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# Bifurcations in the Sakaguchi-Kuramoto model


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**Abstract**

We analyze the Sakaguchi-Kuramoto model of coupled phase oscillators in a continuum limit given by a frequency dependent version of the Ott-Antonsen system. Based on a self-consistency equation, we provide a detailed analysis of partially synchronized states, their bifurcation from the completely incoherent state and their stability properties. We use this method to analyze the bifurcations for various types of frequency distributions and explain the appearance of non-universal synchronization transitions.

# 1 Introduction

The Sakaguchi-Kuramoto model

$$\frac{d\theta_k}{dt} = \omega_k - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_k(t) - \theta_j(t) + \alpha), \quad k = 1, \ldots, N, \tag{1}$$

describes the interaction of $N$ oscillators with phases $\theta_k \in \mathbb{R} \bmod 2\pi$ and natural frequencies $\omega_k$ coupled through their mean field with the strength $K$. It has been formulated in 1986 [1] as a natural generalization of the classical Kuramoto model by including the phase-lag parameter $\alpha$ in the sinusoidal interaction function. Any large system of oscillators that are weakly coupled to their mean field can be reduced to this form by performing a phase reduction and restricting the interaction function to the leading Fourier components [2, 3]. Therefore Eqs. (1) are often used as a paradigm for synchronization processes in neuroscience [4, 5], laser dynamics [6], power grids [7, 8], and coupled nano-oscillators [9, 10, 11] or chemical oscillatory cells [12, 13], see also monographs [2, 3] for further references. For the case $\alpha = 0$ and natural frequencies given by a unimodal distribution $g(\omega)$, (i.e. $g(\omega)$ is an even function with respect to some point $\omega_0$ and non-increasing on $(\omega_0, \infty)$), it is well-known that one observes for increasing coupling strength a universal synchronization transition, where at a value $K = K_c$ a unique stable branch of partially synchronized solutions appears. The synchrony along this branch, measured by the absolute value of the order parameter

$$r(t) = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k(t)}, \tag{2}$$

increases monotonically with increasing coupling strength $K > K_c$ and approaches $1$ when $K \to \infty$, see Fig. 1(a).

It has been common believe that this universal synchronization transition can be found in a similar way also for $|\alpha| < \pi/2$, as long as the frequency distribution $g(\omega)$ remains unimodal. Recently, we reported in [14] that for certain unimodal frequency distributions one can encounter further non-universal synchronization transitions, where synchrony may decay with increasing coupling strength, or multistability between partially synchronized states and/or the incoherent

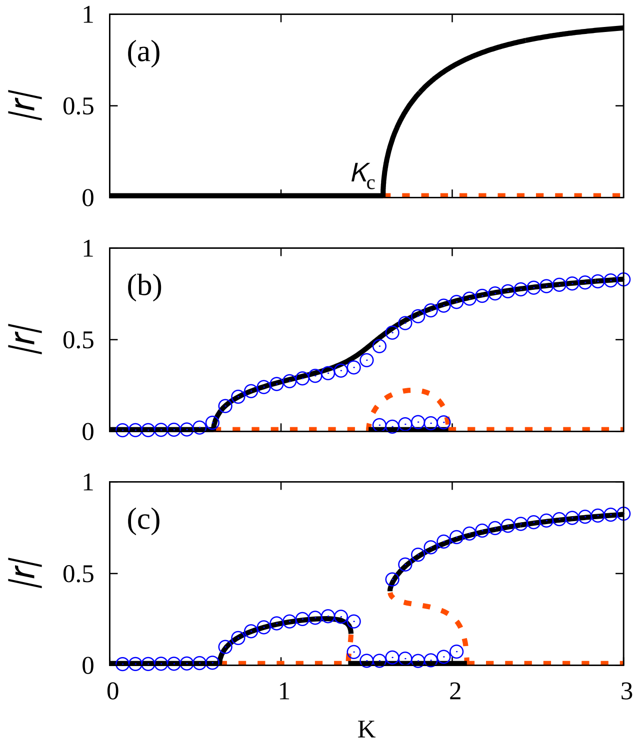


Figure 1: (color online) Examples of synchronization transitions (dependence of the absolute value of order parameter $r$ on the coupling strength $K$) for the Sakaguchi-Kuramoto system (1) with various choices of unimodal frequency distributions $g(\omega)$ and of the phase-lag parameter $\alpha$. Solid (dashed) lines: analytical prediction of stable (unstable) branches, based on the Ott-Antonsen equation. Circles: numerical simulations of system (1) with $N = 20000$ and suitably chosen initial data. (a) Classical synchronization transition for $\alpha = 0$ and unit width Gaussian frequency distribution (80). (b) and (c) Synchronization transitions with bistability and incoherence regaining its stability (frequency distributions and phase-lag parameters are specified in Fig. 2(a)).

state may appear. The results communicated in [14] are based on a new approach to the bifurcation analysis in the limit $N \longrightarrow \infty$ using a frequency dependent version of the Ott-Antonsen equation. This approach turns out to be a universal method to study the qualitative behavior of the Sakaguchi-Kuramoto system for arbitrary frequency distributions. In this paper, we present a detailed description of this approach and demonstrate in several examples the appearance of non-universal synchronization transition in bifurcations of higher codimension.

The paper is organized as follows. In Section 2 we introduce the general framework of a continuum limit $N \longrightarrow \infty$ and recall the Ott-Antonsen reduction [15] that provides an evolution equation for the effective dynamics of Eqs. (1) in this limit. Partially synchronized states of Eqs. (1) can be identified with standing wave solutions of this evolution equation, while the completely incoherent state corresponds to its trivial solution. In Section 3 we derive a universal solution profile providing a description of the set of all partially synchronized states independent from the frequency distribution. Moreover, we derive a self-consistency equation in the form of a function that for a given frequency distribution provides the system parameters $K$ and $\alpha$ for which the different partially synchronized state appear. This function is also used to obtain asymptotic expansions of the branch of partially synchronized states at the onset of partial synchrony. In Section 4 we perform a linear stability analysis for both completely incoherent and partially synchronized states. In particular, we provide explicit expressions for the continuous spectrum and a characteristic function for the eigenvalues. Moreover, we show that the degeneracies of the self-consistency equation can be used to detect certain bifurcations of the solutions. Finally, in Section 5, we show several examples of frequency distributions that display non-trivial bifurcation diagrams with nonuniversal synchronization transitions. In particular, we demonstrate how these non-trivial scenarios arise as a result of bifurcations of higher codimension that can be determined in the framework of our analysis. Discussion and concluding remarks appear in Section 6.

## 2 Continuum limit formulation

For the limit of large $N$, the macroscopic state of a coupled oscillator system can be described by a probability distribution function $f(\theta, \omega, t)$ which for fixed time $t$ gives the relative number of oscillators with $\theta_k(t) \approx \theta$ and $\omega_k \approx \omega$ (see [16, 17, 18]). Conservation of oscillators implies a continuity equation of the form

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}(f\nu) = 0. \tag{3}$$

To obtain the corresponding velocity $\nu$, we rewrite system (1) in the form

$$\frac{d\theta_k}{dt} = \omega_k + \operatorname{Im}\left(Kr(t)e^{-i(\theta_k(t)+\alpha)}\right), \quad k = 1, \ldots, N, \tag{4}$$

and define then

$$\nu(\theta, \omega, t) := \omega + \operatorname{Im}\left(Kr(t)e^{-i(\theta+\alpha)}\right), \tag{5}$$

where we use now the continuum form of the mean field

$$r(t) = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f(\theta, \omega, t)e^{i\theta} d\theta. \tag{6}$$

Assuming that $g(\omega)$ is a non-negative, piecewise-continuous, integrable function on $\mathbb{R}$, satisfying the normalization condition

$$\int_{-\infty}^{\infty} g(\omega)d\omega = 1,$$

we require also for $f$ the normalization condition

$$\int_{0}^{2\pi} f(\theta, \omega, t)d\theta = g(\omega). \tag{7}$$

In particular, $f(\theta, \omega, t) = 0$ for all $\omega$ such that $g(\omega) = 0$. This completes the description of the formal continuum limit for the coupled oscillator system (1) as a closed system (3), (5), (6), for the distribution $f$, represented as a nonlinear integro-differential PDE.

A significant simplification of this formal continuum limit for the coupled oscillator system (1) can be achieved by the Ott–Antonsen reduction. In their seminal papers [15, 19, 20], they showed that Eq. (3) has an invariant attracting manifold given by the ansatz

$$f(\theta, \omega, t) = \frac{g(\omega)}{2\pi}\left(1 + \sum_{n=1}^{\infty}\left[\overline{z}^n(\omega, t)e^{in\theta} + z^n(\omega, t)e^{-in\theta}\right]\right), \tag{8}$$

where $z(\omega, t)$ is a complex-valued function such that $|z| \leq 1$ ($\overline{z}$ denotes the complex conjugate of $z$). Indeed, substituting (8) into the continuity equation (3), we easily verify that it determines a solution to this equation, provided that $z(\omega, t)$ satisfies

$$\frac{dz}{dt} = i\omega z(\omega, t) + \frac{K}{2}e^{-i\alpha}\mathcal{G}z - \frac{K}{2}e^{i\alpha}z^2(\omega, t)\mathcal{G}\overline{z}, \tag{9}$$

where for any $\varphi \in C(\mathbb{R}; \mathbb{C})$ we denote by $\mathcal{G}$ the integral operator

$$\mathcal{G}\varphi := \int_{-\infty}^{\infty} g(\omega)\varphi(\omega)d\omega. \tag{10}$$

The set of admissible solutions $|z(\omega, t)| \leq 1$ turns out to be an invariant set of Eq. (9), see [15], which proves self-consistency of the Ott-Antonsen method.

On the Ott-Antonsen manifold, described by Eq. (9), solutions $z(\omega, t)$ have typically more regularity than corresponding solutions $f$ of Eq. (3). For example, if $|z(\omega, t)| \nearrow 1$ then formula (8) implies $f(\theta, \omega, t) \to g(\omega)\delta(\theta)$, where $\delta(\theta)$ denotes the Dirac delta-function. Hence, Eq. (9) are a more suitable mathematical framework to investigate the synchronization transitions in the continuum limit. Remark, however, that the Ott-Antonsen reduction relies crucially on the sinusoidal coupling function, and cannot be applied to the setting of general coupling functions that has been studied in [17, 18]. Usually, the Ott-Antonsen method includes a further step of reduction: Based on the additional assumption that the frequency distribution $g(\omega)$ is a rational function of $\omega$, the integral term $\mathcal{G}z$ in Eq. (9) can be evaluated as a finite sum of complex residues. Thus Eq. (9) can be further simplified to a finite-dimensional dynamical system. In contrast, we want to address arbitrary choices of $g(\omega)$ and therefore stay with Eq. (9) in its infinite-dimensional form.

The quantity $z(\omega,t)$ in (9) has a clear interpretation as a frequency dependent *local order parameter*, measuring the synchrony of the oscillators with natural frequencies $\omega_k \approx \omega$. Indeed, performing in the definition of the global order parameter (2), only the integration over $\theta$, we obtain from (8) the identity

$$g(\omega)z(\omega,t) = \int_0^{2\pi} f(\theta,\omega,t)e^{i\theta}d\theta. \tag{11}$$

Integrating then with respect to $\omega$, we obtain the relation

$$r(t) = (\mathcal{G}z)(t) = \int_{-\infty}^{\infty} g(\omega)z(\omega,t)d\omega \tag{12}$$

between the global and the local order parameter. Consequently, the completely incoherent state corresponds to the trivial solution $z(\omega,t) \equiv 0$ of (9), while partially synchronous solutions can be represented by standing wave solutions, which we study in the following section.

# 3 Partially synchronized states

Eq. (9) is equivariant with respect to the complex phase shifts $z \mapsto e^{i\phi}z$; therefore we can look for solutions in the form of a standing wave

$$z(\omega,t) = a(\omega)e^{i\Omega t} \tag{13}$$

where $\Omega \in \mathbb{R}$ is a collective frequency and $a \in C(\mathbb{R};\mathbb{C})$ is a time-independent profile. As we will see below, such solutions satisfy $|a(\omega)| = 1$ (synchrony) for some values $\omega$ and $|a(\omega)| < 1$ (asynchrony) for others. Hence, according to the interpretation of $z(\omega,t)$ as a local order parameter, we may refer to them as *partially synchronized states*. Substituting ansatz (13) into Eq. (9) we obtain

$$Ke^{i\alpha}a^2\mathcal{G}\overline{a} - 2i(\omega-\Omega)a - Ke^{-i\alpha}\mathcal{G}a = 0. \tag{14}$$

This equation has to be solved with respect to $\Omega$ and $a(\omega)$ simultaneously. To this end, we rewrite Eq. (14) in a shorter form

$$\overline{Z}a^2 - 2(\omega-\Omega)a + Z = 0, \tag{15}$$

where

$$Z := iKe^{-i\alpha}\mathcal{G}a. \tag{16}$$

Eq. (15) is quadratic with respect to $a(\omega)$ and has two roots

$$a_\pm(\omega) = \begin{cases} \dfrac{\omega-\Omega \pm \sqrt{(\omega-\Omega)^2-|Z|^2}}{|Z|^2}Z & \text{for} \quad |\omega-\Omega| > |Z|, \\[2ex] \dfrac{\omega-\Omega \pm i\sqrt{|Z|^2-(\omega-\Omega)^2}}{|Z|^2}Z & \text{for} \quad |\omega-\Omega| \le |Z|. \end{cases}$$

Now, recall that (13) determines an admissible solution of Eq. (9) only, if inequality $|z(\omega,t)| \le 1$ holds true. For $|\omega-\Omega| > |Z|$ this requirement selects precisely one of two roots $a_-$ or $a_+$, but it does not help to make a decision for $|\omega-\Omega|\leq|Z|$ when $|a_\pm(\omega)|=1$ and hence both roots are admissible. As a result we can construct two composed solutions with $|z(\omega,t)|\leq 1$:

$$a_1(\omega)=h\left(\frac{\omega-\Omega}{|Z|}\right)\frac{Z}{|Z|}, \tag{17}$$

and

$$a_2(\omega)=\overline{h}\left(\frac{\omega-\Omega}{|Z|}\right)\frac{Z}{|Z|}, \tag{18}$$

where

$$h(s)=\begin{cases}\left(1-\sqrt{1-s^{-2}}\right)s & \text{for} \quad |s|>1,\\ s-i\sqrt{1-s^2} & \text{for} \quad |s|\leq 1.\end{cases} \tag{19}$$

Note that due to the phase shift symmetry, the factor $Z/|Z|$ in (17), (18) can without loss of generality be omitted. Eq. (17) shows that indeed all standing wave solutions are of partially synchronized type and moreover can be described by a universal profile $h$. The scaling factor in the argument of $h$, which for convenience we will denote by $p:=|Z|\in(0,\infty)$, describes the size of the synchronization window, while the second parameter $\Omega$ gives the central frequency of the synchronization window. Moreover, combining (12) and (16) we get the relation

$$p=K|r|, \tag{20}$$

to the global order parameter $r$ of the solution (25).

Substituting $a_1(\omega)$ and $a_2(\omega)$ into definition (16) and canceling the non-zero factor $Z$ from both sides we obtain two self-consistency equations

$$1=iKe^{-i\alpha}\frac{1}{p}\mathcal{G}h\left(\frac{\omega-\Omega}{p}\right) \tag{21}$$

and

$$1=iKe^{i\alpha}\frac{1}{p}\mathcal{G}\overline{h}\left(\frac{\omega-\Omega}{p}\right), \tag{22}$$

respectively. Since we are interested in the case $K>0$ and $|\alpha|<\pi/2$, we can discard Eq. (22), which turns out to have no solutions in this parameter range. Inserting (10), we end up with the self-consistency equation

$$\begin{aligned}\frac{1}{K}e^{i\alpha} &= \frac{i}{p}\int_{-\infty}^{\infty}g(\omega)h\left(\frac{\omega-\Omega}{p}\right)d\omega\\ &= i\int_{-\infty}^{\infty}g(\Omega+ps)h(s)ds=:H(p,\Omega).\end{aligned} \tag{23}$$

This equation can be interpreted as follows: For any choice of the solution parameters $(p,\Omega)\in(0,\infty)\times\mathbb{R}$ we obtain system parameters

$$K=|H(p,\Omega)|^{-1}, \quad \alpha=\arg H(p,\Omega). \tag{24}$$

such that by the universal profile $h$ we obtain a solution

$$z(\omega, t) = h\left(\frac{\omega - \Omega}{p}\right) e^{i\Omega t} \tag{25}$$

to Eq. (9) at the system parameters (24). Note that due to the phase-shift symmetry, these solutions appear as continuous families, multiplied with arbitrary phase shifts $e^{i\phi}$, $\phi \in [0, 2\pi]$.

**Remark 1** *It is easy to verify that for* $\Omega = 0$ *and an even frequency distribution* $g(\omega)$*, we get*

$$H(p, 0) = i \int_{-\infty}^{\infty} g(ps)h(s)ds = \int_{-1}^{1} g(ps)\sqrt{1 - s^2}ds.$$

*Hence Eqs. (23) and (20) imply* $\alpha = 0$ *and*

$$\frac{1}{K} = \int_{-1}^{1} g(K|r|s)\sqrt{1 - s^2}ds.$$

*This is the well-known relation between order parameter* $|r|$ *and the coupling strength* $K$ *for the classical Kuramoto model [3], compare with Eq. (4.5) in [21] or Eq. (12) in [22].*

We will discuss now in more details the properties of the partially synchronized states (25) and their relation to the parameters $K$, $\alpha$, given by the self-consistency equation (23). To this end, we recall that Eq. (23) defines a mapping

$$\mathcal{H} \,:\, (p, \Omega) \in (0, \infty) \times \mathbb{R} \mapsto (K, \alpha) \in (0, \infty) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

From its inverse $\mathcal{H}^{-1}$ we would obtain for given system parameters $K$ and $\alpha$ a corresponding partially synchronized state, defined by its parameters $(p, \Omega)$ through (25). However, since $\mathcal{H}$ is not necessarily one-to-one, it may happen that for some parameters $K$ and $\alpha$ the inverse $\mathcal{H}^{-1}$ does not exist at all, or, in contrast, it is multivalued and has different solutions $(p, \Omega)$ for the same parameters $K$ and $\alpha$. In the case of unimodal frequency distribution $g(\omega)$, this situation is of particular interest, since it is an indication for a non-classical synchronization scenario. The parameter regions of non-uniqueness of the inverse $\mathcal{H}^{-1}$ can be detected from the underlying fold bifurcations. Indeed under the mild assumptions on $g(\omega)$ given at the beginning of Section 2, $\mathcal{H}$ turns out to be a local diffeomorphism for any $(p, \Omega) \in (0, \infty) \times \mathbb{R}$, except of those points that satisfy a degeneracy condition of the form

$$\Psi(p, \Omega) := \det \begin{pmatrix} \mathrm{Re}(\partial_p H) & \mathrm{Re}(\partial_\Omega H) \\ \mathrm{Im}(\partial_p H) & \mathrm{Im}(\partial_\Omega H) \end{pmatrix} = \mathrm{Im}\left(\partial_p \overline{H}\, \partial_\Omega H\right) = 0. \tag{26}$$

Eq. (26) determines a curve in the plane of solution parameters $(p, \Omega)$. Applying then the mapping $\mathcal{H}$ we obtain the corresponding fold bifurcation curve in the plane of system parameters $(K, \alpha)$. In Section 5 we will use this approach to study non-classical synchronization transitions for several types of unimodal frequency distributions.

Before we are going to perform a linear stability analysis of complete and partially synchronized solutions, we discuss some further properties of Eq. (23). First, we are going to analyze the two limiting cases $p \to 0$ and $p \to \infty$, corresponding to the onset of partial synchronization and complete synchronization, respectively. Then, we calculate the distribution of averaged phase velocities for a partially synchronized solution (25).

### 3.1 The onset of synchronization: $p \to 0$

The limit $p \to 0$ in Eq. (23) requires a more detailed discussion. Indeed, a branch of partially synchronized states (25) with $p \to 0$ diverges in the $C(\mathbb{R};\mathbb{C})$-norm, and tends to zero only in a week sense. At the same time, the integral in the self-consistency equation (23) becomes singular and $\lim_{p\to 0} H(p,\Omega)$ can be expressed by

$$J(\Omega) := \frac{\pi}{2} g(\Omega) + \frac{i}{2} \lim_{\varepsilon\to+0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{g(\Omega+\omega)}{\omega} d\omega. \tag{27}$$

We prove this statement, together with some expressions for the higher derivatives in the following proposition.

**Proposition 1** *Suppose that $g \in C^3(\mathbb{R}) \cap W^{3,1}(\mathbb{R})$. Then $H \in C^3((0,\infty)\times\mathbb{R})$, and $J \in C^3(\mathbb{R})$. The derivatives of $J(\Omega)$ are given by*

$$J^{(k)}(\Omega) = \frac{\pi}{2} g^{(k)}(\Omega) + \frac{i}{2} \lim_{\varepsilon\to+0} \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{g^{(k)}(\Omega+\omega)}{\omega} d\omega, \quad k = 1,2,3,$$

*and the following identities hold true*

$$\begin{aligned}
\lim_{p\to 0} H(p,\Omega) &= J(\Omega) \\
\lim_{p\to 0} \partial_p H(p,\Omega) &= 0 \\
\lim_{p\to 0} \partial_p^2 H(p,\Omega) &= \frac{1}{4} J''(\Omega) \\
\lim_{p\to 0} \partial_p^3 H(p,\Omega) &= 0.
\end{aligned}$$

**Proof:** The differentiation formulas for the last term in (27) follows from the identity

$$\left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{g^{(k)}(\Omega+\omega)}{\omega} d\omega = \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{g^{(k)}(\Omega+\omega) - g^{(k)}(\Omega)}{\omega} d\omega, \tag{28}$$

and the fact that the right-hand side integral in (28) converges uniformly with respect to $\varepsilon$ and $\Omega$ for $k = 1,2,3$.

Next, we calculate the limits for $p \to 0$ of $H(p,\Omega)$ and its partial derivatives. To this end, we insert definition (19) into (23) and obtain

$$\begin{aligned}
H(p,\Omega) &= \int_{-1}^{1} g(\Omega+ps)\sqrt{1-s^2} ds + i \int_{-1}^{1} g(\Omega+ps) s\, ds \\
&+ i \left( \int_{-\infty}^{-1} + \int_{1}^{\infty} \right) g(\Omega+ps)(1-\sqrt{1-s^{-2}}) s\, ds.
\end{aligned} \tag{29}$$

The first and the second terms in the right-hand side of Eq. (29) are proper integrals that depend regularly on both parameters $p$ and $\Omega$. Hence, passing to the limit we get

$$\lim_{p\to 0}\int_{-1}^{1} g(\Omega+ps)\sqrt{1-s^2}ds = \frac{\pi}{2}g(\Omega) \quad \text{and} \quad \lim_{p\to 0}\int_{-1}^{1} g(\Omega+ps)s\ ds = 0.$$

Similarly we can calculate any partial derivative of these terms with respect to $p$ and $\Omega$, provided that the distribution function $g(\omega)$ is smooth enough.

For the last term of Eq. (29) the calculations become more complex, since an improper integral appears. Expanding

$$(1-\sqrt{1-s^{-2}})s = \frac{1}{2}s^{-1} + \frac{1}{8}s^{-3} + R(s), \tag{30}$$

with $R(s) = O(s^{-5})$ for $s \to \infty$, we rewrite this term as follows

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) g(\Omega+ps)(1-\sqrt{1-s^{-2}})s\ ds = \sum_{k=1}^{3} H_k(p,\Omega), \tag{31}$$

where

$$\begin{aligned} H_1(p,\Omega) &:= \frac{1}{2}\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)\frac{g(\Omega+ps)}{s}ds \\ &= \frac{1}{2}\left(\int_{-\infty}^{-p} + \int_{p}^{\infty}\right)\frac{g(\Omega+\omega)}{\omega}d\omega, \end{aligned} \tag{32}$$

$$\begin{aligned} H_2(p,\Omega) &:= \frac{1}{8}\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right)\frac{g(\Omega+ps)}{s^3}ds \\ &= \frac{p^2}{8}\left(\int_{-\infty}^{-p} + \int_{p}^{\infty}\right)\frac{g(\Omega+\omega)}{\omega^3}d\omega, \end{aligned} \tag{33}$$

$$H_3(p,\Omega) := \left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) g(\Omega+ps)R(s)ds. \tag{34}$$

The decay rate of $R(s)$ ensures that the integral term $H_3(p,\Omega)$ and all its partial derivatives with respect to $p$ up to the third order converge uniformly with respect $p$. Therefore, straightforward calculation yields

$$\begin{aligned} \lim_{p\to 0} H_3(p,\Omega) &= 0, \\ \lim_{p\to 0} \partial_p H_3(p,\Omega) &= \frac{1}{12}g'(\Omega), \\ \lim_{p\to 0} \partial_p^2 H_3(p,\Omega) &= 0, \\ \lim_{p\to 0} \partial_p^3 H_3(p,\Omega) &= \frac{11}{60}g'''(\Omega). \end{aligned}$$

Next, using the right-hand side of formula (32), we calculate

$$
\begin{aligned}
\lim_{p\to 0} \partial_p H_1(p,\Omega) &= -g'(\Omega), \\
\lim_{p\to 0} \partial_p^2 H_1(p,\Omega) &= 0, \\
\lim_{p\to 0} \partial_p^3 H_1(p,\Omega) &= -\frac{1}{3} g'''(\Omega).
\end{aligned}
$$

Finally, in (33), we first integrate twice by parts in order to weaken the integrand singularity and get

$$
\begin{aligned}
H_2(p,\Omega) &= \frac{g(\Omega+p)+pg'(\Omega+p)-g(\Omega-p)+pg'(\Omega-p)}{16} \\
&+ \frac{p^2}{16}\left(\int_{-\infty}^{-p} + \int_p^{\infty}\right)\frac{g''(\Omega+\omega)}{\omega} d\omega.
\end{aligned}
$$

Differentiating the result, we get the identities

$$
\begin{aligned}
\lim_{p\to 0} H_2(p,\Omega) &= 0, \\
\lim_{p\to 0} \partial_p H_2(p,\Omega) &= \frac{1}{4} g'(\Omega), \\
\lim_{p\to 0} \partial_p^2 H_2(p,\Omega) &= \frac{1}{8}\lim_{p\to 0}\left(\int_{-\infty}^{-p} + \int_p^{\infty}\right)\frac{g''(\Omega+\omega)}{\omega} d\omega, \\
\lim_{p\to 0} \partial_p^3 H_2(p,\Omega) &= -\frac{1}{4} g'''(\Omega).
\end{aligned}
$$

Now, calculating explicitly the limits of the first and the second terms in Eq. (29) and using Eq. (31) to evaluate the improper integral there, we can justify all the asserted identities. •

Hence, by $H(0,\Omega) := J(\Omega)$, we can extend $H$ to a function in $C^3([0,\infty)\times\mathbb{R})$. Note that according to (25) the limit corresponds to the onset of partial synchrony. Using the results in Proposition 1 and Eq. (23) we can calculate now the scaling behavior of order parameter $|r|$ along a branch with $p \to 0$ close to the onset of synchronization at $p = 0$.

**Proposition 2** *Suppose that* $(K_c, \alpha_c, \Omega_c) \in (0,\infty)\times(-\pi/2,\pi/2)\times\mathbb{R}$ *is a triple satisfying the self consistency equation for* $p = 0$

$$
\frac{1}{K_c} e^{i\alpha_c} = H(0,\Omega_c) = J(\Omega_c) \tag{35}
$$

*and the non-degeneracy conditions*

$$
\begin{aligned}
\mathrm{Im}\,(\Phi_1) &\neq 0 \qquad (36) \\
\mathrm{Im}\,(\Phi_2) &\neq 0 \qquad (37)
\end{aligned}
$$

*with*

$$\begin{aligned}\Phi_1 &:= \overline{J}(\Omega_\mathrm{c})J'(\Omega_\mathrm{c})\\ \Phi_2 &:= \overline{J}''(\Omega_\mathrm{c})J'(\Omega_\mathrm{c}).\end{aligned}$$

*Then, for* $\alpha = \alpha_\mathrm{c}$ *and all* $K \approx K_\mathrm{c}$ *such that*

$$(K - K_\mathrm{c})\,\mathrm{Im}\,(\Phi_1)\,\mathrm{Im}\,(\Phi_2) < 0,$$

*there exists a solution branch of Eq. (23) with the following asymptotic properties*

$$p = \sqrt{-8\frac{\mathrm{Im}\,(\Phi_1)}{\mathrm{Im}\,(\Phi_2)}\frac{K-K_\mathrm{c}}{K_\mathrm{c}}} + o(|K-K_\mathrm{c}|^{1/2}), \tag{38}$$

$$\Omega = \Omega_\mathrm{c} - \frac{\mathrm{Im}\,(\Phi_0)}{\mathrm{Im}\,(\Phi_2)}\frac{K-K_\mathrm{c}}{K_\mathrm{c}} + o(|K-K_\mathrm{c}|), \tag{39}$$

$$|r| = \sqrt{-8\frac{\mathrm{Im}\,(\Phi_1)}{\mathrm{Im}\,(\Phi_2)}\frac{K-K_\mathrm{c}}{K_\mathrm{c}^3}} + o(|K-K_\mathrm{c}|^{1/2}), \tag{40}$$

*where we denoted*

$$\Phi_0 := \overline{J}''(\Omega_\mathrm{c})J(\Omega_\mathrm{c}).$$

**Proof:** Subtracting (35) from Eq. (23) with $\alpha = \alpha_\mathrm{c}$ we obtain

$$\left(\frac{1}{K} - \frac{1}{K_\mathrm{c}}\right)e^{i\alpha_\mathrm{c}} = H(p,\Omega) - H(0,\Omega_\mathrm{c}),$$

or alternatively

$$-\frac{K-K_\mathrm{c}}{KK_\mathrm{c}^2} = \overline{J}(\Omega_\mathrm{c})\left(H(p,\Omega) - H(0,\Omega_\mathrm{c})\right).$$

The latter can be rewritten as a system of two real equations

$$\mathrm{Im}\left(\overline{J}(\Omega_\mathrm{c})\left(H(p,\Omega) - H(0,\Omega_\mathrm{c})\right)\right) = 0, \tag{41}$$

$$\mathrm{Re}\left(\overline{J}(\Omega_\mathrm{c})\left(H(p,\Omega) - H(0,\Omega_\mathrm{c})\right)\right) = -\frac{K-K_\mathrm{c}}{KK_\mathrm{c}^2}. \tag{42}$$

Using Proposition 1 we can write

$$\begin{aligned}H(p,\Omega) - H(0,\Omega_\mathrm{c}) &= \frac{1}{2}\partial_p^2 H(0,\Omega_\mathrm{c})p^2 + \partial_\Omega H(0,\Omega_\mathrm{c})(\Omega-\Omega_\mathrm{c}) + o(p^2 + |\Omega-\Omega_\mathrm{c}|)\\ &= \frac{1}{8}J''(\Omega_\mathrm{c})p^2 + J'(\Omega_\mathrm{c})(\Omega-\Omega_\mathrm{c}) + o(p^2+|\Omega-\Omega_\mathrm{c}|).\end{aligned} \tag{43}$$

Inserting this Taylor expansion into (41) we obtain

$$\frac{1}{8}\,\mathrm{Im}\left(\overline{\Phi}_0\right)p^2 + \mathrm{Im}\,(\Phi_1)\,(\Omega-\Omega_\mathrm{c}) = o(p^2+|\Omega-\Omega_\mathrm{c}|).$$

Hence, for $\mathrm{Im}\,(\Phi_1) \neq 0$, the Implicit Function Theorem yields

$$\Omega - \Omega_{\mathrm{c}} = -\frac{1}{8}\frac{\mathrm{Im}\left(\overline{\Phi}_0\right)}{\mathrm{Im}\,(\Phi_1)}p^2 + o(p^2). \tag{44}$$

Now we consider Eq. (42). Inserting there the Taylor expansion (43) and then replacing the difference $\Omega - \Omega_{\mathrm{c}}$ by Eq. (44), we obtain

$$-\frac{K - K_{\mathrm{c}}}{KK_{\mathrm{c}}^2} = \frac{1}{8}\left(\mathrm{Re}\left(\overline{\Phi}_0\right) - \mathrm{Re}\,(\Phi_1)\,\frac{\mathrm{Im}\left(\overline{\Phi}_0\right)}{\mathrm{Im}\,(\Phi_1)}\right)p^2 + o(p^2).$$

Using the identity

$$\mathrm{Re}\left(\overline{\Phi}_0\right)\mathrm{Im}\,(\Phi_1) - \mathrm{Re}\,(\Phi_1)\,\mathrm{Im}\left(\overline{\Phi}_0\right) = \mathrm{Im}\,(\Phi_0\Phi_1) = \frac{1}{K_{\mathrm{c}}^2}\,\mathrm{Im}\,(\Phi_2)\,, \tag{45}$$

we rewrite this equation as follows

$$-\frac{K - K_{\mathrm{c}}}{K} = \frac{1}{8}\frac{\mathrm{Im}\,(\Phi_2)}{\mathrm{Im}\,(\Phi_1)}p^2 + o(p^2). \tag{46}$$

Due to assumption $\mathrm{Im}\,(\Phi_2) \neq 0$ Eq. (46) can be solved with respect to $p$ for $K \approx K_{\mathrm{c}}$ that yields asymptotics (38). Then applying (44) and (20) we easily get the other asymptotics (39) and (40), respectively. •

**Remark 2** *Applying Proposition 1 it is easy to verify that for any unimodal distribution $g(\omega)$ centered at zero it holds*

$$J(0) = \frac{\pi}{2}g(0), \quad \mathrm{Re}\ J'(0) = 0, \quad J''(0) = \frac{\pi}{2}g''(0).$$

*Hence,inserting $\Omega_{\mathrm{c}} = 0$ into (35) and (40), we recover Kuramoto's classical result*

$$K_{\mathrm{c}} = \frac{2}{\pi g(0)}$$

*and*

$$|r| = \sqrt{-\frac{16(K - K_{\mathrm{c}})}{\pi K_{\mathrm{c}}^4 g''(0)}} + o(|K - K_{\mathrm{c}}|^{1/2}),$$

*see Eq. (4.6) in [21] or Eq. (14) in [22].*

However, for $\alpha \neq 0$ in Sakaguchi-Kuramoto model, the coefficient of the square root term does not have a definite sign for all parameter values. In this way, the branch of partially synchronized solutions may either protrude to coupling strengths above or below the critical value $K_{\mathrm{c}}$. This will generically change, whenever one of the two degeneracy conditions (36), (37) is violated. The resulting two different types of singularities will be explained by the examples in Section 5.

### 3.2 Complete synchronization limit: $p \to \infty$

Now, we demonstrate that the limit $p \to \infty$ generically implies $|r| \to 1$ and hence corresponds to complete synchronization. To this end, using identity (20) and the rescaled variable $\psi := \Omega/p$ we rewrite Eq. (23) in the following form

$$|r|e^{i\alpha} = i\int_{-\infty}^{\infty} g(\omega)h\left(-\psi + p^{-1}\omega\right) d\omega =: H_\infty(p^{-1},\psi). \tag{47}$$

Then passing formally to the limit $p \to \infty$ and taking into account the normalization condition for $g(\omega)$ we obtain

$$|r|e^{i\alpha} = ih(-\psi).$$

According to formula (19) this implies

$$|r| = 1 \quad \text{and} \quad \psi = -\sin\alpha.$$

In order to rigorously justify this formal derivation we prove the following

**Proposition 3** *Suppose that distribution $g \in C^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ satisfies*

$$|g(\omega)| \le \mathrm{const}\,|\omega|^{-1-\varepsilon} \quad \textit{and} \quad |g'(\omega)| \le \mathrm{const}\,|\omega|^{-2-\varepsilon} \quad \textit{for some} \quad \varepsilon > 0. \tag{48}$$

*Then, given $\alpha \in (-\pi/2, \pi/2)$ for sufficiently large $p$ there exists a solution $(|r|,\psi)$ of Eq. (47) such that*

$$|r| = 1 + o(1), \quad \psi = -\sin\alpha + o(1) \quad \textit{for} \quad p \to \infty.$$

*Respectively, Eq. (23) has a solution with asymptotic properties*

$$K = p + o(p), \quad \Omega = -p\sin\alpha + o(p) \quad \textit{for} \quad p \to \infty,$$

*or alternatively*

$$\Omega = -K\sin\alpha + o(K) \quad \textit{for} \quad K \to \infty.$$

**Proof:** Similar to the proof of Proposition 2 we rewrite Eq. (47) as a system

$$\mathrm{Im}\left(e^{-i\alpha}H_\infty(p^{-1},\psi)\right) = 0, \tag{49}$$

$$\mathrm{Re}\left(e^{-i\alpha}H_\infty(p^{-1},\psi)\right) = |r|. \tag{50}$$

Then, solving Eq. (49) with respect to $\psi$, we calculate $|r|$ explicitly from the Eq. (50). In order to prove the solvability of Eq. (49) we apply the Implicit Function Theorem. To this end we have to verify that

$$H_\infty(\zeta,\psi) \to ih(-\psi) \quad \text{for} \quad |\zeta| + |\psi + \sin\alpha| \to 0, \tag{51}$$

whereas the derivative $\partial_\psi H_\infty(\zeta,\psi)$ remains bounded and non-zero for the same values of $\zeta$ and $\psi$. It is easy to see that for any $g \in L^1(\mathbb{R})$, the function $H_\infty(\zeta,\psi)$ given by formula (47) is globally continuous and hence satisfies (51). However we cannot differentiate (47) directly, since the formally calculated derivative $h'(s)$ turns out to be singular for $s = \pm 1$. On the other hand, if for $\zeta > 0$ we rewrite (47) in the form

$$H_\infty(\zeta, \psi) = \frac{i}{\zeta} \int_{-\infty}^{\infty} g\left(\frac{s+\psi}{\zeta}\right) h(s) ds,$$

then we easily calculate

$$\partial_\psi H_\infty(\zeta, \psi) = \frac{i}{\zeta^2} \int_{-\infty}^{\infty} g'\left(\frac{s+\psi}{\zeta}\right) h(s) ds = \frac{i}{\zeta} \int_{-\infty}^{\infty} g'(\omega) h\left(-\psi + \zeta\omega\right) d\omega.$$

Now we choose some $\Delta > 0$ such that

$$[-\sin\alpha - 2\Delta, -\sin\alpha + 2\Delta] \subset (-1, 1)$$

and rewrite the latter integral as follows

$$\begin{aligned}
\partial_\psi H_\infty(\zeta, \psi) &= \frac{i}{\zeta} \left( \int_{-\infty}^{-\Delta/\zeta} + \int_{\Delta/\zeta}^{\infty} \right) g'(\omega) h\left(-\psi + \zeta\omega\right) d\omega \\
&+ \frac{i}{\zeta} \int_{-\Delta/\zeta}^{\Delta/\zeta} g'(\omega) h\left(-\psi + \zeta\omega\right) d\omega.
\end{aligned}$$

Then using the inequality $|h(s)| \leq 1$ and decay estimates (48) we obtain

$$\frac{i}{\zeta} \left( \int_{-\infty}^{-\Delta/\zeta} + \int_{\Delta/\zeta}^{\infty} \right) g'(\omega) h\left(-\psi + \zeta\omega\right) d\omega = O(\zeta^\varepsilon)$$

and

$$\begin{aligned}
&\left| \int_{-\Delta/\zeta}^{\Delta/\zeta} g(\omega) \Big( h'(-\psi + \zeta\omega) - h'(-\psi) \Big) d\omega \right| \\
&\qquad \leq \max_{|s-\psi| \leq 2\Delta} h''(s) \int_{-\Delta/\zeta}^{\Delta/\zeta} g(\omega) \zeta |w| d\omega = O(\zeta^\varepsilon).
\end{aligned}$$

After the integration by parts

$$\begin{aligned}
&\frac{i}{\zeta} \int_{-\Delta/\zeta}^{\Delta/\zeta} g'(\omega) h\left(-\psi + \zeta\omega\right) d\omega \\
&\qquad = \frac{i}{\zeta} g(\omega) h(-\psi + \zeta\omega) d\omega \Big|_{\omega = -\Delta/\zeta}^{\omega = \Delta/\zeta} - i \int_{-\Delta/\zeta}^{\Delta/\zeta} g(\omega) h'(-\psi + \zeta\omega) d\omega
\end{aligned}$$

this implies

$$\partial_\psi H_\infty(\zeta, \psi) = -ih'(-\psi) + O(\zeta^\varepsilon) \quad \text{for} \quad \zeta \to 0$$

uniformly with respect to $|\psi + \sin\alpha| \leq \Delta$. Taking into account that

$$\operatorname{Im}\left(e^{-i\alpha} \partial_\psi H_\infty(\zeta, \psi)\right) \approx \operatorname{Im}\left(e^{-i\alpha}(-i) h'(\sin\alpha)\right) = -\frac{1}{\cos\alpha} \neq 0$$

we can apply the Implicit Function Theorem to Eq. (49) and finish the proof. ●

### 3.3 Averaged phase velocities

The dynamical mechanism underlying the appearance of partially synchronized solutions (25) can be clarified by calculating the time-averaged phase velocities

$$\omega_{\mathrm{av},k} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{d\theta_k}{dt} dt.$$

In the thermodynamic limit this time-averaged quantity can be replaced by the ensemble average over all oscillators with natural frequencies close to $\omega_k$. To this end, we take for fixed natural frequency $\omega$ the average of the phase velocity $\nu(\theta,\omega,t)$, see definition (5), with respect to the conditional probability $f(\theta,\omega,t)/g(\omega)$ and obtain

$$\omega_{\mathrm{av}}(\omega) = \int_0^{2\pi} \nu(\theta,\omega,t) \frac{f(\theta,\omega,t)}{g(\omega)} d\theta. \tag{52}$$

For a partially synchronized state (25), we obtain from the definition (5) by inserting (12), (23), and (25)

$$\nu(\theta,\omega,t) = \omega + \mathrm{Im}\left(-ipe^{-i\theta}e^{i\Omega t}\right).$$

Inserting this into (52) and using the normalization condition (7) and formula (11), the integral can be evaluated as

$$\begin{aligned} \omega_{\mathrm{av}}(\omega) &= \omega - \mathrm{Re}\left[\, p\, z(\omega,t) e^{-i\Omega t}\right] = \omega - \mathrm{Re}\left[ p\, h\left(\frac{\omega-\Omega}{p}\right)\right] \\ &= \begin{cases} \Omega + (\omega-\Omega)\sqrt{1-\dfrac{p^2}{(\omega-\Omega)^2}} & \text{for} \quad |\omega-\Omega| > p, \\ \Omega & \text{for} \quad |\omega-\Omega| \le p \end{cases} \end{aligned} \tag{53}$$

where we again used explicit form of solution (25) and definition (19). Obviously, the oscillators in the synchronization window $|\omega-\Omega| \le p$ have identical averaged phase velocities equal to the common frequency $\Omega$, whereas the oscillators with $|\omega-\Omega| > p$ have different mean frequencies, which are again given by the universal profile $h$.

## 4 Stability analysis

In this section, we consider the stability of the completely incoherent state and of the partially synchronized states described in Section 3. To this end we perform a linear stability analysis of corresponding solutions $z(\omega,t)$ with respect to the evolution Eq. (9) in the Ott-Antonsen manifold. In order to avoid mathematical difficulties concerned with the treatment of unbounded operators, we suppose here that the frequency distribution $g(\omega)$ has a finite support

$$D := \{\omega \in \mathbb{R} \,:\, g(\omega) \neq 0, \}$$

which is a union of finitely many disjoint intervals. However, we remark that the spectral equations derived below apparently remain valid for any absolutely integrable function $g(\omega)$, although they require more sophisticated methods to justify them.

Our general strategy is as follows. Eq. (9) defines a smooth dynamical system in the Banach space $C(D;\mathbb{C})$. Linearizing it around the trivial solution $z(\omega,t)=0$ (complete incoherence) or a standing wave $z(\omega,t)=a(\omega)e^{i\Omega t}$ (partially synchronized state), we reduce the stability analysis of $z(\omega,t)$ to the spectral analysis of some bounded linear operator $\mathcal{A}$. Following general spectral theory [23], we distinguish two parts in the spectrum $\sigma(\mathcal{A})$:

- point spectrum $\sigma_{\mathrm{pt}}(\mathcal{A})$ consisting of all $\lambda\in\mathbb{C}$ such that the operator $\lambda\mathcal{I}-\mathcal{A}$ is not invertible but is Fredholm of index zero;
- essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{A})$ consisting of all $\lambda\in\mathbb{C}$ such that the operator $\lambda\mathcal{I}-\mathcal{A}$ is neither invertible nor Fredholm of index zero.

By its definition, the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{A})$ includes all $\lambda\in\mathbb{C}$, where the kernel and the range of the operator $\lambda\mathcal{I}-\mathcal{A}$ are both finite-dimensional and

$$\dim\mathrm{Ker}(\lambda\mathcal{I}-\mathcal{A})=\dim\mathrm{Ran}(\lambda\mathcal{I}-\mathcal{A}).$$

Elements of the point spectrum are usually called *eigenvalues*, since they correspond to finite-dimensional degeneracies of operator $\lambda\mathcal{I}-\mathcal{A}$ which can be treated by Lyapunov-Schmidt reduction. In contrast, essential spectrum reflects the infinite-dimensional nature of the operator $\mathcal{A}$.

## 4.1 Stability of complete incoherence

Linearizing formally Eq. (9) around its trivial solution $z(\omega,t)=0$ we obtain

$$\frac{dv}{dt}=i\omega v+\frac{K}{2}e^{-i\alpha}\mathcal{G}v. \tag{54}$$

Note that Eq. (9) contains complex conjugated terms, and hence has to be considered as a system of two equations, comprising the real and imaginary part separately. Introducing the vector-function

$$\mathbf{v}(t)=\begin{pmatrix}\mathrm{Re}\ v(\omega,t)\\ \mathrm{Im}\ v(\omega,t)\end{pmatrix}\in C(D;\mathbb{R}^2). \tag{55}$$

the corresponding form of Eq. (54) is

$$\frac{d\mathbf{v}}{dt}=\mathcal{L}_0\mathbf{v}\quad\text{where}\quad\mathcal{L}_0:=\mathcal{M}_0+\mathcal{K}_0,$$

and $\mathcal{M}_0$ is a multiplication operator in $C(D;\mathbb{R}^2)$ defined by

$$(\mathcal{M}_0\mathbf{v})(\omega)=\mathbf{M}_0(\omega)\mathbf{v}\quad\text{with}\quad\mathbf{M}_0(\omega):=\begin{pmatrix}0&-\omega\\ \omega&0\end{pmatrix},$$

where $\mathcal{K}_0$ is a rank-1 integral operator in $C(D;\mathbb{R}^2)$ given by

$$(\mathcal{K}_0\mathbf{v})(\omega):=\frac{K}{2}\mathbf{Q}\,\mathcal{G}\mathbf{v}=\frac{K}{2}\mathbf{Q}\int_D g\mathbf{v}d\omega$$

with

$$\mathbf{Q} := \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{56}$$

It is well-known [24] that the multiplication operator $\mathcal{M}_0$ has purely essential spectrum consisting of numbers $\lambda \in \mathbb{C}$ which solve equation

$$\det(\lambda\mathbf{I} - \mathbf{M}_0(\omega)) = \det \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix} = 0 \quad \text{for some} \quad \omega \in D.$$

Moreover, the essential spectrum is invariant under finite-rank perturbations (see e.g. [25]). Thus we can easily calculate

$$\sigma_{\mathrm{ess}}(\mathcal{L}_0) = \sigma_{\mathrm{ess}}(\mathcal{M}_0) = \{\lambda = i\omega \ : \ \omega \in D\} \cup \{\mathrm{c.c.}\}.$$

This formula shows that the essential spectrum of complete incoherence is always neutral and therefore cannot induce a linear instability of the completely incoherent state.

The point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L}_0)$ is given by nontrivial solutions of the equation

$$(\lambda\mathcal{I} - \mathcal{M}_0 - \mathcal{K}_0)\mathbf{v} = 0. \tag{57}$$

Substituting here the explicit expressions of the operators $\mathcal{M}_0$ and $\mathcal{K}_0$ and taking into account that $\lambda\mathbf{I} - \mathbf{M}_0(\omega)$ is invertible for $\lambda \notin \sigma_{\mathrm{ess}}(\mathcal{L}_0)$, we transform Eq. (57) into

$$\mathbf{v} = \frac{K}{2}(\lambda\mathbf{I} - \mathbf{M}_0(\omega))^{-1}\mathbf{Q}\,\mathcal{G}\mathbf{v}. \tag{58}$$

Applying now the integral operator $\mathcal{G}$ to both sides of Eq. (58) we arrive at a two dimensional spectral problem for $\mathcal{G}\mathbf{v}$

$$\mathcal{G}\mathbf{v} = \mathbf{B}_0(\lambda)\mathcal{G}\mathbf{v} \quad \text{where} \quad \mathbf{B}_0(\lambda) := \frac{K}{2}\mathbf{Q}\int_D g(\omega)(\lambda\mathbf{I} - \mathbf{M}_0(\omega))^{-1}\,d\omega.$$

This yields the characteristic equation

$$\chi_0(\lambda) := \det(\mathbf{I} - \mathbf{B}_0(\lambda)) = 0. \tag{59}$$

Straightforward calculations show that Eq. (59) is equivalent to the system

$$\frac{K}{2}\int_D \frac{\lambda g(\omega)}{\lambda^2 + \omega^2} d\omega \;=\; \cos\alpha, \tag{60}$$

$$\frac{K}{2}\int_D \frac{\omega g(\omega)}{\lambda^2 + \omega^2} d\omega \;=\; \sin\alpha, \tag{61}$$

which can be written in complex form as

$$\frac{iK}{2}\int_D \frac{g(\omega)}{\omega + i\lambda} d\omega = e^{i\alpha}. \tag{62}$$

To determine the stability boundary, i.e. the appearance of point spectrum $\sigma_{\rm pt}(\mathcal{L}_0)$ in the positive half-plane $\mathrm{Re}\ \lambda > 0$, we substitute $\lambda = i\Omega_0 + \varepsilon$ with $\Omega_0 \in \mathbb{R}$ and $\varepsilon > 0$ into (62) and obtain

$$\frac{1}{K}e^{i\alpha} = \lim_{\varepsilon\to+0}\frac{i}{2}\int_D \frac{g(\omega)d\omega}{\omega - \Omega_0 + i\varepsilon}. \tag{63}$$

Remark that the integral in (63) is singular and has to be calculated via the Cauchy formula. As a result, we recover the self-consistency equation (23) for the case $p = 0$

$$\frac{1}{K}e^{i\alpha} = \frac{\pi}{2}g(\Omega_0) + \frac{i}{2}\lim_{\varepsilon\to+0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{g(\Omega_0+\omega)}{\omega}d\omega = J(\Omega_0). \tag{64}$$

Hence, the onset of instability for the completely incoherent state, which we considered here, coincides with the limit $p \to 0$ for the partially coherent states, analyzed in the previous section. In this way, we identified a bifurcation where an unstable eigenvalue emerges from the neutral continuous spectrum and a branch of solutions bifurcates that becomes singular at the bifurcation point. Some illustrating examples of this phenomenon will be presented in Section 5.

## 4.2 Stability of partially synchronized states

The stability analysis of partially synchronized states described by standing waves (13) can be performed in a similar way. We substitute

$$z(\omega,t) = (a(\omega) + v(\omega,t))e^{i\Omega t}$$

into Eq. (9), where $v \in C(D;\mathbb{C})$ is a small deviation. Formal linearization with respect to $v$ gives

$$\frac{dv}{dt} = \eta(\omega)v + \frac{K}{2}e^{-i\alpha}\mathcal{G}v - \frac{K}{2}e^{i\alpha}a^2(\omega)\mathcal{G}\overline{v} \tag{65}$$

where

$$\eta(\omega) = i(\omega - \Omega) - Ke^{i\alpha}a(\omega)\mathcal{G}\overline{a} = i(\omega - \Omega - \overline{Z}a(\omega)), \tag{66}$$

see formula (16) for the definition of $Z$. As before, we have to switch to a system of two real equations for the vector $\mathbf{v}(t) \in C(D;\mathbb{R}^2)$, see (55). This system has again the form

$$\frac{d\mathbf{v}}{dt} = \mathcal{L}\mathbf{v} \quad \text{with} \quad \mathcal{L} := \mathcal{M} + \mathcal{K}, \tag{67}$$

where the multiplication operator $\mathcal{M} \in C(D;\mathbb{R}^2)$ is now given by

$$(\mathcal{M}\mathbf{v})(\omega) = \mathbf{M}(\omega)\mathbf{v} \quad \text{with} \quad \mathbf{M}(\omega) := \begin{pmatrix} \mathrm{Re}\ \eta(\omega) & -\mathrm{Im}\ \eta(\omega) \\ \mathrm{Im}\ \eta(\omega) & \mathrm{Re}\ \eta(\omega) \end{pmatrix}, \tag{68}$$

and the rank-1 integral operator $\mathcal{K} \in C(D;\mathbb{R}^2)$ has the form

$$(\mathcal{K}\mathbf{v})(\omega) := \frac{K}{2}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))\,\mathcal{G}\mathbf{v} = \frac{K}{2}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))\int_D g(\omega)\mathbf{v}(\omega)d\omega$$

with

$$\mathbf{P}(\omega) := \begin{pmatrix} \mathrm{Re}\left[\left(e^{i\alpha}a(\omega)\right)^2\right] & \mathrm{Im}\left[\left(e^{i\alpha}a(\omega)\right)^2\right] \\ \mathrm{Im}\left[\left(e^{i\alpha}a(\omega)\right)^2\right] & -\mathrm{Re}\left[\left(e^{i\alpha}a(\omega)\right)^2\right] \end{pmatrix} \tag{69}$$

and the constant matrix $\mathbf{Q}$ defined in (56). Again, the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ is identical to the essential spectrum of $\mathcal{M}$ given by all $\lambda \in \mathbb{C}$ such that

$$\det(\lambda\mathbf{I} - \mathbf{M}(\omega)) = 0 \quad \text{for some} \quad \omega \in D.$$

This implies

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{M}) = \{\lambda = \eta(\omega) \;:\; \omega \in D\} \cup \{\mathrm{c.c.}\}. \tag{70}$$

Substituting the expression (17) for $a(\omega)$ with $p = |Z| = Z$ and the expression (19) for $h$ into (66), we obtain

$$\eta(\omega) = \begin{cases} i(\omega - \Omega)\sqrt{1 - \dfrac{p^2}{(\omega-\Omega)^2}} & \text{for} \quad |\omega - \Omega| > p, \\ -\sqrt{p^2 - (\omega-\Omega)^2} & \text{for} \quad |\omega - \Omega| \le p. \end{cases} \tag{71}$$

According to (70) and (71), the partially synchronized solutions (25) have a $T$-shaped essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ with a symmetric interval along the imaginary axis corresponding to drifting oscillators with $|\omega - \Omega| > p$, and another interval on the negative part of the real axis corresponding to frequency-locked oscillators with $|\omega - \Omega| \le p$. Hence, similar to the completely incoherent state, partially synchronized states (25) are always neutrally stable with respect to their essential spectrum. Therefore, instabilities may originate from the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ only. By its definition, $\sigma_{\mathrm{pt}}(\mathcal{L})$ consists of all $\lambda \in \mathbb{C}\backslash\sigma_{\mathrm{ess}}(\mathcal{L})$ such that equation

$$\lambda\mathbf{v} = \mathbf{M}(\omega)\mathbf{v} + \frac{K}{2}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))\mathcal{G}\mathbf{v}$$

has a non-trivial solution $\mathbf{v} \in C(D;\mathbb{R}^2)$. Since $\lambda \notin \sigma_{\mathrm{ess}}(\mathcal{L})$, this equation can be rewritten as follows

$$\mathbf{v} = \frac{K}{2}(\lambda\mathbf{I} - \mathbf{M}(\omega))^{-1}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))\mathcal{G}\mathbf{v}. \tag{72}$$

Next, applying the integral operator $\mathcal{G}$ to both sides of Eq. (72) we obtain a two-dimensional linear system with respect to $\mathcal{G}\mathbf{v} \in \mathbb{R}^2$

$$\mathcal{G}\mathbf{v} = \mathbf{B}(\lambda)\mathcal{G}\mathbf{v}, \tag{73}$$

where

$$\begin{aligned} \mathbf{B}(\lambda) \;&:=\; \frac{K}{2}\int_D g(\omega)(\lambda\mathbf{I} - \mathbf{M}(\omega))^{-1}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))d\omega \\ &=\; \frac{K}{2}\int_{-\infty}^{\infty} g(\omega)(\lambda\mathbf{I} - \mathbf{M}(\omega))^{-1}\mathbf{Q}(\mathbf{I} - \mathbf{P}(\omega))d\omega. \end{aligned}$$

Hence, the characteristic equation for the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ is

$$\chi(\lambda) := \det\left(\mathbf{I} - \mathbf{B}(\lambda)\right) = 0. \tag{74}$$

In general, one can solve Eq. (74) only numerically. However, an important stability criterion for partially synchronized states can be derived analytically by the following properties of $\chi(\lambda)$:

**Proposition 4** *Suppose that $g \in C^2(\mathbb{R})$. Then for any standing wave (25) with parameters $p$, $\Omega$ satisfying Eq. (23) the characteristic equation for real point spectrum $\lambda \in \mathbb{R}$ reduces to*

$$\chi(\lambda) = K^2 \lambda \left( \lambda |F_1(\lambda)|^2 + p \operatorname{Im} \left( F_1(\lambda) \overline{F}_2(\lambda) \right) \right), \tag{75}$$

*where*

$$F_k(\lambda) := \int_{-\infty}^{\infty} g(\Omega + ps) \frac{h^k(s) ds}{\lambda - ip(s - h(s))}, \quad k = 1, 2.$$

**Proof:** Inserting $\mathbf{B}(\lambda)$ into Eq. (74) and taking into account that matrices $\mathbf{Q}$ and $(\lambda \mathbf{I} - \mathbf{M}(\omega))^{-1}$ commute, we obtain

$$\begin{aligned}
\chi(\lambda) &= \det \left( \mathbf{I} - \frac{K}{2} \int_{-\infty}^{\infty} g(\omega)(\lambda \mathbf{I} - \mathbf{M}(\omega))^{-1} \mathbf{Q} (\mathbf{I} - \mathbf{P}(\omega)) d\omega \right) \\
&= \det \left( \frac{K}{2} \mathbf{Q} \right) \det \left( \frac{2}{K} \mathbf{Q}^{-1} - \int_{-\infty}^{\infty} g(\omega)(\lambda \mathbf{I} - \mathbf{M}(\omega))^{-1} (\mathbf{I} - \mathbf{P}(\omega)) d\omega \right) \\
&= \frac{K^2}{4} \det \left( \frac{2}{K} \mathbf{Q}^{-1} - \int_{-\infty}^{\infty} g(\omega)(\lambda \mathbf{I} - \mathbf{M}(\omega))^{-1} (\mathbf{I} - \mathbf{P}(\omega)) d\omega \right).
\end{aligned}$$

Straightforward calculations demonstrate that for real $\lambda$ the latter matrix can be rewritten in the form

$$\begin{aligned}
&\frac{2}{K} \mathbf{Q}^{-1} - \int_{-\infty}^{\infty} g(\omega)(\lambda \mathbf{I} - \mathbf{M}(\omega))^{-1} (\mathbf{I} - \mathbf{P}(\omega)) d\omega \\
&\quad = \begin{pmatrix} \operatorname{Re}\, c(\lambda) & -\operatorname{Im}\, c(\lambda) \\ \operatorname{Im}\, c(\lambda) & \operatorname{Re}\, c(\lambda) \end{pmatrix} + \begin{pmatrix} \operatorname{Re}\, d(\lambda) & \operatorname{Im}\, d(\lambda) \\ \operatorname{Im}\, d(\lambda) & -\operatorname{Re}\, d(\lambda) \end{pmatrix},
\end{aligned}$$

where

$$c(\lambda) := \frac{2}{K} e^{i\alpha} - \int_{-\infty}^{\infty} \frac{g(\omega)}{\lambda - \eta(\omega)} d\omega, \quad d(\lambda) := \int_{-\infty}^{\infty} \frac{g(\omega)(e^{i\alpha} a(\omega))^2}{\lambda - \eta(\omega)} d\omega.$$

Hence,

$$\begin{aligned}
\chi(\lambda) &= \frac{K^2}{4} \left( |c(\lambda)|^2 - |d(\lambda)|^2 \right) \\
&= \frac{K^2}{4} \left( \left| \frac{2}{K} e^{i\alpha} - \int_{-\infty}^{\infty} \frac{g(\omega)}{\lambda - \eta(\omega)} d\omega \right|^2 - \left| \int_{-\infty}^{\infty} \frac{g(\omega) a^2(\omega)}{\lambda - \eta(\omega)} d\omega \right|^2 \right).
\end{aligned} \tag{76}$$

For $a(\omega) = h((\omega - \Omega)/p)$ where $p$, $\Omega$ satisfy Eq. (23), formula (66) implies

$$\eta(\omega) = ip \left( \frac{\omega - \Omega}{p} - h \left( \frac{\omega - \Omega}{p} \right) \right).$$

Inserting this into Eq. (76) and changing the integration variable $\omega \mapsto s = (\omega - \Omega)/p$ we obtain

$$\chi(\lambda) = \frac{K^2}{4}\left(\left|\frac{2}{K}e^{i\alpha} - \int_{-\infty}^{\infty}\frac{pg(\Omega+ps)}{\lambda - ip(s-h(s))}ds\right|^2 - \left|\int_{-\infty}^{\infty}\frac{pg(\Omega+ps)h^2(s)}{\lambda - ip(s-h(s))}ds\right|^2\right).$$

Now, using Eq. (23) and the fact that $h(s)$ solves by definition the quadratic equation

$$h^2(s) - 2sh(s) + 1 = 0, \tag{77}$$

we rewrite $\chi(\lambda)$ as follows:

$$\begin{aligned}\chi(\lambda) &= \frac{K^2}{4}\left(\left|2i\int_{-\infty}^{\infty}g(\Omega+ps)h(s)ds - \int_{-\infty}^{\infty}\frac{pg(\Omega+ps)}{\lambda - ip(s-h(s))}ds\right|^2\right.\\ &- \left.\left|\int_{-\infty}^{\infty}\frac{pg(\Omega+ps)h^2(s)}{\lambda - ip(s-h(s))}ds\right|^2\right) = \frac{K^2}{4}\left(\left|\int_{-\infty}^{\infty}g(\Omega+ps)\frac{2i\lambda h(s) - ph^2(s)}{\lambda - ip(s-h(s))}ds\right|^2\right.\\ &- \left.\left|\int_{-\infty}^{\infty}\frac{pg(\Omega+ps)h^2(s)}{\lambda - ip(s-h(s))}ds\right|^2\right) = \frac{K^2}{4}\left(\left|2i\lambda F_1(\lambda) - pF_2(\lambda)\right|^2 - p^2\left|F_2(\lambda)\right|^2\right)\end{aligned}$$

and obtain the claimed identity. •

Propostion 4 underlines that the characteristic equation (74) always has the zero solution caused by phase-shift symmetry. Moreover, we will give now a sufficient condition for the existence of another positive real eigenvalue in the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$.

**Proposition 5** *If $\Psi(p,\Omega) < 0$, where $\Psi(p,\Omega)$ is the function given by (26) as the degeneracy condition for $H(p,\Omega)$, then the characteristic equation (74) has at least one positive real solution $\lambda$.*

**Proof:** Taking into account formula (71), we see that the integral definitions of $F_1(\lambda)$ and $F_2(\lambda)$ remain correct for all complex $\lambda$ such that $\mathrm{Re}\ \lambda > 0$. Respectively, $F_1(\lambda)$ and $F_2(\lambda)$ are analytic in the right half-plane and thus may have there only isolated zeros of finite multiplicity.

Now, let us consider equation (75) for positive real $\lambda$:

$$\lambda = -p\,\mathrm{Im}\left(F_1(\lambda)\overline{F}_2(\lambda)\right)/|F_1(\lambda)|^2 \tag{78}$$

It is easy to verify that its right-hand side is bounded for all $\lambda \to \infty$. On the other hand, using the identity

$$h'(s) = \frac{h(s)}{h(s) - s}$$

following from Eq. (77), we can find

$$\lim_{\lambda\to+0} F_1(\lambda) = \frac{i}{p}\int_{-\infty}^{\infty}g(\Omega+ps)\frac{h(s)ds}{s-h(s)} = -\frac{i}{p}\int_{-\infty}^{\infty}g(\Omega+ps)h'(s)ds$$

$$
\begin{aligned}
&= i\int_{-\infty}^{\infty} g'(\Omega+ps)h(s)ds = \partial_\Omega H(p,\Omega),\\
\lim_{\lambda\to+0} F_2(\lambda) &= \frac{i}{p}\int_{-\infty}^{\infty} g(\Omega+ps)\frac{h^2(s)ds}{s-h(s)} = -\frac{i}{p}\int_{-\infty}^{\infty} g(\Omega+ps)h(s)h'(s)ds\\
&= \frac{i}{2}\int_{-\infty}^{\infty} g'(\Omega+ps)h^2(s)ds = \frac{i}{2}\int_{-\infty}^{\infty} g'(\Omega+ps)(2sh(s)-1)ds\\
&= \partial_p H(p,\Omega),
\end{aligned}
$$

where in the last formula we also used Eq. (77) to replace $h^2(s)$. This means that

$$
\lim_{\lambda\to+0} \mathrm{Im}\left(F_1(\lambda)\overline{F}_2(\lambda)\right) = \Psi(p,\Omega),
$$

i.e. the right-hand side of Eq. (78) has a positive limiting value. Together with the boundedness for $\lambda \to \infty$ and continuity, this implies the existence of at least one positive real solution. •

# 5 Examples

In the previous sections we developed a theoretical framework that allows for any given frequency distribution $g(\omega)$ to determine the partially synchronous states, their stability and their dependence on the parameters $\alpha$ and $K$. This work was motivated by the recent discovery of non-universal synchronization transitions that can appear for certain unimodal frequency distributions. In this section, we discuss several examples of frequency distributions that display nontrivial bifurcation scenarios. In a second step we also consider families of frequency distributions in order to illustrate, how our theoretical approach can be extended to identify bifurcations of higher codimension at which the nontrivial behavior emerges.

Our first example is a frequency distribution given by a superposition

$$
g(\omega,\delta,\tau) = \tau g_0(\omega,1) + (1-\tau)g_0(\omega,\delta) \tag{79}
$$

of two Gaussian distributions

$$
g_0(\omega,\delta) = \frac{1}{\sqrt{2\pi}\delta} e^{-\omega^2/(2\delta^2)} \tag{80}
$$

where the width of the first Gaussian is fixed to one, while the width $\delta$ of the second is treated as a parameter. First, we fix the parameters of the frequency distribution by setting $\delta = 0.1$ and $\tau = 0.6$. For any partially synchronized state (25), determined by the parameters $p$ and $\Omega$, the self-consistency equation (23) provides the parameters $\alpha$ and $K$, for which this state appears. As we have seen in Section 4.1, the line $p = 0$ defines the location where the completely incoherent state changes its stability and partially synchronized states emerge (red curve in Fig. 2). Within the shaded region, the completely incoherent state is (marginally) stable. According to Proposition 1, along the red curve $p = 0$ we have $\partial_p H(p,\Omega) = 0$ and hence $\Psi(p,\Omega) = 0$. This degeneracy condition is also satisfied at fold curves of partially coherent states with $p > 0$

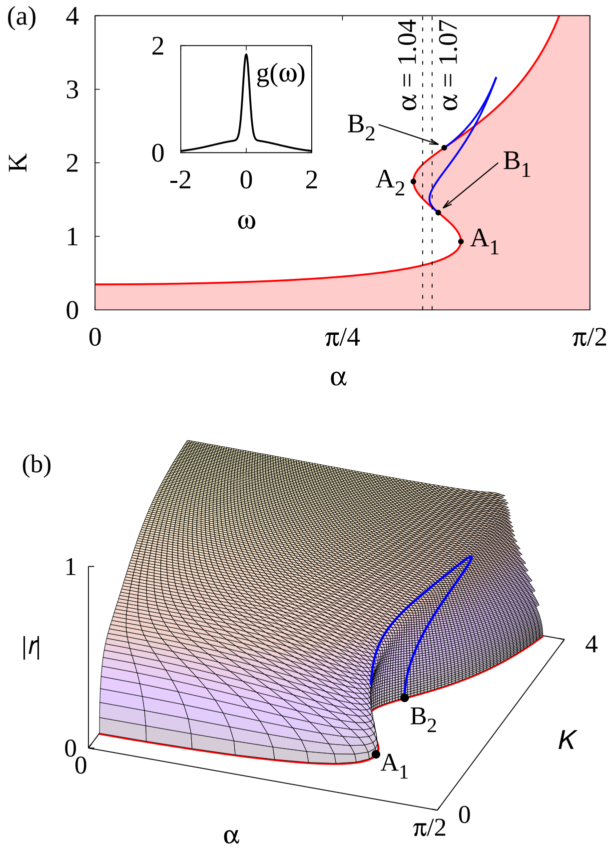


Figure 2: (color online) Bifurcation diagram for a double Gaussian frequency distribution (79), (80) with $\delta = 0.1$ and $\tau = 0.6$ (see insert panel). Stability region (shaded) and stability boundary (red curve) given by (63) for the completely incoherent state. Turning points of this curve at $A_1$, $A_2$, given by (84). Fold of partially synchronized states (blue curve), ending at points $B_1$, $B_2$, given by (82). Cross sections (dashed) for synchronization transitions displayed in Figs. 1(b) and 1(c). Panel (b) shows the order parameter $|r|$ versus $\alpha$ and $K$ calculated by (20) and (23).

(blue curve in Fig. 2). This curve includes a region with multistability ending at the corresponding segment of the red curve.

Fig. 2(a) shows also bifurcation points of codimension two: The blue fold curve contains a cusp point that can be calculated by

$$\left\langle w^*, \frac{\partial^2 H}{\partial w^2} \right\rangle = 0, \tag{81}$$

where $w$ and $w^*$ are the kernel and its adjoint from (26) and the angle brackets $\langle \cdot, \cdot \rangle$ denote usual scalar product in $\mathbb{R}^2$. Further codimension two points are the endpoints $B_1$ and $B_2$ of the blue curve, which are characterized by the degeneracy condition

$$\partial_p \Psi(0, \Omega) = 0. \tag{82}$$

Using the expressions from Proposition 1, this condition turns out to be equivalent to

$$\mathrm{Im}(\overline{J}''(\Omega) J'(\Omega)) = \mathrm{Im}(\Phi_2) = 0, \tag{83}$$

where according to (37) in Proposition 2 the asymptotics $|r| = O(|K - K_\mathrm{c}|^{1/2})$ of the order parameter breaks down.

In terms of synchronization transitions $|r|$ versus $K$ with fixed $\alpha$, as depicted in Fig. 1 (compare vertical dashed lines in Fig. 2(a)), this has the following consequences: At intersection points with the blue line the branch of partially coherent states is folded and changes its stability. Wherever the red curve is not a graph over $\alpha$, we observe incoherence regaining its stability for increasing $K$ and several branches of partially coherent states bifurcating from complete incoherence. The turning points $A_1$ and $A_2$ of the red curve are given by

$$\partial_\Omega \alpha_0(\Omega) = 0, \tag{84}$$

where

$$\alpha_0(\Omega) := \frac{1}{2i} \log \left( \frac{J(\Omega)}{\overline{J}(\Omega)} \right)$$

is the phase lag obtained by solving Eq. (63). Simple calculations demonstrate that (84) is equivalent to

$$\mathrm{Im}(\overline{J}(\Omega) J'(\Omega)) = \mathrm{Im}(\Phi_1) = 0, \tag{85}$$

which is the degeneracy condition (36) in Proposition 2. Note that we have at these points still a usual codimension one fold bifurcation, which however cannot be unfolded by the parameter $K$.

Now, we study the appearance of non-universal synchronization transitions due to changes of the frequency distribution $g(\omega)$, induced by the parameters $\delta$ and $\tau$ in (79), (80). Note that the origin of the non-universal synchronization transitions is twofold: The turning points $A_1$, $A_2$ of the stability boundary (63) with respect to $\alpha$ allow for transitions where incoherence regains its stability for increasing $K$, and the existence of the fold curve (26) with $p > 0$ allows for transitions with bistability.

The points $A_1$ and $A_2$ can merge, when the condition (84) has a double root

$$\partial_\Omega^2 \alpha_0(\Omega) = 0,$$

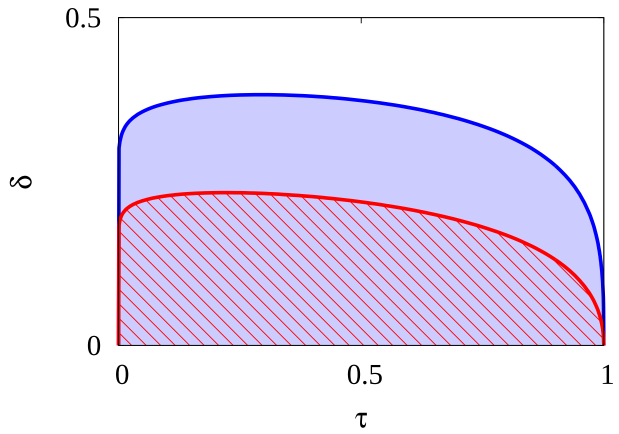


Figure 3: (color online) Bifurcation diagram for the two parameter family of frequency distributions (79), (80). Hatched region: parameters of frequency distributions that allow for the completely incoherent state to regain stability with increasing $K$. Shaded region: parameters of frequency distributions that allow for bistability.

which leads to

$$\mathrm{Im}(\overline{J}''(\Omega)J(\Omega)) = \mathrm{Im}(\Phi_0) = 0. \tag{86}$$

Note that according to the identity (45) the conditions (84) imply (82), i.e. $A_1$ and $A_2$ can merge only at an endpoint of the (blue) fold curve. This implies in particular, that incoherence regaining its stability implies bistability.

Finally, the condition for the points $B_1$ and $B_2$ to merge is that Eq. (82) has a degenerate root $\Omega$ with

$$\partial_\Omega \partial_p \Psi(0, \Omega) = 0, \tag{87}$$

implying

$$\mathrm{Im}(\overline{J}'''(\Omega)J'(\Omega)) = 0. \tag{88}$$

Thus, solving systems (85), (86) and (83), (88) we outline the boundaries of distribution parameters $\delta$ and $\tau$ leading to the appearance of non-universal synchronization transitions, see Fig. 3. Roughly speaking, the non-universality is favoured for heavy tailed distributions with small $\delta$ and almost all values of parameter $\tau$.

With the next example, we demonstrate that our approach can be also applied to non-smooth frequency distributions. For example, using in the superposition (79) the uniform distribution

$$g_0(\omega, \delta) = \begin{cases} 1/(2\delta) & \text{for} \quad |\omega| \le \delta, \\ 0 & \text{for} \quad |\omega| > \delta, \end{cases} \tag{89}$$

we obtain similar results as above, see Fig. 4. However, according to (23) the function $H(p, \Omega)$ loses its smoothness as well, and there appear non-generic features like corners or vertical segments in the bifurcation diagrams.

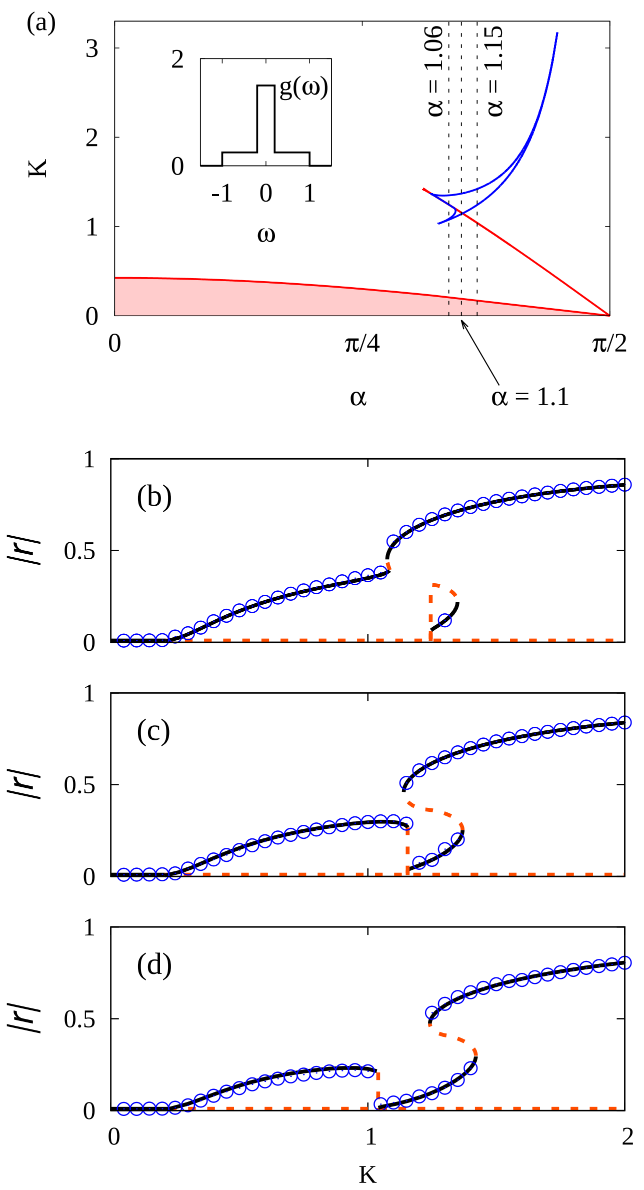


Figure 4: (color online) (a) Bifurcation diagram for a double uniform distribution (79), (89) with $\delta = 0.2$ and $\tau = 0.5$ (see insert panel). Stability region (shaded) and stability boundary (red curve) for the completely incoherent state. Fold of partially synchronized states (blue curve), cf. Fig. 2. Panels (b)–(d): synchronization transitions at cross sections (dashed vertical lines in panel (a)). Circles: numerical simulations of Eqs. (1) with $N = 10000$, cf. Fig. 1.

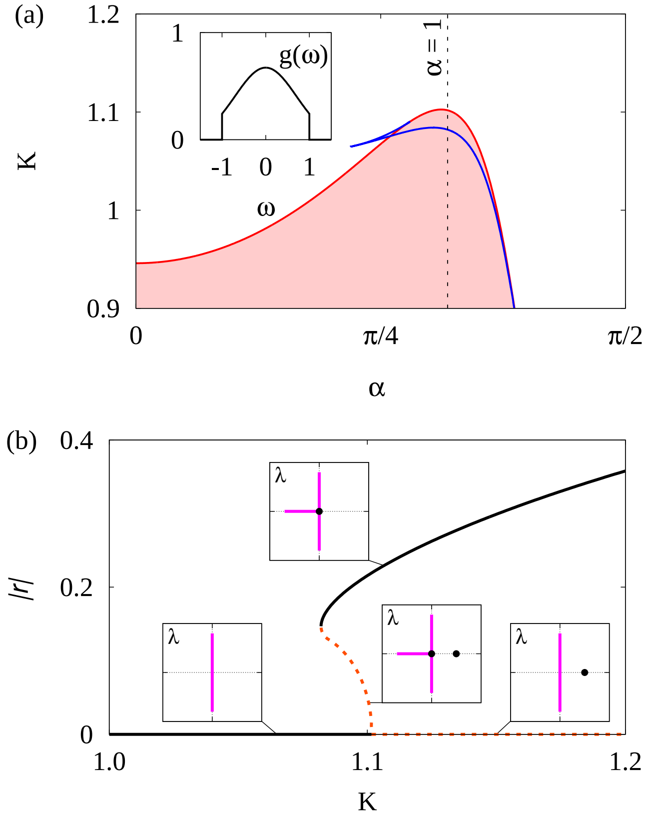


Figure 5: (color online) (a) Bifurcation diagram for a truncated Gaussian distribution (90) with $\kappa = 0.7$ (see insert panel), cf. Fig. 2 and 4. (b) Synchronization transition for $\alpha = 1$ with inserted instances of spectra.

Note that non-classical synchronization transitions may also be found for other distributions that are not given by formula (79). As an example we consider the truncated Gaussian distribution

$$g(\omega,\kappa)=\begin{cases} c_\kappa e^{-\omega^2/(2\kappa^2)} & \text{for} \quad |\omega|\le 1,\\ 0 & \text{for} \quad |\omega|>1, \end{cases} \tag{90}$$

where

$$c_\kappa=\sqrt{\frac{2}{\pi\kappa^2}}\left(\operatorname{erf}\left(\frac{1}{\sqrt{2}\kappa}\right)-\operatorname{erf}\left(-\frac{1}{\sqrt{2}\kappa}\right)\right)^{-1}$$

is a normalization constant. Fig. 5(a) shows the corresponding bifurcation diagram whereas Fig. 5(b) displays the spectra at the different branches. The completely incoherent state has continuous spectrum on the imaginary axis and an unstable eigenvalue that emerges at the stability boundary from the continuous spectrum. Along the partially coherent branch, there is continuous spectrum also on the stable side of the real axis. Moreover, there is an eigenvalue at zero due to symmetry and a positive real eigenvalue at the unstable part of the branch.

# 6 Discussion

We showed that the Sakaguchi-Kuramoto system can display interesting non-trivial bifurcation scenarios even in the case of unimodal and symmetric frequency distributions. Morever, we presented a mathematical approach to study stability and bifurcations in the case of general frequency distributions $g(\omega)$. In this way, we provided a detailed theoretical background for the appearance of non-universal synchronization transitions that have been reported recently in [14]. Further interesting phenomena that appear for more general non-unimodal frequency distributions [26, 27] as well as in systems with distributed phase lags [28], distributed coupling strengths [29] or nonlinear coupling [30, 31] could be also treated within this framework. In this case, one can expect in particular the appearance of Hopf bifurcations which also may involve continuous spectrum. At the other hand, restricting to a superposition of two Lorenzian distributions with different width, similar phenomena could be also discussed in the framework of an ODE system that can be obtained by explicit integration. Indeed, interpreting the superposition of two frequency distributions as a specific case of a two-population model with Lorenzian frequency distributions as introduced in [32], one could also use the reduced ODE system for a straightforward bifurcation analysis. However, by this reduction, the specific properties connected to the existence of neutrally stable continuous spectrum and eigenvalues bifurcating from that are lost.

Concerning the spectral analysis of complete incoherence and partially coherent states, our results are an extension of the results by Mirollo and Strogatz, who studied in [16, 33, 34] the corresponding spectra for the classical Kuramoto system in the continuum limit given by the continuity equation (3). But many important questions arising in this context, e.g about general properties of the bifurcations involving continuous spectrum or about the non-linear stabilization processes that are apparently present in the finite size system, remain still open.

Concerning the relation of finite size systems to the Ott-Antonsen continuum limit there have been recently obtained some interesting results for Sakaguchi-Kuramoto systems with non-local coupling. In this case, with so called chimera states a different type of partial synchrony appears and it turns out that continuous spectrum in the continuum limit can be seen as a limit of Lyapunov spectrum for spatially extended weak chaos [35]. Moreover, in the case of bistability, on can observe spontaneous transitions as a finite size effect induced by the fluctuations with respect to the continuum limit.

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