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Weak solutions to lubrication systems describing the evolution of bilayer thin films

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Abstract

The existence of global nonnegative weak solutions is proved for coupled one-dimensional lubrication systems that describe the evolution of nanoscopic bilayer thin polymer films that take account of Navier-slip or no-slip conditions at both liquid-liquid and liquid-solid interfaces. In addition, in the presence of attractive van der Waals and repulsive Born intermolecular interactions existence of positive smooth solutions is shown.

1 Introduction

During the last decades lubrication theory was successfully applied to modeling of dewetting processes in micro and nanoscopic liquid films on a solid polymer substrates see e.g. [4, 15, 17] to name a few, for a review we refer to [8] and references therein. A typical closed-form one-dimensional lubrication equation derived from the underlying equations for conservation of mass and momentum, together with boundary conditions for the tangential and normal stresses, as well as the kinematic condition at the free boundary, impermeability and a slip condition at the liquid-solid interface has the form:

$$\partial_t h = -\partial_x \Big(M(h) \partial_x \left(\partial_{xx} h - \Pi(h) \right) \Big), \tag{1.1}$$

where function $h(x,t)$ denotes the height profile for the free surface of the film. The mobility function has the form $M(h) = h^3$ or $M(h) = bh^2$ for the no-slip or Navier-slip conditions considered at the solid-liquid interface, respectively, where $b > 0$ denotes the slip-length parameter.

Recently, this model was generalized to a coupled lubrication system describing evolution of a layered system of two viscous, immiscible, nanoscopic Newtonian fluids evolving on a solid substrate [1, 9, 14] and subsequently analysed in [2, 13, 14, 16, 18]. The latter system can be stated in the form:

$$\begin{aligned} u_t &= -\partial_x \left(M_{11} \partial_x p_1 + M_{12} \partial_x p_2 \right), \\ v_t &= -\partial_x \left(M_{12} \partial_x p_1 + M_{22} \partial_x p_2 \right), \end{aligned} \tag{1.2}$$

where $u(x,t)$ and $v(x,t)$ denote the height of the lower liquid and the difference between the heights of the upper and lower liquid, respectively (see Fig. 1). The pressures $p_1(x,t)$ and $p_2(x,t)$ are defined as

$$\begin{aligned} p_1 &= (\sigma + 1)\partial_x^2 u + \partial_x^2 v - \Pi_1(u), \\ p_2 &= \partial_x^2 u + \partial_x^2 v - \Pi_2(v), \end{aligned} \tag{1.3}$$

where $\partial_x^2 u$ and $\partial_x^2 v$ are linearised surface tension terms and potentials $\Pi_1(u)$ and $\Pi_2(v)$ describe the intermolecular interactions of the bottom liquid with the solid surface and of two liquids with each other, respectively. The influence of intermolecular interactions is typically due

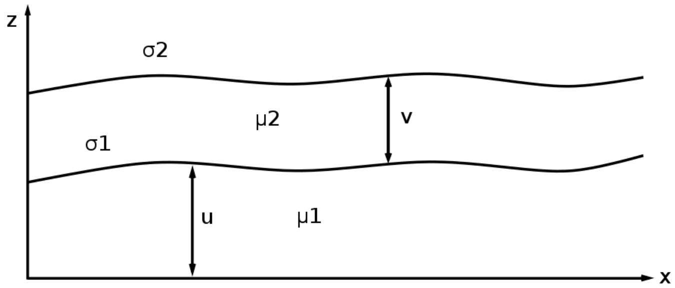

Figure 1: Sketch of a bilayer thin film

to the competition between long-range attractive van der Waals and short-range Born repulsive intermolecular forces, see [10, 17]. In this article we consider two case: the absence of intermolecular interactions, i.e. $\Pi_k(s) \equiv 0$ for $k = 1, 2$ and the case when both van der Waals and Born intermolecular forces are presented in the form

$$\Pi_k(s) = \frac{1}{s^n} - \frac{\gamma_k}{s^m}, \ (n < m, \ \gamma_1, \gamma_2 \ll 1) \tag{1.4}$$

A typical choice for (n, m) is $(3, 12)$ corresponding to the standard Lennard-Jones potential.

As in the case of the single layer lubrication equation (1.1) the form of mobility matrix

$$M(u, v) = \begin{pmatrix} M_{11}(u, v) & M_{12}(u, v) \\ M_{21}(u, v) & M_{22}(u, v) \end{pmatrix}$$

depends on the slip conditions considered at the liquid-liquid and liquid-solid interfaces. In the case of the no-slip at the both interfaces, it has the form

$$M = \frac{1}{\mu} \begin{pmatrix} \frac{1}{3}u^3 & \frac{1}{2}u^2 v \\ \frac{1}{2}u^2 v & \frac{\mu}{3}v^3 + uv^2 \end{pmatrix}. \tag{1.5}$$

The model parameters $\sigma = \sigma_1/\sigma_2$ and $\mu = \mu_1/\mu_2$ in (1.3) and (1.5) are positive constants which denote the ratios of surface tensions and viscosities, respectively. Recently, the lubrication system for the case of Navier-slip conditions considered at both liquid-liquid and liquid-solid interfaces was derived in [14, 18]. The corresponding mobility matrix can be stated in the form

$$M = \frac{1}{\mu} \begin{pmatrix} b_1 u^2 & b_1 uv \\ b_1 uv & b_1 v^2 + b(\mu + 1)v^2 \end{pmatrix},$$

where $b_1 > 0$ and $b \geq 0$ denote the slip lengths at the solid-liquid and liquid-liquid interfaces, respectively. We discuss the origin of the different mobilities in section 5. Note that rescaling time by b_1 and introducing the parameter $\alpha := \frac{b}{b_1}(\mu + 1) > 0$ the latter matrix can be written in the form

$$M = \frac{1}{\mu} \begin{pmatrix} u^2 & uv \\ uv & (1 + \alpha)v^2 \end{pmatrix}. \tag{1.6}$$

In this study we consider the system (1.2) on a time-space domain $Q_T = \Omega \times (0, T)$ where $\Omega = (0, 1)$ with boundary conditions

$$u_x = u_{xxx} = v_x = v_{xxx} = 0 \quad \text{for } x \in \partial\Omega \tag{1.7}$$

and the initial functions

$$u(x, 0) = u_0(x) \geq 0,\ v(x, 0) = v_0(x) \geq 0 \quad u, v \in H^1(\Omega). \tag{1.8}$$

We consider the system (1.2) both with the mobility matrixes (1.5) and (1.6). The system (1.2) can be also generalised to incorporate presence surfactants or temperature-gradient-caused Marangoni flows (see e.g. [7, 9, 16]).

Starting from the seminal work of Bernis and Friedman [3] existence theory of nonnegative weak solutions for single nonlinear parabolic equations in the form (1.1) was successfully developed (see e.g. [5, 6] and references there in). Note that the system (1.2) inherits from the one-layer lubrication equation (1.1) the high order and degeneracy as one of the fluid heights u or v goes to zero. In contrast to (1.1) there are only few analytical results known about the system (1.2). The structure of its stationary solutions were considered in [13, 18]. Existence of nonnegative weak solutions to (1.2) in the no-slip case, i.e. with mobility matrix given by (1.5) was shown recently in [2] using a finite-element approximation under a strong assumption on presence of intermolecular potentials of the form (1.4) between liquid films and between each film and the substrate as well.

In this article, we show existence of nonnegative weak solutions to (1.2) with (1.7)–(1.8) for both no-slip (1.5) and Navier-slip cases (1.6) in the absence of intermolecular forces. In turn in the presence of intermolecular forces between liquid films and just between bottom liquid and the substrate as in (1.3) we show that the observed weak solution becomes positive and smooth.

In our approach we extend ideas introduced in [3] for the single lubrication equation of the form (1.1) to the system (1.2). However, the extension is not straight forward. There are new challenges since the mobility matrix degenerates in more than one way. Beside the case that u and v vanish simultaneously, also the mobility matrix M may degenerates if either u or v becomes zero while the other does not. All these cases have to be treated very carefully. In section 2 we introduce the corresponding regularized version of the system (1.2) for the no-slip case, with mobility matrix (1.5). By deriving the energy dissipation and corresponding a priori estimates, using the theory of uniformly parabolic systems (see [11]), we show global existence of smooth solutions to the regularised problem. Furthermore, we show that the latter converge to a suitably defined weak solution of the original system (1.2). Notice that these results are independent of the presence of intermolecular forces in the equations.

In section 3 we prove nonnegativity of thus obtained weak solutions in the case when intermolecular forces are absent. In the presence of intermolecular forces as in (1.2) the weak solutions turn out to be positive and smooth. Our approach for proving the nonnegativity is based on a definition of suitable analogs of Bernis and Friedman entropies for functions u and v and showing their combined dissipation.

Moreover, in section 4 we show global existence of nonnegative weak solutions to (1.2) in the Navier-slip case (1.6).

We should point out that an alternative proof for existence of weak solution in the no-slip case considered without intermolecular forces was appeared in parallel to our article in [12]. In the section 5 we discuss our results and, in particular, compare them with those of [12].

2 Existence of weak solutions in the no-slip case

In this section we consider the system (1.2) without intermolecular interactions, i.e. with p_1, p_2 in (1.3) given by

$$p_1 = (\sigma+1)u_{xx} + v_{xx}, \;\; p_2 = u_{xx} + v_{xx}; \tag{2.1}$$

and with the no-slip mobility matrix

$$M = \frac{1}{\mu}\begin{pmatrix} \frac{1}{3}|u|^3 & \frac{1}{2}|u|^2|v| \\ \\ \frac{1}{2}|u|^2|v| & \frac{\mu}{3}|v|^3 + |u||v|^2 \end{pmatrix}.$$

Notice that we replaced every u and v in the mobility matrix (1.5) by their absolute values to ensure that the latter is positive semidefinite. We will prove existence of global weak solutions to (1.2), (2.1) considered with boundary and initial conditions (1.7)–(1.8). We begin our analysis with introduction of a regularised version of (1.2), (1.8) and derivation of a priori estimates for its solutions.

2.1 Regularised system and a priori estimates

Since (1.2) is degenerate at $u = 0$ and $v = 0$, we approximate it by a family of non-degenerate equations

$$\begin{aligned} u_t + ((M_{11}+\varepsilon)p_{1,x} + M_{12}p_{2,x})_x = 0 \\ v_t + (M_{21}p_{1,x} + (M_{22}+\varepsilon)p_{2,x})_x = 0 \end{aligned} \quad \text{in } Q_T, \tag{2.2}$$

where $\varepsilon > 0$ is arbitrary. Note that the regularised mobility matrix is positive definite for all u and v. Correspondingly the system (2.2) is uniformly parabolic in Petrovskii sense (see [11] for definition). Furthermore, we approximate u_0 and v_0 in the $H^1(\Omega)$-norm by $C^{4+\alpha}$ functions $u_{0\varepsilon}$ and $v_{0\varepsilon}$ satisfying (1.7),

$$u_{0\varepsilon}(x) \geq u_0(x) \text{ and } v_{0\varepsilon}(x) \geq v_0(x) \text{ for } x \in \Omega, \tag{2.3}$$

and replace (1.8) by

$$u(x,0) = u_{0\varepsilon}(x), \;\; v(x,0) = v_{0\varepsilon}(x). \tag{2.4}$$

By [11, Theorem. 6.3, p.302] the system (2.2) considered with (1.7), (2.4) has a unique local solution $(u_\varepsilon, v_\varepsilon)$ in Q_τ for some small $\tau = \tau(\varepsilon) > 0$.

Everywhere below in this article we denote by C positive constants independent of ε which may vary from line to line. Let us also introduce notations

$$\begin{aligned} &M_\varepsilon = M(u_\varepsilon, v_\varepsilon) \\ &p_{1,\varepsilon} = (\sigma+1)u_{\varepsilon,xx} + v_{\varepsilon,xx}, \text{ and } p_{2,\varepsilon} = u_{\varepsilon,xx} + v_{\varepsilon,xx}. \end{aligned} \tag{2.5}$$

A priori estimates

Let us define an energy (Lyapunov) functional for the system (1.2) coupled with (1.7) as

$$E(u_\varepsilon, v_\varepsilon) = \int\limits_\Omega \left[\sigma u_{\varepsilon,x}^2 + (u_{\varepsilon,x} + v_{\varepsilon,x})^2\right] dx \tag{2.6}$$

Indeed, differentiating the latter in time along solutions of (1.2) with (1.7) one obtains the corresponding energy equality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(u_\varepsilon, v_\varepsilon) + \int\limits_\Omega (M_{11\varepsilon}p_{1\varepsilon,x}^2 + 2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} + M_{22\varepsilon}p_{2\varepsilon,x}^2)dx \\ +\varepsilon \int\limits_\Omega \left(p_{1\varepsilon,x}^2 + p_{2\varepsilon,x}^2\right) dx = 0.\end{aligned} \tag{2.7}$$

Note that the second term in (2.7) is nonnegative since M_ε is positive semidefinite. By the approximation properties of $u_{0\varepsilon}, v_{0\varepsilon}$ one has

$$\begin{aligned}\int\limits_\Omega u_{0\varepsilon,x}^2 dx \leq (1 + \eta(\varepsilon)) \int\limits_\Omega u_{0,x}^2 dx, \\ \int\limits_\Omega v_{0\varepsilon,x}^2 dx \leq (1 + \eta(\varepsilon)) \int\limits_\Omega v_{0,x}^2 dx, ,\end{aligned} \tag{2.8}$$

where $\eta(\varepsilon) \to 0$ if $\varepsilon \to 0$ and therefore $E(u_{0\varepsilon}, v_{0\varepsilon}) \leq C$ holds. This together with (2.7) imply the following a priori estimates:

$$\sup_{t\in(0,\tau)} \int\limits_\Omega u_{\varepsilon,x}^2 dx \leq C, \ \sup_{t\in(0,\tau)} \int\limits_\Omega u_{\varepsilon,x}^2 dx \leq C, \tag{2.9}$$

and

$$\iint\limits_{Q_\tau} (M_{11\varepsilon}p_{1\varepsilon,x}^2 + 2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} + M_{22\varepsilon}p_{2\varepsilon,x}^2)dx \leq C, \tag{2.10}$$

$$\varepsilon \iint\limits_{Q_\tau} \left(p_{1\varepsilon,x}^2 + p_{2\varepsilon,x}^2\right) dx \leq C. \tag{2.11}$$

Integrating (2.2) in time we deduce the conservation of mass law

$$\int\limits_\Omega u_\varepsilon(x,t)dx = \int\limits_\Omega u_{0\varepsilon}dx, \ \int\limits_\Omega v_\varepsilon(x,t)dx = \int\limits_\Omega v_{0\varepsilon}dx \tag{2.12}$$

for all $t \in (0, \tau)$. Using this, (2.8), Poincare's inequality and the Sobolev embedding theorem

$$H^1(\Omega) \subset C^{0,\frac{1}{2}}(\bar{\Omega})$$

one obtains

$$||u_\varepsilon(\,.\,,t)||_{C^{0,\frac{1}{2}}(\bar{\Omega})} \leq C, \ ||v_\varepsilon(\,.\,,t)||_{C^{0,\frac{1}{2}}(\bar{\Omega})} \leq C. \tag{2.13}$$

Next, we obtain uniform Hölder estimates for u_ε and v_ε in time. Let us introduce functions

$$J_{1,\varepsilon} = M_{11\varepsilon} p_{1\varepsilon,x} + M_{12\varepsilon} p_{2\varepsilon,x} \text{ and } J_{2,\varepsilon} = M_{21\varepsilon} p_{1\varepsilon,x} + M_{22\varepsilon} p_{2\varepsilon,x}.$$

Observe that for every $t \in (0,\tau)$

$$\begin{aligned}
&\iint\limits_{Q_t} J_{1,\varepsilon}^2 dxdt \\
&\leq C \iint\limits_{Q_t} M_{11\varepsilon} \left(M_{11\varepsilon} p_{1\varepsilon,x}^2 + 2M_{12\varepsilon} p_{1\varepsilon,x} p_{2\varepsilon,x} + M_{22\varepsilon} p_{2\varepsilon,x}^2 \right) dxdt \\
&\leq C, \qquad (2.14)
\end{aligned}$$

where we use $M_{12\varepsilon}^2 \leq M_{11\varepsilon} M_{22\varepsilon}$, (2.10) and (2.13). Analogously,

$$\iint\limits_{Q_t} J_{2,\varepsilon}^2 dxdt \leq C \qquad (2.15)$$

holds. Now, using (2.14)–(2.15) and the relations

$$\iint\limits_{Q_t} u_\varepsilon \phi_t = - \iint\limits_{Q_t} J_{1,\varepsilon} \phi_x \text{ and } \iint\limits_{Q_t} v_\varepsilon \phi_t = - \iint\limits_{Q_t} J_{2,\varepsilon} \phi_x$$

considered with the special test function ϕ taken exactly as in the analogous proof for the single layer lubrication equation (1.1) in [3, Lemma 2.1] one obtains that for all $x \in \bar{\Omega}$ and $t_1,\ t_2$ in $(0,\tau)$ the following holds

$$\begin{aligned}
|u_\varepsilon(x,t_2) - u_\varepsilon(x,t_1)| &\leq C|t_2 - t_1|^{\frac{1}{8}}, \\
|v_\varepsilon(x,t_2) - v_\varepsilon(x,t_1)| &\leq C|t_2 - t_1|^{\frac{1}{8}}
\end{aligned} \qquad (2.16)$$

Conclusion

The relations (2.13) and (2.16) imply upper bounds on the $C_{x,t}^{\frac{1}{2},\frac{1}{8}}$-norms of u_ε and v_ε in Q_τ, which are independent of τ, ε. These a priori bounds allows us to conclude that $(u_\varepsilon, v_\varepsilon)$ can be extended step-by-step to a solution of (2.2) considered with (1.7), (2.4) in Q_T for any positive $T > 0$ (see [11, Theorem. 9.3, p.316]), and that

$$\begin{gathered}
\text{the sequences } \{u_\varepsilon\} \text{ and } \{v_\varepsilon\} \text{ are a uniformly bounded} \\
\text{and equi-continuous families in } \bar{Q}_T.
\end{gathered} \qquad (2.17)$$

2.2 Convergence to global weak solutions

Here we show that solutions $u_\varepsilon,\ v_\varepsilon$ of the regularised system (2.2) converge to suitably defined global weak solutions of the initial system (1.2). By (2.17), every sequence $\varepsilon \longrightarrow 0$ has a subsequence (for short both not labeled) such that

$$u_\varepsilon \longrightarrow u,\ v_\varepsilon \longrightarrow v \quad \text{uniformly in } \bar{Q}_T. \qquad (2.18)$$

Note that, due to uniform bounds (2.13) and (2.16), any such limits u and v can be defined globally in time using a standard Cantor diagonal argument (choosing a sequence $T_n \to \infty$).

Theorem 2.1. *Any pair of functions (u,v) obtained as in (2.18) satisfies for any $T > 0$ the following properties:*

$$u, v \in C^{1/2,1/8}_{x,t}(\bar{Q}_T),\ u, v \in C^{4,1}_{x,t}(P), \tag{2.19}$$

$$M_{11}p_{1,x} + M_{12}p_{2,x},\ M_{21}p_{1,x} + M_{22}p_{2,x} \in L^2(P), \tag{2.20}$$

$$|u|^3 p_{1,x} \in L^2(R), |v|^3 p_{2,x} \in L^2(S); \tag{2.21}$$

where $P = \bar{Q}_T \backslash (\{u = 0\} \cup \{v = 0\} \cup \{t = 0\})$, $R = \bar{Q}_T \cap \{v = 0\} \cap \{|u| > 0\}$ and $S = \bar{Q}_T \cap \{u = 0\} \cap \{|v| > 0\}$. Furthermore, there exists a function $w \in L^2(R)$, such that (u,v) satisfies (1.2) in the following sense:

$$\iint\limits_{Q_T} u\phi_t + \iint\limits_{P} \left(M_{11}p_{1,x} + M_{12}p_{2,x}\right)\phi_x + \iint\limits_{R} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x = 0 \tag{2.22}$$

$$\iint\limits_{Q_T} v\phi_t + \iint\limits_{P} \left(M_{21}p_{1,x} + M_{22}p_{2,x}\right)\phi_x + \iint\limits_{S} \frac{1}{3}|v|^3 p_{2,x}\phi_x = 0 \tag{2.23}$$

for all $\phi \in Lip(\bar{Q}_T), \phi = 0$ near $t = 0$ and $t = T$;

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \bar{\Omega}, \tag{2.24}$$

$$||u(\cdot,t)||_{L^1(\Omega)} = ||u_0||_{L^1(\Omega)},\ ||v(\cdot,t)||_{L^1(\Omega)} = ||u_0||_{L^1(\Omega)}, \tag{2.25}$$

$$u_x(\cdot,t) \to u_{0,x} \text{ and } v_x(\cdot,t) \to v_{0,x} \text{ strongly in } L^2(\Omega) \text{ as } t \to 0, \tag{2.26}$$

and

$$u \text{ and } v \text{ satisfy (1.7) at all points of the lateral boundary, where } u \neq 0 \text{ and } v \neq 0. \tag{2.27}$$

Proof. By the properties of the constructed solutions u_ε, v_ε to the regularised system and their uniform convergence to u, v the first assertion in (2.19) and also (2.24)–(2.25) follow immediately. Using (2.10), we observe

$$\iint\limits_{Q_T} (M_{11\varepsilon}p^2_{1,\varepsilon,x} + M_{22\varepsilon}p^2_{2\varepsilon,x})dxdt \leq C - 2\iint\limits_{Q_T} M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x}dxdt = C - \frac{1}{\mu}\iint\limits_{Q_T} |u_\varepsilon|^2|v_\varepsilon|p_{1\varepsilon,x}p_{2\varepsilon,x}dxdt.$$

From this applying Young's inequality

$$|u_\varepsilon|^2|v_\varepsilon|p_{1\varepsilon,x}p_{2\varepsilon,x} \leq \frac{7}{24}|u_\varepsilon|^3 p^2_{1\varepsilon,x} + \frac{6}{7}|u_\varepsilon||v_\varepsilon|^2 p^2_{2\varepsilon,x}$$

one obtains

$$\iint\limits_{Q_T} (M_{11\varepsilon}p^2_{1,\varepsilon,x} + M_{22\varepsilon}p^2_{2\varepsilon,x})dxdt \leq C \tag{2.28}$$

and therefore estimates

$$\iint\limits_{Q_T} |u_\varepsilon|^3 p_{1\varepsilon,x}^2 dxdt \le C, \ \iint\limits_{Q_T} |v_\varepsilon|^3 p_{2\varepsilon,x}^2 dxdt \le C, \ \iint\limits_{Q_T} |u_\varepsilon||v_\varepsilon|^2 p_{2\varepsilon,x}^2 dxdt \le C. \tag{2.29}$$

For any ϕ as in (2.22) one has

$$\iint\limits_{Q_T} u_\varepsilon \phi_t dxdt + \iint\limits_{Q_T} \left((M_{11\varepsilon}+\varepsilon)p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt = 0, \tag{2.30}$$

$$\iint\limits_{Q_T} v_\varepsilon \phi_t dxdt + \iint\limits_{Q_T} \left(M_{21\varepsilon}p_{1\varepsilon,x} + (M_{22\varepsilon}+\varepsilon)p_{2\varepsilon,x}\right)\phi_x dxdt = 0. \tag{2.31}$$

By (2.14) and (2.15) there exist $J_1, J_2 \in L^2(Q_T)$ and a subsequence as $\varepsilon \to 0$ such that

$$\begin{aligned}&\left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right) \rightharpoonup J_1\\ \text{and }\ &\left(M_{21\varepsilon}p_{1\varepsilon,x} + M_{22\varepsilon}p_{2\varepsilon,x}\right) \rightharpoonup J_2 \text{ weakly in } L^2(Q_T).\end{aligned} \tag{2.32}$$

Additionally by (2.11)

$$\varepsilon\iint\limits_{Q_T} p_{1\varepsilon,x}\phi_x dxdt \to 0, \ \varepsilon\iint\limits_{Q_T} p_{2\varepsilon,x}\phi_x dxdt \to 0 \text{ as } \varepsilon \to 0.$$

By regularity theory of uniformly parabolic systems and the uniform Hölder continuity of the u_ε and v_ε we deduce that $u, v \in C^{4,1}_{x,t}(P)$ and

$$J_1 = M_{11}p_{1,x} + M_{12}p_{2,x}, \ J_2 = M_{21}p_{1,x} + M_{22}p_{2,x} \ \text{ in } \ P. \tag{2.33}$$

Next, for a fixed $\delta > 0$ define a set $I_{1,\delta} = \{|v| \le \delta < |u|\}$. From the estimates (2.29) it follows that there exists $w \in L^2(I_{1,\delta})$ such that $p_{1\varepsilon,x} \rightharpoonup p_{1,x}$ and $v_\varepsilon p_{2\varepsilon,x} \rightharpoonup w$ weakly in $L^2(I_{1,\delta})$ as $\varepsilon \to 0$. Therefore, one obtains

$$\iint\limits_{I_{1,\delta}} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt \to \iint\limits_{I_{1,\delta}} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt \tag{2.34}$$

as $\varepsilon \to 0$. On the other hand, one has the estimate

$$\begin{aligned}&\iint\limits_{|u|\le\delta}(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x})\phi_x dxdt\\ &\le C\left(\iint\limits_{|u|\le\delta} M_{11\varepsilon}\left(M_{11\varepsilon}p_{1\varepsilon,x}^2 + 2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} + M_{22\varepsilon}p_{2\varepsilon,x}^2\right)dxdt\right)^{1/2}\\ &\le C\delta^{3/2}\end{aligned} \tag{2.35}$$

Let us decompose the second term in (2.30) as follows

$$\begin{aligned}&\iint\limits_{|u|>\delta,\ |v|>\delta} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt + \iint\limits_{|u|\le\delta} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt\\ &\quad+\varepsilon\iint\limits_{Q_T} p_{1\varepsilon,x}\phi_x dxdt + \iint\limits_{|u|>\delta,\ |v|\le\delta} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt\end{aligned} \tag{2.36}$$

and take the limit $\delta \to 0$ extracting a proper diagonal subsequence $\varepsilon \to 0$ as follows. Using (2.33) one has

$$\lim_{\delta\to 0} \Big| \iint\limits_{|u|>\delta,\ |v|>\delta} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt - \iint\limits_{P} J_1\phi_x dxdt\Big| \leq$$

$$\lim_{\delta\to 0} \Big| \iint\limits_{|u|>\delta,\ |v|>\delta} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt - \iint\limits_{|u|>\delta,\ |v|>\delta} J_1\phi_x dxdt\Big| +$$

$$+ \lim_{\delta\to 0} \Big| \iint\limits_{|u|>\delta,\ |v|>\delta} J_1\phi_x dxdt - \iint\limits_{P} J_1\phi_x dxdt\Big| = 0,$$

where in the last line we used that $J_1\phi_x$ is a bounded continuous function in P. Furthermore, from (2.34) one obtains

$$\lim_{\delta\to 0} \Big| \iint\limits_{I_{1,\delta}} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt$$

$$- \iint\limits_{v=0,\ |u|>0} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt\Big| \leq$$

$$\lim_{\delta\to 0} \Big| \iint\limits_{I_{1,\delta}} \left(M_{11\varepsilon}p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x}\right)\phi_x dxdt$$

$$- \iint\limits_{I_{1,\delta}} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt\Big| +$$

$$+ \lim_{\delta\to 0} \Big| \iint\limits_{|v|\leq\delta,\ |u|>\delta} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt$$

$$- \iint\limits_{v=0,\ |u|>0} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt\Big| \leq$$

$$\lim_{\delta\to 0} \Big| \iint\limits_{\delta\geq|v|>0} \left(\frac{1}{3\mu}|u|^3 p_{1,x} + \frac{1}{2\mu}|u|^2 w\right)\phi_x dxdt\Big| \leq$$

$$\lim_{\delta\to 0} C\left(\left(\iint\limits_{\delta\geq|v|>0} |u|^3 dxdt \iint\limits_{\delta\geq|v|>0} |u|^3 p_{1,x}^2 dxdt\right)^{\frac{1}{2}} + \left(\iint\limits_{\delta\geq|v|>0} |u|^4 dxdt \iint\limits_{\delta\geq|v|>0} w^2 dxdt\right)^{\frac{1}{2}}\right) \leq$$

$$\lim_{\delta\to 0} C\left(\iint\limits_{\delta\geq|v|>0} 1 dxdt\right)^{\frac{1}{2}} = 0,$$

where in the last inequality we used (2.19) and (2.29). Therefore, the last two estimates together with (2.35) imply that there exists a subsequence $\varepsilon \to 0$ such that (2.30) converge to (2.22).

Similarly, defining for a fixed $\delta \geq 0$ the set $I_{2,\delta} = \{|u| \leq \delta < |v|\}$ one can estimate

$$\iint\limits_{|v|\leq\delta} (M_{21\varepsilon}p_{1\varepsilon,x} + M_{22\varepsilon}p_{2\varepsilon,x})\phi_x dxdt \leq C\delta, \tag{2.37}$$

and

$$
\begin{aligned}
&\iint\limits_{I_{2,\delta}} \left(M_{21\varepsilon} p_{1\varepsilon,x} + M_{22\varepsilon} p_{2\varepsilon,x} - \frac{1}{3}|v_\varepsilon|^3 p_{2\varepsilon,x} \right) \phi_x dxdt \\
&\le C \left(\iint\limits_{I_{2,\delta}} \left(M_{21\varepsilon}^2 p_{1\varepsilon,x}^2 + 2M_{21\varepsilon} \left(M_{22\varepsilon} - \frac{1}{3}|v_\varepsilon|^3 \right) p_{1\varepsilon,x} p_{2\varepsilon,x} \right.\right. \\
&\qquad \left.\left. + \left(M_{22\varepsilon} - \frac{1}{3}|v_\varepsilon|^3 \right)^2 p_{2\varepsilon,x}^2 \right) dxdt \right)^{1/2} \\
&\le C \left(\iint\limits_{I_{2,\delta}} \frac{1}{\mu} |u_\varepsilon||v_\varepsilon|^2 \left(M_{11\varepsilon} p_{1\varepsilon,x}^2 + 2M_{21\varepsilon} p_{1\varepsilon,x} p_{2\varepsilon,x} + M_{22\varepsilon} p_{2\varepsilon,x}^2 \right) dxdt \right)^{1/2} \\
&\le C\delta^{1/2}.
\end{aligned} \tag{2.38}
$$

Moreover, again from (2.29) it follows that $p_{2\varepsilon,x} \rightharpoonup p_{2,x}$ weakly in $L^2(I_{2,\delta})$ as $\varepsilon \to 0$ and therefore, we deduce that

$$
\iint\limits_{I_{2,\delta}} \left(\frac{1}{3}|v_\varepsilon|^3 p_{2\varepsilon,x} \right) \phi_x dxdt \to \iint\limits_{I_{2,\delta}} \left(\frac{1}{3}|v|^3 p_{2,x} \right) \phi_x dxdt \tag{2.39}
$$

as $\varepsilon \to 0$. Taking the limit $\delta \to 0$ and the corresponding diagonal sequence $\varepsilon \to 0$ in (2.31) (as was done before for (2.30)) shows that it converges to (2.23) in view of (2.37)–(2.39).

To prove (2.26) notice that from $u_{0,\varepsilon} \to u_0$, $v_{0,\varepsilon} \to v_0$ in $H^1(\Omega)$ and (2.7) we get

$$
\limsup_{t\to 0} \int\limits_{\Omega} \left(\sigma u_x^2(x,t) + (u_x + v_x)^2(x,t) \right) dx \le \int\limits_{\Omega} \left(\sigma u_{0,x}^2 + (u_{0,x} + v_{0,x})^2 \right) dx.
$$

Since also

$$
u_x(.,t) \to u_{0,x} \text{ and } v_x(.,t) \to v_{0,x} \text{ weakly in } L^2(\Omega)
$$

as $t \to 0$, the assertion (2.26) follows.

The proof of the theorem is complete. $\square$

3 Nonnegativity of solutions

In this section we prove that the global weak solutions constructed in the previous section are nonnegative provided the initial data u_0 and v_0 are nonnegative. Furthermore, for the system (2.1) considered with intermolecular potentials $\Pi_1(u)$ and $\Pi_2(v)$ as in (1.3)–(1.4) we show existence of positive smooth solutions.

3.1 Nonnegativity in the absence of intermolecular forces

Following ideas of [3], we define a suitable entropy in order to show nonnegativity of the weak solutions u and v from the Theorem 2.1, provided (1.8) holds.

For $n \in \{2,3\}$ we set

$$g_{\varepsilon,n}(s) = -\int_s^A \frac{dr}{(|r|^n+\varepsilon)^{1/2}},\ G_{\varepsilon,n}(s) = -\int_s^A g_{\varepsilon,n}(r)dr \tag{3.1}$$

with a constant A such that $A \geq \max\{|u_\varepsilon|,\,|v_\varepsilon|\}$ for all sufficiently small ε. Then one has

$$G'_{\varepsilon,n}(s) = g_{\varepsilon,n}(s),\ G''_{\varepsilon,n}(s) = g'_{\varepsilon,n}(s) = \frac{1}{(|s|^n+\varepsilon)^{1/2}}.$$

Also,

$$g_{\varepsilon,n}(s) \leq 0,\ G_{\varepsilon,n}(s) \geq 0 \text{ if } s \leq A$$

and

$$G_{\varepsilon,n}(s) \leq G_0(s) \text{ for all } s \in \mathbb{R}^1 \tag{3.2}$$

where $G_{0,n} = \lim_{\varepsilon\to 0} G_{\varepsilon,n}$ such that for $0 \leq s \leq A$

$$G_0(s) = \begin{cases} (A - s - s\log\left(\frac{A}{s}\right)), & n=2 \\ 2(\sqrt{A} + \sqrt{\frac{1}{A}}s - 2\sqrt{s}), & n=3 \end{cases}. \tag{3.3}$$

Since the structure of (1.2) is not symmetric with respect to u and v we use two entropies: one depending on u and the other on v. For a fixed $\delta > 0$ one has

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(G_{\varepsilon,3}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\right)dx = \int_\Omega \Big(G'_{\varepsilon,3}(u_\varepsilon)u_{\varepsilon,t} + G'_{\varepsilon,2}(v_\varepsilon)v_{\varepsilon,t}\Big)dx \\
&= \int_\Omega \Big(G''_{\varepsilon,3}(u_\varepsilon)u_{\varepsilon,x}((M_{11\varepsilon}+\varepsilon)p_{1\varepsilon,x} + M_{12\varepsilon}p_{2\varepsilon,x})\Big)dx \\
&\quad + \int_\Omega \left(G''_{v\varepsilon}(v_\varepsilon)v_{\varepsilon,x}(M_{21\varepsilon}p_{1\varepsilon,x} + (M_{22\varepsilon}+\varepsilon)p_{2\varepsilon,x})\right)dx \\
&\leq \frac{\delta}{2}\int_\Omega \Big((G''_{\varepsilon,3}(u_\varepsilon))^2(M_{11\varepsilon}+\varepsilon)((M_{11\varepsilon}+\varepsilon)p^2_{1\varepsilon,x} + 2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} \\
&\quad +(M_{22\varepsilon}+\varepsilon)p^2_{2\varepsilon,x})\Big)dx + \frac{\delta}{2}\int_\Omega \Big((G''_{v\varepsilon}(v_\varepsilon))^2(M_{22\varepsilon}+\varepsilon)((M_{11\varepsilon}+\varepsilon)p^2_{1\varepsilon,x} \\
&\quad +2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} + (M_{22\varepsilon}+\varepsilon)p^2_{2\varepsilon,x})\Big)dx + \frac{1}{2\delta}\int_\Omega \Big(u^2_{\varepsilon,x} + v^2_{\varepsilon,x}\Big)dx.
\end{aligned} \tag{3.4}$$

By definition (3.1) it follows that

$$(G''_{\varepsilon,3}(u_\varepsilon))^2(M_{11\varepsilon}+\varepsilon) \leq C, \text{ and } (G''_{\varepsilon,2}(v_\varepsilon))^2(M_{22\varepsilon}+\varepsilon) \leq C, \tag{3.5}$$

where in the last inequality we used the fact that there exists a constant C such that $M_{22,\varepsilon} \leq C|v_\varepsilon|^2$ holds. Combining (3.4) with the energy inequality (2.7) and taking $\delta < 1$ one obtains

$$\frac{d}{dt}\int_\Omega \big(G_{\varepsilon,3}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\big)\,dx + \frac{1}{2}\frac{d}{dt}E(u_\varepsilon, v_\varepsilon) \tag{3.6}$$

$$+ (1-\delta)\int_\Omega \Big((M_{11\varepsilon}+\varepsilon)p_{1\varepsilon,x}^2 + 2M_{12\varepsilon}p_{1\varepsilon,x}p_{2\varepsilon,x} + (M_{22\varepsilon}+\varepsilon)p_{2\varepsilon,x}^2\Big)\,dx \tag{3.7}$$

$$\leq \frac{1}{2\delta}\int_\Omega \Big(u_{\varepsilon,x}^2 + v_{\varepsilon,x}^2\Big)\,dx \leq \frac{C}{2\delta}\int_\Omega \Big(\sigma u_{\varepsilon,x}^2 + (u_{\varepsilon,x}+v_{\varepsilon,x})^2\Big)\,dx. \tag{3.8}$$

This implies using Gronwall inequality

$$\int_\Omega \big(G_{\varepsilon,3}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\big)\,dx \leq \exp\left(\frac{t}{\delta}\right)\int_\Omega \big(G_{\varepsilon,3}(u_{0\varepsilon}) + G_{\varepsilon,2}(v_{0\varepsilon})\big)\,dx + \frac{1}{2}E(u_{0\varepsilon}, v_{0\varepsilon}).$$

On the other hand by (1.8), (2.3) and (3.2)–(3.3) one has

$$\begin{aligned}\int_\Omega \big(G_{\varepsilon,3}(u_{0\varepsilon}) + G_{\varepsilon,2}(v_{0\varepsilon})\big)\,dx &\leq \int_\Omega \big(G_{0,3}(u_{0\varepsilon}) + G_{0,2}(v_{0\varepsilon})\big)\,dx \\ &\leq \int_\Omega \big(G_{0,3}(u_0) + G_{0,2}(v_0)\big)\,dx \leq C.\end{aligned}$$

Therefore, the last two estimates imply that for all $t \leq T$

$$\int_\Omega \big(G_{\varepsilon,3}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\big)\,dx \leq C. \tag{3.9}$$

Finally, we prove the nonnegativity of u and v by contradiction. Assume there is a point $(x_0, t_0)\in Q_T$ such that $u(x_0, t_0) < 0$. Since $u_\varepsilon \to u$ uniformly there exist $\gamma > 0,\ \varepsilon_0 > 0$ such that

$$u_\varepsilon(x, t_0) < -\gamma \text{ if } |x - x_0| < \gamma,\ x \in \bar{\Omega},\ \varepsilon < \varepsilon_0.$$

For such x

$$\begin{aligned}G_{\varepsilon,3}(u_\varepsilon(x,t_0)) = -\int_{u_\varepsilon(x,t_0)}^{A} g_{\varepsilon,3}(s)ds &\geq -\int_{-\gamma}^{0} g_{\varepsilon,3}(s)ds \\ &\to -\int_{-\gamma}^{0} g_{0,3}(s)ds \text{ as } \varepsilon \to 0\end{aligned}$$

by monotone convergence theorem, where $g_{0,n}(s) = \lim_{\varepsilon\to 0} g_{\varepsilon,n}(s)$. Since by (3.1)

$$g_{0,n}(s) = -\infty \text{ if } s < 0,\ n \geq 2 \tag{3.10}$$

it follows that

$$\lim_{\varepsilon\to 0} G_{\varepsilon,3}(u_\varepsilon(x,t_0)) = \infty,$$

which is a contradiction to (3.9). A completely analogous argument shows $v \geq 0$ using (3.9) and (3.10) with $n = 2$.

3.2 The case including intermolecular forces

In this section we consider the system (1.2)-(1.3) in the presence of the intermolecular forces given as in (1.4) considered with (1.7) and the initial data satisfying

$$\int_{\Omega} [U_1(u_0) + U_2(v_0)]\, dx \leq C_1, \tag{3.11}$$

where by definition

$$U_k(s) = -\int_s^{\infty} \Pi_k(\tau) d\tau.$$

Theorem 3.1. *Assume that* $0 < n < m$ *and* $m \geq 3$ *in (1.4). Then a positive smooth solution to (1.2)-(1.4) coupled with (1.7), (3.11) exists for all* $t \in (0, T)$.

Proof. Taking a suitable Hölder continuous regularisation of potentials $\Pi_k(s)$ and proceeding as in the section 2 one can show existence of regularised solutions u_ε and v_ε to (2.2) considered now with (1.2)-(1.4) and (1.7), (2.4) that satisfy the regularity properties as before. Note that the energy functional (2.6) transforms in this case to

$$E(u_\varepsilon, v_\varepsilon) = \int_{\Omega} \left[\sigma u_{\varepsilon,x}^2 + (u_{\varepsilon,x} + v_{\varepsilon,x})^2 + 2U_1(u_\varepsilon) + 2U_2(u_\varepsilon)\right] dx$$

for which the energy inequality (2.7) still holds. Therefore, using the fact that $U_k(s),\ k = 1, 2$ are bounded from below, and hence

$$-\int_{\Omega} [U_1(u_\varepsilon) + U_2(v_\varepsilon)]\, dx \leq C \tag{3.12}$$

together with (3.11) imply again the estimates (2.9)–(2.11).

We show additionally that there exists a constant δ independent of ε such that

$$u_\varepsilon \geq \delta > 0,\ v_\varepsilon \geq \delta > 0 \text{ hold in } Q_T. \tag{3.13}$$

Then proceeding to the limit $\varepsilon \to 0$ as in Theorem 2.1 the smoothness of the positive limits u and v will follow from the uniform parabolic theory and (3.13) and the statement of the theorem will be shown. Indeed, observe from (2.7) that

$$\sup_{t \in (0,T)} \int_{\Omega} (U_1(u_\varepsilon(\cdot, t)) + U_2(v_\varepsilon(\cdot, t)))\, dx \leq C. \tag{3.14}$$

Since U_2 is bounded from below one has also

$$\sup_{t \in (0,T)} \int_{\Omega} U_1(u_\varepsilon(\cdot, t)) dx \leq C.$$

Let $u_\varepsilon(x_0, t) = \min_\Omega u_\varepsilon(\cdot, t)$. By Hölder continuity of u_ε we get

$$u_\varepsilon(x, t) \leq u_\varepsilon(x_0, t) + C|x - x_0|^{1/2}.$$

Analogously to the proof for the single layer equation (1.1) in [4] one obtains for $0 < n < m$

$$C \geq \int\limits_{\Omega} U_1(u_\varepsilon(\cdot,,t))dx \geq C_2\eta(u_\varepsilon(x_0,t)) + C_3,$$

where $\eta(s) = -\log s$ for $m = 3$, $\eta(s) = s^{3-m}$ for $m > 3$. Hence $u_\varepsilon(x_0,t) > 0$ holds for all $t \in (0,T)$. The same argument works for $\min_\Omega v_\varepsilon(\cdot,t)$. Therefore (3.13) is true. $\square$

4 Existence of nonnegative weak solutions in the Navier-slip case

In this section we show that solutions u_ε and v_ε to the regularised system (2.2)–(2.4) considered now with the Navier-slip mobility matrix

$$M = \begin{pmatrix} |u|^2 + \varepsilon & |u||v| \\ & \\ |u||v| & (1+\alpha)|v|^2 + \varepsilon \end{pmatrix}.$$

converge to global nonnegative weak solutions to (1.2) considered with (1.6), (2.1) and (1.7)–(1.8). Note that the case when intermolecular forces are present i.e. for (1.2) considered with (1.3)–(1.4) and (1.6) proceeds then exactly as in Theorem 3.2 for the no-slip case.

The dissipation (2.7) of the energy functional (2.6) and the corresponding a priori estimates (2.9)–(2.13) and (2.16) stay true in the Navier-slip case as well. Therefore, (2.18) holds again up to a subsequence as $\varepsilon \to 0$. The following theorems that thus obtained limits u and v are nonnegative global weak solutions.

Theorem 4.1. *Functions (u,v) satisfy for any $T > 0$ the following properties:*

$$u, v \in C^{1/2,1/8}_{x,t}(\bar{Q}_T),\ u, v \in C^{4,1}_{x,t}(P), \tag{4.1}$$

$$M_{11}p_{1,x} + M_{12}p_{2,x},\ M_{21}p_{1,x} + M_{22}p_{2,x} \in L^2(P), \tag{4.2}$$

$$|u|^2 p_{1,x} \in L^2(R), |v|^2 p_{2,x} \in L^2(S); \tag{4.3}$$

where $P = \bar{Q}_T \backslash (\{u = 0\} \cup \{v = 0\} \cup \{t = 0\})$, $R = \bar{Q}_T \cap \{v = 0\} \cap \{|u| > 0\}$ and $S = \bar{Q}_T \cap \{u = 0\} \cap \{|v| > 0\}$. Furthermore, there exist functions $w_1 \in L^2(R)$ and $w_2 \in L^2(S)$, such that (u,v) satisfies (1.2) in the following sense:

$$\begin{aligned} \iint\limits_{Q_T} u\phi_t + \iint\limits_{P} \left(M_{11}p_{1,x} + M_{12}p_{2,x}\right)\phi_x \\ + \iint\limits_{R} \Big(|u|^2 p_{1,x} + |u|w_1\Big)\phi_x = 0, \end{aligned} \tag{4.4}$$

$$\iint\limits_{Q_T} v\phi_t + \iint\limits_{P} \left(M_{21}p_{1,x} + M_{22}p_{2,x}\right)\phi_x + \iint\limits_{S} (|v|w_2 + (1+\alpha)|v|^2 p_{2,x})\phi_x = 0 \tag{4.5}$$

for all $\phi \in Lip(\bar{Q}_T), \phi = 0$ near $t = 0$ and $t = T$;

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \quad x \in \bar{\Omega}, \tag{4.6}$$

$$||u(\cdot,t)||_{L^1(\Omega)} = ||u_0||_{L^1(\Omega)},\ ||v(\cdot,t)||_{L^1(\Omega)} = ||u_0||_{L^1(\Omega)}, \tag{4.7}$$

$$u_x(\cdot,t) \to u_{0,x} \text{ and } v_x(\cdot,t) \to v_{0,x} \text{ strongly in } L^2(\Omega) \text{ as } t \to 0, \tag{4.8}$$

and

$$\begin{gathered} u \text{ and } v \text{ satisfy (1.7) at all points of the lateral} \\ \text{boundary, where } u \neq 0 \text{ and } v \neq 0. \end{gathered} \tag{4.9}$$

Proof. The assertions (4.1)–(4.2) and (4.6)–(4.8) follow exactly as in the proof of Theorem 2.1. Using (2.10) one observes

$$\begin{aligned} \iint\limits_{Q_T} (M_{11\varepsilon} p_{1,\varepsilon,x}^2 + M_{22\varepsilon} p_{2\varepsilon,x}^2) dxdt &\leq C - 2 \iint\limits_{Q_T} M_{12\varepsilon} p_{1\varepsilon,x} p_{2\varepsilon,x} dxdt \\ &= C - 2 \iint\limits_{Q_T} (|u_\varepsilon||v_\varepsilon| p_{1\varepsilon,x} p_{2\varepsilon,x}) dxdt \end{aligned}$$

By Young's inequality

$$2|u_\varepsilon||v_\varepsilon| p_{1\varepsilon,x} p_{2\varepsilon,x}) \leq \frac{2}{2+\alpha} |u_\varepsilon|^2 p_{1\varepsilon,x}^2 + \left(1 + \frac{\alpha}{2}\right) |v_\varepsilon|^2 p_{2\varepsilon,x}^2$$

and hence one obtains

$$\begin{aligned} &\iint\limits_{Q_T} |u_\varepsilon|^2 p_{1\varepsilon,x}^2 dx \leq C, \qquad \iint\limits_{Q_T} |v_\varepsilon|^2 p_{2\varepsilon,x}^2 dx \leq C, \\ &\iint\limits_{Q_T} (M_{11\varepsilon} p_{1,\varepsilon,x}^2 + M_{22\varepsilon} p_{2\varepsilon,x}^2) dxdt \leq C. \end{aligned} \tag{4.10}$$

Next, for ϕ as in (4.5)–(4.5) one writes again (2.30)–(2.31). Considering the set $I_{1,\delta}$ and $I_{2,\delta}$ as in the proof of Theorem 2.1 and using the estimates (4.10) one can show in the analogous manner that (2.30) and (2.31) converge up to a diagonal subsequence as $\delta \to 0$ and $\varepsilon \to 0$ to (4.4) and (4.5) respectively. ☐

Theorem 4.2. *The global weak solutions u and v constructed in the previous theorem are non-negative.*

Proof. The proof proceeds similarly to the argument for nonnegativity of the weak solutions in the no-slip case presented in the section 3.1. The following estimates

$$(G''_{\varepsilon,2}(u_\varepsilon))^2 (M_{11\varepsilon} + \varepsilon) \leq C, \text{ and } (G''_{\varepsilon,2}(v_\varepsilon))^2 (M_{22\varepsilon} + \varepsilon) \leq C,$$

and

$$\begin{aligned} &\frac{d}{dt} \int\limits_\Omega \big(G_{\varepsilon,3}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\big)\, dx + \frac{1}{2}\frac{d}{dt} E(u_\varepsilon, v_\varepsilon) \\ &\quad + (1-\delta) \int\limits_\Omega \Big((M_{11\varepsilon} + \varepsilon) p_{1\varepsilon,x}^2 + 2M_{12\varepsilon} p_{1\varepsilon,x} p_{2\varepsilon,x} + (M_{22\varepsilon} + \varepsilon) p_{2\varepsilon,x}^2\Big)\, dx \\ &\leq \frac{1}{2\delta} \int\limits_\Omega \Big(u_{\varepsilon,x}^2 + v_{\varepsilon,x}^2\Big)\, dx. \end{aligned}$$

are used as analogs to (3.5)–(3.6) in this case in order to obtain the crucial estimate

$$\int_\Omega \big(G_{\varepsilon,2}(u_\varepsilon) + G_{\varepsilon,2}(v_\varepsilon)\big)\, dx \le C.$$

The rest of the proof proceeds proceeds exactly as in the last paragraph of the section 3.1 but now using (3.10) with $n = 2$ for both functions u and v. ☐

5 Conclusion and discussions

In this article we showed existence of nonnegative global weak solutions for the coupled lubrication systems corresponding to the cases of no-slip and Navier-slip conditions at both liquid-liquid and liquid-solid interfaces. Our results can be generalised in a straight forward way to the system (1.2) considered with the mobility matrix

$$M = \frac{1}{\mu}\begin{pmatrix} \frac{1}{3}u^3 + b_1 u^2 & \frac{1}{2}u^2 v + b_1 uv \\ \frac{1}{2}u^2 v + b_1 uv & \frac{\mu}{3}v^3 + uv^2 + b_1 v^2 + b(\mu+1)v^2 \end{pmatrix}. \tag{5.1}$$

corresponding to the weak-slip conditions at the both interfaces. As it was shown recently in [14] the latter model incorporates both the no-slip and the Navier-slip models (1.5),(1.6) as limiting cases as the slip lengths b, b_1 tend simultaneously to zero or infinity, respectively.

One needs to point out that we obtained a slight difference between the weak formulations in the no-slip and Navier-slip cases (compare Theorems 2.1 and 4.1). Due to the fact that M_{11} and M_{22} components in (1.6) depend only on u or v, respectively, in contrast to the no-slip case (1.5) there is no an analog of estimate (2.37) in the Navier-slip case. Therefore, an additional (so far not identified in terms of solutions u and v) function w_2 appears on the singular set S in the latter case. The same problem persists also in the weak-slip case because the leading orders of the mobility matrix components on the set S coincide with those for the Navier-slip case.

In this sense we have “more regularity“ for the weak solutions in the no-slip case then for ones the in weak- or Navier-slip cases. This interesting observation should be understood better in future in view of the fact that for the single lubrication equation (1.1) the no-slip case is known to be more singular from both physical and analytical points of view then the slip cases. At the same time we’ve become aware of an alternative proof for the existence of weak solutions in the no-slip case in [12] for which the authors have shown the same regularity as we in the Navier-slip case. But also in our weak formulation for the no-slip case remains an open question weather not yet identified in terms of the solutions function w_1 vanishes on the singular set R. Another observation appearing as well due to different component structures of the mobility matrices (1.5) and (1.6) is that we have stronger entropy for u then for v in the case of the former matrix whereas the entropies are the same for the latter one.

Additionally in contrast to the existing results for the single lubrication equation (1.1) (see e.g. [3, 5]) we are not aware if the constructed weak solutions for the systems (1.2) should necessarily posses zero contact angles. This is due to an absence so far of the strict entropy dissipation inequality for the two layered systems (1.2) which was shown before to hold for (1.1). Combined energy-entropy dissipation inequalities derived here (as e.g. (3.6)) do not imply H^2 a priori estimates on the solutions.

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