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Multiple scattering of electromagnetic waves by a finite number of point-like obstacles

Durga Prasad Challa[1], Guanghui Hu[2], Mourad Sini[1]

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[1] Johann Radon Institute
for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenbergerstr. 69
A–4040 Linz
Austria
E-Mail: durga.challa@oeaw.ac.at
mourad.sini@oeaw.ac.at

[2] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: guanghui.hu@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper is concerned with the time-harmonic electromagnetic scattering problem for a finite number M of point-like obstacles in $\mathbb{R}^3$. First, we give a rigorous justification of the Foldy method and describe the intermediate levels of scattering between the Born and Foldy models. Second, we study the problem of detecting the scatterers and the scattering strengths from the far-field measurements and discuss the effect of multiple scattering related to each of these models.

1 Introduction

We consider the scattering of a time-harmonic electromagnetic plane wave from an inhomogeneous isotropic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x) > 0$, magnetic permeability $\mu = \mu_0 > 0$ and electric conductivity $\sigma = \sigma(x)$. It is supposed that the inhomogeneous medium occupies a bounded domain such that $\epsilon(x) = \epsilon_0 > 0$ and $\sigma(x) = 0$ for x outside of some sufficiently large ball. Assume that the time-harmonic incident plane waves (with the time-form $\exp(-i\omega t)$) take the form

$$E^i(x;\theta,p) = p\exp(i\kappa\, x\cdot\theta), \quad H^i(x;\theta,p) = (\theta\times p)\exp(i\kappa\, x\cdot\theta), \quad \theta\perp p,$$

where $\kappa := \omega\sqrt{\epsilon_0\mu_0} > 0$ is the wavenumber corresponding to the background medium and $\theta, p \in \mathbb{S}^2 := \{x : |x| = 1\}$ stand for the propagation and polarization directions, respectively. Then, the total electric and magnetic fields E, H satisfy the reduced time-harmonic Maxwell equations

$$\operatorname{curl} E - i\kappa\, H = 0, \quad \operatorname{curl} H + i\kappa\, n(x)E = 0, \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where the refractive index $n = n(x)$ is given by

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right).$$

The scattered fields $E^s := E - E^i, H^s := H - H^i$ are required to satisfy the Silver-Müller radiation condition

$$\lim_{|x|\to\infty}(H^s\times x - |x|E^s) = 0,$$

uniformly in all directions $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^2$, leading to the electric and magnetic far-field patterns E^∞, H^∞ given by the asymptotic behavior

$$E^s(x) = \frac{e^{i\kappa\,|x|}}{|x|}\left\{E^\infty(\hat{x}) + O(\frac{1}{|x|})\right\}, \quad H^s(x) = \frac{e^{i\kappa\,|x|}}{|x|}\left\{H^\infty(\hat{x}) + O(\frac{1}{|x|})\right\} \tag{1.2}$$

as $|x| \to \infty$. It is well known that E^∞ and H^∞ are both analytic functions defined on $\mathbb{S}^2$, satisfying $E^\infty\perp H^\infty$, and that they are both normal to $\mathbb{S}^2$.

In this paper we assume that the inhomogeneous medium consists of a finite number of components and that the wavelength of incidence is much larger than the diameter of each component. The inhomogeneous medium in this situation can be regarded as the collection of a finite number of point-like obstacles centered at $y_j, j = 1, 2, \cdots, M$. These point-like obstacles are treated as isotropic, so we can write the refractive index function in the form

$$\kappa^2(n(x) - 1) = \sum_{j=1}^{M} a_j \, \delta(x - y_j), \tag{1.3}$$

where $a_j \in \mathbb{C}$ is the scattering strength attached to the scatterer located at y_j, see [12]. The value of a_j can be viewed as the limit of the potential coefficients for approximating the idealized δ-function $\delta(x-y_j)$. Eliminating the magnetic field from (1.1) and making use of (1.3), we find

$$\operatorname{curl}\operatorname{curl} E - \kappa^2 E = \sum_{j=1}^{M} a_j \, \delta(x - y_j) E, \tag{1.4}$$

which models the electromagnetic scattering by M point-like obstacles.

We refer the reader to the interesting book [9] for a comprehensive study of the multiple scattering in general and the scattering by point-like scatterers in particular, where practical motivations of the corresponding models and historical facts are discussed. Another close reference to our work is the seminal book [1] concerning the scattering by point-like potentials in quantum mechanics, where applications to many different areas and historical references are provided. Regarding the scattering by point-like scatterers for the Maxwell models, we use [12] as a key reference where an overview of the applications related to the model (1.2)-(1.4) is given.

The contributions of this paper are twofold. First, we give sense to the scattering problem (1.2)-(1.4). For this, we follow the *regularization approach* developed in the frame work of the solvable models in quantum mechanics, see [1]. A main difficulty regarding the Maxwell system, compared to the acoustic model or the elastic system, is the fact that the corresponding Green's tensor has a higher (and non integrable) singularity. To overcome this difficulty, we use an idea from [12], where the problem (1.2)-(1.4) is studied in the case of one point-scatterer and a formal computation of the scattering matrix is shown. This idea consists of decomposing this Green's tensor into its longitudinal and transversal parts, see (2.23)-(2.24) and then regularize them (in the Fourier variables), see (2.32). Using this regularization step, we generalize the method in [1] to the Maxwell system and we derive the explicit form of the Green's tensor of the problem (1.4) from which we deduce the representation of far fields corresponding to plane incident waves, see (2.40). This representation is nothing but the model (2.13) obtained using the formal Foldy method after adjusting accordingly the scattering strengths, compare (2.13) with (2.40). In particular, we retrieve the results in [12] as a special case, see Remark 2.6. In addition, this representation takes into account the multiple scattering between the point scatterers. Based on this model, we then describe the intermediate scattering models and the Born model as well. The analogue results for the Lamé system are shown in [7]. Second, we study the inverse scattering problem consisting of recovering the point-scatterers as well as the attached scattering strengths from the far fields corresponding to incident plane waves. For this, we use the three different models given by Born, Foldy and intermediate levels and discuss the effect of the multiple scattering on the resolution of the reconstructions in terms of the used wavelength, the distance between the scatterers and the scattering strengths. This study is a continuation of the one provided in [4] for the acoustic and elastic cases.

The rest of the paper is organized as follows. In Section 2, we study the forward problem where we justify the Foldy model and derive the intermediate levels of scattering. In Section 3, we study the inverse problems related to these models while in Section 4, we justify some technical identities used in our analysis.

Throughout the paper the notation $(\cdot)^\top$ means the transpose of a vector or a matrix, and $e_j, j = 1, 2, 3,$ denote the Cartesian unit vectors in $\mathbb{R}^3$.

2 The forward problem

2.1 The far-field patterns corresponding to Born, Foldy and intermediate models

We introduce the dyadic Green's function Π_κ for the Maxwell equations in the homogeneous isotropic background. It is well-known that Π_κ takes the form (see e.g. [10, Chapter 12] and [11, Theorem 5.2.1])

$$\Pi_\kappa(x, y) = \Phi_\kappa(x, y)\,\mathbf{I} + \frac{1}{\kappa^2}\nabla_y\nabla_y\Phi_\kappa(x, y) \in \mathbb{C}^{3\times 3}, \quad x \neq y, \tag{2.1}$$

satisfying

$$\mathrm{curl}\,_y\mathrm{curl}\,_y\Pi_\kappa(x, y) - \kappa^2\Pi_\kappa(x, y) = \delta(x - y)\,\mathbf{I}, \quad x \neq y, \tag{2.2}$$

where the notation $\mathbf{I}$ stands for the 3×3 identity matrix, $\Phi_\kappa(x, y) := (4\pi)^{-1}\exp(i\kappa|x - y|)/|x - y|$ is the fundamental function to the Helmholtz equation $(\Delta + \kappa^2)u = 0$ in $\mathbb{R}^3$, and $\nabla_y\nabla_y\Phi_\kappa(x, y)$ is the Hessian matrix for Φ_κ defined by

$$(\nabla_y\nabla_y\Phi_\kappa(x, y))_{j,l} = \frac{\partial^2\Phi}{\partial y_j\partial y_l}, \quad 1 \leq j, l \leq 3.$$

Note that $\mathrm{curl}\,\Pi_\kappa$ is understood as the application of curl to each column of Π_κ. A simple calculation shows that each column of Π_κ satisfies the Silver-Müller radiation condition, leading to the far-field matrix $\Pi^\infty(\hat{x}; y)$ of $\Pi_\kappa(x, y)$ as $|x| \to \infty$ given by

$$\Pi^\infty(\hat{x}; y) = \frac{e^{-i\kappa\hat{x}\cdot y}}{4\pi}(\mathbf{I} - \hat{x} \otimes \hat{x}), \tag{2.3}$$

where $\hat{x} \otimes \hat{x} := \hat{x}\,\hat{x}^\top \in \mathbb{R}^{3\times 3}$. For the incident electric field E^i and the scattered field E^s, we have (see e.g. [10, Theorem 12.2])

$$E^i(x) = -\int_{|y|=R}\left\{\Pi_\kappa^\top(x, y)(\nu \times \mathrm{curl}\,E^i)(y) + (\mathrm{curl}\,_y\Pi_\kappa)^\top(x, y)(\nu \times E^i)(y)\right\}ds(y) \tag{2.4}$$

for $|x| < R$, and

$$0 = \lim_{R\to\infty}\int_{|y|=R}\left\{\Pi_\kappa^\top(x, y)(\nu \times \mathrm{curl}\,E^s)(y) + (\mathrm{curl}\,_y\Pi_\kappa^\top)(x, y)\,(\nu \times E^s)(y)\right\}ds(y), \tag{2.5}$$

where ν is the unit normal vector to the sphere $|y| = R$ directed outside. Note that each term in (2.4) and (2.5) is understood as the matrix-vector multiplication. Multiplying Π_κ to both sides of (1.4) and using integration by parts with the aid of (2.4) and (2.5), we obtain

$$E(x) = E^i(x) + \sum_{j=1}^{M} a_j\,\Pi_\kappa(x, y_j)E(y_j), x \neq y_j, j = 1, \ldots, M. \tag{2.6}$$

However, it is not easy to evaluate $E(x)$ by calculating the values of $E(y_j)$'s, since the Green's function on the right hand side of (2.6) is singular at y_j. Below we describe several methods for approximating solutions of (2.6).

2.1.1 Born approximation

In the Born approximation, we only need to replace $E(y_j)$ by the value $E^i(y_j)$ of the incident field. Therefore, $E(x)$ can be represented as

$$E(x) = E^i(x) + \sum_{j=1}^{M} a_j \, \Pi_\kappa(x, y_j) E^i(y_j), \tag{2.7}$$

and the far-field pattern of the scattered field is given by

$$E^\infty(\hat{x}) = \frac{1}{4\pi} \left(\sum_{j=1}^{M} a_j \, e^{i\kappa\,(\theta - \hat{x})\cdot y_j} \right) (\mathbf{I} - \hat{x} \otimes \hat{x})\, p, \ \hat{x} \in \mathbb{S}^2. \tag{2.8}$$

The Born (weak) approximation neglects the multiple scattering between the point-like obstacles. Hence (2.7) is a good approximation only if the distance between y_j and y_l ($l \neq j$) is relatively large with the wave length.

2.1.2 Foldy method

In the Foldy model, the total field $E(x)$ has the form

$$E(x) = E^i(x) + \sum_{j=1}^{M} a_j \, \Pi_\kappa(x, y_j) \, \Lambda_j, \quad \Lambda_j \in \mathbb{C}^{3\times 1}, \tag{2.9}$$

where the approximating terms $\Lambda_j := \Lambda_j(y_j)$ can be calculated from the Foldy linear algebraic system given by

$$\Lambda_j = E^i(y_j) + \sum_{\substack{l=1 \\ l \neq j}}^{M} a_l \, \Pi_\kappa(y_j, y_l) \, \Lambda_l, \ \forall\, j = 1, \ldots, M. \tag{2.10}$$

Remark that (2.10) is obtained from (2.6), tending x to y_j and deleting the singular part $\Pi_\kappa(y_j, y_j)$. The equations in (2.10) can be written as the system

$$[\tilde{\Gamma}]_{3M\times 3M} \, [\Lambda]_{3M\times 1} = [\mathbf{E}^I]_{3M\times 1}, \tag{2.11}$$

with $\Lambda := (\Lambda_1^\top, \Lambda_2^\top, \cdots, \Lambda_M^\top)^\top \in \mathbb{C}^{3M\times 1}$, $\mathbf{E}^I := (E^i(y_1)^\top, \cdots, E^i(y_M)^\top)^\top \in \mathbb{C}^{3M\times 1}$ and

$$\tilde{\Gamma} := \begin{pmatrix} \mathbf{I} & -a_2\Pi_\kappa(y_1, y_2) & -a_3\Pi_\kappa(y_1, y_3) & \cdots & -a_M\Pi_\kappa(y_1, y_M) \\ -a_1\Pi_\kappa(y_2, y_1) & \mathbf{I} & -a_3\Pi_\kappa(y_2, y_3) & \cdots & -a_M\Pi_\kappa(y_2, y_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1\Pi_\kappa(y_M, y_1) & -a_2\Pi_\kappa(y_M, y_2) & -a_3\Pi_\kappa(y_M, y_3) & \cdots & \mathbf{I} \end{pmatrix}.$$

Assuming $\det(\tilde{\Gamma}) \neq 0$ and denoting the 3-by-3 blocks of $\tilde{\Gamma}^{-1} \in \mathbb{C}^{3M\times 3M}$ by $[\tilde{\Gamma}^{-1}]_{lj}$ for $l, j = 1, 2, \cdots M$, we deduce from (2.9) that the scattered field takes the form

$$E^s(x) = \sum_{l,j=1}^{M} a_j \, \Pi_\kappa(x, y_j) \, [\tilde{\Gamma}^{-1}]_{jl} \, E^i(y_l), \tag{2.12}$$

with the far-field pattern

$$E^\infty(\hat{x}) = \frac{1}{4\pi} \sum_{l,j=1}^{M} a_j \, e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\,\theta\cdot y_l} (\mathbf{I} - \hat{x} \otimes \hat{x}) \, [\tilde{\Gamma}^{-1}]_{jl} \, p, \ \hat{x} \in \mathbb{S}^2. \tag{2.13}$$

We will rigorously justify the validity of the Foldy method in Section 2.2, extending the renormalization techniques in quantum mechanics for modeling M-particle interactions to the electromagnetic case. In particular, this justifies the choice of (2.10) and gives sense to the comments made just after (2.10). Following the seminal paper [5] by Foldy, we call the system (2.10) the fundamental system of multiple scattering.

2.1.3 Intermediate levels of approximations

Between the Born and Foldy models, we can define the k-th ($k \in \mathbb{N}$) level of the total field $E^{(k)}$ as follows

$$E^{(k)}(x) = E^i(x) + \sum_{j=1}^{M} a_j \, \Pi_\kappa(x, y_j) \, \Lambda_j^{(k)} \tag{2.14}$$

where the value $\Lambda_j^{(k)}$ can be computed recursively via

$$\Lambda_j^{(k+1)} := E^i(y_j) + \sum_{\substack{l=1 \\ l\neq j}}^{M} a_j \, \Pi_\kappa(y_j, y_l) \, \Lambda_l^{(k)}, \quad j = 1, 2, \cdots, M, \tag{2.15}$$

with the 0-th level approximations $\Lambda_j^{(0)}$ given by

$$\Lambda_j^{(0)} := E^i(y_j), \quad j = 1, 2 \cdots, M.$$

Thus $E^{(0)}$ is just the Born approximation (2.7). When $k = +\infty$, we define $\Lambda_j^{(\infty)}$ as

$$\Lambda_j^{(\infty)} := E^i(y_j) + \sum_{\substack{l=1 \\ l\neq j}}^{M} a_j \, \Pi_\kappa(y_j, y_l) \, \Lambda_l^{(\infty)}, \quad j = 1, 2, \cdots, M.$$

Then, we see that for $k = +\infty$ the total field in (2.14) coincides with the total field in (2.9), i.e. the Foldy regime. The k-th level approximation $E^{(k)}$ only takes into account k time interactions between the scatterers, and thus is considered as the intermediate level.
Remark that, the system (2.15) is nothing but the $k + 1^{th}$ iteration of the Foldy algebraic system (2.10). From (2.14), we write the following form of the scattered field in k^{th} level:

$$E^{(k,s)}(x) = \sum_{j=1}^{M} a_j \, \Pi_\kappa(x, y_j) \, \Lambda_j^{(k)} \tag{2.16}$$

To write (2.16) and the corresponding far fields in a more useful form, we define the vector $\Lambda^{(k)} \in \mathbb{C}^{3M}$ with components $\Lambda_j^{(k)}$ arranged elementwise as in the pattern of Λ in (2.11). Define $\tilde{\mathbf{I}} \in \mathbb{C}^{3M\times 3M}$ as an identity matrix, then the 3×3 diagonal blocks of $\tilde{\mathbf{I}}$, $\tilde{\mathbf{I}}_{jl}$, are $\mathbf{I}$ and the non diagonal blocks are zero matrices. Set $\tilde{\mathbf{M}} := \tilde{\Gamma} - \tilde{\mathbf{I}}$,[1] then (2.15) can be written in a compact form as

$$\Lambda^{(k)} = \sum_{l=0}^{k} (-\tilde{\mathbf{M}})^l \mathbf{E}^I \text{ for } k = 0, 1, \dots . \tag{2.17}$$

Define the matrix $\tilde{C}_k \in \mathbb{C}^{3M\times 3M}$ by $\tilde{C}_k := \sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$ for $k = 0, 1, \dots$. Denote the 3-by-3 blocks of $\tilde{C}_k \in \mathbb{C}^{3M\times 3M}$ by $[\tilde{C}_k]_{lj}$ for $l, j = 1, 2, \cdots M$. With this setting we deduce from (2.15),(2.16) that the scattered field in k^{th} intermediate level takes the form

$$E^{(k,s)}(x) = \sum_{l,j=1}^{M} a_j\, \Pi_\kappa(x, y_j)\, [\tilde{C}_k]_{jl}\, E^I(y_l), \tag{2.18}$$

and so the far-field pattern of the scattered field in the k^{th} intermediate level is

$$E^{(k,\infty)}(\hat{x}) = \frac{1}{4\pi} \sum_{l,j=1}^{M} a_j\, e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\,\theta\cdot y_l} (\mathbf{I} - \hat{x}\otimes\hat{x})\, [\tilde{C}_k]_{jl}\, p,\ \hat{x} \in \mathbb{S}^2. \tag{2.19}$$

2.2 Justification of the Foldy model

Define the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) := (2\pi)^{-3/2} \lim_{R\to\infty} \int_{|x|\le R} f(x)\, e^{-ix\cdot\xi}\, dx, \quad \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3.$$

Its inverse transform is given by

$$(\mathcal{F}^{-1}g)(x) := (2\pi)^{-3/2} \lim_{R\to\infty} \int_{|\xi|\le R} g(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

For $u = (u_1, u_2, u_3)^\top \in L^2(\mathbb{R}^3)^3$, a simple calculation shows

$$\mathcal{F}(\operatorname{curl}\operatorname{curl} u) = (|\xi|^2\mathbf{I} - \xi\otimes\xi)\hat{u} = |\xi|^2(\mathbf{I} - \hat{\xi}\otimes\hat{\xi})\hat{u}, \quad \hat{\xi} = \xi/|\xi|,$$

where $\hat{u} := \mathcal{F}u = (\hat{u}_1, \hat{u}_2, \hat{u}_3)^\top$. Define

$$\mathcal{M}_\kappa(\xi) := |\xi|^2(\mathbf{I} - \hat{\xi}\otimes\hat{\xi}) - \kappa^2\mathbf{I} = (|\xi|^2 - \kappa^2)\mathbf{I} - \xi\otimes\xi \in \mathbb{R}^{3\times 3}.$$

It is easy to check that the inverse matrix of $\mathcal{M}_\kappa$ takes the form

$$\mathcal{M}_\kappa^{-1}(\xi) = \frac{1}{|\xi|^2 - \kappa^2}(\mathbf{I} - \frac{1}{\kappa^2}\xi\otimes\xi) \tag{2.20}$$

and that (cf. (2.2))

$$(2\pi)^{-3/2}\mathcal{F}^{-1}[\mathcal{M}_\kappa(\xi)] = \left(\mathbf{I} + \frac{1}{\kappa^2}\nabla_x\nabla_x\right)\frac{e^{i\kappa|x|}}{4\pi|x|} = \Pi_\kappa(x, 0), \quad x \ne 0. \tag{2.21}$$

[1] In the case that the norm of $\mathbf{M}$ is less than one, the inverse of Γ can be approximated by the truncated Neumann series.

For the purpose of analyzing the solvability of electromagnetic scattering by M point-like obstacles, we adapt the regularization procedures proposed in [12]. To do this, we decompose $\mathcal{M}_\kappa^{-1}(\xi)$ into the sum

$$\mathcal{M}_\kappa^{-1}(\xi) = T_\kappa(\xi) + L_\kappa(\xi), \quad T_\kappa := \frac{1}{|\xi|^2-\kappa^2}(\mathbf{I} - \hat{\xi}\otimes\hat{\xi}),\ L_\kappa := -\frac{1}{\kappa^2}\hat{\xi}\otimes\hat{\xi}, \tag{2.22}$$

where T_κ, L_κ denote the transverse and longitudinal parts with respect to ξ, respectively. Accordingly, the dyadic Green's function $\Pi_\kappa(x,0)$ admits the decomposition (see [12, Part II])

$$\Pi_\kappa(x,0) = \Pi_\kappa^T(x) + \Pi_\kappa^L(x), \quad x \neq 0,$$

with (see [12, Part II] or the appendix of the present paper)

$$\Pi_\kappa^L(x) := (2\pi)^{-3/2}\mathcal{F}^{-1}[L_\kappa(\xi)] = -\frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}, \tag{2.23}$$

$$\begin{aligned}\Pi_\kappa^T(x) &:= (2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)] \\ &= \frac{\mathbf{I} - 3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} + \frac{e^{i\kappa|x|}}{4\pi|x|}\left[P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\hat{x}\otimes\hat{x}\right],\end{aligned} \tag{2.24}$$

where $P(t) = 1 - 1/t + 1/t^2$, and $Q(t) = -1 + 3/t - 3/t^2$.

Now, we introduce a new operator

$$H_\kappa E := \operatorname{curl}\operatorname{curl} E - \kappa^2 E - \sum_{j=1}^M a_j\,\delta(x-y_j)\,E, \quad y_j = (y_{j_1}, y_{j_2}, y_{j_3})^\top \in \mathbb{R}^3.$$

The objective of subsequent sections is to give a mathematically rigorous meaning of this operator and derive the associated Green's function. To start, we set $\widetilde{H}(f) := \mathcal{F}H_\kappa\mathcal{F}^{-1}(f)$ for $f = (f_1, f_2, f_3)^\top \in L^2(\mathbb{R}^3)^3$. Then, we have

$$[\mathcal{F}(\operatorname{curl}\operatorname{curl} - \kappa^2)\mathcal{F}^{-1}]\,\hat{f} = \mathcal{M}_\kappa(\xi)\hat{f},$$

and formally

$$\begin{aligned}(\mathcal{F}\delta(x-y_j)\,\mathcal{F}^{-1}\,\hat{f})(\xi) &= (\mathcal{F}\delta(x-y_j)f)(\xi) \\ &= (2\pi)^{-3/2}f(y_j)e^{-iy_j\cdot\xi} \\ &= (2\pi)^{-3/2}e^{-iy_j\cdot\xi}\left(\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^2}\hat{f}(\xi)e^{iy_j\cdot\xi}d\xi\right) \\ &= \sum_{m=1}^3\left\langle\hat{f}, \varphi_{y_j}^m\right\rangle\varphi_{y_j}^m(\xi),\end{aligned}$$

where

$$\varphi_{y_j}^m(\xi) := \phi_{y_j}(\xi)\,e_m \in \mathbb{C}^{3\times 1},\ m = 1,2,3, \quad \phi_{y_j}(\xi) := (2\pi)^{-3/2}e^{-iy_j\cdot\xi}.$$

Here we used the inner product

$$\langle f, g\rangle := \int_{\mathbb{R}^2} f(\xi)\cdot\overline{g(\xi)}\,d\xi, \quad \text{for } f, g \in L^2(\mathbb{R}^3)^3.$$

Therefore, formally we have

$$\widetilde{H}(f)(\xi) = \mathcal{M}_\kappa(\xi) f - \sum_{j=1}^{M} \sum_{m=1}^{3} \left\langle a_j\, f, \varphi_{y_j}^m \right\rangle \varphi_{y_j}^m(\xi),$$

which is a finite rank perturbation of the multiplication operator $\mathcal{M}_\kappa(\xi)$. Our aim is to prove the existence of $\widetilde{H}^{-1}$ and deduce an explicit expression of the Green's tensor to $\widetilde{H}^{-1}$. To make the computations rigorous, we introduce the cut-off function

$$\chi_\epsilon(\xi) = \begin{cases} 1, & \text{if} \quad \epsilon \le |\xi| \le 1/\epsilon, \\ 0, & \text{if} \quad |\xi| < \epsilon \ \text{ or } \ |\xi| > 1/\epsilon, \end{cases} \qquad \text{for some } 0 < \epsilon < 1,$$

and define the operator

$$\widetilde{H}^\epsilon_\alpha f := \mathcal{M}_{\kappa_\alpha}(\xi)\, f - \sum_{j=1}^{M} \sum_{m=1}^{3} \left\langle a_j(\epsilon)\, f, \varphi_{y_j}^{\epsilon,m} \right\rangle \varphi_{y_j}^{\epsilon,m}(\xi), \quad \varphi_{y_j}^{\epsilon,m}(\xi) := \chi_\epsilon(\xi)\, \varphi_{y_j}^m(\xi), \tag{2.25}$$

where $\kappa_\alpha := \kappa + i\alpha$ with $\alpha > 0$. We will choose the coupling constants $a_j(\epsilon)$ in a suitable way such that $\widetilde{H}^\epsilon$ has a reasonable limit as ϵ tends to zero. Let us first recall the Weinstein-Aronszajn determinant formula from [1, Lemma B.5], which is our main tool for analyzing the inverse of $\widetilde{H}^\epsilon_\alpha$.

Lemma 2.1. *Let $\mathcal{H}$ be a (complex) separable Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. Let A be a closed operator in $\mathcal{H}$ and $\Phi_j,\ \Psi_j \in \mathcal{H}$, $j = 1, ..., m$. Then*

$$\Big(A + \sum_{j=1}^{m} \langle \cdot, \Phi_j \rangle \Psi_j - z\Big)^{-1} = \big(A - z\big)^{-1} - \sum_{j=1}^{m} \big[\Gamma(z)\big]^{-1}_{j,j'} \big\langle \cdot,\ [\big(A - z\big)^{-1}]^* \Phi_{j'} \big\rangle \big(A - z\big)^{-1} \Psi_j \tag{2.26}$$

for z in the resolvent of A such that $\det \big[\Gamma(z)\big] \neq 0$, with the entries of $\Gamma(z)$ given by

$$\big[\Gamma(z)\big]_{j,j'} := \delta_{j,j'} + \big\langle \big(A - z\big)^{-1} \Psi_{j'}, \Phi_j \big\rangle. \tag{2.27}$$

Note that in Lemma 2.1, the notation $\big[\Gamma(z)\big]^{-1}_{j,j'}$ denotes the (j, j')-th entry of the matrix $\big[\Gamma(z)\big]^{-1}$, and $[\,]^*$ stands for the adjoint operator of $[\,]$. To apply Lemma 2.1, we take

$$\mathcal{H} := L^2(\mathbb{R}^3)^3,\ A(f) := |\xi|^2(\mathbf{I} - \hat{\xi} \otimes \hat{\xi}) f(\xi),\ m := 3M,\ z = \kappa_\alpha^2$$

and $\Phi_j := \Phi_j^\epsilon$, $\Psi_j = -\tilde{a}_j \Phi_j^\epsilon$ for $j = 1, \cdots, 3M$, with $\tilde{a}_j$ and Φ_j^ϵ defined as follows:

$$\tilde{a}_j(\epsilon) = a_l(\epsilon) \quad \text{if} \quad j \in \{3l-1, 3l-2, 3l\}, \quad \Phi_j^\epsilon := \begin{cases} \varphi_{y_l}^{\epsilon,1} & \text{if } j = 3l-2, \\ \varphi_{y_l}^{\epsilon,2} & \text{if } j = 3l-1, \\ \varphi_{y_l}^{\epsilon,3} & \text{if } j = 3l, \end{cases}$$

for some $l \in \{1, 2, \cdots, M\}$. The multiplication operator A is closed with a dense domain

$$\mathcal{D}(A) := \Big\{ f(\xi) \in L^2(\mathbb{R}^3)^3 : \quad |\xi|^2(\mathbf{I} - \hat{\xi} \otimes \hat{\xi}) f(\xi) \in L^2(\mathbb{R}^3)^3 \Big\}$$

in $L^2(\mathbb{R}^3)^3$ hence $\widetilde{H}^{\epsilon}_{\alpha}$, with $\epsilon > 0, \alpha > 0$, is also closed with the same domain. For a complex-valued number $\kappa + i\alpha$, one can observe that $\det(\mathcal{M}_{\kappa_\alpha}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}^3$, so that $(\mathcal{M}_{\kappa_\alpha})^{-1}(\xi)$ always exists. Further, it holds that

$$[(\mathcal{M}_{\kappa_\alpha})^{-1}]^* = [(\mathcal{M}_{\overline{\kappa_\alpha}})]^{-1},$$

where $\overline{\kappa_\alpha} := \kappa - i\alpha$ denotes the conjugate of κ_α. Simple calculations show

$$\begin{aligned} (A - z)^{-1}\Psi_j &= -\tilde{a}_j(\mathcal{M}_{\kappa_\alpha})^{-1}\Phi^{\epsilon}_j, \\ \delta_{j,j'} + \left\langle (A - z)^{-1}\Psi_{j'}, \Phi_j \right\rangle &= \tilde{a}_j \left[\tilde{a}_j^{-1}\, \delta_{j,j'} - \left\langle (\mathcal{M}_{\kappa_\alpha})^{-1}\Phi^{\epsilon}_{j'}, \Phi^{\epsilon}_{j'} \right\rangle\right]. \end{aligned}$$

Therefore, by Lemma 2.1 we arrive at an explicit expression of the inverse of $\widetilde{H}^{\epsilon}_{\alpha}$, given by

$$(\widetilde{H}^{\epsilon}_{\alpha})^{-1} f = (\mathcal{M}_{\kappa_\alpha})^{-1} f + \sum_{j,j'=1}^{3M} [\Gamma_\epsilon(\kappa_\alpha)]^{-1}_{j,j'} \left\langle f, \chi_\epsilon\, F^{(j')}_{-\overline{\kappa_\alpha}} \right\rangle \chi_\epsilon\, F^{(j)}_{\kappa_\alpha}, \quad \alpha > 0, \tag{2.28}$$

with

$$\Gamma_\epsilon(\kappa_\alpha) := \left[\tilde{a}_j^{-1}\delta_{j,j'} - \left\langle (\mathcal{M}_{\kappa_\alpha})^{-1}\Phi^{\epsilon}_{j'},\ \Phi^{\epsilon}_j \right\rangle\right]_{j,j'=1}^{3M}, \quad \chi_\epsilon F^{(j)}_{\kappa_\alpha} := (\mathcal{M}_{\kappa_\alpha})^{-1}\Phi^{\epsilon}_j, \tag{2.29}$$

provided $\det[\Gamma_\epsilon(\kappa_\alpha)] \neq 0$.

In order to get $\widetilde{H}^{-1}$ for the complex wavenumber κ_α, we need to remove the cut-off function in (2.28) by evaluating the limits of $\Gamma_\epsilon(\kappa_\alpha)$ and $\left\langle f, \chi_\epsilon F^{(j')}_{-\overline{\kappa_\alpha}} \right\rangle \chi_\epsilon F^{(j)}_{\kappa_\alpha}$ as $\epsilon \to 0$. This will be done in the subsequent lemmas 2.2 and 2.4.

Lemma 2.2. *The coefficients $\tilde{a}_j(\epsilon)$ can be chosen in such a way that the limit $\Gamma_{B,Y}(\kappa_\alpha) = \lim_{\epsilon\to 0}\Gamma_\epsilon(\kappa_\alpha)$ exists and takes the form*

$$\Gamma_{B,Y}(\kappa_\alpha) = \begin{pmatrix} (-a_1^{-1} + b_1)\, \boldsymbol{I} & -\Pi_{\kappa_\alpha}(y_1, y_2) & \cdots & -\Pi_{\kappa_\alpha}(y_1, y_M) \\ -\Pi_{\kappa_\alpha}(y_2, y_1) & (-a_2^{-1} + b_2)\, \boldsymbol{I} & \cdots & -\Pi_{\kappa_\alpha}(y_2, y_M) \\ \vdots & \vdots & \ddots & \vdots \\ -\Pi_{\kappa_\alpha}(y_M, y_1) & -\Pi_{\kappa_\alpha}(y_M, y_2) & \ldots & (-a_M^{-1} + b_M)\, \boldsymbol{I} \end{pmatrix}, \tag{2.30}$$

where a_j's are the scattering strengths and $B := (b_1, b_2, \cdots, b_M)$ with

$$b_j := b_j(\beta_{T,j}, \beta_{L,j}, \kappa_\alpha) = \frac{\beta_{T,j} + i\kappa_\alpha}{6\pi} \frac{\beta^2_{T,j}}{\beta^2_{T,j} + \kappa^2_\alpha} - \frac{(\beta_{L,j}/\sqrt{2})^3}{6\pi\kappa^2_\alpha}, \quad \beta_{T,j}, \beta_{L,j} \in \mathbb{R}. \tag{2.31}$$

If in addition we assume that $a_j \in \mathbb{R}$, then $[\Gamma_{B,Y}(\kappa_\alpha)]^* = \Gamma_{B,Y}(-\overline{\kappa_\alpha})$.

Remark 2.3. *For $j = 1, \ldots, M$, the coefficient a_j can be absorbed by b_j, through the coefficients $\beta_{T,j}$ or $\beta_{L,j}$. We kept these three coefficients a_j, $\beta_{T,j}$ and $\beta_{L,j}$ just to make our final form of the scattering amplitude consistent with the formula* (2.31) *of [12] concerning the scattering by a single point-like scatterer, see also Remark 2.6.*

Proof : The proof will be carried out in the following three cases of $j, j' \in \{1, \cdots, 3M\}$.

- Case 1: $|j' - j| = 1$, and $j, j' \in \{3l-2, 3l-1, 3l\}$ for some $l \in \{1, \cdots, M\}$.

  Assume firstly that $j = 3l-2$, $j' = 3l-1$ for some $l = 1, \cdots, M$. Then, we have $\Phi^\epsilon_j = \chi_\epsilon \phi_{y_l} e_1$, $\Phi^\epsilon_{j'} = \chi_\epsilon \phi_{y_l} e_2$. Hence

$$\left\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi^\epsilon_j, \Phi^\epsilon_{j'} \right\rangle = (2\pi)^{-3} \left\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \chi_\epsilon e_1,\, \chi_\epsilon e_2 \right\rangle$$

  since $\varphi_{y_j}(\xi)\, \overline{\varphi_{y_j}(\xi)} = (2\pi)^{-3}$. Consequently, it holds that

$$\left\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi^\epsilon_j, \Phi^\epsilon_{j'} \right\rangle = (2\pi)^{-3} \int_{\epsilon<|\xi|<1/\epsilon} (\mathcal{M}_{\kappa_\alpha})^{-1} e_1 \cdot e_2 d\xi = 0$$

  because the scalar function $(\mathcal{M}_{\kappa_\alpha})(\xi)^{-1} e_1 \cdot e_2$ is odd in ξ_j, $j = 1, 2, 3$ (see (2.20)). By symmetry, we have also $\left\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi^\epsilon_{j'}, \Phi^\epsilon_j \right\rangle = 0$. The other cases for $j \neq j'$ and $j, j' \in \{3l-2, 3l-1, 3l\}$ can be proved analogously.

- Case 2: $j = j' \in \{3l-2, 3l-1, 3l\}$ for some $l \in \{1, 2, \cdots, M\}$.

  In this case, we set

$$\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l}) := \frac{1}{(2\pi)^3} \int_{\epsilon<|\xi|<1/\epsilon} \left[T_{\kappa_\alpha}(\xi) \frac{|\xi|^2 + 2\beta_{T,l}^2}{\beta_{T,l}^2 + |\xi|^2} + L_{\kappa_\alpha}(\xi) \frac{|\xi|^4 + 2\beta_{L,l}^4}{\beta_{L,l}^4 + |\xi|^4} \right] d\xi \tag{2.32}$$

  for some $\beta_{T,l}, \beta_{L,l} \in \mathbb{R}$, and define

$$\tilde{a}_j^{-1}(\epsilon) := -a_l^{-1} + \begin{cases} [\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l})]_{1,1} & \text{if } j = 3l-2, \\ [\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l})]_{2,2} & \text{if } j = 3l-1, \quad l = 1, 2, \cdots, M. \\ [\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l})]_{3,3} & \text{if } j = 3l, \end{cases} \tag{2.33}$$

  Employing (2.22) and the definition of the inverse Fourier transformation, we see

$$\begin{aligned} & \lim_{\epsilon \to 0} \left[\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l}) - \frac{1}{(2\pi)^3} \int_{\epsilon<|\xi|<1/\epsilon} (\mathcal{M}_{\kappa_\alpha})^{-1}(\xi) d\xi \right] \\ = & \lim_{\epsilon \to 0} \left[\Theta(\epsilon, \kappa_\alpha, \beta_{T,l}, \beta_{L,l}) - \frac{1}{(2\pi)^3} \int_{\epsilon<|\xi|<1/\epsilon} (T_{\kappa_\alpha}(\xi) + L_{\kappa_\alpha}(\xi))\ d\xi \right] \\ = & \lim_{\epsilon \to 0} \frac{1}{(2\pi)^3} \int_{\epsilon<|\xi|<1/\epsilon} \left[T_{\kappa_\alpha}(\xi) \frac{\beta_{T,l}^2}{\beta_{T,l}^2 + |\xi|^2} + L_{\kappa_\alpha}(\xi) \frac{\beta_{L,l}^4}{\beta_{L,l}^4 + |\xi|^4} \right] d\xi \\ = & \ b(\kappa_\alpha, \beta_{T,l}, \beta_{L,l})\, \mathbf{I}, \end{aligned} \tag{2.34}$$

  where the constant $b(\kappa_\alpha, \beta_{T,l}, \beta_{L,l})$, which is given in (2.31), follows from the arguments in [12]; see also the Appendix for the details. This implies that, for $j \in \{3l-2, 3l-1, 3l\}$,

$$\lim_{\epsilon \to 0} \left[\tilde{a}_j^{-1}(\epsilon) - \left\langle (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi^\epsilon_j,\ \Phi^\epsilon_j \right\rangle \right] = -a_l^{-1} + b(\kappa_\alpha, \beta_{T,l}, \beta_{L,l}). \tag{2.35}$$

To sum up Cases 1 and 2, we conclude that the 3×3 diagonal blocks of the matrix $\Gamma_{B,Y} := \lim_{\epsilon \to 0} \Gamma_\epsilon(\omega)$ coincide with the 3×3 matrix $(-a_l^{-1} + b(\kappa_\alpha, \beta_{T,l}, \beta_{L,l}))\, \mathbf{I}$.

- Case 3: $j \in \{3l-2, 3l-1, 3l\}$, $j' \in \{3l'-2, 3l'-1, 3l'\}$ for some $l, l' \in \{1, \cdots, M\}$ such that $|l-l'| \geq 1$, i.e. the element $[\Gamma_{B,Y}]_{j,j'}$ lies in the off diagonal-by-3×3-blocks of $\Gamma_{B,Y}$.

  Without loss of generality we assume $j = 3l-2, j' = 3l'-2$. Define the 3×3 matrix $\Upsilon_l := (\Phi_j^\epsilon, \Phi_{j+1}^\epsilon, \Phi_{j+2}^\epsilon) = \chi_\epsilon \phi_{y_l}$ **I**. A short computation shows

$$
\begin{aligned}
& \big\langle (\mathcal{M}_{\kappa_\alpha})^{-1}(\xi) \Upsilon_l(\xi),\ \Upsilon_{l'}(\xi) \big\rangle \\
= & \int_{\epsilon < |\xi| < 1/\epsilon} (\mathcal{M}_{\kappa_\alpha})^{-1}(\xi)\, \phi_{y_l}(\xi)\, \overline{\phi_{y_{l'}}(\xi)}\, d\xi \\
= & \frac{1}{(2\pi)^3} \int_{\epsilon < |\xi| < 1/\epsilon} (\mathcal{M}_{\kappa_\alpha})^{-1}(\xi) \exp[i(y_{l'} - y_l) \cdot \xi]\, d\xi \\
\to & \Big[\Pi_{\kappa_\alpha}(y_{l'}, y_l) \Big] \quad \text{as } \epsilon \to 0,
\end{aligned}
$$

  where the last step follows from the inverse Fourier transformation.

Finally, combining Cases 1-3 gives the matrix (2.30). □

We next prove the convergence of the operator $K_{j,j'}^\epsilon : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ defined by

$$
K_{j,j'}^\epsilon(f) := \Big\langle f,\ \chi_\epsilon F_{-\kappa_\alpha}^{(j')} \Big\rangle \chi_\epsilon F_{\kappa_\alpha}^{(j)}, \quad f \in L^2(\mathbb{R}^3)^3.
$$

To be consistent with the definitions of Φ_j^ϵ and $\chi_\epsilon F_\omega^{(j)}$, we introduce the functions

$$
\Phi_j(\xi) := \begin{cases} (2\pi)^{-3/2} e^{-i\xi \cdot y_l} e_1 & \text{if} \quad j = 3l-2, \\ (2\pi)^{-3/2} e^{-i\xi \cdot y_l} e_2 & \text{if} \quad j = 3l-1, \\ (2\pi)^{-3/2} e^{-i\xi \cdot y_l} e_3 & \text{if} \quad j = 3l \end{cases}, \ F_{\kappa_\alpha}^{(j)} := (\mathcal{M}_{\kappa_\alpha})^{-1} \Phi_j(\xi), \tag{2.36}
$$

for $l = 1, ..., M$.

Lemma 2.4. *For $\alpha > 0$, the operator $K_{j,j'}^\epsilon$ converges in the operator norm to $K_{j,j'} : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$, defined by*

$$
K_{j,j'}(f) := \Big\langle f,\ F_{-\kappa_\alpha}^{(j')} \Big\rangle F_{\kappa_\alpha}^{(j)}.
$$

Proof : The proof is similar to [7, Lemma II. 5]. □

Combining Lemma 2.2 and Lemma 2.4, we obtain the convergence in the operator norm of $(\widetilde{H}_\alpha^\epsilon)^{-1}$ to

$$
\mathcal{G}(\kappa_\alpha)\hat{f} := (\mathcal{M}_{\kappa_\alpha})^{-1}\hat{f} + \sum_{j,j'=1}^{3M} \Big[\Gamma_{B,Y}(\kappa_\alpha) \Big]_{j,j'}^{-1} \Big\langle \hat{f}, F_{-\kappa_\alpha}^{(j')} \Big\rangle F_{\kappa_\alpha}^{(j)}, \quad \forall\, \hat{f} \in L^2(\mathbb{R}^3)^3, \tag{2.37}
$$

for $\alpha > 0$ such that $\det[\Gamma_{B,Y}(\kappa_\alpha)] \neq 0$. The main theorem of this section is stated as follows.

Theorem 2.5. *Suppose that the constant $\tilde{a}_j(\epsilon)$ is given by (2.33), with $\beta_{T,l}, \beta_{L,l} \in \mathbb{R}$ for $j \in \{3l-2, 3l-1, 3l\}$, $l = 1, 2, \cdots, M$. Let $\Gamma_{B,Y}, F_{\kappa_\alpha}^{(j)}$ be defined by (2.30), (2.36) respectively. Then we have the following properties.*

(i) *The operator $\widetilde{H}_\alpha^\epsilon$ converges in norm resolvent sense to a closed and self-adjoint operator $\hat{\Xi}_{B,Y}$ as $\epsilon \to 0$, where the resolvent of $\hat{\Xi}_{B,Y}$ is given by (2.37). That is, for $\alpha > 0$ such that $\det[\Gamma_{B,Y}(\kappa_\alpha)] \neq 0$,*

$$(\hat{\Xi}_{B,Y} - \kappa_\alpha^2)^{-1} = (\mathcal{M}_{\kappa_\alpha})^{-1} + \sum_{j,j'=1}^{3M} \Big[\Gamma_{B,Y}(\kappa_\alpha)\Big]^{-1}_{j,j'} \Big\langle \cdot\,, F^{(j')}_{-\overline{\kappa_\alpha}} \Big\rangle F^{(j)}_{\kappa_\alpha},$$

where $\Big[\Gamma_{B,Y}(\kappa_\alpha)\Big]^{-1}_{j,j'}$ denotes the (j,j')-th entry of the matrix $\Big[\Gamma_{B,Y}(\kappa_\alpha)\Big]^{-1}$.

(ii) *For $\kappa^2 > 0$ such that $\det[\Gamma_{B,Y}(\kappa)] \neq 0$, the resolvent of $\Xi_{B,Y}$ reads*

$$(\Xi_{B,Y} - \kappa^2)^{-1} = \Pi_\kappa + \sum_{l,l'=1}^{M} \Pi_\kappa(\cdot, y_l) \left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'} \langle \cdot\,, \overline{\Pi_\kappa(\cdot, y_{l'})} \rangle,$$

with the Green's tensor

$$(\Xi_{B,Y} - \kappa^2)^{-1}(x,y) = \Pi_\kappa(x,y) + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l) \left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'} \Pi_\kappa(y_{l'}, y), \tag{2.38}$$

for $x \neq y$ and $x, y \neq y_l$. Here $\left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'}$ denote the 3-by-3 blocks of the matrix $\left[\Gamma_{B,Y}(\kappa)\right]^{-1}$.

Proof : Based on (2.37), the proof of Theorem 2.5 can be carried out analogously to [7, Theorem II.6]. □

In classical scattering theory, (2.38) describes the total field by the collection of point like scatterers $Y = \{y_1, y_2, \ldots, y_M\}$ corresponding to the incident point source $\Pi_\kappa(x,y)$ located at y. We are also interested in the case of plane wave incidence. By making use of (2.3) in (2.38), we obtain

$$U(x,\theta) = \frac{e^{i\kappa\, x\cdot\theta}}{4\pi}(\mathbf{I} - \theta\otimes\theta) + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)\left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'} \frac{e^{i\kappa\, y_{l'}\cdot\theta}}{4\pi}(\mathbf{I} - \theta\otimes\theta),$$

with $\theta := -\hat{y}$ and $U(x,\theta) := \lim_{|y|\to\infty} |y|(\Xi_{B,Y} - \kappa^2)^{-1}(x,y)$. In particular, multiplying the previous identity by the polarization direction $p \in \mathbb{S}^2$ ($p \perp \theta$) gives the total field

$$E(x,\theta,p) = E^i(x,\theta,p) + \sum_{l,l'=1}^{M} \Pi_\kappa(x, y_l)\left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'} E^i(y_{l'},\theta,p), \tag{2.39}$$

corresponding to a plane wave incidence $E^i(x,\theta,p) = p\exp(i\kappa\, x\cdot\theta)$ with $E(x,\theta,p) := U(x,\theta)\cdot p$. The far-field corresponding to the scattered field is then given by

$$E^\infty(\hat{x};\theta,p) = \frac{1}{4\pi}\sum_{l,l'=1}^{M} \exp(-i\kappa\hat{x}\cdot y_l)(\mathbf{I} - \hat{x}\otimes\hat{x}) \left[\Gamma^{-1}_{B,Y}(\kappa)\right]_{l,l'} E^i(y_{l'},\theta,p), \tag{2.40}$$

which is reminiscent to the representation (2.13) in the Foldy method.

Remark 2.6. *If we choose, as in [12], the regularization parameters $\beta_{T,j}$ sufficiently large (compared to the fixed wavenumber κ), then the coefficient b_j in Lemma 2.2 takes the form*

$$b_j := b_j(\beta_{T,j}, \beta_{L,j}, \kappa) = \frac{\beta_{T,j} + i\kappa}{6\pi} - \frac{(\beta_{L,j}/\sqrt{2})^3}{6\pi\kappa^2} + O(\frac{\kappa}{\beta_{T,j}}),\ \frac{\kappa}{\beta_{T,j}} \ll 1. \tag{2.41}$$

Additionally, suppose that there exists only one point-like scatterer located at the origin (i.e. $M = 1, y_1 = O$). Then by neglecting the term $O(\frac{\kappa}{\beta_{T,j}})$ in (2.41)*, the identity (2.38) becomes*

$$(\Xi_{B,Y} - \kappa^2)^{-1}(x, y) = \Pi_\kappa(x, y) + t\,\Pi_\kappa(x, 0)\Pi_\kappa(0, y),$$

where t is given by (see (2.31) in Lemma 2.2)

$$t = (b_1 - a_1^{-1})^{-1} = \frac{1}{-a_1^{-1} + (\beta_T + i\kappa)(6\pi)^{-1} - \beta_L^3(6\pi\kappa^2)^{-1}\,2^{-3/2}}, \quad \beta_T, \beta_L \in \mathbb{R}.$$

The number t is exactly the one characterizing to the scattering T matrix in [12] (cf. [12, Section III, A (33) and (19)]), remarking that in [12], $\frac{\beta_L}{\sqrt{2}}$ is taken as β_L, i.e. $\frac{1}{\sqrt{2}}$ is absorbed in β_L, and $\kappa^2\alpha_B = a_1$.

3 The inverse problems for the Born, Foldy and intermediate models

From (2.8),(2.13),(2.19), we can write the far-field corresponding to scattered field in various models as

$$E^\infty(\hat{x}; \theta, p) = \frac{1}{4\pi}\sum_{l,j=1}^{M} a_j\, e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\,\theta\cdot y_l} (\mathbf{I} - \hat{x}\otimes\hat{x})\,[\tilde{\mathbf{T}}]_{jl}\, p. \tag{3.1}$$

with

$$\tilde{\mathbf{T}} := \begin{cases} \tilde{\mathbf{I}}, & \text{Born approximation,} \\ \tilde{\Gamma}^{-1}, & \text{Foldy method,} \\ \tilde{C}_k, & k^{th} \text{ intermediate level.} \end{cases} \tag{3.2}$$

The above mentioned far-field patterns are vectors. We define the following scalar far field, denoted by $\dot{E}^\infty(\hat{x})$, which will be useful in the statement and the justification of the MUSIC algorithm, see [4, 6] for instance.

$$\dot{E}^\infty(\hat{x}; \theta, p) := \hat{x}^\perp \cdot E^\infty(\hat{x}) = \frac{1}{4\pi}\sum_{l,j=1}^{M} a_j\, e^{-i\kappa\hat{x}\cdot y_j} e^{i\kappa\,\theta\cdot y_l} \hat{x}^{\perp\top}\,[\tilde{\mathbf{T}}]_{jl}\, p. \tag{3.3}$$

Here, $\hat{x}^\perp \in \mathbb{S}^2$ is any orthogonal direction to the observational direction $\hat{x} \in \mathbb{S}^2$. Since p is any direction in $\mathbb{S}^2$ perpendicular to θ, it has two orthogonal components called horizontal and vertical polarization directions denoted by p^h and p^v respectively. So, $p := \theta^\perp = \underline{\theta^\perp}/|\underline{\theta^\perp}|$ with $\underline{\theta^\perp} := c_1\,p^h + c_2\,p^v$ for arbitrary constants c_1 and c_2. To give the explicit forms of p^h and p^v, we recall the Euclidean basis $\{e_1, e_2, e_3\}$ where $e_1 := (1, 0, 0)^\top$, $e_2 := (0, 1, 0)^\top$ and $e_3 := (0, 0, 1)^\top$, write $\theta := (\theta_x, \theta_y, \theta_z)^\top$ and set $r^2 := \theta_x^2 + \theta_y^2$. Let $\mathcal{R}_3$ be the rotation map transforming θ to e_3. Then in the basis $\{e_1, e_2, e_3\}, \mathcal{R}_3 = \mathcal{R}_3(\theta)$ is given by the matrix

$$\mathcal{R}_3 = \frac{1}{r^2} \begin{bmatrix} \theta_y^2 + \theta_x^2\theta_z & -\theta_x\theta_y(1-\theta_z) & -\theta_x r^2 \\ -\theta_x\theta_y(1-\theta_z) & \theta_x^2 + \theta_y^2\theta_z & -\theta_y r^2 \\ \theta_x r^2 & \theta_y r^2 & \theta_z r^2 \end{bmatrix}. \tag{3.4}$$

It satisfies $\mathcal{R}_3^T\mathcal{R}_3 = I$ and $\mathcal{R}_3\,\theta = e_3$. Correspondingly, we write $p^h := \mathcal{R}_3^T\, e_1$ and $p^v := \mathcal{R}_3^T\, e_2$. These two directions represent the horizontal and the vertical directions of the polarized direction and they are given by

$$p^h := \theta^{\perp_h} = \frac{1}{r^2}(\theta_y^2 + \theta_x^2\theta_z, \theta_x\theta_y(\theta_z - 1), -r^2\theta_x)^\top, \quad p^v = \theta^{\perp_v} = \frac{1}{r^2}(\theta_x\theta_y(\theta_z - 1), \theta_x^2 + \theta_y^2\theta_z, -r^2\theta_y)^\top. \tag{3.5}$$

Hence, p can be written in terms of θ and then we can write $\dot{E}^\infty$ as a function of $\hat{x}$ and θ only.

Our concern in this section is to study the following inverse problem.

Inverse Problem : *Given the far-field pattern $\dot{E}^\infty(\hat{x}, \theta)$ for several incident and observation directions θ and $\hat{x}$, find the locations $y_1, y_2, \ldots, y_M$ and the scattering strengths $a_1, a_2, \ldots, a_M$.*

In the next sections, we deal with the mentioned models, (3.2)-(3.3), providing a detailed study of the resolution of the reconstruction depending on the distance between the scatterers, the frequency used and the scattering strengths.

3.1 MUSIC algorithm for the Maxwell system

The MUSIC algorithm is a method to determine the locations $y_j, j = 1, 2, \ldots, M$, of the scatterers from the measured far-field pattern $\dot{E}^\infty(\hat{x}, \theta)$ for a finite set of incidence and observation directions, i.e. $\hat{x}, \theta \in \{\theta_j, j = 1, \ldots, N\} \subset \mathbb{S}^2$. We refer the reader to the monographs [2] and [8] for more information about this algorithm. We follow the way presented in [4, 8]. We assume that the number of scatterers is not larger than the number of incident and observation directions, in particular $N \geq 3M$. We define the response matrix $F \in \mathbb{C}^{N \times N}$ by

$$F_{st} := \dot{E}^\infty(\theta_s, \theta_t). \tag{3.6}$$

Then by making use of (3.3), the response matrix F can be factorized as

$$F = H^* T H \tag{3.7}$$

with the matrices $T \in \mathbb{C}^{3M \times 3M}$ and $H \in \mathbb{C}^{3M \times N}$ are given by

$$T := \mathbf{a}\,\tilde{\mathbf{T}}, \mathbf{a} := Diag(a_1\mathbf{I}, a_2\mathbf{I}, \ldots, a_M\mathbf{I}), \tag{3.8}$$

and

$$H := \begin{pmatrix} \theta_1^\perp e^{i\kappa\theta_1\cdot y_1} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_1} & \ldots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_1} \\ \theta_1^\perp e^{i\kappa\theta_1\cdot y_2} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_2} & \ldots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_2} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^\perp e^{i\kappa\theta_1\cdot y_M} & \theta_2^\perp e^{i\kappa\theta_2\cdot y_M} & \ldots & \theta_N^\perp e^{i\kappa\theta_N\cdot y_M} \end{pmatrix}.$$

In order to determine the locations y_j, we consider a grid of sampling points $z \in \mathbb{R}^3$ in a region containing the scatterers $y_1, y_2, \dots, y_M$. For each point z, we define the vectors $\phi_z^u \in \mathbb{C}^M$ by

$$\phi_z^m := \left(\left(\theta_1^\perp \cdot e_m\right) e^{-i\kappa\theta_1\cdot z}, \left(\theta_2^\perp \cdot e_m\right) e^{-i\kappa\theta_2\cdot z}, \dots, \left(\theta_N^\perp \cdot e_m\right) e^{-i\kappa\theta_N\cdot z}\right)^\top, \forall m = 1,2,3. \quad (3.9)$$

MUSIC characterization of the scatterers: Recall that MUSIC is essentially based on characterizing the range of the response matrix F, forming projections onto its null space, and computing its singular value decomposition. Under the assumption that the matrix T in the factorization (3.7) of F is non-singular, the standard linear algebraic argument yields that, $\mathcal{R}(H^*)$ and $\mathcal{R}(F)$ coincide for $N \geq 3M$, if the matrix H has maximal rank $3M$. So, let us discuss the invertibility of the matrix T. From the definition of T in (3.8), its invertibility depends only on the non-singularity of $\tilde{\mathbf{T}}$.

- In case of the Born approximation, it is clear that T is invertible as $\tilde{\mathbf{T}} = \tilde{\mathbf{I}}$ from the definition (3.2) of $\tilde{\mathbf{T}}$.
- In case of the Foldy's method, from (3.2), we have $\tilde{\mathbf{T}} = \tilde{\Gamma}^{-1}$. So, the invertibility of T depends on the existence of $\tilde{\Gamma}^{-1}$. It can be observed that $\tilde{\Gamma}$ can be factorized as $\tilde{\Gamma} = \bar{\Gamma}\mathbf{a}$ with
$$\bar{\Gamma} = \begin{pmatrix} \frac{1}{a_1}\mathbf{I} & -\Pi_\kappa(y_1,y_2) & -\Pi_\kappa(y_1,y_3) & \cdots & -\Pi_\kappa(y_1,y_M) \\ -\Pi_\kappa(y_2,y_1) & \frac{1}{a_2}\mathbf{I} & -\Pi_\kappa(y_2,y_3) & \cdots & -\Pi_\kappa(y_2,y_M) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Pi_\kappa(y_M,y_1) & -\Pi_\kappa(y_M,y_2) & -\Pi_\kappa(y_M,y_3) & \cdots & \frac{1}{a_M}\mathbf{I} \end{pmatrix}.$$
Then $\tilde{\Gamma}$ is invertible when $\bar{\Gamma}$ is invertible and $T = \bar{\Gamma}^{-1}$. Let us assume it holds and postpone this issue to Section 3.2.
- In case of the approximation by intermediate level k, we have $\tilde{\mathbf{T}} = \tilde{C}_k = \sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$. One can observe that the norm of $\tilde{\mathbf{M}}$ less than half is a sufficient condition for the invertibility of T in every level of scattering.

Hence, under the assumption of the non-singularity of T, we can state the MUSIC related theorem for the Electromagnetic wave scattering by point-like scatterers as follows.

Theorem 3.1. *Let $\{\theta_s : s \in \mathbb{N}\} \subset \mathbb{S}^2$ be a countable set of directions such that any analytic function on $\mathbb{S}^2$ that vanishes in θ_s for all $s \in \mathbb{N}$ vanishes identically. Let $\mathbf{K}$ be a compact subset of $\mathbb{R}^3$ containing $\{y_j : j = 1, \dots, M\}$. Then there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ the following characterization holds for every $z \in \mathbf{K}$:*

$$z \in \{y_1, \dots, y_M\} \iff \phi_z^m \in \mathcal{R}(H^*), \text{ for some } m = 1,2,3. \quad (3.10)$$

Furthermore, the ranges of H^* and F coincide and thus

$$z \in \{y_1, \dots, y_M\} \iff \phi_z^m \in \mathcal{R}(F) \iff \mathcal{P}\phi_z^m = 0, \text{ for some } m = 1,2,3, \quad (3.11)$$

where $\mathcal{P} : \mathbb{C}^N \to \mathcal{R}(F)^\perp = \mathcal{N}(F^*)$ is the orthogonal projection onto the null space $\mathcal{N}(F^*)$ of F^*.

Proof : One can prove this theorem in the lines of the proofs of Theorem 4.1 in [8] concerning the Born-approximation for the acoustic model and more closely Theorem 3.1 and Theorem 3.2 in [4] concerning the acoustic and elastic wave scattering respectively related to the Born, Foldy and the intermediate models, by proving the maximal rank property of the matrix H and using the test vector ϕ_z^m. □

Remark 3.2. *We can observe in (3.3) that either horizontal polarized directions, p^h, or vertical polarized directions, p^v, are sufficient for the reconstruction. In addition, either the horizontal observation directions or the vertical observation directions are also sufficient for the reconstruction.*

3.2 Invertibility of the matrix $\bar{\Gamma}$

To discuss the invertibility of $\bar{\Gamma}$, we distinguish two situations.
Case 1 (Diagonally dominant condition): As mentioned in [4], concerning acoustic and elastic cases, when the scatterers are relatively far away from each other comparing to the scattering strengths, then the invertibility condition of $\bar{\Gamma}$ is the diagonally dominant condition and it is given by

$$\sum_{\substack{j=1 \\ j \neq l}}^{M} ||\Pi_\kappa(y_j, y_l)||_1 < \frac{1}{|a_l|}, \forall\, l = 1, 2, \dots, M. \tag{3.12}$$

Here $||\cdot||_1$ is the 1-norm and it is defined for a matrix $\mathbf{L} = [L_{nm}] \in \mathbb{C}^{N \times M}$, as $||\mathbf{L}||_1 := \max\limits_{1 \leq m \leq M} \sum\limits_{n=1}^{N} |L_{nm}|$.
Case 2 (Non-diagonally dominant condition): Using the form (2.1), we can write $\Pi_\kappa(x, y)$ explicitly as

$$\Pi_\kappa(x, y) \quad = \quad \Phi_\kappa(x, y)\mathbf{I} + \frac{1}{\kappa^2}\frac{\Phi_\kappa(x, y)}{r^2}\left[-\kappa^2\, R \otimes\, R + (1 - i\kappa\, r)(3\hat{R} \otimes \hat{R} - \mathbf{I})\right], \tag{3.13}$$

where $R = x - y, r = |x - y|$ and $\hat{R} = \frac{R}{r}$. We remark that the entries of $\Pi_\kappa(x, y)$ are analytic in terms of the variables $\eta_{jlm} = (y_j - y_l)_m$, $j, l = 1, \dots, M$ and $m = 1, 2, 3$ for $\eta_{jlm} \in \mathbb{R}\backslash\{0\}$. Remark also that the determinant of $\bar{\Gamma}$, $det\bar{\Gamma}$, is given by the products and sums of a_j^{-1} and the entries of $\Pi_\kappa(y_j, y_l)$ for $j, l = 1, \dots, M$. Then $det\bar{\Gamma}$ is analytic in terms of the $3\frac{M(M-1)}{2}$ real variables η_{jlm} for $j, l = 1, \dots, M$ with $j < l$, $m = 1, 2, 3$. Fix the wavenumber κ and the scattering strengths a_j, for $j = 1, \dots, M$, we deduce then that except for few distributions of the scatterers, $y_1, \dots, y_M$, the matrix $\bar{\Gamma}$ is invertible. The wavenumbers κ for which $\bar{\Gamma}$ is not invertible are called resonances, see [1] for a study of these resonances concerning the acoustic case.

3.3 Recovering the scattering strengths a_j's

In this section we discuss how one can recover the scattering strengths a_j of the scatterers y_j for $j = 1, \dots, M$ for the given far-field pattern, i.e. response matrix F. Recall that F has the factorization $F = H^*TH$ where H and T are defined as in Section 3.1. As the matrix H is of maximal rank $3M$ and $N \geq 3M$ the matrix $HH^* \in \mathbb{C}^{3M \times 3M}$ is invertible. Let us denote this inverse by I_H. Once we locate the scatterers $y_1, y_2, \dots, y_M$ by using MUSIC algorithm for the given far-field patterns, we can recover the matrix $T \in C^{3M \times 3M}$ as $T = I_H HFH^* I_H$, $I_H H$ is the pseudo inverse of H^*. Then, we can recover the scattering strengths $a_1, \dots, a_M$ from the entries of the matrix T. We explain how this procedure is going to work in each model.

- In the Born approximation we have $T = \mathbf{a}$, and hence the diagonal entries of T give the scattering strengths $a_1, \dots, a_M$.
- In the Foldy's method we have $T = \bar{\Gamma}^{-1}$, and hence the reciprocals of the diagonal entries of T^{-1} produces the scattering strengths $a_1, \dots, a_M$.
- In the intermediate level, k, approximation we have $T = \mathbf{a}\sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$. We have already seen how one can recover the scattering strengths for $k = 0$ (Born) and for $k = \infty$ (Foldy). In the case

$k = 1$, we have $T = \mathbf{a} - \mathbf{a}\tilde{\mathbf{M}}$. As we know that $\mathbf{a}$ is a diagonal matrix and the 3×3 diagonal blocks of $\tilde{\mathbf{M}}$ are zero, the diagonal entries of T are equal to the scattering strengths of the M scatterers. But for intermediate level approximation $k > 1$, it is difficult to recover the scattering strengths due to the complicated structure of the matrices $(-\tilde{\mathbf{M}})^l$, for $l = 2, \dots,$ and hence of T. For this reason, as in the acoustic and elastic cases of [4], we restrict ourselves to the special case of two point-like obstacles y_1, y_2 with the corresponding scattering strengths a_1, a_2. In this case using the symmetry relation of the fundamental matrix $\Pi_\kappa(x,y)$, i.e. $\Pi_\kappa(x,y) = [\Pi_\kappa(y,x)]^\top$, we have the explicit form of $(-\tilde{\mathbf{M}})^l$ for $l = 0, 1, 2, \dots$ as follows

$$(-\tilde{\mathbf{M}})^l = \begin{cases} \begin{bmatrix} a_1^{\frac{l}{2}} a_2^{\frac{l}{2}} \Pi_\kappa^l(y_1,y_2) & \mathbf{0} \\ \mathbf{0} & a_1^{\frac{l}{2}} a_2^{\frac{l}{2}} \Pi_\kappa^l(y_1,y_2) \end{bmatrix}, & l \in 2\mathbb{N} \cup \{0\} \\ \begin{bmatrix} \mathbf{0} & a_1^{\frac{l-1}{2}} a_2^{\frac{l+1}{2}} \Pi_\kappa^l(y_1,y_2) \\ a_1^{\frac{l+1}{2}} a_2^{\frac{l-1}{2}} \Pi_\kappa^l(y_1,y_2) & \mathbf{0} \end{bmatrix}, & l \in 2\mathbb{N} - 1 \end{cases}$$

Here, $\mathbf{0}$ is the zero matrix of order 3. The matrix $(-\tilde{\mathbf{M}})^l$ is either diagonal or anti-diagonal by blocks of the size 3×3. This structure is not valid anymore for the case of more than two scatterers. From this structure, we obtain the explicit form of $T = \mathbf{a}\sum_{l=0}^{k}(-\tilde{\mathbf{M}})^l$ in the k^{th} order scattering as follows

$$T = \begin{cases} \begin{bmatrix} a_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & a_2\mathbf{I} \end{bmatrix}, & k = 0, \\ \begin{bmatrix} a_1 \sum\limits_{l=0}^{\frac{k}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2) & \sum\limits_{l=1}^{\frac{k}{2}} a_1^l a_2^l \Pi_\kappa^{2l-1}(y_1,y_2) \\ \sum\limits_{l=1}^{\frac{k}{2}} a_1^l a_2^l \Pi_\kappa^{2l-1}(y_1,y_2) & a_2 \sum\limits_{l=0}^{\frac{k}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2) \end{bmatrix}, & k \in 2\mathbb{N}, \\ \begin{bmatrix} a_1 \sum\limits_{l=0}^{\frac{k-1}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2) & \sum\limits_{l=0}^{\frac{k-1}{2}} a_1^{l+1} a_2^{l+1} \Pi_\kappa^{2l+1}(y_1,y_2) \\ \sum\limits_{l=1}^{\frac{k-1}{2}} a_1^{l+1} a_2^{l+1} \Pi_\kappa^{2l+1}(y_1,y_2) & a_2 \sum\limits_{l=0}^{\frac{k-1}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2) \end{bmatrix}, & k \in 2\mathbb{N} - 1, \\ \begin{bmatrix} \frac{1}{a_1}\mathbf{I} & -\Pi_\kappa(y_1,y_2) \\ -\Pi_\kappa(y_1,y_2) & \frac{1}{a_2}\mathbf{I} \end{bmatrix}^{-1}, & k = \infty. \end{cases}$$

From the above explicit form of T, we observe the following points.

- The diagonal entries of T give the scattering strengths in the Born approximation, i.e. $k = 0$.
- Substituting the non diagonal entries in the diagonal entries give the scattering strengths in every even level scattering k, i.e. $k \in 2\mathbb{N}$. Indeed, define $\breve{a} := \sum\limits_{l=1}^{\frac{k}{2}} a_1^l a_2^l \Pi_\kappa^{2l-1}(y_1,y_2)$ then the non-diagonal entries of T are equal to $\breve{a}$. Also the diagonal entries T_{11} and T_{22} of T are

equal to $a_1(1+\Pi_\kappa(y_1,y_2)\breve{a})$ and $a_2(1+\Pi_\kappa(y_1,y_2)\breve{a})$ respectively. Now, with the knowledge of the scatterers y_1 and y_2 from the MUSIC algorithm and by substituting the value of $\breve{a}$ in the diagonal entries, we can evaluate the scattering strengths a_1 and a_2.

- Substituting the diagonal entries in the non diagonal entries give the scattering strengths in every odd level scattering k, i.e. $k \in 2\mathbb{N}-1$. Indeed, define $\breve{b}_1 := a_1 \sum_{l=0}^{\frac{k-1}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2)$ and $\breve{b}_2 := a_2 \sum_{l=0}^{\frac{k-1}{2}} a_1^l a_2^l \Pi_\kappa^{2l}(y_1,y_2)$ then the diagonal entries T_{11} and T_{22} of T are equal to $\breve{b}_1$ and $\breve{b}_2$ respectively. Also the non-diagonal entries T_{12} and T_{21} of T are the same and are equal to $a_1\breve{b}_2\Pi_\kappa(y_1,y_2) = a_2\breve{b}_1\Pi_\kappa(y_1,y_2)$. Now again with the knowledge of the scatterers y_1 and y_2 from the MUSIC algorithm and by substituting the diagonal entries in the non diagonal entries of T, we can evaluate the scattering strengths a_1 and a_2.
- The diagonal entries of T^{-1} give the scattering strengths in the method of Foldy. i.e. $k=\infty$.

3.3.1 Numerical results and discussions

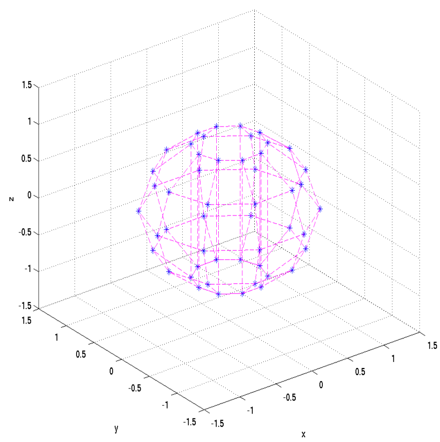

Figure 1: Incident and the observational directions

For the convenience of visualization, we show the results for the scatterers in XY-plane. For our calculations, we consider 50 incident and observational directions and the point-like scatterers of the same scattering strength located at the points $y_1 = (0,0,0),\ y_2 = (0,0.5,0),\ y_3 = (0.5,0,0),\ y_4 = (0.5,0.5,0),\ y_5 = (1,1,0),\ y_6 = (1,-1,0),\ y_7 = (-1,-1,0),\ y_8 = (-1,1,0),\ y_9 = (1,-1.5,0)$, $y_{10} = (1.5,0.5,0)$ and $y_{11} = (-1.5,1,0)$. Let d_{GL} stands for the degree of Gauss-Legendre polynomial. We used the $2d_{GL}^2(=50)$ incident and the observational directions obtained from the Gauss-Legendre polynomial of degree $d_{GL}(=5)$, i.e. if we denote the zeros of the Gauss-Legendre polynomial of degree by GL_k, for $k=1,\dots,d_{GL}$ then the azimuth and the zenith angles θ and ϕ are given by

$$\phi = \cos^{-1}(GL_k),\ k=1,\dots,d_{GL}$$
$$\theta = j * \frac{\pi}{d_{GL}},\ j=0,1,\dots,2d_{GL}-1.$$

Combinations of these spherical coordinates will allow us to find the incident and the observational directions given by $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. These directions are shown in Fig.1. To show numerically that horizontal, p^h, or vertical, p^v, polarization direction is enough for the reconstruction, we used the directions p^h and p^v as per the definition (3.5).

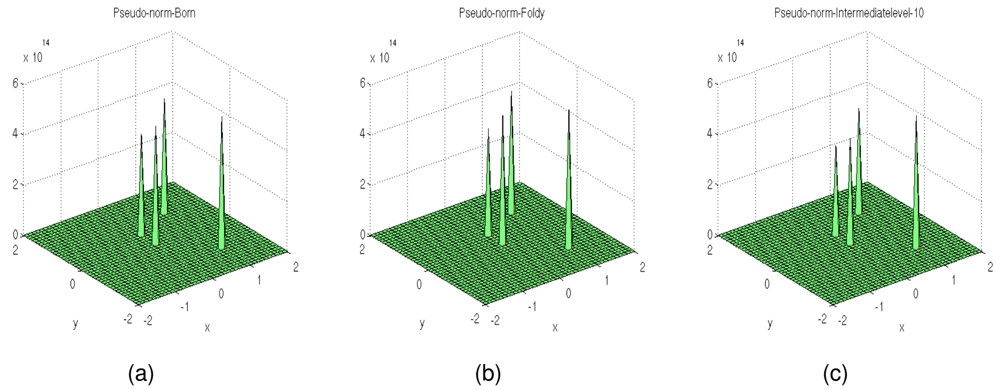

Figure 2: (a) Born , (b) Foldy , (c) $10^{th} level$ - based reconstructions using p^h with 0% noise, a_j=1 and $\kappa = \pi$ for 4 scatterers.

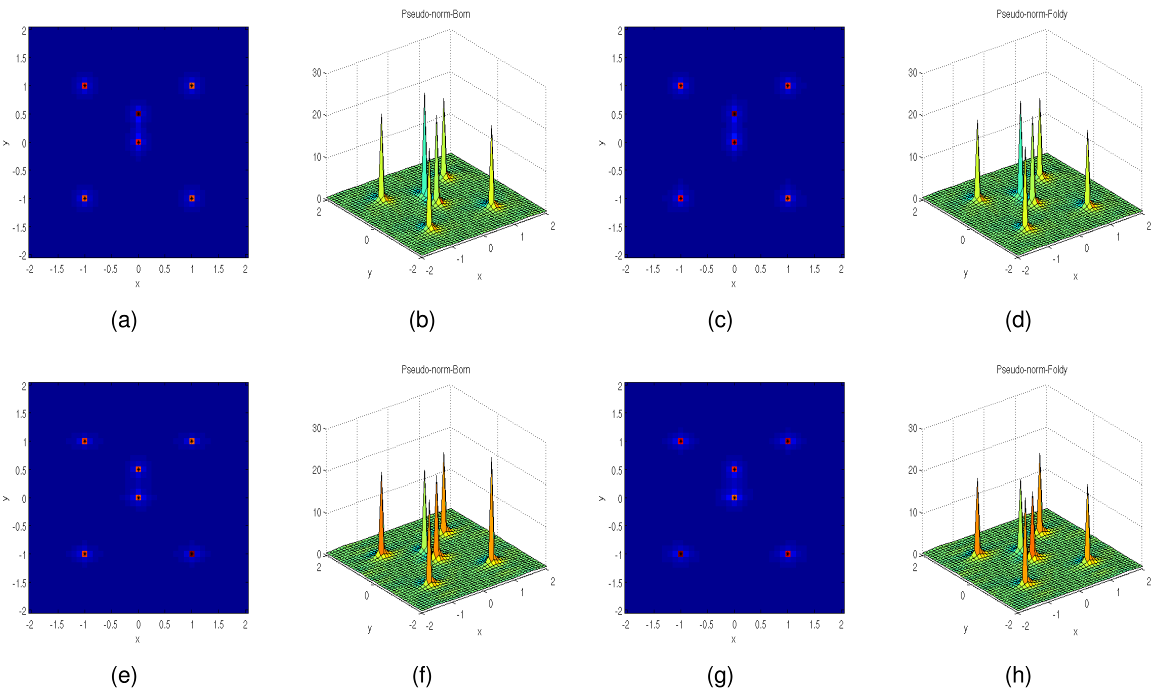

Figure 3: Born (a,b,e,f) and Foldy (c,d,g,h) based reconstructions with 1% noise, a_j=1 and $\kappa = 2\pi$ for 6 scatterers. Upper part (a,b,c,d) - p^h, lower part(e,f,g,h) -p^v.

Since MUSIC algorithm is an exact method, the reconstruction is very accurate in the absence of noise in measured data, for Born, Foldy and intermediate models. It can be observed in Fig.2, from the pseudo spectrum of the scatterers located at the points $y_1,\ y_2,\ y_5,\ y_6$ having scattering strengths 1 for each with the wavenumber $\kappa = \pi$ (i.e. minimum distance between the scatterers is quarter of the wavelength) with respect to the Born, the Foldy and the intermediate models.

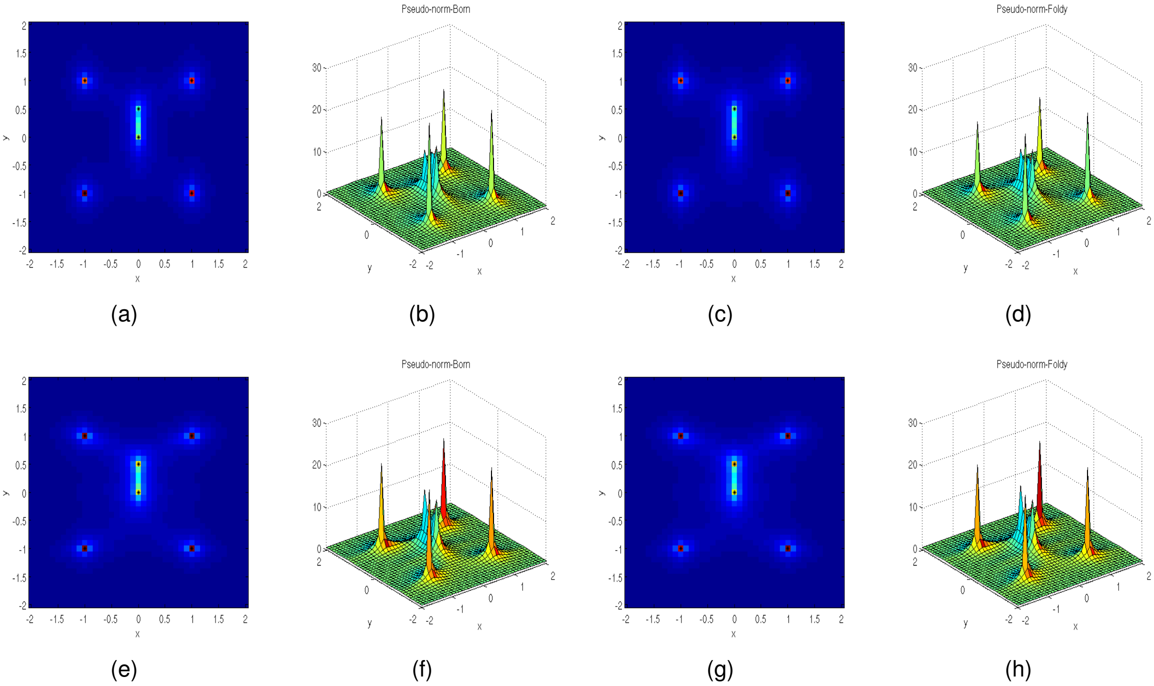

Figure 4: Born (a,b,e,f) and Foldy (c,d,g,h) based reconstructions with 1% noise, a_j=1 and $\kappa = \pi$ for 6 scatterers. Upper part (a,b,c,d) - p^h, lower part(e,f,g,h) -p^v.

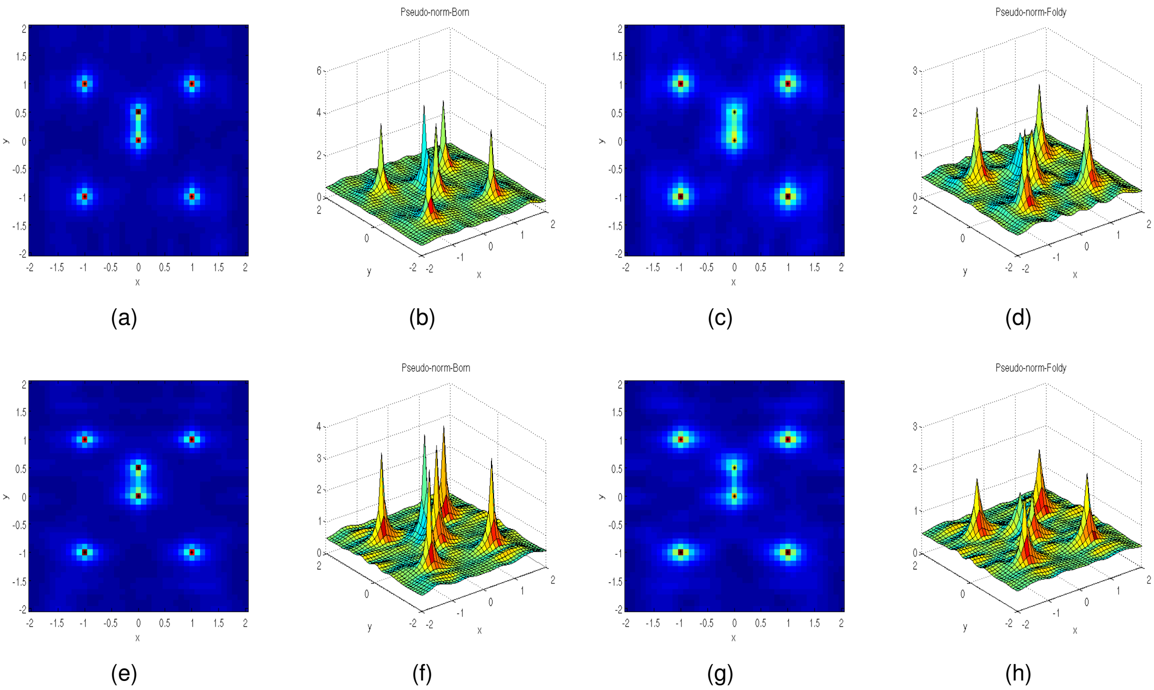

Figure 5: Born (a,b,e,f) and Foldy (c,d,g,h) based reconstructions with 6% noise, a_j=10 and $\kappa = 2\pi$ for 6 scatterers. Upper part (a,b,c,d) - p^h, lower part(e,f,g,h) -p^v.

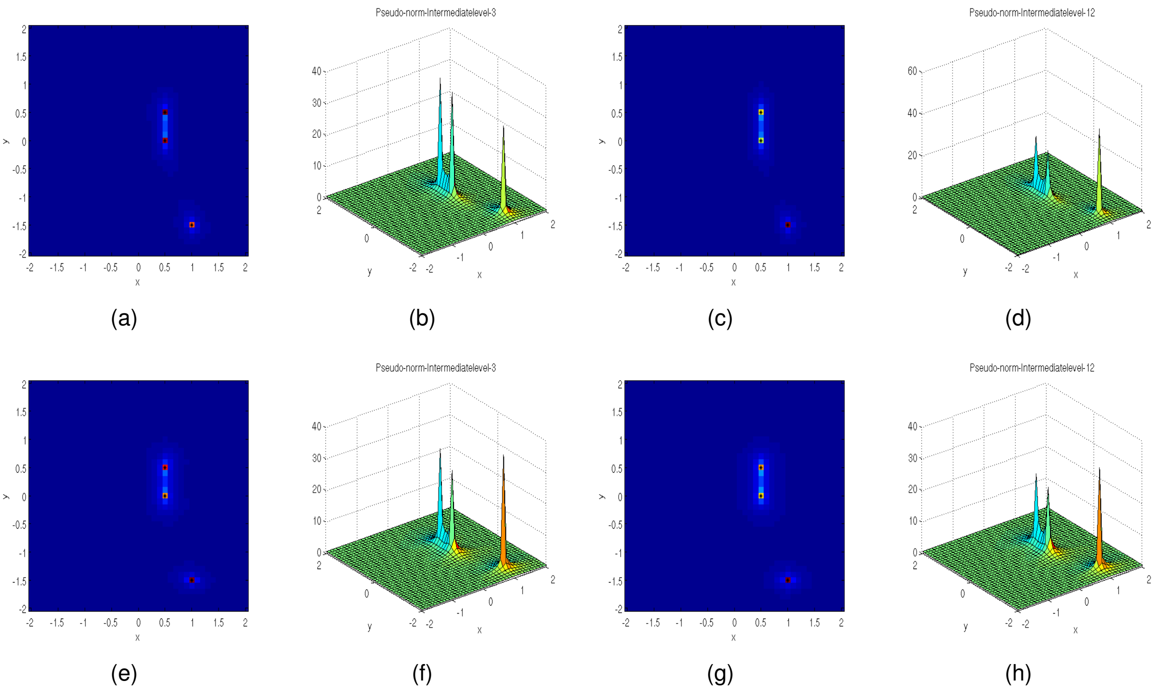

Figure 6: Reconstruction of 3 scatterers with 1% noise, a_j=7 and $\kappa = \pi$. 3^{rd} level (a,b,e,f) and 12^{th} level (c,d,g,h) approximations. Upper part (a,b,c,d) - p^h, lower part(e,f,g,h) -p^v.

To analyze the effect of the noise level on the resolution of the algorithm, different noise levels are used. To distinguish the differences between the Born approximation and the Foldy model, we used different scattering strengths, noise levels and distance between the scatterers.

Fig.3 and Fig.4 are related to the 6 scatterers located at the points $y_1,\ y_2,\ y_5, y_6,\ y_7$ and y_8 having scattering strength 1 for each with 1% random noise in the measured far-field pattern. Fig.3 shows the pseudo spectrum of the mentioned 6 scatterers for the wavenumber $\kappa = 2\pi$ whereas figure Fig.4 shows the pseudo spectrum for the wavenumber $\kappa = \pi$. i.e., minimum distance between the scatterers is half of the wave length and quarter of the wavelength respectively. We observe that due to the higher wave number, Fig.3 has the better reconstruction comparing to Fig.4 w.r.t to p^h and p^v respectively. Also, we can observe that the scatterers satisfy largely the condition (3.12) and the reconstruction looks similar in both the Born approximation and the Foldy model. Hence, if the scatterers are well separated with low scattering strengths there is no much difference in the reconstruction between the Born approximation and the Foldy model.

Now, we present an example where the scatterers do not satisfy the condition (3.12). Fig.5 shows the pseudo spectrum of the 6 scatterers again located at $y_1,\ y_2,\ y_5, y_6,\ y_7$ and y_8 of each having scattering strength 10 for $\kappa = 2\pi$ with 6% random noise in the measured far field patterns with respect to the Born approximation and the Foldy method. Compared to Fig.3 and Fig.4, we see in Fig.5 how the reconstruction deteriorates due to the effect of multiple scattering created by the close obstacles. In this case, we can see the differences between the Born approximation and the Foldy model.

As a conclusion, we have seen that if the condition (3.12) is satisfied largely then the effect of the multiple scattering is quite low and the reconstruction is similar in both Born and Foldy but above the condition (3.12) the use of the Born approximation gives better reconstruction than the use of the Foldy method. However in the latter case, Born approximation is not valid as the scatterers are relatively close. It is observed that, in general, increase of the noise level and decrease of the distance between the scatterers make the reconstruction worse in both the approximations. It is also observed that when the scatterers have different scattering strengths and if they are not well separated, the visibility of the scatterer is proportional to the scattering strength of the respective scatterer.

We have similar kind of difference between the intermediate level approximations as the level k increases with respect to the condition (3.12). We can observe this in Fig.6 which shows the numerical reconstruction of the 3 scatterers, based on 3^{rd} and 12^{th} level approximations, located at $y_3,\ y_4$ and y_9 and having scattering strength 7 with $\kappa = \pi$ and of 1% random noise in the measured far-field pattern. Finally, let us remind that the reconstruction depends on the choice of the signal and noise subspaces of the multiscale response matrix, see for instance [4] for a discussion on this issue concerning the acoustic and elastic cases.

4 Conclusion

We justified the Foldy method to model the electromagnetic scattering by point-like scatterers by generalizing the regularization method, known in the quantum mechanics [1], to the Maxwell case. Then we described the intermediate levels of scattering between the Born and the Foldy models. We showed how we can locate the scatterers, using MUSIC type algorithms, and then how to recover the scattering strengths from far fields corresponding to incident plane waves. We demonstrated by several numerical tests that the accuracy of the reconstruction is proportional to the distance between the scatterers but inversely proportional to the wavelength and the noise in measured far-field patterns. In addition, the point-like scatterers with high scattering strengths are more visible compared to the ones with less scattering strength. Finally, we have seen that either horizontal polarized directions, or vertical polarized directions, and either the horizontal observation directions, or the vertical observation directions, are enough for the reconstruction. This is true for Born, Foldy or any of the intermediate levels of scattering.

5 Appendix

For the reader's convenience we show the proofs of (2.23), (2.24) and (2.34).

Proofs of (2.24) and (2.23) : The Green's tensor $\Pi_\kappa(x,0)$ can be written as

$$\Pi_\kappa(x,0) = \frac{e^{i\kappa\,|x|}}{4\pi|x|}\left\{P(i\kappa\,|x|)\mathbf{I} + Q(i\kappa\,|x|)\,\hat{x}\otimes\hat{x}\right\}, \tag{5.1}$$

where the functions P and Q are defined in Section 2.2.

In the following we will only prove (2.24), since (2.23) follows automatically from (2.24), (2.21) and (5.1).

By the definitions of T_κ and the inverse Fourier transform,

$$\begin{aligned}
(2\pi)^{-3/2}\mathcal{F}^{-1}(T_\kappa(\xi)) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\kappa^2}e^{i\xi\cdot x}d\xi\,\mathbf{I} - \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\kappa^2}\hat{\xi}\otimes\hat{\xi}e^{i\xi\cdot x}d\xi \\
&= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^3}\nabla_x\nabla_x\int_{\mathbb{R}^3}\frac{1}{(\kappa^2-|\xi|^2)|\xi|^2}\,e^{i\xi\cdot x}d\xi \\
&= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^3\kappa^2}\nabla_x\nabla_x\int_{\mathbb{R}^3}\left(\frac{1}{\kappa^2-|\xi|^2}+\frac{1}{|\xi|^2}\right)e^{i\xi\cdot x}d\xi \\
&= \frac{e^{i\kappa|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}(\frac{1}{|\xi|^2}) - \mathcal{F}^{-1}(\frac{1}{|\xi|^2-\kappa^2})\right).
\end{aligned}$$

Employing $(2\pi)^{-3/2}\mathcal{F}^{-1}(\frac{1}{|\xi|^2-\kappa^2})(x) = \frac{e^{i\kappa|x|}}{4\pi|x|}$, we get

$$\begin{aligned}
(2\pi)^{-3/2}\mathcal{F}^{-1}(T_\kappa(\xi)) &= \frac{e^{ik|x|}}{4\pi|x|}\mathbf{I} - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\left(\frac{1-e^{i\kappa|x|}}{|x|}\right) \\
&= (\mathbf{I}+\frac{1}{\kappa^2}\nabla_x\nabla_x)\frac{e^{i\kappa|x|}}{4\pi|x|} - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{1}{|x|} \\
&= \Pi_\kappa(x,0) - \frac{1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{1}{|x|}.
\end{aligned}$$

Simple calculations show

$$\nabla_x\nabla_x\frac{1}{|x|} = -\frac{1}{|x|^3}\left\{\mathbf{I}-3\hat{x}\otimes\hat{x}\right\}. \tag{5.2}$$

With the help of (5.1), we finally obtain

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\kappa^2}(\mathbf{I}-\hat{\xi}\otimes\hat{\xi})e^{i\xi\cdot x}d\xi \\
=\ &\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3} + \frac{e^{i\kappa|x|}}{4\pi|x|}\left\{P(i\kappa|x|)\mathbf{I} + Q(i\kappa|x|)\,\hat{x}\otimes\hat{x}\right\}.
\end{aligned}$$

The identity (2.24) is thus proven. □

Proof of (2.34) : We first compute the integral $(2\pi)^{-3}\int_{\mathbb{R}^3}T_\kappa(\xi)f(\beta_T,\xi)\,d\xi$ with $f(\beta_T,\xi) = \frac{\beta_T^2}{\beta_T^2+|\xi|^2}$. Again using the definition of T_κ, we find

$$\begin{aligned}
T_\kappa(\xi)\,f(\beta_T,\xi) &= T_\kappa(\xi)\,f(\beta_T,\kappa) + T_\kappa(\xi)\,\left(f(\beta_T,\xi)-f(\beta_T,\kappa)\right) \\
&= \left[T_\kappa(\xi)\,+T_\kappa(\xi)\,\frac{\kappa^2-\xi^2}{\beta_T^2+\xi^2}\right]f(\beta_T,\kappa) \\
&= \left[\frac{1}{\xi^2-\kappa^2}(\mathbf{I}-\hat{\xi}\otimes\hat{\xi}) - \frac{1}{\xi^2+\beta_T^2}(\mathbf{I}-\hat{\xi}\otimes\hat{\xi})\right]f(\beta_T,\kappa).
\end{aligned}$$

Recalling that $\Pi_\kappa^T(x) = (2\pi)^{-3/2}\mathcal{F}^{-1}[\frac{1}{\xi^2-\kappa^2}(\mathbf{I}-\hat{\xi}\otimes\hat{\xi})]$, we have

$$(2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)\,f(\beta_T,\xi)] = [\Pi_\kappa^T(x) - \Pi_{i\beta_T}^T(x)]\,f(\beta_T,\kappa).$$

It is easy to see

$$\left[\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}-\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{-4\pi\beta_T^2|x|^3}\right]\frac{\beta_T^2}{\beta_T^2+\kappa^2}=\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}.$$

Using the previous identity, it follows from the expression of $\Pi_\kappa^T(x)$ that

$$\begin{aligned}(2\pi)^{-3/2}\mathcal{F}^{-1}[T_\kappa(\xi)\,f(\beta_T,\xi)] &= \frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}+\Big\{\frac{e^{i\kappa|x|}}{4\pi|x|}\left[P(i\kappa|x|)\mathbf{I}+Q(i\kappa|x|)\hat{x}\otimes\hat{x}\right]\\ &\quad-\frac{e^{-\beta_T|x|}}{4\pi|x|}\left[P(-\beta_T|x|)\mathbf{I}+Q(-\beta_T|x|)\hat{x}\otimes\hat{x}\right]\Big\}\frac{\beta_T^2}{\beta_T^2+\kappa^2}.\end{aligned}$$

Now, by the inverse Fourier transformation and the expressions for P and Q,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}T_\kappa(\xi)f(\beta_T,\xi)\,d\xi &= (2\pi)^{-3/2}\lim_{|x|\to 0}\mathcal{F}^{-1}[T_\kappa(\xi)\,f(\beta_T,\xi)](x)\\ &= \frac{\beta_T+i\kappa}{6\pi}\frac{\beta_T^2}{\beta_T^2+\kappa^2}\mathbf{I}.\end{aligned}\tag{5.3}$$

Indeed, recall that

$$P(z)=1-\frac{1}{z}+\frac{1}{z^2},\; Q(z)=-1+\frac{3}{z}-\frac{3}{z^2}.$$

Now,

- Consider the term $\frac{e^{i\kappa|x|}}{4\pi|x|}Q(i\kappa|x|)-\frac{e^{-\beta_T|x|}}{4\pi|x|}Q(-\beta_T|x|)=:A_Q$, then

$$A_Q = \frac{e^{i\kappa|x|}}{4\pi|x|}\left[-1+\frac{3}{i\kappa|x|}+\frac{3}{\kappa^2|x|^2}\right]-\frac{e^{-\beta_T|x|}}{4\pi|x|}\left[-1-\frac{3}{\beta_T|x|}-\frac{3}{\beta_T^2|x|^2}\right].$$

  From the Taylor series, we have

$$\begin{aligned}e^{i\kappa|x|} &= 1+i\kappa|x|-\frac{\kappa^2|x|^2}{2}-\frac{i\kappa^3|x|^3}{6}+o(|x|^4),\\ e^{-\beta_T|x|} &= 1-\beta_T|x|+\frac{\beta_T^2|x|^2}{2}-\frac{\beta_T^3|x|^3}{6}+o(|x|^4).\end{aligned}$$

  Then by using the Taylor series, we obtain the following after few computations,

$$\begin{aligned}\frac{e^{i\kappa|x|}}{4\pi|x|}\left[-1+\frac{3}{i\kappa|x|}+\frac{3}{\kappa^2|x|^2}\right] &= \frac{1}{8\pi|x|}+\frac{3}{4\pi\kappa^2|x|^3}+o(|x|),\\ \frac{e^{-\beta_T|x|}}{4\pi|x|}\left[-1-\frac{3}{\beta_T|x|}-\frac{3}{\beta_T^2|x|^2}\right] &= \frac{1}{8\pi|x|}-\frac{3}{4\pi\beta_T^2|x|^3}+o(|x|).\end{aligned}$$

  By substituting the above expressions in A_Q, we obtain

$$\begin{aligned}A_Q &= \frac{3}{4\pi|x|^3}\left[\frac{1}{\kappa^2}+\frac{1}{\beta_T^2}\right]+o(|x|)\\ &= \frac{3}{4\pi\kappa^2|x|^3}\frac{1}{f(\beta_T,\kappa)}+o(|x|);\, f(\beta_T,\kappa):=\frac{\beta_T^2}{\beta_T^2+\kappa^2}.\end{aligned}\tag{5.4}$$

■ Consider the term $\frac{e^{i\kappa|x|}}{4\pi|x|}P(i\kappa\,|x|) - \frac{e^{-\beta_T|x|}}{4\pi|x|}P(-\beta_T\,|x|) =: A_P$, then

$$A_P \;=\; \frac{e^{i\kappa\,|x|}}{4\pi|x|}\left[1-\frac{1}{i\kappa\,|x|}-\frac{1}{\kappa^2\,|x|^2}\right]-\frac{e^{-\beta_T\,|x|}}{4\pi|x|}\left[1+\frac{1}{\beta_T\,|x|}+\frac{1}{\beta_T^2\,|x|^2}\right].$$

Again by using the Taylor series, we obtain the following after few computations,

$$\begin{aligned}\frac{e^{i\kappa\,|x|}}{4\pi|x|}\left[1-\frac{1}{i\kappa\,|x|}-\frac{1}{\kappa^2\,|x|^2}\right] &= \frac{i\kappa}{6\pi}+\frac{1}{8\pi\,|x|}-\frac{1}{4\pi\kappa^2\,|x|^3}+o(|x|),\\ \frac{e^{-\beta_T\,|x|}}{4\pi|x|}\left[1+\frac{1}{\beta_T\,|x|}+\frac{1}{\beta_T^2\,|x|^2}\right] &= -\frac{\beta_T}{6\pi}+\frac{1}{8\pi\,|x|}+\frac{1}{4\pi\beta_T^2\,|x|^3}+o(|x|).\end{aligned}$$

By substituting the above expressions in A_P, we obtain

$$\begin{aligned}A_P &= \frac{\beta_T+i\kappa}{6\pi}-\frac{1}{4\pi\,|x|^3}\left[\frac{1}{\kappa^2}+\frac{1}{\beta_T^2}\right]+o(|x|)\\ &= \frac{\beta_T+i\kappa}{6\pi}-\frac{1}{4\pi\kappa^2\,|x|^3}\frac{1}{f(\beta_T,\kappa)}+o(|x|).\end{aligned}\tag{5.5}$$

Gathering all :

$$\begin{aligned}\tilde{g}_0^T(\kappa,|x|) &:= \frac{\mathrm{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}+\Big\{\frac{e^{i\kappa\,|x|}}{4\pi|x|}\left[P(i\kappa\,|x|)\mathrm{I}+Q(i\kappa\,|x|)\hat{x}\otimes\hat{x}\right]\\ &\quad-\frac{e^{-\beta_T|x|}}{4\pi|x|}\left[P(-\beta_T|x|)\mathrm{I}+Q(-\beta_T|x|)\hat{x}\otimes\hat{x}\right]\Big\}\frac{\beta_T^2}{\beta_T^2+\kappa^2}\\ &= \frac{\mathrm{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2\,|x|^3}+\Big\{\frac{\beta_T+i\kappa}{6\pi}\mathrm{I}+\frac{3\hat{x}\otimes\hat{x}-\mathrm{I}}{4\pi\kappa^2\,|x|^3}\frac{1}{f(\beta_T,\kappa)}\Big\}f(\beta_T,\kappa)+o(|x|)\\ &= \frac{\beta_T+i\kappa}{6\pi}f(\beta_T,\kappa)\mathrm{I}+o(|x|).\end{aligned}$$

Hence

$$\lim_{|x|\to 0}\tilde{g}_0^T(\kappa,|x|) \;=\; \frac{\beta_T+i\kappa}{6\pi}f(\beta_T,\,\kappa)\mathrm{I}.\tag{5.6}$$

It remains to check the relation

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}L_\kappa(\xi)\frac{\beta_L^4}{\beta_L^4+\xi^4}\,d\xi = -\frac{\tilde{\beta}_L^3}{6\pi\kappa^2}\mathrm{I},\quad \tilde{\beta}_L=\beta_L/\sqrt{2}.\tag{5.7}$$

From the definition of L_k, we see

$$\begin{aligned}&\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}[L_\kappa(\xi)\frac{\beta_L^4}{\beta_L^4+|\xi|^4}](x)\\ &= \frac{-1}{(2\pi)^3\kappa^2}\int_{\mathbb{R}^3}\frac{\beta_L^4}{\beta_L^4+|\xi|^4}\hat{\xi}\otimes\hat{\xi}\,e^{i\xi\cdot x}\,d\xi\\ &= \frac{1}{(2\pi)^3\kappa^2}\nabla_x\nabla_x\int_{\mathbb{R}^3}\frac{\beta_L^4}{(\beta_L^4+|\xi|^4)|\xi|^2}\,e^{i\xi\cdot x}\,d\xi\\ &= \frac{1}{(2\pi)^3\kappa^2}\nabla_x\nabla_x\int_{\mathbb{R}^3}\left(\frac{1}{|\xi|^2}-\frac{|\xi|^2}{\beta_L^4+|\xi|^4}\right)\,e^{i\xi\cdot x}\,d\xi\\ &= \frac{-1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}[\frac{|\xi|^2}{\beta_L^4+|\xi|^4}](x)-\mathcal{F}^{-1}[\frac{1}{|\xi|^2}](x)\right).\end{aligned}\tag{5.8}$$

In view of (5.2),

$$\frac{1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\left(\mathcal{F}^{-1}[\frac{1}{|\xi|^2}](x)\right)=\frac{1}{\kappa^2}\nabla_x\nabla_x\frac{1}{4\pi|x|}=-\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2|x|^3}.$$

To evaluate the first term on the right hand side of (5.8), we need the integral identity

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{|\xi|^2}{\beta_L^4+|\xi|^4}e^{i\xi\cdot x}d\xi &= \int_0^\infty\int_0^{2\pi}\int_0^\pi\frac{|\xi|^4}{\beta_L^4+|\xi|^4}\sin\theta\, e^{i|\xi|\,|x|\cos\theta}d\theta\, d\phi\, d|\xi| \\ &= 4\pi\int_0^\infty\frac{|\xi|^3\sin(|\xi|\,|x|)}{(\beta_L^4+|\xi|^4)\,|x|}\,d|\xi| \\ &= 2\pi^2\frac{e^{-\tilde{\beta}_L|x|}\,\cos(\tilde{\beta}_L|x|)}{|x|}\end{aligned}$$

with $\tilde{\beta}_L=\beta_L/\sqrt{2}$, where the last equality follows from the Fourier sine transform of the odd function $t^3/(\beta_L^4+t^4)$. It then follows that

$$\begin{aligned}\frac{-1}{(2\pi)^{3/2}\kappa^2}\nabla_x\nabla_x\mathcal{F}^{-1}[\frac{|\xi|^2}{\beta_L^4+|\xi|^4}](x) &= \frac{-1}{4\pi\kappa^2}\nabla_x\nabla_x\frac{e^{-\tilde{\beta}_L|x|}\,\cos(\tilde{\beta}_L|x|)}{|x|} \\ &= \frac{1}{4\pi\kappa^2}\left\{g(|x|)\,\mathbf{I}+|x|\,g'(|x|)\,\hat{x}\otimes\hat{x}\right\}\end{aligned}$$

where $g(t)=\{e^{-\tilde{\beta}_Lt}[\cos(\tilde{\beta}_Lt)+\tilde{\beta}_Lt(\cos(\tilde{\beta}_Lt)+\sin(\tilde{\beta}_Lt))]\}/t^3$. After elementary calculations, we obtain

$$\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}[L_\kappa(\xi)\frac{\beta_L^4}{\beta_L^4+|\xi|^4}](x)=-\frac{\mathbf{I}-3\hat{x}\otimes\hat{x}}{4\pi\kappa^2}\{1-g(|x|)\}-\frac{\tilde{\beta}_L^2e^{-\tilde{\beta}_L|x|}\sin(\tilde{\beta}_L|x|)}{2\pi\kappa^2|x|}\hat{x}\otimes\hat{x},$$

and arguing similarly to the justification of (5.3), we obtain

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}L_\kappa(\xi)\frac{\beta_L^4}{\beta_L^4+\xi^4}\,d\xi=\lim_{|x|\to 0}\frac{1}{(2\pi)^{3/2}}\mathcal{F}^{-1}[L_\kappa(\xi)\frac{\beta_L^4}{\beta_L^4+|\xi|^4}](x)=-\frac{\tilde{\beta}_L^3}{6\pi\kappa^2}\,\mathbf{I}.$$

This proves (5.7). Finally, combining (5.7) and (5.3) yields (2.34). □

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