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A quasi-incompressible diffuse interface model with phase transition

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Abstract

This work introduces a new thermodynamically consistent diffuse model for two-component flows of incompressible fluids. For the introduced diffuse interface model, we investigate physically admissible sharp interface limits by matched asymptotic techniques. To this end, we consider two scaling regimes where in one case we recover the Euler equations and in the other case the Navier-Stokes equations in the bulk phases equipped with admissible interfacial conditions. For the Navier-Stokes regime, we further assume the densities of the fluids are close to each other in the sense of a small parameter which is related to the interfacial thickness of the diffuse model.

1 Introduction

In recent years, for the description of flows of multi-component fluid mixtures, diffuse interface models have emerged as a powerful tool for both theoretical and numerical treatments. In this work, we introduce a new thermodynamically consistent diffuse interface model describing flows of two incompressible fluids, which might be different substances or two phases of one substance. The model permits the transfer of mass between the phases due to diffusion and phase transitions. A basic diffuse interface model for two incompressible, viscous Newtonian fluids having the same densities has been introduced by Hohenberg and Halperin in [HH77]. That model has been modified in a thermodynamically consistent way in several works, see e.g. [GPV96, LT98]. The thermodynamically consistent versions have been investigated analytically in [Abe09, Abe12], where existence of strong local-in-time solutions and weak solutions has been shown. In the present work, we will derive a diffuse interface model for two incompressible constituents, in which phase transitions may occur. The densities of the fluids may be different, which leads to quasi–incompressibility of the mixture. In addition, we study sharp interface limits for two different scalings, which we deduce by using formally matched asymptotic expansions. For one scaling, we recover the incompressible Euler system in the bulk and Young–Laplace and generalized Gibbs–Thomson laws at the interfaces between the two fluids. In the other scaling, we obtain the incompressible Navier–Stokes system in the bulk phases and Young–Laplace and generalized Gibbs–Thomson laws at the interfaces between the two fluids.

The newly introduced diffuse interface model is of Navier-Stokes-Korteweg/Cahn–Hilliard/ Allen–Cahn type. It is given by the following system of PDEs in $[0, T_f) \times \Omega$:

$$\begin{cases} \partial_t\varphi + \mathrm{div}(\mathbf{v}\varphi) = c_+\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda), \\ \rho(\varphi)(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \nabla p(\varphi) + \nabla\lambda \\ \qquad\qquad = \nabla(\eta(\varphi)\mathrm{div}\mathbf{v}) + \mathrm{div}(\hat{\eta}(\varphi)(\nabla\mathbf{v} + \nabla\mathbf{v}^T)) + \gamma\varphi\nabla\Delta\varphi, \\ \mathrm{div}\mathbf{v} = c_-\big(m_\mathrm{j}\Delta - m_\mathrm{r}\big)(c_+\mu(\varphi) + c_-\lambda), \end{cases} \tag{NSK-CH-AC}$$

where the chemical potential μ and the non-monotone pressure p are given by

$$\mu(\varphi) = W'(\varphi) - \gamma\Delta\varphi \qquad \text{and} \qquad p(\varphi) = \varphi W'(\varphi) - W(\varphi). \tag{1}$$

Here, the basic variables are the (volumetric) phase fraction φ, the (barycentric) velocity $\mathbf{v}$, and the Lagrange multiplier λ which takes care for the incompressibility of the constituents. The phase fraction φ is an order parameter such that the pure constituents correspond to $\varphi = \pm 1$. The total density of the mixture is determined by φ via

$$\rho(\varphi) = \tfrac{\tilde{\rho}_1}{2}(1+\varphi) + \tfrac{\tilde{\rho}_2}{2}(1-\varphi)\,. \tag{2}$$

Here, $\tilde{\rho}_1 > 0$ and $\tilde{\rho}_2 > 0$ denote the constant densities of the incompressible constituents. The constant coefficients $c_+ > 0$ and $c_- \geq 0$ in (NSK-CH-AC) are given by the following relation

$$c_\pm = \tfrac{1}{\tilde{\rho}_1} \pm \tfrac{1}{\tilde{\rho}_2} \qquad \text{with} \quad \tilde{\rho}_2 \geq \tilde{\rho}_1.$$

We like to emphasize that the order parameter φ is related to the volumetric fraction of the constituents unlike the approach in [Wit10] which relates it to the concentration of the fluids.

The Lagrange multiplier λ is the analogue of the hydrodynamic pressure in the case of a single incompressible fluid. We assume that the local part of the free energy of the system, i.e. $W(\varphi)$, is a double-well potential of φ (see Section 4 for further assumptions on W). The free energy then reads

$$\rho\psi(\varphi, \nabla\varphi) = W(\varphi) + \frac{\gamma}{2}|\nabla\varphi|^2,$$

where the gradient term $\frac{\gamma}{2}|\nabla\varphi|^2$ models capillarity effects. Although the details of the derivation of the model are given later, we like to note here that the φ-dependent Navier-Stokes viscosity coefficients of the mixture, $\eta(\varphi)$ and $\hat{\eta}(\varphi)$, are given as the interpolation between the two constant viscosities of the individual fluids. The further coefficients $m_{\mathrm{j}} \geq 0$ and $m_{\mathrm{r}} \geq 0$ are the mobility constant of the diffusion flux and the production rate of φ.

The work is organized as follows. In the upcoming section we derive the thermodynamically consistent model for two-phase flows with phase transitions. The third section is devoted to the non-dimensionalization and the introduction of two interesting scaling regimes of the system. Then, in Section 4, we present the setting of asymptotic analysis. Finally, in Sections 5 and 6, we determine the sharp interface models for the two different scaling regimes derived previously.

2 Derivation of the Quasi–incompressible Two-phase Model

In this section, we will derive the model (NSK-CH-AC) starting from the balances of mass, momentum and energy for the constituents. Note that the modeling includes the non–isothermal case, although we only consider the sharp interface limits in the isothermal case. The core of the modeling is the dissipation inequality, and a suitable decomposition for the determination of the constitutive relations.

We consider a binary mixture in an open set $\Omega \subset \mathbb{R}^d$ whose constituents may either be different substances or two phases of a single substance. The constituents are described by two mass densities $(\rho_i)_{i\in\{1,2\}} : [0, T_f) \times \Omega \to \mathbb{R}_+$, two velocities $(\mathbf{v}_i)_{i\in\{1,2\}} : [0, T_f) \times \Omega \to \mathbb{R}^d$ and a common temperature $T : [0, T_f) \times \Omega \to \mathbb{R}_+$. These quantities are the basic variables that describe the *thermodynamic state of the mixture*.

2.1 Equations of balance

The field equations for the basic variables rely on the partial balance equations for mass, momentum and energy.

2.1.1 Partial balances of mass, momentum and energy.

When external forces are neglected, these equations read for $i \in \{1, 2\}$, cf. [Mül85b]:

$$\begin{aligned} \partial_t\rho_i + \operatorname{div}(\rho_i\mathbf{v}_i) &= r_i, &(3)\\ \partial_t(\rho_i\mathbf{v}_i) + \operatorname{div}(\rho_i\mathbf{v}_i \otimes \mathbf{v}_i - \boldsymbol{\sigma}_i) &= \mathbf{f}_i, &(4)\\ \partial_t(\rho_i e_i + \tfrac{1}{2}\rho_i\mathbf{v}_i^2) + \operatorname{div}((\rho_i e_i + \tfrac{1}{2}\rho_i\mathbf{v}_i^2)\mathbf{v}_i + \mathbf{q}_i - \boldsymbol{\sigma}_i\mathbf{v}_i) &= h_i. &(5)\end{aligned}$$

The partial balance equations contain mass production rates r_i, stresses $\boldsymbol{\sigma}_i$, momentum production rates $\mathbf{f}_i$, internal energies e_i, heat fluxes $\mathbf{q}_i$, and energy production rates h_i.

2.1.2 Description of the total mixture.

The quantities that are assigned to the constituents are used to define the corresponding quantities that are assigned to the total mixture.

The total mass density and the barycentric velocity are given by

$$\rho = \sum_{i=1}^{2} \rho_i \quad \text{and} \quad \mathbf{v} = \frac{1}{\rho}\sum_{i=1}^{2} \rho_i \mathbf{v}_i. \tag{6}$$

The internal motion of the constituents is described by the diffusion velocities $\mathbf{u}_i = \mathbf{v}_i - \mathbf{v}$.

The total stress has the form

$$\tilde{\boldsymbol{\sigma}} = \sum_{i=1}^{2}(\boldsymbol{\sigma}_i - \rho_i \mathbf{u}_i \otimes \mathbf{u}_i) = \boldsymbol{\sigma} - \sum_{i=1}^{2} \rho_i \mathbf{u}_i \otimes \mathbf{u}_i,$$

and the total internal energy density and total heat flux are given by

$$\begin{aligned} \rho\tilde{e} &= \sum_{i=1}^{2}(\rho_i e_i + \frac{1}{2}\rho_i \mathbf{u}_i^2) = \rho e + \sum_{i=1}^{2}\frac{1}{2}\rho_i \mathbf{u}_i^2, \\ \tilde{\mathbf{q}} &= \sum_{i=1}^{2}(\mathbf{q}_i + \rho_i e_i \mathbf{u}_i - \boldsymbol{\sigma}_i \mathbf{u}_i + \frac{1}{2}\rho_i \mathbf{u}_i^2 \mathbf{u}_i) = \mathbf{q} + \sum_{i=1}^{2}\frac{1}{2}\rho_i \mathbf{u}_i^2 \mathbf{u}_i. \end{aligned} \tag{7}$$

2.1.3 Conservation laws of mass, momentum and energy.

Total mass, momentum and energy are conserved quantities. Thus, we have

$$\sum_{i=1}^{2} r_i = 0, \qquad \sum_{i=1}^{2} \mathbf{f}_i = 0, \quad \text{and} \quad \sum_{i=1}^{2} h_i = 0.$$

2.1.4 Balance equations for the total mixture.

We sum up the partial balances and obtain the balance equations for the total mixture:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{8}$$

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} - \tilde{\boldsymbol{\sigma}}) = 0, \tag{9}$$

$$\partial_t(\rho\tilde{e} + \tfrac{1}{2}\rho \mathbf{v}^2) + \operatorname{div}((\rho\tilde{e} + \tfrac{1}{2}\rho \mathbf{v}^2)\mathbf{v} + \tilde{\mathbf{q}} - \tilde{\boldsymbol{\sigma}}\mathbf{v}) = 0. \tag{10}$$

Remark 2.1. *The given setting forms the basis of our treatment of two-phase flows. It relies, in particular, on the following meta-rules given by Truesdell, [Tru68]:*

1. *So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.*

2. *The motion of the mixture is governed by the same equations as a single body.*

2.1.5 Incompressibility of constituents.

We are interested in a mixture whose constituents are incompressible. However, the total mixture is not incompressible. For illustration of that fact, we consider a small homogeneous portion of the total volume V of the mixture which contains the masses m_1 and m_2 of the two constituents. The partial mass densities are then defined by $\rho_1 = m_1/V$ and $\rho_2 = m_2/V$. Note that partial mass densities are defined via the total volume of the mixture such that additivity of mass densities is guaranteed. The notion of incompressibility is not related to the partial mass densities but to the real densities of the constituents, namely $\tilde{\rho}_1 = m_1/V_1$ and $\tilde{\rho}_2 = m_1/V_2$ with $V = V_1 + V_2$. By definition, the volumes V_1 and V_2 are the volumes of the pure constituents if they were isolated under the pressure of the mixture. By means of the volume fraction $V_1/V \equiv (1+\varphi)/2$, we may write $\rho_1 = \tilde{\rho}_1(1+\varphi)/2$ and $\rho_2 = \tilde{\rho}_2(1-\varphi)/2$.

If the densities $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are constants, we call the constituents incompressible. In this case the two variables ρ_1 and ρ_2 reduce to a single variable $\varphi : [0, T_f) \times \Omega \to \mathbb{R}$. In the following we have to take care for that constraint.

2.1.6 Partial mass balance equations for incompressible constituents.

We replace the two partial densities ρ_1 and ρ_2 in (3) by the volume fraction φ. Then we subtract and add, respectively, the two balance equations to obtain two equivalent equations that may be written as

$$\mathrm{div}\mathbf{v} = c_-(r - \mathrm{div}\mathbf{j}) \quad \text{and} \quad \partial_t\varphi + \mathrm{div}(\varphi\mathbf{v} + c_+\mathbf{j}) = c_+ r, \tag{11}$$

where $c_\pm = 1/\tilde{\rho}_1 \pm 1/\tilde{\rho}_2$ are constants, $r = r_1$, and $\mathbf{j} = \mathbf{j}_1$ denotes the diffusion flux, where $\mathbf{j}_i = \rho_i \mathbf{u}_i$.

For equal mass densities, i.e. $\tilde{\rho}_1 = \tilde{\rho}_2$, equation $(11)_1$ reduces to the classical constraint $\mathrm{div}\mathbf{v} = 0$ of a single incompressible liquid.

Remark 2.2. *Recently, a similar theory for two incompressible fluids was introduced in [AGG12] and [Boy99]. The main difference to the model at hand is the different definition of the mixture velocity. In contrast to* (6)*, the authors of [AGG12] define the mixture velocity (in our notation) according to*

$$\mathbf{v} = \frac{1}{2}(1+\varphi)\mathbf{v}_1 + \frac{1}{2}(1-\varphi)\mathbf{v}_2$$

which is the volume averaged velocity. This changes the mass and momentum balance, such that they can deal with $\mathrm{div}\mathbf{v} = 0$ *even for unequal mass densities* $\tilde{\rho}_1$ *and* $\tilde{\rho}_2$.

2.1.7 Relevant equations of balance.

The various equations of balance from above play different roles in our treatment of a two-constituent mixture. We use the constraint $(11)_1$, the partial mass balance $(11)_2$, the barycentric momentum balance (9) and the energy balance (10) as the PDE system to determine the reduced set of variables φ, $\mathbf{v}$ and T plus a Lagrange multiplier, which we will introduce below.

The balance equations for partial energies do not occur here because a common temperature is assigned to the constituents. However, in order to derive the constitutive model we need to know the decompositions (7) of internal energy and energy flux. Accordingly, the partial momentum balance equations (4) are also only needed to establish the constitutive model.

An inspection of the relevant balance equations shows that we need constitutive equations for the reaction rate r, the stress $\boldsymbol{\sigma}$ and the diffusion flux $\mathbf{j}$. These will be derived in the following subsection.

2.2 The 2nd law of thermodynamics.

The constitutive model for an incompressible two-constituent system is restricted by the 2nd law of thermodynamics. It consists of five axioms, cf. [DK10].

1. There exists an entropy density/entropy flux pair $(\rho\eta, \mathbf{\Phi})$ that satisfies an equation of balance

$$\partial_t(\rho\eta) + \mathrm{div}(\rho\eta\mathbf{v} + \mathbf{\Phi}) = \xi, \tag{12}$$

   where ξ is the entropy production.

2. The entropy flux $\mathbf{\Phi}$ is an objective vector.

3. The flux has to be determined such that

   (i) the entropy production ξ is non-negative for every solution of the system of balance equations,

   (ii) ξ consists of a sum of binary products *flux* $\times$ *driving force*:

$$\xi = \sum_{A=1}^{N} F_A D_A \geq 0,$$

   (iii) the entropy production is zero in equilibrium.

4. The entropy density $\rho\eta$ is an objective scalar. For an incompressible two-constituent system it is given by a concave constitutive function of the general form

$$\rho\eta = h(\rho e, \varphi, \nabla\varphi), \tag{13}$$

   where ρe is the thermal part of the internal energy density $\rho\tilde{e}$.

5. The (absolute) temperature and chemical potential are defined by

$$\frac{1}{T} = \frac{\partial h}{\partial \rho e} \quad \text{and} \quad \frac{\mu}{T} = -(\frac{\partial h}{\partial \varphi} - \nabla \cdot \frac{\partial h}{\partial \nabla\varphi}). \tag{14}$$

Remark 2.3. *The entropy principle is slightly alternative to those versions given in Rational Thermodynamics by Müller in [Mül85a] and Alt in [Alt09]. The axiom on the representation of the entropy production requires preliminary knowledge on equilibrium thermodynamics, where the driving forces are to be identified as those quantities that vanish in equilibrium. In addition we have to identify in advance the various mechanisms that drive a body to equilibrium. Each mechanism contributes with a binary product to the entropy production.*

2.2.1 Exploitation of the 2nd law, I: Balance of thermal energy and diffusion approximation of partial momenta.

According to (13), the entropy function depends on the thermal energy ρe rather than on the internal energy $\rho\tilde{e}$. For this reason we need the corresponding balance equation for ρe. It results from (10) by the following two steps: 1. Eliminate the kinetic energy $\rho\mathbf{v}^2/2$ by means of the balance equations of mass and barycentric momentum. 2. Eliminate the kinetic energy of the diffusion velocity $\sum_{i=1}^{2} \rho_i \mathbf{u}_i^2/2$ by the partial balance equations of mass and momentum. This strategy leads to

$$\partial_t(\rho e) + \mathrm{div}(\rho e\mathbf{v} + \mathbf{q}) = \nabla\mathbf{v} : \boldsymbol{\sigma} - \sum_{i=1}^{2} \mathbf{u}_i(\mathbf{f}_i - r_i\mathbf{v}_i + \mathrm{div}\boldsymbol{\sigma}_i + \tfrac{1}{2} r_i\mathbf{u}_i).$$

Next, we eliminate in (4) the time derivatives of the partial mass densities by (3) and divide the resulting equations by the corresponding mass densities. After subtraction of the equations from each other we obtain

$$(\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1) - (\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2) = \frac{1}{\rho_1}\mathrm{div}\boldsymbol{\sigma}_1 - \frac{1}{\rho_2}\mathrm{div}\boldsymbol{\sigma}_2 + \frac{1}{\rho_1}(\mathbf{f}_1 - r_1\mathbf{v}_1) - \frac{1}{\rho_2}(\mathbf{f}_2 - r_2\mathbf{v}_2). \tag{15}$$

The *diffusion approximation* ignores the acceleration of the relative motion on the left hand side of (15). Its cancellation removes sound waves due to diffusional motion. In [BD12] the reader finds detailed conditions that guarantee the validity of the approximation

$$\frac{1}{\rho_1}\mathrm{div}\boldsymbol{\sigma}_1 - \frac{1}{\rho_2}\mathrm{div}\boldsymbol{\sigma}_2 + \frac{1}{\rho_1}(\mathbf{f}_1 - r_1\mathbf{v}_1) - \frac{1}{\rho_2}(\mathbf{f}_2 - r_2\mathbf{v}_2) = 0.$$

2.2.2 Exploitation of the 2nd law, II: Calculation of the entropy inequality.

We insert the entropy function (13) in the equation of balance (12), then we apply the chain rule and eliminate the time derivatives $\partial_t \rho e$, $\partial_t \varphi$ and $\partial_t \nabla\varphi$ by the corresponding balance equations. Moreover, we decompose the stress into the Navier-Stokes and the Korteweg part: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{NS}} + \boldsymbol{\sigma}^{\mathrm{K}}$, in such a way that $\boldsymbol{\sigma}^{\mathrm{K}}$ neither depends on $\mathbf{v}$ nor on its derivatives. Finally, we take care of the incompressibility constraint $(11)_1$ by introducing a Lagrange multiplier λ/T, i.e. we add

$$\frac{\lambda}{T}(\mathrm{div}\mathbf{v} - c_{\_}r) + c_{\_}\mathrm{div}(\frac{\lambda}{T}\mathbf{j}) - c_{\_}\mathbf{j}\cdot\nabla(\frac{\lambda}{T}) = 0$$

to the entropy production. After some rearrangements, we obtain

$$\begin{aligned}
\xi = \mathrm{div}\left(\boldsymbol{\Phi} - \frac{\mathbf{q}}{T} + \frac{(c_+\mu + c_-\lambda)\,\mathbf{j}}{T} + (\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi)\frac{\partial h}{\partial\nabla\varphi}\right) \\
+ \frac{1}{T}\nabla\mathbf{v} : \left(\boldsymbol{\sigma}^{\mathrm{K}} - T\nabla\varphi\otimes\frac{\partial h}{\partial\nabla\varphi} - (\rho e - Th - \varphi\mu)\mathbf{1}\right) \\
+ \frac{1}{T}\nabla\mathbf{v} : (\boldsymbol{\sigma}^{\mathrm{NS}} + \lambda\mathbf{1}) + \mathbf{q}\cdot\nabla\frac{1}{T} - r\left(\frac{c_+\mu + c_-\lambda}{T} + \frac{\mathbf{u}_1^2}{2} - \frac{\mathbf{u}_2^2}{2}\right) \\
- \mathbf{j}\cdot\left(\nabla\frac{c_+\mu + c_-\lambda}{T} - (e_1\mathbf{1} - \frac{\boldsymbol{\sigma}_1}{\rho_1} - (e_2\mathbf{1} - \frac{\boldsymbol{\sigma}_2}{\rho_2}))\cdot\nabla\frac{1}{T}\right) \geq 0.
\end{aligned}$$

2.2.3 Exploitation of the 2nd law, III: Interpretation of the results.

The entropy inequality consists of six different terms. The first line is used to define the entropy flux as

$$\boldsymbol{\Phi} = \frac{\mathbf{q}}{T} - \frac{(c_+\mu + c_-\lambda)\,\mathbf{j}}{T} - (\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi)\frac{\partial h}{\partial\nabla\varphi}.$$

The second line is linear in $\nabla\mathbf{v}$, as $\boldsymbol{\sigma}^{\mathrm{K}}$, $\frac{\partial h}{\partial\nabla\varphi}$ and μ do not depend on $\nabla\mathbf{v}$. For that reason its factor must be zero, because otherwise it would become possible to violate the restriction of a non-negative entropy production for arbitrary solutions of the balance equations that will constitute the PDE system. There results a representation of the Korteweg stress, namely

$$\boldsymbol{\sigma}^{\mathrm{K}} = T\nabla\varphi\otimes\frac{\partial h}{\partial\nabla\varphi} + (\rho e - Th - \varphi\mu)\mathbf{1}. \tag{16}$$

The remaining lines give the sum of four binary products that form the non-negative entropy production. In other words: there are four mechanisms leading to dissipation. They are due to *(i)* viscosity, *(ii)* heat flux, *(iii)* phase transition, and *(iv)* diffusion.

2.2.4 Exploitation of the 2nd law, IV: Constitutive model for the isothermal case.

From now on we exclusively consider isothermal processes, i.e. we have $\nabla T = 0$. The simplest possibility to identically satisfy the non-negative entropy production is to propose linear relations between fluxes and driving forces. In this setting the constitutive model contains the following laws:

1. The Navier-Stokes stress

$$\boldsymbol{\sigma}^{\mathrm{NS}} = -\lambda \mathbf{1} + \eta(\varphi)\mathrm{div}\mathbf{v} + \hat{\eta}(\varphi)(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$$

with interpolated viscosities, see (A3) in Section 4.

2. The diffusion law

$$\mathbf{j} = -m_{\mathrm{j}} \nabla \left(\frac{c_+ \mu + c_- \lambda}{T} \right)$$

with the diffusion mobility $m_{\mathrm{j}} \geq 0$.

3. The phase transition law

$$r = -m_{\mathrm{r}} \left(\frac{c_+ \mu + c_- \lambda}{T} + \frac{\mathbf{u}_1^2}{2} - \frac{\mathbf{u}_2^2}{2} \right)$$

with the transition mobility $m_{\mathrm{r}} \geq 0$.

These three laws and the representation (16) of the Korteweg stress form our constitutive model for an isothermal two-phase system.

2.2.5 Exploitation of the 2nd law, V: Legendre transform and free energy density.

If the entropy function $h(\rho e, \varphi, \nabla\varphi)$ were explicitly given, the constitutive laws would become explicit functions of the variables ρe, φ and of the Lagrange multiplier λ/T. However, what usually is known is not the entropy function but the free energy density $\rho\psi = \rho e - Th$. By means of a Legendre transform, that substitutes the energy density as a variable by the temperature, the entropy function $h(\rho e, \varphi, \nabla\varphi)$ and the free energy function $\rho\hat{\psi}(T, \varphi, \nabla\varphi)$ are related to each other by

$$h = -\frac{\partial \rho\hat{\psi}}{\partial T}, \qquad \frac{\partial \rho\hat{\psi}}{\partial \varphi} = -T\frac{\partial h}{\partial \varphi}, \qquad \frac{\partial \rho\hat{\psi}}{\partial \nabla\varphi} = -T\frac{\partial h}{\partial \nabla\varphi}.$$

2.2.6 Summary of the isothermal incompressible two-phase model.

For the application of the model we choose a free energy function

$$\rho\psi = W(T, \varphi) + \frac{\gamma}{2}|\nabla\varphi|^2, \tag{17}$$

and consider a temperature where W is a double–well function of φ. As a further simplification we ignore the quadratic dependence of the stress $\boldsymbol{\sigma}$ and the reaction law r, on the diffusion flux, which occurs in both laws via the quadratic dependence on the diffusion velocities. In this case the summary of the isothermal incompressible two–constituent model is as follows.

The variables are the volume–fraction φ, the barycentric velocity $\mathbf{v}$ and the Lagrange multiplier λ. The system of coupled PDEs reads

$$\mathrm{div}\mathbf{v} = c_-(r - \mathrm{div}\mathbf{j}), \quad \partial_t\varphi + \mathrm{div}(\varphi\mathbf{v} + c_+\mathbf{j}) = c_+ r, \quad \rho(\varphi)(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \mathrm{div}\boldsymbol{\sigma}^{\mathrm{NS}} = \mathrm{div}\boldsymbol{\sigma}^{\mathrm{K}}. \tag{18}$$

The constitutive laws are given by the stresses

$$\boldsymbol{\sigma}^{\mathrm{NS}} = -\lambda \mathbf{1} + \eta(\varphi)\mathrm{div}\mathbf{v} + \hat{\eta}(\varphi)(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \quad \boldsymbol{\sigma}^{\mathrm{K}} = -\nabla\varphi \otimes \frac{\partial \rho\psi}{\partial \nabla\varphi} + (\rho\psi - \varphi\mu)\mathbf{1}, \tag{19}$$

and the diffusion and phase transition rate laws

$$\mathbf{j} = -m_{\mathrm{j}} \nabla \left(c_+ \mu + c_- \lambda\right) \quad \text{and} \quad r = -m_{\mathrm{r}} \left(c_+\mu + c_-\lambda\right). \tag{20}$$

The special choice (17) gives explicit expressions for the chemical potential (14) and the Korteweg stress:

$$\mu = W'(\varphi) - \gamma\Delta\varphi, \qquad \boldsymbol{\sigma}^{\mathrm{K}} = -\gamma\nabla\varphi \otimes \nabla\varphi + (W(\varphi) - \varphi W'(\varphi) + \gamma\varphi\Delta\varphi + \frac{\gamma}{2}|\nabla\varphi|^2)\mathbf{1}.$$

Moreover a straightforward calculation gives

$$\mathrm{div}\boldsymbol{\sigma}^{\mathrm{K}} = -\nabla p(\varphi) + \gamma\varphi\nabla\Delta\varphi \qquad \text{with} \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

Remark 2.4. *Concerning boundary conditions on $\partial\Omega$ an energy argument shows that*

$$\nabla\varphi \cdot \boldsymbol{\nu} = 0, \quad \nabla\lambda \cdot \boldsymbol{\nu} = 0, \quad \mathbf{v} = \mathbf{0}$$

is a reasonable choice, see e.g. [ADD$^+$12].

3 Non-dimensionalization and Scaling

In order to obtain a physically relevant scaling for the system derived in the previous section, we introduce the dimensionless quantities as follows:

$$\begin{aligned} &\mathbf{x}^* = \tfrac{\mathbf{x}}{x_c}, \quad t^* = \tfrac{t}{t_c}, \quad \mathbf{v}^* = \tfrac{\mathbf{v}}{v_c}, \quad \rho^* = \tfrac{\rho}{\rho_c}, \quad \eta_i^* = \tfrac{\eta_i}{\eta_c}, \quad \hat{\eta}_i^* = \tfrac{\hat{\eta}_i}{\eta_c}, \quad p^* = \tfrac{p}{p_c}, \\ &\gamma^* = \tfrac{\gamma}{\gamma_c}, \quad \mu^* = \tfrac{\mu}{p_c}, \quad \lambda^* = \tfrac{\lambda}{\lambda_c}, \quad m_{\mathrm{j}}^* = \tfrac{m_{\mathrm{j}}}{m_{\mathrm{j},c}}, \quad m_{\mathrm{r}}^* = \tfrac{m_{\mathrm{r}}}{m_{\mathrm{r},c}} \end{aligned}$$

for $i \in \{1, 2\}$, where the subscript c refers to the corresponding characteristic unit of the physical quantity. We use p_c to non-dimensionalize W and μ and choose $v_c = \frac{x_c}{t_c}$ as the characteristic velocity. Note that the phase field variable φ, interpolating the density between $\tilde{\rho}_1, \tilde{\rho}_2$, is already a non-dimensional quantity, see (2). We expect that φ is near ± 1. Moreover, we set

$$c_\pm^* = \rho_c c_\pm$$

and like to mention that, while the quantities $c_\pm^*$ are dimensionless, they are not necessarily of order 1. In particular, c_-^* is small in case the densities $\tilde{\rho}_1, \tilde{\rho}_2$ are similar.

Then we rewrite the system (18)-(20) in terms of the dimensionless quantities. We obtain

$$\begin{aligned} &\partial_t\varphi + \mathrm{div}(\mathbf{v}\varphi) = c_+ \tfrac{t_c p_c}{\rho_c^2} \big(\tfrac{m_{\mathrm{j},c}}{x_c^2} m_{\mathrm{j}} \Delta - m_{\mathrm{r},c} m_{\mathrm{r}} \big) (c_+ \mu(\varphi) + c_- \tfrac{\lambda_c}{p_c} \lambda), \\ &\rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \tfrac{p_c}{v_c^2 \rho_c} \nabla p(\varphi) + \tfrac{\lambda_c}{v_c^2\rho_c} \nabla\lambda = \tfrac{\eta_c}{v_c x_c \rho_c} \left(\nabla(\eta(\varphi)\mathrm{div}\mathbf{v}) \right. \\ &\qquad\qquad \left. + \mathrm{div}(\hat{\eta}(\varphi)(\nabla\mathbf{v} + \nabla\mathbf{v}^T)) \right) + \tfrac{\gamma_c}{v_c^2 x_c^2 \rho_c} \gamma\varphi\nabla\Delta\varphi, \\ &\mathrm{div}\mathbf{v} = c_- \tfrac{t_c p_c}{\rho_c^2} \big(\tfrac{m_{\mathrm{j},c}}{x_c^2} m_{\mathrm{j}} \Delta - m_{\mathrm{r},c} m_{\mathrm{r}} \big)(c_+\mu(\varphi) + c_- \tfrac{\lambda_c}{p_c}\lambda), \end{aligned} \tag{21}$$

where

$$\mu(\varphi) = W'(\varphi) - \tfrac{\gamma_c}{x_c^2 p_c} \gamma\Delta\varphi, \quad \text{and} \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi)$$

and the superscript $*$ is omitted for the sake of brevity.

There are several possibilities for scaling the characteristic units. Some scalings with $m_{\mathrm{r}} = 0$ have been treated in [ADD$^+$12]. In this work, we will consider different scalings with $m_r > 0$. In particular, we study two scalings: the so-called *Euler* regime where the viscosity is small, and the *Navier-Stokes* regime for similar densities. In both cases, we obtain physically admissible sharp interface models.

3.1 Euler regime

In this regime, we aim to recover the Euler equations in the bulk with admissible interface conditions in the sharp interface limit. To this end, we consider a small parameter $\varepsilon > 0$ and scale the characteristic units such that

$$M = \sqrt{\tfrac{\rho_c}{p_c}}v_c \sim \sqrt{\varepsilon}, \quad \mathrm{Re} = \tfrac{\rho_c}{\eta_c}v_c x_c \sim \frac{1}{\varepsilon}, \quad \frac{\lambda_c}{p_c} \sim \varepsilon, \quad \frac{t_c p_c m_{j,c}}{\rho_c^2 x_c^2} \sim \frac{1}{\varepsilon}, \quad \frac{m_{j,c}}{m_{r,c}} \sim 1, \quad \frac{\gamma_c}{v_c^2 x_c^2 \rho_c} = \varepsilon, \tag{22}$$

where M and Re are the Mach and Reynolds number, respectively. Moreover, we assume here that c_-^* is independent of ε.

A particular choice of the characteristic units satisfying (22) is given by:

$$t_c \sim \frac{1}{\varepsilon}, \quad m_{\mathrm{j,c}}, m_{\mathrm{r,c}} \sim \frac{1}{\varepsilon}, \quad x_c \sim 1, \quad \rho_c \sim 1, \quad p_c \sim \varepsilon, \quad \eta_c \sim \varepsilon^2, \quad \lambda_c \sim \varepsilon^2, \quad \gamma_c \sim \varepsilon^3.$$

Note that this choice of scaling addresses the long time limit, i.e. $t \to \varepsilon t$.

Using (22) and (21), we obtain the following scaled system:

$$\begin{cases} \partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \frac{1}{\varepsilon} c_+ \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + \varepsilon c_- \lambda), \\ \rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \frac{1}{\varepsilon}\nabla p(\varphi) + \nabla\lambda = \varepsilon\nabla(\eta(\varphi)\operatorname{div}\mathbf{v}) + \varepsilon \operatorname{div}(\hat{\eta}(\varphi)(\nabla\mathbf{v} + \nabla\mathbf{v}^T)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \gamma\varepsilon\varphi\nabla\Delta\varphi, \\ \operatorname{div}\mathbf{v} = \frac{1}{\varepsilon} c_- \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + \varepsilon c_- \lambda), \end{cases} \tag{E}$$

where

$$\mu(\varphi) = W'(\varphi) - \gamma\varepsilon^2\Delta\varphi \quad \text{and} \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

3.2 Navier-Stokes regime with similar densities

In this case, we scale the characteristic units in such a way that we recover the Navier-Stokes system in the bulk. The scaling is based on the assumption that the densities are similar, i.e. $c_-^* \sim \varepsilon$. Further, we scale the characteristic quantities such that

$$M = \sqrt{\tfrac{\rho_c}{p_c}}v_c \sim \sqrt{\varepsilon}, \quad \mathrm{Re} = \tfrac{\rho_c}{\eta_c}v_c x_c \sim 1, \quad \frac{\lambda_c}{p_c} \sim 1, \quad \frac{t_c p_c m_{j,c}}{\rho_c^2 x_c^2} \sim \frac{1}{\varepsilon}, \quad \frac{m_{j,c}}{m_{r,c}} \sim 1, \quad \frac{\gamma_c}{v_c^2 x_c^2 \rho_c} = \varepsilon. \tag{23}$$

For example, a possible choice fulfilling (23) is:

$$t_c \sim \frac{1}{\varepsilon}, \quad m_{\mathrm{j,c}}, m_{\mathrm{r,c}} \sim \frac{1}{\varepsilon}, \quad x_c \sim 1, \quad \rho_c \sim 1, \quad p_c \sim \varepsilon, \quad \eta_c \sim \varepsilon, \quad \lambda_c \sim \varepsilon, \quad \gamma_c \sim \varepsilon^3.$$

Thus, the similar densities scaling leads to the following system:

$$\begin{cases} \partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) = \frac{1}{\varepsilon} c_+ \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + \varepsilon \lambda), \\ \rho(\varphi)(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \frac{1}{\varepsilon}\left(\nabla p(\varphi) + \nabla\lambda\right) = \nabla(\eta(\varphi)\operatorname{div}\mathbf{v}) + \operatorname{div}(\hat{\eta}(\varphi)(\nabla\mathbf{v} + \nabla\mathbf{v}^T)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \gamma\varepsilon\varphi\nabla\Delta\varphi, \\ \operatorname{div}\mathbf{v} = \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu(\varphi) + \varepsilon \lambda), \end{cases} \tag{N-S}$$

where

$$\mu(\varphi) = W'(\varphi) - \gamma\varepsilon^2\Delta\varphi \quad \text{and} \quad p(\varphi) = \varphi W'(\varphi) - W(\varphi).$$

4 Asymptotic Analysis

In this section, we state fundamental assumptions and definitions which we will use when we perform (formal) matched asymptotic expansions for solutions $(\varphi_\varepsilon, \mathbf{v}_\varepsilon,\, \lambda_\varepsilon)$ of the systems (E) and (N-S) in order to derive their sharp interface limits in the forthcoming sections. The technique of matched asymptotic expansions is a powerful tool to understand the asymptotic behavior of the phase field variables when the small parameter ε tends to zero, e.g. [CF88, FP95]. It has been employed in fluid dynamics, see e.g. [AGG12, AW11, DGKR12, HKK11, Wit10], where the phase field models, i.e. Cahn-Hilliard or Allen-Cahn equations, are coupled to the Navier-Stokes systems in both compressible and incompressible cases.

For the convenience of the reader we summarize the properties and the assumptions of the model:

(A1) The mixture density is given by $\rho(\varphi) = \frac{\tilde{\rho}_1}{2}(1+\varphi) + \frac{\tilde{\rho}_2}{2}\,(1-\varphi)$.

(A2) The constant mobility coefficients of the diffusion flux and the production rate are denoted by $m_{\mathrm{j}} \geq 0$ and $m_{\mathrm{r}} \geq 0$, respectively.

(A3) The interpolation function η is given by $\eta(\varphi) = \frac{\eta_1}{2}(1+\varphi) + \frac{\eta_2}{2}(1-\varphi)$ and accordingly for $\hat{\eta}(\varphi)$. The bulk viscosity satisfies $\eta_i + \frac{2}{3}\hat{\eta}_i \geq 0$ and the shear viscosity fulfills $\hat{\eta}_i > 0$ for the pure constituents, $i \in \{1, 2\}$.

(A4) The capillarity constant satisfies $\gamma > 0$.

(A5) $W : \mathbb{R} \to [0, \infty)$ is a double-well potential with $W(-1) = W(1) = 0$, and $\exists\; a_1, a_2 \in (-1, 1)$ such that $W'' > 0 \text{ in } (-\infty, a_1) \cup (a_2, \infty)$ and $W'' < 0 \text{ in } (a_1, a_2)$.

4.1 Outer setting

We define the two bulk phases for $t \in [0, T_f)$ by

$$\Omega^-(t;\varepsilon) := \{\mathbf{x} \in \Omega :\ \varphi_\varepsilon(t,\mathbf{x}) < 0\} \quad \text{and} \quad \Omega^+(t;\varepsilon) := \{\mathbf{x} \in \Omega :\ \varphi_\varepsilon(t,\mathbf{x}) > 0\}.$$

We assume that the solution $(\varphi_\varepsilon, \mathbf{v}_\varepsilon,\, \lambda_\varepsilon)$ to the system (E) or (N-S) admits an expansion in ε in the outer regions $\Omega^+(t;\varepsilon)$ and $\Omega^-(t;\varepsilon)$:

$$\varphi_\varepsilon(t,\mathbf{x}) = \sum_{i=0}^{\infty} \varepsilon^i \varphi_i(t,\mathbf{x}), \qquad \mathbf{v}_\varepsilon(t,\mathbf{x}) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{v}_i(t,\mathbf{x}), \quad \text{and} \quad \lambda_\varepsilon(t,\mathbf{x}) = \sum_{i=0}^{\infty} \varepsilon^i \lambda_i(t,\mathbf{x}). \tag{24}$$

Therefore, we may expand $\mu(\varphi_\varepsilon)$ and $p(\varphi_\varepsilon)$ into their Taylor series, i.e.

$$\begin{aligned}
\mu(\varphi_\varepsilon) &= W'(\varphi_0) + W''(\varphi_0)\varphi_1\varepsilon + (W''(\varphi_0)\varphi_2 + \tfrac{1}{2}W'''(\varphi_0)\varphi_1^2 - \gamma\Delta\varphi_0)\varepsilon^2 + \mathcal{O}(\varepsilon^3),\\
p(\varphi_\varepsilon) &= p(\varphi_0) + p'(\varphi_0)\varphi_1\varepsilon + (p'(\varphi_0)\varphi_2 + \tfrac{1}{2}p''(\varphi_0)\varphi_1^2)\varepsilon^2 + \mathcal{O}(\varepsilon^3),
\end{aligned}$$

as $\varepsilon \to 0$. This motivates the following abbreviations:

$$\mu_0 = W'(\varphi_0), \qquad\qquad p_0 = \varphi_0 W'(\varphi_0) - W(\varphi_0), \tag{25}$$

$$\mu_1 = W''(\varphi_0)\varphi_1, \qquad\qquad p_1 = \varphi_0 W''(\varphi_0)\varphi_1. \tag{26}$$

4.2 Inner setting

We assume that $\Gamma(t;\varepsilon)$ defined by

$$\Gamma(t;\varepsilon) := \{\mathbf{x} \in \Omega \ : \ \varphi_\varepsilon(t,\mathbf{x}) = 0\}$$

is a set of smoothly evolving $C^{1,2}$-hypersurfaces in $\mathbb{R}^{\mathrm{d}}$. Moreover, we assume that a limiting curve $\Gamma = \Gamma(t)$ exists when ε tends to zero. This corresponds to the zeroth order of the interface. The limiting bulk regions are denoted by $\Omega^+(t)$ and $\Omega^-(t)$. Further orders of $\Gamma(t;\varepsilon)$ are not required in our treatment. They would be needed if we considered higher order jump conditions, see [DGKR12].

In a neighborhood of Γ, we introduce a new coordinate system. To this end, we consider a local parametrization $\boldsymbol{\varrho}$ of Γ:

$$\boldsymbol{\varrho} : [0,T_f) \times U \to \mathbb{R}^{\mathrm{d}},$$

where $[0,T_f) \subset \mathbb{R}$ and $U \subset \mathbb{R}^{\mathrm{d}-1}$ are the time interval and the spatial parameter domain, respectively. We denote by $\boldsymbol{\nu}$ the unit normal to Γ pointing towards Ω^+. For details on the assumptions on Γ and $\boldsymbol{\nu}$ we refer to the appendix.

Next, we introduce a local parametrization of a neigborhood of $\boldsymbol{\varrho}([0,T_f)\times U)$ in $[0,T_f)\times\mathbb{R}^{\mathrm{d}}$ as follows:

$$(t,\mathbf{x}) = (t, \boldsymbol{\varrho}(t,\mathbf{s}) + \varepsilon z \boldsymbol{\nu}(t,\mathbf{s})) \tag{27}$$

with $0 \le \varepsilon \le \varepsilon_0$ for some $\varepsilon_0 > 0$. The normal and tangential velocity of the interface Γ are given by

$$\mathbf{w}_{\boldsymbol{\nu}} = w_{\boldsymbol{\nu}}\boldsymbol{\nu} = (\partial_t\boldsymbol{\varrho}\cdot\boldsymbol{\nu})\boldsymbol{\nu} \quad \text{and} \quad \mathbf{w}_{\boldsymbol{\tau}} = \partial_t\boldsymbol{\varrho} - (\partial_t\boldsymbol{\varrho}\cdot\boldsymbol{\nu})\boldsymbol{\nu}. \tag{28}$$

Let f be a generic function depending on outer variables. The corresponding function in inner variables is denoted by capital F, i.e.

$$F(t,\mathbf{s},z) = f(t,\mathbf{x}).$$

The partial derivatives of these functions transform as follows:

$$\begin{pmatrix} \nabla f \\ \partial_t f \end{pmatrix} = \begin{pmatrix} (1+\varepsilon z\kappa)\frac{1}{|\mathbf{T}|^2}\mathbf{T} & \varepsilon^{-1}\boldsymbol{\nu} & 0 \\ -(1+\varepsilon z\kappa)(\mathbf{w}_{\boldsymbol{\tau}} - \varepsilon z\frac{1}{|\mathbf{T}|^2}(\partial_t\boldsymbol{\nu})^{\mathrm{T}}\mathbf{T}) & -\varepsilon^{-1}w_{\boldsymbol{\nu}} & 1 \end{pmatrix} \begin{pmatrix} \nabla_\Gamma F \\ \partial_z F \\ \partial_t F \end{pmatrix} + \mathcal{O}(\varepsilon^2)$$

where $\mathbf{T}$ is a $d\times(d-1)$-matrix whose columns are given by a basis of tangent vectors. Furthermore, we have

$$\begin{aligned} \operatorname{div}\mathbf{f} &= \tfrac{1}{\varepsilon}\partial_z\mathbf{F}\cdot\boldsymbol{\nu} + \operatorname{div}_\Gamma\mathbf{F} + \mathcal{O}(\varepsilon), \\ \Delta f &= \tfrac{1}{\varepsilon^2}\partial_{zz}F - \tfrac{1}{\varepsilon}\kappa\partial_z F - z|\kappa|^2\partial_z F + \Delta_\Gamma F + \mathcal{O}(\varepsilon), \end{aligned}$$

where $\nabla_\Gamma, \operatorname{div}_\Gamma, \Delta_\Gamma$ are the surface gradient, the surface divergence, and the surface Laplacian on Γ, and κ is the mean curvature, respectively, cf. the appendix.

For the inner counterpart $(\Phi_\varepsilon, \mathbf{V}_\varepsilon, \Lambda_\varepsilon)$ of the outer functions $(\varphi_\varepsilon, \mathbf{v}_\varepsilon, \lambda_\varepsilon)$, we assume:

$$\Phi_\varepsilon(t,\mathbf{s},z) = \sum_{i=0}^\infty \varepsilon^i\Phi_i(t,\mathbf{s},z), \quad \mathbf{V}_\varepsilon(t,\mathbf{s},z) = \sum_{i=0}^\infty \varepsilon^i\mathbf{V}_i(t,\mathbf{s},z), \quad \Lambda_\varepsilon(t,\mathbf{s},z) = \sum_{i=0}^\infty \varepsilon^i\Lambda_i(t,\mathbf{s},z). \tag{29}$$

Remark 4.1. *Note that by our definitions of $\Gamma(t;\varepsilon)$ and $\Gamma(t)$ we cannot expect $\Phi_0(t,\mathbf{s},0) = 0$ but there will be a translational quantity dependending on t and $\mathbf{s}$, see Lemma 5.7. Alternatively, we could expand the interface position in ε which would automatically ensure $\Phi_0(t,\mathbf{s},0) = 0$. However, we prefer the first possibilty as in the orders studied here no interfacial mass density appears. This is in contrast to [DGKR12], where an interfacial mass density occurs.*

4.3 Matching relations

In matched asymptotic techniques, inner and outer quantities are linked together by certain matching conditions, see e.g. [CF88]. We impose the following asymptotic behavior for a generic quantity f as $z \to \pm\infty$ at $\mathbf{x} = \boldsymbol{\varrho}(\mathbf{s})$:

$$F_0(t, \mathbf{s}, z) - f_0^{\pm} = o(1/z), \tag{30}$$
$$F_1(t, \mathbf{s}, z) - f_1^{\pm} - (\nabla f_0^{\pm} \cdot \boldsymbol{\nu}(t, \mathbf{s}))z = o(1/z), \tag{31}$$
$$F_2(t, \mathbf{s}, z) - f_2^{\pm} - (\nabla f_1^{\pm} \cdot \boldsymbol{\nu}(t, \mathbf{s}))z - \tfrac{1}{2}(\boldsymbol{\nu}(t, \mathbf{s}) \cdot D^2 f_0^{\pm} \boldsymbol{\nu}(t, \mathbf{s}))z^2 = o(1/z), \tag{32}$$
$$\partial_z F_2(t, \mathbf{s}, z) - \nabla f_1^{\pm} \cdot \boldsymbol{\nu}(t, \mathbf{s}) - (\boldsymbol{\nu}(t, \mathbf{s}) \cdot D^2 f_0^{\pm} \boldsymbol{\nu}(t, \mathbf{s}))z = o(1/z), \tag{33}$$

where the superscript $\pm$ denotes $\lim_{\delta \searrow 0} f(t, \boldsymbol{\varrho}(t, \mathbf{s}) \pm \delta \boldsymbol{\nu}(t, \mathbf{s}))$. Moreover, we have

$$\lim_{z \to \pm\infty} \partial_z F_0(t, \mathbf{s}, z) = 0, \tag{34}$$
$$\lim_{z \to \pm\infty} \partial_{zz} F_0(t, \mathbf{s}, z) = 0, \tag{35}$$
$$\lim_{z \to \pm\infty} \partial_z F_1(t, \mathbf{s}, z) = \nabla f_0^{\pm} \cdot \boldsymbol{\nu}(t, \mathbf{s}), \tag{36}$$
$$\lim_{z \to \pm\infty} \nabla_\Gamma F_0(t, \mathbf{s}, z) = \nabla f_0^{\pm} - (\nabla f_0^{\pm} \cdot \boldsymbol{\nu}(t, \mathbf{s}))\boldsymbol{\nu}(t, \mathbf{s}), \tag{37}$$

where we assume the convergence is superlinearly fast. The idea behind this matching method is that the large-z behavior (for very small ε) of the inner quantities should coincide with the traces of the outer quantities, see e.g. [Lag88]. To this end, a formal term-by-term matching of the ε-expansion of the inner quantities to the Taylor polynomials of the outer ones is made, see [CF88, GS06].

5 Sharp Interface Limit of the Euler Regime

The outer equations are obtained by inserting (24) into (E) and considering the equations order by order.

Definition 5.1. *(***Outer solution***) A tuple* $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0)$ *such that*

$$\begin{aligned} \varphi_0 &\in C^1([0, T_f), C^0(\bar{\Omega}^{\pm})) \cap C^0([0, T_f), C^2(\bar{\Omega}^{\pm})), \\ \varphi_1, \lambda_0 &\in C^0([0, T_f), C^2(\bar{\Omega}^{\pm})), \\ \mathbf{v}_0 &\in C^1([0, T_f), C^0(\bar{\Omega}^{\pm}, \mathbb{R}^d)) \cap C^0([0, T_f), C^1(\bar{\Omega}^{\pm})) \end{aligned}$$

satisfying

$$\begin{aligned} \big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)\mu_0 &= 0, \\ \nabla p_0 &= 0, \\ \partial_t \varphi_0 + \operatorname{div}(\varphi_0 \mathbf{v}_0) &= c_+\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0), \\ \rho(\varphi_0)(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0) + \nabla p_1 + \nabla\lambda_0 &= 0, \\ \operatorname{div}\mathbf{v}_0 &= c_-\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0) \end{aligned} \tag{38}$$

in $\Omega^+ \cup \Omega^-$ *and the boundary condition*

$$\nabla\varphi_0 \cdot \boldsymbol{\nu} = 0 \qquad \text{on} \quad \partial\Omega,$$

where $p_0, p_1, \mu_0,$ *and* μ_1 *are given by* (25) *and* (26)*, is called an* **outer solution** *of the Euler regime.*

The inner equations are obtained from (E) by performing the coordinate change (27), inserting (29) and gathering terms of the same order.

Definition 5.2. *(***Inner solution***) A tuple* $(\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0)$ *such that*

$$\begin{aligned}
\Phi_0 &\in C^0([0,T_f), C^2(\bar{U}, C^2(\mathbb{R}))) \cap C^0([0,T_f), C^0(\bar{U}, C^4(\mathbb{R}))),\\
\Phi_1 &\in C^0([0,T_f), C^0(\bar{U}, C^4(\mathbb{R}))),\\
\Lambda_0 &\in C^0([0,T_f), C^0(\bar{U}, C^2(\mathbb{R}))),\\
\mathbf{V}_0 &\in C^0([0,T_f), C^0(\bar{U}, C^2(\mathbb{R}, \mathbb{R}^d)))
\end{aligned}$$

is called an **inner solution** *with the normal velocity* $w_{\boldsymbol{\nu}}$ *of the Euler regime provided it satisfies*

$$\partial_{zz} M_0 = 0, \qquad \partial_{zz}(c_+ M_1 + c_- \Lambda_0) - \kappa c_+ \partial_z M_0 = 0, \tag{39}$$

$$\partial_z P_0 - \gamma \Phi_0 \partial_{zzz} \Phi_0 = 0, \tag{40}$$

$$\begin{aligned}
-\partial_z(\rho(\Phi_0)\mathbf{V}_0) w_{\boldsymbol{\nu}} + \partial_z(\rho(\Phi_0)(\mathbf{V}_0 \otimes \mathbf{V}_0))\boldsymbol{\nu} + \partial_z P_1 \boldsymbol{\nu} + \nabla_\Gamma P_0 + \partial_z \Lambda_0 \boldsymbol{\nu}&\\
= \partial_z(\eta(\Phi_0)\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} + \partial_z\big(\hat{\eta}(\Phi_0)(\partial_z \mathbf{V}_0 + (\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu})\big)&\\
+\gamma \Phi_1 \partial_{zzz}\Phi_0 \boldsymbol{\nu} + \gamma \Phi_0 \partial_{zzz} \Phi_1 \boldsymbol{\nu} - \kappa\gamma \Phi_0 \partial_{zz}\Phi_0 \boldsymbol{\nu} + \gamma \Phi_0 \nabla_\Gamma(\partial_{zz}\Phi_0),&
\end{aligned} \tag{41}$$

$$\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} = \tfrac{c_-}{c_+} \partial_z(\Phi_0(\mathbf{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})), \tag{42}$$

$$\begin{aligned}
\partial_z(\Phi_0(\mathbf{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})) = c_+ m_{\mathrm{j}} \partial_{zz}(c_+ M_2 + c_- \Lambda_1) - \kappa c_+ m_{\mathrm{j}} \partial_z(c_+ M_1 + c_- \Lambda_0)&\\
+ c_+^2 m_{\mathrm{j}}(\Delta_\Gamma M_0 - z|\kappa|^2 \partial_z M_0) - c_+^2 m_{\mathrm{r}} M_0,&
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
P_0 &= \Phi_0 W'(\Phi_0) - W(\Phi_0), &\qquad P_1 &= \Phi_0 W''(\Phi_0)\Phi_1, &(44)\\
M_0 &= W'(\Phi_0) - \gamma \partial_{zz}\Phi_0, &\qquad M_1 &= W''(\Phi_0)\Phi_1 - \gamma\partial_{zz}\Phi_1 + \kappa\gamma\partial_z \Phi_0, &(45)
\end{aligned}$$

and

$$M_2 = W''(\Phi_0)\Phi_2 + \frac{1}{2}W'''(\Phi_0)\Phi_1^2 - \gamma\partial_{zz}\Phi_2 + \gamma\kappa\partial_z\Phi_1 + \gamma|\kappa|^2 z \partial_z \Phi_0 - \gamma\Delta_\Gamma \Phi_0. \tag{46}$$

Definition 5.3. *(***Matching solution***) We call a tuple* $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0)$ *a* **matching solution** *with the normal velocity* $w_{\boldsymbol{\nu}}$ *of the Euler regime when* $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0)$ *is an outer solution and* $(\Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0)$ *is an inner solution with the normal velocity* $w_{\boldsymbol{\nu}}$ *and both are linked by the matching conditions* (30)-(36).

Now we are in the position to formulate one of our main results:

Theorem 5.4. *Let* $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \Phi_0, \Phi_1, \mathbf{V}_0, \Lambda_0)$ *be a matching solution with the normal velocity* $w_{\boldsymbol{\nu}}$ *of the Euler regime then the following equations hold in the bulk:*

$$\begin{aligned}
&\varphi_0 = \pm 1 \quad \text{in} \quad \Omega^\pm, &(47)\\
&\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + c_-\lambda_0) = 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-, &(48)\\
&\rho(\varphi_0)(\partial_t \mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0) + \nabla(p_1 + \lambda_0) = 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-, &(49)\\
&\mathrm{div}\mathbf{v}_0 = 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-. &(50)
\end{aligned}$$

In addition, the following jump conditions are satisfied at the interface:

$$[\![\rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = 0, \tag{51}$$

$$[\![j_0 \mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]\!] = \kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \tag{52}$$

$$[\![\mathbf{v}_0 \cdot \boldsymbol{\nu}]\!] = \tfrac{c_-}{c_+} 2\langle \mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}} \rangle, \tag{53}$$

$$[\![c_- \lambda_0 + c_+ \mu_1]\!] = 0, \tag{54}$$

$$[\![\mu_1]\!] = \tfrac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \Big(\tfrac{1}{2} c_- c_+ \, j_0^2\big|_\Gamma + j_0|_\Gamma \int_{-\infty}^{\infty} (\eta(\Phi_0) + 2\hat{\eta}(\Phi_0)) \partial_z \left(\tfrac{1}{\rho(\Phi_0)} \right)^2 \mathrm{d}z \Big), \tag{55}$$

$$[\![\mathbf{v}_0 - (\mathbf{v}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu}]\!] = 0, \tag{56}$$

where $j_0 := \rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})$. Moreover,

$$w_{\boldsymbol{\nu}} = \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle - \tfrac{m_{\mathrm{j}}}{2} c_+ [\![c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}]\!]. \tag{57}$$

Here, we denote the jump of a quantity f across the interface by $[\![f]\!] = f^+ - f^-$, whereas $\langle f \rangle = \frac{1}{2}(f^+ + f^-)$ denotes the mean value.

Remark 5.5. *Equations* (51)*,* (52) *and* (56) *correspond to the conservation of mass across the interface, the Young-Laplace law for the normal componet of the momentum flux and the continuity of the tangential velocity, respectively. Note that $p_1 + \lambda_0$ appears as the (hydrodynamic) pressure in bulk equation* (49)*. Equation* (53) *is just an algebraic reformulation of* (51)*. Equation* (54) *states the continuity of the diffusion potential. According to* (55) *the jump of the chemical potential is related to the mass flux across the interface. This is the kinetic relation of our model in the Euler regime. Equation* (57) *can be interpreted as the Stefan law for the phase fraction.*

The case of a flow without mass transfer across the interface, which is described by $[\![\mathbf{v}_0 \cdot \boldsymbol{\nu}]\!] = 0$ and $w_{\boldsymbol{\nu}} = \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle$, is included in Theorem 5.4.

Corollary 5.6. *If $j_0 = 0$ the interface conditions from Theorem 5.4 reduce to*

$$\begin{aligned}
&w_{\boldsymbol{\nu}} = \mathbf{v}_0|_\Gamma \cdot \boldsymbol{\nu}, \\
&[\![\mathbf{v}_0]\!] = 0, \quad [\![\mu_1]\!] = 0, \quad [\![\lambda_0]\!] = 0, \\
&[\![p_1]\!] = \kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \\
&\mu_1|_\Gamma = \tfrac{1}{2}\kappa\gamma \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z, \\
&[\![c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}]\!] = 0.
\end{aligned}$$

Proof. Conditions $w_{\boldsymbol{\nu}} = \mathbf{v}_0|_\Gamma \cdot \boldsymbol{\nu}$, $[\![\mathbf{v}_0]\!] = 0$, $[\![\mu_1]\!] = 0$, and $[\![\lambda_0]\!] = 0$ are obvious. For the pressure jump we compute

$$[\![p_1]\!] = [\![\varphi_0 W''(\varphi_0)\varphi_1]\!] = 2\langle W''(\varphi_0)\varphi_1 \rangle = 2\langle \mu_1 \rangle = 2\mu_1|_\Gamma.$$

The continuity of $c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}$ follows from (57).

We proceed with the proof of Theorem 5.4, which is based on several lemmata. Our first lemma shows that we have pure phases in the bulk.

Lemma 5.7. *Let φ_0 be given as in Definition 5.3, then*

$$\varphi_0 \in \{-1, 1\}.$$

Furthermore, for every $\bar{z} \in \mathbb{R}$ there exists a uniquely determined monotonically increasing function $\Phi_0 \in C^1([0, T_f) \times U; C^3(\mathbb{R}))$ satisfying

$$W'(\Phi_0) - \gamma \partial_{zz} \Phi_0 = 0 \tag{58}$$

with $\partial_z \Phi_0 \to 0, \Phi_0 \to \pm 1$ as $z \to \pm\infty$ and $\Phi_0(t, \mathbf{s}, \bar{z}) = 0$ independently of $\mathbf{s}$ and t. In particular, all Φ_0 as in Definition 5.3 are given by the one parameter family

$$\Phi_0(t, \mathbf{s}, \cdot) = \bar{\Phi}_0(\cdot - \bar{z}(t, \mathbf{s})), \quad \bar{z} \in \mathbb{R},$$

where $\bar{\Phi}_0$ is the unique solution of (58) *satisfying $\bar{\Phi}_0(0) = 0$.*

Proof. We consider $(39)_1$ and obtain

$$M_0(\mathbf{s}, z) = C_0(t, \mathbf{s})z + C_1(t, \mathbf{s}) \quad \text{with} \quad C_0(t, \mathbf{s}) = 0,\ C_1(t, \mathbf{s}) = W'(\varphi_0^+) = W'(\varphi_0^-)$$

by the matching conditions. Next, we multiply equation (45) by $\partial_z \Phi_0$ and integrate over z to obtain

$$W(\varphi_0^+) - W(\varphi_0^-) = W'(\varphi_0^\pm)(\varphi_0^+ - \varphi_0^-). \tag{59}$$

Due to the assumptions on the double-well potential W, we derive that

$$\varphi_0^+ = 1, \quad \varphi_0^- = -1, \qquad \text{and} \qquad C_1(t, \mathbf{s}) = 0. \tag{60}$$

From (59), we obtain that φ_0^+ and φ_0^- are the Maxwell points. Thus, we have $\mu_0^\pm = \mu(\varphi_0^\pm) = 0$. Now we use $(38)_1$ and the boundary condition on $\partial\Omega$ which gives

$$\begin{aligned} (m_{\mathrm{j}}\Delta - m_{\mathrm{r}})\mu_0 &= 0 \qquad \text{in} \quad \Omega^- \cup \Omega^+, \\ \nabla \mu_0 \cdot \boldsymbol{\nu} &= 0 \qquad \text{on} \quad \partial\Omega, \\ \mu_0|_\Gamma &= 0 \qquad \text{on} \quad \Gamma. \end{aligned} \tag{61}$$

Equation (61) is uniquely solvable with $\mu_0 = W'(\varphi_0) \equiv 0$ in Ω. Now $\varphi_0 \in \{-1, 1\}$ follows from $\varphi_0^\pm = \pm 1$ and the continuity of φ_0 on $\bar{\Omega}^\pm$.

The result on Φ_0 follows from $M_0 = 0$, i.e.

$$\gamma \partial_{zz} \Phi_0 - W'(\Phi_0) = 0 \qquad \text{with} \quad \partial_z \Phi_0 \to 0 \quad \text{for} \quad z \to \pm\infty,$$

and a phase portrait analysis, which can be found in [BDDJ07].

Our second lemma deals with the normal velocities of the fluids and the interfacial velocity.

Lemma 5.8. *Let $\mathbf{v}_0$ and $w_{\boldsymbol{\nu}}$ be elements of a matching solution as in Definition 5.3, then*

$$[\![\mathbf{v}_0 \cdot \boldsymbol{\nu}]\!] = 2\tfrac{c_-}{c_+} \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}} \rangle, \tag{62}$$

which is equivalent to the continuity of the mass flux across the interface, i.e.

$$[\![\rho_0(\mathbf{v}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})]\!] = 0 \qquad \textit{with} \qquad \rho_0^+ = \tilde{\rho}_1,\ \rho_0^- = \tilde{\rho}_2, \tag{63}$$

where $\rho_0 = \rho(\varphi_0)$. In fact, $J_0 := \rho_0(\mathbf{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})$ satisfies

$$\partial_z J_0 = 0. \tag{64}$$

Furthermore,

$$w_{\boldsymbol{\nu}} = \langle \mathbf{v}_0 \cdot \boldsymbol{\nu} \rangle - \tfrac{m_{\mathrm{j}}}{2} c_+ [\![c_+ \nabla \mu_1 \cdot \boldsymbol{\nu} + c_- \nabla \lambda_0 \cdot \boldsymbol{\nu}]\!]. \tag{65}$$

Proof. Equation (62) follows from (42) by integration and applying the matching conditions. The continuity of the mass flux (63) is a simple algebraic consequence of (62) using the definition of $c_{\pm}$. Similarly, (64) follows from (42). To prove (65), note that $(39)_2$ reduces to $\partial_z(c_+M_1 + c_-\Lambda_0) = C(t,\mathbf{s})$, see the proof of Lemma 5.7. We can deduce that $C(t,\mathbf{s}) = 0$ by the matching conditions, i.e $\partial_z M_1 \to (\nabla\mu_0)^{\pm}\cdot\boldsymbol{\nu} = 0$ and $\partial_z\Lambda_0 \to 0$. Thus, (43) becomes

$$\partial_z(\Phi_0(\mathbf{V}_0\cdot\boldsymbol{\nu} - w_{\boldsymbol{\nu}})) = c_+ m_{\mathrm{j}}\partial_{zz}(c_+M_2 + c_-\Lambda_1).$$

Finally, we obtain (65) by integration, keeping in mind that $D^2\mu_0 \equiv 0$, see Lemma 5.7.

Lemma 5.9. *Let the assumptions be given as in Theorem 5.4. Then*

$$[\![c_+\mu_1 + c_-\lambda_0]\!] = 0. \tag{66}$$

Proof. From Lemma 5.7 we obtain $M_0 = 0$ and $(\nabla\mu_0)^{\pm}\cdot\boldsymbol{\nu} = 0$. By integrating $(39)_2$, the claim follows.

Remark 5.10. *We note that the lowest order momentum equation* (40) *does not yield any further information, because it is satisfied due to* (58)*.*

Now we use the next order of the balance of momentum to derive the continuity of the tangential fluid velocity

Lemma 5.11. *Let the assumptions of Theorem 5.4 be given. Then*

$$[\![\mathbf{v}_0 - (\mathbf{v}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu}]\!] = 0.$$

Proof. Let $\boldsymbol{\tau}\in\mathbb{R}^d$ with $\boldsymbol{\tau}\cdot\boldsymbol{\nu} = 0$ be arbitrary. Then multiplying (41) by $\boldsymbol{\tau}$ yields

$$\partial_z(\rho(\Phi_0)(\mathbf{V}_0\cdot\boldsymbol{\nu} - w_{\boldsymbol{\nu}})\mathbf{V}_0)\cdot\boldsymbol{\tau} + (\nabla_\Gamma P_0)\cdot\boldsymbol{\tau} = \partial_z(\hat{\eta}(\Phi_0)(\partial_z\mathbf{V}_0)\cdot\boldsymbol{\tau}) + \gamma\Phi_0\nabla_\Gamma(\partial_{zz}\Phi_0)\cdot\boldsymbol{\tau}. \tag{67}$$

We have

$$(\nabla_\Gamma P_0) - \gamma\Phi_0\nabla_\Gamma(\partial_{zz}\Phi_0) = \Phi_0\nabla_\Gamma(W'(\Phi_0) - \gamma\partial_{zz}\Phi_0) = 0$$

and recall $\partial_z J_0 = 0$. Thus, (67) becomes

$$J_0\partial_z\mathbf{V}_0\cdot\boldsymbol{\tau} = \partial_z(\hat{\eta}(\Phi_0)(\partial_z\mathbf{V}_0\cdot\boldsymbol{\tau})). \tag{68}$$

For $J_0 \neq 0$ integrating (68) yields $[\![\mathbf{v}_0\cdot\boldsymbol{\tau}]\!] = 0$. In case $J_0 = 0$, we infer the same by integrating (68) twice and noting $\hat{\eta}(\Phi_0) > 0$.
It remains to deduce the solvability conditions for the normal part of the first order momentum balance, equation (41).

Lemma 5.12. *Let the assumptions of Theorem 5.4 be given. In addition, let $\tilde{\Phi}_1 \in C^\infty(\mathbb{R})$ fulfilling $\tilde{\Phi}_1(z) = \varphi_1^{\pm}$ for $|z| > 1$.*

(i) The operator $L : W^{3,1} \to L^1$

$$L\Psi := -(c_+ - c_-\Phi_0)\partial_z(W''(\Phi_0)\Psi - \gamma\partial_{zz}\Psi) \tag{69}$$

is well defined and $\Psi := \Phi_1 - \tilde{\Phi}_1$ satisfies

$$L\Psi = f(\Phi_0, V_0, \tilde{\Phi}_1) \tag{70}$$

with

$$\begin{aligned} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1) = &-c_-\partial_z\big(\rho(\Phi_0)(\mathbf{V}_0\cdot\boldsymbol{\nu} - w_{\boldsymbol{\nu}})\mathbf{V}_0\cdot\boldsymbol{\nu}\big) + c_-\partial_z\big((\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu}\big) \\ &+ \kappa\gamma(c_+ - c_-\Phi_0)\partial_{zz}\Phi_0 + (c_+ - c_-\Phi_0)\partial_z(W''(\Phi_0)\tilde{\Phi}_1 - \gamma\partial_{zz}\tilde{\Phi}_1). \end{aligned}$$

(ii) Equation (70) *admits a solution if and only if*

$$\int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)\mathrm{d}z = 0, \qquad \int_{-\infty}^{\infty} \frac{f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)}{c_+ - c_-\Phi_0}\mathrm{d}z = 0. \tag{71}$$

(iii) The following jump conditions are satisfied:

$$[\![j_0\mathbf{v}_0 \cdot \boldsymbol{\nu} + p_1 + \lambda_0]\!] = \kappa\gamma \int (\partial_z\Phi_0)^2 \mathrm{d}z, \tag{72}$$

$$[\![\mu_1]\!] = \tfrac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big(\tfrac{1}{2}c_- c_+\, j_0^2\big|_\Gamma + j_0|_\Gamma \int_{-\infty}^{\infty} (\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\partial_z \left(\tfrac{1}{\rho(\Phi_0)}\right)^2 \mathrm{d}z\Big). \tag{73}$$

Proof. We start with the first assertion:
Proof of (i) Multiplying (41) by $\boldsymbol{\nu}$, we obtain

$$\begin{aligned}\partial_z(\rho(\Phi_0)(\mathbf{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})\mathbf{V}_0 \cdot \boldsymbol{\nu}) + \partial_z P_1 + \partial_z \Lambda_0 = \partial_z((\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu}) \\ + \gamma\Phi_1\partial_{zzz}\Phi_0 + \gamma\Phi_0\partial_{zzz}\Phi_1 - \kappa\gamma\Phi_0\partial_{zz}\Phi_0.\end{aligned} \tag{74}$$

Next substitute P_1, M_1 using $(44)_2$, $(45)_2$ and Λ_0 via (39), which we integrate. Here we use once more that $M_0 = 0$ by (58). This yields

$$\begin{aligned}c_-\partial_z(\Phi_0 W''(\Phi_0)\Phi_1) - c_+\partial_z(\Phi_1 W''(\Phi_0)) + c_+\gamma\partial_{zzz}\Phi_1 - c_-\gamma(\Phi_1\partial_{zzz}\Phi_0 + \Phi_0\partial_{zzz}\Phi_1) \\ = -c_-\partial_z(\rho(\Phi_0)(\mathbf{V}_0 \cdot \boldsymbol{\nu} - w_{\boldsymbol{\nu}})\mathbf{V}_0 \cdot \boldsymbol{\nu}) + c_-\partial_z((\eta(\Phi_0) + 2\hat{\eta}(\Phi_0))\partial_z\mathbf{V}_0 \cdot \boldsymbol{\nu}) \\ + \kappa\gamma(c_+ - c_-\Phi_0)\partial_{zz}\Phi_0.\end{aligned} \tag{75}$$

Here Φ_0 and $\mathbf{V}_0 \cdot \boldsymbol{\nu}$ are known by Lemma 5.7 and (42), respectively. Now (70) follows from (75) by inserting $\Phi_1 = \Psi + \tilde{\Phi}_1$ in (75) and recalling the relation $W''(\Phi_0)\partial_z\Phi_0 = \gamma\partial_{zzz}\Phi_0$.

It only remains to show that $f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1) \in L^1(\mathbb{R})$. This is an easy consequence of the matching conditions as $\partial_z\tilde{\Phi}_1(z) = 0$ for $|z| > 1$.
Proof of (ii) This assertion essentially follows from the Fredholm alternative theorem which states that equation (70) is solvable if and only if the right hand side satisfies

$$\int_{-\infty}^{\infty} f(\Phi_0, \mathbf{V}_0, \tilde{\Phi}_1)\phi \; \mathrm{d}z = 0$$

for all solutions $\phi \in L^\infty(\mathbb{R})$ of the homogeneous problem for the adjoint operator $L^* : L^\infty(\mathbb{R}) \to (W^{3,1})^*(\mathbb{R})$ given by

$$L^*\phi = (W''(\Phi_0)\partial_z - \gamma\partial_{zzz})\big((c_+ - c_-\Phi_0)\phi\big).$$

Therefore, we need to find all linearly independent solutions to $L^*\phi = 0$ which are in $L^\infty(\mathbb{R})$. As L^* is a third order operator there are three linearly independent solutions in $C^\infty(\mathbb{R})$. Thus, there are at most three linearly independent solutions in $L^\infty(\mathbb{R})$.

First we observe that due to the relation $W''(\Phi_0)\partial_z\Phi_0 = \gamma\partial_{zzz}\Phi_0$ via (58) every constant is a solution, i.e. $\phi_1 =$ constant, which already implies $(71)_1$. Next we find that $L^*\phi = 0$ can be written as

$$0 = (c_+ - c_-\Phi_0)(W''(\Phi_0)g - \gamma\partial_{zz}g) + 3\gamma c_-(\partial_z g\partial_z\Phi_0 + \partial_{zz}\Phi_0 g), \tag{76}$$

for $g = \partial_z\phi$. Thus, it remains to determine two linearly independent solutions to (76), whose primitives are in $L^\infty(\mathbb{R})$. To this end, we employ the ansatz $g(z) = h(\Phi_0(z))\partial_z\Phi_0(z)$ to find that (76) turns into

$$0 = \gamma(\partial_z\Phi_0)^3\Big((c_+ - c_-\Phi_0)h''(\Phi_0) - 3c_- h'(\Phi_0)\Big) \tag{77}$$

$$+ 3\gamma\partial_z\Phi_0\partial_{zz}\Phi_0\Big((c_+ - c_-\Phi_0)h'(\Phi_0) - 2c_-h(\Phi_0)\Big),$$

where we use the relation $W''(\Phi_0)\partial_z\Phi_0 = \gamma\partial_{zzz}\Phi_0$. The solution to equation (77) is given by

$$h(\Phi_0) = \frac{1}{(c_+ - c_-\Phi_0)^2} \qquad \text{with} \qquad \sup_{x\in[-1,1]} h(x) < C,$$

where C is a positive constant. Thus we have a solution g to (76) satisfying

$$g(z) = h(\Phi_0(z))\partial_z\Phi_0(z) \qquad \text{and} \qquad \phi_2(z) = -\frac{1}{c_-(c_+ - c_-\Phi_0(z))}.$$

This implies $\phi_2 \in L^\infty(\mathbb{R})$, as we know $\Phi_0(z) \in [-1,1]$ and $c_+ > c_-$. This implies $(71)_2$. It remains to show that there is no further solvability condition.

By the d'Alembert reduction principle we construct the second linearly independent solution $\tilde{g}$ to (76). We define $r := \partial_z\tilde{g}$, then (76) turns into a system of ODEs for $(\tilde{g}, r)$ as follows

$$\begin{pmatrix}\partial_z\tilde{g}\\ \partial_z r\end{pmatrix} = \begin{pmatrix}0 & 1\\ \frac{1}{\gamma}W''(\Phi_0) + \frac{3c_-\partial_{zz}\Phi_0}{c_+ - c_-\Phi_0} & \frac{3c_-\partial_z\Phi_0}{c_+ - c_-\Phi_0}\end{pmatrix}\begin{pmatrix}\tilde{g}\\ r\end{pmatrix}.$$

We make the following ansatz for the solutions $\tilde{g}$ and r:

$$\begin{aligned}\tilde{g}(z) &= q(z)g(z),\\ r(z) &= q(z)\partial_z g(z) + \hat{r}(z).\end{aligned}$$

In consequence, we obtain

$$\begin{cases}\partial_z q = \dfrac{\hat{r}}{g},\\ \partial_z\hat{r} = -\dfrac{\hat{r}\partial_z g}{g} + \dfrac{3c_-\partial_z\Phi_0}{c_+ - c_-\Phi_0}\hat{r},\end{cases}$$

which admits the following solutions:

$$\hat{r}(z) = \frac{1}{g(z)}\frac{1}{(c_+ - c_-\Phi_0(z))^3}, \qquad q(z) = \int_0^z \frac{1}{g^2(\tilde{z})}\frac{1}{(c_+ - c_-\Phi_0(\tilde{z}))^3}\mathrm{d}\tilde{z}.$$

We recall that $g = \partial_z\Phi_0/(c_+ - c_-\Phi_0)^2$. Therefore, $\tilde{g}$ satisfies

$$|\tilde{g}(z)| = \left|g(z)\int_0^z \frac{c_+ - c_-\Phi_0}{(\partial_z\Phi_0)^2}\mathrm{d}\tilde{z}\right| \longrightarrow +\infty \qquad \text{as} \quad z \to \pm\infty$$

by a slight modification of Lemma 7.3 in [DGKR12]. Consequently, there exists no third linearly independent solution in $L^\infty(\mathbb{R})$.

Proof of (iii) Condition $(71)_1$ implies

$$\begin{aligned}0 = -c_-[\![j_0\mathbf{v}_0\cdot\boldsymbol{\nu}]\!] + c_-\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z + c_-\int_{-\infty}^{\infty}(\partial_z\Phi_0)(W''(\Phi_0)\tilde{\Phi}_1 - \gamma\partial_{zz}\tilde{\Phi}_1)\mathrm{d}z\\ +[\![(c_+ - c_-\varphi_0)W''(\varphi_0)\varphi_1]\!],\end{aligned}$$

where we recall that the asymptotic behavior of $\tilde{\Phi}_1$ is the same as Φ_1. Next, we observe that $\int_{-\infty}^{\infty} \partial_z\Phi_0\partial_{zz}\tilde{\Phi}_1 \mathrm{d}z = \int_{-\infty}^{\infty} \partial_{zzz}\Phi_0\tilde{\Phi}_1 \mathrm{d}z$ via integration by parts with vanishing boundary values. Then the third term on the right hand side vanishes due to (58). It remains

$$c_-[\![j_0\mathbf{v}_0\cdot\boldsymbol{\nu}]\!] - [\![(c_+ - c_-\varphi_0)W''(\varphi_0)\varphi_1]\!] = c_-\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z.$$

Therefore, recalling $\mu_1^\pm = W''(\varphi_0^\pm)\varphi_1^\pm$ and $p_1^\pm = \varphi_0^\pm W''(\varphi_0^\pm)\varphi_1^\pm$, we have

$$c_-[\![j_0\mathbf{v}_0\cdot\boldsymbol{\nu}]\!] - c_+[\![\mu_1]\!] + c_-[\![p_1]\!] = c_-\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z.$$

This is equivalent to the Young-Laplace law (72) using Lemma 5.9.

Condition $(71)_2$ can be written as follows:

$$\begin{aligned}0 = \int_{-\infty}^{\infty}\Big(&-c_-\partial_z\big(J_0\mathbf{V}_0\cdot\boldsymbol{\nu} - (\eta(\Phi_0)+2\hat{\eta}(\Phi_0))\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu}\big) + \kappa\gamma(c_+ - c_-\Phi_0)\partial_{zz}\Phi_0\\ &+ (c_+ - c_-\Phi_0)\partial_z(W''(\Phi_0)\tilde{\Phi}_1 - \gamma\partial_{zz}\tilde{\Phi}_1)\Big)(c_+ - c_-\Phi_0)^{-1}\mathrm{d}z.\end{aligned}$$

By inserting

$$\mathbf{V}_0\cdot\boldsymbol{\nu} = \frac{J_0}{\rho(\Phi_0)} + w_{\boldsymbol{\nu}} \qquad \text{and} \qquad c_+ - c_-\Phi_0 = \tfrac{2}{\tilde{\rho}_2\tilde{\rho}_1}\rho(\Phi_0),$$

we find

$$0 = -\tfrac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\int_{-\infty}^{\infty}\partial_z\left(\tfrac{J_0^2}{\rho(\Phi_0)} - (\eta(\Phi_0)+2\hat{\eta}(\Phi_0))\partial_z\left(\tfrac{J_0}{\rho(\Phi_0)}\right)\right)\tfrac{1}{\rho(\Phi_0)}\mathrm{d}z + [\![W''(\varphi_0)\varphi_1]\!],$$

where $\partial_z\Phi_0$ vanishes at the boundary. Now using $(1/\rho)_z 1/\rho = (1/(2\rho^2))_z$ and partial integration for the stress term, we find

$$[\![\mu_1]\!] = \tfrac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\left(\tfrac{1}{2}c_-c_+\, j_0^2\big|_\Gamma + \int_{-\infty}^{\infty}(\eta(\Phi_0)+2\hat{\eta}(\Phi_0))J_0\partial_z\left(\tfrac{1}{\rho(\Phi_0)}\right)^2\mathrm{d}z\right),$$

where we recall $\mu_1^\pm = W''(\varphi_0^\pm)\varphi_1^\pm$. This finishes the proof of Lemma 5.12.

Proof of Theorem 5.4. Let us start with the equations in the bulk. Equation (47) follows from Lemma 5.7. Inserting $(38)_3$ into $(38)_5$, we find, using Lemma 5.7,

$$\frac{c_-}{c_+}\mathrm{div}\mathbf{v}_0 = \mathrm{div}\mathbf{v}_0,$$

which implies (50). Moreover, inserting (50) into $(38)_3$ gives (48). Finally, (49) is $(38)_3$. The interface conditions follow from Lemmata 5.7–5.12.

6 Sharp Interface Limit of the Similar Densities Regime

Let us start by defining outer, inner, and matching solutions. The outer equations are obtained by inserting (24) into (NS) and gathering terms of the same order in ε.

Definition 6.1. *(**Outer solution**) A tuple* $(\varphi_0, \varphi_1, \mathbf{v}_0, \lambda_0, \lambda_1)$ *such that*

$$\begin{aligned}
\varphi_0 &\in C^1([0,T_f), C^0(\bar{\Omega}^{\pm})) \cap C^0([0,T_f), C^2(\bar{\Omega}^{\pm})),\\
\varphi_1, \lambda_0 &\in C^0([0,T_f), C^2(\bar{\Omega}^{\pm})),\\
\lambda_1 &\in C^0([0,T_f), C^1(\bar{\Omega}^{\pm})),\\
\mathbf{v}_0 &\in C^1([0,T_f), C^0(\bar{\Omega}^{\pm}, \mathbb{R}^d)) \cap C^0([0,T_f), C^2(\bar{\Omega}^{\pm}, \mathbb{R}^d))
\end{aligned}$$

satisfying

$$\begin{aligned}
\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)\mu_0 &= 0,\\
\nabla p_0 + \nabla\lambda_0 &= 0,\\
\partial_t\varphi_0 + \operatorname{div}(\varphi_0\mathbf{v}_0) &= c_+\big(m_{\mathrm{j}}\Delta - m_{\mathrm{r}}\big)(c_+\mu_1 + \lambda_0),\\
\rho_0(\partial_t\mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0) + \nabla p_1 + \nabla\lambda_1 &= \operatorname{div}(\hat{\eta}(\varphi_0)(\nabla\mathbf{v}_0 + \nabla{\mathbf{v}_0}^T))\\
\operatorname{div}\mathbf{v}_0 &= 0,
\end{aligned} \tag{78}$$

in $\Omega^+ \cup \Omega^-$ *and the boundary condition*

$$\nabla\varphi_0 \cdot \boldsymbol{\nu} = 0 \qquad \text{on} \quad \partial\Omega,$$

where $p_0, p_1, \mu_0,$ *and* μ_1 *are given by* (25) *and* (26)*, is called an* **outer solution** *of the similar densities regime.*

We like to point out that the evolution system is derived by assuming that the density $\rho(\varphi)$ is constant in the leading order ρ_0. The inner equations are obtained by changing the variables according to (27) and inserting the expansions (29) into (N-S).

Definition 6.2. *(**Inner solution**) A tuple* $(\Phi_0, \Phi_1, \mathbf{V}_0, \mathbf{V}_1, \Lambda_0, \Lambda_1)$ *such that*

$$\begin{aligned}
&\Phi_0 \in C^0([0,T_f), C^2(U, C^2(\mathbb{R}))) \cap C^0([0,T_f), C^0(U, C^4(\mathbb{R}))),\\
&\Phi_1 \in C^0([0,T_f), C^0(U, C^4(\mathbb{R}))),\\
&\mathbf{V}_0 \in C^0([0,T_f), C^1(U, C^1(\mathbb{R}, \mathbb{R}^d))) \cap C^0([0,T_f), C^0(U, C^2(\mathbb{R}, \mathbb{R}^d))),\\
&\mathbf{V}_1 \in C^0([0,T_f), C^0(U, C^2(\mathbb{R}, \mathbb{R}^d))),\\
&\Lambda_0 \in C^0([0,T_f), C^1(U, C^0(\mathbb{R}))) \cap C^0([0,T_f), C^0(U, C^2(\mathbb{R}))),\\
&\Lambda_1 \in C^0([0,T_f), C^0(U, C^2(\mathbb{R}))),
\end{aligned}$$

is called an **inner solution** *with the normal velocity* $w_{\boldsymbol{\nu}}$ *of the similar densities regime provided it satisfies* (39)*,* (43) *and*

$$(\partial_z P_0 + \partial_z\Lambda_0)\boldsymbol{\nu} = \partial_z(\eta(\Phi_0)\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \partial_z\big(\hat{\eta}(\Phi_0)(\partial_z\mathbf{V}_0 + (\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu})\big) + \gamma\Phi_0\partial_{zzz}\Phi_0\boldsymbol{\nu}, \tag{79}$$

$$\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu} = \partial_{zz}(c_+M_1 + \Lambda_0) - \kappa c_+\partial_z M_0 = 0, \tag{80}$$

$$\partial_z\left(\Phi_0(\mathbf{V}_0\cdot\boldsymbol{\nu} - w_{\boldsymbol{\nu}})\right) = c_+\left(\operatorname{div}_\Gamma\mathbf{V}_0 + \partial_z\mathbf{V}_1\cdot\boldsymbol{\nu}\right), \tag{81}$$

$$\begin{aligned}
&-\partial_z(\rho_0\mathbf{V}_0)w_{\boldsymbol{\nu}} + \partial_z(\rho_0(\mathbf{V}_0\otimes\mathbf{V}_0))\boldsymbol{\nu} + \partial_z P_1\boldsymbol{\nu} + \nabla_\Gamma P_0 + \partial_z\Lambda_1\boldsymbol{\nu} + \nabla_\Gamma\Lambda_0\\
&= \partial_z\Big(\eta'(\Phi_0)\Phi_1(\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \eta(\Phi_0)(\partial_z\mathbf{V}_1\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \eta(\Phi_0)\operatorname{div}_\Gamma\mathbf{V}_0\,\boldsymbol{\nu}\Big) + \nabla_\Gamma\big(\eta(\Phi_0)\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu}\big)
\end{aligned} \tag{82}$$

$$+\partial_z\Big(\hat{\eta}'(\Phi_0)\Phi_1\big(\partial_z\mathbf{V}_0+(\partial_z\mathbf{V}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu}\big)+\hat{\eta}(\Phi_0)\big(\partial_z\mathbf{V}_1+(\boldsymbol{\nu}\otimes\partial_z\mathbf{V}_1)\boldsymbol{\nu}+(\nabla_\Gamma\mathbf{V}_0)^{\mathrm{T}}\boldsymbol{\nu}\big)\Big)$$
$$+\mathrm{div}_\Gamma\Big(\hat{\eta}(\Phi_0)\big(\partial_z\mathbf{V}_0\otimes\boldsymbol{\nu}+\boldsymbol{\nu}\otimes\partial_z\mathbf{V}_0\big)\Big)$$
$$+\gamma(\Phi_0\partial_{zzz}\Phi_1+\Phi_1\partial_{zzz}\Phi_0-\kappa\Phi_0\partial_{zz}\Phi_0)\boldsymbol{\nu}+\gamma\Phi_0\nabla_\Gamma(\partial_{zz}\Phi_0),$$

where we have used (39) *to obtain the second equality in* (80). *In addition, the quantities* P_0, P_1, M_0, M_1 *and* M_2 *are given by* (44)-(46).

As in the outer setting, the evolution system is deduced by the fact that the leading order density ρ_0 is constant.

Definition 6.3. *(***Matching solution***) A tuple* $(\varphi_0,\varphi_1,\mathbf{v}_0,\lambda_0,\lambda_1,\Phi_0,\Phi_1,\Lambda_0,\Lambda_1,\mathbf{V}_0,\mathbf{V}_1)$ *is called a* **matching solution** *with the normal velocity* $w_{\boldsymbol{\nu}}$ *of the similar densities regime provided* $(\varphi_0,\varphi_1,\mathbf{v}_0,\lambda_0,\lambda_1)$ *is an outer solution,* $(\Phi_0,\Phi_1,\Lambda_0,\Lambda_1,\mathbf{V}_0,\mathbf{V}_1)$ *is an inner solution with the normal velocity* $w_{\boldsymbol{\nu}}$ *and both are linked by the matching conditions.*

Now we have all prerequisites to formulate our second main result:

Theorem 6.4. *Let* $(\varphi_0,\varphi_1,\mathbf{v}_0,\lambda_0,\lambda_1,\Phi_0,\Phi_1,\mathbf{V}_0,\mathbf{V}_1,\Lambda_0,\Lambda_1)$ *be a* **matching solution** *of the similar densities regime, then the following equations are satisfied in the bulk:*

$$\varphi_0=\pm1 \quad \textit{in} \quad \Omega^\pm, \tag{83}$$
$$\nabla\lambda_0=0 \quad \textit{in} \quad \Omega^+\cup\Omega^-, \tag{84}$$
$$\mathrm{div}\mathbf{v}_0=0 \quad \textit{in} \quad \Omega^+\cup\Omega^-, \tag{85}$$
$$\big(m_{\mathrm{j}}\Delta-m_{\mathrm{r}}\big)(c_+\mu_1+\lambda_0)=0 \quad \textit{in} \quad \Omega^+\cup\Omega^-, \tag{86}$$
$$\rho(\varphi_0)(\partial_t\mathbf{v}_0+(\mathbf{v}_0\cdot\nabla)\mathbf{v}_0)+\nabla p_1+\nabla\lambda_1=\mathrm{div}(\hat{\eta}(\varphi_0)(\nabla\mathbf{v}_0+\nabla{\mathbf{v}_0}^T)) \quad \textit{in} \quad \Omega^+\cup\Omega^-. \tag{87}$$

Moreover, the following conditions are fulfilled at the interface:

$$[\![\lambda_0]\!]=0, \tag{88}$$
$$[\![\mathbf{v}_0]\!]=0, \tag{89}$$
$$[\![\mu_1]\!]=0, \tag{90}$$
$$\mu_1|_\Gamma=\frac{1}{2}\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z, \tag{91}$$
$$[\![p_1+\lambda_1]\!]\boldsymbol{\nu}-[\![\hat{\eta}(\varphi_0)(\nabla\mathbf{v}_0+\nabla\mathbf{v}_0^{\mathrm{T}})]\!]\boldsymbol{\nu}=\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z\boldsymbol{\nu}, \tag{92}$$
$$[\![p_1]\!]=\kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z, \tag{93}$$
$$w_{\boldsymbol{\nu}}=\mathbf{v}_0|_\Gamma\cdot\boldsymbol{\nu}-c_+^2\frac{m_j}{2}[\![\nabla\mu_1]\!]\cdot\boldsymbol{\nu}. \tag{94}$$

Remark 6.5. *Equation* (92) *is the Young–Laplace law corresponding to the bulk equation* (87). *Note that the acceleration terms in* (87) *are not reflected in the jump conditions because we have* $[\![\mathbf{v}_0]\!]=0$ *by* (89). *Nevertheless, according to* (94) *a mass flux across the interface is possible because the leading order densities are equal in both phases in the regime at hand.*

The kinetic relation reduces here to the classical Gibbs–Thomson law (90), (91). *The jump of* p_1 *in* (93) *follows from* (91) *because of the Gibbs relation between* μ *and* p.

We proceed with the proof of Theorem 6.4 for which we need several lemmata to deduce the interface conditions. Let us note that all requirements of Lemma 5.7 are also satisfied in the similar densities regime. First we establish the continuity of the Lagrange multiplier and the velocity.

Lemma 6.6. *Under the assumptions of Theorem 6.4 the following equations are satisfied:*

$$\partial_z \mathbf{V}_0 = 0 \quad \textit{and} \quad \partial_z \Lambda_0 = 0. \tag{95}$$

In particular,

$$[\![\mathbf{v}_0]\!] = 0 \quad \textit{and} \quad [\![\lambda_0]\!] = 0. \tag{96}$$

Proof. Multiplying (79) by $\boldsymbol{\nu}$ we obtain

$$\partial_z \Lambda_0 = 0 \tag{97}$$

because of (58) and (80). Integrating (97) gives $[\![\lambda_0]\!] = 0$. Now we return to (79), which due to (80), (97) and (58), yields

$$0 = \partial_z(\hat{\eta}(\Phi_0)\partial_z \mathbf{V}_0).$$

By the matching condition (34) and $\hat{\eta}(\Phi_0) > 0$, we derive $\partial_z \mathbf{V}_0 = 0$ which implies $(96)_1$ because of the matching condition (30).

Lemma 6.7. *Under the assumptions of Theorem 6.4 the chemical potential is continuous across the interface*

$$[\![\mu_1]\!] = 0, \tag{98}$$

and its value is determined by

$$\mu_1|_\Gamma = \frac{1}{2}\gamma\kappa \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z. \tag{99}$$

Moreover,

$$[\![p_1]\!] = \gamma\kappa \int_{-\infty}^{\infty} (\partial_z \Phi_0)^2 \mathrm{d}z. \tag{100}$$

In addition, the normal velocity of the interface is given by

$$w_{\boldsymbol{\nu}} = \mathbf{v}_0 \cdot \boldsymbol{\nu} - c_+^2 \frac{m_j}{2} [\![\nabla \mu_1]\!] \cdot \boldsymbol{\nu}. \tag{101}$$

Proof.

Because of (58), which implies $M_0 = 0$, and Lemma 6.6 equation $(39)_2$ simplifies to $\partial_z M_1 = (\nabla \mu_0)^\pm \cdot \boldsymbol{\nu} = 0$. Hence, we have

$$W''(\Phi_0)\Phi_1 - \gamma\partial_{zz}\Phi_1 + \gamma\kappa\partial_z\Phi_0 = W''(\varphi_0^+)\varphi_1^+ = W''(\varphi_0^-)\varphi_1^-. \tag{102}$$

This, in particular, implies (98) by definition of μ_1. Now we are going to determine the value of μ_1 on the interface. We multiply (102) by $\partial_z \Phi_0$ and integrate which yields

$$\int_{-\infty}^{\infty} W''(\Phi_0)\partial_z\Phi_0\Phi_1 - \gamma\partial_{zz}\Phi_1\partial_z\Phi_0 \mathrm{d}z = \int_{-\infty}^{\infty} W''(\varphi_0^\pm)\varphi_1^\pm\partial_z\Phi_0 - \gamma\kappa(\partial_z\Phi_0)^2 \mathrm{d}z.$$

Using integration by parts, (58), and the matching conditions, we infer

$$0 = 2W''(\varphi_0^\pm)\varphi_1^\pm - \gamma\kappa \int_{-\infty}^{\infty} (\partial_z\Phi_0)^2 \mathrm{d}z.$$

This implies (99). Now, applying $(26)_2$ and Lemma 5.7, we obtain

$$[\![p_1]\!] = [\![\varphi_0 W''(\varphi_0)\varphi_1]\!] = W''(\varphi_0^+)\varphi_1^+ + W''(\varphi_0^-)\varphi_1^- = 2\mu_1|_\Gamma$$

which proves (100) because of (99).

Due to $\partial_z M_1 = 0$, (58) and Lemma 6.6 equation (43) becomes

$$-w_{\boldsymbol{\nu}}\partial_z\Phi_0 + \partial_z(\Phi_0\mathbf{V}_0\cdot\boldsymbol{\nu}) = c_+ m_j \partial_{zz}\left(c_+M_2 + \Lambda_1\right) \tag{103}$$

and integration gives (101) because of $(78)_2$ and the matching condition (33).

Lemma 6.8. *Let the conditions of Theorem 6.4 be satisfied, then*

$$[\![p_1 + \lambda_1]\!]\boldsymbol{\nu} - [\![\hat\eta(\varphi_0)(\nabla\mathbf{v}_0 + \nabla\mathbf{v}_0^{\mathrm{T}})]\!]\boldsymbol{\nu} = \kappa\gamma\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\mathrm{d}z.$$

Proof. Because of Lemma 5.7, $(95)_2$, $(78)_2$ and the matching relation (37) we find

$$\nabla_\Gamma\Lambda_0 = 0 \qquad \text{and} \qquad \nabla_\Gamma P_0 - \gamma\Phi_0\nabla_\Gamma(\partial_{zz}\Phi_0) = 0. \tag{104}$$

Due to $(95)_1$ and (104), equation (82) becomes

$$\begin{aligned}\partial_z P_1\boldsymbol{\nu} + \partial_z\Lambda_1\boldsymbol{\nu} =& \partial_z\Big(\eta(\Phi_0)\big((\partial_z\mathbf{V}_1\cdot\boldsymbol{\nu})\boldsymbol{\nu} + \mathrm{div}_\Gamma\mathbf{V}_0\ \boldsymbol{\nu}\big)\Big)\\ &+ \partial_z\Big(\hat\eta(\Phi_0)\big(\partial_z\mathbf{V}_1 + (\partial_z\mathbf{V}_1\otimes\boldsymbol{\nu})^{\mathrm{T}}\boldsymbol{\nu} + (\nabla_\Gamma\mathbf{V}_0)^{\mathrm{T}}\boldsymbol{\nu}\big)\Big)\\ &+ \gamma(\Phi_0\partial_{zzz}\Phi_1 + \Phi_1\partial_{zzz}\Phi_0 - \kappa\Phi_0\partial_{zz}\Phi_0)\boldsymbol{\nu}.\end{aligned} \tag{105}$$

From the matching relations (36) and (37), we observe that for $z\to\pm\infty$:

$$\partial_z\mathbf{V}_1\cdot\boldsymbol{\nu} + \mathrm{div}_\Gamma\mathbf{V}_0 - (\mathrm{div}\mathbf{v}_0)^\pm = o(1), \tag{106}$$

$$\partial_z\mathbf{V}_1\otimes\boldsymbol{\nu} + \nabla_\Gamma\mathbf{V}_0 - (\nabla\mathbf{v}_0)^\pm = o(1). \tag{107}$$

Therefore, integrating (105) and recalling $\mathrm{div}\mathbf{v}_0 = 0$, we find

$$[\![p_1+\lambda_1]\!]\boldsymbol{\nu} = [\![\hat\eta(\varphi_0)(\nabla\mathbf{v}_0 + (\nabla\mathbf{v}_0)^{\mathrm{T}})]\!]\boldsymbol{\nu} + \gamma\kappa\int_{-\infty}^{\infty}(\partial_z\Phi_0)^2\,\mathrm{d}z\ \boldsymbol{\nu}$$

which is the assertion of the Lemma.

Remark 6.9. *Equation* (81) *does not contribute any new information to the leading order terms. Applying the matching condition* (106) *to* (81) *yields*

$$(\mathrm{div}\mathbf{v}_0)^\pm = 0$$

which is already known from $(78)_5$.

Proof of Theorem 6.4. We know $\varphi_0 = \pm1$ in $\Omega^\pm$ from Lemma 5.7. The other bulk equations follow by inserting $\varphi_0 = \pm1$ into (78). The interface conditions are already known from Lemma 6.6, Lemma 6.7 and Lemma 6.8.

Appendix

Moving surfaces. A family $(\Gamma(t))_{t\in(0,T_f)}$ is called an oriented $C^{1,2}$-family of hypersurfaces if for each point $(t_0,\mathbf{x}_0)\in(0,T_f)\times\mathbb{R}^{\mathrm{d}}$ with $\mathbf{x}_0\in\Gamma(t_0)$ the following properties are satisfied:

(i) There exists an open subset $O\subset\mathbb{R}^{\mathrm{d}}$ containing $\mathbf{x}_0$, $\delta>0$ and a function $u\in C^{1,2}((t_0-\delta,t_0+\delta)\times O)$ such that

$$O\cap\Gamma(t)=\{\mathbf{x}\in O\mid u(t,\mathbf{x})=0\}\quad\text{and}\quad\nabla u(t,\mathbf{x})\neq 0,\ \mathbf{x}\in O\cap\Gamma(t).$$

(ii) There exists a unit normal field $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}\in C^0\left(\bigcup_{0<t<T_f}(\{t\}\times\Gamma(t),\mathbb{R}^{\mathrm{d}})\right)$ and $\boldsymbol{\nu}(t,\cdot)\in C^1(\Gamma(t),\mathbb{R}^{\mathrm{d}})$.

For the representation of the hypersurface $\Gamma(t)$, we define the signed distance function d by

$$d(t,\mathbf{x})=\begin{cases}\operatorname{dist}(\mathbf{x},\Gamma(t)), & \mathbf{x}\in\Omega^+(t),\\ 0, & \mathbf{x}\in\Gamma,\\ -\operatorname{dist}(\mathbf{x},\Gamma(t)), & \mathbf{x}\in\Omega^-(t).\end{cases}$$

Note that $d(t,\cdot)$ is Lipschitz continuous and that there exists a $\delta>0$ such that

$$d(t,\cdot)\in C^2(\Gamma^\delta(t)),\quad\text{where}\quad\Gamma^\delta(t)=\{\mathbf{x}\in\mathbb{R}^{\mathrm{d}}\mid |d(t,x)|<\delta\}.$$

Now, for each point $\mathbf{x}(t)\in\Gamma^\delta(t)$, $t\in(0,T_f)$, there exists a unique point $\mathbf{x}_0(t)\in\Gamma(t)$ such that $|\mathbf{x}(t)-\mathbf{x}_0(t)|=d(t,\mathbf{x}(t))$, where the points satisfy

$$\mathbf{x}(t)=\mathbf{x}_0(t)+\boldsymbol{\nu}(t,\mathbf{x}_0(t))\,d(t,\mathbf{x}(t)).$$

Hence, $\nabla d(t,\mathbf{x}(t))=\boldsymbol{\nu}(t,\mathbf{x}_0(t))$ with $|d(t,\mathbf{x}(t))|=1$ for $\mathbf{x}(t)\in\Gamma^\delta(t)$, $t\in(0,T_f)$.

Let $\boldsymbol{\gamma}\in C^1((t_0-h,t_0+h),\mathbb{R}^{\mathrm{d}})$ for some $h>0$ fulfill $\boldsymbol{\gamma}(t_0)=\mathbf{x}_0$ and $\boldsymbol{\gamma}(t)\in\Gamma(t)$ for $|t-t_0|<h$. Then the normal velocity at the point $(t_0,\mathbf{x}_0)$, $\mathbf{x}_0\in\Gamma(t_0)$, is given by

$$w_{\boldsymbol{\nu}}(t_0,\mathbf{x}_0)=\partial_t\boldsymbol{\gamma}(t_0)\cdot\boldsymbol{\nu}(t_0,\mathbf{x}_0).$$

New coordinate system. In this paragraph, we give a short sketch how the spatial derivatives translate into a local coordinate system around the interface Γ, cf. [AGG12]. We consider a local parameterization of Γ, i.e.

$$\boldsymbol{\varrho}:(0,T_f)\times U,$$

where $U\subset\mathbb{R}^{d-1}$ is the spatial parameter domain. Now, we introduce a local parameterization of a neighborhood of $\boldsymbol{\varrho}((0,T_f)\times U)$:

$$\Upsilon^\varepsilon(t,\mathbf{s},z)=\big(t,\boldsymbol{\varrho}(t,\mathbf{s})+\varepsilon z\boldsymbol{\nu}(t,\mathbf{s})\big)$$

By this transformation, a new locally covariant basis in a neighborhood of Γ forms as

$$(\mathbf{g}_i^\varepsilon)_{i=1,\ldots,d}=\begin{pmatrix}D_{\mathbf{s}}\boldsymbol{\varrho}+\varepsilon zD_{\mathbf{s}}\boldsymbol{\nu}\\ \varepsilon\boldsymbol{\nu}\end{pmatrix},\qquad\text{where } D_{\mathbf{s}}\mathbf{f}:=(\partial_{s_1}\mathbf{f},\ldots,\partial_{s_{d-1}}\mathbf{f})^{\mathrm{T}}.$$

Therefore, an ε-dependent metric is defined by

$$g_{ij}^\varepsilon=\mathbf{g}_i^\varepsilon\cdot\mathbf{g}_j^\varepsilon\quad\text{for}\quad i,j=1,\ldots,d,$$

with $g^\varepsilon_{id} = g^\varepsilon_{di} = 0$ for $i = 1, \dots, d-1$ and $g_{dd} = \varepsilon^2$. Next, we denote the inverse matrix by (g_ε^{ij}) with $g_\varepsilon^{dd} = \varepsilon^{-2}$ and $g_\varepsilon^{id} = g_\varepsilon^{di} = 0$ for $i = 1, \dots, d-1$. The ∇-operator in the new coordinates is then given by

$$\nabla = \sum_{i=1}^{d-1} \mathbf{g}_\varepsilon^i \partial_{s_i} + \tfrac{1}{\varepsilon} \boldsymbol{\nu} \partial_z,$$

where the elements of contravariant basis are given by $\mathbf{g}_\varepsilon^i = \sum_{j=1}^d g_\varepsilon^{ij} \mathbf{g}_j^\varepsilon$. Hence, the gradient of a scalar function $f(t, \mathbf{x}) = F(t, \mathbf{s}, z)$ is given by

$$\nabla f = \nabla_{\Gamma_\varepsilon} F + \tfrac{1}{\varepsilon} \partial_z F \, \boldsymbol{\nu},$$

where $\nabla_{\Gamma_\varepsilon}$ denotes the surface (tangential) gradient on $\Gamma_\varepsilon(t; z) := \{\boldsymbol{\varrho}(t, \mathbf{s}) + \varepsilon z \boldsymbol{\nu}(t, \mathbf{s}) : \ \mathbf{s} \in U\}$, $t \in (0, T_f)$. Similarly, for a vector valued quantity $\mathbf{f}(t, \mathbf{x}) = \mathbf{F}(t, \mathbf{s}, z)$ we have

$$\operatorname{div} \mathbf{f} = \operatorname{div}_{\Gamma_\varepsilon} \mathbf{F} + \tfrac{1}{\varepsilon} \partial_z \mathbf{F} \cdot \boldsymbol{\nu},$$

where $\operatorname{div}_{\Gamma_\varepsilon}$ is the surface divergence on Γ_ε. Now, we compute the Laplace operator:

$$\begin{aligned}
\Delta f = \operatorname{div}(\nabla f) &= \operatorname{div}\Big(\nabla_{\Gamma_\varepsilon} F + \tfrac{1}{\varepsilon} \partial_z F \, \boldsymbol{\nu}\Big) \\
&= \operatorname{div}_{\Gamma_\varepsilon} \Big(\nabla_{\Gamma_\varepsilon} F + \tfrac{1}{\varepsilon} \partial_z F \, \boldsymbol{\nu}\Big) \\
&\quad + \tfrac{1}{\varepsilon} \partial_z \Big(\nabla_{\Gamma_\varepsilon} F + \tfrac{1}{\varepsilon} \partial_z F \, \boldsymbol{\nu}\Big) \cdot \boldsymbol{\nu} \\
&= \Delta_{\Gamma_\varepsilon} F + \tfrac{1}{\varepsilon} \kappa_\varepsilon \partial_z F + \tfrac{1}{\varepsilon^2} \partial_{zz} F,
\end{aligned}$$

where $\kappa_\varepsilon = \operatorname{div}_{\Gamma_\varepsilon} \boldsymbol{\nu}$ is the mean curvature of Γ_ε. Note that for the last equation we have used the following identities for a function $B(\mathbf{s}, z)$:

$$\partial_z (\nabla_{\Gamma_\varepsilon} B \cdot \boldsymbol{\nu}) = 0 \quad \text{and} \quad (\partial_z \nabla_{\Gamma_\varepsilon} B) \cdot \boldsymbol{\nu} = 0$$

which follow from $\nabla_{\Gamma_\varepsilon} B \cdot \boldsymbol{\nu} = 0$ and $\partial_z \boldsymbol{\nu} = 0$. The explicit expression for the operator $\Delta_{\Gamma_\varepsilon}$ is given by

$$\Delta_{\Gamma_\varepsilon} = \tfrac{1}{\sqrt{\det(g_{ij}^\varepsilon)}} \sum_{i,j=1}^{d-1} \partial_{s_i} \big(\sqrt{\det(g_{ij}^\varepsilon)} g_\varepsilon^{ij} \partial_{s_j}\big).$$

Next we identify the expansions in ε for the operators $\nabla_{\Gamma_\varepsilon}, \operatorname{div}_{\Gamma_\varepsilon}, \Delta_{\Gamma_\varepsilon}$ and for the mean curvature κ_ε. Since $g_{ij}^\varepsilon = g_{ij} + \mathcal{O}(\varepsilon)$ with

$$g_{ij}(\mathbf{s}) = \partial_{s_i} \boldsymbol{\varrho}(\mathbf{s}) \cdot \partial_{s_j} \boldsymbol{\varrho}(\mathbf{s}) \quad \text{for} \quad i, j = 1, \dots, d-1$$

we obtain the following expansions:

$$\nabla_{\Gamma_\varepsilon} = \nabla_\Gamma + \mathcal{O}(\varepsilon), \qquad \operatorname{div}_{\Gamma_\varepsilon} = \operatorname{div}_\Gamma + \mathcal{O}(\varepsilon), \qquad \Delta_{\Gamma_\varepsilon} = \Delta_\Gamma + \mathcal{O}(\varepsilon)$$

The mean curvature $\kappa_\varepsilon = \Delta d|_{\Gamma_\varepsilon}$ is given by

$$\begin{aligned}
\kappa_\varepsilon = \sum_{i=1}^{d-1} \frac{-\kappa^i}{1 - \varepsilon \kappa^i z} &= -\sum_{i=1}^{d-1} \kappa^i - \varepsilon \sum_{i=1}^{d-1} (\kappa^i)^2 z + \mathcal{O}(\varepsilon^2) \\
&= -\kappa - \varepsilon |\kappa|^2 z + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

where κ^i are the principal curvatures, κ and $|\kappa|$ are the mean curvature and the root mean square curvature of Γ, respectively, see for instance [GT98].

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