Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Path-wise approximation of the Cox-Ingersoll-Ross process

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submitted: January 16, 2013

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No. 1763
Berlin 2013

2010 *Mathematics Subject Classification.* 65C30, 60H35.

Key words and phrases. Cox-Ingersoll-Ross process, Heston model, Doss-Sussmann formalism, exact simulation.

Edited by
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Abstract

The Doss-Sussmann (DS) approach is used for simulating the Cox-Ingersoll-Ross (CIR) process. The DS formalism allows for expressing trajectories of the CIR process by solutions of some ordinary differential equation (ODE) that depend on realizations of the Wiener process involved. Via simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving an ODE, we approximately construct the trajectories of the CIR process. From a conceptual point of view the proposed method may be considered as an exact simulation approach.

1 Introduction

The Cox-Ingersoll-Ross process $V(t)$ is determined by the following stochastic differential equation (SDE)

$$dV(t) = k(\lambda - V(t))dt + \sigma\sqrt{V}dw(t),\ V(t_0) = V_0, \tag{1}$$

where $k,\ \lambda,\ \sigma$ are positive constants, and w is a scalar Brownian motion (see [5]). It is known ([13], [14]) that for $V_0 > 0$ there exists a unique strong solution $V_{t_0,V_0}(t)$ of (1) for all $t \geq t_0 \geq 0$. The CIR process $V(t) = V_{t_0,V_0}(t)$ is positive in the case $2k\lambda \geq \sigma^2$ and nonnegative in the case $2k\lambda < \sigma^2$.

The CIR process is used in financial modelling, in particular, as volatility process in the Heston model. As a matter of fact, (1) does not satisfy the global Lipschitz assumption. The difficulties arising in a simulation method for (1) are connected with this fact, and with the natural requirement of preserving nonnegative approximations. A lot of approximation methods for the CIR process are proposed. For an extensive list of articles on this subject we refer to [3] and [6]. Besides [3] and [6] we also refer to [1, 2, 10, 11], where a number of discretization schemes for the CIR process can be found. Further we note that in [16] a weakly convergent fully implicit method is implemented for the Heston model. Exact simulation of (1) is considered in [4, 8] (see [3] as well).

In this paper we consider path-wise approximation of $V(t)$ on an interval $[t_0, t_0 + T]$ using the Doss-Sussmann transformation ([7], [18], [17]) which allows for expressing any trajectory of $V(t)$ by the solution of some ordinary differential equation that depends on the realization of $w(t)$. By simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving this ODE, we approximately construct a generic trajectory of $V(t)$. Such kind of simulation is more simple than the one proposed in [4] and moreover has some other advantages. That said, it should be noted that the original DS results rely on a global Lipschitz assumption that is not fulfilled for (1). We therefore have introduced a local DS formalism that yields a corresponding ODE which solutions are defined on random time intervals. If V gets close to zero however, the ODE becomes intractable for numerical integration and so for the parts of a trajectory $V(t)$ that are close to zero we are forced to use some other (non DS based) approach. For such parts we here propose to use exact simulation. Another restriction is connected with the condition $\alpha := 4k\lambda - \sigma^2 \geq 0$ (we note that the case $\alpha \geq 0$ is implied by the above mentioned case $2k\lambda \geq \sigma^2$ that ensures positivity of $V(t)$). Apparently the results here obtained for $\alpha \geq 0$ can be carried over to the case where $\alpha < 0$. However, the case $\alpha < 0$ will be considered in a subsequent work.

The next two sections are devoted to the DS formalism in connection with (1.1) and to some auxiliary propositions. In Sections 4 and 5 we deal with the one-step approximation and the convergence of the proposed method, respectively.

2 The Doss-Sussmann transformation

2.1. Due to the Doss-Sussmann approach ([7], [18], [17]), the solution of (1) may be expressed in the form

$$V(t) = F(X(t), w(t)), \tag{2}$$

where $F = F(x, y)$ is some deterministic function and $X(t)$ is the solution of some ordinary differential equation depending on the part $w(s),\ 0 \le s \le t$, of the realization $w(\cdot)$ of the Wiener process $w(t)$.

Let us recall the Doss-Sussmann formalism according to [17], V.28. In [17] one considers the Stratonovich SDE

$$dV(t) = b(V)dt + \gamma(V) \circ dw(t). \tag{3}$$

The function $F = F(x, y)$ is found from the equation

$$\frac{\partial F}{\partial y} = \gamma(F),\ F(x, 0) = x, \tag{4}$$

and $X(t)$ is found from the ODE

$$\frac{dX}{dt} = \frac{1}{\partial F/\partial x(X(t), w(t))} b(F(X(t), w(t)),\ X(0) = V(0). \tag{5}$$

Obviously, the solution $X(t)$ of (5) depends on $X(0)$ and on $w(s),\ 0 \le s \le t$, but for convenience we shall suppress this dependence and write $X(t)$ for short. Equation (1) has the following Stratonovich form

$$dV = (k(\lambda - V) - \frac{\sigma^2}{4})dt + \sigma\sqrt{V} \circ dw(t), \tag{6}$$

i.e., in (3)-(5) we have

$$\gamma(V) = \sigma\sqrt{V},\ b(V) = k(\lambda - V) - \frac{\sigma^2}{4}.$$

Equation (4) thus becomes

$$\frac{\partial F}{\partial y} = \sigma\sqrt{F},\ F(x, 0) = x,\ x > 0,$$

whence

$$F(x, y) = (\sqrt{x} + \frac{\sigma}{2}y)^2. \tag{7}$$

Due to (5) we get

$$\frac{dX}{dt} = [-\frac{\sigma^2}{4} + k(\lambda - (\sqrt{X} + \frac{\sigma}{2}w(t))^2)]\frac{\sqrt{X}}{\sqrt{X} + \frac{\sigma}{2}w(t)},\ X(0) = V(0). \tag{8}$$

From (2), (7), and the solution of (8), we formally obtain the solution $V(t)$ of (1):

$$V(t) = (\sqrt{X(t)} + \frac{\sigma}{2}w(t))^2. \tag{9}$$

2.2. Since the Doss-Sussmann results rely on a global Lipschitz assumption that is not fulfilled for (1), solution (9) has to be considered only formally. In this section we therefore give a direct proof of the following more precise result.

Proposition 1 *Let $X(0) = V(0) > 0$. There exists a stopping time $\tau > 0$ such that equation (8) has a unique positive solution $X(t)$ on the interval $[0,\tau)$ and the solution $V(t)$ of (1) is expressed by formula (9) on this interval.*

Proof. Let $(w(t),\ V(t))$ be the solution of the SDE system

$$dw = dw(t),\ dV = k(\lambda - V)dt + \sigma\sqrt{V}dw(t), \tag{10}$$

which satisfies the initial conditions $w(0) = 0,\ V(0) > 0$. Let us introduce the stopping time $\tau > 0$ defined by

$$\tau := \inf\{t : V(t)\cdot(\sqrt{V(t)} - \frac{\sigma}{2}w(t)) = 0\}.$$

We note that in the case $2k\lambda \geq \sigma^2$ the stopping time is equal to

$$\tau = \inf\{t : \sqrt{V(t)} - \frac{\sigma}{2}w(t) = 0\}$$

because $V(t) > 0$ for any $t \geq 0$.

Now let us consider the function

$$Z(t) = (\sqrt{V(t)} - \frac{\sigma}{2}w(t))^2,\ 0 \leq t < \tau.$$

Clearly, $Z(t) > 0$ on $[0,\tau)$, and

$$\sqrt{Z(t)} = \sqrt{V(t)} - \frac{\sigma}{2}w(t) > 0,$$

whence

$$\sqrt{V(t)} = \sqrt{Z(t)} + \frac{\sigma}{2}w(t) > 0. \tag{11}$$

Due to Itô's formula, we get

$$dZ(t) = [-\frac{\sigma^2}{4} + k(\lambda - V(t))]\cdot\frac{\sqrt{V(t)} - \frac{\sigma}{2}w(t)}{\sqrt{V(t)}}dt. \tag{12}$$

By substituting (11) in (12) we see that $Z(t)$ is a positive solution of (8) on $[0,\tau)$. The formula (9) then follows from (11). The uniqueness of $X(t)$ follows from the uniqueness of $V(t)$. ∎

2.3. So far we were starting at the moment $t = 0$. It is useful to consider the Doss-Sussmann transformation with an arbitrary initial time $t_0 > 0$ (which even may be a stopping time, for example, $0 \leq t_0 < \tau$). In this case we obtain instead of (8) for $X = X(t;t_0)$,

$$\begin{aligned}\frac{dX}{dt} &= [-\frac{\sigma^2}{4} + k(\lambda - (\sqrt{X} + \frac{\sigma}{2}(w(t) - w(t_0)))^2)]\frac{\sqrt{X}}{\sqrt{X} + \frac{\sigma}{2}(w(t) - w(t_0))}, \\ &t_0 \leq t < t_0 + \tau,\ X(t_0;t_0) = V(t_0),\end{aligned} \tag{13}$$

and

$$V(t) = (\sqrt{X(t;t_0)} + \frac{\sigma}{2}(w(t) - w(t_0)))^2,\ t_0 \le t < t_0 + \tau, \tag{14}$$

with

$$t_0 + \tau := \inf\{t : V(t) \cdot (\sqrt{V(t)} - \frac{\sigma}{2}(w(t) - w(t_0))) = 0,\ t > t_0\}.$$

2.4. Let us return to equation (8). It becomes "bad" (in the sense of numerical integration) for t close to τ (of course, under $t < \tau$). If in addition $V(t)$ is close to zero, then the Doss-Sussmann approach has rather limited potential. However, if $V(t)$ is not close to zero, then we can pass to equation (13) for next constructing $V(t)$ according to (14).

So, we have established that

$$X(t) = (\sqrt{V(t)} - \frac{\sigma}{2}w(t))^2,$$

where $(w(t),\ V(t))$ is a solution of (10), is differentiable function for any realization $w(\cdot)$. Clearly, for any appropriate function $f(x)$ the function $f(X(t))$ is differentiable in t as well. In particular, let us consider the function

$$Y(t) := \sqrt{X(t)} = \sqrt{V(t)} - \frac{\sigma}{2}w(t),\ 0 \le t < \tau.$$

From (8) we get

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}w(t)} - \frac{k}{2}(Y + \frac{\sigma}{2}w(t)),\ Y(0) = \sqrt{V(0)}, \tag{15}$$
$$\alpha := \frac{4k\lambda - \sigma^2}{8}.$$

Analogously, in connection with (13), we get for

$$Y = Y(t;t_0) := \sqrt{X(t;t_0)} = \sqrt{V(t)} - \frac{\sigma}{2}(w(t) - w(t_0))),\ t_0 \le t < t_0 + \tau,$$

the equation

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_0))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_0))), \tag{16}$$
$$Y(t_0;t_0) = \sqrt{V(t_0)},\ t_0 \le t < t_0 + \tau.$$

Clearly,

$$V(t) = (Y(t;t_0) + \frac{\sigma}{2}(w(t) - w(t_0)))^2,\ t_0 \le t < t_0 + \tau. \tag{17}$$

Remark 2 *One can derive equations (15) and (16) after applying the Lamperti transformation $U(t) = \sqrt{V(t)}$ to (1) (see [6]), that yields the following SDE with additive noise*

$$dU = (\frac{\alpha}{U} - \frac{k}{2}U)dt + \frac{\sigma}{2}dw,\ U(0) = \sqrt{V(0)}, \tag{18}$$

and then apply the Doss-Sussmann transformation to (18).

3 Auxiliary propositions

3.1. Let us consider the set of solutions of (16) under $\sigma = 0$. That is, the solutions of the ordinary differential equations

$$\frac{dy_0}{dt} = \frac{\alpha}{y_0} - \frac{k}{2} y_0,\ y_0(t_0) = y_0 > 0,\ t \geq t_0 \geq 0, \tag{19}$$

which are given by

$$y_0(t) = y_{t_0,y_0}(t) = [y_0^2 e^{-k(t-t_0)} + \frac{2\alpha}{k}(1 - e^{-k(t-t_0)})]^{1/2},\ t \geq t_0. \tag{20}$$

In the case $\alpha > 0$, i.e., $4k\lambda > \sigma^2$, we have: If $y_0 > \sqrt{2\alpha/k}$ then $y_{t_0,y_0}(t) \downarrow \sqrt{2\alpha/k}$ as $t \to \infty$, and if $0 < y_0 < \sqrt{2\alpha/k}$ then $y_{t_0,y_0}(t) \uparrow \sqrt{2\alpha/k}$ as $t \to \infty$. Further, $y_0(t) = \sqrt{2\alpha/k}$ is a solution of (19).

In the case $\alpha = 0$, the solution $y_{t_0,y_0}(t) \downarrow 0$ under $t \to \infty$ for any $y_0 > 0$. We note that the case $\alpha \geq 0$ is more general than the case $2k\lambda \geq \sigma^2$ (we recall that in the latter case $V(t) > 0, t \geq t_0$).

In the case $\alpha < 0$, the solution $y_{t_0,y_0}(t)$ is convexly decreasing, attains zero at the moment $\bar{t}$ given by

$$\bar{t} = t_0 + \frac{1}{k} \ln \frac{y_0^2 - 2\alpha/k}{-2\alpha/k}, \tag{21}$$

and $y'_{t_0,y_0}(\bar{t}) = \infty$.

3.2. Our next goal is to obtain some estimates for solutions of the equation

$$\frac{dy}{dt} = \frac{\alpha}{y + \frac{\sigma}{2}\varphi(t)} - \frac{k}{2}(y + \frac{\sigma}{2}\varphi(t)),\ y(t_0) = y_0,\ t_0 \leq t \leq t_0 + \theta \tag{22}$$

(cf. (16)), where $\varphi(t)$ is a continuous function satisfying the inequality

$$\mid \varphi(t) \mid \leq r,\ t_0 \leq t \leq t_0 + \theta \leq t_0 + T. \tag{23}$$

In what follows we deal with the case

$$\alpha = \frac{4k\lambda - \sigma^2}{8} \geq 0. \tag{24}$$

Lemma 3 *Let* $\alpha \geq 0$. *Let* $y_i(t),\ i = 1, 2,$ *be two solutions of (22), (23) such that* $y_i(t) + \frac{\sigma}{2}\varphi(t) > 0$ *on* $[t_0, t_0 + \theta]$. *Then*

$$\mid y_1(t) - y_0(t) \mid \leq \mid y_1(t_0) - y_0(t_0) \mid,\ t_0 \leq t \leq t_0 + \theta. \tag{25}$$

Proof. We have

$$d(y_1(t) - y_0(t))^2 = 2(y_1(t) - y_0(t))$$
$$\times \left(\frac{\alpha}{y_1(t) + \frac{\sigma}{2}\varphi(t)} - \frac{k}{2}(y_1(t) + \frac{\sigma}{2}\varphi(t)) - \frac{\alpha}{y_0(t) + \frac{\sigma}{2}\varphi(t)} + \frac{k}{2}(y_0(t) + \frac{\sigma}{2}\varphi(t)) \right), \quad \text{so}$$

$$(y_1(t)-y_0(t))^2=(y_1(t_0)-y_0(t_0))^2$$
$$+2\int_{t_0}^{t}[-\alpha\frac{(y_1(s)-y_0(s))^2}{(y_0(s)+\frac{\sigma}{2}\varphi(s))(y_1(s)+\frac{\sigma}{2}\varphi(s))}-\frac{k}{2}(y_1(s)-y_0(s))^2]ds$$
$$\leq (y_1(t_0)-y_0(t_0))^2,$$

from which (25) follows. ■

3.3. Let us consider along with (19) and (22) the equations

$$\frac{dy}{dt}=\frac{\alpha}{y+\frac{\sigma}{2}r}-\frac{k}{2}(y+\frac{\sigma}{2}r),\ y(t_0)=y_0, \tag{26}$$
$$\frac{dy}{dt}=\frac{\alpha}{y-\frac{\sigma}{2}r}-\frac{k}{2}(y-\frac{\sigma}{2}r),\ y(t_0)=y_0. \tag{27}$$

Let us further suppose that

$$y_0\geq\eta\geq\sigma r>0, \tag{28}$$

and let us denote the solutions of (19), (22), (26), and (27) respectively by $y_0(t),\ y(t),\ y^-(t)$, and $y^+(t)$. The solution $y_0(t)$ is given by (20). Using (20) we derive straightforwardly that

$$y^-(t)=[(y_0+\frac{\sigma}{2}r)^2e^{-k(t-t_0)}+\frac{2\alpha}{k}(1-e^{-k(t-t_0)})]^{1/2}-\frac{\sigma}{2}r,\ t_0\leq t\leq t_0+\theta, \tag{29}$$
$$y^+(t)=[(y_0-\frac{\sigma}{2}r)^2e^{-k(t-t_0)}+\frac{2\alpha}{k}(1-e^{-k(t-t_0)})]^{1/2}+\frac{\sigma}{2}r,\ t_0\leq t\leq t_0+\theta. \tag{30}$$

Due to the comparison theorem for ODEs (see, e.g., [9], Ch. 3), the inequality

$$\frac{\alpha}{y+\frac{\sigma}{2}r}-\frac{k}{2}(y+\frac{\sigma}{2}r)\leq\frac{\alpha}{y+\frac{\sigma}{2}\varphi(t)}-\frac{k}{2}(y+\frac{\sigma}{2}\varphi(t))\leq\frac{\alpha}{y-\frac{\sigma}{2}r}-\frac{k}{2}(y-\frac{\sigma}{2}r),$$

which is fulfilled on $[t_0,t_0+\theta]$ in view of (23), implies that

$$y^-(t)\leq y(t)\leq y^+(t),\ t_0\leq t\leq t_0+\theta. \tag{31}$$

The same inequality holds for $y(t)$ replaced by $y_0(t)$. We thus get

$$\mid y(t)-y_0(t)\mid\leq y^+(t)-y^-(t),\ t_0\leq t\leq t_0+\theta. \tag{32}$$

Proposition 4 *Let* $\alpha=\dfrac{4k\lambda-\sigma^2}{8}\geq 0$*, the inequalities (23) and (28) be fulfilled, and a positive number* η *in (28) be fixed. Let* $\theta\leq T$*. Then there exists a number* $C>0$*, which is independent of* $t_0,\ y_0$*, and of* r *(provided* $r\leq\eta/\sigma$*) such that*

$$\mid y(t)-y_0(t)\mid\leq Cr\theta,\ t_0\leq t\leq t_0+\theta. \tag{33}$$

Proof. We estimate the difference $y^+(t)-y^-(t)$. It holds,

$$y^+(t)=z^-(t)+\frac{\sigma}{2}r,\ y^-(t)=z^+(t)-\frac{\sigma}{2}r,$$
$$y^+(t)-y^-(t)=\sigma r-(z^+(t)-z^-(t)), \tag{34}$$

where

$$z^{\pm}(t) = [(y_0 \pm \frac{\sigma}{2}r)^2 e^{-k(t-t_0)} + \frac{2\alpha}{k}(1 - e^{-k(t-t_0)})]^{1/2}.$$

Further,

$$z^+(t) - z^-(t) = \frac{(z^+(t))^2 - (z^-(t))^2}{z^+(t) + z^-(t)} = \frac{2y_0\sigma r e^{-k(t-t_0)}}{z^+(t) + z^-(t)}. \tag{35}$$

Using the inequality $(a^2 + b)^{1/2} \leq a + b/2a$ for any $a > 0$ and $b \geq 0$, we get

$$z^+(t) \leq (y_0 + \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{(y_0 + \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)}},$$

$$z^-(t) \leq (y_0 - \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{(y_0 - \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)}},$$

whence

$$z^+(t) + z^-(t) \leq 2y_0 e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{e^{-\frac{k}{2}(t-t_0)}}\frac{2y_0}{(y_0^2 - \dfrac{\sigma^2}{4}r^2)}.$$

Therefore

$$\frac{1}{z^+(t) + z^-(t)} \geq \frac{1}{2y_0 e^{-\frac{k}{2}(t-t_0)}}\left(1 - \frac{\alpha}{k(y_0^2 - \dfrac{\sigma^2}{4}r^2)}(e^{k(t-t_0)} - 1)\right).$$

From (35) we have that

$$z^+(t) - z^-(t) \geq \sigma r e^{-\frac{k}{2}(t-t_0)}\left(1 - \frac{\alpha}{k(y_0^2 - \dfrac{\sigma^2}{4}r^2)}(e^{k(t-t_0)} - 1)\right)$$

and so due to (34) we get,

$$0 \leq y^+(t) - y^-(t) \leq \sigma r(1 - e^{-\frac{k}{2}(t-t_0)}) + \frac{\alpha\sigma r}{k(y_0^2 - \dfrac{\sigma^2}{4}r^2)}(e^{\frac{k}{2}(t-t_0)} - e^{-\frac{k}{2}(t-t_0)}).$$

Since $1 - e^{-q\vartheta} \leq q\vartheta$ for any $q \geq 0,\ \vartheta \geq 0$, and $y_0^2 - \dfrac{\sigma^2}{4}r^2 \geq \dfrac{3}{4}\eta^2$ under (28), we obtain

$$0 \leq y^+(t) - y^-(t) \leq \frac{\sigma r k}{2}(t - t_0) + \frac{4\alpha\sigma r}{3k\eta^2}e^{\frac{k}{2}(t-t_0)}k(t - t_0).$$

From this and (32), (33) follows with $C = \dfrac{\sigma k}{2} + \dfrac{4\alpha\sigma}{3\eta^2}e^{\frac{k}{2}T}$. ■

4 One-step approximation

Let $t_m,\ t_0 \leq t_m < t_0 + T$, be a stopping time, and $V(t_m)$ be a known value. Knowing $Y(t_m;t_m) = \sqrt{V(t_m)}$, consider the following analogue of equation (16), the equation for $Y(t;t_m)$ on some interval $[t_m, t_m + \theta_m]$,

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_m))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_m))), \tag{36}$$
$$Y(t_m;t_m) = \sqrt{V(t_m)},\ t_m \leq t \leq t_m + \theta_m.$$

It is supposed that

$$\sqrt{V(t_m)} \geq \eta \geq \sigma r. \tag{37}$$

Due to (17), the solution $V(t)$ of (1) on $[t_m, t_m + \theta_m]$ is expressed by the formula

$$\sqrt{V(t)} = Y(t;t_m) + \frac{\sigma}{2}(w(t) - w(t_m)),\ t_m \leq t \leq t_m + \theta_m. \tag{38}$$

Though equation (36) is (just) an ODE, it is not easy to solve because of the non-smoothness of $w(t)$. We obtain an approximation $y_m(t)$ of $Y(t;t_m)$ according to Proposition 4. To this aim we simulate the point $(t_m + \theta_m, w(t_m + \theta_m) - w(t_m))$, where $t_{m+1} := t_m + \theta_m$ is the first-passage time of the Wiener process $w(t) - w(t_m)$ to the boundary of the interval $[-r, r]$. Therefore $\mid w(t) - w(t_m) \mid \leq r$ on $[t_m, t_m + \theta_m]$ and the value $w(t_{m+1}) - w(t_m)$ is either equal to $-r$ or to $+r$. For simulating θ_m we use the distribution function

$$\mathcal{P}(t) := P(\tau < t),$$

where τ is the first-passage time of a Wiener process $W(t)$ to the boundary of the interval $[-1, 1]$ (see details in [15], Ch. 5, Sect. 3 and Appendix A3).

It may happen that $t_m + \theta_m > t_0 + T$, i.e., the m-th step is the last one. In such a case we simulate $w(t_0 + T) - w(t_m)$ using the known conditional probability (see [15], Ch. 5, Sect. 3 and Appendix A3)

$$\mathcal{Q}(\beta;t) := P(W(t) < \beta / \mid W(s) \mid < 1,\ 0 < s < t,$$

where $-1 < \beta \leq 1$.

We set $t_{\nu+1} = t_{m+1} = t_0 + T$, where $\nu + 1$ is a random number of steps for approximating $V(t)$ on the whole interval $[t_0, t_0 + T]$, and (without confusion) we may introduce a new $\theta_m := t_0 + T - t_m$.

Application of Proposition 4 yields

$$\mid Y(t;t_m) - y_m(t) \mid \leq Cr\theta_m,\ t_m \leq t \leq t_m + \theta_m = t_{m+1}, \tag{39}$$

where $y_m(t)$ is the solution of the problem

$$\frac{dy_m}{dt} = \frac{\alpha}{y_m} - \frac{k}{2}y_m,\ y_m(t_m) = Y(t_m;t_m) = \sqrt{V(t_m)},$$

that is given by (20) with $(t_0, y_0) = (t_m, \sqrt{V(t_m)})$. We so have

$$\sqrt{V(t)} = Y(t;t_m) + \frac{\sigma}{2}(w(t) - w(t_m)) = y_m(t) + \frac{\sigma}{2}(w(t) - w(t_m)) + \rho_m(t),$$

where due to (39),

$$| \rho_m(t) | \leq Cr\theta_m. \tag{40}$$

Now introduce the approximation $\sqrt{\bar{V}(t)}$ of $\sqrt{V(t)}$ on $[t_m, t_m + \theta_m]$,

$$\sqrt{\bar{V}(t)} := y_m(t) + \frac{\sigma}{2}(w(t) - w(t_m)),\ t_m \leq t \leq t_m + \theta_m. \tag{41}$$

We have that $| \ w(t) - w(t_m) \ | < r$ for $t_m \leq t < t_m + \theta_m = t_{m+1}$, $| \ w(t_{m+1}) - w(t_m) \ | = r$ if $t_{m+1} < t_0 + T$, and $| \ w(t_{m+1}) - w(t_m) \ | < r$ if $t_{m+1} = t_0 + T$. Thus, the one-step approximation (41) is constructed for $t = t_{m+1}$ by

$$\sqrt{\bar{V}(t_{m+1})} := y_m(t_{m+1}) + \frac{\sigma}{2}(w(t_{m+1}) - w(t_m)). \tag{42}$$

Note that using the conditional probability $\mathcal{Q}$, one obtains from (41) $\sqrt{\bar{V}(t')}$ for any $t' \in [t_m, t_m + \theta_m]$.

Since

$$\sqrt{V(t)} = \sqrt{\bar{V}(t)} + \rho_m(t),\ t_m \leq t \leq t_m + \theta_m = t_{m+1}, \tag{43}$$

we get the following proposition (see (40)).

Proposition 5 *The one-step error $\rho_m(t)$ of the one-step approximation (41) is estimated by $Cr\theta_m$.*

In the next section we show that the global error on the interval $[t_0, t_0 + T]$ is evaluated by the sum of these errors, i.e., by $\sum_{m=0}^{\nu} Cr\theta_m = Cr\sum_{m=0}^{\nu}\theta_m = CrT$, hence the global error tends to zero when $r \to 0$. We further note that $E\theta_v = r^2$, the random number of steps ν is finite with probability one, and $E\nu = O(1/r^2)$.

5 Convergence theorem

Let us now proceed with simulating the solution of (1) on the interval $[t_0, t_0 + T]$ in the case $\alpha \geq 0$. Suppose we are given $V(t_0),\ \eta$, and r such that

$$\sqrt{V(t_0)} \geq \eta \geq \sigma r.$$

Moreover, we suppose that the inequality

$$\sqrt{\bar{V}(t_m)} \geq \eta \geq \sigma r \tag{44}$$

will be fulfilled for the next approximations $\bar{V}(t_m)$, so that Proposition 4 may be applied.

For the initial step we use the one-step approximation according to the previous section. We obtain (see (42) and (43))

$$\sqrt{\bar{V}(t_1)} = y_0(t_1) + \frac{\sigma}{2}(w(t_1) - w(t_0)),$$
$$\sqrt{V(t_1)} = \sqrt{\bar{V}(t_1)} + \rho_0(t_1),$$

where $| \ \rho_0(t_1) \ | \leq Cr\theta_0$.

For the first step we use the expression

$$\sqrt{V(t)} = Y(t;t_1) + \frac{\sigma}{2}(w(t) - w(t_1)), \tag{45}$$

where $Y(t;t_1)$ is the solution of the problem (see (36))

$$\begin{aligned} \frac{dY}{dt} &= \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_1))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_1))), \\ Y(t_1;t_1) &= \sqrt{V(t_1)},\ t_1 \le t \le t_1 + \theta_1. \end{aligned} \tag{46}$$

However, in contrast to the initial step, the value $\sqrt{V(t_1)}$ is unknown and we are forced to use $\sqrt{\bar{V}(t_1)}$ instead. Therefore we introduce $\bar{Y}(t;t_1)$ as a solution of the equation (46) with initial data $\bar{Y}(t_1;t_1) = \sqrt{\bar{V}(t_1)}$. We have that $| Y(t_1;t_1) - \bar{Y}(t_1;t_1) | = | \sqrt{V(t_1)} - \sqrt{\bar{V}(t_1)} | = | \rho_0(t_1) | \le Cr\theta_0$. Hence, due to Lemma 3,

$$| Y(t;t_1) - \bar{Y}(t;t_1) | \le | \rho_0(t_1) | \le Cr\theta_0,\ t_1 \le t \le t_1 + \theta_1. \tag{47}$$

Denote $t_2 := t_1 + \theta_1$, where $t_1 + \theta_1$ is the first-passage time of the Wiener process $w(t) - w(t_1)$ to the boundary of the interval $[-r, r]$. To find $\bar{Y}(t;t_1)$ for $t_1 \le t \le t_2$ let us consider together with equation (46) the following equation

$$\frac{dy_1}{dt} = \frac{\alpha}{y_1} - \frac{k}{2}y_1,\ y_1(t_1) = \bar{Y}(t_1;t_1) = \sqrt{\bar{V}(t_1)}.$$

Due to Proposition 4 it holds

$$| \bar{Y}(t;t_1) - y_1(t) | \le Cr\theta_1,\ t_1 \le t \le t_1 + \theta_1,$$

and so by (47) we have

$$| Y(t;t_1) - y_1(t) | \le Cr(\theta_0 + \theta_1),\ t_1 \le t \le t_1 + \theta_1.$$

We also have (see (45))

$$\sqrt{V(t)} = Y(t;t_1) + \frac{\sigma}{2}(w(t) - w(t_1)) = y_1(t) + \frac{\sigma}{2}(w(t) - w(t_1)) + \rho^1(t),$$

where

$$| \rho^1(t) | \le Cr(\theta_0 + \theta_1),\ t_1 \le t \le t_1 + \theta_1.$$

Next, we introduce

$$\sqrt{\bar{V}(t)} := y_1(t) + \frac{\sigma}{2}(w(t) - w(t_1)),\ t_1 \le t \le t_1 + \theta_1.$$

We have

$$\sqrt{V(t)} = \sqrt{\bar{V}(t)} + \rho^1(t),\ t_1 \le t \le t_1 + \theta_1,$$

and

$$\sqrt{\bar{V}(t_2)} = y_1(t_2) + \frac{\sigma}{2}(w(t_2) - w(t_1)).$$

We thus get $\sqrt{\bar{V}(t_2)}$ such that

$$| \sqrt{V(t_2)} - \sqrt{\bar{V}(t_2)} | = | \rho^1(t_2) | \le Cr(\theta_0 + \theta_1).$$

Knowing $\sqrt{\bar{V}(t_2)}$ we are ready for the second step. Proceeding just in the same way we get $\sqrt{\bar{V}(t_{m+1})}$ after m-th steps such that

$$| \sqrt{V(t_{m+1})} - \sqrt{\bar{V}(t_{m+1})} |=| \rho^m(t_{m+1}) |\leq Cr \sum_{i=0}^{m} \theta_i \leq CrT, \tag{48}$$

where $\rho^m(t)$ is the accumulated error after m-th steps (we note that $\rho_m(t)$ is the one-step error on the m-th step assuming $\bar{V}(t_m) = V(t_m)$).

So, under the assumption (44) we obtain the estimate (48) and thus have justified the convergence of the proposed method. Assumption (44) cannot always be realized however. The next theorem shows that if a trajectory of $V(t)$ under consideration is positive on $[t_0, t_0 + T]$, then (44) can be realized. We recall that in the case $2k\lambda \geq \sigma^2$ almost all trajectories are positive.

Theorem 6 *Let $4k\lambda \geq \sigma^2$ (i.e., $\alpha \geq 0$). Then for any positive trajectory $V(t) > 0$ on $[t_0, t_0 + T]$ there exist η and r_0 such that (44) is fulfilled for this η and any $r \leq r_0$. Hence the estimate (48) holds and in the case $2k\lambda \geq \sigma^2$ the proposed method is convergent.*

Proof. Let us define

$$\mu := \min_{t_0<t<t_0+T} \sqrt{V(t)}, \qquad \text{and set}$$

$$\eta = \frac{\mu}{2},\ r_0 = \min(\frac{\mu}{2\sigma}, \frac{\mu}{2CT}).$$

Let $r \leq r_0$. Clearly, $r \leq \mu/2\sigma = \eta/\sigma$. Due to (48), we have

$$| \sqrt{V(t_{m+1})} - \sqrt{\bar{V}(t_{m+1})} |\leq CrT \leq \frac{\mu}{2}.$$

Since $\sqrt{V(t_{m+1})} \geq \mu$, we get $\sqrt{\bar{V}(t_{m+1})} \geq \mu/2 = \eta$, i.e., (44) is fulfilled. ■

Remark 7 *In practice it is important to develop a way for continuing simulations in cases where $\bar{V}(t_{m+1})$ is very small. In this respect one can propose different procedures. For instance, one can proceed with exact simulations keeping in mind existence of some relieved schemes for small V (see [3]). Of course, any of such proposals require special consideration.*

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