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A vanishing diffusion limit in a nonstandard system of phase field equations

Pierluigi Colli[1], Gianni Gilardi[1], Pavel Krejčí[2], Jürgen Sprekels[3]

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[1] Dipartimento di Matematica "F. Casorati"
Università di Pavia
via Ferrata 1
27100 Pavia
Italy
E-Mail: pierluigi.colli@unipv.it
E-Mail: gianni.gilardi@unipv.it

[2] Institute of Mathematics
Czech Academy of Sciences
Žitná 25
11567 Praha 1
Czech Republic
E-Mail: krejci@math.cas.cz

[3] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: Juergen.Sprekels@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We are concerned with a nonstandard phase field model of Cahn–Hilliard type. The model, which was introduced by Podio-Guidugli (Ric. Mat. 2006), describes two-species phase segregation and consists of a system of two highly nonlinearly coupled PDEs. It has been recently investigated by Colli, Gilardi, Podio-Guidugli, and Sprekels in a series of papers: see, in particular, SIAM J. Appl. Math. 2011, and Boll. Unione Mat. Ital. 2012. In the latter contribution, the authors can treat the very general case in which the diffusivity coefficient of the parabolic PDE is allowed to depend nonlinearly on both variables. In the same framework, this paper investigates the asymptotic limit of the solutions to the initial-boundary value problems as the diffusion coefficient σ in the equation governing the evolution of the order parameter tends to zero. We prove that such a limit actually exists and solves the limit problem, which couples a nonlinear PDE of parabolic type with an ODE accounting for the phase dynamics. In the case of a constant diffusivity, we are able to show uniqueness and to improve the regularity of the solution.

1 Introduction

In this paper, we consider the following system

$$\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\,g'(\rho)\,\partial_t\rho - \mathrm{div}\big(\kappa(\mu,\rho)\nabla\mu\big) = 0 \tag{1.1}$$

$$\partial_t\rho - \sigma\Delta\rho + f'(\rho) = \mu\,g'(\rho) \tag{1.2}$$

$$\big(\kappa(\mu,\rho)\nabla\mu\big)\cdot\nu|_\Gamma = 0 \quad \text{and} \quad \partial_\nu\rho|_\Gamma = 0 \tag{1.3}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0, \tag{1.4}$$

in the unknown fields μ and ρ, where the partial differential equations (1.1)–(1.2) are meant to hold in a three-dimensional bounded domain Ω, endowed with a smooth boundary Γ, and in some time interval $(0,T)$. Relations (1.4) specify the initial conditions for μ and ρ, while (1.3) are nothing but homogeneous boundary conditions of Neumann type, involving precisely those boundary operators that match the elliptic differential operators in (1.1)–(1.2).

This system has been recently addressed in the paper [6]: the existence of solutions has been proved, thus complementing and extending the results of the papers [3, 4, 5] concerned with simpler or reduced versions of the problem.

Here, we are interested to investigate the asymptotic behavior of the above initial-boundary value problem (1.1)–(1.4) as the positive diffusion coefficient σ appearing in (1.2) tends to 0.

Let us briefly explain the modelling background for (1.1)–(1.4). Such a system comes from a generalization of the phase-field model of viscous Cahn-Hilliard type originally proposed in [14], and it aims to describe the phase segregation of two species (atoms and vacancies, say) on a lattice in presence of diffusion. The state variables are the *order parameter* ρ, interpreted as the volume density of one of the two species, and the *chemical potential* μ. For physical reasons, μ is required to be nonnegative, while the phase parameter ρ must of course take values in the domain of f'.

We also recall the features of [3] and what has been generalized in [5, 6]. Firstly, the nonlinearity f considered in [3] is a double-well potential defined in $(0,1)$, whose derivative f' diverges at the endpoints $\rho = 0$ and $\rho = 1$: e.g., for $f = f_1 + f_2$ with f_2 smooth, one can take

$$f_1(\rho) = c\,(\rho\,\log(\rho) + (1-\rho)\,\log(1-\rho)), \tag{1.5}$$

with c a positive constant. In this paper, we let $f_1 : \mathbb{R} \to [0, +\infty]$ be a convex, proper and lower semicontinuous function so that its subdifferential (and not the derivative) is a maximal monotone graph from $\mathbb{R}$ to $\mathbb{R}$. Then, we rewrite equation (1.2) as a differential inclusion, in which the derivative of the convex part f_1 of f is replaced by the subdifferential $\beta := \partial f_1$, i.e.,

$$\partial_t \rho - \sigma\Delta\rho + \xi + f_2'(\rho) = \mu g'(\rho) \quad \text{with} \quad \xi \in \beta(\rho). \tag{1.6}$$

Note that f_1 needs not be differentiable in its domain, so that its possibly nonsmooth and multivalued subdifferential $\beta := \partial f_1$ appears in (1.2) in place of f_1'. In general, β is only a graph, not necessarily a function, and it may include vertical (and horizontal) lines, as for example when

$$f_1(\rho) = I_{[0,1]}(\rho) = \begin{cases} 0 & \text{if } 0 \le \rho \le 1 \\ +\infty & \text{elsewhere} \end{cases} \tag{1.7}$$

and $\beta = \partial I_{[0,1]}$ is specified by

$$\xi \in \beta(\rho) \quad \text{if and only if} \quad \xi \begin{cases} \le 0 & \text{if } \rho = 0 \\ = 0 & \text{if } 0 < \rho < 1 \\ \ge 0 & \text{if } \rho = 1 \end{cases}. \tag{1.8}$$

Secondly, while in [3] g was simply taken as the identity map $g(\rho) = \rho$, in [5, 6] g is allowed be any nonnegative smooth function, defined (at least) in the domain where f_1 and its subdifferential live. The presence of such a function g allows for a more general behavior of (the related term in) the free energy, which reads

$$\psi(\rho, \nabla\rho, \mu) = -\frac{\mu}{2} - \mu\, g(\rho) + f(\rho) + \frac{\sigma}{2}|\nabla\rho|^2. \tag{1.9}$$

Indeed, in particular $g(\rho)$ is not obliged, as it was instead for $g(\rho) = \rho$, to take its minimum value at $\rho = 0$, be increasing and with maximum value at $\rho = 1$ (when $D(f_1) = [0,1]$), but we may have many other instances like, e.g., a specular behavior of g around the extremal points of the domain of f. Here, we have to impose an additional restriction on g, which however looks reasonable from the modelling point of view: we postulate that g is a (smooth) concave function, which in turn implies convexity with respect to ρ of the term $-\mu\, g(\rho)$ in the free energy (1.9). However, let us recall that f may stand for a multi-well potential in which the nonconvex perturbations are incorporated into f_2, so that ψ in its entirety needs not be convex with respect to ρ.

An important generalization that is considered in this paper concerns the diffusivity κ. In [3], κ was just assumed to be a constant function, but it can be a positive-valued, continuous, bounded, and nonlinear function of μ (and this was the setting of [5]), or of μ and ρ as it is postulated in [6]. For simplicity, we confine ourselves to study of the convergence properties of the solution under an assumption that guarantees uniform parabolicity, i.e., $\kappa \ge \kappa_* > 0$. We point out that [5] treats the situation of κ depending only on μ and possibly degenerating somewhere.

Therefore, the system

$$\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \mathrm{div}\big(\kappa(\mu,\rho)\nabla\mu\big) = 0 \tag{1.10}$$

$$\partial_t\rho - \sigma\Delta\rho + \xi + f_2'(\rho) = \mu g'(\rho) \quad \text{with} \quad \xi \in \beta(\rho), \tag{1.11}$$

$$\big(\kappa(\mu,\rho)\nabla\mu\big)\cdot\nu|_\Gamma = 0 \quad \text{and} \quad \partial_\nu\rho|_\Gamma = 0 \tag{1.12}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0, \tag{1.13}$$

turns out to be the initial and boundary value problem for a nonstandard and highly nonlinear phase field system in which however the role usually played by the temperature is here conducted by the chemical potential μ. In the study of phase field systems, it has been always considered rather important to analyze the behavior of the problem as the coefficient σ of the diffusion term in the phase parameter equation tends to 0. The limiting case $\sigma = 0$ corresponds indeed to a pointwise ordinary differential equation (or inclusion)

$$\partial_t\rho + \xi + f_2'(\rho) = \mu g'(\rho), \quad \xi \in \beta(\rho), \tag{1.14}$$

in place of (1.11), and to an expression for the free energy (1.9) in which the last quadratic term accounting for nonlocal interactions is removed.

In fact, especially for the choice (1.7)–(1.8), the limiting problem can be formulated in terms of hysteresis operators: in particular, the so-called *stop* and *play* operators are involved; the interested reader can find some discussion and various results on this class of problems in [7, 8, 9, 10, 11, 12, 13].

By collecting a number of estimates independent of σ for the solution $(\mu_\sigma, \rho_\sigma)$ to the problem (1.10)–(1.13), by weak and weak star compactness we prove that any limit in a suitable topology of a convergent subsequence of $\{(\mu_\sigma, \rho_\sigma)\}$ yields a solution to the limiting problem in which (1.11) is replaced by (1.14). Furthermore, under natural compatibility conditions on the nonlinearities and the initial data, we show boundedness for all the components of any solution to the limit problem. Finally, in the special case of a constant mobility κ in (1.10), we prove that the solution is unique and more regular than required.

The paper is organized as follows. In the next section, we state precise assumptions along with our results. The basic a priori estimates independent of σ are proved in Section 3 and they allow us to pass to the limit by compactness and monotonicity techniques. Finally, the last section is devoted to the study of the limit problem and our boundedness, uniqueness, and further regularity properties are proved.

2 Assumptions and results

The aim of this section is to introduce precise assumptions on the data for the mathematical problem under investigation, and establish our main result. We assume Ω to be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ (treating lower-dimensional cases would require only minor changes) and let $T \in (0, +\infty)$ stand for a final time. We introduce the spaces

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v \in H^2(\Omega) : \ \partial_\nu v = 0 \text{ on } \Gamma\} \tag{2.1}$$

and endow them with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$. For powers of these spaces, norms are denoted by the same symbols. We remark that the embeddings

$W \subset V \subset H$ are compact, because Ω is bounded and smooth. The symbol $\langle \cdot , \cdot \rangle$ denotes the duality product between V^*, the dual space of V, and V itself. Moreover, for $p \in [1, +\infty]$, we write $\| \cdot \|_p$ for the usual norm in $L^p(\Omega)$; as no confusion can arise, the symbol $\| \cdot \|_p$ is used for the norm in $L^p(Q)$ as well, where $Q := \Omega \times (0, T)$.

Now, we present the structural assumptions we make. It is useful to fix an upper bound for σ, that is,

$$0 < \sigma \leq 1. \tag{2.2}$$

Then, for the diffusivity coefficient κ we assume that

$$\kappa : (m, r) \mapsto \kappa(m, r) \text{ is continuous from } [0, +\infty) \times \mathbb{R} \text{ to } \mathbb{R}, \tag{2.3}$$

$$\text{the partial derivatives } \partial_r \kappa \text{ and } \partial_r^2 \kappa \text{ exist and are continuous,} \tag{2.4}$$

$$\kappa_*, \, \kappa^* \in (0, +\infty), \tag{2.5}$$

$$\kappa_* \leq \kappa(m, r) \leq \kappa^*, \;\; |\partial_r \kappa(m, r)| \leq \kappa^*, \;\; |\partial_r^2 \kappa(m, r)| \leq \kappa^* \quad \text{for } m \geq 0 \text{ and } r \in \mathbb{R}, \tag{2.6}$$

and for the other nonlinearities we require that

$$f = f_1 + f_2 \,, \quad f_1 : \mathbb{R} \to [0, +\infty], \quad f_2 : \mathbb{R} \to \mathbb{R}, \tag{2.7}$$

$$f_1 \text{ is convex, proper, l.s.c. and } f_2 \text{ is a } C^2 \text{ function,} \tag{2.8}$$

$$g \in C^2(\mathbb{R}), \quad g(r) \geq 0 \;\text{ and }\; g''(r) \leq 0 \;\text{ for } r \in \mathbb{R}, \tag{2.9}$$

$$f_2', \, g, \text{ and } g' \text{ are Lipschitz continuous.} \tag{2.10}$$

It is convenient to introduce the notations

$$\kappa' := \partial_r \kappa, \quad \kappa'' := \partial_r^2 \kappa, \quad \beta := \partial f_1 \,, \quad \text{and} \quad \pi := f_2' \tag{2.11}$$

$$K(m, r) := \int_0^m \kappa(s, r) \, ds, \;\; K_1(m, r) := \int_0^m \kappa'(s, r) \, ds, \;\; K_2(m, r) := \int_0^m \kappa''(s, r) \, ds$$
$$\text{for } m \geq 0 \text{ and } r \in \mathbb{R}. \tag{2.12}$$

We denote by $D(f_1)$ and $D(\beta)$ the effective domains of f_1 and β, respectively. Thanks to (2.6), it is clear that

$$\max\{|K(m, r)|, |K_1(m, r)|, |K_2(m, r)|\} \leq \kappa^* m \quad \text{for every } m \geq 0 \text{ and } r \in \mathbb{R}. \tag{2.13}$$

We also note that the structural assumptions of [5] are fulfilled if κ only depends on m, and that, due to the presence of $\beta(\rho)$, a strong singularity in equation (1.11) is allowed. On the other hand, equation (1.10) is uniformly parabolic, since g is nonnegative and κ is bounded away from zero.

Remark 2.1. Let us recall that any convex, proper, l.s.c. function is bounded from below by an affine function (cf., e.g., [1, Prop. 2.1, p. 51]), whence the assumption $f_1 \geq 0$ looks reasonable, as one can suitably modify the smooth perturbation f_2. Moreover, we point out that the sign conditions $g \geq 0$ and $g'' \leq 0$ are just needed on the set $D(\beta)$, for g can be extended outside of $D(\beta)$ accordingly.

Concerning the initial data, we require that

$$\mu_0 \in V, \quad \mu_0 \geq 0 \quad \text{a.e. in } \Omega, \tag{2.14}$$

$$\rho_0 \in V, \quad \rho_0 \in D(f_1) \quad \text{a.e. in } \Omega, \quad f_1(\rho_0) \in L^1(\Omega) \tag{2.15}$$

and point out that the above assumptions regard the initial data for the limiting problem, i.e., the one with (1.14) in place of (1.11). On the other hand, let us consider a family of initial data $\mu_{0\sigma},\ \rho_{0\sigma}$ with

$$\mu_{0\sigma} \in V \cap L^\infty(\Omega), \quad \mu_{0\sigma} \geq 0 \quad \text{a.e. in } \Omega, \tag{2.16}$$

$$\rho_{0\sigma} \in W, \quad \text{there is } \ \xi_{0\sigma} \in H \ \text{ such that } \ \rho_{0\sigma} \in D(\beta),\ \xi_{0\sigma} \in \beta(\rho_{0\sigma}) \ \text{ a.e. in } \Omega, \tag{2.17}$$

that approximate $\mu_0,\ \rho_0$ in the sense that

$$\mu_{0\sigma} \to \mu_0 \ \text{ and } \ \rho_{0\sigma} \to \rho_0 \ \text{ weakly in } \ V, \tag{2.18}$$

$$\|f_1(\rho_{0\sigma})\|_1 \ \text{ is bounded independently of } \ \sigma. \tag{2.19}$$

For the reader's convenience, we show that such a family $\{\mu_{0\sigma},\ \rho_{0\sigma}\}$ actually exists. Of course, if $\mu_0 \notin L^\infty(\Omega)$ we can take as $\mu_{0\sigma}$ some truncation of μ_0, e.g., $\mu_{0\sigma} = \min\{\mu_0, 1/\sigma\}$. Concerning $\rho_{0\sigma}$, one possible choice is letting $\rho_{0\sigma} \in W$ denote the solution to

$$\rho_{0\sigma} - \sigma\Delta\rho_{0\sigma} + \sigma\xi_{0\sigma} = \rho_0, \quad \text{with} \quad \xi_{0\sigma} \in \beta(\rho_{0\sigma}), \ \text{ a.e. in } \Omega. \tag{2.20}$$

Indeed, the elliptic problem (2.20) has a unique solution for all $\sigma > 0$, since $-\Delta + \beta$ is a maximal monotone graph in $H \times H$ with effective domain

$$\{v \in W : \ \exists\, \eta \in H \text{ such that } v \in D(\beta),\ \eta \in \beta(v) \text{ a.e. in } \Omega\}.$$

Thus, $\rho_{0\sigma}$ is nothing but the outcome of the application of the resolvent of $-\Delta + \beta$ to ρ_0 (let us refer to [1] and [2] for basic definitions and properties of maximal monotone operators). A formal test of the equality in (2.20) by $\xi_{0\sigma}$ and the definition of subdifferential lead us to the estimate

$$\int_\Omega f_1(\rho_{0\sigma}) + \sigma\|\xi_{0\sigma}\|_H^2 \leq \int_\Omega f_1(\rho_0), \tag{2.21}$$

which ensures (2.17) and (2.19), thanks to the nonnegativity of f_1. A rigorous way of proving the existence of $\rho_{0\sigma}$ and estimate (2.21) passes through the use of the Yosida approximation β_σ (see, e.g., [2, p. 28]) in place of β.

Now, we recall the result proved in [6] that allows us to specify a solution to the problem (1.10)–(1.12), with $\sigma > 0$, which fulfills the appropriate initial conditions.

Proposition 2.2. *Assume that both* (2.3)–(2.12) *and* (2.16)–(2.17) *hold. Then, there exists at least one triplet* $(\mu_\sigma, \rho_\sigma, \xi_\sigma)$ *satisfying*

$$\rho_\sigma \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{2.22}$$

$$\xi_\sigma \in L^\infty(0,T;H), \tag{2.23}$$

$$\mu_\sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^\infty(Q), \quad \mu_\sigma \geq 0 \quad \textit{a.e. in } Q, \tag{2.24}$$

$$\mathrm{div}\big(\kappa(\mu_\sigma,\rho_\sigma)\nabla\mu_\sigma\big) \in L^2(Q) \quad \textit{and} \quad \big(\kappa(\mu_\sigma,\rho_\sigma)\nabla\mu\big)\cdot\nu = 0 \quad \textit{a.e. on } \Sigma, \tag{2.25}$$

and solving the system of equations and conditions in the following strong form

$$\big(1 + 2g(\rho_\sigma)\big)\partial_t\mu_\sigma + \mu_\sigma\, g'(\rho_\sigma)\,\partial_t\rho_\sigma - \mathrm{div}\big(\kappa(\mu_\sigma,\rho_\sigma)\nabla\mu_\sigma\big) = 0 \qquad \textit{a.e. in } Q, \tag{2.26}$$

$$\partial_t\rho_\sigma - \sigma\Delta\rho_\sigma + \xi_\sigma + \pi(\rho_\sigma) = \mu_\sigma\, g'(\rho_\sigma) \quad \textit{and} \quad \xi_\sigma \in \beta(\rho_\sigma) \qquad \textit{a.e. in } Q, \tag{2.27}$$

$$\mu_\sigma(0) = \mu_{0\sigma} \quad \textit{and} \quad \rho_\sigma(0) = \rho_{0\sigma} \qquad \textit{a.e. in } \Omega. \tag{2.28}$$

Let us point out that equation (2.26) can be rewritten as

$$\partial_t u_\sigma - \mathrm{div}\big(\kappa(\mu_\sigma,\rho_\sigma)\nabla\mu_\sigma\big) = \mu_\sigma\, g'(\rho_\sigma)\,\partial_t\rho_\sigma, \quad \text{where} \quad u_\sigma = (1+2g(\rho_\sigma))\mu_\sigma, \quad \text{a.e. in } Q, \tag{2.29}$$

and the auxiliary variable u_σ has been added. Now, we take advantage of a variational formulation of (2.29) which also accounts for the boundary condition in (2.25), that is,

$$\langle \partial_t u_\sigma(t), v\rangle + \int_\Omega \big(\kappa(\mu_\sigma,\rho_\sigma)\nabla\mu_\sigma\big)(t)\cdot\nabla v = \int_\Omega \mu_\sigma\, g'(\rho_\sigma)\,\partial_t\rho_\sigma\, v \quad \text{for all } v\in V \text{ and a.a. } t\in(0,T). \tag{2.30}$$

The main result of this paper reads as follows.

Theorem 2.3. *Assume that* (2.3)–(2.12) *and* (2.14)–(2.19) *hold. For any* $\sigma\in(0,1]$ *let* $(\mu_\sigma,\rho_\sigma,\xi_\sigma)$ *be the triplet defined by Proposition 2.2 and let* $u_\sigma := (1+2g(\rho_\sigma))\mu_\sigma$*. Then, there exists a subsequence, still labelled by the parameter* σ*, and a quadruplet* (μ,ρ,ξ,u) *such that*

$$\mu_\sigma \to \mu \quad \textit{weakly star in } L^\infty(0,T;H)\cap L^2(0,T;V), \tag{2.31}$$
$$\rho_\sigma \to \rho \quad \textit{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V), \tag{2.32}$$
$$\xi_\sigma \to \xi \quad \textit{weakly in } L^2(Q), \tag{2.33}$$
$$u_\sigma \to u \quad \textit{weakly in } W^{1,4/3}(0,T;V^*)\cap L^2(0,T;W^{1,3/2}(\Omega)) \tag{2.34}$$

as $\sigma\searrow 0$*. Moreover, any quadruplet* (μ,ρ,ξ,u) *that is found as limit of converging subsequences yields a solution to the following limit problem*

$$\langle \partial_t u(t), v\rangle + \textstyle\int_\Omega \kappa(\mu,\rho)\nabla\mu(t)\cdot\nabla v = \int_\Omega \mu\, g'(\rho)\,\partial_t\rho\, v \quad \textit{for all } v\in V \textit{ and a.a. } t\in(0,T), \tag{2.35}$$
$$u = (1+2g(\rho))\mu \quad \textit{a.e. in } Q, \tag{2.36}$$
$$\partial_t\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \textit{and} \quad \xi\in\beta(\rho) \quad \textit{a.e. in } Q, \tag{2.37}$$
$$\mu(0) = \mu_0 \quad \textit{and} \quad \rho(0) = \rho_0 \quad \textit{a.e. in } \Omega. \tag{2.38}$$

Remark 2.4. The nonnegativity property $\mu\ge 0$ a.e. in Q plainly follows from (2.24) and (2.31).

Remark 2.5. One standard situation for the limit problem (2.35)–(2.38) is obtained for $\beta = \partial I_{[0,1]}$ (cf. (1.7)–(1.8)). In this case (2.37) becomes

$$-\pi(\rho) + \mu\, g'(\rho) - \partial_t\rho \in \partial I_{[0,1]}(\rho) \quad \text{a.e. in } Q. \tag{2.39}$$

Then, if one introduces the generalized "freezing index"

$$w(x,t) := \int_0^t (-\pi(\rho) + \mu\, g'(\rho))(x,s)ds, \quad (x,t)\in Q,$$

we thus have $\partial_t w - \partial_t\rho \in \partial I_{[0,1]}(\rho)$, or equivalently, $\rho = \mathcal{S}_K[w]$, where $\mathcal{S}_K$ is the stop hysteresis operator associated with the closed convex set $K=[0,1]$ (see, e.g., [10, 11, 12]). Hence, we may rewrite (2.39) as

$$\partial_t w = -\pi(\mathcal{S}_K[w]) + \mu\, g'(\mathcal{S}_K[w]) \quad \text{a.e. in } Q.$$

In addition to the convergence result stated in Theorem 2.3, one can derive boundedness for both the components ρ and ξ of any solution to the limit problem, provided that special additional requirements are satisfied, namely, by assuming that there exist real constants $\rho_*,\ \rho^*,\ \xi_*,\ \xi^*$ such that

$$\rho_*,\ \rho^* \in D(\beta), \quad \xi_* \in \beta(\rho_*), \quad \xi^* \in \beta(\rho^*), \tag{2.40}$$

$$\xi_* + \pi(\rho_*) \le 0, \quad \xi^* + \pi(\rho^*) \ge 0, \tag{2.41}$$

$$g'(\rho_*) \ge 0, \quad g'(\rho^*) \le 0. \tag{2.42}$$

Theorem 2.6. *In addition to the assumptions of Theorem 2.3, suppose that* (2.40)–(2.42) *and*

$$\rho_* \le \rho_0 \le \rho^* \quad \text{a.e. in } \Omega \tag{2.43}$$

hold. Then, the components ρ and ξ of any solution (μ, ρ, ξ, u) to problem (2.35)–(2.38) *satisfy*

$$\rho_* \le \rho \le \rho^* \quad \text{and} \quad \xi_* \le \xi \le \xi^* \quad \text{a.e. in } Q. \tag{2.44}$$

If moreover

$$\mu_0 \in L^\infty(\Omega) \tag{2.45}$$

and $\kappa = \kappa_0$ is constant, then the solution of Problem (2.35)–(2.38) *is unique and*

$$\mu \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W). \tag{2.46}$$

Remark 2.7. We observe that the above result is very general. Indeed, assumptions (2.40)–(2.42) are fulfilled with suitable constants for any graph β with bounded domain that generalizes the examples (1.5) or (1.7). Of course, the decreasing function g' (cf. (2.9)) should not assume a definite sign on $D(\beta)$.

Now, we list a number of tools and notations we owe to throughout the paper. We repeatedly use the elementary Young inequalities

$$a\,b \le \gamma a^2 + \frac{1}{4\,\gamma}\,b^2 \quad \text{and} \quad a\,b \le \vartheta a^{\frac{1}{\vartheta}} + (1-\vartheta) b^{\frac{1}{1-\vartheta}}$$
$$\text{for every } a, b \ge 0,\ \gamma > 0, \text{ and } \vartheta \in (0,1) \tag{2.47}$$

as well as the Hölder and Sobolev inequalities. The precise form of the latter we use is the following

$$W^{1,p}(\Omega) \subset L^q(\Omega) \quad \text{and} \quad \|v\|_q \le C_{p,q} \|v\|_{W^{1,p}(\Omega)} \quad \text{for every } v \in W^{1,p}(\Omega),$$
$$\text{provided that} \quad 1 \le p < 3 \quad \text{and} \quad 1 \le q \le p^* := \frac{3p}{3-p} \tag{2.48}$$

with a constant $C_{p,q}$ in (2.48) depending only on Ω, p, and q, since $\Omega \subset \mathbb{R}^3$. Moreover

$$\text{the embedding} \quad W^{1,p}(\Omega) \subset L^q(\Omega) \quad \text{is compact if} \quad 1 \le q < p^*. \tag{2.49}$$

The particular case $p = 2$ of (2.48) becomes

$$V \subset L^q(\Omega) \quad \text{and} \quad \|v\|_q \le C \|v\|_V \quad \text{for every } v \in V \text{ and } q \in [1,6] \tag{2.50}$$

where C depends only on Ω. Moreover, the compactness inequality

$$\|v\|_q \le \varepsilon \|\nabla v\|_2 + C_{q,\varepsilon} \|v\|_2 \quad \text{for every } v \in V,\ q \in [1,6), \text{ and } \varepsilon > 0 \tag{2.51}$$

holds for some constant $C_{q,\varepsilon}$ depending on Ω, q, and ε, only. We also recall the interpolation inequalities, which hold for any $\vartheta \in [0,1]$,

$$\|v\|_r \le \|v\|_p^{\vartheta} \, \|v\|_q^{1-\vartheta} \quad \forall \, v \in L^p(\Omega) \cap L^q(\Omega),$$
$$\text{where } p, q, r \in [1, +\infty] \quad \text{and} \quad \frac{1}{r} = \frac{\vartheta}{p} + \frac{1-\vartheta}{q} \,. \tag{2.52}$$

$$\|v\|_{L^{r_1}(0,T;L^{r_2}(\Omega))} \le \|v\|^{\vartheta}_{L^{p_1}(0,T;L^{p_2}(\Omega))} \, \|v\|^{1-\vartheta}_{L^{q_1}(0,T;L^{q_2}(\Omega))}$$
$$\forall \, v \in L^{p_1}(0,T;L^{p_2}(\Omega)) \cap L^{q_1}(0,T;L^{q_2}(\Omega)),$$
$$\text{where} \quad p_i \, , q_i \, , r_i \in [1, +\infty] \quad \text{and} \quad \frac{1}{r_i} = \frac{\vartheta}{p_i} + \frac{1-\vartheta}{q_i} \qquad \text{for } i = 1, 2. \tag{2.53}$$

We observe that (2.52) implies $\|v\|_r \le \vartheta \|v\|_p + (1-\vartheta)\|v\|_q$ for every $v \in L^p(\Omega) \cap L^q(\Omega)$ thanks to the Young inequality, and a similar remark holds for (2.53). Thus, we have the continuous embeddings

$$L^p(\Omega) \cap L^q(\Omega) \subset L^r(\Omega) \quad \text{and} \quad L^{p_1}(0,T;L^{p_2}(\Omega)) \cap L^{q_1}(0,T;L^{q_2}(\Omega)) \subset L^{r_1}(0,T;L^{r_2}(\Omega)).$$

We stress the important case of the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^6(\Omega))$, which occurs several times in the sequel and corresponds to $p_1 = \infty$, $p_2 = 2$, $q_1 = 2$, and $q_2 = 6$. In particular, the choices $\vartheta = 2/5$ and $\vartheta = 1/7$ yield the inequalities (for every v of the above space) and the continuous embeddings

$$\|v\|_{L^{10/3}(Q)} \le \|v\|_X^{2/5} \|v\|_Y^{3/5} \quad \text{and} \quad X \cap Y \subset L^{10/3}(Q) \tag{2.54}$$
$$\|v\|_{L^{7/3}(0,T;L^{14/3}(\Omega))} \le \|v\|_X^{1/7} \|v\|_Y^{6/7} \quad \text{and} \quad X \cap Y \subset L^{7/3}(0,T;L^{14/3}(\Omega)) \tag{2.55}$$
$$\text{where} \quad X := L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad Y := L^2(0,T;L^6(\Omega)).$$

Notice that we can take $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$ in (2.54)–(2.55), since $V \subset L^6(\Omega)$. Finally, we set

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T], \tag{2.56}$$

and, again throughout the paper, we use a small-case italic c for different constants, that may only depend on Ω, the final time T, the shape of the nonlinearities f and g, and the properties of the data involved in the statements at hand; a notation like c_ε signals a constant that depends also on the parameter ε. The reader should keep in mind that the meaning of c and c_ε might change from line to line and even in the same chain of inequalities, whereas those constants we need to refer to are always denoted by capital letters, just like C in (2.50).

3 The asymptotic analysis

In this section, we prove Theorem 2.3, which ensures the existence of a solution to problem (2.35)–(2.38) along with the convergence properties stated in (2.31)–(2.34).

Then, for any $\sigma \in (0,1]$ we let $(\mu_\sigma, \rho_\sigma, \xi_\sigma)$ denote the triplet defined by Proposition 2.2 and set $u_\sigma := (1 + 2g(\rho_\sigma))\mu_\sigma$. The existence of $(\mu_\sigma, \rho_\sigma, \xi_\sigma)$ has been proved in [6]: we follow in parts the arguments developed there in order to recover useful estimates independent of σ. Before that, let us remark that the property $\mu_\sigma \ge 0$ can be verified by simply multiplying equation (2.26) by $-\mu_\sigma^-$,

the negative part of μ_σ, and integrate over Q_t. In principle, in this computation one has to define κ everywhere, e.g., by taking an even extension $\bar\kappa$ with respect to the first variable. We observe that

$$\big[\big(1+2g(\rho_\sigma(t))\big)\,\partial_t\mu_\sigma+\mu_\sigma\,g(\rho_\sigma)\,\partial_t\rho_\sigma\big]\,(-\mu_\sigma^-)=\frac12\,\partial_t\big((1+2g(\rho_\sigma(t)))\,|\mu_\sigma^-|^2\big).$$

Hence, by using $\mu_{0\sigma}\ge0$ and owing to the boundary condition in (2.25), we have

$$\frac12\int_\Omega(1+2g(\rho_\sigma(t)))\,|\mu_\sigma^-(t)|^2+\int_{Q_t}\bar\kappa(\mu_\sigma,\rho_\sigma)|\nabla\mu_\sigma^-|^2=0\quad\text{for a.a. } t\in(0,T).$$

As both g and $\bar\kappa$ are nonnegative, this implies $\mu_\sigma^-=0$, that is, $\mu_\sigma\ge0$ a.e. in Q.

First a priori estimate. We test (2.26) by μ_σ and point out that

$$\big[\big(1+2g(\rho_\sigma)\big)\,\partial_t\mu_\sigma+\mu_\sigma\,g'(\rho_\sigma)\,\partial_t\rho_\sigma\big]\mu_\sigma=\frac12\,\partial_t\big[(1+2g(\rho_\sigma)\mu_\sigma^2\big].\tag{3.1}$$

Thus, by integrating over $(0,t)$, where $t\in[0,T]$ is arbitrary, we obtain

$$\int_\Omega\big(1+2g(\rho_\sigma(t))\big)|\mu_\sigma(t)|^2+2\int_{Q_t}\kappa(\mu_\sigma(s),\rho_\sigma(s))|\nabla\mu_\sigma|^2=\int_\Omega(1+2g(\rho_{0\sigma}))\mu_{0\sigma}^2\,.$$

We recall that g is nonnegative and Lipschitz continuous (cf. (2.9)–(2.10)). Moreover, $\rho_{0\sigma}$, $\mu_{0\sigma}$ are both uniformly bounded in V by (2.18), whence

$$\int_\Omega(1+2g(\rho_{0\sigma}))\mu_{0\sigma}^2\le c\,\big(\|\mu_{0\sigma}\|_2^2+\|\rho_{0\sigma}\|_2\|\mu_{0\sigma}\|_4^2\big)\le c$$

owing to the Hölder and Sobolev inequalities (see (2.50)). Then, in view of $g\ge0$ and $\kappa\ge\kappa_*>0$, from (3.1) it follows that

$$\|\mu_\sigma\|_{L^\infty(0,T;H)}+\|\mu_\sigma\|_{L^2(0,T;V)}\le c.\tag{3.2}$$

Second a priori estimate. We add ρ_σ to both sides of (2.27) and test by $\partial_t\rho_\sigma$. On account of (2.7)–(2.8) and (2.11), we obtain

$$\begin{aligned}&\int_{Q_t}|\partial_t\rho_\sigma|^2+\frac12\,\|\rho_\sigma(t)\|_H^2+\frac\sigma2\,\|\nabla\rho_\sigma(t)\|_H^2+\int_\Omega f_1(\rho_\sigma(t))\\&=\frac\sigma2\int_\Omega|\nabla\rho_{0\sigma}|^2+\int_\Omega f(\rho_{0\sigma})+\frac12\int_\Omega\big(\rho_\sigma^2(t)-2f_2(\rho_\sigma(t))\big)+\int_{Q_t}\mu_\sigma g'(\rho_\sigma)\partial_t\rho_\sigma\end{aligned}$$

for every $t\in[0,T]$. Then, thanks to the Lipschitz continuity of f_2' and g, and owing to the bounds entailed by (2.18)–(2.19), we find out that

$$\begin{aligned}&\int_{Q_t}|\partial_t\rho_\sigma|^2+\frac12\,\|\rho_\sigma(t)\|_H^2+\frac\sigma2\,\|\nabla\rho_\sigma(t)\|_H^2+\int_\Omega f_1(\rho_\sigma(t))\\&\le c+c\int_\Omega|\rho_\sigma(t)|^2+\frac14\int_{Q_t}|\partial_t\rho_\sigma|^2+c\|\mu_\sigma\|^2_{L^\infty(0,T;H)}.\end{aligned}$$

On the other hand, by the chain rule and the Young inequality (2.47) we have that

$$c\int_\Omega|\rho_\sigma(t)|^2\le c\int_\Omega|\rho_{0\sigma}|^2+\frac14\int_{Q_t}|\partial_t\rho_\sigma|^2+c\int_0^t\|\rho_\sigma(s)\|_H^2\,ds.$$

Then, as f_1 is nonnegative, by accounting for (3.2), with the help of the Gronwall lemma we infer that

$$\int_{Q_t} |\partial_t \rho_\sigma|^2 + \|\rho_\sigma(t)\|_H^2 + \sigma \|\nabla \rho_\sigma(t)\|_H^2 \le c \quad \text{for all } t \in [0,T].$$

Thus, we conclude that

$$\|\rho_\sigma\|_{H^1(0,T;H)} + \sigma^{1/2} \|\rho_\sigma\|_{L^\infty(0,T;V)} \le c. \tag{3.3}$$

Third a priori estimate. We proceed formally and test (2.27) by $-\Delta \rho_\sigma$. Hence, integrating by parts and with respect to time, we deduce that

$$\begin{aligned} &\frac{1}{2} \|\nabla \rho_\sigma(t)\|_H^2 + \sigma \int_{Q_t} |\Delta \rho_\sigma|^2 + \int_{Q_t} \beta'(\rho_\sigma) |\nabla \rho_\sigma|^2 \\ &\le \frac{1}{2} \int_\Omega |\nabla \rho_{0\sigma}|^2 - \int_{Q_t} \pi'(\rho_\sigma) |\nabla \rho_\sigma|^2 + \int_{Q_t} g'(\rho_\sigma) \nabla \mu_\sigma \cdot \nabla \rho_\sigma + \int_{Q_t} g''(\rho_\sigma) \mu_\sigma |\nabla \rho_\sigma|^2, \end{aligned} \tag{3.4}$$

where the equality $\xi_\sigma = \beta(\rho_\sigma)$ has been used along with the smoothness of β, according to our formal procedure. In fact, what is important is that the related term on the left-hand side is nonnegative, i.e.,

$$\int_{Q_t} \beta'(\rho_\sigma) |\nabla \rho_\sigma|^2 \ge 0.$$

Concerning the right-hand side of (3.4), we have that $\frac{1}{2} \int_\Omega |\nabla \rho_{0\sigma}|^2 \le c$ due to (2.18), and the estimate

$$- \int_{Q_t} \pi'(\rho_\sigma) |\nabla \rho_\sigma|^2 + \int_{Q_t} g'(\rho_\sigma) \nabla \mu_\sigma \cdot \nabla \rho_\sigma \le c \int_0^t \|\nabla \rho_\sigma(s)\|_H^2 ds + c \|\mu_\sigma\|_{L^2(0,T;V)}^2$$

owing to the boundedness of π' and g' (see (2.10)–(2.11)). About the last term, (2.9) and (2.24) imply

$$\int_{Q_t} g''(\rho_\sigma) \mu_\sigma |\nabla \rho_\sigma|^2 \le 0,$$

so that the sign properties of g'' and μ_σ become crucial to control this term. Then, in view of (3.2), from (3.4) it follows that

$$\frac{1}{2} \|\nabla \rho_\sigma(t)\|_H^2 + \sigma \int_{Q_t} |\Delta \rho_\sigma|^2 \le c + c \int_0^t \|\nabla \rho_\sigma(s)\|_H^2 \, ds \quad \text{for all } t \in [0,T],$$

and the Gronwall lemma and (3.3) allow us to deduce that

$$\|\rho_\sigma\|_{L^\infty(0,T;V)} + \sigma^{1/2} \|\rho_\sigma\|_{L^2(0,T;W)} \le c. \tag{3.5}$$

Note that here we have used the regularity theory for elliptic equations, owing to the bound on $\sigma \|\Delta \rho_\sigma\|_2^2$ and to the homogeneous Neumann boundary condition satisfied by ρ_σ (cf. (2.22)). Finally, an easy consequence of (3.3) and (3.5) comes out from a comparison of terms in (2.27), which yields

$$\|\xi_\sigma\|_{L^2(0,T;H)} \le c. \tag{3.6}$$

Fourth a priori estimate. As $u_\sigma = (1 + 2g(\rho_\sigma))\mu_\sigma$, by (2.10) we have that

$$|u_\sigma| \le c\,(1 + |\rho_\sigma|)\,|\mu_\sigma|,$$
$$|\nabla u_\sigma| = |2g'(\rho_\sigma)\mu_\sigma\nabla\rho_\sigma + (1 + 2g(\rho_\sigma))\nabla\mu_\sigma| \le c\,|\mu_\sigma|\,|\nabla\rho_\sigma| + c\,(1 + |\rho_\sigma|)\,|\nabla\mu_\sigma|.$$

Now, taking (3.2) into account, we see that $|\nabla\mu_\sigma|$ is bounded in $L^2(0,T;L^2(\Omega))$, while $|\mu_\sigma|$ is bounded in $L^2(0,T;L^6(\Omega))$ thanks to the Sobolev inequality (2.50). On the other hand, (3.5) provides a bound for $|\nabla\rho_\sigma|$ in $L^\infty(0,T;L^2(\Omega))$ and for $|\rho_\sigma|$ in $L^\infty(0,T;L^6(\Omega))$. Hence, using Hölder's inequality, it is not difficult to check that the products $|\mu_\sigma|\,|\nabla\rho_\sigma|$ and $|\rho_\sigma|\,|\nabla\mu_\sigma|$ are bounded in $L^2(0,T;L^{3/2}(\Omega))$, whereas $|\rho_\sigma|\,|\mu_\sigma|$ is even bounded in $L^2(0,T;L^3(\Omega))$. Therefore, we conclude that

$$\|u_\sigma\|_{L^2(0,T;W^{1,3/2}(\Omega))} \le c\,. \tag{3.7}$$

Fifth a priori estimate. Let us recall that (3.2) and (2.50) imply the boundedness of $\{\mu_\sigma\}$ in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^6(\Omega))$. Then, we can apply (2.52) with $p = 2$, $q = 6$, $\vartheta = 1/2$, $r = 3$ to see that

$$\|\mu_\sigma(t)\|_3^2 \le \|\mu_\sigma(t)\|_2\|\mu_\sigma(t)\|_6 \quad \text{for a.a. } t \in (0,T),$$

whence squaring and integrating with respect to t lead to

$$\|\mu_\sigma\|^4_{L^4(0,T;L^3(\Omega))} \le \|\mu_\sigma\|^2_{L^\infty(0,T;L^2(\Omega))}\,\|\mu_\sigma\|^2_{L^2(0,T;L^6(\Omega))} \le c. \tag{3.8}$$

Consider now (2.30) which turns out to be a variational formulation of (2.26). As we want to prove that

$$\|\partial_t u_\sigma\|_{L^{4/3}(0,T;V^*)} \le c\,, \tag{3.9}$$

we use (2.30) and let v vary in $L^4(0,T;V)$. By integrating with respect to time and invoking (2.6), the boundedness of g' and Hölder's inequality, we obtain

$$\left|\int_0^T \langle \partial_t u_\sigma(t), v(t)\rangle\,dt\right|$$
$$\le \kappa^*\|\nabla\mu_\sigma\|_{L^2(0,T;H)}\|\nabla v\|_{L^2(0,T;H)} + c\int_0^T \|\mu_\sigma(t)\|_3\|\partial_t\rho_\sigma(t)\|_2\|v(t)\|_6\,dt.$$

Hence, in view of (3.2), by applying the Hölder and Sobolev inequalities (see (2.50)) in the time integral, we infer that

$$\left|\int_0^T \langle \partial_t u_\sigma(t), v(t)\rangle\,dt\right|$$
$$\le c\|v\|_{L^2(0,T;V)} + c\|\mu_\sigma\|_{L^4(0,T;L^3(\Omega))}\|\partial_t\rho_\sigma\|_{L^2(0,T;H)}\|v\|_{L^4(0,T;V)}.$$

Now, the continuous embedding $L^4(0,T;V) \subset L^2(0,T;V)$, (3.8) and (3.3) allow us to conclude that

$$\left|\int_0^T \langle \partial_t u_\sigma(t), v(t)\rangle\,dt\right| \le c\|v\|_{L^4(0,T;V)},$$

whence (3.9) follows.

Passage to the limit. By the above estimates, there are a quadruplet (μ, ρ, ξ, u), with $\mu \ge 0$ a.e. in Q, and a function k such that (2.31)–(2.34) are satisfied as long as

$$\kappa(\mu_\sigma, \rho_\sigma) \to k \quad \text{weakly star in } L^\infty(Q) \tag{3.10}$$

at least for a subsequence $\sigma = \sigma_i \searrow 0$. By the weak convergence of time derivatives, the Cauchy conditions (2.28) hold for the limit pair (ρ, u). By (2.32), (2.34), and the compact embedding (2.49), we can apply well-known strong compactness results (see, e.g., [15, Sect. 8, Cor. 4]) and infer that (possibly taking another subsequence)

$$\rho_\sigma \to \rho \qquad \text{strongly in } C^0([0,T]; L^p(\Omega)) \text{ for } p < 6 \text{ and a.e. in } Q \tag{3.11}$$

$$u_\sigma \to u \qquad \text{strongly in } L^2(0,T; L^p(\Omega)) \text{ for } p < 3 \text{ and a.e. in } Q. \tag{3.12}$$

The weak convergence (2.33), together with (3.11) with $p = 2$, implies that $\xi \in \beta(\rho)$ a.e. in Q (see, e.g., [2, Prop. 2.5, p. 27]), due to the maximal monotonicity of the operator induced by β on $L^2(Q)$. Now, we deal with the other nonlinear terms and the products. We first observe that (3.11) also entails that

$$\phi(\rho_\sigma) \to \phi(\rho) \quad \text{strongly in } C^0([0,T]; L^p(\Omega)) \text{ for } p < 6 \text{ and a.e. in } Q \tag{3.13}$$

for $\phi = g, g', \pi, 1/(1+2g)$, thanks to the Lipschitz continuity of such functions. This is sufficient to establish equation (2.37). Indeed, by accounting for (2.31), we see that the product $\mu_\sigma g(\rho_\sigma)$ converges to $\mu g(\rho)$ weakly (e.g.) in $L^2(Q)$. On the other hand, (3.5) implies that $\sigma \Delta \rho_\sigma$ converges to zero strongly in $L^2(Q)$. Now, we prove equations (2.35)–(2.36), which involve the whole triplet (μ, ρ, u). The first step is showing strong convergence for μ_σ and relation (2.36). By combining (3.13) with (3.12), we see that

$$\mu_\sigma = \frac{u_\sigma}{1 + 2g(\rho_\sigma)} \to \frac{u}{1 + 2g(\rho)} \quad \text{a.e. in } Q. \tag{3.14}$$

This and (2.31) imply $\mu = u/(1 + 2g(\rho))$ and (2.36) is proved. Moreover, as $\{\mu_\sigma\}$ is bounded in $L^{10/3}(Q)$ by (3.2), the Sobolev embedding $V \subset L^6(\Omega)$, and (2.54), we can also deduce a strong convergence. We summarize as follows:

$$\mu_\sigma \to \mu \quad \text{strongly in } L^p(Q) \text{ for every } p < 10/3 \text{ and a.e. in } Q. \tag{3.15}$$

From this, we immediately infer that $\kappa(\mu_\sigma, \rho_\sigma)$ converges to $\kappa(\mu, \rho)$ a.e. in Q, just by continuity. Then, (3.10) implies $k = \kappa(\mu, \rho)$ and

$$\kappa(\mu_\sigma, \rho_\sigma) \to \kappa(\mu, \rho) \quad \text{strongly in } L^p(Q) \text{ for every } p < +\infty. \tag{3.16}$$

Therefore, $\kappa(\mu_\sigma, \rho_\sigma)\nabla\mu_\sigma$ converges to $\kappa(\mu, \rho)\nabla\mu$ weakly in $L^p(Q)$ for every $p < 2$, thanks to (2.31), and the choice $p = 3/2$ yields

$$\int_Q \kappa(\mu_\sigma, \rho_\sigma)\nabla\mu_\sigma \cdot \nabla v \to \int_Q \kappa(\mu, \rho)\nabla\mu \cdot \nabla v \quad \text{for every } v \in L^3(0,T; W^{1,3}(\Omega)).$$

On the other hand, $\mu_\sigma g'(\rho_\sigma)\partial_t\rho_\sigma$ converges to $\mu g'(\rho)\partial_t\rho$ weakly at least in $L^1(Q)$, as one can easily see by combining (2.32), (3.13), and (3.15). It follows that

$$\int_Q \mu_\sigma g'(\rho_\sigma)\partial_t\rho_\sigma\, v \to \int_Q \mu g'(\rho)\partial_t\rho\, v \quad \text{for every } v \in L^\infty(Q).$$

Moreover, (2.34) holds. Hence, we can conclude that

$$\int_0^T \langle \partial_t u(t), v(t) \rangle \, dt + \int_Q \kappa(\mu, \rho)\nabla\mu \cdot \nabla v = \int_Q \mu g'(\rho)\partial_t\rho\, v$$
$$\text{for every } v \in L^3(0,T; W^{1,3}(\Omega)) \cap L^\infty(Q). \tag{3.17}$$

Now, we observe that $\partial_t u \in L^{4/3}(0,T; V^*)$ by (2.34) and that $\kappa(\mu, \rho)\nabla v \in L^2(0,T; H)$ by (2.31) and the boundedness of κ. Finally, $\mu g'(\rho)\partial_t\rho \in L^{4/3}(0,T; L^{6/5}(\Omega))$, since g' is bounded, $\partial_t\rho \in L^2(0,T; H)$, and $\mu \in L^4(0,T; L^3(\Omega))$ as a consequence of (2.31), $V \subset L^6(\Omega)$, and (3.8)). Therefore, we can improve (3.17) by a density argument and see that the variational equation still holds for any $v \in L^4(0,T; V)$. What we obtain is equivalent to (2.35), and the proof is complete.

4 Properties of the limit problem

In this section, we prove Theorem 2.6. In the whole section, it is understood that the assumptions of Theorem 2.6 are satisfied, and sometimes we do not remind the reader about that. As far as the first part of Theorem 2.6 is concerned, the true result regards ordinary variational inequalities and we present it in the form of a lemma. For convenience, we use the same notation ρ, etc., even though it is clear that everything is independent of x: the dot over the variable ρ denotes the (time) derivative, here.

Lemma 4.1. *Let* (2.40)–(2.42) *hold and* $\rho_* \le \rho_0 \le \rho^*$*. Then for every nonnegative function* $\mu \in L^1(0,T)$*, the differential inclusion*

$$\dot\rho(t) + \beta(\rho(t)) + \pi(\rho(t)) - \mu(t) g'(\rho(t)) \ni 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad \rho(0) = \rho_0 \tag{4.1}$$

has a unique solution $\rho \in W^{1,1}(0,T)$ *such that*

$$\rho_* \le \rho(t) \le \rho^* \quad \text{and} \quad \xi_* \le \xi(t) \le \xi^* \quad \text{for a.a. } t \in (0,T), \tag{4.2}$$

where

$$\xi(t) := -\big(\dot\rho(t) + \pi(\rho(t)) - \mu(t) g'(\rho(t))\big) \in \beta(\rho(t)).$$

Moreover, there exists a constant $C > 0$ *such that if* $\mu_1, \mu_2 \in L^1(0,T)$ *and* ρ_0^1, ρ_0^2 *are two inputs and* ρ_1, ρ_2 *are the corresponding solutions of* (4.1)*, then for every* $t \in [0,T]$ *we have*

$$\begin{aligned} &|\rho_1 - \rho_2|(t) + \int_0^t |\dot\rho_1 - \dot\rho_2|(\tau)\, d\tau \\ &\le C \left(|\rho_0^1 - \rho_0^2| + \int_0^t \big((1+\mu_1)|\rho_1 - \rho_2| + |\mu_1 - \mu_2| \big)(\tau)\, d\tau \right). \end{aligned} \tag{4.3}$$

Proof. The existence of a unique solution can easily be proved, e.g., by the iterated Banach Contraction Principle, due to the monotonicity of β and to the Lipschitz continuity of the other nonlinearities. In (4.2), we only prove the upper inequalities since the proof of the lower ones is quite similar. It suffices to prove the desired inequalities for the solution (ρ, ξ) of the cut-off problem

$$\dot\rho(t) + \xi(t) + \pi^*(\rho(t)) - \mu(t) g^*(\rho(t)) = 0, \quad \xi(t) \in \beta(\rho(t)) \quad \text{for a.a. } t \in (0,T), \tag{4.4}$$

$$\rho(0) = \rho_0\,, \tag{4.5}$$

where π^* and g^* are defined by

$$\pi^*(r) := \pi(\min\{r, \rho^*\}) \quad \text{and} \quad g^*(r) := g'(\min\{r, \rho^*\})\,.$$

We test (4.1) by $(\rho - \rho^*)^+$ and integrate. Recalling (2.40)–(2.42) and noting that $\xi \ge \xi^*$ and $g^*(\rho) = g'(\rho^*)$ where $\rho > \rho^*$, we obtain

$$\begin{aligned} \frac{1}{2}|(\rho(t) - \rho^*)^+|^2 \;&\le\; -\int_0^t \big(\xi - \xi^*\big)(\rho - \rho^*)^+ - \int_0^t \big(\xi^* + \pi^*(\rho^*)\big)(\rho - \rho^*)^+ \\ &\quad + \int_0^t \big(\pi(\rho^*) - \pi(\rho)\big)(\rho - \rho^*)^+ + \int_0^t \mu\, g^*(\rho)(\rho - \rho^*)^+ \\ &\le\; \int_0^t \big(\pi(\rho^*) - \pi(\rho)\big)(\rho - \rho^*)^+ \le c \int_0^t |(\rho - \rho^*)^+|^2 \end{aligned}$$

and the assertion is obtained by the Gronwall argument. The second inequality follows from the monotonicity of β. Moreover, the lower bounds can be checked in a similar way. To prove (4.3), we set $w_i(t) = \mu_i(t)g'(\rho_i(t)) - \pi(\rho_i(t)), \xi_i(t) = w_i(t) - \dot\rho_i(t), i = 1, 2$. We have $(\xi_1 - \xi_2)(\rho_1 - \rho_2) \geq 0$ almost everywhere. The function $\mathrm{sign}(\xi_1 - \xi_2)$ (with $\mathrm{sign}(0) = 0$) is bounded and measurable, and so is $\mathrm{sign}(\rho_1 - \rho_2)$. We now claim that by testing the identity

$$(\xi_1 - \xi_2) + (\dot\rho_1 - \dot\rho_2) = w_1 - w_2 \tag{4.6}$$

by $\mathrm{sign}(\xi_1 - \xi_2)$, we infer that

$$|\xi_1 - \xi_2| + \frac{d}{dt}|\rho_1 - \rho_2| \leq |w_1 - w_2| \quad \text{a.e. in } (0, T). \tag{4.7}$$

Indeed, this is obvious for all t such that $\mathrm{sign}(\xi_1 - \xi_2)(t) = \mathrm{sign}(\rho_1 - \rho_2)(t)$ or such that $\xi_1(t) = \xi_2(t)$. The remaining case is $\mathrm{sign}(\xi_1 - \xi_2)(t) \neq 0, \mathrm{sign}(\rho_1 - \rho_2)(t) = 0$. For almost all t with this property, we have $\dot\rho_1(t) = \dot\rho_2(t)$, $\frac{d}{dt}|\rho_1 - \rho_2|(t) = 0$, and (4.7) follows. Using the Lipschitz continuity properties in (2.10) and integrating (4.7) over $(0, t)$, we obtain for $t \in (0, T)$

$$\int_0^t |\xi_1 - \xi_2|(s)\, ds + |\rho_1 - \rho_2|(t) \leq c \left(|\rho_0^1 - \rho_0^2| + \int_0^t \big((1 + \mu_1)|\rho_1 - \rho_2| + |\mu_1 - \mu_2|\big)(\tau)\, d\tau \right).$$

On the other hand, (4.6) yields

$$\int_0^t |\dot\rho_1 - \dot\rho_2|(s)\, ds \leq \int_0^t \big(|w_1 - w_2| + |\xi_1 - \xi_2|\big)(s)\, ds$$

and (4.3) follows from the sum of the last two inequalities. □

Next, if (μ, ρ, ξ, u) is a solution to problem (2.35)–(2.38), it is clear that, for almost all $x \in \Omega$, the functions $\mu(x, \cdot)$ and $\rho(x, \cdot)$, and the constant $\rho_0(x)$ satisfy the assumptions of Lemma 4.1. Thus, the first part of Theorem 2.6 concerning bounds (2.44) is proved. We derive an interesting consequence.

Corollary 4.2. *Under the assumptions of Theorem 2.6, let (μ, ρ, ξ, u) be a solution to problem* (2.35)–(2.38) *satisfying the regularity conditions specified in Theorem 2.3. Then*

$$\mu \in L^\infty(Q) \quad \textit{and} \quad \partial_t \rho \in L^\infty(Q). \tag{4.8}$$

Proof. We already know that both ξ and $\pi(\rho)$ are bounded. Moreover, $\mu g'(\rho)$ belongs to $L^\infty(0, T; H) \cap L^2(0, T; L^6(\Omega))$ since μ does so and $g'(\rho)$ is bounded. We see that also $\partial_t \rho$ belongs to such a space, just by comparison in (2.37). It follows that $\partial_t \rho \in L^{7/3}(0, T; L^{14/3}(\Omega))$ by (2.55). From this and assumption (2.45), we derive the boundedness of μ. Indeed, we can reproduce the proof carried out in [6, Fifth a priori estimate], since that proof acts only on the equation for μ and works provided that an estimate of $\partial_t \rho$ in $L^{7/3}(0, T; L^{14/3}(\Omega))$ is known. At this point, by comparing in (2.37) once more, we conclude that $\partial_t \rho$ is bounded as well. □

Remark 4.3. The analogous estimate

$$\rho_* \leq \rho_\sigma \leq \rho^* \quad \text{a.e. in } Q \tag{4.9}$$

for the solution to problem (2.26)–(2.28) also holds provided that

$$\rho_* \leq \rho_{0\sigma} \leq \rho^* \quad \text{a.e. in } \Omega. \tag{4.10}$$

We prove one of the inequalities (4.9), the other one being similar. We proceed as in the proof of Lemma 4.1, testing (2.27) by $(\rho_\sigma-\rho^*)^+$ and integrating. By accounting for the second inequality (4.10), we easily obtain

$$\begin{aligned}
&\frac12 \int_\Omega |(\rho_\sigma-\rho^*)^+(t)|^2 + \sigma \int_{Q_t} |\nabla(\rho_\sigma-\rho^*)^+|^2 \\
&\quad + \int_{Q_t} \big(\xi_\sigma-\xi^*\big)(\rho_\sigma-\rho^*)^+ + \int_{Q_t} \big(\xi^*+\pi(\rho^*)\big)(\rho_\sigma-\rho^*)^+ \\
&\le \int_{Q_t} \big(\pi(\rho^*)-\pi(\rho_\sigma)\big)(\rho_\sigma-\rho^*)^+ + \int_{Q_t} \mu_\sigma\, g'(\rho_\sigma)(\rho_\sigma-\rho^*)^+ .
\end{aligned}$$

Now, we observe that all the terms on the left-hand side are nonnegative, the third one thanks to (2.40) and the monotonicity of β (as before, the integrand vanishes whenever $\rho_\sigma \le \rho^*$), the last one due to (2.41). Concerning the right-hand side, we show that the last integrand is nonpositive. Indeed, g' is decreasing (see (2.9)), whence $g'(\rho_\sigma) \le g'(\rho^*) \le 0$ if $\rho_\sigma > \rho^*$, and $\mu_\sigma \ge 0$. By taking all this into account and owing to the Lipschitz continuity of π (cf. (2.11)), we can apply the Gronwall lemma and conclude that $(\rho_\sigma-\rho^*)^+ = 0$, i.e., $\rho \le \rho^*$ a.e. in Q.

Remark 4.4. A sufficient condition for (4.10) to hold at least for small σ is that $\rho_{0\sigma}$ is given by (2.20) and the hypotheses of Theorem 2.6 are reinforced by also assuming that

$$\text{either} \qquad \inf\operatorname{ess}\rho_0 > \rho_* \ \text{ and } \ \sup\operatorname{ess}\rho_0 < \rho^* \qquad \text{or} \qquad \xi_* \le 0 \le \xi^*. \tag{4.11}$$

The proof is rather simple and we show just one of the desired inequalities since the other one is quite similar. We test (2.20) by $(\rho_{0\sigma}-\rho^*)^+$. We easily obtain

$$\begin{aligned}
&\int_\Omega |(\rho_{0\sigma}-\rho^*)^+|^2 + \sigma \int_\Omega |\nabla(\rho_{0\sigma}-\rho^*)^+|^2 + \sigma \int_\Omega (\xi_{0\sigma}-\xi^*)(\rho_{0\sigma}-\rho^*)^+ \\
&= \int_\Omega (\rho_0-\rho^*-\sigma\xi^*)(\rho_{0\sigma}-\rho^*)^+ .
\end{aligned} \tag{4.12}$$

In the first case (4.11), we set $\delta := \rho^* - \sup\operatorname{ess}\rho_0$ and take $\sigma^* > 0$ such that $\sigma^*\,|\xi^*| \le \delta$. Then, for $\sigma \le \sigma^*$, we have $\rho_0-\rho^*-\sigma\xi^* \le -\delta + \sigma^*|\xi^*| \le 0$ a.e. in Ω, so that the right-hand side of (4.12) is nonpositive. In the second case (4.11), the same conclusion trivially holds. As the last two terms on the left-hand side are nonnegative (since (2.40) holds, β is monotone, and the third integrand vanishes whenever $\rho_{0\sigma} \le \rho^*$), we conclude that $(\rho_{0\sigma}-\rho^*)^+ = 0$, whence $\rho_{0\sigma} \le \rho^*$.

Proof of the second part of Theorem 2.6. Assume thus that $\kappa(\mu,\rho) = \kappa_0$ and set for simplicity $\kappa_0 = 1$. The system now reads

$$\langle \partial_t u(t), v\rangle + \int_\Omega \nabla\mu(t)\cdot\nabla v = \int_\Omega \mu\, g'(\rho)\,\partial_t\rho\, v \qquad \text{for all } v \in V \text{ and a.a. } t \in (0,T), \tag{4.13}$$

$$u = (1+2g(\rho))\mu \quad \text{a.e. in } Q, \tag{4.14}$$

$$\partial_t\rho + \xi + \pi(\rho) = \mu\, g'(\rho) \quad \text{and} \quad \xi \in \beta(\rho) \quad \text{a.e. in } Q, \tag{4.15}$$

$$\mu(0) = \mu_0 \quad \text{and} \quad \rho(0) = \rho_0 \quad \text{a.e. in } \Omega. \tag{4.16}$$

Let $(\mu_i, \rho_i, \xi_i, u_i)$, $i = 1, 2$ be two solutions of (4.13)–(4.16). We integrate (4.13) in time from 0 to t and subtract the equation with index 2 from the one with index 1. We test the result by $v = (\mu_1 - \mu_2)(t)$ and obtain, by virtue of Corollary 4.2, that

$$\int_\Omega (u_1 - u_2)(\mu_1 - \mu_2)(t) + \frac{1}{2}\frac{d}{dt}\int_\Omega \left| \int_0^t \nabla(\mu_1 - \mu_2) d\tau \right|^2 \\ \leq c \int_\Omega \left(|\mu_1 - \mu_2|(t) \int_0^t \left(|\mu_1 - \mu_2| + |\rho_1 - \rho_2| + |\partial_t \rho_1 - \partial_t \rho_2| \right)(\tau) d\tau \right). \quad (4.17)$$

In addition, from Lemma 4.1 (see, in particular, (4.3)) and Hölder's inequality it follows that

$$\int_\Omega \left(\int_0^t |\partial_t \rho_1 - \partial_t \rho_2|(\tau)\, d\tau \right)^2 \leq c \int_\Omega \left(\int_0^t (|\rho_1 - \rho_2| + |\mu_1 - \mu_2|)(\tau)\, d\tau \right)^2, \quad (4.18)$$

$$\int_\Omega |\rho_1 - \rho_2|^2(s) \leq D \int_0^s \int_\Omega \big(|\rho_1 - \rho_2|^2 + |\mu_1 - \mu_2|^2 \big)(\tau)\, d\tau \quad (4.19)$$

for every $t, s \in [0, T]$, thanks to the boundedness for μ_1 ensured by Corollary 4.2. Note that the constant D in (4.19) is marked for later reference.

Now, we observe that the inequalities

$$(u_1 - u_2)(\mu_1 - \mu_2) \geq |\mu_1 - \mu_2|^2 - 2\mu_1 \big(g(\rho_1) - g(\rho_2) \big)(\mu_1 - \mu_2) \geq \frac{1}{2} |\mu_1 - \mu_2|^2 - c|\rho_1 - \rho_2|^2$$

hold a.e. in Q. Thus, by integrating (4.17) from 0 to s, $s \in (0, T)$, and ignoring a positive term on the left-hand side, we obtain

$$\int_0^s \int_\Omega |\mu_1 - \mu_2|^2(t)\, dt \leq c \int_0^s \int_\Omega |\rho_1 - \rho_2|^2(t)\, dt + c \left(\int_0^s \int_\Omega |\mu_1 - \mu_2|^2(t)\, dt \right)^{1/2} \\ \times \left(\int_0^s \int_\Omega \left(\int_0^t \left(|\mu_1 - \mu_2| + |\rho_1 - \rho_2| + |\partial_t \rho_1 - \partial_t \rho_2| \right)(\tau) d\tau \right)^2 dt \right)^{1/2}. \quad (4.20)$$

Hence, using Young's inequality and (4.18), we have that

$$\int_0^s \int_\Omega |\mu_1 - \mu_2|^2(t)\, dt \leq c \int_0^s \int_\Omega |\rho_1 - \rho_2|^2(t)\, dt \\ + c \int_0^s \int_\Omega \left(\int_0^t \left(|\mu_1 - \mu_2| + |\rho_1 - \rho_2| \right)(\tau) d\tau \right)^2 dt. \quad (4.21)$$

We now multiply (4.21) by $2D$ and add it to (4.19). Thus, we obtain an inequality of the form $\Phi(s) \leq c \int_0^s \Phi(t) dt$, with

$$\Phi(s) = \int_\Omega |\rho_1 - \rho_2|^2(s) + \int_0^s \int_\Omega |\mu_1 - \mu_2|^2(t)\, dt.$$

From the Gronwall argument, it is straightforward to deduce that $\Phi(s) = 0$ for all s, hence, $\mu_1 = \mu_2$, $\rho_1 = \rho_2$, which implies uniqueness.

The L^2 bound for $\partial_t \mu$ can be established in the following way. Assume first that $\mu_0 \in W$. We extend μ by μ_0 and ρ by ρ_0 for $t < 0$. Then, equation (4.13) can be written as

$$\langle \partial_t u(t), v \rangle + \int_\Omega \nabla \mu(t) \cdot \nabla v = \int_\Omega \psi(t)\, v \quad \text{for all } v \in V \text{ and a.a. } t \in (0, T), \quad (4.22)$$

where ψ is defined by $\psi(t) = \big(\mu g'(\rho)\partial_t\rho\big)(t)$ for $t > 0$ and $\psi(t) = -\Delta\mu_0$ for $t < 0$. We observe that $\psi \in L^\infty(-T, T; H)$ thanks to Corollary 4.2 and to our assumption on μ_0. Next, we integrate (4.22) in time from $(t-h)$ to t for any fixed $t \in (0, T)$ and a small $h > 0$, with the intention to let h tend to zero, and test the resulting equality by $\mu(t) - \mu(t-h)$. We obtain

$$\begin{aligned}
&\int_\Omega \big(u(t) - u(t-h)\big)\big(\mu(t) - \mu(t-h)\big) + \frac{1}{2}\int_\Omega \frac{d}{dt}\left|\int_{t-h}^t \nabla\mu(\tau)\,d\tau\right|^2 \\
&= \int_\Omega \left(\int_{t-h}^t \psi(\tau)\,d\tau\right)\big(\mu(t) - \mu(t-h)\big) \\
&\le \frac{1}{4}\int_\Omega |\mu(t) - \mu(t-h)|^2 + \left\|\int_{t-h}^t \psi(\tau)\,d\tau\right\|_H^2 \\
&\le \frac{1}{4}\int_\Omega |\mu(t) - \mu(t-h)|^2 + c\,h^2
\end{aligned} \tag{4.23}$$

Now, we recall that (4.14) holds, that g is nonnegative and Lipschitz continuous, and that μ and $\partial_t\rho$ are bounded by Corollary 4.2. Hence, we easily derive that

$$\begin{aligned}
&\big(u(t) - u(t-h)\big)\big(\mu(t) - \mu(t-h)\big) \\
&\ge |\mu(t) - \mu(t-h)|^2 - 2\mu(t)\,|g(\rho(t)) - g(\rho(t-h))|\,|\mu(t) - \mu(t-h)| \\
&\ge |\mu(t) - \mu(t-h)|^2 - c\,h\,|\mu(t) - \mu(t-h)| \ge \frac{1}{2}\,|\mu(t) - \mu(t-h)|^2 - c\,h^2.
\end{aligned}$$

Therefore, by integrating (4.23) from 0 to T, forgetting the nonnegative term that involves $\nabla\mu$, and rearranging, we obtain

$$\int_0^T\int_\Omega |\mu(t) - \mu(t-h)|^2\,dt \le c\,h^2 + c\int_\Omega \left|\int_{-h}^0 \nabla\mu_0\,d\tau\right|^2 \le c\,h^2.$$

As $h > 0$ is arbitrarily small, this implies that $\partial_t\mu \in L^2(Q)$. At this point, we are allowed to use the Leibniz rule for the time derivative $\partial_t u$; then, from (4.13)–(4.14) we infer that the equation

$$\big(1 + 2g(\rho)\big)\partial_t\mu + \mu g'(\rho)\partial_t\rho - \Delta\mu = 0 \tag{4.24}$$

holds at least in the sense of distributions. By comparison, we deduce that $\Delta\mu \in L^2(Q)$, whence $\mu \in L^2(0, T; W)$. Using the identity

$$-\int_\Omega \partial_t\mu\,\Delta\mu = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mu|^2 \quad \text{a.e. in } (0, T),$$

we see that $\nabla\mu \in L^\infty(0, T; L^2(\Omega))$. Thus, the regularity (2.46) is established if $\mu_0 \in W$.

Let now $\mu_0 \in V \cap L^\infty(\Omega)$ be arbitrary, and consider a sequence $\{\mu_k^0\} \subset W$ bounded in $L^\infty(\Omega)$ and converging to μ_0 in V as $k \to \infty$. Let $(\mu_k, \rho_k, \xi_k, u_k)$ be the corresponding solutions to (4.13)–(4.16). Then, we can use equation (4.24) written with the index k and test it by $\partial_t\mu_k$. We obtain

$$\int_\Omega |\partial_t\mu_k(t)|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mu_k(t)|^2 \le \int_\Omega |\psi_k(t)|\,|\partial_t\mu_k(t)|, \tag{4.25}$$

with an obvious choice of $\psi_k \in L^2(Q)$ bounded in this space (even better) independently of k. By time integration, it is straightforward to obtain a bound for $\|\partial_t\mu_k\|_{L^2(Q)}$ and for $\|\nabla\mu_k\|_{L^\infty(0,T;H)}$ independent of k. Then, by weak star compactness we infer that

$$\mu_k \to \tilde\mu \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V)$$

at least for a subsequence, which implies (see, e.g., [15, Cor. 4, p. 85]) strong convergence in $C^0([0,T];H)$. In particular, $\tilde{\mu}(0) = \mu_0$. On the other hand, $(\mu_k, \rho_k, \xi_k, u_k)$ satisfies the estimates stated in Lemma 4.1 and the boundedness properties for μ_k and $\partial_t \rho_k$ given by Corollary 4.2, which are uniform with respect to k. This yields weak or weak star limits $\tilde{\rho}$ and $\tilde{\xi}$. Moreover, strong convergence in $L^1(Q)$ for $\{\rho_k\}$ and $\{\partial_t \rho_k\}$ is ensured via a Cauchy sequence argument based on (4.3), integration over Ω, and Gronwall's lemma. Hence, $\{\mu_k\}$, $\{\rho_k\}$, $\{\partial_t \rho_k\}$ converge strongly in $L^p(Q)$ for every $p \in [1, \infty)$. At this point, it is not difficult to verify that $(\tilde{\mu}, \tilde{\rho}, \tilde{\xi}, \tilde{u})$, with the corresponding $\tilde{u}$, actually solves problem (2.35)–(2.38) and thus coincides with the unique solution (μ, ρ, ξ, u). Therefore, the proof is complete.

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