BOUNDARY INTEGRAL OPERATORS FOR PLATE BENDING IN DOMAINS WITH CORNERS

GUNTHER SCHMIDT

WEIERSTRASS INSTITUTE FOR APPLIED ANALYSIS AND STOCHASTICS, MOHRENSTR. 39, D-10117 BERLIN, GERMANY

Abstract. The paper studies boundary integral operators of the bi–Laplacian on piecewise smooth curves with corners and describes their mapping properties in the trace spaces of variational solutions of the biharmonic equation. We formulate a direct integral equation method for solving mixed boundary value problems for the biharmonic equation on a nonsmooth plane domain, analyse the solvability of the corresponding systems of integral equations and prove their strong ellipticity.

Keywords: biharmonic equation, plate bending, nonsmooth curve, boundary integral operators, boundary integral equations.
AMS Subject Classifications: 31A30, 47G10, 65N38

1. Introduction

The present paper is devoted to the study of boundary integral operators of the bi–Laplacian on piecewise smooth curves with corners and to the analysis of a direct integral equation method for solving the biharmonic equation with mixed boundary conditions on a nonsmooth plane domain Ω with boundary Γ. Although boundary element methods offer important advantages over domain type methods and are frequently used for solving plate bending or related problems for fourth–order equations (cf [2], [11] and also the references therein), their theoretical foundation is very limited compared with the results for second–order equations.

For the case of smooth boundary quite satisfactory results are available by using nowadays standard tools from the theory of integral and pseudodifferential equations and of approximation methods. In connection with indirect boundary integral equation methods we mention Chapter 8 of the book [2], where a detailed analysis of the mapping properties of biharmonic boundary integral operators and of indirect formulations for four types of boundary value problems can be found. As a rule indirect methods are designed for specific classes of problems, but their application to other types of plate bending problems, for example to mixed boundary conditions, is complicated both in analytical and numerical respect. The study of direct methods can be based on the approach of Costabel and Wendland, which was developed in [4], [9] and results in a complete description of the mapping properties of boundary integral operators and the strong ellipticity of systems of first kind integral equations corresponding to various types of boundary conditions. This can be used to consider different numerical methods for solving the corresponding integral equations, to analyse stability and error estimates similar to well–established techniques for second–order equations.

If the boundary of the domain has corners then the situation is quite different. The boundary integral operators are no longer classical pseudodifferential operators and biharmonic boundary value problems have in general only weak solutions. Thus the extension of similar considerations concerning direct methods for second–order equations requires the study of the behaviour of biharmonic boundary integral operators applied to the Cauchy data of H^2-functions. In [8], the first paper devoted to the study of boundary integral equations for the biharmonic equation in nonsmooth domains, Costabel, Stephan and Wendland considered an indirect method for the solution of the boundary value problem $grad\, u|_\Gamma = \mathbf{f}$. Using a layer potential ansatz with the gradient of the biharmonic fundamental solution as integral kernel they obtained a system of two integral equations of the first kind with logarithmic principal part. Thus they avoided the above mentioned problem of dealing with the application of biharmonic integral operators to the Cauchy data of weak solutions. This was first done by Bourlard in [1], where the biharmonic Dirichlet problem on a polygonal domain was transformed into the variational formulation for the first kind boundary integral equation with the biharmonic single layer potential. It was shown that the variational problem is coercive on the dual of the space of Dirichlet data of H^2-functions (the boundary values of the function and its normal derivative), that means the single layer potential operator is a symmetric and strongly elliptic mapping from this dual into the trace space. Similar results were obtained in [15] by applying some methods for second–order equations from [5] and [7] in order to define and to study biharmonic boundary integral operators. These operators were associated with the bilinear form

$$\int_\Omega \Delta u \, \Delta v \, dx \, , \tag{1.1}$$

which is positive definite on $H_0^2(\Omega)$ and corresponds to the biharmonic Dirichlet problem. The simple idea was to consider the two functions of the Dirichlet datum of a H^2-function, which obviously are subjected to some compatibility conditions at the corner points of Γ, as one element of a trace space and to define Neumann data of H^2-functions u with $\Delta^2 u \in L^2$ by using the form (1.1) similar to the case of second–order equations. It turns out that the Neumann data belong to the dual of the trace space. Then the biharmonic layer potentials are simply the values of the duality functional applied to the Neumann datum (single layer) or the Dirichlet datum (double layer) of the biharmonic fundamental solution and to an element of the corresponding dual space, which becomes the density. Now the setting is the same as for potentials of second–order equations, and by using the approach of Costabel [5] we prove jump relations for the potentials, introduce the boundary integral operators and analyse their mapping properties in the trace spaces of variational solutions. The obtained results were used to derive boundary integral equations for interior and exterior biharmonic Dirichlet problems in nonsmooth domains and to analyse their solvability.

In the present paper we extend the approach of [15] in order to treat other types of boundary conditions, which appear in thin plate bending as free, simply supported or roller-supported plate. To this end the bilinear form (1.1) has to be replaced by another form

$$a^\sigma(u,v) = \int_\Omega \Big(\sigma \Delta u \, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j \partial_k u \, \partial_j \partial_k v \Big) \, dx$$

connected with the bending strain energy of a Kirchhoff plate if $0 < \sigma < 1/2$. In Section 2 we provide the analogous construction as in [15] to define the Neumann data of H^2-functions u with $\Delta^2 u \in L^2$, which now depend on σ and contain, even for smooth u, Dirac–functionals supported at the corner points of the boundary. Further we consider the existence of variational solutions of interior and exterior Dirichlet and Neumann problems. In Section 3 we introduce the biharmonic layer potentials associated with a^σ, characterize their behaviour at infinity and prove the jump relations and representation formulas for biharmonic functions. The corresponding boundary integral operators will be studied in Section 4. Here we see that for $0 \le \sigma < 1$ these operators behave like the boundary integral operators of the Laplacian. In Section 5 we transform biharmonic boundary value problems into equivalent systems of boundary integral equations. If the boundary value problem allows a coercive variational formulation then the corresponding system of integral equation is strongly elliptic. We study the solvability of this system, which leads immediately to stability results for Galerkin boundary element methods.

In this paper we restrict the analysis of continuity problems to the trace spaces of variational solutions, i.e. to energy norms. By using the calculus of Mellin convolutions it is possible to consider the continuity of the biharmonic boundary integral operators in other than energy norms and to study the regularity of solutions of the obtained strongly elliptic systems of integral equations. Additionally, other boundary integral formulations and their approximate solution can be analysed. This will be the topic of another paper.

2. Traces of H^2–functions on piecewise smooth boundaries

For the following let Γ be a simple closed curve in the plane (x_1, x_2) composed of m smooth arcs Γ_i. Adjacent arcs Γ_{i-1} and Γ_i meet at the corner points x^i, $i = 1, \dots m$, with the interior angles α_i, $0 < \alpha_i < 2\pi$. The interior of Γ we denote by Ω_1, the exterior $\mathbb{R}^2 \backslash \overline{\Omega}_1$ by Ω_2, and let the unit normal $n = (n_1, n_2)$ on Γ be directed into Ω_2. In the following we denote by ∂_j, $j = 1, 2$, the partial derivative with respect to x_j, by $\partial_n = n_1\,\partial_1 + n_2\,\partial_2$ the normal derivative and by $\partial_\tau = -n_2\,\partial_1 + n_1\,\partial_2$ the tangential derivative along Γ. Using a general result of Jakovlev [12] we can characterize the traces of the Sobolev space $H^2(\Omega_1)$, which we equip with the norm

$$\|u\|_{H^2(\Omega_1)} = \left(\|u\|^2_{L^2(\Omega_1)} + |u|^2_{H^2(\Omega_1)}\right)^{1/2}, \quad \text{where } |u|^2_{H^2(\Omega_1)} := \sum_{j,k=1}^{2} \|\partial_j \partial_k u\|^2_{L^2(\Omega_1)} .$$

Lemma 2.1. ([12]) *There exists a constant $c > 0$ not depending on $u \in H^2(\Omega_1)$ such that*

$$\sum_{i=1}^{m} \left(\|u\|_{H^{3/2}(\Gamma_i)} + \|\partial_n u\|_{H^{1/2}(\Gamma_i)} \right) + \|\partial_1 u\|_{H^{1/2}(\Gamma)} + \|\partial_2 u\|_{H^{1/2}(\Gamma)} \le c\,\|u\|_{H^2(\Omega)} .$$

In order to define the trace space of $H^2(\Omega_1)$ and the corresponding trace mapping we make the following conventions. We identify functions on Γ with periodic functions depending on arc length s and denote the derivative with respect to s by $du/ds = u'$. Since with exception of the corner points x^i there holds

$$\partial_1 u|_\Gamma = n_1\,\partial_n u - n_2\,\partial_\tau u\ ,\ \ \partial_2 u|_\Gamma = n_2\,\partial_n u + n_1\,\partial_\tau u \ \text{ and } \ \partial_\tau = \frac{d}{ds}$$

Lemma 2.1 suggests the definition of the trace space

$$V(\Gamma) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : \ u_1 \in H^1(\Gamma),\ n_1 u_2 - n_2 u_1',\ n_2 u_2 + n_1 u_1' \in H^{1/2}(\Gamma) \right\}$$

equipped with the canonical norm. We introduce the generalized trace mapping

$$\gamma u := \begin{pmatrix} u|_\Gamma \\ \partial_n u|_\Gamma \end{pmatrix} : H^2(\Omega_1) \to V(\Gamma) .$$

Lemma 2.2. ([12]) *The linear mapping*

$$\gamma : H^2_{loc}(\mathbb{R}^2) \to V(\Gamma)$$

is continuous and has a continuous right inverse

$$\gamma^- : V(\Gamma) \to H^2_{loc}(\mathbb{R}^2) .$$

In particular, γ maps $C_0^\infty(\mathbb{R}^2)$ onto a dense subspace of $V(\Gamma)$.

If we define the duality form

$$\left[\begin{pmatrix} v_4 \\ v_3 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] := -\langle v_4, u_1 \rangle_\Gamma + \langle v_3, u_2 \rangle_\Gamma\ , \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the extension of the L^2–scalar product on Γ, then the dual space of $V(\Gamma)$ can be described as follows.

Lemma 2.3. *The vector $\begin{pmatrix} v_4 \\ v_3 \end{pmatrix}$ belongs to $(V(\Gamma))'$ iff there exist $z_1, z_2 \in H^{-1/2}(\Gamma)$ such that*

$$v_3 = n_1 z_1 + n_2 z_2 \quad \textit{and} \quad \langle \varphi|_\Gamma, v_4 \rangle_\Gamma = -\big\langle (\varphi|_\Gamma)', n_2 z_1 - n_1 z_2 \big\rangle_\Gamma\,, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2) .$$

In the following we will consider boundary integral equations connected with plate bending problems. To this end we introduce the bilinear form

$$a^\sigma(u,v) = a^\sigma_{\Omega_1}(u,v) := \int\limits_{\Omega_1} \Big(\sigma \Delta u \, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j \partial_k u \, \partial_j \partial_k v \Big) \, dx \tag{2.2}$$

well-known in the variational formulation of bending problems for a thin plate with Poisson ratio $\sigma = \lambda/2(\lambda+\mu)$, λ and μ are the Lamé constants of the material. If u represents the deflection function on Ω_1 corresponding to suitable loading and boundary conditions then the value of

$$a^\sigma(u,u) = \sigma \, \|\Delta u\|^2_{L^2(\Omega_1)} + (1-\sigma) \, |u|^2_{H^2(\Omega_1)} \tag{2.3}$$

is exactly twice the bending strain energy of the plate.

The analysis of boundary value problems for the biharmonic equation and of corresponding boundary integral equations is based on the fact that the bilinear form a^σ is coercive on appropriate function spaces for certain values of the parameter σ. By (2.3) the form a^σ is coercive on $H^2(\Omega_1)$ at least for $0 \le \sigma < 1$. We mention that in the case of a smooth boundary Γ the form a^σ is coercive on $H^2(\Omega_1)$ if and only if $-3 < \sigma < 1$, as stated in [10].

Furthermore, if $u, v \in C_0^\infty(\Omega_1)$ then we have

$$\int\limits_{\Omega_1} \partial_j \partial_k u \, \partial_j \partial_k v \, dx = \int\limits_{\Omega_1} \partial_j \partial_j u \, \partial_k \partial_k v \, dx \ ,$$

hence for $u, v \in H_0^2(\Omega_1)$ the value of $a^\sigma(u,v)$ does not depend on σ and the seminorm $(a^\sigma(u,u))^{1/2} = |u|_{H^2(\Omega_1)}$ is a norm on $H_0^2(\Omega_1)$ equivalent to $\|\cdot\|_{H^2(\Omega_1)}$. Thus for given $f \in L^2(\Omega_1)$, $\psi \in V(\Gamma)$ the problem

$$\gamma u = \psi \quad , \quad a^\sigma(u,v) = \langle f, v \rangle_{\Omega_1} \ , \quad \forall\, v \in H_0^2(\Omega_1) \ , \tag{2.4}$$

has a unique solution $u \in H^2(\Omega_1)$ being the weak solution of the Dirichlet problem

$$\Delta^2 u = f \quad \text{in } \Omega_1 \, , \qquad \gamma u = \psi \, . \tag{2.5}$$

It is obvious that the solution operator defined by $u = T(f,\psi)$ is a continuous mapping

$$T : L^2(\Omega_1) \times V(\Gamma) \to H^2(\Omega_1, \Delta^2) := \big\{ u \in H^2(\Omega_1) : \Delta^2 u \in L^2(\Omega_1) \big\} \, . \tag{2.6}$$

In order to consider other boundary value problems we define besides the operators ∂_n and ∂_τ of normal and tangential differentiation along Γ the differential operators ∂_{nn}, $\partial_{n\tau}$ and $\partial_{\tau\tau}$ by the relations

$$\begin{aligned} \partial_{nn} u &= n_1^2 \, \partial_1^2 u + 2 n_1 n_2 \, \partial_1 \partial_2 u + n_2^2 \, \partial_2^2 u \ , \\ \partial_{\tau n} u &= (n_1^2 - n_2^2) \partial_1 \partial_2 u - n_1 n_2 (\partial_1^2 u - \partial_2^2 u) \ , \\ \partial_{\tau\tau} u &= n_2^2 \, \partial_1^2 u - 2 n_1 n_2 \, \partial_1 \partial_2 u + n_1^2 \, \partial_2^2 u \ . \end{aligned} \tag{2.7}$$

Lemma 2.4. *Let $u \in H^2(\Omega_1, \Delta^2)$ and $\sigma \in I\!R$. The mapping*

$$\delta_\sigma u : \, \psi \to [\delta_\sigma u, \psi] := a^\sigma(u, \gamma^- \psi) - \int\limits_{\Omega_1} \gamma^- \psi \, \Delta^2 u \, dx \tag{2.8}$$

is a continuous linear functional on $V(\Gamma)$ that coincides for sufficiently smooth u and for $\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma)$ with the functional

$$[\delta_\sigma u, \psi] = - \int\limits_\Gamma \big(v_1 \, \partial_n \Delta u - (1-\sigma) \, v_1' \, \partial_{\tau n} u \big) \, ds + \int\limits_\Gamma v_2 \, \big(\sigma \Delta u + (1-\sigma) \, \partial_{nn} u \big) \, ds \, . \tag{2.9}$$

Moreover, the linear operator $\delta_\sigma : H^2(\Omega_1, \Delta^2) \to (V(\Gamma))'$ is continuous.

Proof: Since

$$\sigma\Delta u\,\Delta v + (1-\sigma)\sum_{j,k=1}^{2}\partial_j\partial_k u\,\partial_j\partial_k v$$
$$= \Delta u\Delta v + (1-\sigma)\big(2\,\partial_1\partial_2 u\,\partial_1\partial_2 v - \partial_1^2 u\,\partial_2^2 v - \partial_2^2 u\,\partial_1^2 v\big)$$

we use for $u \in H^4(\Omega_1)$ and $v \in H^2(\Omega_1)$ Green's formula to get

$$\int_{\Omega_1}(\Delta u\,\Delta v - v\,\Delta^2 u)\,dx = \int_{\Gamma}(\Delta u\,\partial_n v - v\,\partial_n\Delta u)\,ds$$

and

$$\int_{\Omega_1}\big(2\,\partial_1\partial_2 u\,\partial_1\partial_2 v - \partial_1^2 u\,\partial_2^2 v - \partial_2^2 u\,\partial_1^2 v\big)\,dx = \int_{\Gamma}(\partial_\tau v\,\partial_{\tau n} u - \partial_n v\,\partial_{\tau\tau} u)\,ds\;.$$

Thus the value of the domain integrals

$$a^{\sigma}_{\Omega_1}(u,v) - \int_{\Omega_1} v\,\Delta^2 u\,dx$$

depends only on $\gamma v \in V(\Gamma)$ and we obtain formula (2.9) known as Rayleigh–Green formula. Since

$$|a^{\sigma}(u,v)| \le |\sigma|\|\Delta u\|_{L^2(\Omega_1)}\|\Delta v\|_{L^2(\Omega_1)} + |1-\sigma|\;|u|_{H^2(\Omega_1)}|v|_{H^2(\Omega_1)}$$

there exists a constant depending only on σ such that

$$\big|[\delta_\sigma u,\psi]\big| \le \|\Delta^2 u\|_{L^2(\Omega_1)}\|\gamma^-\psi\|_{L^2(\Omega_1)} + c_\sigma\,|u|_{H^2(\Omega_1)}|\gamma^-\psi|_{H^2(\Omega_1)}\;. \tag{2.10}$$

Hence the assertion follows by continuity from Lemmas 2.2 and the fact that $C^\infty(\overline{\Omega}_1)$ is dense in $H^2(\Omega_1,\Delta^2)$ (see [15]).

□

Corollary 2.1. *For $u, v \in H^2(\Omega_1,\Delta^2)$ there holds Green' second formula*

$$\int_{\Omega_1}(v\,\Delta^2 u - u\,\Delta^2 v)dx = [\delta_\sigma v,\gamma u] - [\delta_\sigma u,\gamma v]\;.$$

For $\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma)$ we write formula (2.9) in the form

$$[\delta_\sigma u,\psi] = -\langle v_1,\widetilde{N}_\sigma u\rangle_\Gamma + \langle v_2, M_\sigma u\rangle_\Gamma\;, \tag{2.11}$$

where for sufficiently smooth u, say $u \in H^4(\Omega_1)$, we have

$$M_\sigma u := \sigma\Delta u + (1-\sigma)\,\partial_{nn} u \quad,\quad \widetilde{N}_\sigma u := \partial_n\Delta u + \frac{d}{ds}\big(T_\sigma u\big)\;, \tag{2.12}$$

and the derivative of

$$T_\sigma u := (1-\sigma)\,\partial_{\tau n} u \tag{2.13}$$

is understood in distributional sense. In plate bending $M_\sigma u$ corresponds to the bending moment, $T_\sigma u$ to the twisting moment and $\widetilde{N}_\sigma u$ is known as transverse force. In general the twisting moment $T_\sigma u$ is discontinuous at the corner points of Γ. Therefore

$$\widetilde{N}_\sigma u = N_\sigma u + \sum_{i=1}^{m}\delta(\cdot - x^i)\big(T_\sigma u(x^i_+) - T_\sigma u(x^i_-)\big)\;,$$

where $\delta(x)$ is the Dirac functional, $T_\sigma u(x^i+) - T_\sigma u(xi^-)$ is the corner force at x^i and the function $N_\sigma u$, known as Kirchhoff shear, is equal to

$$N_\sigma u = \partial_n\Delta u + \frac{d}{ds}\big(T_\sigma u\big) \qquad \text{on the arcs } \Gamma_i\;. \tag{2.14}$$

If we use that adjacent arcs meet at the corner point x^i with the interior angle α_i, then from (2.7) follows easily that

$$T_\sigma u(x^i_+) - T_\sigma u(x^i_-) = (1-\sigma)\sin\alpha_i\left(\partial_{\tau^i\tau^i}u(x^i) - \partial_{n^in^i}u(x^i)\right). \tag{2.15}$$

Here the unit vector

$$n^i = \left(\cos(\varphi_i + \frac{\pi-\alpha_i}{2}), \sin(\varphi_i + \frac{\pi-\alpha_i}{2})\right) = \left(-\sin(\varphi_i - \frac{\alpha_i}{2}), \cos(\varphi_i - \frac{\alpha_i}{2})\right)$$

is directed like the bisector of the angle between $n(x^i_-)$ and $n(x^i_+)$, φ_i denotes the angle between the x_1–axis and $n(x^i_-)$, and

$$\tau^i = -\left(\cos(\varphi_i - \frac{\alpha_i}{2}), \sin(\varphi_i - \frac{\alpha_i}{2})\right).$$

Hence we get

$$\widetilde{N}_\sigma u = N_\sigma u + (1-\sigma)\sum_{i=1}^{m}\delta(\cdot - x^i)\sin\alpha_i\left(\partial_{\tau^i\tau^i}u(x^i) - \partial_{n^in^i}u(x^i)\right). \tag{2.16}$$

The vector composed of the components of the Dirichlet and Neumann data

$$\begin{pmatrix}\gamma u\\ \delta_\sigma u\end{pmatrix} = \begin{pmatrix} u\\ \partial_n u\\ M_\sigma u\\ \widetilde{N}_\sigma u\end{pmatrix} \tag{2.17}$$

will be called *Cauchy datum* of $u \in H^2(\Omega_1, \Delta^2)$ associated with the bilinear form a^σ.

Let us now consider the problem to find $u \in H^2(\Omega_1)$ such that for given $\chi \in (V(\Gamma))'$

$$a^\sigma(u,v) = [\chi, \gamma v], \quad \forall\, v \in H^2(\Omega_1). \tag{2.18}$$

By (2.8) this is equivalent to the Neumann problem for the biharmonic equation

$$\Delta^2 u = 0 \quad \text{in } \Omega_1, \qquad \delta_\sigma u = \chi. \tag{2.19}$$

We denote by $I\!P_1$ the space of linear functions on $I\!R^2$ and introduce the factor space $\mathcal{H}^2(\Omega_1) := H^2(\Omega_1)/I\!P_1$. It is well known that

$$\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)} := |u|_{H^2(\Omega_1)}$$

gives a norm on the Hilbert space $\mathcal{H}^2(\Omega_1)$ equivalent to the quotient norm

$$\inf_{p\in I\!P_1}\|u-p\|_{H^2(\Omega_1)}.$$

Further we denote by $l(\Gamma)$ the traces of linear functions, $l(\Gamma) := \gamma(I\!P_1)$, consider the space $W(\Gamma) := V(\Gamma)/l(\Gamma)$ equipped with the factor norm and the adjoint $(W(\Gamma))'$ with respect to (2.1), which can be identified with the polar set

$$l(\Gamma)^\perp := \{\chi \in (V(\Gamma))' : [\chi, \psi] = 0,\ \forall\,\psi \in l(\Gamma)\}.$$

Obviously the assertions of Lemma 2.2 remain true for the mapping $\gamma : \mathcal{H}^2(\Omega_1) \to W(\Gamma)$.

Lemma 2.5. *Let $\dot{u} \in \mathcal{H}^2(\Omega_1)$ with $\Delta^2\dot{u} = 0$ and $0 \le \sigma < 1$. There exist constants not depending on $\dot{u}$ such that*

$$c_1\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)} \le \|\delta_\sigma\dot{u}\|_{(W(\Gamma))'} \le c_2\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)}.$$

Proof: Since $\delta_\sigma p = 0$, $p \in I\!P_1$, the mapping δ_σ is defined on the equivalence classes $\dot{u} \in \mathcal{H}^2(\Omega_1)$ with $\Delta^2 u \in L^2(\Omega_1)$. Further, for any $u \in H^2(\Omega_1)$ with $\Delta^2 u = 0$ there holds

$$[\delta_\sigma u, \gamma p] = 0,\ p \in I\!P_1, \quad \text{i.e.,} \quad \delta_\sigma u \in l(\Gamma)^\perp. \tag{2.20}$$

From (2.10) we get

$$\left|[\delta_\sigma u, \psi]\right| \le c_\sigma\,|u|^2_{H^2(\Omega_1)}|\gamma^-\psi|^2_{H^2(\Omega_1)} \le c\,\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)}\|\dot{\psi}\|^2_{W(\Gamma)},$$

hence δ_σ maps $\{\dot{u} \in \mathcal{H}^2(\Omega_1) : \Delta^2\dot{u} = 0\}$ into $(W(\Gamma))'$ and

$$\|\delta_\sigma\dot{u}\|_{(W(\Gamma))'} \le c_2\,\|\dot{u}\|_{\mathcal{H}^2(\Omega_1)}.$$

On the other hand, for $u \in H^2(\Omega_1)$ with $\Delta^2 u = 0$ we have

$$[\delta_\sigma u, \gamma u] = a^\sigma_{\Omega_1}(u,u) = \sigma \, \|\Delta u\|^2_{L^2(\Omega_1)} + (1-\sigma)\, |u|^2_{H^2(\Omega_1)} ,$$

such for $0 \le \sigma < 1$

$$[\delta_\sigma \dot{u}, \gamma \dot{u}] \ge (1-\sigma)\, \|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)} . \tag{2.21}$$

Hence we derive

$$\|\delta_\sigma \dot{u}\|_{(W(\Gamma))'} \|\gamma \dot{u}\|_{W(\Gamma)} \ge (1-\sigma)\, \|\dot{u}\|^2_{\mathcal{H}^2(\Omega_1)} \ge c_1 \|\dot{u}\|_{\mathcal{H}^2(\Omega_1)} \|\gamma \dot{u}\|_{W(\Gamma)} .$$

□

Corollary 2.2. *Let $0 \le \sigma < 1$. The Neumann problem (2.19) has a solution $u \in H^2(\Omega_1)$ if and only if $\chi \in l\,(\Gamma)^\perp$. The corresponding equivalence class $\dot{u} \in \mathcal{H}^2(\Omega_1)$ is unique.*

Lemma 2.6. *The set $\{(\gamma\varphi, \delta_\sigma\varphi) : \varphi \in C_0^\infty(I\!R^2)\}$ is dense in $V(\Gamma) \times (V(\Gamma))'$.*

Proof: The assertion is proved if we show that for $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ the relation

$$[\delta_\sigma\varphi, \psi] - [\chi, \gamma\varphi] = 0 , \quad \forall\, \varphi \in C_0^\infty(I\!R^2) \tag{2.22}$$

implies $\psi = \chi = 0$.

Choosing arbitrary $f \in L^2(\Omega_1)$ we obtain by applying Corollary 2.1 and (2.6)

$$\begin{aligned}
[\delta_\sigma T(f,0), \psi] &= [\delta_\sigma T(f,0), \gamma T(0,\psi)] - [\delta_\sigma T(0,\psi), \gamma T(f,0)] \\
&= \int\limits_{\Omega_1} \big(T(f,0)\, \Delta^2 T(0,\psi) - T(0,\psi)\, \Delta^2 T(f,0)\big)\, dx \\
&= -\int\limits_{\Omega_1} f\, T(0,\psi)\, dx .
\end{aligned}$$

Since $C^\infty(\overline{\Omega}_1)$ is dense in $H^2(\Omega_1, \Delta^2)$ relation (2.22) holds also for $\varphi = T(f,0)$, such that

$$\int\limits_{\Omega_1} f\, T(0,\psi)\, dx = 0 \quad \text{for all } \ f \in L^2(\Omega_1) .$$

Thus $T(0,\psi) = 0$ yielding $\psi = \gamma T(0,\psi) = 0$. From (2.22) it follows now that

$$[\chi, \gamma\varphi] = 0 \quad \text{for all } \ \varphi \in H^2(\Omega_1, \Delta^2) ,$$

which together with Lemma 2.2 implies $\chi = 0$.

□

Let us consider some properties of boundary value problems in the exterior domain Ω_2. The traces of functions given outside of Ω_1 are defined such that for any $\varphi \in C_0^\infty(I\!R^2)$ there holds

$$\gamma(\varphi|_{\Omega_2}) = \gamma(\varphi|_{\Omega_1}) \quad , \quad \delta_\sigma(\varphi|_{\Omega_2}) = \delta_\sigma(\varphi|_{\Omega_1}) .$$

Hence, if $\widetilde{\Omega}$ denotes a domain containing $\overline{\Omega}_1$, $u \in H^2(\widetilde{\Omega}\backslash\Omega_1, \Delta^2)$ and $v \in H^2(\widetilde{\Omega}\backslash\Omega_1)$ then we have

$$[\delta_\sigma u, \gamma v] := \int\limits_{\widetilde{\Omega}\backslash\Omega_1} \Big((\varphi v)\, \Delta^2 u - \sigma\, \Delta(\varphi v)\, \Delta u - (1-\sigma) \sum_{j,k=1}^{2} \partial_j\partial_k u\, \partial_j\partial_k(\varphi v) \Big) dx ,$$

where $\varphi \in C_0^\infty(\widetilde{\Omega})$ with $\varphi \equiv 1$ on a neighbourhood of $\overline{\Omega}_1$.

We define the Hilbert space $W^2(\Omega_2)$ which is a special case in a family of weighted Sobolev spaces studied in [13] and allows the variational formulation of exterior boundary value problems for the biharmonic equation. We denote $\rho(r) = \log(2 + r^2)$ and introduce

$$W^2(\Omega_2) := \Big\{ u : \frac{u}{(1+|x|^2)\rho(|x|)} , \frac{\partial_j u}{(1+|x|^2)^{1/2}\rho(|x|)} , \partial_j\partial_k u \in L^2(\Omega_2) , \, j,k = 1,2 \Big\} ,$$

$$W_0^2(\Omega_2) := \ \text{closure of } C_0^\infty(\Omega_2) \text{ in } W^2(\Omega_2) .$$

equipped with the canonical norm.

It is proved in [13] that the seminorm

$$|u|_{W^2(\Omega_2)} = \Big(\sum_{j,k=1}^{2} \|\partial_j \partial_k u\|^2_{L^2(\Omega_2)} \Big)^{1/2}$$

is a norm on $W_0^2(\Omega_2)$ and on the factor space $W^2(\Omega_2)/\mathbb{P}_1$ equivalent to the corresponding induced norms. Hence the bilinear form

$$a^\sigma_{\Omega_2}(u,v) := \int\limits_{\Omega_2} \Big(\sigma \Delta u \, \Delta v + (1-\sigma) \sum_{j,k=1}^{2} \partial_j \partial_k u \, \partial_j \partial_k v \Big) \, dx \tag{2.23}$$

is positive definite on $W_0^2(\Omega_2)$ and, for $0 \le \sigma < 1$, on $\mathcal{H}^2(\Omega_2)$, where we use the notations $\mathcal{H}^2(\Omega_2) = W^2(\Omega_2)/\mathbb{P}_1$ and $\|\dot{u}\|_{\mathcal{H}^2(\Omega_2)} := |u|_{W^2(\Omega_2)}$. Furthermore, for $u \in W^2(\Omega_2)$ with $\Delta^2 u = 0$ and $0 \le \sigma < 1$ there holds

$$[\delta_\sigma u, \gamma u] = - \, a^\sigma_{\Omega_2}(u,u) \le (\sigma - 1) \, |u|^2_{W^2(\Omega_2)} \, , \tag{2.24}$$

such that analogously to Lemma 2.5 we obtain

Lemma 2.7. *Let $\dot{u} \in \mathcal{H}^2(\Omega_2)$ with $\Delta^2 \dot{u} = 0$ and $0 \le \sigma < 1$. There exist constants not depending on $\dot{u}$ such that*

$$c_1 \, \|\dot{u}\|^2_{\mathcal{H}^2(\Omega_2)} \le \|\delta_\sigma \dot{u}\|_{(W(\Gamma))'} \le c_2 \, \|\dot{u}\|^2_{\mathcal{H}^2(\Omega_2)} \, .$$

Quite analogously to the interior problems the following assertions holds.

Lemma 2.8. *For any $\psi \in V(\Gamma)$ the weak formulation of the Dirichlet problem*

$$\gamma u = \psi \quad , \quad a^\sigma_{\Omega_2}(u,v) = 0 \, , \quad \forall \, v \in W_0^2(\Omega_2) \, ,$$

has a unique solution $u \in W^2(\Omega_2)$. The exterior Neumann problem

$$a^\sigma_{\Omega_2}(u,v) = -[\chi, \gamma v] \, , \quad \forall \, v \in W^2(\Omega_2) \, ,$$

has a solution $u \in W^2(\Omega_2)$ if and only if $\chi \in l\,(\Gamma)^\perp \subset (V(\Gamma))'$. The corresponding equivalence class $\dot{u} \in \mathcal{H}^2(\Omega_2)$ is unique.

3. Layer potentials for the bi-Laplacian

Here we consider the biharmonic layer potentials, which are based on the fundamental solution of the bi–Laplacian Δ^2

$$G(x) := \frac{1}{8\pi} |x|^2 \log |x| \, , \quad x \in \mathbb{R}^2 \, ,$$

and are associated with the form a^σ. Note that the operator

$$\mathcal{G}u(x) := \langle G(x,\cdot), u \rangle_{\mathbb{R}^2} \quad \text{ with } \; G(x,y) = G(x-y)$$

is the two–sided inverse of Δ^2 on the space of compactly supported distributions on $\mathbb{R}^2$. Furthermore,

$$\mathcal{G} : H^s_{comp}(\mathbb{R}^2) \to H^{s+4}_{loc}(\mathbb{R}^2) \, , \quad s \in \mathbb{R} \, , \tag{3.1}$$

is continuous. We have the following representation formula which follows immediately from the special case $\sigma = 1$.

Lemma 3.1. *Let $u \in L^2(\mathbb{R}^2)$ be a function with compact support such that the restrictions $u|_{\Omega_1} \in H^2(\Omega_1)$, $u|_{\Omega_2} \in H^2_{loc}(\Omega_2)$ and $f = \Delta^2 u|_{\mathbb{R}^2 \setminus \Gamma} \in L^2(\mathbb{R}^2)$. Then for $x \in \mathbb{R}^2 \setminus \Gamma$ the representation*

$$u(x) = \mathcal{G}f(x) - [\{\delta_\sigma u\}, \gamma G(x,\cdot)] + [\delta_\sigma G(x,\cdot), \{\gamma u\}]$$

holds, where

$$\{\gamma u\} := \gamma(u|_{\Omega_2}) - \gamma(u|_{\Omega_1}) \quad , \quad \{\delta_\sigma u\} := \delta_\sigma(u|_{\Omega_2}) - \delta_\sigma(u|_{\Omega_1}) \, ,$$

denote the jumps of the Cauchy data across Γ.

This leads to the definition of the layer potentials for $x \in I\!R^2 \backslash \Gamma$

$$\begin{aligned} \mathcal{V}\chi(x) &:= [\chi, \gamma G(x,\cdot)]\,, \qquad && \chi \in (V(\Gamma))'\,, \\ \mathcal{K}_\sigma\psi(x) &:= [\delta_\sigma G(x,\cdot), \psi]\,, \qquad && \psi \in V(\Gamma)\,. \end{aligned} \tag{3.2}$$

Lemma 3.2. *The layer potentials*

$$\mathcal{V} : (V(\Gamma))' \to H^2_{loc}(I\!R^2) \quad \textit{and} \quad \mathcal{K}_\sigma : V(\Gamma) \to H^2(\Omega_1)\,,$$

are continuous.

Proof: Because of

$$\mathcal{V}\chi(x) = \langle G(x,\cdot), \gamma'\chi\rangle_{I\!R^2}$$

we can write

$$\mathcal{V}\chi = \mathcal{G}\gamma'\chi\,. \tag{3.3}$$

The adjoint of the trace map $\gamma' : (V(\Gamma))' \to H^{-2}_{comp}(I\!R^2)$ is continuous, therefore the assertion for $\mathcal{V}$ follows from (3.1).

Due to Lemma 3.1 the solution $u = T(0,\psi)$ of the Dirichlet problem (2.5) can be represented by

$$T(0,\psi) = \mathcal{V}\delta_\sigma T(0,\psi) - \mathcal{K}_\sigma\psi\,,$$

such that Lemma 2.4 and the continuity of T imply

$$\|\mathcal{K}_\sigma\psi\|_{H^2(\Omega_1)} \le c\|\psi\|_{V(\Gamma)}\,.$$

□

Note that the definitions (2.1) and (3.2) lead to the known representations of $\mathcal{V}$ and $\mathcal{K}_\sigma$ as integral operators (cf [16], [11]). If the components of the vector $\chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ are integrable functions then we have

$$\begin{aligned} \mathcal{V}\chi(x) = &-\frac{1}{8\pi}\int_\Gamma v_1(y)\,|x-y|^2\,\log|x-y|\,ds_y \\ &+\frac{1}{8\pi}\int_\Gamma v_2(y)\,(n_y, y-x)\big(2\log|x-y|+1\big)\,ds_y\,. \end{aligned} \tag{3.4}$$

Using (2.16) we obtain that for $\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma)$ the potential $\mathcal{K}_\sigma\psi$ is the sum of two integrals and of a finite number of functions depending on the point values at the corners $v_1(x^i)$

$$\begin{aligned} \mathcal{K}_\sigma\psi(x) = &\int_\Gamma v_2(y)\,M_{\sigma,y}G(x,y)\,ds_y - \int_\Gamma v_1(y)\,N_{\sigma,y}G(x,y)\,ds_y \\ &-\frac{1-\sigma}{4\pi}\sum_{i=1}^m v_1(x^i)\sin\alpha_i\left(1 - \frac{2(n_i, x-x^i)^2}{|x-x^i|^2}\right)\,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} M_{\sigma,y}G(x,y) &= \frac{1+\sigma}{4\pi}\big(\log|x-y|+1\big) + \frac{1-\sigma}{4\pi}\left(\frac{(n_y,y-x)^2}{|x-y|^2} - \frac{1}{2}\right)\,, \\ N_{\sigma,y}G(x,y) &= \frac{1+\sigma}{4\pi}\,\frac{(n_y,y-x)}{|x-y|^2} + \frac{1-\sigma}{2\pi}\left(\frac{(n_y,y-x)^3}{|x-y|^4} - \kappa_y\left(\frac{(n_y,y-x)^2}{|x-y|^2} - \frac{1}{2}\right)\right)\,. \end{aligned}$$

Here κ_y denotes the curvature of Γ at the boundary point y, $\kappa = d\varphi/ds$, where φ is the angle between the x_1–axis and n_y.

Let us define the linear spaces

$$L^\sigma_j := \{u(x) = \mathcal{V}\chi(x) - \mathcal{K}_\sigma\psi(x) : (\psi,\chi) \in V(\Gamma)\times(V(\Gamma))',\, x \in \Omega_j\}\,,$$

of biharmonic functions representable via layer potentials. From Lemmas 3.1 and 3.2 we conclude that the space L_1^σ corresponding to the interior domain is independent of σ and coincides with the set of functions $u \in H^2(\Omega_1)$ satisfying $\Delta^2 u = 0$. Moreover, for $u \in L_1$ there holds the representation formula

$$\mathcal{V}\delta_\sigma u(x) - \mathcal{K}_\sigma \gamma u(x) = \begin{cases} u(x) & , \ x \in \Omega_1 \ , \\ 0 & , \ x \in \Omega_2 \ . \end{cases} \tag{3.6}$$

The space L_2^σ consists of functions $u \in H^2_{loc}(\Omega_2)$ characterized by $\Delta^2 u = 0$ and a special asymptotic behaviour at infinity which will be described in the following lemma. To this end we take the functions on $\mathbb{R}^2$

$$g_1(x,y) = 1 \ , \qquad g_2(x,y) = (x,y) \ ,$$
$$g_3(x,y) = |y|^2 \ , \qquad g_4(x,y) = \frac{|y|^2}{2} + (x,y)^2 \ ,$$

denote by $\hat{x} = x/|x|$ the direction of x and introduce

$$\begin{aligned} I_j\chi(x) &= [\chi, \gamma g_j(\hat{x}, \cdot)] \ , \quad \chi \in (V(\Gamma))' \ , \\ J_j^\sigma \psi(x) &= [\delta_\sigma g_j(\hat{x}, \cdot), \psi] \ , \quad \psi \in V(\Gamma) \ , \quad j = 1, \ldots, 4 \ . \end{aligned} \tag{3.7}$$

Note that J_1^σ and J_2^σ vanish, I_1, I_3 and J_3^σ are constants, while I_2, I_4 and J_4^σ depend on the direction of x. Since the asymptotics of the fundamental solution for $|x| = R \to \infty$ can be written in the form (cf. [3])

$$\begin{aligned} G(x,y) = \frac{1}{8\pi}\big(R^2 \log R - g_2(\hat{x},y)(2R\log R + R) + g_3(\hat{x},y)\log R + g_4(\hat{x},y)\big) \\ + O(R^{-1}) \ , \end{aligned} \tag{3.8}$$

the definition (3.2) of the layer potentials implies

Lemma 3.3. *For given $(\psi, \chi) \in V(\Gamma) \times (V(\Gamma))'$ the function*

$$u(x) = \mathcal{K}_\sigma \psi(x) - \mathcal{V}\chi(x)$$

behaves for large $|x| = R$ as

$$\begin{aligned} u(x) = -\frac{1}{8\pi}\Big(&I_1\chi\, R^2 \log R - I_2\chi(x)(2R\log R + R) + (I_3\chi - J_3^\sigma\psi)\log R \\ &+ I_4\chi(x) - J_4^\sigma\psi(x)\Big) + O(R^{-1}) \ . \end{aligned} \tag{3.9}$$

Corollary 3.1. *The operator $\mathcal{K}_\sigma : V(\Gamma) \to W^2(\Omega_2)$ is continuous.*

Now one can prove the representation formula for functions $u \in L_2^\sigma$.

Lemma 3.4. *For $u \in L_2^\sigma$ with Cauchy data $(\gamma u, \delta_\sigma u)$ there holds*

$$\mathcal{K}_\sigma \gamma u(x) - \mathcal{V}\delta_\sigma u(x) = \begin{cases} u(x) & , \ x \in \Omega_2 \ , \\ 0 & , \ x \in \Omega_1 \ . \end{cases} \tag{3.10}$$

Proof. We enclose Ω_1 by a ball B_R with radius $R > |x|$. Then the representation formula (3.6) is valid for the bounded domain $\Omega_2 \cap B_R$ yielding

$$\begin{aligned} u(x) &= \mathcal{K}_\sigma \gamma u(x) - \mathcal{V}\delta_\sigma u(x) \\ &+ \int_{S_R} \big(u\, N_{\sigma,z} G(x,z) - M_{\sigma,z} G(x,z)\, \partial_n u + M_\sigma u\, \partial_{n_z} G(x,z) - G(x,z)\, N_\sigma u\big)\, ds_z \ . \end{aligned}$$

Using the asymptotics (3.9) of $u(z)$ as $R = |z| \to \infty$ and the asymptotics (3.8) of the fundamental solution it was shown in [15] that the integral

$$\int_{S_R} \big(u\, \partial_{n_z} \Delta G(x,z) - \Delta G(x,z)\, \partial_n u + \Delta u\, \partial_{n_z} G(x,z) - G(x,z)\, \partial_n \Delta u\big)\, ds_z \ .$$

converges to 0 as $R \to \infty$. By the same technique one obtains after some lengthy computations that the remaining integral converges to 0, too.

$\square$

Corollary 3.2. *The function $u \in L_2^\sigma$ belongs to the weighted Sobolev space $W^2(\Omega_2)$ if and only if $\delta_\sigma u \in l\,(\Gamma)^\perp$.*

Corollary 3.3. *Let $0 \le \sigma < 1$. If the exterior Neumann problem*

$$\Delta^2 u = 0 \quad \text{in } \Omega_2\,, \qquad \delta_\sigma u = \chi \in (V(\Gamma))'\,, \tag{3.11}$$

has a solution $u \in L_2^\sigma$ then this solution is unique.

Proof: Obviously it suffices to show that $\delta_\sigma u = 0$ for $u \in L_2^\sigma$ implies $u = 0$. Due to Lemma 2.7 we have $\|\dot{u}\|^2_{\mathcal{H}^2(\Omega_2)} = 0$, hence $u \in \mathbb{P}_1$. But in view of the asymptotics (3.9) this is only possible if $u = 0$.

$\square$

We note that the exterior Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_2\,, \qquad \gamma u = \psi \in V(\Gamma)\,, \tag{3.12}$$

is not uniquely solvable in L_2^σ, in general. For example, the two biharmonic functions

$$u_j(x_1, x_2) = x_j\big(2\log|x| + 1 + e^{-2}\,|x|^{-2}\big)$$

have vanishing trace $\gamma u_j = 0$ on the circle Γ with radius e^{-1}, whereas for any circle Γ with radius $r \ne e^{-1}$ the problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_2\,, \qquad \gamma u = 0\,, \tag{3.13}$$

has only the trivial solution.

In the following we say that the curve Γ satisfies the assumption $\mathbf{A}_I$ if the corresponding exterior homogeneous Dirichlet problem (3.13) has only the trivial solution, or equivalently

$$\mathbf{A}_I: \qquad u \in L_2^\sigma \quad \text{with} \quad \gamma u = 0 \quad \text{implies} \quad \delta_\sigma u = 0\,.$$

Recently Costabel and Dauge proved in [6] that for any general curve Γ there exist between 1 and 4 values of the scaling factor $\rho > 0$ such that the scaled curve $\rho\Gamma = \{\rho x \in \mathbb{R}^2,\ x \in \Gamma\}$ violates assumption $\mathbf{A}_I$.

The layer potentials provide the following jump relations:

Lemma 3.5.

$$\begin{aligned}
\{\gamma \mathcal{V}\chi\} &= 0\,, \quad \{\delta_\sigma \mathcal{V}\chi\} = -\chi && \text{for all } \chi \in (V(\Gamma))'\,,\\
\{\gamma \mathcal{K}_\sigma\psi\} &= \psi\,, \quad \{\delta_\sigma \mathcal{K}_\sigma\psi\} = 0 && \text{for all } \psi \in V(\Gamma)\,.
\end{aligned}$$

Proof: Since $u = \mathcal{V}\chi \in H^2_{loc}(\mathbb{R}^2)$ we have $\gamma(u|_{\Omega_1}) = \gamma(u|_{\Omega_2})$. Further, from (3.3) we obtain that $\Delta^2 u = \gamma'\chi$ in distributional sense, i.e.

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi\,dx = \langle\gamma'\chi, \varphi\rangle_{\mathbb{R}^2} = [\chi, \gamma\varphi]$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$. On the other hand

$$\begin{aligned}
\int_{\Omega_1} u\,\Delta^2\varphi\,dx &= a^\sigma_{\Omega_1}(u,\varphi) - [\delta_\sigma\varphi, \gamma u] = [\delta_\sigma(u|_{\Omega_1}), \gamma\varphi] - [\delta_\sigma\varphi, \gamma u]\,,\\
\int_{\Omega_2} u\,\Delta^2\varphi dx &= -a^\sigma_{\Omega_2}(u,\varphi) + [\delta_\sigma\varphi, \gamma u] = -[\delta_\sigma(u|_{\Omega_2}), \gamma\varphi] + [\delta_\sigma\varphi, \gamma u]\,.
\end{aligned}$$

Thus

$$[\chi, \gamma\varphi] = -[\delta_\sigma(\mathcal{V}\chi|_{\Omega_2}) - \delta_\sigma(\mathcal{V}\chi|_{\Omega_1}), \gamma\varphi]\,, \quad \forall\varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Let now $u = \mathcal{K}_\sigma\psi$, $\psi \in V(\Gamma)$, and again $\varphi \in C_0^\infty(\mathbb{R}^2)$. The second Green formula yields

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi\,dx = -[\{\delta_\sigma u\}, \gamma\varphi] + [\delta_\sigma\varphi, \{\gamma u\}]\,. \tag{3.14}$$

The definition of $\mathcal{K}_\sigma$ provides

$$u = \mathcal{K}_\sigma\psi = \mathcal{G}\delta'_\sigma\psi\,, \tag{3.15}$$

where $\delta'_\sigma\psi$ denotes the compactly supported distribution on $\mathbb{R}^2$ defined by

$$\langle\varphi, \delta'_\sigma\psi\rangle_{\mathbb{R}^2} = [\delta_\sigma\varphi, \psi] \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

So $\Delta^2 u = \delta'_\sigma\psi$ in distributional sense, therefore

$$\int_{\mathbb{R}^2} u\,\Delta^2\varphi\,dx = [\delta_\sigma\varphi, \psi]\,. \tag{3.16}$$

Comparing (3.14) and (3.16) we obtain

$$[\delta_\sigma\varphi, \psi - \{\gamma u\}] = -[\{\delta_\sigma u\}, \gamma\varphi] \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^2)\,.$$

Thus from (2.22) we conclude that

$$\{\gamma\mathcal{K}_\sigma\psi\} - \psi = 0 = \{\delta_\sigma\mathcal{K}_\sigma\psi\}\,.$$

□

4. Boundary integral operators for the bi-Laplacian

In this sction we study some basic properties of boundary integral operators connected with the biharmonic layer potentials. These operators are defined as the traces

$$\begin{array}{ll} \mathcal{A}\chi := 2\,\gamma\mathcal{V}\chi \quad , & \mathcal{B}_\sigma\chi := 2\,\delta_\sigma(\mathcal{V}\chi|_{\Omega_1})\,, \\ \mathcal{C}_\sigma\psi := 2\,\gamma(\mathcal{K}_\sigma\psi|_{\Omega_1})\,, & \mathcal{D}_\sigma\psi := -2\,\delta_\sigma(\mathcal{K}_\sigma\psi|_{\Omega_1})\,. \end{array}$$

Formally this definition is the same as for second order equations given in [5]. We will show that the biharmonic boundary integral operators have analogous properties as the corresponding operators of the Laplacian.

Lemma 4.1. ([6],[15]) *The operator $\mathcal{A} : (V(\Gamma))' \to V(\Gamma)$ is continuous, symmetric and strongly elliptic, it is positive definite on $(W(\Gamma))'$, i.e. for any $\chi \in (W(\Gamma))' = l\,(\Gamma)^\perp$ there holds*

$$[\chi, \mathcal{A}\chi] \geq c\,\|\chi\|^2_{(V(\Gamma))'}$$

with a positive constant not depending on χ. If additionally the curve Γ satisfies the assumption $\mathbf{A}_I$ then $\mathcal{A}$ is bijective.

Here and in the following the adjoints of boundary integral operators are taken of course with respect to the duality (2.1).

Lemma 4.2. *Let $0 \leq \sigma < 1$. The continuous operator $\mathcal{D}_\sigma : V(\Gamma) \to (V(\Gamma))'$ is symmetric and strongly elliptic,* ker $\mathcal{D}_\sigma = l\,(\Gamma)$ *and* im $\mathcal{D}_\sigma = l\,(\Gamma)^\perp$*. Moreover, the isomorphism $\mathcal{D}_\sigma : W(\Gamma) \to (W(\Gamma))'$ is positive definite.*

Proof : Note first that the boundedness and symmetry of $\mathcal{D}_\sigma$ follows immediately from Lemmas 2.2, 2.4, 3.2 and the symmetry of the kernel function G. To prove that $\mathcal{D}_\sigma$ is positive definite we take $\psi \in V(\Gamma)$ and set $u_1 = -\mathcal{K}_\sigma\psi|_{\Omega_1}$, $u_2 = -\mathcal{K}_\sigma\psi|_{\Omega_2}$. The jump relations lead to

$$\delta_\sigma u_1 = \delta_\sigma u_2 = \frac{1}{2}\mathcal{D}_\sigma\psi\,, \quad \gamma u_2 - \gamma u_1 = -\psi\,.$$

Due to Corollary 3.1 we have $|u|^2_{W^2(\Omega_2)} < \infty$, such that by (2.21) and (2.24)

$$\frac{1}{2}\left[\mathcal{D}_\sigma\psi,\psi\right] = [\delta_\sigma u_1, \gamma u_1] - [\delta_\sigma u_2, \gamma u_2] = a^\sigma_{\Omega_1}(u_1,u_1) + a^\sigma_{\Omega_2}(u_2,u_2)$$
$$\geq (1-\sigma)\big(\|u_1\|^2_{\mathcal{H}^2(\Omega_1)} + \|u_2\|^2_{\mathcal{H}^2(\Omega_2)}\big)\ .$$

Since

$$\|\psi\|_{W(\Gamma)} \leq \|\gamma u_1\|_{W(\Gamma)} + \|\gamma u_2\|_{W(\Gamma)} \leq \|u_1\|_{\mathcal{H}^2(\Omega_1)} + \|u_2\|_{\mathcal{H}^2(\Omega_2)}$$

we obtain

$$[\mathcal{D}_\sigma\psi,\psi] \geq c_\sigma\,\|\psi\|^2_{W(\Gamma)}\ ,$$

hence $\mathcal{D}_\sigma$ is strongly elliptic in $V(\Gamma)$. From (2.20) it is clear that $\ker\mathcal{D}_\sigma = l\,(\Gamma)$.

□

Lemma 4.3. *The boundary operators $\mathcal{C}_\sigma : V(\Gamma) \to V(\Gamma)$ and $\mathcal{B}_\sigma : (V(\Gamma))' \to (V(\Gamma))'$ are continuous and connected by the relation $\mathcal{B}'_\sigma = \mathcal{C}_\sigma + 2\,I$.*

Proof: For any $(\psi,\chi) \in V(\Gamma) \times (V(\Gamma))'$ we obtain

$$\begin{aligned}[\mathcal{B}_\sigma\chi,\psi] &= [\delta_\sigma(\mathcal{V}\chi|_{\Omega_1}) + \delta_\sigma(\mathcal{V}\chi|_{\Omega_2}) + \chi,\psi] \\ &= \langle \mathcal{G}\gamma'\chi|_{\Omega_1} + \mathcal{G}\gamma'\chi|_{\Omega_2}, \delta'_\sigma\psi\rangle_{\mathbb{R}^2} + [\chi,\psi] = \langle \mathcal{G}\gamma'\chi, \delta'_\sigma\psi\rangle_{\mathbb{R}^2} + [\chi,\psi] \\ &= \langle \gamma'\chi, \mathcal{K}_\sigma\psi\rangle_{\mathbb{R}^2} + [\chi,\psi] = [\chi, \gamma(\mathcal{K}_\sigma\psi|_{\Omega_1}) + \gamma(\mathcal{K}_\sigma\psi|_{\Omega_2})] + [\chi,\psi] \\ &= [\chi, 2\gamma(\mathcal{K}_\sigma\psi|_{\Omega_1}) + \psi] + [\chi,\psi] = [\chi,\mathcal{C}_\sigma\psi] + 2[\chi,\psi]\ ,\end{aligned}$$

where the jump relations for $\mathcal{V}\chi$ and $\gamma\mathcal{K}_\sigma\psi$ as well as (3.15) are used.

□

If we introduce the operator $\mathcal{W}_\sigma := I + \mathcal{C}_\sigma$ then $\mathcal{B}_\sigma = I + \mathcal{W}'_\sigma$ and from Lemma 3.5 we derive for $j = 1,2$

$$\gamma(\mathcal{K}_\sigma\psi|_{\Omega_j}) = \frac{1}{2}\left(\mathcal{W}_\sigma + (-1)^j I\right)\psi\ \ ,\ \ \delta_\sigma(\mathcal{V}\chi|_{\Omega_j}) = \frac{1}{2}\left(\mathcal{W}'_\sigma - (-1)^j I\right)\chi\ . \tag{4.1}$$

Let us mention that in the special case $\sigma = 1$, where the form a^σ is not coercive, we obtained the following characterizations in [15]:

- The operator $\frac{1}{2}\left(I - \mathcal{W}_1\right) = -\frac{1}{2}\,\mathcal{C}_1$ is the Calderon projection onto the traces γu of harmonic functions $u \in H^2(\Omega_1)$;
- The operator $\frac{1}{2}\left(I + \mathcal{W}_1\right) = \frac{1}{2}\,\mathcal{B}'_1$ projects onto the traces γu of all harmonic functions $u \in H^2_{loc}(\Omega_2)$ with the asymptotics $u(x) = a(\log|x| + 1) + O(|x|^{-1})$, $|x| \to \infty$, for some real a;
- The operator $\mathcal{D}_1 = 0$.

Now we introduce the bounded linear operator

$$\mathfrak{B}_\sigma := \begin{pmatrix} -\mathcal{W}_\sigma & \mathcal{A} \\ \mathcal{D}_\sigma & \mathcal{W}'_\sigma \end{pmatrix} : \begin{matrix} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{matrix} \quad\to\quad \begin{matrix} V(\Gamma) \\ \times \\ (V(\Gamma))' \end{matrix} \tag{4.2}$$

and define two mappings

$$\mathfrak{C}_{\sigma,j} := \frac{1}{2}\left(I - (-1)^j\mathfrak{B}_\sigma\right)\ ,\quad j = 1,2\ . \tag{4.3}$$

Lemma 4.4. *The operators $\mathfrak{C}_{\sigma,j}$ are the Calderon projections in $V(\Gamma) \times (V(\Gamma))'$ mapping onto the set of Cauchy data $(\gamma u, \delta_\sigma u)$ of functions $u \in L^\sigma_j$.*

Proof: For arbitrary $(\psi,\chi) \in V(\Gamma) \times (V(\Gamma))'$ and $u = (-1)^j(\mathcal{K}_\sigma\psi - \mathcal{V}\chi) \in L^\sigma_j$ the jump relations of Lemma 3.5 and (4.1) imply

$$\begin{pmatrix}\gamma u \\ \delta_\sigma u\end{pmatrix} = (-1)^j\begin{pmatrix}\gamma(\mathcal{K}_\sigma\psi|_{\Omega_j}) - \gamma(\mathcal{V}\chi|_{\Omega_j}) \\ \delta_\sigma(\mathcal{K}_\sigma\psi|_{\Omega_j}) - \delta_\sigma(\mathcal{V}\chi|_{\Omega_j})\end{pmatrix} = \frac{(-1)^j}{2}\begin{pmatrix}(\mathcal{W}_\sigma + (-1)^j I)\psi - \mathcal{A}\chi \\ -\mathcal{D}_\sigma\psi - (\mathcal{W}'_\sigma - (-1)^j I)\chi\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} I + (-1)^j\mathcal{W}_\sigma & -(-1)^j\mathcal{A} \\ -(-1)^j\mathcal{D}_\sigma & I - (-1)^j\mathcal{W}'_\sigma\end{pmatrix}\begin{pmatrix}\psi \\ \chi\end{pmatrix} = \frac{1}{2}\left(I - (-1)^j\mathfrak{A}_\sigma\right)\begin{pmatrix}\psi \\ \chi\end{pmatrix} = \mathfrak{C}_{\sigma,j}\begin{pmatrix}\psi \\ \chi\end{pmatrix}\ .$$

Let now $u \in L_j^\sigma$. Then the representation formula (3.6) or (3.10), respectively, yields

$$u(x) = (-1)^j (\mathcal{K}_\sigma \gamma u(x) - \mathcal{V}\delta_\sigma u(x)) , \quad x \in \Omega_j ,$$

after applying the jump relations we obtain

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \mathfrak{C}_{\sigma,j} \begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} ,$$

showing that the mappings $\mathfrak{C}_{\sigma,j}$ are bounded projections and that the Cauchy data of all functions from L_j^σ belong to the image of $\mathfrak{C}_{\sigma,j}$.

□

Since the Calderon projections for the interior and exterior problems are conjugate

$$\mathfrak{C}_{\sigma,1} + \mathfrak{C}_{\sigma,2} = I ,$$

the space $V(\Gamma) \times (V(\Gamma))'$ can be decomposed as the direct sum of closed subspaces

$$V(\Gamma) \times (V(\Gamma))' = \{(\gamma u, \delta_\sigma u) : u \in L_1\} \dot{+} \{(\gamma u, \delta_\sigma u) : u \in L_2^\sigma\} . \tag{4.4}$$

Further, from $\mathfrak{C}_{\sigma,j}^2 = \mathfrak{C}_{\sigma,j}$ we derive the relations

Corollary 4.1.

$$\mathcal{W}_\sigma \mathcal{A} = \mathcal{A} \mathcal{W}'_\sigma \quad , \quad \mathcal{W}'_\sigma \mathcal{D}_\sigma = \mathcal{D}_\sigma \mathcal{W}_\sigma \quad , \quad \mathcal{A} \mathcal{D}_\sigma = I - \mathcal{W}_\sigma^2 . \tag{4.5}$$

Lemma 4.5. *Let* $0 \le \sigma < 1$. *The operator* $(I - \mathcal{W}_\sigma) : V(\Gamma) \to V(\Gamma)$ *is bijective, whereas* $(I + \mathcal{W}_\sigma) : V(\Gamma) \to V(\Gamma)$ *is Fredholm with index zero. Furthermore,* $\ker\,(I + \mathcal{W}_\sigma) = l\,(\Gamma)$ *and* $\mathrm{im}\,(I + \mathcal{W}_\sigma) = \mathcal{A}(l\,(\Gamma)^\perp)$.

Proof: From (4.5) we have

$$\mathcal{A} \mathcal{D}_\sigma = (I + \mathcal{W}_\sigma)(I - \mathcal{W}_\sigma) = (I - \mathcal{W}_\sigma)(I + \mathcal{W}_\sigma) . \tag{4.6}$$

Since $\mathcal{A}$ and $\mathcal{D}_\sigma$, $0 \le \sigma < 1$, are strongly elliptic the operator $\mathcal{A}\mathcal{D}_\sigma$ is Fredholm with index zero and by well-known arguments (cf. [14], Thms. 1.3.1 and 1.3.3) the operators $(I \pm \mathcal{W}_\sigma)$ are Fredholm itself. Based on the relations (4.1) one can use as in the case of Laplace's equation the uniqueness of the interior Dirichlet problem in L_1 and of the exterior Neumann problem in L_2^σ to derive that

$$\ker\,(I - \mathcal{W}_\sigma) = \ker\,(I - \mathcal{W}'_\sigma) = 0 . \tag{4.7}$$

Therefore $(I + \mathcal{W}_\sigma)$ is Fredholm with index 0, from (4.6) its kernel and image can be determined by using Lemmas 4.1 and 4.2.

□

5. Boundary integral equations for plate bending problems

Using the layer potentials and boundary integral operators it is now quite easy to transform biharmonic boundary value problems into integral equations over the boundary. For example, the results of Sections 2 and 3 and certain layer potential representations lead immediately to equivalent integral equations for Dirichlet and Neumann problems. However, the analysis of indirect methods for other types of boundary conditions seems to be more involved. Here we fix attention on a direct method which produces strongly elliptic systems of boundary integral equations equivalent to mixed biharmonic boundary value problems. Having the properties of boundary integral operators at hand the analysis of the proposed method extends simply the well–studied approach for second order equations to our situation.

We introduce the bounded bilinear form on $V(\Gamma) \times (V(\Gamma))'$

$$\left\langle \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \begin{pmatrix} \rho \\ \tau \end{pmatrix} \right\rangle_{V(\Gamma)\times(V(\Gamma))'} := [\tau, \psi] + [\chi, \rho] . \tag{5.1}$$

From (4.2) we see that for any $(\psi,\chi)\in V(\Gamma)\times(V(\Gamma))'$ the equality

$$\left\langle \mathfrak{B}_\sigma \begin{pmatrix}\psi\\ \chi\end{pmatrix}, \begin{pmatrix}\psi\\ \chi\end{pmatrix}\right\rangle_{V(\Gamma)\times(V(\Gamma))'} = -[\chi,\mathcal{W}_\sigma\psi]+[\chi,\mathcal{A}\chi]+[\mathcal{D}_\sigma\psi,\psi]+[\mathcal{W}'_\sigma\chi,\psi] \\ = [\chi,\mathcal{A}\chi]+[\mathcal{D}_\sigma\psi,\psi] \tag{5.2}$$

holds. Let us denote by $\mathcal{P}: V(\Gamma)\to V(\Gamma)$ a bounded projection, set $\mathcal{Q}=I-\mathcal{P}$ and introduce the projection $\mathfrak{P}$ in $V(\Gamma)\times(V(\Gamma))'$ by

$$\mathfrak{P} := \begin{pmatrix}\mathcal{P} & 0\\ 0 & \mathcal{Q}'\end{pmatrix} : \begin{matrix} V(\Gamma)\\ \times \\ (V(\Gamma))'\end{matrix} \rightarrow \begin{matrix}\text{im }\mathcal{P}\\ \times\\ \text{im }\mathcal{Q}'\end{matrix} . \tag{5.3}$$

Note that

$$\text{im }\mathcal{P}\times\text{im }\mathcal{Q}' = \text{im }\mathfrak{P} \quad\text{and}\quad \text{im }\mathcal{Q}\times\text{im }\mathcal{P}' = \text{im }(I-\mathfrak{P})$$

are closed subspaces of $V(\Gamma)\times(V(\Gamma))'$ which are in duality with respect to (5.1). Since $(\text{im }\mathcal{Q})^\perp = (\text{ker }\mathcal{P})^\perp = \text{im }\mathcal{P}'$ the equality (5.2) leads to

$$\left\langle \mathfrak{C}_{\sigma,j}\begin{pmatrix}\mathcal{Q}\psi\\ \mathcal{P}'\chi\end{pmatrix}, \begin{pmatrix}\mathcal{Q}\psi\\ \mathcal{P}'\chi\end{pmatrix}\right\rangle_{V(\Gamma)\times(V(\Gamma))'} = \frac{1}{2}\left\langle (I-(-1)^j\mathfrak{B}_\sigma)\begin{pmatrix}\mathcal{Q}\psi\\ \mathcal{P}'\chi\end{pmatrix}, \begin{pmatrix}\mathcal{Q}\psi\\ \mathcal{P}'\chi\end{pmatrix}\right\rangle_{V(\Gamma)\times(V(\Gamma))'} \\ = \frac{(-1)^{j+1}}{2}\big([\mathcal{A}\mathcal{P}'\chi,\mathcal{P}'\chi]+[\mathcal{D}_\sigma\mathcal{Q}\psi,\mathcal{Q}\psi]\big) .$$

Hence for any projection $\mathcal{P}$ the mappings

$$\mathfrak{A}^{\mathcal{P}}_\sigma := (-1)^{j+1}\,\mathfrak{P}\,\mathfrak{C}_{\sigma,j}\,(I-\mathfrak{P}) : \begin{matrix}\text{im }\mathcal{Q}\\ \times\\ \text{im }\mathcal{P}'\end{matrix} \longrightarrow \begin{matrix}\text{im }\mathcal{P}\\ \times\\ \text{im }\mathcal{Q}'\end{matrix} \tag{5.4}$$

do not depend on $j=1,2$. If $0\le\sigma<1$ then in view of Lemmas 4.1 and 4.2 the operator $\mathfrak{A}^{\mathcal{P}}_\sigma$ satisfies a Gårding inequality

$$\left\langle (\mathfrak{A}^{\mathcal{P}}_\sigma+\mathfrak{T})\begin{pmatrix}\psi\\ \chi\end{pmatrix}, \begin{pmatrix}\psi\\ \chi\end{pmatrix}\right\rangle_{V(\Gamma)\times(V(\Gamma))'} \ge c\left(\|\psi\|^2_{V(\Gamma)}+\|\chi\|^2_{(V(\Gamma))'}\right) , \\ \forall\,(\psi,\chi)\in\text{im }\mathcal{Q}\times\text{im }\mathcal{P}' ,$$

with a positive constant c and some compact operator $\mathfrak{T}$. Since the adjoint of $\mathfrak{A}^{\mathcal{P}}_\sigma$ with respect to the form (5.1) provides the same property we derive

Lemma 5.1. *Let $0\le\sigma<1$ and $\mathcal{P}$ be a bounded projection in $V(\Gamma)$. Then $\mathfrak{A}^{\mathcal{P}}_\sigma$ defined in* (5.4) *is a Fredholm operator with index zero from* im $(I-\mathfrak{P})$ *into* im $\mathfrak{P}$ *and strongly elliptic with respect to the form* (5.1).

Note that the two trivial cases $\mathcal{P}=I$ and $\mathcal{P}=0$ are treated in Lemmas 4.1 and 4.2, respectively.

The mapping $\mathfrak{A}^{\mathcal{P}}_\sigma$ is closely connected with the biharmonic boundary value problem:

$$\text{Find}\quad u\in L^\sigma_j \quad\text{such that}\quad \mathcal{P}\gamma u=\rho \quad\text{and}\quad \mathcal{Q}'\delta_\sigma u=\tau , \tag{5.5}$$

where $(\rho,\tau)\in\text{im }\mathcal{P}\times\text{im }\mathcal{Q}'$ are the given boundary values. Indeed, for $u\in L^\sigma_j$ we know from Lemma 5.1 that

$$\mathfrak{C}_{\sigma,j}\begin{pmatrix}\gamma u\\ \delta_\sigma u\end{pmatrix} = \begin{pmatrix}\gamma u\\ \delta_\sigma u\end{pmatrix} .$$

To solve (5.5) we decompose

$$\begin{pmatrix}\gamma u\\ \delta_\sigma u\end{pmatrix} = \begin{pmatrix}\mathcal{P}\gamma u\\ \mathcal{Q}'\delta_\sigma u\end{pmatrix} + \begin{pmatrix}\mathcal{Q}\gamma u\\ \mathcal{P}'\delta_\sigma u\end{pmatrix} ,$$

such that the unknowns $\psi=\mathcal{Q}\gamma u$ and $\chi=\mathcal{P}'\delta_\sigma u$ have to satisfy the equation

$$(I-\mathfrak{C}_{\sigma,j})\begin{pmatrix}\psi\\ \chi\end{pmatrix} = (\mathfrak{C}_{\sigma,j}-I)\begin{pmatrix}\rho\\ \tau\end{pmatrix} . \tag{5.6}$$

In particular, applying the projection $\mathfrak{P}$ to both sides we get the equation

$$\mathfrak{A}_\sigma^{\mathcal{P}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = (-1)^j \, \mathfrak{P} \, (\mathfrak{C}_{\sigma,j} - I) \begin{pmatrix} \rho \\ \tau \end{pmatrix} . \tag{5.7}$$

Lemma 5.2. *Let* $(\rho, \tau) \in \mathrm{im}\ \mathcal{P} \times \mathrm{im}\ \mathcal{Q}'$.

(i) *If* $u \in L_j^\sigma$ *satisfies* (5.5) *then* $\psi = \mathcal{Q}\gamma u$ *and* $\chi = \mathcal{P}'\delta_\sigma u$ *solve the equation* (5.7).

(ii) *If* $(\psi, \chi) \in \mathrm{im}\ \mathcal{Q} \times \mathrm{im}\ \mathcal{P}'$ *is a solution of* (5.7) *then the function* u *given in* Ω_j *by*

$$u = (-1)^j \big(\mathcal{K}_\sigma(\psi + \rho) - \mathcal{V}(\chi + \tau) \big) \tag{5.8}$$

solves the boundary value problem (5.5).

Proof: It remains to show (ii). For u from (5.8) there holds in view of Lemma 4.4

$$\begin{pmatrix} \gamma u \\ \delta_\sigma u \end{pmatrix} = \mathfrak{C}_{\sigma,j} \begin{pmatrix} \psi + \rho \\ \chi + \tau \end{pmatrix} .$$

Since the equation (5.7) is fulfilled we have

$$\mathfrak{P} \, (I - \mathfrak{C}_{\sigma,j}) \begin{pmatrix} \psi + \rho \\ \chi + \tau \end{pmatrix} = 0 ,$$

implying

$$\mathcal{P}\gamma u = \mathcal{P}(\psi + \rho) = \rho \quad \text{and} \quad \mathcal{Q}'\delta_\sigma u = \mathcal{Q}'(\chi + \tau) = \tau .$$

□

Thus any solution of the boundary value problem (5.5) can be obtained by solving the system of boundary integral equations (5.7). Note that in general this system has more linear independent solutions than (5.5).

Lemma 5.3. *Let* $0 \le \sigma < 1$ *and* β_j, $j = 1, 2$, *be the dimension of the null–space of the corresponding homogeneous problem* (5.5) *with* $\rho = \tau = 0$. *Then*

$$\dim \ker \mathfrak{A}_\sigma^{\mathcal{P}} = \beta_1 + \beta_2 \le 3 \quad \textit{and} \quad \beta_1 = \dim\ \mathcal{Q}(l(\Gamma)) .$$

Proof: Since $u \in L_j^\sigma$ with $(\gamma u, \delta_\sigma u) \in \mathrm{im}\ \mathcal{Q} \times \mathrm{im}\ \mathcal{P}'$, i.e. $\mathcal{P}\gamma u = \mathcal{Q}'\delta_\sigma u = 0$, determines an element $(\gamma u, \delta_\sigma u) \in \ker\ \mathfrak{A}_\sigma^{\mathcal{P}}$ and and by (4.4)

$$\{(\gamma u, \delta_\sigma u) : u \in L_1\} \, \cap \, \{(\gamma u, \delta_\sigma u) : u \in L_2^\sigma\} = \emptyset$$

it is clear that $\dim \ker \mathfrak{A}_\sigma^{\mathcal{P}} \ge \beta_1 + \beta_2$.

On the other hand, since $V(\Gamma) \times (V(\Gamma))'$ is the direct sum of these subspaces there exists a basis in $\ker\ \mathfrak{A}_\sigma^{\mathcal{P}}$ consisting of elements of the subspaces. Due to Lemma 5.2 (ii) and the representation formulas (3.6) and (3.10) we get therefore $\dim \ker \mathfrak{A}_\sigma^{\mathcal{P}} \le \beta_1 + \beta_2$.

Let now $u \in L_1$ with $\mathcal{P}\gamma u = \mathcal{Q}'\delta_\sigma u = 0$. Then

$$a_{\Omega_1}^\sigma(u, u) = [\delta_\sigma u, \gamma u] = [\delta_\sigma u, \mathcal{P}\gamma u] + [\mathcal{Q}'\delta_\sigma u, \gamma u] = 0$$

and Lemma 2.5 yields $u \in \mathbb{P}_1$, i.e. $\gamma u \in l\,(\Gamma)$ and $\delta_\sigma u = 0$. Hence the homogeneous boundary conditions can be satisfied by $\beta_1 = \dim\ \mathcal{Q}\,(l(\Gamma))$ linear independent elements of L_1.

Using (2.24), Corollaries 3.2 and 3.3 it can be seen quite similarly that

$$u \in L_2^\sigma \cap W^2(\Omega_2) \quad \text{with} \quad \mathcal{P}\gamma u = \mathcal{Q}'\delta_\sigma u = 0 \quad \text{implies} \quad u = 0 .$$

Hence any nontrivial solution of the homogeneous boundary value problem in the outer domain Ω_2 satisfies $\delta_\sigma u \notin l\,(\Gamma)^\perp$ or more precisely, the corresponding equivalence class $\delta_\sigma \dot{u}$ in the factor space $(V(\Gamma))'/l\,(\Gamma)^\perp$ is different from zero, $\delta_\sigma \dot{u} \ne 0$.

Consequently, if $(\psi, \chi) \in (\mathrm{im}\ \mathcal{Q} \times \mathrm{im}\ \mathcal{P}') \cap \ker \mathfrak{A}_\sigma^{\mathcal{P}}$ and $\chi \ne 0$ then the equivalence class $\dot{\chi} \ne 0$ in $(V(\Gamma))'/l\,(\Gamma)^\perp$. This means that β_2 is not greater than the number of linear independent elements $\chi \in \mathrm{im}\ \mathcal{P}'$ with $\dot{\chi} \ne 0$ which equals to

$$\dim\ \mathcal{P}\,(l(\Gamma)) = 3 - \dim\ \mathcal{Q}\,(l(\Gamma)) = 3 - \beta_1 .$$

□

Now we introduce the assumption

$$\mathbf{A}_{\mathcal{P}}: \text{ If } u \in L_2^\sigma \text{ satisfies } \mathcal{P}\gamma u = 0 \text{ and } (I - \mathcal{P}')\delta_\sigma u = 0 \text{ then } u = 0 .$$

and consider the boundary value problem (5.5) for $j = 2$.

Theorem 5.1. *Suppose that Γ satisfies assumption $\mathbf{A}_{\mathcal{P}}$, let $(\rho, \tau) \in \operatorname{im} \mathcal{P} \times \operatorname{im} \mathcal{Q}'$ and $0 \leq \sigma < 1$. Then the boundary value problem for the bi–Laplacian*

$$\begin{aligned} \Delta^2 u &= 0 \qquad in \ \Omega_2 \\ \mathcal{P}\gamma u &= \rho \ , \ (I - \mathcal{P}')\, \delta_\sigma u = \tau \end{aligned} \tag{5.9}$$

has in the space L_2^σ a unique solution given by

$$u = \mathcal{K}_\sigma(\psi + \rho) - \mathcal{V}(\chi + \tau) ,$$

where $(\psi, \chi) \in \operatorname{im} \mathcal{Q} \times \operatorname{im} \mathcal{P}'$ solve the system of boundary integral equations

$$\mathfrak{A}_\sigma^{\mathcal{P}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = -\mathfrak{P}\, \mathfrak{C}_{\sigma,1} \begin{pmatrix} \rho \\ \tau \end{pmatrix} . \tag{5.10}$$

If additionally the projection $\mathcal{P}$ reproduces the traces of linear functions, $\mathcal{P}\gamma p = \gamma p$ for all $p \in I\!P_1$, then (5.10) is uniquely solvable.

For $j = 1$ the boundary value problem (5.5) admits the variational formulation:

$$\begin{aligned} &\text{Find} \quad u \in H^2(\Omega_1) \quad \text{such that} \quad \mathcal{P}\gamma u = \rho \quad \text{and} \\ &a_{\Omega_1}^\sigma(u, v) = [\tau, \mathcal{Q}\gamma v] , \quad \forall\, v \in H_{\mathcal{P}}^2(\Omega_1) := \{u \in H^2(\Omega_1) : \mathcal{P}\gamma u = 0\} . \end{aligned} \tag{5.11}$$

It is clear that (5.11) is uniquely solvable for $0 \leq \sigma < 1$ iff the only linear function p satisfying $\mathcal{P}\gamma p = 0$ is the trivial function $p = 0$.

Theorem 5.2. *Suppose that the projection $\mathcal{P}$ satisfies $\mathcal{P}(l(\Gamma)) = l(\Gamma)$, let $0 \leq \sigma < 1$ and $(\rho, \tau) \in \operatorname{im} \mathcal{P} \times \operatorname{im} \mathcal{Q}'$. The unique weak solution of the boundary value problem for the bi–Laplacian*

$$\begin{aligned} \Delta^2 u &= 0 \qquad in \ \Omega_1 \\ \mathcal{P}\gamma u &= \rho \ , \ (I - \mathcal{P}')\, \delta_\sigma u = \tau \end{aligned} \tag{5.12}$$

can be obtained by the formula

$$u = \mathcal{V}(\chi + \tau) - \mathcal{K}_\sigma(\psi + \rho) ,$$

where $(\psi, \chi) \in \operatorname{im} \mathcal{Q} \times \operatorname{im} \mathcal{P}'$ solve the system of boundary integral equations

$$\mathfrak{A}_\sigma^{\mathcal{P}} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \mathfrak{P}\, \mathfrak{C}_{\sigma,2} \begin{pmatrix} \rho \\ \tau \end{pmatrix} . \tag{5.13}$$

If Γ satisfies assumption $\mathbf{A}_{\mathcal{P}}$ then (5.13) is uniquely solvable.

Roughly spoken, if the boundary conditions are such that the biharmonic boundary value problem can be transformed into a coercive variational problem then it is equivalent to a strongly elliptic system of boundary integral equations.

As an example we now consider the choice of the projection $\mathcal{P}$ for mixed boundary conditions. We assume that the boundary Γ is composed of four disjoint parts Γ_c, Γ_h, Γ_r, and Γ_f such that

$$\Gamma = \overline{\Gamma_c \cup \Gamma_h \cup \Gamma_r \cup \Gamma_f} ,$$

and consider a bounded projection $\mathcal{P}$ in $V(\Gamma)$ providing

$$\mathcal{P}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in V(\Gamma) \quad \text{with} \quad \begin{cases} w_1 = v_1 \ , \ w_2 = v_2 & \text{on } \Gamma_c \ , \\ w_1 = v_1 & \text{on } \Gamma_h \ , \\ w_2 = v_2 & \text{on } \Gamma_r \ , \end{cases} \tag{5.14}$$

whereas the functions w_j are extended to the other parts of Γ in some specific way. Clearly there exists of variety of projections giving (5.14), which differ only in the method of extending w_j. But the concrete form of the projection $\mathcal{P}$ is not important, we need only the existence of bounded projections, $\|\mathcal{P}\psi\|_{V(\Gamma)} \le c\,\|\psi\|_{V(\Gamma)}$, which is obvious. Since for the adjoint of $\mathcal{Q} = I - \mathcal{P}$ we have

$$\mathcal{Q}'\begin{pmatrix} v_4 \\ v_3 \end{pmatrix} = \begin{pmatrix} z_4 \\ z_3 \end{pmatrix} \in (V(\Gamma))' \quad \text{with} \left\{ \begin{array}{l} \langle v_4, w_1\rangle_\Gamma = \langle z_4, w_1\rangle_\Gamma \\ \langle v_3, w_2\rangle_\Gamma = \langle z_3, w_2\rangle_\Gamma \end{array} \right.$$

$$\text{for all } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \ker \mathcal{P} = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma) : v_1|_{\Gamma_c \cup \Gamma_h} = 0 \;,\; v_2|_{\Gamma_c \cup \Gamma_r} = 0 \right\},$$

we conclude that in weak sense

$$\mathcal{Q}'\begin{pmatrix} v_4 \\ v_3 \end{pmatrix} = \begin{pmatrix} z_4 \\ z_3 \end{pmatrix} \in (V(\Gamma))' \quad \text{with} \left\{ \begin{array}{ll} z_3 = v_3 \;,\; z_4 = v_4 & \text{on } \Gamma_f \;, \\ z_3 = v_3 & \text{on } \Gamma_h \;, \\ z_4 = v_4 & \text{on } \Gamma_r \;. \end{array} \right. \tag{5.15}$$

We note that the space $\ker \mathcal{P} \times \ker \mathcal{Q}'$ in which the unknowns (ψ, χ) of the system (5.6) have to be sought is independent of the concrete choice of $\mathcal{P}$. Moreover, the definition of the trace spaces together with the description of $\ker \mathcal{P} \times \ker \mathcal{Q}'$ imposes certain compatibility conditions for the components of ψ and χ at singular boundary points, i.e. corners and points at which the type of boundary conditions changes. We will not go into detail, we mention only that it is important to take into account these compatibility conditions in choosing the approximation spaces for the numerical solution of (5.6).

If we formulate the boundary conditions in (5.5)

$$\mathcal{P}\gamma u = \rho = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \operatorname{im} \mathcal{P} \quad , \quad \mathcal{Q}'\delta_\sigma u = \tau = \begin{pmatrix} g_4 \\ g_3 \end{pmatrix} \in \operatorname{im} \mathcal{Q}'$$

in terms of the Cauchy data of u, which are defined in (2.17), then we obtain from (5.14) and (5.15) the well–known mixed boundary conditions of plate bending

(i) clamped

$$u = g_1 \;,\; \partial_n u = g_2 \qquad \text{on } \Gamma_c \;;$$

(ii) hinged or simply supported

$$u = g_1 \;,\; M_\sigma u = g_3 \qquad \text{on } \Gamma_h \;;$$

(iii) roller–supported

$$\partial_n u = g_2 \;,\; \widetilde{N}_\sigma u = g_4 \qquad \text{on } \Gamma_r \;;$$

(iv) free

$$M_\sigma u = g_3 \;,\; \widetilde{N}_\sigma u = g_4 \qquad \text{on } \Gamma_f \;.$$

To state the stability result of the Galerkin method for solving the system of integral equations derived from the mixed boundary conditions (i)-(iv) we introduce sequences of finite dimensional spaces of approximating functions $X_h \subset \ker \mathcal{P}$ and $Y_h \subset \ker \mathcal{Q}'$, $h \to 0$, such that

$$\bigcup_h X_h \times Y_h \text{ is dense in } \ker \mathcal{P} \times \ker \mathcal{Q}' \;.$$

Theorem 5.3. *Suppose that $0 \le \sigma < 1$ and that the interior and the exterior boundary value problem for the biharmonic equation with homogeneous boundary conditions* (i)-(iv), *i.e. $g_i = 0$, have only trivial solutions. Then the Galerkin equations*

$$\left\langle \mathfrak{B}_\sigma \begin{pmatrix} \psi_h \\ \chi_h \end{pmatrix}, \begin{pmatrix} \varphi_h \\ \phi_h \end{pmatrix} \right\rangle_{V(\Gamma)\times(V(\Gamma))'} = 2(-1)^j \left\langle (\mathfrak{C}_{\sigma,j} - I) \begin{pmatrix} \rho \\ \tau \end{pmatrix}, \begin{pmatrix} \varphi_h \\ \phi_h \end{pmatrix} \right\rangle_{V(\Gamma)\times(V(\Gamma))'},$$
$$\forall \begin{pmatrix} \varphi_h \\ \phi_h \end{pmatrix} \in X_h \times Y_h \tag{5.16}$$

are uniquely solvable for all sufficiently small h and the approximate solutions

$$u_h = (-1)^j \big(\mathcal{K}_\sigma(\psi_h + \rho) - \mathcal{V}(\chi_h + \tau)\big) \;,$$

converge quasioptimally to the biharmonic function u in Ω_j, $j = 1, 2$, satisfying the boundary conditions (i)-(iv)*, for example, for any $x \in \Omega_j$ the estimate*

$$|u(x) - u_h(x)| \leq c\,\Big(\inf_{\varphi_h \in X_h} \|\mathcal{Q}\gamma u - \varphi_h\|_{V(\Gamma)} + \inf_{\phi_h \in Y_h} \|\mathcal{P}'\delta_\sigma u - \phi_h\|_{(V(\Gamma))'} \Big)$$

holds with some constant not depending on u and h.

References

[1] M. Bourlard, *Problème de Dirichlet pour le bilaplacien dans un polygone: résolution par éléments finis frontières raffinés,* C.R.Acad.Sci.Paris, Sér. I, **306** (1988), 461-466.

[2] G. Chen, J. Zhou, *Boundary element methods,* Academic Press, London et al, 1992.

[3] S. Christiansen, P. Hougaard, *An investigation of a pair of integral equations for the biharmonic problem,* J.Inst.Maths.Applics **22** (1978) 15-27.

[4] M. Costabel, *Starke Elliptizität von Randintegraloperatoren erster Art,* Habilitationsschrift, Darmstadt 1984.

[5] M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results,* SIAM J. Math. Anal. **19** (1988), No.3, 613-625.

[6] M. Costabel, M. Dauge, *Invertibility of the biharmonic single layer potential operator*, Institut de Recherche Mathématique de Rennes, Prépublication 95-13, Rennes 1995.

[7] M. Costabel, E. Stephan, *A direct boundary integral equation method for transmission problems,* J. Math. Anal. Appl. **106** (1985), 367-413.

[8] M. Costabel, E. Stephan, W. L. Wendland, *On boundary integral equations of the first kind for the bi-Laplacian in a polygonal plane domain,* Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4), **10** (1983), 197–241.

[9] M. Costabel, W. L. Wendland, *Strong ellipticity of boundary integral operators,* J. Reine Angew. Math. **372** (1986), 39-63.

[10] R. D. Grigorieff, *Randwertaufgaben für elliptische Bilinearformen auf $W^{m,2}(\Omega)$*, IV §4 in F. Stummel, *Rand- und Eigenwertaufgaben in Sobolewschen Räumen*, Lecture Notes in Mathematics 102, Springer, Berlin-Heidelberg-New York 1969.

[11] F. Hartman, R. Zotemantel, *The direct boundary element method in plate bending*, Int. J. Num. Methods Engng. **23** (1986), 2049-2069.

[12] G. N. Jakovlev, Boundary properties of functions of the class $W_p^{(l)}$ on domains with corners, Dokl. Akad. Nauk SSSR **140** (1961), No. 1, 73–76 (in Russian).

[13] J.C. Nedelec, *Approximation des équations intégrales en mécanique et en physique*, Lecture Notes, Centre de Mathématiques Appliquées. Ecole Polytechnique, Palaiseau 1977.

[14] , S. Prößdorf, *Some classes of singular equations*, North-Holland, Amsterdam-New York-Oxford, 1978.

[15] G. Schmidt, B. N. Khoromskij, *Boundary integral equations for the biharmonic Dirichlet problem on nonsmooth domains*, Preprint WIAS No. 129, Berlin 1994.

[16] M. Stern, *Boundary integral equations for bending of thin plates*, in C. A. Brebbia (ed.), *Progress in Boundary Element Methods, Vol.* 2, Pentech Press London, Plymouth, 1983, 158-181.