big)} \frac{\varphi^2(y)}{|x-y|^{n+2s}}\,dx &\leqslant \int_{\mathbb{R}^n\setminus B_{r_\varepsilon}(y)} \frac{\varphi^2(y)}{|x-y|^{n+2s}}\,dx \\ &\leqslant 2^{2s} C\,\varepsilon^{2s(n+3s)/(n+2s)}\,|\log\varepsilon|^{2s}\,\varphi^2(y).\end{aligned} \tag{5.26}$$

Now we use that

$$(\varphi(x)-\varphi(y))^2 \leqslant (|\varphi(x)|+|\varphi(y)|)^2 \leqslant 4(\varphi^2(x)+\varphi^2(y)),$$

so that by (5.25) and (5.26) we obtain

$$\int_{\mathbb{R}^n\setminus\big(B_{r_\varepsilon}(x_0)\cup B_{r_\varepsilon}(y)\big)} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}\,dx \leqslant C\,\varepsilon^{2s(n+3s)/(n+2s)}\,|\log\varepsilon|^{2s}\,\varphi^2(y), \tag{5.27}$$

up to renaming constants once again. In view of (5.21), (5.22) and (5.27), the proof of (5.16) is finished.

As a consequence of (5.16) we obtain that

$$I \leqslant \varepsilon' \int_{\mathbb{R}^n} u^2(y)\varphi^2(y)\,dy = \varepsilon' \int_{\mathbb{R}^n} u^2\varphi^2\,dx.$$

So we take ε so small that $\varepsilon' \leqslant c_0/4$, we plug the estimate above into (5.8) and we reabsorb one term into the left hand side (this fixes ε now once and for all): we conclude that

$$\frac{c_0}{4} \int_{\mathbb{R}^n} u^2\varphi^2\,dx \leqslant \frac{1}{2c_0} \int_{\mathbb{R}^n} g^2\varphi^2\,dx.$$

Hence, from (5.11),

$$\frac{c_0}{4} \int_{\mathbb{R}^n} u^2\varphi^2\,dx \leqslant \frac{C_\varepsilon}{2c_0} R^{-\gamma}.$$

Now we use that $\varphi \geqslant 1/2$ in $B_1(x_0)$ to deduce from this that

$$\fint_{B_1(x_0)} u^2\,dx \leqslant CR^{-\gamma},$$

for some $C > 0$. Then, by the Hölder inequality, (5.10) and (5.12), for any $x_0 \in \mathbb{R}^n$ such that $|x_0| > 20$ we have that

$$\fint_{B_1(x_0)} u\,dx \leqslant \sqrt{\fint_{B_1(x_0)} u^2\,dx} \leqslant \sqrt{CR^{-\gamma}} = \sqrt{C}\,R^{-\vartheta} = 2^\vartheta\sqrt{C}|x_0|^{-\vartheta}.$$

Since u is bounded, a similar estimate holds for $|x_0| \leqslant 20$ as well, by possibly changing the constants (also in dependence of $\|u\|_{L^\infty(B_{20})}$). This proves (5.3) and concludes the proof of Proposition 5.1. □

Remark 5.2. In the sequel, we will only use Proposition 5.1 for the proof of Theorem 1.3 when $n = 1$ and $s \in (0, 1/2)$. Though the statement of Proposition 5.1 remains valid for the whole parameter range $s \in (0, 1)$, in general the exponent ϑ found in (5.4) would not be sufficiently accurate (indeed, we think it is an interesting open problem to find a sharp value for the exponent ϑ in general).

The sensitivity of the decay estimates on the fractional parameter s is the main reason for which different methods are needed to prove Theorem 1.3 when $s \in (0, 1/2)$ and $s \in [1/2, 1)$: in a sense, when $s \in (0, 1/2)$, the integral contributions coming from far are predominant and they strongly affect the available bounds on the asymptotic behaviour of the solution at infinity.

6. Proof of Theorem 1.3

Let v be as in Theorem 1.3. We prove that

$$v(x) \leqslant \frac{M_0}{(1 + |x|)^\vartheta} \tag{6.1}$$

for any $x \in \mathbb{R}$, where $M_0 > 0$ is a universal constant (the bound from below follows by exchanging v with $-v$). To this goal, fixed any $\varepsilon > 0$, we use (1.11) to find $R_\varepsilon > 0$ such that

$$|v(x)| \leqslant \varepsilon/2 \text{ for all } |x| \geqslant R_\varepsilon. \tag{6.2}$$

We claim that

$$v(x) < \frac{M}{(1 + |x|)^\vartheta} + \varepsilon \tag{6.3}$$

for any $x \in \mathbb{R}$, as long as

$$M \geqslant \|v\|_{L^\infty(\mathbb{R})}\,(1 + R_\varepsilon)^\vartheta.$$

To check this, we distinguish two cases. If $|x| \leqslant R_\varepsilon$, then

$$v(x) \leqslant \frac{|v(x)|\,(1+R_\varepsilon)^\vartheta}{(1+|x|)^\vartheta} \leqslant \frac{M}{(1+|x|)^\vartheta} < \frac{M}{(1+|x|)^\vartheta} + \varepsilon,$$

proving (6.3) in this case. Conversely if $|x| \geqslant R_\varepsilon$, then $v(x) < \varepsilon$ and so (6.3) holds true in this case too.

Hence, we can take the smallest $M := M_\varepsilon \geqslant 0$ for which (6.3) is satisfied. If $M_\varepsilon = 0$ for a sequence of $\varepsilon \searrow 0$ then (6.3) gives that $v(x) \leqslant \varepsilon$ and so, in the limit, $v \leqslant 0$, which proves (6.1). Thus, without loss of generality, we can suppose that $M_\varepsilon > 0$. In this case, by (6.2) and a simple compactness argument, there exists $x_\varepsilon \in \mathbb{R}$ for which

$$v(x_\varepsilon) = \frac{M_\varepsilon}{(1+|x_\varepsilon|)^\vartheta} + \varepsilon. \tag{6.4}$$

Our goal is to show that

$$M_\varepsilon \leqslant M_0 \tag{6.5}$$

for a suitable $M_0 > 0$ independent of ε. For this, we observe that, by (6.3), (6.4) and Proposition 5.1 (with $\alpha := 4s$), we have that the hypotheses of Corollary 4.2 are satisfied (by taking $u := v$ and $x_0 := x_\varepsilon$). Therefore, by (4.11), if M_ε were too large we would have that

$$L_s v(x_\varepsilon) \leqslant -\frac{M_\varepsilon\,|B_1|}{20\,(1+|x_\varepsilon|)^\vartheta}. \tag{6.6}$$

On the other hand, by (6.4), (1.12), and (1.13), we have

$$\begin{aligned} L_s v(x_\varepsilon) &= L_s v(x_\varepsilon) - c v(x_\varepsilon) + c\left(\frac{M_\varepsilon}{(1+|x_\varepsilon|)^\vartheta} + \varepsilon\right) \\ &\geqslant L_s v(x_\varepsilon) - c v(x_\varepsilon) \\ &= -g(x_\varepsilon) \\ &\geqslant -\frac{C}{(1+|x_\varepsilon|)^{4s}} \\ &\geqslant -\frac{C}{(1+|x_\varepsilon|)^\vartheta} \end{aligned} \tag{6.7}$$

(recall that $\vartheta \leqslant \alpha = 4s$, see (5.4)). Hence (6.7) and (6.6) show that M_ε is universally bounded, proving (6.5).

From (6.5) we deduce that

$$v(x) \leqslant \frac{M_\varepsilon}{(1+|x|)^\vartheta} + \varepsilon \leqslant \frac{M_0}{(1+|x|)^\vartheta} + \varepsilon$$

for any $x \in \mathbb{R}$, and so, by letting $\varepsilon \searrow 0$, we obtain (6.1). This concludes[1] the proof of Theorem 1.3.

[1] We remark that ϑ, as defined in (5.4), satisfies

$$\vartheta = \begin{cases} 4s & \text{if } s \in (0, 1/6], \\ \frac{1+2s}{2} & \text{if } s \in (1/6, 1/2). \end{cases}$$

In any case, since $s \in (0, 1/2)$, we have that

$$2s < \vartheta < 1 + 2s.$$

7. Proof of Theorem 1.1

The proof of Theorem 1.1 is now analogous to the one of Proposition 7.2 in [4], up to the following modifications, needed in the case $s \in (0, 1/2)$:

- the exponent $1 + 2s$ in formulas (7.9) and the previous one in [4] must be replaced by ϑ (the rest of the argument remains unchanged, since $\vartheta \in (2s, 1 + 2s]$),
- the use of Corollary 7.1 of [4] is replaced here by Theorem 1.3.

8. L^∞ bounds

The goal of this section is to state some uniform regularity estimates that will be needed in the subsequent Section 9.

We introduce the norm

$$\|f\|_{H_0^s(\mathbb{R}^n)} := \sqrt{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}}\, dx\, dy} \tag{8.1}$$

and we provide an auxiliary estimate:

Lemma 8.1. *Let $s \in (0, 1)$. There exists a constant $C = C(n, s) > 0$ such that, if $f \in H^s(\mathbb{R}^n)$, then*

$$\|f\|_{L^2(\mathbb{R}^n)} \leqslant C\|f\|_{H_0^s(\mathbb{R}^n)}^{n/(n+2s)}\, \|f\|_{L^1(\mathbb{R}^n)}^{2s/(n+2s)}. \tag{8.2}$$

Also, if $f \geqslant 0$ then

$$\|f\|_{L^2(\mathbb{R}^n)} \leqslant C\|f\|_{H_0^s(\mathbb{R}^n)}\, |\{f > 0\}|^{s/n}. \tag{8.3}$$

Proof. We start by proving (8.2), which is a variation of the classical Nash inequality. Without loss of generality, we suppose that $f \in L^1(\mathbb{R}^n)$, otherwise the right hand side of (8.2) is infinite and there is nothing to prove. Given $\rho > 0$, we have

$$\int_{\mathbb{R}^n\setminus B_\rho} |\hat f(\xi)|^2\, d\xi \leqslant \int_{\mathbb{R}^n\setminus B_\rho} \frac{|\xi|^{2s}}{\rho^{2s}} |\hat f(\xi)|^2\, d\xi \leqslant C\rho^{-2s}\|f\|_{H_0^s(\mathbb{R}^n)}^2. \tag{8.4}$$

Here we have used the notation of the norm $\|\cdot\|_{H_0^s(\mathbb{R}^n)}$, as introduced in (8.1) and its equivalent in Fourier spaces (see e.g. Proposition 3.4 in [3]). On the other hand, $|\hat f(\xi)| \leqslant \|f\|_{L^1(\mathbb{R}^n)}$ for any $\xi \in \mathbb{R}^n$, and so by integrating over B_ρ we obtain

$$\int_{B_\rho} |\hat f(\xi)|^2\, d\xi \leqslant |B_1|\, \rho^n \|f\|_{L^1(\mathbb{R}^n)}^2.$$

By adding this to (8.4) we obtain

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = \|\hat f\|_{L^2(\mathbb{R}^n)}^2 \leqslant C\rho^{-2s}\|f\|_{H_0^s(\mathbb{R}^n)}^2 + |B_1|\, \rho^n \|f\|_{L^1(\mathbb{R}^n)}^2.$$

Since this estimate is valid for any $\rho > 0$, we now choose

$$\rho := \left(\|f\|_{H_0^s(\mathbb{R}^n)}/\|f\|_{L^1(\mathbb{R}^n)}\right)^{2/(n+2s)}$$

to obtain

$$\|f\|_{L^2(\mathbb{R}^n)}^2 \leqslant (C + |B_1|)\, \|f\|_{H_0^s(\mathbb{R}^n)}^{2n/(n+2s)}\, \|f\|_{L^1(\mathbb{R}^n)}^{4s/(n+2s)},$$

which gives (8.2).

Now we prove (8.3) by using (8.2) and the Hölder inequality: we have

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^n)}^{n+2s} &\leqslant C\|f\|^n_{H^s_0(\mathbb{R}^n)}\,\|f\|^{2s}_{L^1(\mathbb{R}^n)}\\ &\leqslant C\|f\|^n_{H^s_0(\mathbb{R}^n)}\,\left[\|f\|_{L^2(\mathbb{R}^n)}\,|\{f>0\}|^{1/2}\right]^{2s}\\ &= C\|f\|^n_{H^s_0(\mathbb{R}^n)}\,\|f\|^{2s}_{L^2(\mathbb{R}^n)}\,|\{f>0\}|^{s},\end{aligned}$$

which implies (8.3). □

We can now prove a uniform pointwise estimate using a De Giorgi-type argument. For the sake of generality, we prove it for any $s \in (0,1)$ and any $n \geqslant 1$ (though we only need it here for $n = 1$ and $s \in (0,1/2)$).

Theorem 8.2. *Let $s \in (0,1)$ and let $\psi \in H^s(\mathbb{R}^n)$ be a weak solution to*

$$-L_s\psi = \lambda\psi + b \quad \text{in } \mathbb{R}^n,$$

with $b, \lambda \in L^\infty(\mathbb{R}^n)$. Then $\psi \in L^\infty(\mathbb{R}^n)$ and

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant C$$

where the constant $C > 0$ depends only on n, s, $\|\psi\|_{L^2(\mathbb{R}^n)}$, $\|\lambda\|_{L^\infty(\mathbb{R}^n)}$, and $\|b\|_{L^\infty(\mathbb{R}^n)}$.

Proof. First, for any $0 < \delta << 1$ (we will choose later a suitable δ, see formula (8.15) below), we consider the function ϕ defined as

$$\phi(x) := \frac{\delta\psi(x)}{\|\psi\|_{L^2(\mathbb{R}^n)}}, \quad \text{for any } x \in \mathbb{R}^n.$$

By construction,

$$\|\phi\|_{L^2(\mathbb{R}^n)} = \delta,$$

and

$$-L_s\phi = \lambda\phi + \delta b/\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{8.5}$$

In order to prove the theorem, it will suffice to prove that

$$\|\phi\|_{L^\infty(\mathbb{R}^n)} \leqslant 1, \tag{8.6}$$

since this implies that

$$\|\psi\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{\|\psi\|_{L^2(\mathbb{R}^n)}}{\delta}\|\phi\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{\|\psi\|_{L^2(\mathbb{R}^n)}}{\delta}$$

and δ is fixed.

Now, for any integer $k \in \mathbb{N}$, we consider the function w_k defined as follows

$$w_k(x) := (\phi(x) - (1-2^{-k}))^+, \quad \text{for any } x \in \mathbb{R}^n.$$

By construction, $w_k \in H^s(\mathbb{R}^n)$, $w_k(\pm\infty) = 0$, and

$$w_{k+1}(x) \leqslant w_k(x) \quad \text{a.e. in } \mathbb{R}^n. \tag{8.7}$$

The following inclusion

$$\left\{w_{k+1} > 0\right\} \subseteq \left\{w_k > 2^{-(k+1)}\right\} \tag{8.8}$$

holds true for all $k \in \mathbb{N}$. Indeed, if $x \in \left\{w_{k+1} > 0\right\}$, then

$$0 < w_{k+1}(x) = \phi(x) - 1 + 2^{-k-1}$$

hence

$$\phi(x) - (1-2^{-k}) > 2^{-k} - 2^{-k-1} = 2^{-k-1}$$

and so $w_k(x) > 2^{-k-1}$, thus proving (8.8). Moreover, we have the inequality

$$\phi(x) < 2^{k+1} w_k(x) \quad \text{for any } x \in \big\{w_{k+1} > 0\big\}. \tag{8.9}$$

Indeed, if $x \in \big\{w_{k+1} > 0\big\}$ then

$$w_k(x) \geqslant w_{k+1}(x) = \phi(x) - (1 - 2^{-k-1}),$$

which together with (8.8) implies

$$\begin{aligned} \phi(x) &\leqslant w_k(x) + (1 - 2^{-k-1}) = w_k(x) + (2^{k+1} - 1)2^{-k-1} \\ &< w_k(x) + (2^{k+1} - 1)w_k(x) = 2^{k+1} w_k(x). \end{aligned}$$

This proves (8.9).

Also, we remark that for any $v \in H^s(\mathbb{R}^n)$ we have

$$\big(v^+(x) - v^+(y)\big)\big(v(x) - v(y)\big) \geqslant |v^+(x) - v^+(y)|^2, \tag{8.10}$$

for all $x, y \in \mathbb{R}^n$. In order to check this, let assume that $v(x) \geqslant v(y)$. There is no loss of generality in such assumption, since the roles of x and y can be interchanged. Then, one can reduce to the case when $x \in \{v > 0\}$ and $y \in \{v \leqslant 0\}$, as otherwise the inequality in (8.10) plainly follows. Finally, we notice that in such a case (8.10) becomes

$$(v(x) - v(y))v(x) \geqslant v(x)^2$$

which does hold since $v(y) \leqslant 0$ and $v(x) > 0$. This proves (8.10).

We now prove (8.6) by a standard iterative argument based on estimating the decay of the quantity

$$U_k := \|w_k\|^2_{L^2(\mathbb{R}^n)}.$$

First, in view of (8.10) with $v := \phi - (1 - 2^{-k})$, we have

$$\begin{aligned} \|w_{k+1}\|^2_{H^s_0(\mathbb{R}^n)} &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w_{k+1}(x) - w_{k+1}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(\phi(x) - \phi(y)\big)\big(w_{k+1}(x) - w_{k+1}(y)\big)}{|x-y|^{n+2s}}\, dx\, dy. \end{aligned}$$

Thus, plugging w_{k+1} as a test function in (8.5), we obtain

$$\|w_{k+1}\|^2_{H^s_0(\mathbb{R}^n)} \leqslant \int_{\{w_{k+1}>0\}} \left(\lambda(x)\phi(x) + \frac{\delta\, b(x)}{\|\psi\|_{L^2(\mathbb{R}^n)}}\right) w_{k+1}(x)\, dx.$$

Notice that if $x \in \{w_{k+1} > 0\}$ then $\phi(x) > 0$, and therefore, using (8.9) and (8.7), we get

$$\begin{aligned} \|w_{k+1}\|^2_{H^s_0(\mathbb{R}^n)} &\leqslant \int_{\{w_{k+1}>0\}} \left(\sup_{\mathbb{R}^n}|\lambda|\, \phi(x)\, w_{k+1}(x) + \frac{\delta \sup_{\mathbb{R}^n}|b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} w_{k+1}(x)\right) dx \\ &\leqslant \int_{\{w_{k+1}>0\}} \left(\sup_{\mathbb{R}^n}|\lambda|\, 2^{k+1}\, w_k(x)\, w_{k+1}(x) + \frac{\delta \sup_{\mathbb{R}^n}|b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} w_{k+1}(x)\right) dx \\ &\leqslant \int_{\{w_{k+1}>0\}} \left(\sup_{\mathbb{R}^n}|\lambda|\, 2^{k+1}\, w^2_k(x) + \frac{\delta \sup_{\mathbb{R}^n}|b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} w_k(x)\right) dx \\ &\leqslant \sup_{\mathbb{R}^n}|\lambda|\, 2^{k+1} U_k + \frac{\delta\ \sup_{\mathbb{R}^n}|b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \sqrt{|\{w_{k+1} > 0\}|}\ U_k^{\frac{1}{2}}, \end{aligned} \tag{8.11}$$

where we have also used the Hölder inequality.

Also, by (8.8) and Chebychev's inequality, one has

$$|\{w_{k+1} > 0\}| \leqslant |\{w_k > 2^{-(k+1)}\}| \leqslant 2^{2(k+1)} U_k, \tag{8.12}$$

so that (8.11) becomes

$$\|w_{k+1}\|^2_{H^s_0(\mathbb{R}^n)} \leqslant \left(\sup_{\mathbb{R}^n} |\lambda| + \frac{\delta \sup_{\mathbb{R}^n} |b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \right) 2^{k+1} U_k. \tag{8.13}$$

On the other hand, using (8.3) (with $f := w_{k+1}$ here) we have

$$U_{k+1} \leqslant c \|w_{k+1}\|^2_{H^s_0(\mathbb{R}^n)} \big|\{w_{k+1} > 0\}\big|^{\frac{2s}{n}}, \tag{8.14}$$

where the constant $c > 0$ only depends on n and s.

Combining (8.13) with (8.14) and using (8.12), we get

$$\begin{aligned}
U_{k+1} &\leqslant c \left(\sup_{\mathbb{R}^n} |\lambda| + \frac{\delta \sup_{\mathbb{R}^n} |b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \right) 2^{k+1} U_k \left(2^{2(k+1)}\right)^{\frac{2s}{n}} U_k^{\frac{2s}{n}} \\
&= c \left(\sup_{\mathbb{R}^n} |\lambda| + \frac{\delta \sup_{\mathbb{R}^n} |b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \right) 2^{(1+\frac{4s}{n})(k+1)} U_k^{1+\frac{2s}{n}} \\
&\leqslant \left[1 + c \left(\sup_{\mathbb{R}^n} |\lambda| + \frac{\delta \sup_{\mathbb{R}^n} |b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \right) \right] 2^{(1+\frac{4s}{n})(k+1)} U_k^{1+\frac{2s}{n}} \\
&\leqslant \left[\left(1 + c \left(\sup_{\mathbb{R}^n} |\lambda| + \frac{\delta \sup_{\mathbb{R}^n} |b|}{\|\psi\|_{L^2(\mathbb{R}^n)}} \right) \right) 2^{1+\frac{4s}{n}} \right]^{k+1} U_k^{1+\frac{2s}{n}} \\
&= \bar{C}^{k+1} U_k^{1+\frac{2s}{n}},
\end{aligned}$$

for some constant $\bar{C} > 1$ depending on $\sup_{\mathbb{R}^n} |\lambda|$, $\sup_{\mathbb{R}^n} |b|$, $\|\psi\|_{L^2(\mathbb{R}^n)}$, n, and s. Hence, an estimate of the form

$$U_{k+1} \leqslant \bar{C}^{k+1} U_k^{1+\alpha} \quad \text{for any } k \in \mathbb{N},$$

holds for suitable $\bar{C} > 1$ and $\alpha > 0$.

Now we perform our choice of δ, that is we assume that

$$\delta^{2\alpha} = \frac{1}{\bar{C}^{(1/\alpha)+1}}. \tag{8.15}$$

We set

$$\eta := \frac{1}{\bar{C}^{1/\alpha}}. \tag{8.16}$$

Since $\bar{C} > 1$ and $\alpha > 0$, we have that

$$\eta \in (0, 1). \tag{8.17}$$

We claim that

$$U_k \leqslant \delta^2 \eta^k. \tag{8.18}$$

We show (8.18) by induction. Indeed, we notice that

$$U_0 := \|w_0\|^2_{L^2(\mathbb{R}^n)} = \|\phi^+\|^2_{L^2(\mathbb{R}^n)} \leqslant \|\phi\|^2_{L^2(\mathbb{R}^n)} = \delta^2,$$

which is (8.18) for $k = 0$. Now, suppose that (8.18) is true for k and let us prove it for $k+1$:

$$U_{k+1} \leqslant \bar{C}^{k+1} U_k^{1+\alpha} \leqslant \bar{C}^{k+1} (\delta^2 \eta^k)^{1+\alpha} = \delta^2 \eta^k (\bar{C}\eta^\alpha)^k \bar{C} \delta^{2\alpha} = \delta^2 \eta^{k+1},$$

where we have used (8.15) and (8.16). Then, by (8.17) and (8.18) we have that

$$\lim_{k\to\infty} U_k = 0. \tag{8.19}$$

Noticing that

$$0 \leqslant w_k = \big(\phi - (1-2^{-k})\big)^+ \leqslant |\phi| \in L^2(\mathbb{R}^n)$$

and

$$w_k \to (\phi - 1)_+ \quad \text{a.e. in } \mathbb{R}^n \quad \text{as } k \to +\infty,$$

by the Dominated Convergence Theorem we get

$$\lim_{k\to+\infty} U_k = \|(\phi-1)^+\|^2_{L^2(\mathbb{R}^n)}. \tag{8.20}$$

Hence, from (8.19) and (8.20) we have that $(\phi - 1)^+ = 0$ almost everywhere in $\mathbb{R}^n$, and so $\phi \leqslant 1$ almost everywhere in $\mathbb{R}^n$. By replacing ϕ with $-\phi$ we get (8.6), which concludes the proof. ☐

9. The corrector equation

Now we consider the equation

$$\begin{cases} L_s\psi - W''(u)\psi = u' + \eta\,(W''(u) - W''(0)) \ \text{in } \mathbb{R}, \\ \psi \in H^s(\mathbb{R}), \end{cases} \tag{9.1}$$

where u is the solution of (1.5) and

$$\eta = \frac{\displaystyle\int_{\mathbb{R}} (u'(x))^2\,dx}{W''(0)}. \tag{9.2}$$

For a detailed heuristic motivation of such an equation see Section 3.1 of [6].

Theorem 9.1. *There exists a unique solution $\psi \in H^s(\mathbb{R})$ to* (9.1)*. Furthermore*

$$\psi \in C^{1,\alpha}_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ for some } \alpha \in (0,1), \text{ and } \|\psi'\|_{L^\infty(\mathbb{R})} < +\infty. \tag{9.3}$$

Proof. The proof is analogous to the one of Theorem 5.2 in [4], where the result was obtained for $s \in (1/2, 1)$, except for the modifications listed below.

The proof of Theorem 5.2 in [4] uses the condition $s \in (1/2,1)$ only twice, namely before formula (5.26) and at the end of Section 5. In the first occasion, such condition was used to obtain that

$$\begin{aligned} &\text{a weak solution of } L_s v_0 = W''(u)v_0 \text{ is } C^{2s+\alpha}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \\ &\text{and, in particular, it is a classical solution.} \end{aligned} \tag{9.4}$$

In the second occasion, the condition on s was used to obtain (9.3). In both the cases, the condition $s \in (1/2,1)$ permitted to obtain the desired results as an easy consequence of the fractional Morrey-Sobolev embedding (see e.g. Theorem 8.2 in [3]), and this embedding is not available in the present case.

Hence, we prove (9.3) and (9.4) directly from the regularity theory developed in Section 8, thus obtaining that Theorem 9.1 also holds when $s \in (0, 1/2)$.

To prove (9.4), we first use Theorem 8.2 to obtain that $v_0 \in L^\infty(\mathbb{R})$. Hence, from Proposition 5 in [9] we deduce that $v_0 \in C^\alpha(\mathbb{R})$ for any $0 < \alpha < 2s$. In particular v_0 is a viscosity solution, and since

$W''(u)v_0 \in C^\alpha(\mathbb{R})$, by Proposition 2.8 in [11] we deduce that $v_0 \in C^{\alpha+2s}(\mathbb{R})$. Thus v_0 is a classical solution, proving (9.4).

To show (9.3), we use Theorem 8.2 and Proposition 5 in [9] to obtain that ψ is a viscosity solution to (9.1) such that

$$\psi \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R}) \tag{9.5}$$

for any $0 < \alpha < 2s$.

Now, we define the incremental quotient of ψ as

$$\psi_h(x) := \frac{\psi(x+h) - \psi(x)}{h} \quad \text{for any } x, h \in \mathbb{R}.$$

From (9.1) we have that ψ_h satisfies

$$L_s \psi_h(x) = W''(u(x+h))\psi_h(x) + W''_h(u(x))\psi(x) + u'_h(x) + \eta W''_h(u(x)) \tag{9.6}$$

where, for any $x \in \mathbb{R}$,

$$u'_h(x) := \frac{u'(x+h) - u'(x)}{h}$$

and

$$W''_h(u(x)) := \frac{W''(u(x+h)) - W''(u(x))}{h}.$$

From (1.2), (9.5), and Lemma 6 in [8], we have that

$$W''(u) \in L^\infty(\mathbb{R}) \quad \text{and} \quad W''_h(u)\psi + u'_h + \eta W''_h(u) \in L^\infty(\mathbb{R}),$$

and so we can apply Theorem 8.2 to the solution of (9.6) to obtain that $\psi_h \in L^\infty(\mathbb{R})$. Using Proposition 5 in [9], this gives that $\psi_h \in C^\alpha(\mathbb{R})$ for any $\alpha < 2s$.

So we have proved that, for any $x, y, h \in \mathbb{R}$,

$$|\psi_h(x)| \leqslant C_1 \quad \text{and} \quad |\psi_h(x) - \psi_h(y)| \leqslant C_2 |x-y|^\alpha,$$

for some positive constants C_1, C_2. Letting $h \searrow 0$ we obtain that $\psi' \in L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$, concluding the proof of (9.3). $\square$

Remark 9.2. Thanks to (9.1) and (9.3), we have that $\psi \in H^s(\mathbb{R})$ is uniformly continuous, and this implies that

$$\lim_{x \to \pm\infty} \psi(x) = 0. \tag{9.7}$$

10. Proof of Theorem 1.2

The proof is now conceptually similar to the one given in Section 8 of [4], but some quantitative estimates of Proposition 8.4 there need to be modified when $s \in (0, 1/2)$. For the facility of the reader, we provide the details of the proof of Proposition 8.4 of [4] in our case (this will be done in Proposition 10.1 here below).

To this goal, we recall some of the notation of [6, 4] needed for our purposes. We take an auxiliary parameter $\delta > 0$ and define $(\overline{x}_i(t))_{i=1,\dots,N}$ to be the solution of the system

$$\begin{cases} \dot{\overline{x}}_i = \gamma \left(-\delta - \sigma(t, \overline{x}_i) + \displaystyle\sum_{j \neq i} \frac{\overline{x}_i - \overline{x}_j}{2s\, |\overline{x}_i - \overline{x}_j|^{1+2s}} \right) \text{ in } (0, +\infty), \\ \overline{x}_i(0) = x_i^0 - \delta. \end{cases} \tag{10.1}$$

Moreover, we set

$$\overline{c}_i(t) := \dot{\overline{x}}_i(t) \tag{10.2}$$

$$\tilde{\sigma} := \frac{\delta+\sigma}{\beta}, \quad \text{where } \beta = W''(0) \text{ was introduced in (1.10),} \tag{10.3}$$

$$\overline{v}_\varepsilon(t,x) := \varepsilon^{2s}\tilde{\sigma}(t,x) + \sum_{i=1}^{N}\left\{u\left(\frac{x-\overline{x}_i(t)}{\varepsilon}\right) - \varepsilon^{2s}\overline{c}_i(t)\psi\left(\frac{x-\overline{x}_i(t)}{\varepsilon}\right)\right\}, \tag{10.4}$$

where u is given in Theorem 1.1 and ψ in Theorem 9.1. We set

$$\tilde{u}_i := u\left(\frac{x-\overline{x}_i(t)}{\varepsilon}\right) - H\left(\frac{x-\overline{x}_i(t)}{\varepsilon}\right), \tag{10.5}$$

where H is the Heaviside function,

$$\psi_i := \psi\left(\frac{x-\overline{x}_i(t)}{\varepsilon}\right).$$

and

$$I_\varepsilon := \varepsilon(\overline{v}_\varepsilon)_t + \frac{1}{\varepsilon^{2s}}\left(W'(\overline{v}_\varepsilon) - \varepsilon^{2s}L_s\overline{v}_\varepsilon - \varepsilon^{2s}\sigma\right). \tag{10.6}$$

With this notation we have that (see Lemma 8.3 in [4]), for every $i_0 \in \{1,\dots,N\}$,

$$I_\varepsilon = e_\varepsilon^{i_0} + (\beta\tilde{\sigma}-\sigma) + O(\tilde{u}_{i_0})\Bigg(\eta\,\overline{c}_{i_0} + \tilde{\sigma} + \sum_{\substack{1\leqslant i\leqslant N\\ i\neq i_0}}\frac{\tilde{u}_i}{\varepsilon^{2s}}\Bigg), \tag{10.7}$$

where the error $e_\varepsilon^{i_0}$ is given by

$$e_\varepsilon^{i_0} := O(\varepsilon^{2s}) + \sum_{\substack{1\leqslant i\leqslant N\\ i\neq i_0}} O(\psi_i) + \sum_{\substack{1\leqslant i\leqslant N\\ i\neq i_0}} O(\tilde{u}_i) + \sum_{\substack{1\leqslant i\leqslant N\\ i\neq i_0}} O\left(\frac{\tilde{u}_i^2}{\varepsilon^{2s}}\right). \tag{10.8}$$

Now we can state the following result, which replaces Proposition 8.4 in [4]:

Proposition 10.1. *There exists $\delta_0>0$ such that, for any $0<\delta\leqslant\delta_0$ and $T>0$, we have*

$$(\overline{v}_\varepsilon)_t \geqslant \frac{1}{\varepsilon}\Bigg(L_s\overline{v}_\varepsilon - \frac{1}{\varepsilon^{2s}}W'(\overline{v}_\varepsilon) + \sigma\Bigg) \quad \text{in } (0,T)\times\mathbb{R},$$

for $\varepsilon>0$ sufficiently small.

Proof. Recalling the definition of I_ε in (10.6), our goal is to show that

$$I_\varepsilon \geqslant 0 \tag{10.9}$$

for ε small enough. For this, we make a preliminary observation: recalling the definition of $\tilde{u}_i$ in (10.5) and using Theorem 1.1, we obtain that, for any $i \in \{1,\dots,N\}$,

$$\left|\tilde{u}_i + \frac{\varepsilon^{2s}}{2sW''(0)}\frac{x-\overline{x}_i(t)}{|x-\overline{x}_i(t)|^{1+2s}}\right| \leqslant \frac{C\,\varepsilon^{\vartheta}}{|x-\overline{x}_i(t)|^{\vartheta}}. \tag{10.10}$$

Since $\vartheta > 2s$, we can choose γ such that

$$0<\gamma<\frac{\vartheta-2s}{\vartheta}. \tag{10.11}$$

Now we divide the proof of (10.9) by dealing with two separate cases.
Case 1: Suppose that there exists $i_0 \in \{1,\dots,N\}$ such that

$$|x-\overline{x}_{i_0}(t)| \leqslant \varepsilon^{\gamma}. \tag{10.12}$$

Therefore, since the $\overline{x}_i$'s are well-separated, for ε sufficiently small we have that

$$|x-\overline{x}_i(t)| \geqslant \kappa > 0, \quad \text{for any } \ i \neq i_0, \tag{10.13}$$

where κ is a constant independent of ε.

Hence, thanks to (10.10) and (10.13),

$$\left|\sum_{i\neq i_0}\left(\frac{\tilde{u}_i}{\varepsilon^{2s}}+\frac{1}{2sW''(0)}\frac{x-\overline{x}_i(t)}{|x-\overline{x}_i(t)|^{1+2s}}\right)\right| \leqslant \frac{C\,\varepsilon^{\vartheta}}{\varepsilon^{2s}}\sum_{i\neq i_0}\frac{1}{|x-\overline{x}_i(t)|^{\vartheta}} \leqslant C\,\varepsilon^{\vartheta-2s}.$$

Therefore, from (10.7), we deduce that

$$\begin{aligned} I_\varepsilon &= e_\varepsilon^{i_0}+\beta\tilde{\sigma}-\sigma+O(\tilde{u}_{i_0})\left(\eta\,\overline{c}_{i_0}+\tilde{\sigma}+\sum_{i\neq i_0}\frac{\tilde{u}_i}{\varepsilon^{2s}}\right)\\ &= e_\varepsilon^{i_0}+\beta\tilde{\sigma}-\sigma+O(\tilde{u}_{i_0})\left(\eta\,\overline{c}_{i_0}+\tilde{\sigma}-\frac{1}{2sW''(0)}\sum_{i\neq i_0}\frac{x-\overline{x}_i(t)}{|x-\overline{x}_i(t)|^{1+2s}}\right)+O(\varepsilon^{\vartheta-2s}). \end{aligned} \tag{10.14}$$

Now, we Taylor expand the function $\frac{x-\overline{x}_i(t)}{|x-\overline{x}_i(t)|^{1+2s}}$ for x in a neighborhood of the point $\overline{x}_{i_0}(t)$, and we use (10.12) to get

$$\begin{aligned} &\left|\sum_{i\neq i_0}\frac{x-\overline{x}_i(t)}{|x-\overline{x}_i(t)|^{1+2s}}-\sum_{i\neq i_0}\frac{\overline{x}_{i_0}(t)-\overline{x}_i(t)}{|\overline{x}_{i_0}(t)-\overline{x}_i(t)|^{1+2s}}\right|\\ &\quad= \left|\sum_{i\neq i_0}\left(\frac{1}{|\xi-\overline{x}_i(t)|^{1+2s}}-(1+2s)\frac{(\xi-\overline{x}_i(t))^2}{|\xi-\overline{x}_i(t)|^{3+2s}}\right)(x-\overline{x}_{i_0}(t))\right|\\ &\quad\leqslant \sum_{i\neq i_0}\frac{2+2s}{|\xi-\overline{x}_i(t)|^{1+2s}}\,\varepsilon^{\gamma}\\ &\quad\leqslant C\,\varepsilon^{\gamma}, \end{aligned} \tag{10.15}$$

where ξ is a suitable point lying on the segment joining x to $\overline{x}_{i_0}(t)$ (and hence $|\xi-\overline{x}_i(t)| \geqslant \kappa/2$ thanks to (10.12)). Therefore, using (10.15) in (10.14), we have

$$\begin{aligned} I_\varepsilon &= e_\varepsilon^{i_0}+\beta\tilde{\sigma}-\sigma+O(\tilde{u}_{i_0})\left(\eta\,\overline{c}_{i_0}+\tilde{\sigma}-\frac{1}{2sW''(0)}\sum_{i\neq i_0}\frac{\overline{x}_{i_0}(t)-\overline{x}_i(t)}{|\overline{x}_{i_0}(t)-\overline{x}_i(t)|^{1+2s}}\right)\\ &\quad+O(\varepsilon^{\vartheta-2s})+O(\varepsilon^{\gamma}). \end{aligned} \tag{10.16}$$

Now, we compute the term in parenthesis. From the definitions of η, $\overline{c}_{i_0}$ and $\tilde{\sigma}$ given in (9.2), (10.2), and (10.3) respectively, and recalling (1.8), we obtain

$$\begin{aligned} &\eta\,\overline{c}_{i_0}+\tilde{\sigma}-\frac{1}{2sW''(0)}\sum_{i\neq i_0}\frac{\overline{x}_{i_0}(t)-\overline{x}_i(t)}{|\overline{x}_{i_0}(t)-\overline{x}_i(t)|^{1+2s}}\\ &= \frac{1}{\gamma\,W''(0)}\dot{\overline{x}}_{i_0}(t)+\frac{\delta}{W''(0)}+\frac{\sigma(t,x)}{W''(0)}-\frac{1}{2sW''(0)}\sum_{i\neq i_0}\frac{\overline{x}_{i_0}(t)-\overline{x}_i(t)}{|\overline{x}_{i_0}(t)-\overline{x}_i(t)|^{1+2s}}\\ &= \frac{1}{W''(0)}\left(\frac{\dot{\overline{x}}_{i_0}(t)}{\gamma}+\delta+\sigma(t,\overline{x}_{i_0}(t))-\frac{1}{2s}\sum_{i\neq i_0}\frac{\overline{x}_{i_0}(t)-\overline{x}_i(t)}{|\overline{x}_{i_0}(t)-\overline{x}_i(t)|^{1+2s}}\right)\\ &\quad+\frac{\sigma(t,x)-\sigma(t,\overline{x}_{i_0}(t))}{W''(0)}. \end{aligned} \tag{10.17}$$

Recalling (10.1), we have that

$$\frac{\dot{\overline{x}}_{i_0}(t)}{\gamma} + \delta + \sigma(t, \overline{x}_{i_0}(t)) - \frac{1}{2s} \sum_{i \neq i_0} \frac{\overline{x}_{i_0}(t) - \overline{x}_i(t)}{|\overline{x}_{i_0}(t) - \overline{x}_i(t)|^{1+2s}} = 0,$$

and so the term in parenthesis in (10.17) vanishes. Therefore (10.17) becomes

$$\begin{aligned} \eta\, \overline{c}_{i_0} + \tilde{\sigma} - \frac{1}{2sW''(0)} \sum_{i \neq i_0} \frac{\overline{x}_{i_0}(t) - \overline{x}_i(t)}{|\overline{x}_{i_0}(t) - \overline{x}_i(t)|^{1+2s}} &= \frac{\sigma(t,x) - \sigma(t, \overline{x}_{i_0}(t))}{W''(0)} \\ &= O(x - \overline{x}_{i_0}(t)) \\ &= O(\varepsilon^{\gamma}), \end{aligned}$$

thanks to (1.3) and (10.12). Hence (10.16) reads

$$I_\varepsilon = e_\varepsilon^{i_0} + \beta\tilde{\sigma} - \sigma + O(\varepsilon^{\gamma}) + O(\varepsilon^{\vartheta - 2s}) + O(\varepsilon^{\gamma}). \tag{10.18}$$

Also, in the light of (10.3), we see that

$$\beta\tilde{\sigma} - \sigma = \delta > 0. \tag{10.19}$$

Now, we claim that

(10.20) the error $e_\varepsilon^{i_0}$ (that was defined in (10.8)) tends to zero as $\varepsilon \to 0$.

For this, we notice that $\psi_i = \psi\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right)$, with $i \neq i_0$, tends to zero because of the behavior of the corrector at infinity (recall (9.7) and (10.13)). Moreover, thanks to (1.6) and (10.13) we have that, for $i \neq i_0$,

$$\tilde{u}_i = u\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right) - H\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right) = O\left(\frac{\varepsilon^{2s}}{|x - \overline{x}_i(t)|^{2s}}\right) = O(\varepsilon^{2s})$$

and

$$\frac{(\tilde{u}_i)^2}{\varepsilon^{2s}} = \frac{O(\varepsilon^{4s})}{\varepsilon^{2s}} = O(\varepsilon^{2s}),$$

thus proving (10.20).

Hence, from (10.18), (10.19) and (10.20) we obtain that for ε sufficiently small

$$I_\varepsilon \geqslant \frac{\delta}{2} > 0,$$

which implies (10.9) in this case.

Case 2: Suppose that $|x - \overline{x}_i(t)| > \varepsilon^{\gamma}$ for every $i \in \{1, \ldots, N\}$. In this case, we can fix i_0 arbitrarily, say $i_0 := 1$ for concreteness. We use (10.10) to obtain

$$\begin{aligned} \left| \sum_{i \neq i_0} \left(\frac{\tilde{u}_i}{\varepsilon^{2s}} + \frac{1}{2sW''(0)} \frac{x - \overline{x}_i(t)}{|x - \overline{x}_i(t)|^{1+2s}} \right) \right| &\leqslant \frac{C\, \varepsilon^{\vartheta}}{\varepsilon^{2s}} \sum_{i \neq i_0} \frac{1}{|x - \overline{x}_i(t)|^{\vartheta}} \\ &\leqslant C \frac{\varepsilon^{\vartheta - 2s}}{\varepsilon^{\gamma\vartheta}} = C\, \varepsilon^{\vartheta - 2s - \gamma\vartheta}. \end{aligned}$$

Therefore, by formula (10.7) and the definition of $\tilde{\sigma}$ in (10.3) we have

$$I_\varepsilon = e_\varepsilon^{i_0} + \delta + O(\tilde{u}_{i_0}) \left(\eta\, \overline{c}_{i_0} + \tilde{\sigma} - \frac{1}{2sW''(0)} \sum_{i \neq i_0} \frac{x - \overline{x}_i(t)}{|x - \overline{x}_i(t)|^{1+2s}} \right) + O(\varepsilon^{\vartheta - 2s - \gamma\vartheta}). \tag{10.21}$$

Now we observe that, for any $i \neq i_0$,

$$\left| \frac{x - \overline{x}_i(t)}{|x - \overline{x}_i(t)|^{1+2s}} \right| \leqslant \frac{1}{|x - \overline{x}_i(t)|^{2s}} \leqslant \frac{1}{\varepsilon^{2\gamma s}} = O(\varepsilon^{-2\gamma s}). \tag{10.22}$$

Notice that this term is divergent as ε tends to zero. Therefore, from (10.22) we conclude that

$$\eta\,\overline{c}_{i_0} + \tilde{\sigma} - \frac{1}{2sW''(0)} \sum_{i \neq i_0} \frac{x - \overline{x}_i(t)}{|x - \overline{x}_i(t)|^{1+2s}} = O(\varepsilon^{-2\gamma s}),$$

since the other terms are bounded. By plugging this into (10.21) we obtain

$$I_\varepsilon = e_\varepsilon^{i_0} + \delta + O(\tilde{u}_{i_0}) \cdot O(\varepsilon^{-2\gamma s}) + O(\varepsilon^{\vartheta - 2s - \gamma\vartheta}). \tag{10.23}$$

Now we observe that for every $i \in \{1, \ldots, N\}$,

$$\begin{aligned} \tilde{u}_i &= u\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right) - H\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right) \\ &= O\left(\frac{\varepsilon^{2s}}{|x - \overline{x}_i(t)|^{2s}}\right) = O\left(\frac{\varepsilon^{2s}}{\varepsilon^{2\gamma s}}\right) = O\left(\varepsilon^{2s(1-\gamma)}\right). \end{aligned} \tag{10.24}$$

As a consequence

$$\frac{(\tilde{u}_i)^2}{\varepsilon^{2s}} = O\left(\varepsilon^{2s(1-2\gamma)}\right) \quad \text{and} \quad O(\tilde{u}_{i_0}) \cdot O(\varepsilon^{-2\gamma s}) = O\left(\varepsilon^{2s(1-2\gamma)}\right). \tag{10.25}$$

We observe that, since $\vartheta \leqslant 4s$ (see (5.4) and recall that $\alpha = 4s$), from (10.11) we have

$$1 - 2\gamma > 1 - \frac{2(\vartheta - 2s)}{\vartheta} = \frac{4s - \vartheta}{\vartheta} \geqslant 0. \tag{10.26}$$

Also, notice that, thanks again to (10.11),

$$\vartheta - 2s - \gamma\vartheta > 0. \tag{10.27}$$

By inserting (10.25) into (10.23) and recalling (10.26) and (10.27) we get

$$I_\varepsilon = e_\varepsilon^{i_0} + \delta + O(\varepsilon^\alpha), \tag{10.28}$$

for some $\alpha > 0$. Now we check that

$$\text{the error term } e_\varepsilon^{i_0} \text{ tends to zero as } \varepsilon \to 0. \tag{10.29}$$

For this, we remark that, in this case,

$$\frac{|x - \overline{x}_i(t)|}{\varepsilon} \geqslant \frac{\varepsilon^\gamma}{\varepsilon} = \varepsilon^{\gamma - 1},$$

which diverges for small ε, since $\gamma < 1$. Therefore, for x fixed as in the assumption of Case 2, we have that

$$\psi_i(x) = \psi\left(\frac{x - \overline{x}_i(t)}{\varepsilon}\right) \longrightarrow 0$$

as $\varepsilon \to 0$, due to the infinitesimal behavior of ψ at infinity (see (9.7)). Using this, (10.24), (10.25) and the definition of the error term given in (10.8), we obtain (10.29).

Hence, by using (10.29) inside (10.28) and recalling that $\delta > 0$, we conclude that

$$I_\varepsilon \geqslant \frac{\delta}{2} > 0$$

for ε sufficiently smooth, thus proving (10.9) in this case too. $\square$

REFERENCES

[1] X. CABRÉ, Y. SIRE: Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.*, to appear.
Available at http://arxiv.org/abs/1111.0796

[2] L. A. CAFFARELLI, X. CABRÉ: *Fully nonlinear elliptic equations*. American Mathematical Society (1995), pp. 104.

[3] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. math.* **136** (2012), no. 5, 521–573.

[4] S. DIPIERRO, G. PALATUCCI, E. VALDINOCI: Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting. *Preprint*.
Available at http://www.wias-berlin.de/preprint/1847/wias_preprints_1847.pdf

[5] N. FORCADEL, C. IMBERT, R. MONNEAU: Homogenization of some particle systems with two-body interactions and of the dislocation dynamics. *Discrete Contin. Dyn. Syst.* **23** (2009), no. 3, 785–826.

[6] M. D. M. GONZÁLEZ, R. MONNEAU: Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one. *Discrete Contin. Dyn. Syst.* **32** (2012), no. 4, 1255–1286.

[7] F. R. N. NABARRO: Fifty-year study of the Peierls–Nabarro stress. *Mat. Sci. Eng. A* **234–236** (1997), 67–76.

[8] G. PALATUCCI, O. SAVIN, E. VALDINOCI: Local and global minimizers for a variational energy involving a fractional norm. *Ann. Mat. Pura Appl. (4)* **192** (2013), no. 4, 673–718.

[9] R. SERVADEI, E. VALDINOCI: Weak and viscosity solutions of the fractional Laplace equation. *Publ. Mat.* **58** (2014), no. 1.

[10] L. SILVESTRE: *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Ph.D. Thesis, University of Texas at Austin (2005).

[11] L. SILVESTRE: Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.