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Some inverse problems arising from elastic scattering by rigid obstacles

Guanghui Hu[1], Andreas Kirsch[2], Mourad Sini[3]

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[1] Weierstrass Institute
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: guanghui.hu@wias-berlin.de

[2] Department of Mathematics, KIT
76131 Karlsruhe
Germany
E-Mail: andreas.kirsch@kit.edu

[3] Johann Radon Institute
for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenbergerstr. 69
A–4040 Linz
Austria
E-Mail: mourad.sini@oeaw.ac.at

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

In the first part, it is proved that a C^2-regular rigid scatterer in $\mathbb{R}^3$ can be uniquely identified by the shear part (i.e. S-part) of the far-field pattern corresponding to all incident shear waves at any fixed frequency. The proof is short and it is based on a kind of decoupling of the S-part of scattered wave from its pressure part (i.e. P-part) on the boundary of the scatterer. Moreover, uniqueness using the S-part of the far-field pattern corresponding to only one incident plane shear wave holds for a ball or a convex Lipschitz polyhedron. In the second part, we adapt the factorization method to recover the shape of a rigid body from the scattered S-waves (resp. P-waves) corresponding to all incident plane shear (resp. pressure) waves. Numerical examples illustrate the accuracy of our reconstruction in $\mathbb{R}^2$. In particular, the factorization method also leads to some uniqueness results for all frequencies excluding possibly a discrete set.

1 Introduction

1.1 Direct elastic scattering problems

Consider a time-harmonic elastic plane wave u^{in} (with the time variation of the form $e^{-i\omega t}$, with a fixed frequency $\omega > 0$) incident on a rigid scatterer $D \subset \mathbb{R}^3$ embedded in an infinite isotropic and homogeneous elastic medium in $\mathbb{R}^3$. This can be modeled by the reduced Navier equation (or Lamé system)

$$(\Delta^* + \omega^2)u = 0, \quad \text{in} \quad \mathbb{R}^3\backslash\overline{D}, \quad \Delta^* := \mu\Delta + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div} \tag{1.1}$$

where u denotes the total displacement field, and λ, μ are the Lamé constants satisfying $\mu > 0$, $3\lambda + 2\mu > 0$. Throughout the paper we suppose that $D \subset \mathbb{R}^3$ is a bounded open set such that $\mathbb{R}^3\backslash\overline{D}$ is connected, and that the unit normal vector ν to ∂D always points into $\mathbb{R}^3\backslash\overline{D}$. Denote the linearized strain tensor by

$$\varepsilon(u) \;:=\; \frac{1}{2}\big(\nabla u + \nabla u^\top\big) \;\in\; \mathbb{R}^{3\times 3}, \tag{1.2}$$

where $\nabla u \in \mathbb{R}^{3\times3}$ and $\nabla u^\top$ stand for the derivative of u and its adjoint, respectively. By Hooke's law the strain tensor is related to the stress tensor via the identity

$$\sigma(u) \;=\; \lambda\,(\mathrm{div}\,u)\,I \;+\; 2\mu\,\varepsilon(u) \;\in\; \mathbb{R}^{3\times3}, \tag{1.3}$$

where I denotes the 3×3 identity matrix. The surface traction (or the stress operator) on ∂D is given by

$$T_\nu u := \sigma(u)\nu = (2\mu\nu\cdot\mathrm{grad} + \lambda\,\nu\,\mathrm{div} + \mu\nu\times\mathrm{curl}\,)u. \tag{1.4}$$

As usual, $a\cdot b$ denotes the scalar product and $a\times b$ denotes the vector product of $a, b \in \mathbb{R}^3$. In this paper the incident wave is allowed to be either a plane shear wave taking the form

$$u^{in} = u_s^{in} := q\exp(ik_s x\cdot d), \quad q, d \in \mathbb{S}^2 := \{x\in\mathbb{R}^3 : |x| = 1\}, \tag{1.5}$$

with the incident direction d and the polarization direction q satisfying $q \perp d$, or a plane pressure wave taking the form

$$u^{in} = u_p^{in} := d \exp(ik_p x \cdot d), \quad d \in \mathbb{S}^2. \tag{1.6}$$

Here, $k_s := \omega/\sqrt{\mu}$ and $k_p := \omega/\sqrt{\lambda + 2\mu}$ denote the shear wave number and the compressional wave number, respectively. For a rigid body D, the total field u satisfies the first kind (Dirichlet) boundary condition

$$u = 0 \quad \text{on} \quad \partial D. \tag{1.7}$$

Since the scattered field $u^{sc} := u - u^{in}$ also satisfies the Navier equation (1.1), it can be decomposed into the sum

$$u^{sc} := u_p^{sc} + u_s^{sc}, \quad u_p^{sc} := -\frac{1}{k_p^2} \operatorname{grad} \operatorname{div} u^{sc}, \ u_s^{sc} := \frac{1}{k_s^2} \operatorname{curl} \operatorname{curl} u^{sc},$$

where the vector functions u_p^{sc} and u_s^{sc} are referred to as the pressure (longitudinal) and shear (transversal) parts of u^{sc} respectively, satisfying

$$\begin{aligned} (\Delta + k_p^2) u_p^{sc} = 0, \quad \operatorname{curl} u_p^{sc} = 0, \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \\ (\Delta + k_s^2) u_s^{sc} = 0, \quad \operatorname{div} u_s^{sc} = 0, \quad \text{in } \mathbb{R}^3 \backslash \overline{D}. \end{aligned}$$

Moreover, the scattered field u^{sc} is required to satisfy the Kupradze radiation condition (see, e.g. [1])

$$\lim_{r \to \infty} \left(\frac{\partial u_p^{sc}}{\partial r} - ik_p u_p^{sc} \right) = 0, \quad \lim_{r \to \infty} \left(\frac{\partial u_s^{sc}}{\partial r} - ik_s u_s^{sc} \right) = 0, \quad r = |x|, \tag{1.8}$$

uniformly in all directions $\hat{x} = x/|x| \in \mathbb{S}^2$. The radiation conditions in (1.8) lead to the P-part (longitudinal part) u_p^∞ and the S-part (transversal part) u_s^∞ of the far-field pattern of u^{sc}, given by the asymptotic behavior

$$u^{sc}(x) = \frac{\exp(ik_p|x|)}{4\pi(\lambda + \mu)|x|} u_p^\infty(\hat{x}) + \frac{\exp(ik_s|x|)}{4\pi\mu|x|} u_s^\infty(\hat{x}) + \mathcal{O}(\frac{1}{|x|^2}), \quad |x| \to +\infty, \tag{1.9}$$

where $u_p^\infty(\hat{x})$ is normal to $\mathbb{S}^2$ and $u_s^\infty(\hat{x})$ is tangential to $\mathbb{S}^2$. In this paper, we define the far-field pattern u^∞ of the scattered field u^{sc} as the sum of u_p^∞ and u_s^∞, that is,

$$u^\infty(\hat{x}) := u_p^\infty(\hat{x}) + u_s^\infty(\hat{x}).$$

The direct scattering problem (DP) is stated as follows.

(DP): Given a scatterer $D \subset \mathbb{R}^3$ and an incident plane wave u^{in}, find the total field $u = u^{in} + u^{sc}$ in $\mathbb{R}^3 \backslash \overline{D}$ such that the Dirichlet boundary condition (1.7) holds on ∂D and that the scattered field u^{sc} satisfies Kupradze's radiation condition (1.8).

We refer to the monograph [21] for a comprehensive treatment of the boundary value problems of elasticity, including the boundary conditions of the third and fourth kinds. It is well-known that (see [21]) the direct scattering problem admits one solution $u \in C^2(\mathbb{R}^3 \backslash \overline{D})^3 \cap C^1(\mathbb{R}^3 \backslash D)^3$ if ∂D is C^2-smooth, while $u \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{D})^3$ if ∂D is Lipschitz.

1.2 Inverse elastic scattering problems

We are interested in the following inverse problems arising from elastic scattering.

(IP): Determine the shape of the scatterer D from the knowledge of the transversal far-field pattern $u_s^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^2$ corresponding to one or more incident plane shear waves at a fixed frequency.

(IP'): Determine ∂D from the longitudinal far-field pattern $u_p^\infty(\hat{x})$ for all $\hat{x} \in \mathbb{S}^2$ associated with all incident plane pressure waves at a fixed frequency.

There is already a vast literature on inverse elastic scattering problems using the full far-field pattern u^∞. We refer to the first uniqueness result proved in [12], the sampling type methods for impenetrable elastic bodies developed in [1, 2] and those for penetrable ones in [4, 25]. Note that in the above works, not only the pressure part of far-field pattern for all plane shear and pressure waves are needed, but also the shear part of far-field pattern. The aim of this paper is to reduce these measurement data to only the S- or P-part of the far-field pattern over all directions of measurement corresponding to the same type of plane elastic waves. We will study uniqueness issues and inversion algorithms for both (IP) and (IP').

The first uniqueness results using only one type of elastic waves was proved in [11] by D. Gintides and M. Sini. The authors proved that a C^4-smooth obstacle can be uniquely determined from the S-part of the far-field pattern corresponding to all incident plane pressure (or shear) waves. Moreover, the same uniqueness result remains valid using the shear part of the far-field pattern. This shows that any of the two different types of waves is enough to detect obstacles at a fixed frequency. The arguments in [11], which are applicable for both the two and three dimensions and also for different boundary or transmission conditions, mainly rely on the asymptotic analysis, near the boundary of the obstacle, of the pressure and shear parts of reflected solutions when the P-part or S-part of the fundamental solution to the Navier equation (1.1) is taken as an incident field. This analysis requires the C^4-smoothness assumption mentioned above. We also refer to [10] for a MUSIC type algorithm applied to the detection of point-like scatterers using only one type of scattered elastic waves. However, apart from the inversion scheme proposed in [11], no inversion algorithms have been proposed and tested for identifying an extended obstacle using one type of elastic waves.

In the first half of this paper, we present a new uniqueness proof to (IP) for C^2-smooth obstacles, following Isakov's idea of using singular solutions (see [15]). Since only the S-part of scattered fields can be reconstructed from the transversal far-field pattern, a boundary condition (see (2.16) or (2.31)) will be derived in order to couple the incident shear wave and the S-part of scattered waves on ∂D. This shows some kind of decoupling of the S-part of the scattered waves from the P-parts. Based on this observation, our proof seems more straightforward than the arguments used in [11] and can be extended to Lipschitz scatterers as well as the fourth kind boundary conditions. Moreover, we prove that a ball or a convex polyhedron can be uniquely identified from the S-part of the far-field pattern corresponding to only one incident shear wave. However, our approach (essentially the boundary condition (2.31)) is only valid for problem (IP) in 3D and cannot be generalized to problem (IP'); see Remarks 2.4 and 2.5 for a brief discussion of what goes wrong in these cases.

In the second half, we adapt the factorization method to recover ∂D from the scattered S-waves (resp. P-waves) for all incident plane shear (resp. pressure) waves. In particular, the factorization method also implies some uniqueness results provided ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D. It is well known that such eigenvalues form a discrete set with the only accumulating point at infinity. Our numerical experiments demonstrate satisfactory results from the S-part or P-part of the far-field pattern compared to the reconstruction from the full far-field pattern.

2 Uniqueness using S-part of far-field pattern

Concerning the regularity of the boundary ∂D, it is supposed that either ∂D is of class C^2 or D is a convex polyhedron defined as below.

Definition 2.1. *A scatterer $D \subset \mathbb{R}^3$ is called a convex polyhedron if D is the intersection of a finite number of half spaces with connected, non-void and bounded interior.*

Note that the boundary of a convex polyhedron consists of a finite number of cells without any cracks. Here a cell is defined as the closure of an open connected subset of a two-dimensional plane. As a notation convention we shall employ the symbol

$$U(\cdot) = U(\cdot; d, q, D), \quad U = u^{sc}, u, u_p^\infty, u_s^\infty, u_p^{sc}, u_s^{sc},$$

to indicate the dependence of $U(\cdot)$ on the obstacle D, the incident direction d and the polarization q. In some cases we write $U(\cdot; d, q, D) = U(\cdot; d, q)$ for brevity. Here is our main result on the uniqueness of (IP).

Theorem 2.2. *If there are two scatterers D and $\tilde{D}$ such that*

$$u_s^\infty(\hat{x}; d, q) = \tilde{u}_s^\infty(\hat{x};\ d, q), \quad \textit{for all} \quad \hat{x},\ d,\ q \in \mathbb{S}^2,\ q \perp d, \tag{2.1}$$

then $D = \tilde{D}$. Moreover, if D and $\tilde{D}$ are both balls or convex polyhedral scatterers, then the relation

$$u_s^\infty(\hat{x}; d_0, q_0) = \tilde{u}_s^\infty(\hat{x}; d_0, q_0), \quad \textit{for all} \quad q_0 \perp d_0,\ \hat{x} \in \mathbb{S}^2, \tag{2.2}$$

with one incident direction $d_0 \in \mathbb{S}^2$ and one polarization $q_0 \in \mathbb{S}^2$ is enough to imply that $D = \tilde{D}$.

Theorem 2.2 will be proved in Section 2.1 for general C^2-smooth scatterers, in Section 2.2 for balls and in Section 2.3 for convex polyhedral scatterers. To prove Theorem 2.2, we need the fundamental solution (Green's tensor) to the Navier equation given by

$$\Pi(x, y) = \frac{k_s^2}{4\pi\omega^2} \frac{e^{ik_s|x-y|}}{|x-y|} I + \frac{1}{4\pi\omega^2} \operatorname{grad}_x \operatorname{grad}_x^\top \left[\frac{e^{ik_s|x-y|}}{|x-y|} - \frac{e^{ik_p|x-y|}}{|x-y|} \right]. \tag{2.3}$$

Let the vector $a \in \mathbb{S}^2$ be fixed. Denote by $G^{in}(x; y) = G^{in}(x; y, a)$ the shear part of $\Pi(x, y)a$, i.e., for $x \neq y$,

$$\begin{aligned} G^{in}(x; y, a) \quad &:= \quad \frac{1}{k_s^2} \operatorname{curl}_x \operatorname{curl}_x [\Pi(x, y)a] \\ &= \quad \frac{k_s^2}{4\pi\omega^2} \left[\frac{e^{ik_s|x-y|}}{|x-y|} I + \frac{1}{k_s^2} \operatorname{grad}_x \operatorname{grad}_x^\top \frac{e^{ik_s|x-y|}}{|x-y|} \right] a. \end{aligned} \tag{2.4}$$

In the sequel, we view $G^{in}(x; y, a)$ as an incident point source wave, and correspondingly, denote by $G^{sc}(x; y, a), G(x; y, a), G_s^\infty(\hat{x}; y, a)$ the scattered, total waves and the transversal far-field pattern associated with G^{in}, respectively. Since the function $G^{in}(x; y, a)$ satisfies (see e.g. [23])

$$\operatorname{curl}_x \operatorname{curl}_x G^{in}(x; y, a) - k_s^2 G^{in}(x; y, a) = \frac{k_s^2}{4\pi\omega^2} \delta(x - y)a,$$

for $x \neq y$, it is easy to check that

$$(\Delta^* + \omega^2) G^{in} = -\delta(x - y)a, \tag{2.5}$$

i.e., $G^{in}(x; y, a)$ is one of the Green's functions to the Navier equation (1.1). The relation (2.5) will be used in Section 2.1 below.

2.1 Uniqueness for a general scatterer

The aim of this section is to prove the first assertion of Theorem 2.2, i.e., the unique determination of a C^2-smooth scatterer using only the S-part of the far-field pattern for all incident shear waves. Our proof is based on the mixed reciprocity relation between the transversal far-field pattern $G_s^\infty(\hat{x}; y, a)$ corresponding to G^{in} and the S-part $u_s^{sc}(x; d, q)$ of the scattered field corresponding to u_s^{in}. The following Lemma 2.3 extends the mixed reciprocity relations of R. Potthast in acoustic and electromagnetic scattering (see [24]) to the elastic case.

Lemma 2.3. *For $y \in \mathbb{R}^3 \backslash \overline{D}$, we have*

$$q \cdot G^\infty(-d; y, a) = q \cdot G_s^\infty(-d; y, a) = a \cdot u_s^{sc}(y; d, q) \quad \text{for all} \quad q \perp d. \tag{2.6}$$

Proof. Since G^{sc} and u^{sc} both fulfill the Kupradze radiation condition, there holds

$$\int_{\partial D} G^{sc}(z) \cdot T_{\nu(z)} u^{sc}(z) - T_{\nu(z)} G^{sc}(z) \cdot u^{sc}(z) \, \mathrm{d}s(z) = 0, \tag{2.7}$$

where T_ν is the stress operator defined in (1.4). Note that in (2.7) we wrote $G^{sc}(z; y, a) = G^{sc}(z)$ and $u^{sc}(z; d, q) = u^{sc}(z)$ for simplicity. This notational rule also applies to the total fields $G(z), u(z)$ and the transversal far-field patterns $G_s^\infty(\hat{x}), u_s^\infty(\hat{x})$. From Betti's integral theorem, for radiating solutions $G^{sc} \in C^2(\mathbb{R}^3 \backslash \overline{D})^3 \cap C^1(\mathbb{R}^3 \backslash D)^3$ to the Navier equation, one can derive the integral representation

$$G^{sc}(x) = \int_{\partial D} [T_{\nu(z)} \Pi(x, z)]^\top G^{sc}(z) - \Pi(x, z) T_{\nu(z)} G^{sc}(z) \, \mathrm{d}s(z), \quad x \in \mathbb{R}^3 \backslash \overline{D}, \tag{2.8}$$

where $T_\nu \Pi = (T_\nu \Pi_1, T_\nu \Pi_2, T_\nu \Pi_3)$ with Π_j being the j-th column of Π. Letting $|x| \to \infty$ in (2.8) and using the definitions of u_p^∞ and u_s^∞ in (1.9), it follows that (see also [1])

$$G_p^\infty(\hat{x}) = \int_{\partial D} \left\{ [T_{\nu(z)} \, \{\hat{x}\hat{x}^\top e^{-ik_p \hat{x} \cdot z}\}]^\top G^{sc}(z) - \hat{x}\hat{x}^\top e^{-ik_p \hat{x} \cdot z} \, T_{\nu(z)} G^{sc}(z) \right\} \, \mathrm{d}s(z), \tag{2.9}$$

and

$$\begin{aligned} G_s^\infty(\hat{x}) \;\; = \;\; & \int_{\partial D} \big\{ [T_{\nu(z)} \{(I - \hat{x}\hat{x}^\top) e^{-ik_s \hat{x} \cdot z}\}]^\top G^{sc}(z) \\ & - (I - \hat{x}\hat{x}^\top) e^{-ik_s \hat{x} \cdot z} T_{\nu(z)} G^{sc}(z) \big\} \mathrm{d}s(z). \end{aligned} \tag{2.10}$$

Since $q \cdot d = 0$, we deduce from (2.9), (2.10) with $\hat{x} = -d$ that

$$q \cdot G_p^\infty(-d) = 0, \tag{2.11}$$

$$q \cdot G_s^\infty(-d) = \textstyle\int_{\partial D} \left\{ G^{sc}(z) \cdot T_{\nu(z)} \, [q e^{ik_s d \cdot z}] - q e^{ik_s d \cdot z} \cdot T_{\nu(z)} G^{sc}(z) \right\} \, \mathrm{d}s(z). \tag{2.12}$$

Combining (2.11), (2.12) and (2.7) gives

$$\begin{aligned} & q \cdot G^\infty(-d; y, a) \\ = \;\; & q \cdot G_s^\infty(-d; y, a) \\ = \;\; & \int_{\partial D} \left\{ G^{sc}(z) \cdot T_{\nu(z)} \, u(z; d, q) - u(z; d, q) \cdot T_{\nu(z)} G^{sc}(z) \right\} \mathrm{d}s(z). \end{aligned} \tag{2.13}$$

This proves the first identity in Lemma 2.3. Again using Betti's integral theorem, we have (cf. (2.8))

$$u^{sc}(x) = \int_{\partial D} \left\{ [T_{\nu(z)} \Pi(x, z)]^\top u^{sc}(z) - \Pi(x, z) T_{\nu(z)} u^{sc}(z) \right\} \, \mathrm{d}s(z), \quad x \in \mathbb{R}^3 \backslash \overline{D},$$

implying that

$$\begin{aligned}
& a\cdot u_s^{sc}(x;d,q)\\
= \;& a\cdot \frac{1}{k_s^2}\mathrm{curl}\,_x\mathrm{curl}\,_x\left\{u^{sc}(x;d,q)\right\}\\
= \;& a\cdot \frac{1}{k_s^2}\mathrm{curl}\,_x\mathrm{curl}\,_x\left\{\int_{\partial D}[T_{\nu(z)}\Pi(x,z)]^\top u^{sc}(z)-\Pi(x,z)T_{\nu(z)}u^{sc}(z)\,\mathrm{d}s(z)\right\}\\
= \;& \int_{\partial D}\left\{u^{sc}(z)\cdot T_{\nu(z)}G^{in}(x;z)-T_{\nu(z)}u^{sc}(z)\cdot G^{in}(x;z)\right\}\,\mathrm{d}s(z).
\end{aligned}$$

Moreover, applying Betti's second integral theorem to u^{in} and G^{in} in D yields

$$0=\int_{\partial D}\left\{u^{in}(z)\cdot T_{\nu(z)}G^{in}(x;z)-T_{\nu(z)}u^{in}(z)\cdot G^{in}(x;z)\right\}\,\mathrm{d}s(z),\quad x\in\mathbb{R}^3\backslash\overline{D}.$$

Adding up the previous two equalities with $x=y$, we arrive at

$$\begin{aligned}
a\cdot u_s^{sc}(y;d,q) &= \int_{\partial D}\left\{u(z)\cdot T_{\nu(z)}G^{in}(y;z)-T_{\nu(z)}u(z)\cdot G^{in}(y;z)\right\}\,\mathrm{d}s(z)\\
&= \int_{\partial D}\left\{u(z)\cdot T_{\nu(z)}G^{in}(z;y)-T_{\nu(z)}u(z)\cdot G^{in}(z;y)\right\}\,\mathrm{d}s(z), \qquad (2.14)
\end{aligned}$$

where the last equality sign follows from the symmetry of $G^{in}(z;y)$ in z and y. Combining (2.14) and (2.13), we find

$$\begin{aligned}
& q\cdot G_s^{\infty}(-d;y,a)-a\cdot u_s^{sc}(x;d,q)\\
= \;& \int_{\partial D}\left\{[T_{\nu(z)}\,u(z;d,q)]^\top\cdot G(z)-u(z;d,q)\cdot T_{\nu(z)}G(z)\right\}\mathrm{d}s(z)\\
= \;& 0.
\end{aligned}$$

This proves the second identity in (2.6). □

Our proof of the first assertion of Theorem 2.2 relies on a refinement of the arguments in [15, 18, 12] using singular solutions and the simplified version (see e.g. [23, Theorem 14.6]) using the mixed reciprocity relations. Note that in our proof only the S-part of scattered fields can be uniquely determined from the transversal far-field pattern.

Proof of Theorem 2.2 for a general obstacle. Let D and $\tilde{D}$ be the two rigid obstacles in Theorem 2.2 satisfying (2.1). Let Ω denote the unbounded connected component of $\mathbb{R}^3\backslash\overline{D\cup\tilde{D}}$, and define the incident point source waves $G^{in}(x;y,a)$ as in (2.4) for $y\in\Omega$, with some polarization vector $a\in\mathbb{S}^2$ to be determined later. From the identity (2.1) and the Rellich lemma it follows that

$$a\cdot u_s^{sc}(y;d,q)=a\cdot\tilde{u}_s^{sc}(y;d,q),\quad \text{for all}\quad d,q\in\mathbb{S}^2,\ d\perp q,\ y\in\Omega,$$

which, combined with the reciprocity relation in Lemma 2.3, gives

$$q\cdot G_s^{\infty}(-d;y,a)=q\cdot\tilde{G}_s^{\infty}(-d;y,a),\quad \text{for all}\quad d,q\in\mathbb{S}^2,d\perp q,\ y\in\Omega.$$

Together with the relation $d\cdot G_s^{\infty}(-d;y,a)=d\cdot\tilde{G}_s^{\infty}(-d;y,a)=0$, the previous identity implies

$$G_s^{\infty}(\hat{x};y,a)=\tilde{G}_s^{\infty}(\hat{x};y,a),\quad \text{for all}\quad \hat{x}\in\mathbb{S}^2,y\in\Omega.$$

Again applying the Rellich lemma, we have the coincidence of the shear parts of G^{sc} and $\tilde{G}^{sc}$,

$$G_s^{sc}(z; y, a) = \tilde{G}_s^{sc}(z; y, a), \quad \text{for all} \quad z, y \in \Omega. \tag{2.15}$$

Since the compressional part G_p^{sc} of G^{sc} is irrotational and the total field $G(z; y) = G^{in}(z; y) + G_p^{sc}(z; y) + G_s^{sc}(z; y) = 0$ on ∂D for all $y \in \mathbb{R}^3 \backslash \overline{D}$, we get

$$\begin{aligned} \nu(z) \cdot \operatorname{curl}_z(G^{in}(z; y) + G_s^{sc}(z; y)) &= \nu(z) \cdot \operatorname{curl}_z G(z; y) \\ &= -\mathrm{Div}_z\,(\nu(z) \times G(z; y)) \\ &= 0 \end{aligned} \tag{2.16}$$

for $z \in \partial D$, where $\mathrm{Div}(\cdot)$ stands for the surface divergence operator which is well-defined on the C^2-smooth boundary ∂D. Analogously, there holds $\nu(z) \cdot \operatorname{curl}_z(G^{in}(z; y) + \tilde{G}_s^{sc}(z; y)) = 0$ for $z \in \partial \tilde{D}$.

Assuming that $D \neq \tilde{D}$, we next derive a contradiction from (2.15) and the boundary condition (2.16). Without loss of generality, we may choose a point $y^* \in \partial\Omega$ and a vector $a^* = a^*(y^*) \in \mathbb{S}^2$ such that $y^* \in \partial D$, $y^* \notin \partial \tilde{D}$ and $a^* \times \nu(y^*) \neq 0$. In particular, for sufficiently large $N \in \mathbb{N}^+$, we may assume $y_n = y^* + a^*(y^*)/n \in \Omega, n \geq N$. Choose the polarization vector $a \in \mathbb{S}^2$ such that $a \cdot (\nu(y^*) \times a^*) \neq 0$. Taking $z = y^*, y = y_n$ in (2.15) and setting $\nu = \nu(y^*)$, we find

$$\nu(y^*) \cdot \operatorname{curl}_z(G_s^{sc}(z; y_n))|_{z=y^*} = \nu(y^*) \cdot \operatorname{curl}_z(\tilde{G}_s^{sc}(z; y_n))|_{z=y^*}, \quad \text{for all} \quad n \geq N. \tag{2.17}$$

On the one hand, the right hand side of (2.17) is uniformly bounded, due to the well-posedness of the forward scattering problem and the fact that $y^* \in \mathbb{R}^3 \backslash \overline{\tilde{D}}$. On the other hand, it follows from (2.16) and the definition of $G^{in}(z, y)$ that

$$\begin{aligned} \nu(y^*) \cdot \operatorname{curl}_z(G_s^{sc}(z; y_n))|_{z=y^*} &= -\nu(y^*) \cdot \operatorname{curl}_z(G^{in}(z; y_n))|_{z=y^*} \\ &= \frac{k_s^2}{4\pi\omega^2}\, \nu(y^*) \cdot (a \times \nabla_z \frac{e^{ik_s|z-y_n|}}{|z-y_n|})|_{z=y^*} \\ &= \frac{k_s^2}{4\pi\omega^2}\, (\nu(y^*) \times a) \cdot [\nabla_z \frac{e^{ik_s|z-y_n|}}{|z-y_n|}]|_{z=y^*} \\ &= \frac{k_s^2}{4\pi\omega^2} (ik_s n - n^2) \exp(ik_s/n)\, (\nu(y^*) \times a) \cdot a^* \end{aligned}$$

which tends to infinity as $n \to \infty$, since $(\nu(y^*) \times a) \cdot a^* \neq 0$. This contradiction implies that $D = \tilde{D}$. $\square$

Remark 2.4. *Our arguments do not work using the longitudinal far-field pattern of plane shear waves. To see this, we need the compressional part of* $\Pi(x, y)a$ *given by*

$$H^{in}(x; y, a) \quad = \quad -\frac{1}{k_p^2} \operatorname{grad}_x \operatorname{div}_x [\Pi(x, y)a] = -\frac{1}{4\pi\omega^2} \operatorname{grad}_x \operatorname{grad}_x^\top \frac{e^{ik_p|x-y|}}{|x-y|} a, \quad x \neq y.$$

Similarly, denote by $H_s^\infty(x; y, a), H_p^\infty(\hat{x}; y, a)$ *the transversal and longitudinal far-field pattern associated with the incident wave* H^{in}. *Analogously to Lemma 2.3, one can prove the reciprocity relation*

$$q \cdot H^\infty(-d; y, a) = q \cdot H_s^\infty(-d; y, a) = a \cdot u_p^{sc}(y; d, q), \quad y \in \mathbb{R}^3 \backslash \overline{D}, \tag{2.18}$$

for all $d \perp q$. *However, since* $\operatorname{curl}_x H^{in}(x; z) = 0$ *for* $x \neq z$ *we cannot generalize the arguments from Section 2.1 to the present case by employing the boundary condition (2.16) with* G_s^{sc} *replaced by* H_s^{sc}. *Further, we point out that our proof cannot be applied to the second (Neumann) or third kind boundary conditions, even in the case of two-dimensional elastic scattering.*

Remark 2.5. *Our approach for proving the first assertion of Theorem 2.2 cannot apply to incident plane pressure waves. Given the incident pressure wave u_p^{in} defined in (1.6), we can establish the following mixed reciprocity relations*

$$d \cdot G^\infty(-d; y, a) = d \cdot G_p^\infty(-d; y, a) = a \cdot u_s^{sc}(y; d), \tag{2.19}$$
$$d \cdot H^\infty(-d; y, a) = d \cdot H_p^\infty(-d; y, a) = a \cdot u_p^{sc}(y; d), \tag{2.20}$$

for all $d \perp q$, where u_s^{sc} and u_p^{sc} denote the S-part and P-part of the scattered field $u^{sc}(y; d)$ corresponding to u_p^{in}. Again the boundary condition (2.16) cannot be employed with the transversal part G_s^{sc} replaced by the irrotational longitudinal part G_p^{sc} or H_p^{sc}.

Note that the mixed reciprocity relations (2.6), (2.18), (2.19) and (2.20) remain true for other boundary conditions and penetrable scatterers. It is worth mentioning the more general identity established in [3, Theorem 7] between full far-field patterns, the proof of which is based on the reciprocity relation for two point source incidences. Our Lemma 2.3 provides a more straightforward proof of these mixed reciprocity relations. See also [3, 7, 8] for the reciprocity principles due to two incident plane elastic waves.

2.2 Uniqueness for balls

Continue of the proof of Theorem 2.2. To prove the second assertion of Theorem 2.2 for balls, we will follow Kress' arguments from [19] for proving uniqueness in inverse electromagnetic scattering by perfectly conducting balls. Let $\mathbb{Q}$ be a rotation matrix in $\mathbb{R}^3$. We have the following relation between $u^\infty(\hat{x}; d, q, D)$ and $u^\infty(\hat{x}; d, q, \mathbb{Q}D)$ (see [22, Section 5])

$$\mathbb{Q}\, u_\alpha^\infty(\hat{x}; d, q, D) = u_\alpha^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d, \mathbb{Q}q, \mathbb{Q}D), \quad \text{for all} \quad \hat{x}, d, q \in \mathbb{S}^2,\ q \perp d,\ \alpha = p \text{ or } s. \tag{2.21}$$

If D is a ball centered at the origin, then the relation (2.21) with $\alpha = s$ reduces to

$$\mathbb{Q}\, u_s^\infty(\hat{x}; d, q, D) = u_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d, \mathbb{Q}q, D), \quad \text{for all} \quad \hat{x}, d, q \in \mathbb{S}^2,\ d \perp q. \tag{2.22}$$

Letting D and $\tilde{D}$ be the two balls given in Theorem 2.2, one can conclude from the explicit expression of the S-part of the scattered field that $u_s^{sc}(x; d, q)$ resp. $\tilde{u}_s^{sc}(x; d, q)$ can be analytically extended into the interior of D resp. $\tilde{D}$ with the exception of its center; see Appendix for the proof in elasticity. Since $u_s^{sc}(x; d, q)$ and $\tilde{u}_s^{sc}(x; d, q)$ are radiating solutions to the Helmholtz equation, by the Rellich lemma the centers of D and $\tilde{D}$ must coincide. Without loss of generality we may assume the center is located at the origin. Thus $\mathbb{Q}D = D$ and $\mathbb{Q}\tilde{D} = \tilde{D}$ for any rotation matrix $\mathbb{Q}$. We claim that the identity (2.2) with one incident direction d_0 and one polarization direction q_0 implies the relation (2.1) for all incident and polarization directions. Therefore, the uniqueness for balls with one incident wave follows directly from the first assertion in Theorem 2.2. In fact, given $d_1, q_1 \in \mathbb{S}^2$ such that $d_1 \perp q_1$, there exists a rotation $\mathbb{Q}$ satisfying either $\mathbb{Q}q_0 = q_1, \mathbb{Q}d_0 = d_1$ or $\mathbb{Q}q_0 = -q_1, \mathbb{Q}d_0 = d_1$. Applying the rotation $\mathbb{Q}$ to both sides of (2.2) and making use of (2.22) for D and $\tilde{D}$, we obtain

$$u_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d_0, \mathbb{Q}q_0, D) = \tilde{u}_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d_0, \mathbb{Q}q_0, \tilde{D}), \quad \text{for all} \quad \hat{x} \in \mathbb{S}^2,$$

which implies

$$u_s^\infty(\mathbb{Q}\hat{x}; d_1, q_1, D) = \tilde{u}_s^\infty(\mathbb{Q}\hat{x}; d_1, q_1, \tilde{D}), \quad \text{for all} \quad \hat{x} \in \mathbb{S}^2. \tag{2.23}$$

Note that u_s^∞ and $\tilde{u}_s^\infty$ depend linearly on the polarization. By the arbitrariness of $\hat{x}, d_1, q_1 \in \mathbb{S}^2$, we arrive at the identity (2.1). Therefore, we obtain $D = \tilde{D}$ as a consequence of the uniqueness result in Theorem 2.2 for all incident and polarization directions. This completes the proof of Theorem 2.2 for balls. □

The well-known Karp's theorem in two-dimensional acoustics states that, a scatterer is a disc if the far-field pattern only depends on the angle between the incident and observation directions; see e.g. [6, Chapter 5.1] in acoustics and [6, Chapter 7.1] in electromagnetics. The elastodynamic analogue of Karp's theorem was proved in [22] by applying the uniqueness results in inverse elastic scattering with an infinite number of incident waves. Following [22], we next prove another elastodynamic analogue of Karp's theorem using only the transversal far-field pattern.

Corollary 2.6. *Suppose that $u_s^\infty(\hat{x}; d_0, p_0, D)$ is the S-part of the far-field pattern associated with the incident shear wave (1.5) with $d = d_0, q = q_0$. Then, D is a ball if there holds for any rotational matrix $\mathbb{Q}$ that*

$$\mathbb{Q}\, u_s^\infty(\hat{x}; d_0, q_0, D) = u_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d_0, \mathbb{Q}q_0, D), \quad \textit{for all} \quad \hat{x} \in \mathbb{S}^2. \tag{2.24}$$

Proof. It follows from (2.22) and (2.24) that

$$u_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d_0, \mathbb{Q}q_0, \mathbb{Q}D) = u_s^\infty(\mathbb{Q}\hat{x}; \mathbb{Q}d_0, \mathbb{Q}q_0, D), \quad \text{for all} \quad \hat{x} \in \mathbb{S}^2. \tag{2.25}$$

With the similar arguments used to derive (2.23), we deduce from (2.25) that

$$u_s^\infty(\hat{x}; d, q, \mathbb{Q}D) = u_s^\infty(\hat{x}; d, q, D), \quad \text{for all} \quad \hat{x}, q, d \in \mathbb{S}^2,\ q \perp d. \tag{2.26}$$

Applying Theorem 2.2, we conclude that $\mathbb{Q}D = D$ for all rotational matrices $\mathbb{Q}$. Hence D is a ball. □

Remark 2.7. *The results in Section 2.2 are also true for other boundary conditions using P- or S-part of the scattered waves corresponding to P- or S-incident waves, since the arguments are based on the corresponding results using many incident waves.*

2.3 Uniqueness for convex polyhedrons

Continue of the proof of Theorem 2.2. Suppose that (2.2) holds for two different convex Lipschitz polyhedral obstacles D and $\tilde{D}$. Without loss of generality, we may always assume that there exists a corner point $A \in \mathbb{R}^3$ of ∂D such that $A \notin \partial\tilde{D}$. Denote by $\tilde{\Gamma}_j$, $j = 1, 2, 3$ three cells of ∂D meeting at A, and by Γ_j the extension of $\tilde{\Gamma}_j$ to $\mathbb{R}^3 \backslash \overline{\tilde{D}}$. Obviously, each cell $\tilde{\Gamma}_j$ can be extended to infinity in $\mathbb{R}^3 \backslash \overline{\tilde{D}}$ due to the convexity of both D and $\tilde{D}$. Since the total field $u = u_s^{in} + u_s^{sc} + u_p^{sc} = 0$ on ∂D, there holds the boundary condition

$$\nu_j \cdot \operatorname{curl} u = -\operatorname{Div}(\nu_j \times u) = 0 \quad \text{on} \quad \tilde{\Gamma}_j,\ j = 1, 2, 3, \tag{2.27}$$

with ν_j being the normal direction of Γ_j. Note that the differential operators in (2.27) make sense, because $\tilde{\Gamma}_j$ is flat so that u is smooth up to the boundary except for a finite number of corner points and edges. Analogously to (2.16), we have

$$\nu_j \cdot \operatorname{curl}(u_s^{in} + u_s^{sc}) = 0 \quad \text{on} \quad \tilde{\Gamma}_j,\ j = 1, 2, 3, \tag{2.28}$$

since $\operatorname{curl} u_p^{sc} = 0$ in $\mathbb{R}^3 \backslash \overline{D}$. By the Rellich lemma, the relation (1.9) yields $u_s^{sc}(x) = \tilde{u}_s^{sc}(x)$ for all $x \in \Omega := \mathbb{R}^3 \backslash \overline{(D \cup \tilde{D})}$. In particular, we have $\operatorname{curl} u_s^{sc} = \operatorname{curl} \tilde{u}_s^{sc}$ on $\tilde{\Gamma}_j \backslash \overline{\tilde{D}}$ using standard elliptic regularity of $u_s^{sc}(x)$ and $\tilde{u}_s^{sc}(x)$. Thus, the identity (2.28) implies the relation

$$\nu_j \cdot \operatorname{curl}(u_s^{in} + \tilde{u}_s^{sc}) = 0 \quad \text{on} \quad \tilde{\Gamma}_j \backslash \overline{\tilde{D}},\ j = 1, 2, 3, \tag{2.29}$$

which combined with the analyticity of the function $U := u^{in} + \tilde{u}_s^{sc}$ in $\mathbb{R}^3\backslash\overline{\tilde{D}}$ gives

$$\nu_j \cdot \mathrm{curl}\,(u_s^{in} + \tilde{u}_s^{sc}) = 0 \quad \text{on} \quad \Gamma_j,\ j = 1,2,3,$$

that is,

$$ik_s\nu_j \cdot (d_0 \times q_0)\exp(ik_s x \cdot d_0) + \nu_j \cdot \mathrm{curl}\,\tilde{u}_s^{sc}(x) = 0 \quad \text{on} \quad \Gamma_j,\ j = 1,2,3. \tag{2.30}$$

Letting $|x| \to +\infty$ in (2.30) for $x \in \Gamma_j$, we obtain $\nu_j \cdot (d_0 \times q_0) = 0, j = 1,2,3,$ since $\mathrm{curl}\,\tilde{u}_s^{sc}(x)$ decays uniformly in all directions (see the radiation condition in (1.8)). Noting that ν_j are three linearly independent vectors, we get $d_0 \times q_0 = 0$. However, this is impossible because $d_0 \perp q_0$. This contradiction gives $D = \tilde{D}$. □

Remark 2.8. *We have no uniqueness for non-convex polyhedrons. If D and $\tilde{D}$ are not necessarily convex polyhedrons, one can only conclude that the convex hulls of D and $\tilde{D}$ coincide. For global uniqueness results within non-convex polyhedral obstacles, we refer to [9] by J. Elschner and M. Yamamoto using the full far-field patten of one or several incident plane elastic waves. Their proofs were based on the reflection principle for the Navier equation under the third or fourth kind boundary conditions. However, there seems no reflection principle for the Navier equation under the Dirichlet boundary condition.*

To sum up, our uniqueness proofs for (IP) are essentially based on the identity

$$\nu \cdot \mathrm{curl}\,(u_s^{in} + u_s^{sc}) = -\mathrm{Div}(\nu \times u) = 0 \quad \text{on} \quad \partial D. \tag{2.31}$$

One may observe further that the relation (2.31) is still true under the fourth kind boundary conditions (see e.g. [21])

$$\nu \times u = 0,\ \nu \cdot Tu = 0 \quad \text{on} \quad \partial D, \tag{2.32}$$

where T is the stress operator given in (1.4). Hence, we have

Corollary 2.9. *The uniqueness results in Theorem 2.2 remain valid if the Dirichlet boundary condition (1.7) is replaced by the fourth kind boundary conditions (2.32).*

Relying on the boundary condition (2.31), some existing numerical methods, e.g., linear sampling method [6], probe method [14] or singular sources method [24] can be utilized to recover the shape of a rigid scatterer from only the transversal far-field pattern associated with all incident shear waves. We next adapt the factorization method established in [16] (see also the monograph [17]) to this case. Moreover, the factorization method also allows us to handle the problem (IP') using only pressure waves.

3 Factorization method

We first review the F^*F-method in inverse elastic scattering problems (see [1]) involving the full far-field pattern, and then use a modified version to reconstruct ∂D using only one part of the far-field pattern. In this section, the boundary ∂D of $D \subset \mathbb{R}^3$ is allowed to be Lipschitz, and D may consist of several components.

For $g(d) \in L^2(\mathbb{S}^2)^3$, $d \in \mathbb{S}^2$, there holds the decomposition

$$g(d) = g_s(d) + g_p(d), \quad g_s(d) := d \times g(d) \times d, \quad g_p(d) := (g(d) \cdot d)\, d, \tag{3.1}$$

where $g_s(d)$ belongs to the space

$$L_s^2(\mathbb{S}^2) := \{g_s : \mathbb{S}^2 \to \mathbb{C}^3 : g_s(d) \cdot d = 0,\ |g_s| \in L^2(\mathbb{S}^2)\} \tag{3.2}$$

of transversal vector fields on $\mathbb{S}^2$, while $g_p(d)$ belongs to the space

$$L_p^2(\mathbb{S}^2) := \{g_p : \mathbb{S}^2 \to \mathbb{C}^3 : g_p(d) \times d = 0,\ |g_p| \in L^2(\mathbb{S}^2)\}$$

of longitudinal vector fields on $\mathbb{S}^2$. For $g \in L^2(\mathbb{S}^2)^3$, introduce the incident field

$$v_g^{in}(x) := \int_{\mathbb{S}^2} \left[g_s(d)e^{ik_s x\cdot d} + g_p(d)e^{ik_p x\cdot d}\right] \mathrm{d}s(d).$$

The far-field pattern v_g^∞ corresponding to the incident wave v_g^{in} defines the far-field operator F from $L^2(\mathbb{S}^2)^3$ into itself by $Fg = v_g^\infty$. Denote by $v_{g,s}^\infty \in L_s^2(\mathbb{S}^2)$ and $v_{g,p}^\infty \in L_p^2(\mathbb{S}^2)$ the S-part and P-part of v_g^∞, which are defined in the same way as in (3.1). The following properties of F have been derived in [1].

Lemma 3.1. **(i)** *The far-field operator F is compact, normal with dense range in $L^2(\mathbb{S}^2)^3$, and the scattering matrix $I + \frac{i}{2\pi}F$ is unitary. Here I denotes the identity operator.*

(ii) *If ω^2 is not the Dirichlet eigenvalue of the operator $-\Delta^*$ in D, then F is injective and its normalized eigenfunctions form a complete orthonormal system in $L^2(\mathbb{S}^2)^3$.*

Let the Herglotz operator $H : L^2(\mathbb{S}^2)^3 \to H^{1/2}(\partial D)^3$ be defined by $Hg := v_g^{in}(x)|_{\partial D}$. Then, the adjoint operator $H^* : H^{-1/2}(\partial D)^3 \to L^2(\mathbb{S}^2)^3$ of H is given by

$$H^*\varphi(\hat{x}) := \int_{\partial D} \left[\varphi_s(y)e^{-ik_s y\cdot\hat{x}} + \varphi_p(y)e^{-ik_p y\cdot\hat{x}}\mathrm{d}s(y)\right], \quad \hat{x} \in \mathbb{S}^2,$$

which is, by our normalization, just the far-field pattern of the function

$$w(x) = \int_{\partial D} \Pi(x,y)\varphi(y)\,\mathrm{d}s(y), \quad x \notin D,$$

where $\Pi(x,y)$ is the Green's tensor to the Navier equation (see (2.3)). Note that, for some non-trivial vector $a \in \mathbb{S}^2$, the far-field pattern of the function $x \to \Pi(x,y)a$ is given by

$$\Pi_y^\infty(\hat{x}) = e^{-ik_s\hat{x}\cdot y}\left[\hat{x} \times (a \times \hat{x})\right] + e^{-ik_p\hat{x}\cdot y}\left(\hat{x}\cdot a\right)\hat{x}, \quad \text{for all} \quad y \in \mathbb{R}^3. \tag{3.3}$$

Define the data-to-pattern operator $G : H^{1/2}(\partial D)^3 \to L^2(\mathbb{S}^2)^3$ by $f \to v^\infty$, where v^∞ is the far-field pattern of the radiating solution v which satisfies the Navier equation (1.1) in $\mathbb{R}^3\backslash\overline{D}$ with the boundary data $f \in H^{1/2}(\partial D)^3$. With these definitions, we have the decomposition

$$Fg = -GHg, \quad H^*(\varphi) = G(w|_{\partial D}) = GS(\varphi), \tag{3.4}$$

where $S :\ H^{-1/2}(\partial D)^3 \to H^{1/2}(\partial D)^3$ denotes the classical single-layer operator

$$(S\varphi)(x) = \int_{\partial D} \Pi(x,y)\varphi(y)\,\mathrm{d}s(y), \quad x \in \partial D.$$

From (3.4), it follows the factorization

$$F = -GS^*G^*. \tag{3.5}$$

The sampling method developed in [1] is based on the factorization (3.5) combined with some properties of the single-layer operator S (see [1, Lemmas 6.1, 6.2]), which extends Kirsch's results [16] from acoustic scattering to the elastic case in 3D. It is seen from [1, Section 6] that all the assumptions of [17, Theorem 1.23] are fulfilled, so that we have (see [1, Theorem 6.3])

Lemma 3.2. *If ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, then the ranges of G and $(F^*F)^{1/4}$ coincide.*

Further, it is proved in [1, Theorem 6.4] that the function $\Pi_y^\infty(\hat{x})$ given in (3.3) belongs to the range of G if and only if $y \in D$. Thus, by Lemma 3.2 we can characterize the scatterer D in terms of the range of $(F^*F)^{1/4}$. Using the orthogonal system of eigenfunctions of F, it follows from Picard's theorem that

Theorem 3.3. *If ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, then*

$$y \in D \quad \Longleftrightarrow \quad W(y) := \left[\sum_{n=1}^{\infty} \frac{|(g_n, \Pi_y^\infty)_{L^2(\mathbb{S}^2)}|^2}{|\eta_n|} \right]^{-1} > 0, \tag{3.6}$$

where $\eta_n \in \mathbb{C}$ denote the eigenvalues of F with the corresponding orthonormal eigenfunctions $g_n \in L^2(\mathbb{S}^2)^3$, and $(\cdot,\cdot)_{L^2(\mathbb{S}^2)}$ denotes the usual inner product in the space $L^2(\mathbb{S}^2)^3$.

Remark 3.4. *Analogously to the factorization method in acoustics, the eigensystem (η_n, g_n) in Theorem 3.3 can be replaced by the eigensystem of $F_\#$ defined by*

$$F_\# := |\mathrm{Re}\, F| + |\mathrm{Im}\, F|, \quad \mathrm{Re}\, F := \frac{1}{2}[F + F^*], \quad \mathrm{Im}\, F := \frac{1}{2i}[F - F^*].$$

This is mainly due to the inequality

$$\frac{1}{\sqrt{2}}[\,|\mathrm{Re}\,\eta_n| + |\mathrm{Im}\,\eta_n|\,] \le |\eta_n| \le |\mathrm{Re}\,\eta_n| + |\mathrm{Im}\,\eta_n|.$$

We note that the eigensystem of F used in Theorem 3.3 are determined by both the P-part and S-part of the far-field pattern for all incident pressure and shear plane waves. Relying on the previous analysis, we now turn to the study of the factorization method for (IP) and (IP') where the incident fields consist of plane shear or pressure waves only.

Introduce the orthogonal projection operator $\mathcal{P}_s : L^2(\mathbb{S}^2)^3 \to L^2_s(\mathbb{S}^2)^3$, where $L^2_s(\mathbb{S}^2)^3$ is given in (3.2), i.e. $\mathcal{P}_s\, g(d) = g_s(d)$. The adjoint $\mathcal{P}_s^* : L^2_s(\mathbb{S}^2)^3 \to L^2(\mathbb{S}^2)^3$ of $\mathcal{P}_s$ is just the inclusion from $L^2_s(\mathbb{S}^2)^3$ to $L^2(\mathbb{S}^2)^3$. Therefore, the operator $F_s := \mathcal{P}_s\, F\, \mathcal{P}_s^*$, which maps $L^2_s(\mathbb{S}^2)^3$ to $L^2_s(\mathbb{S}^2)^3$, is the projection of the restriction of F to $L^2_s(\mathbb{S}^2)^3$. By (3.5), it has the factorization

$$F_s := \mathcal{P}_s\, F\, \mathcal{P}_s^* = -(\mathcal{P}_s G)\, S^*\, (\mathcal{P}_s G)^*. \tag{3.7}$$

In contrast to F the operator F_s fails to be normal. Therefore, Theorem 1.23 of [17] is not applicable. We further note that the characterization (3.6) is essentially based on the normality of the far-field operator F and the unitarity of the scattering operator $I + \frac{i}{2\pi}F$. Instead of [17, Theorem 1.23], we now apply the range identity [17, Theorem 2.15] to the operator F_s.

Lemma 3.5. *If ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, then the ranges of $\mathcal{P}_s G$ and $F_{s\#}^{1/2}$ coincide, where $F_{s\#} := |\mathrm{Re}\, F_s| + |\mathrm{Im}\, F_s|$.*

Proof. We need to justify all the conditions in [17, Theorem 2.15]. Obviously, the operator

$$\mathcal{P}_s G : H^{1/2}(\partial D)^3 \to L^2_s(\mathbb{S}^2)^3$$

is compact with dense range, since the data-to-pattern operator $G : H^{1/2}(\partial D)^3 \to L^2(\mathbb{S}^2)^3$ is compact with dense range. We collect the following properties of the single-layer operator S from [1, Section 6].

(a) $\operatorname{Im} S = \frac{1}{2i}(S - S^*)$ is non-negative, that is,

$$\operatorname{Im}(S\varphi, \varphi)_{L^2(\partial D)} \geq 0 \quad \text{for all } \varphi \in C^\alpha(\partial D)^3 .$$

More generally, it holds that

$$\operatorname{Im}\langle \varphi, S\varphi \rangle \leq 0 \quad \text{for all } \varphi \in H^{-1/2}(\partial D)^3 ,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $\langle H^{-1/2}(\partial D)^3, H^{1/2}(\partial D)^3 \rangle$.

(b) The strict inequality in the assertion (a) holds for all $\varphi \neq 0$ provided ω^2 is not an interior Dirichlet eigenvalue.

(c) Let S_i be the single-layer operator corresponding to $\omega = i$. Then S_i is self-adjoint and coercive; that is, there exists $c > 0$ such that

$$\langle \varphi, S_i\varphi \rangle \geq c\|\varphi\|^2_{H^{-1/2}(\partial D)} \quad \text{for all } \varphi \in H^{-1/2}(\partial D)^3 .$$

Furthermore, $S - S_i$ is compact from $H^{-1/2}(\partial D)^3$ into $H^{1/2}(\partial D)^3$.

Now, the range identity of Theorem 2.15 of [17] yields that the ranges of $\mathcal{P}_s G$ and $F_{s\#}^{1/2}$ coincide. $\square$

To characterize the scatterer D in terms of the operator F_s, we need the following lemma.

Lemma 3.6. *Let the function $\Pi_y^\infty(\hat{x})$ be given as in (3.3). The function $\mathcal{P}_s(\Pi_y^\infty)$ belongs to the range of $\mathcal{P}_s\, G$ if and only if $y \in D$.*

Proof. If $y \in D$, then the trace of the function $x \to \Pi(x, y)a$ on ∂D belongs to $H^{1/2}(\partial D)^3$. Thus $\mathcal{P}_s(\Pi_y^\infty) = \mathcal{P}_s G(f)$ with $f = \Pi(x, y)a|_{x\in\partial D}$. Assume that $\mathcal{P}_s(\Pi_y^\infty) = \mathcal{P}_s G(\tilde{f})$ for some $\tilde{f} \in H^{1/2}(\partial D)^3$, that is, the S-part $\Pi_{y,s}^\infty$ of Π_y^∞ coincides with the S-part v_s^∞ of v^∞, where v^∞ is the far field pattern of the radiating solution v in $H^1_{loc}(\mathbb{R}^3\backslash\overline{D})^3$ with $v = f$ on ∂D. Denote by $[\Pi(x, y)a]_s$ and v_s the shear parts of $\Pi(x, y)a$ and v, respectively. If $y \notin D$, it follows from the Rellich identity and the unique continuity of solutions of the Helmholtz equation that $v_s(x) = [\Pi(x, y)a]_s$ for all $x \in \mathbb{R}^3\backslash\{\overline{D} \cup \{y\}\}$. Therefore,

$$\operatorname{curl}_x [\Pi(x, y)a]_s = \operatorname{curl} v_s = \operatorname{curl} v \in L^2_{loc}(\mathbb{R}^3\backslash\overline{D})^3. \tag{3.8}$$

However, it follows from (2.4) that

$$\operatorname{curl}_x [\Pi(x, y)a]_s = \operatorname{curl}_x G^{in}(x; y, a) \sim \mathcal{O}(|x - y|^{-2}), \quad \text{as} \quad x \to y \quad \text{in} \quad \mathbb{R}^3\backslash\{\overline{D} \cup \{y\}\},$$

which contradicts (3.8). Thus $y \in D$. $\square$

Combining the previous two lemmas, we get

Theorem 3.7. *If ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, then*

$$y \in D \iff W_s(y) := \left[\sum_{n=1}^{\infty} \frac{|(g_n, \Pi_{y,s}^\infty)_{L^2(\mathbb{S}^2)}|^2}{\eta_n}\right]^{-1} > 0, \tag{3.9}$$

where $\Pi_{y,s}^\infty := \mathcal{P}_s(\Pi_y^\infty) = \exp(-ik_s\hat{x} \cdot y)\,[\hat{x} \times (a \times \hat{x})]$ for some $a \in \mathbb{S}^2$, and $\{\eta_n, g_n\}$ is an eigensystem of the positive operator $F_{s\#}$ defined in Lemma 3.5.

The same technique is also applicable to the inverse problem (IP'). Let $\mathcal{P}_p$ denote the orthogonal projection operator from $L^2(\mathbb{S}^2)^3$ to $L^2_p(\mathbb{S}^2)^3$, i.e., $\mathcal{P}_p g(d) = g_p(d)$. Define the operator $F_p := \mathcal{P}_p\, F\, \mathcal{P}_p^*$. We then have the factorization

$$F_p = -(\mathcal{P}_p G)\, S^*\, (\mathcal{P}_p G)^*. \tag{3.10}$$

The arguments in Lemmas 3.5 and 3.6 can be immediately applied to the operator $\mathcal{P}_p$. As a consequence, we obtain

Theorem 3.8. *If ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, then*

$$y \in D \quad \Longleftrightarrow \quad W_p(y) := \left[\sum_{n=1}^{\infty} \frac{|(g_n, \Pi^\infty_{y,p})_{L^2(\mathbb{S}^2)}|^2}{\eta_n}\right]^{-1} > 0, \tag{3.11}$$

where $\Pi^\infty_{y,p} := \mathcal{P}_p(\Pi^\infty_y) = \exp(-ik_p\hat{x}\cdot y)(\hat{x}\cdot a)\hat{x}$ for some $a \in \mathbb{S}^2$, and $\{\eta_n, g_n\}$ is an eigensystem of the (positive) operator $F_{p\#} := |\mathrm{Re}\, F_p| + |\mathrm{Im}\, F_p|$.

Obviously, Theorems 3.7 and 3.8 provide new uniqueness results by using only one type of elastic waves.

Corollary 3.9. *Assume that the rigid body $D \in \mathbb{R}^3$ has a Lipschitz boundary, and that ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D. Then D can be uniquely identified from the S-part of the far-field pattern for all incident plane shear waves. The uniqueness is also true by employing the P-part of the far-field pattern for all incident plane pressure waves.*

4 Numerical examples

We suppose that $D = \Omega \times \mathbb{R} \subset \mathbb{R}^3$ is an infinitely long cylinder, and turn to the presentation of some numerical simulations in $\mathbb{R}^2$ for constructing the boundary $\partial\Omega \subset \mathbb{R}^2$. We refer to [2] for the linear sampling method and F^*F-method in the two-dimensional inverse elastic scattering where the full far-field pattern is involved.

Recall that in $\mathbb{R}^2$, the Green's tensor of the Navier equation is given by

$$\Gamma(x,y) := \frac{1}{4\mu} H_0^{(1)}(k_s|x-y|)\, I + \frac{i}{4\omega^2}\mathrm{grad}_x \mathrm{grad}_x^\top [H_0^{(1)}(k_s|x-y|) - H_0^{(1)}(k_p|x-y|)]$$

for $x, y \in \mathbb{R}^2, x \neq y$, where $H_0^{(1)}(t)$ denotes the Hankel function of the first kind and of order zero. To be consistent with the presentation in $\mathbb{R}^3$ we define the far-field pattern

$$u^\infty := \hat{x}\, u_p^\infty(\hat{x}) + \hat{x}^\perp\, u_s^\infty(\hat{x}), \quad (x_1, x_2)^\perp := (-x_2, x_1).$$

Here, note that $u_p^\infty(\hat{x}) = u^\infty(\hat{x}) \cdot \hat{x}, u_s^\infty(\hat{x}) = u^\infty(\hat{x}) \cdot \hat{x}^\perp$ are two scalar functions given by the asymptotic behavior

$$u^{sc}(x) = \frac{\exp(ik_p x + i\pi/4)}{\sqrt{8\pi k_p|x|}} u_p^\infty(\hat{x})\, \hat{x} + \frac{\exp(ik_s x + i\pi/4)}{\sqrt{8\pi k_s|x|}} u_s^\infty(\hat{x})\, \hat{x}^\perp + \mathcal{O}(|x|^{-3/2})$$

as $|x| \to \infty$. With this normalization, for a fixed vector $a \in \mathbb{C}^2$ the far-field pattern $\Gamma^\infty_y(\hat{x})$ of the function $x \to \Gamma(x,y)a$ is given by

$$\Gamma_y^\infty(\hat{x}) = \exp(-ik_p\hat{x}\cdot y)(\hat{x}\cdot a)\,\hat{x} + \exp(-ik_s\hat{x}\cdot y)(\hat{x}^\perp\cdot a)\,\hat{x}^\perp =: \Gamma^\infty_{y,p}(\hat{x})\,\hat{x} + \Gamma^\infty_{y,s}(\hat{x})\,\hat{x}^\perp.$$

We make the ansatz for the scattered field u^{sc} in the form

$$u^{sc}(x) = \int_{\partial\Omega} \Gamma(x,y)\,\phi(y)\,\mathrm{d}s(y), \quad x \in \mathbb{R}^3\backslash\overline{\Omega},$$

with some function $\phi(y) \in L^2(\partial\Omega)^2$. Assume that $\partial\Omega$ can be parameterized by $(r_1(t), r_2(t))$, $t \in [0, 2\pi]$. Then the P-part and S-part of the far-field pattern of u^{sc} are given by

$$u_p^\infty(\hat{x}) = \int_0^{2\pi} e^{-ik_p\hat{x}\cdot(r_1(t),r_2(t))^\top}\,[\hat{x}\cdot\phi(r_1(t),r_2(t))]\,\sqrt{r_1'(t)^2 + r_2'(t)^2}\,\mathrm{d}t,$$
$$u_s^\infty(\hat{x}) = \int_0^{2\pi} e^{-ik_s\hat{x}^\perp\cdot(r_1(t),r_2(t))^\top}\,[\hat{x}^\perp\cdot\phi(r_1(t),r_2(t))]\,\sqrt{r_1'(t)^2 + r_2'(t)^2}\,\mathrm{d}t,$$

respectively, in terms of the density function ϕ.

Now, let N plane pressure waves $d_j\exp(ik_px\cdot d_j)$ or N plane shear waves $d_j^\perp\exp(ik_sx\cdot d_j)$ be given at equidistantly distributed directions, that is, $d_j = (\cos\theta_j, \sin\theta_j)$ with $\theta_j = (2\pi j)/N$, $j = 1, 2, \cdots, N$. Denote by $u_p^\infty(\hat{x}, d_j), u_s^\infty(\hat{x}, d_j)$ the P-part, S-part of the far-field pattern corresponding to the incident pressure wave $d_j\exp(ik_px\cdot d_j)$, and by $u_p^\infty(\hat{x}, d_j^\perp), u_s^\infty(\hat{x}, d_j^\perp)$ the counterparts associated with the incident shear wave $d_j^\perp\exp(ik_sx\cdot d_j)$. We perform our numerical experiments in three cases.

SS case: Reconstruct ∂D from $u_s^\infty(d_k, d_j^\perp)$ for N incident plane shear waves $d_j^\perp\exp(ik_sx\cdot d_j)$.

PP case: Reconstruct ∂D from $u_p^\infty(d_k, d_j)$ for N incident plane pressure waves $d_j\exp(ik_px\cdot d_j)$.

FF case: Reconstruct ∂D from the full far-field pattern $u^\infty(d_k, d_j)d_k + u^\infty(d_k, d_j^\perp)d_k^\perp$ for N incident plane elastic waves of the form $d_j\exp(ik_px\cdot d_j) + d_j^\perp\exp(ik_sx\cdot d_j)$.

The operators $F_s := \mathcal{P}_sF\mathcal{P}_s^*$ (in the SS case) and $F_p := \mathcal{P}_pF\mathcal{P}_p^*$ (in the PP case) can be approximated by the $N\times N$ matrices given by $M = (u_s^\infty(d_k, d_j^\perp))_{k,j}$ and $M = (u_p^\infty(d_k, d_j))_{k,j}$, respectively. Let (τ_n, V_n) be the eigensystem of the matrix M, where $\tau_n \in \mathbb{C}, V_n = (v_{n,1}, \cdots, v_{n,N}) \in \mathbb{C}^{1\times N}$. Define $\eta_n := |\mathrm{Re}\,\tau_n| + |\mathrm{Im}\,\tau_n| \in \mathbb{R}$ and $b = (b_1, b_2, \cdots, b_N) \in \mathbb{C}^{1\times N}$ with b_k given by

$$\begin{array}{llll} \text{SS case:} & b_k = b_k^{(s)} := \exp(-ik_sd_k\cdot y)(d_k^\perp\cdot a) \in \mathbb{C}, & \\ \text{PP case:} & b_k = b_k^{(p)} := \exp(-ik_pd_k\cdot y)(d_k\cdot a) \in \mathbb{C}, & \end{array} \quad k = 1, 2, \cdots, N. \tag{4.1}$$

By Theorems 3.7 and 3.8, for each sampling point $y \in \mathbb{R}^2$ and some fixed polarization vector $a \in \mathbb{S}$, we need to compute the indicator function

$$W(y) = \left[\sum_{n=1}^N \frac{|\rho_n^{(y)}|^2}{\eta_n}\right]^{-1}, \quad \rho_n^{(y)} := b\cdot\overline{V}_n^\top = \sum_{k=1}^N b_k\overline{v}_{n,k}$$

and plot the contour (or the level) lines of the function $y \to W(y)$. The values of $W(y)$ should be much smaller for $y \notin D$ than for $y \in D$.

In the FF case, discretizing the far-field operator F gives rise to the $2N\times 2N$ matrix (see also [2])

$$M = \begin{pmatrix} (u_p^\infty(d_k, d_j))_{k,j} & (u_p^\infty(d_k, d_j^\perp))_{k,j} \\ (u_s^\infty(d_k, d_j))_{k,j} & (u_s^\infty(d_k, d_j^\perp))_{k,j} \end{pmatrix}, \quad k, j = 1, 2, \cdots, N.$$

In this case we only need to redefine $b := (b^{(p)}, b^{(s)}) \in \mathbb{C}^{1\times 2N}$, where

$$b^{(p)} = (b_1^{(p)}, b_2^{(p)}\cdots, b_N^{(p)}) \in \mathbb{C}^{1\times N}, \quad b^{(s)} = (b_1^{(s)}, b_2^{(s)}\cdots, b_N^{(s)}) \in \mathbb{C}^{1\times N}$$

are defined in terms of $b_j^{(p)}$, $b_j^{(s)}$, $j = 1, 2, \cdots, N$ given in (4.1).

Figure 1 shows the two obstacles to be recovered through the factorization method. In both examples, we compare the reconstruction results in the SS case, PP case and FF case; see Figures 2 and 3. Using the S-part or P-part of the far-field pattern still produces satisfactory reconstruction, but it is less reliable compared to the FF case. A possible explanation for the worse reconstruction lies in the stronger singularities of the S-part and P-part of the scattered field $G^{sc}(z; z)$ than itself as z approaches $\partial\Omega$ from $\mathbb{R}^2\backslash\overline{\Omega}$, where $G^{sc}(x; z)$ denotes the scattered field due to the point source wave generated by the Green's tensor. Let r denote the distance between z and $\partial\Omega$. Following the arguments in [11], developed for the 3D case, we can show that $|G^{sc}(z; z)| \sim \mathcal{O}(r^{-2})$ as $r \to 0^+$, while the P-part and S-part of $G^{sc}(z; z)$ behave as $\mathcal{O}(r^{-4})$. This suggests that the level curves, corresponding to the same level, will be closer to $\partial\Omega$ in the FF case than in the SS or PP cases. From Figures 2 and 3, it seems hard to conclude which one is better in the SS case and PP case. We use the kite-shaped obstacle to test the sensitivity of the method to the polarization vector $a \in \mathbb{S}$, and employ the peanut-shaped obstacle to examine noisy effects. It is seen from Figures 2 and 3 that the reconstructions in the SS case and PP case are more sensitive than the FF case to the polarization vector a and to the white noise of level δ.

It remains an interesting question to investigate the mixed PS case (resp. SP case), i.e., to reconstruct $\partial\Omega$ from the S-part (resp. P-part) of the far-field pattern corresponding to all incident plane pressure (resp. shear) waves. In our experiments, the $F_{\#}$-method fails if we apply the same inversion procedure to the SP or PS case. This is understandable, because the factorization of the corresponding far-field operators in the mixed case (see (3.7) in the SS case and (3.10) in the PP case) is no longer symmetric and thus the range identity of [17, Theorem 2.15] is not applicable. A further investigation of these cases will be written in a future work.

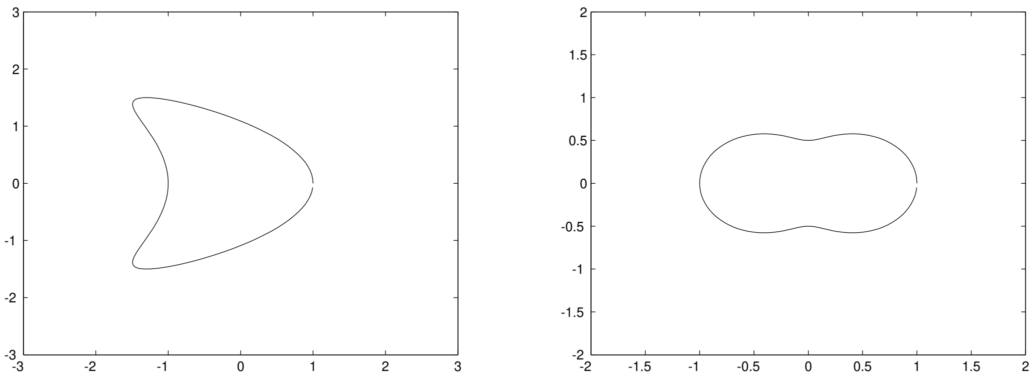

Figure 1: The obstacles to be reconstructed.

5 Appendix

In this section, we give an explicit solution of the S-part of the scattered field for a ball $D = B_R := \{|x| \le R\}$ in terms of radiating spherical vector wave functions and prove its analytical extension to $\mathbb{R}^3\backslash\{0\}$. Let j_n and y_n be the spherical Bessel and Neumann functions of order n, and recall that the linear combination $h_n^{(1)} := j_n + i\, y_n$ are known as spherical Hankel functions of the first kind of order n. In our calculations it is more convenient to employ spherical coordinates. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let

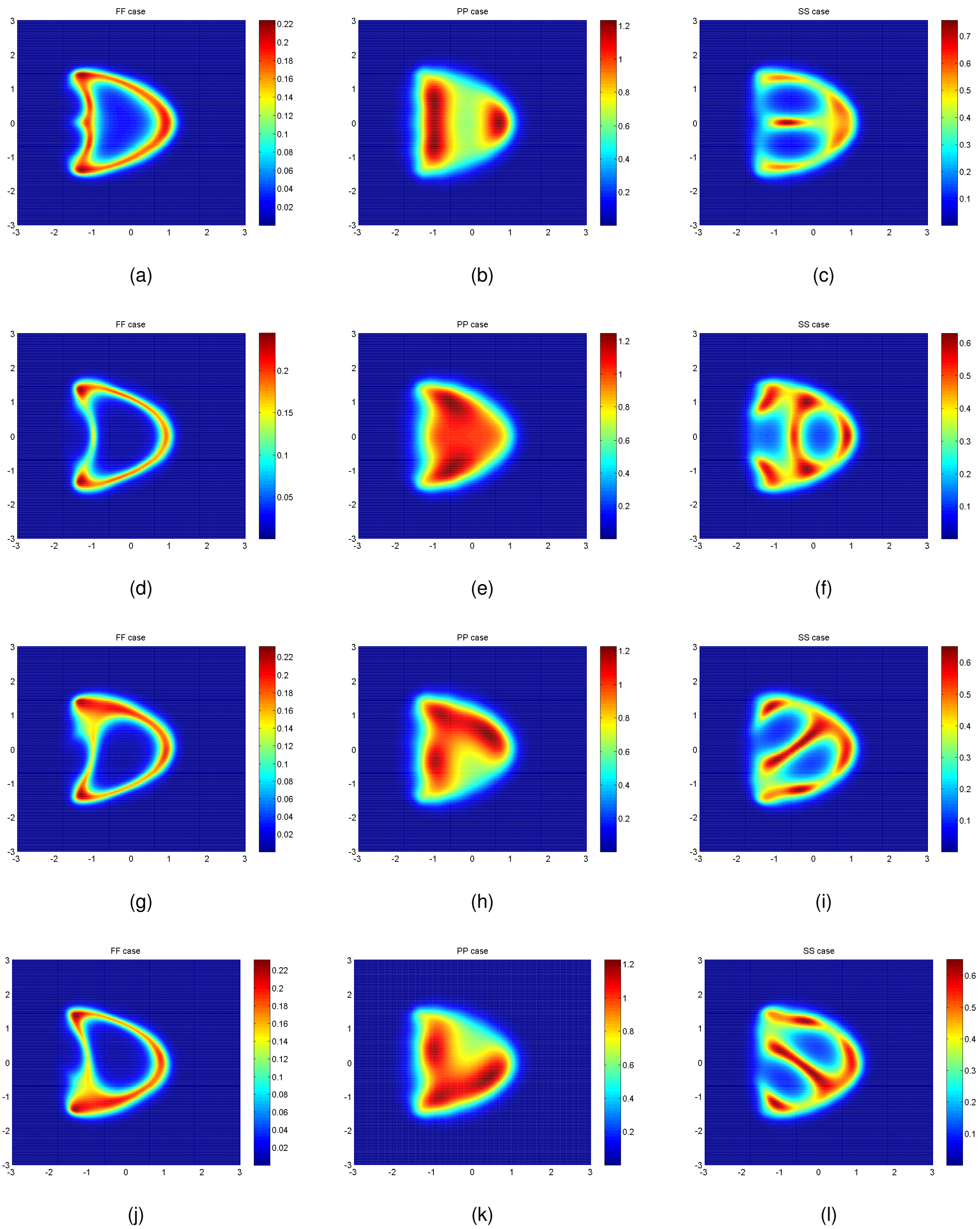

Figure 2: Reconstruction of a kite-shaped obstacle for $N = 64, \mu = 1, \lambda = 1, \omega = 2\sqrt{2}$ with different polarization vectors $a = (\cos\alpha, \sin\alpha)$. $\alpha = 0$ in (2a), (2b) and (2c), $\alpha = \pi/2$ in (2d), (2e) and (2f), $\alpha = 5\pi/4$ in (2g),(2h) and (2i), and $\alpha = 7\pi/4$ in (2j), (2k) and (2l). We used unpolluted far-field data.

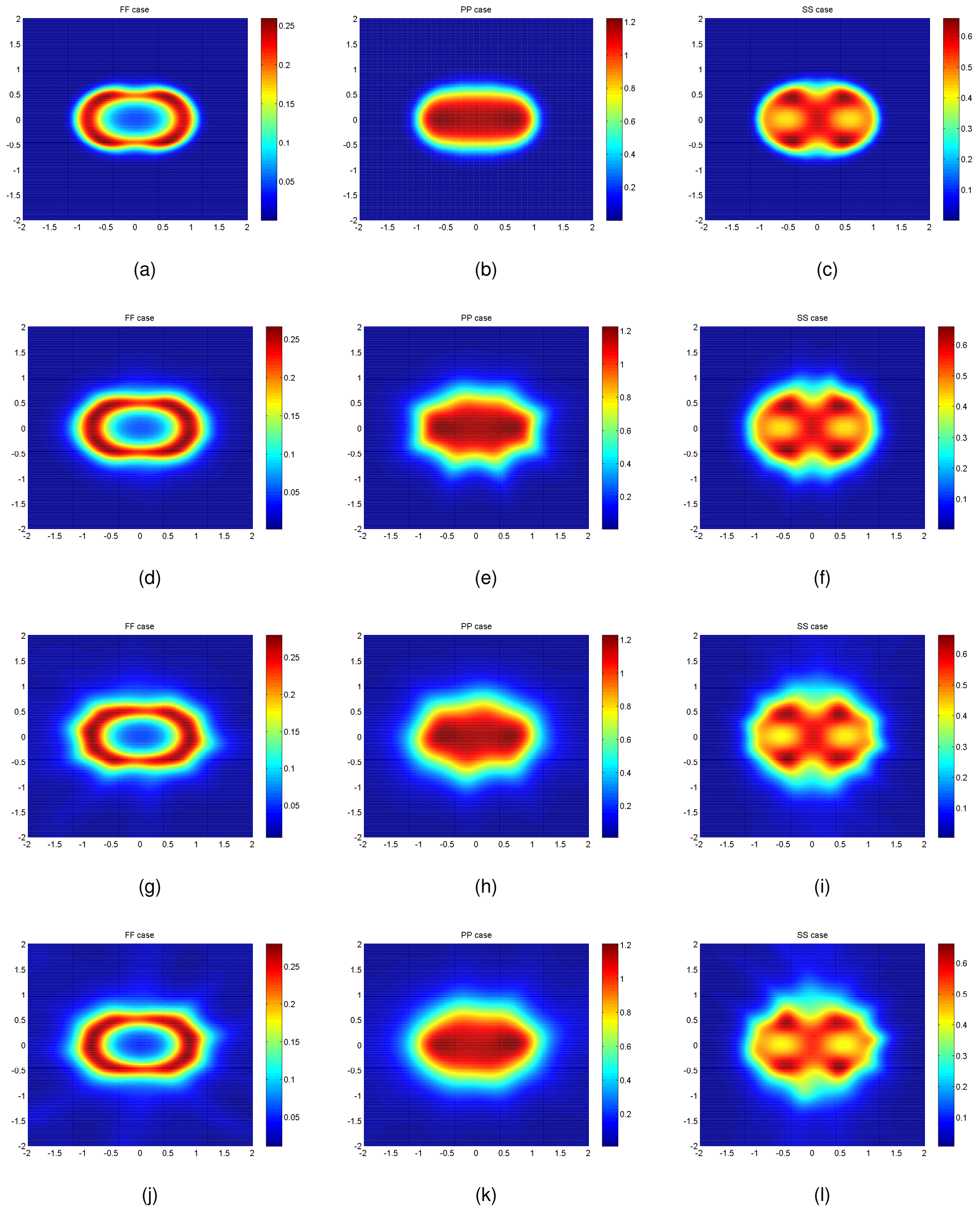

Figure 3: Reconstruction of a peanut-shaped obstacle for $N = 64, \mu = 1, \lambda = 1, \omega = 3\sqrt{2}$ from noised far-field pattern with the noise level δ. In (3a),(3b) and (3c), $\delta = 0$. In (3d),(3e) and (3f), $\delta = 1\%$. In (3g),(3h) and (3i), $\delta = 5\%$. In (3j),(3k) and (3l), $\delta = 8\%$. We used a fixed polarization vector $a = (1, 0)$.

$r = |x|$, $x_1 = r\sin\theta\cos\phi$, $x_2 = r\sin\theta\cos\phi$, $x_3 = r\cos\theta$, and set

$$\overrightarrow{e_r} = \hat{x} := x/r,\ \overrightarrow{e_\theta} := (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),\ \overrightarrow{e_\phi} := (-\sin\phi, \cos\phi, 0).$$

Suppose that u^{in} is a plane wave to the Navier equation (1.1) in $\mathbb{R}^3$. Then there holds the expansion

$$\begin{aligned} u^{in} \;=\; & \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \{A_n^m \frac{1}{k_p}\nabla_x\,[j_n(k_p r)Y_n^m(\hat{x})] + B_n^m \frac{1}{\sqrt{n(n+1)}}\,\mathrm{curl}\,_x[x j_n(k_s r)Y_n^m(\hat{x})] \\ & + C_n^m \frac{1}{\sqrt{n(n+1)}\,k_s}\,\mathrm{curl}\,_x\mathrm{curl}\,_x[x j_n(k_s r)Y_n^m(\hat{x})]\big\} \end{aligned} \tag{5.2}$$

for some constants $A_n^m, B_n^m, C_n^m \in \mathbb{C}$, where Y_n^m denote the spherical harmonics. The first term on the right hand side of (5.2) stands for the longitudinal mode, while the second and third terms represent the two transverse modes. Elementary calculations show that on $|x| = R$ there holds

$$\begin{aligned} \frac{1}{k_p}\nabla_x\,[j_n(k_p r)Y_n^m(\hat{x})]|_{r=R} &= j_n'(t_p)Y_n^m(\hat{x})\hat{x} + j_n(t_p)\frac{1}{t_p}\,\mathrm{Grad}\,Y_n^m(\hat{x}), \\ \mathrm{curl}\,_x[x j_n(k_s r)Y_n^m(\hat{x})]|_{r=R} &= j_n(t_s)\mathrm{Grad}\,Y_n^m(\hat{x})\times\hat{x}, \\ \frac{1}{k_s}\,\mathrm{curl}\,_x\mathrm{curl}\,_x[x j_n(k_s r)Y_n^m(\hat{x})]|_{r=R} &= \frac{n(n+1)}{t_s}Y_n^m(\hat{x})\hat{x} + \frac{1}{t_s}[j_n(t_s) + t_s j_n'(t_s)]\mathrm{Grad}\,Y_n^m(\hat{x}), \end{aligned}$$

where $t_p = k_p R, t_s = k_s R$ and $\mathrm{Grad}\,Y_n^m := \overrightarrow{e_\theta}\,\partial_\theta Y_n^m + (\sin\theta)^{-1}\overrightarrow{e_\phi}\,\partial_\phi Y_n^m$ denotes the surface gradient of Y_n^m over the unit sphere. Note that the tangential fields $\mathrm{Grad}\,Y_n^m(\hat{x}), \mathrm{Grad}\,Y_n^m(\hat{x})\times\hat{x}$ are called vector spherical harmonics of order n. Inserting the previous three identities into (5.2) gives

$$\begin{aligned} u^{in}|_{r=R} = \quad & \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \{\,[j_n'(t_p)A_n^m + t_s^{-1}\sqrt{n(n+1)}C_n^m]\,Y_n^m(\hat{x})\,\hat{x} \\ & + \frac{1}{\sqrt{n(n+1)}}B_n^m j_n(t_s)\mathrm{Grad}\,Y_n^m(\hat{x})\times\hat{x} \\ & + [t_p^{-1}j_n(t_p)A_n^m + \frac{1}{\sqrt{n(n+1)}\,t_s}(j_n(t_s) + t_s j_n'(t_s))\,C_n^m]\mathrm{Grad}\,Y_n^m(\hat{x})\big\}. \end{aligned} \tag{5.3}$$

Since the scattered field $u^{sc} = u_p^{sc} + u_s^{sc}$ satisfies the radiation condition (1.8), the P-part u_p^{sc} and the S-part u_s^{sc} can be expanded into

$$\begin{aligned} u_p^{sc} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \widetilde{A}_n^m \frac{1}{k_p}\nabla_x\,[h_n^{(1)}(k_p r)Y_n^m(\hat{x})], \\ u_s^{sc} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \{\widetilde{B}_n^m \frac{1}{\sqrt{n(n+1)}}\,\mathrm{curl}\,_x[x h_n^{(1)}(k_s r)Y_n^m(\hat{x})] + \\ & \qquad \widetilde{C}_n^m \frac{1}{\sqrt{n(n+1)}\,k_s}\,\mathrm{curl}\,_x\mathrm{curl}\,_x[x h_n^{(1)}(k_s r)Y_n^m(\hat{x})]\} \end{aligned}$$

for $|x| \geq R$ with some complex valued constants $\widetilde{A}_n^m, \widetilde{B}_n^m, \widetilde{C}_n^m$. On the surface $|x| = R$ the scattered field u^{sc} admits an expansion analogous to (5.3) only with j_n, A_n^m, B_n^m, C_n^m replaced by $h_n^{(1)}, \widetilde{A}_n^m, \widetilde{B}_n^m, \widetilde{C}_n^m$, respectively. Taking into account the Dirichlet boundary condition, we obtain

$$\widetilde{B}_n^m h_n^{(1)}(t_s) = -B_n^m j_n(t_s), \quad \widetilde{\mathcal{M}}_n(t_p, t_s)\begin{pmatrix}\widetilde{A}_n^m \\ \widetilde{C}_n^m\end{pmatrix} = -\mathcal{M}_n(t_p, t_s)\begin{pmatrix}A_n^m \\ C_n^m\end{pmatrix}, \tag{5.4}$$

where $\widetilde{\mathcal{M}}_n(t_p, t_s)$ is the 2×2 complex valued matrix given by

$$\widetilde{\mathcal{M}}_n(t_p, t_s) = \begin{pmatrix} h_n^{(1)\prime}(t_p) & t_s^{-1}\sqrt{n(n+1)} \\ t_p^{-1}\, h_n^{(1)}(t_p) & \frac{1}{t_s\sqrt{n(n+1)}}\,[h_n^{(1)}(t_s) + t_s h_n^{(1)\prime}(t_s)] \end{pmatrix}$$

and $\mathcal{M}_n(t_p, t_s)$ is defined analogously to $\widetilde{\mathcal{M}}_n(t_p, t_s)$ only with $h_n^{(1)}, h_n^{(1)\prime}$ replaced by j_n, j_n'. By the uniqueness of the forward elastic scattering problem, the above system (5.4) for $\widetilde{A}_n^m, \widetilde{B}_n^m, \widetilde{C}_n^m$ is uniquely solvable.

Suppose that $u^{in} = u_s^{in}$ is an incident plane shear wave taking the form (1.5). The vector analogue of the Jacobi-Anger expansion yields the expression (5.2) for u_s^{in} with (see [20])

$$A_n^m = 0,\ B_n^m = \frac{-4\pi i^n}{\sqrt{n(n+1)}}(d\times \mathrm{Grad}\,\overline{Y_n^m(d)}\cdot q),\ C_n^m = \frac{-4\pi i^{n+1}}{\sqrt{n(n+1)}}\mathrm{Grad}\,\overline{Y_n^m(d)}\cdot q.$$

Consequently, we derive from (5.4) that

$$\widetilde{B}_n^m = \frac{i^n 4\pi j_n(k_pR)}{\sqrt{n(n+1)}\,h_n^{(1)}(k_pR)}(d\times \mathrm{Grad}\,\overline{Y_n^m(d)}\cdot q), \quad \widetilde{C}_n^m = \widetilde{D}_n^m\frac{i^{n+1}4\pi}{\sqrt{n(n+1)}}\mathrm{Grad}\,\overline{Y_n^m(d)}\cdot q,$$

with the coefficient

$$\widetilde{D}_n^m = \frac{f_n(k_pR) - [j_n(k_sR) + k_sR\,j_n'(k_sR)]}{f_n(k_pR) - [h_n^{(1)}(k_sR) + k_sR\,h_n^{(1)\prime}(k_sR)]},\ f_n(k_pR) := \frac{n(n+1)h_n^{(1)}(k_pR)}{k_pR\,h_n^{(1)\prime}(k_pR)}.$$

Therefore we arrive at an explicit representation of the S-part u_s^{sc} of u^{sc} with the coefficients $\widetilde{B}_n^m$ and $\widetilde{C}_n^m$ given above. By the addition theorem (see e.g. [6, Theorem 2.8]) it can be further concluded that u_s^{sc} is a function of $\cos\beta$, where β denotes the angle between $\hat{x}$ and the incident direction d.

Since u_s^{sc} satisfies the Maxwell equation, to prove its analytical extension into $\{x : |x| < R, x\neq 0\}$ we only need to justify the convergence of the tangential component of u_s^{sc} in the mean square sense on the sphere $|x| = R_0$ for any $0 < R_0 < R$; see [6, Theorem 6.26]. Using Parseval's equality,

$$\begin{aligned}\int_{|x|=R_0} |\nu\times u_s^{sc}|^2 ds(x) &= R_0^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}|\widetilde{B}_n^m|^2\,|h_n^{(1)}(k_sR_0)|^2 \\ &\quad + R_0^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{|\widetilde{C}_n^m|^2\,\left|h_n^{(1)\prime}(k_sR_0) + (k_sR_0)^{-1}\,h_n^{(1)}(k_sR_0)\right|^2\right\}.\end{aligned}$$

By the asymptotic behavior of the spherical Bessel and Hankel functions as $n\to\infty$ and their differential formulas (see e.g. [6, Chapter 2.4] or [23]), we have

$$\begin{aligned} j_n(t) &= \frac{(2t)^n\,n!}{(2n+1)!}\left(1+\mathcal{O}(\frac{1}{n})\right), \\ tj_n'(t) &= \frac{n(2t)^n\,n!}{(2n+1)!}\left(1+\mathcal{O}(\frac{1}{n})\right), \\ h_n^{(1)}(t) &= \frac{(2n-1)!}{i\,2^{n-1}(n-1)!\,t^n}\left(1+\mathcal{O}(\frac{1}{n})\right), \\ th_n^{(1)\prime}(t) &= -\frac{n\,(2n-1)!}{i\,2^{n-1}(n-1)!\,t^n}\left(1+\mathcal{O}(\frac{1}{n})\right).\end{aligned}$$

Then we can check that indeed $||\nu\times u_s^{sc}||^2_{L^2(B_{R_0})^3} < \infty$ for any $0 < R_0 < R$.

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