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Nonuniversal transitions to synchrony in the Sakaguchi-Kuramoto model

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Abstract

We investigate the transition to synchrony in a system of phase oscillators that are globally coupled with a phase lag (Sakaguchi-Kuramoto model). We show that for certain unimodal frequency distributions there appear unusual types of synchronization transitions, where synchrony can decay with increasing coupling, incoherence can regain stability for increasing coupling, or multistability between partially synchronized states and/or the incoherent state can appear. Our method is a bifurcation analysis based on a frequency dependent version of the Ott-Antonsen method and allows for a universal description of possible synchronization transition scenarios for any given distribution of natural frequencies.

The Sakaguchi-Kuramoto model [1]

$$\frac{d\theta_k}{dt} = \omega_k - \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_k(t) - \theta_j(t) + \alpha), \tag{1}$$

is a fundamental paradigm for the emergence of collective behavior (synchrony) in a system of non-identical oscillators, described here by their phases $\theta_k \in \mathbb{R} \bmod 2\pi$, $k = 1, \ldots, N$, that are weakly coupled by the mean field. It has been used to explain the behavior of a huge variety of systems in physics, biology, and other fields [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In the most simple setting, where the natural frequencies ω_k are drawn from a unimodal probability distribution $g(\omega)$, a classical synchronization transition is observed, where at a critical coupling strength K_c a stable branch of partially coherent states appears and the synchrony, measured by the magnitude of the global order parameter [3, 13, 14], increases gradually with further increasing coupling strength K, see Fig 1(a). According to the classical paper of Sakaguchi and Kuramoto [1] also a phase-lag parameter $\alpha < \pi/2$, which has to be taken into account in many cases, *"...does not seem to change the result in an essential way"*. At the other hand, it is known that changing the system, e.g. to a bimodal frequency distribution [15], distributed shears [12], or to more complicated coupling schemes [16, 17, 18, 19, 20, 21], results in a wealth of interesting dynamical phenomena.

In this paper, we demonstrate that also in the classical Sakaguchi-Kuramoto system (1) for certain unimodal frequency distributions and carefully chosen phase-lag parameter α, one can observe several nontrivial synchronization transitions, see Fig. 1(b)–(c) and Fig. 3(b)–(c):

- decreasing synchronization with increasing coupling strength,
- incoherence regaining stability for increasing coupling,
- coexistence of stable incoherence with a partially synchronized state
- coexistence of two stable partially synchronized state.

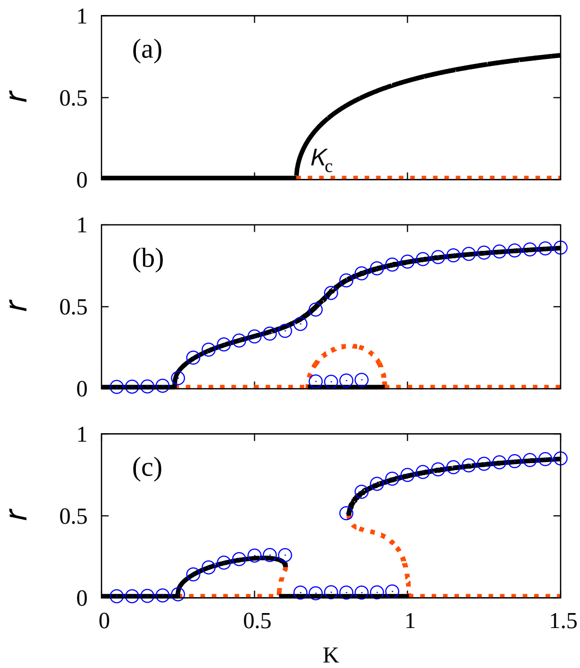

Figure 1: (color online) Examples of synchronization transitions (dependence of the order parameter r on the coupling strength K) for the Sakaguchi-Kuramoto system (1) with various choices of unimodal frequency distributions $g(\omega)$ and of the phase-lag parameter α. Solid (dashed) lines: analytical prediction of stable (unstable) branches, based on the Ott-Antonsen equation. Circles: numerical simulation of system (1) with $N = 10000$ and suitably chosen initial data. (a): classical synchronization transition for $\alpha = 0$ and Lorentzian frequency distribution. (b) and (c): synchronization transitions with bistability and incoherence regaining its stability (frequency distributions and phase-lag parameter are specified in Fig. 2).

Our method is a detailed bifurcation analysis that can be carried out in the limit of large N for a frequency dependent version of a corresponding Ott-Antonsen equation. An equation of this type has already been used in [22] to demonstrate the appearance of nonuniversal synchronization transitions for certain globally coupled excitable systems.

The Ott-Antonsen approach. In the thermodynamic limit, i.e. for $N \to \infty$, different partially synchronized regimes observed in Eq. (1) can be effectively analyzed with the help of the Ott-Antonsen method [23]. Following this approach, we describe the state of the oscillator system (1) at time t by a probability distribution function $f(\omega, \theta, t)$. Since $g(\omega)$ is time independent, the function f satisfies

$$\int_0^{2\pi} f(\omega, \theta, t) d\theta = g(\omega).$$

To derive the evolution equation for f, we first rewrite Eq. (1) in the complex form

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{2i} \left(e^{-i\alpha} r(t) e^{-i\theta_k(t)} - e^{i\alpha} r^*(t) e^{i\theta_k(t)} \right), \qquad (2)$$

where

$$r(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} \tag{3}$$

is the Kuramoto's order parameter and the star $*$ denotes complex conjugation. Then, expressing the conservation of oscillators law for Eq. (2), we get a continuity equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}(fv) = 0, \tag{4}$$

where

$$v(\theta,\omega,t) := \omega + \frac{K}{2i}\left(e^{-i\alpha} r(t) e^{-i\theta} - e^{i\alpha} r^*(t) e^{i\theta}\right) \tag{5}$$

and $r(t)$ is now given by

$$r(t) = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f(\omega,\theta,t) e^{i\theta} d\theta. \tag{6}$$

In their seminal papers [23, 24, 25], Ott and Antonsen pointed out that all the attractors of Eq. (4) can be found using the following ansatz

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi}\left(1 + \sum_{n=1}^{\infty} [z^*(\omega,t)]^n e^{in\theta} + \mathrm{c.c.}\right), \tag{7}$$

where $\mathrm{c.c.}$ stands for the complex conjugate of the preceding sum. Indeed, by a direct substitution one can easily verify that formula (7) represents a solution to Eq. (4) if $z(\omega,t)$ satisfies

$$\frac{dz}{dt} = i\omega z(\omega,t) + \frac{K}{2} e^{-i\alpha} \mathcal{G} z - \frac{K}{2} e^{i\alpha} z^2(\omega,t) \mathcal{G} z^*, \tag{8}$$

where for any $\varphi \in C(\mathbb{R};\mathbb{C})$ we denote by $\mathcal{G}\varphi$ the integral operator

$$\mathcal{G}\varphi := \int_{-\infty}^{\infty} g(\omega)\varphi(\omega) d\omega. \tag{9}$$

Equation (8) will be our main tool to study the partially synchronized regime in the original Sakaguchi-Kuramoto system (1). As shown in [23], one has to restrict to solutions of (8) satisfying $|z(\omega,t)| \le 1$. There, the summation in (7) yields

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi} \frac{1 - |z|^2}{1 + |z|^2 - 2|z|\cos(\theta - \arg z)}, \tag{10}$$

degenerating into a delta function $f(\omega,\theta,t) = \delta(\theta - \arg z)$ for $|z(\omega,t)|$ approaching 1 from below. Indeed, the Ott-Antonsen ansatz (7) describes a probability density function f, which for any (ω,t) looks as a bunch in phases θ. In particular, $\arg z(\omega,t)$ reflects the location of the bunch peak, whereas $|z(\omega,t)|$ characterizes the degree to which the oscillators are bunched. Based on this interpretation, we say that oscillators with natural frequencies ω are *coherent* or *incoherent* at time t, if $|z(\omega,t)| = 1$ or $|z(\omega,t)| < 1$, respectively. Note that the function f is

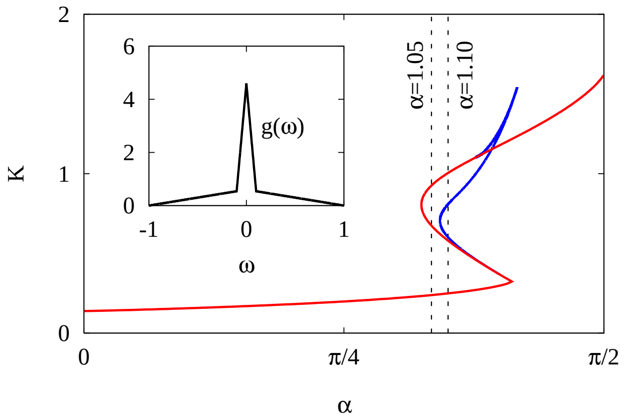

Figure 2: (color online) Bifurcation diagram for a piecewise linear frequency distribution (see insert panel or formulae (21), (23) with $\delta = 0.1$ and $\tau = 0.6$). Red/gray: stability boundary of the completely incoherent state. Blue/dark: fold of partially synchronized states. Dashed vertical lines indicate the cross-sections given in Fig. 1(b)–(c).

uniquely determined for any complex-valued functions z with $|z(\omega,t)| \leq 1$, since (7) implies in particular that

$$g(\omega)z(\omega,t) = \int_0^{2\pi} f(\omega,\theta,t)e^{i\theta}d\theta. \tag{11}$$

Stability of the incoherent state. Equation (8) always has the trivial solution $z(\omega,t) = 0$, which after substitution in (7) results in the *completely incoherent state*

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi}.$$

Linearizing Eq. (8) around zero, we obtain the linear equation

$$\frac{du}{dt} = i\omega u + \frac{K}{2}e^{-i\alpha}\mathcal{G}u \tag{12}$$

that gives for $u = u_0(\omega)e^{\lambda t}$ the spectral equation

$$\frac{1}{K}e^{i\alpha} = \frac{i}{2}\int_{-\infty}^{\infty}\frac{g(\omega)d\omega}{\omega + i\lambda}.$$

The completely incoherent state looses its stability if the above equation has a solution $\lambda = i\Omega + 0$ with $\Omega \in \mathbb{R}$, i.e. the stability boundary for incoherence is given parametrically by the singular integral

$$\frac{1}{K}e^{i\alpha} = J(\Omega) := \lim_{\varepsilon\to+0}\frac{i}{2}\int_{-\infty}^{\infty}\frac{g(\omega)d\omega}{\omega - \Omega + i\varepsilon} \tag{13}$$

that, for a given distribution $g(\omega)$, can be calculated by standard methods.

Partially coherent states. Since Eq. (8) is equivariant with respect to the complex phase shifts $z \mapsto e^{i\phi}z$, we may expect solutions of the form

$$z(\omega,t) = \hat{z}(\omega)e^{i\Omega t}, \tag{14}$$

where Ω is a real common frequency and $\hat{z}(\omega)$ is a continuous, complex-valued function of ω alone. According to (10), functions $\hat{z}(\omega)$ with $|\hat{z}(\omega)| = 1$ for all $\omega \in \mathrm{Supp}\, g$ correspond to *completely synchronized states*, whereas those with $|\hat{z}(\omega)| = 1$ on a proper subset of $\mathrm{Supp}\, g$ only, correspond to *partially synchronized states*. In order to find all possible partially and completely synchronized states described by ansatz (14), we substitute (14) into Eq. (8) and obtain an integral self-consistency equation

$$Ke^{i\alpha}\hat{z}^2\mathcal{G}\hat{z}^* - 2i(\omega-\Omega)\hat{z} - Ke^{-i\alpha}\mathcal{G}\hat{z} = 0,$$

which connects the collective frequency Ω with corresponding complex amplitudes $\hat{z}(\omega)$. It is convenient to rewrite this equation in a shorter form

$$Z^*\hat{z}^2 - 2(\omega-\Omega)\hat{z} + Z = 0, \tag{15}$$

where

$$Z = iKe^{-i\alpha}\mathcal{G}\hat{z}. \tag{16}$$

Solving Eq. (15) with respect to $\hat{z}(\omega)$ we find two roots

$$\hat{z}_\pm(\omega) = \frac{(\omega-\Omega) \pm \sqrt{(\omega-\Omega)^2 - |Z|^2}}{Z^*},$$

which are complex conjugate for $|\omega-\Omega| < |Z|$. Recalling that we are restricted to solutions of Eq. (8) with $|z(\omega,t)| \le 1$, we can construct by

$$\hat{z}(\omega) = h\left(\frac{\omega-\Omega}{p}\right) \tag{17}$$

with

$$h(s) = \begin{cases} \left(1-\sqrt{1-s^{-2}}\right)s & \text{for} \quad |s|>1, \\ s - i\sqrt{1-s^2} & \text{for} \quad |s| \le 1. \end{cases} \tag{18}$$

a suitable solution of (15), where (16) leads to the self-consistency condition

$$1 = iKe^{-i\alpha}\frac{1}{p}\mathcal{G}h\left(\frac{\omega-\Omega}{p}\right).$$

Using (9) we can rewrite it as

$$\begin{aligned} \frac{1}{K}e^{i\alpha} &= \frac{i}{p}\int_{-\infty}^{\infty} g(\omega)h\left(\frac{\omega-\Omega}{p}\right)d\omega \\ &= i\int_{-\infty}^{\infty} g(\Omega+ps)h(s)ds =: H(p,\Omega). \end{aligned} \tag{19}$$

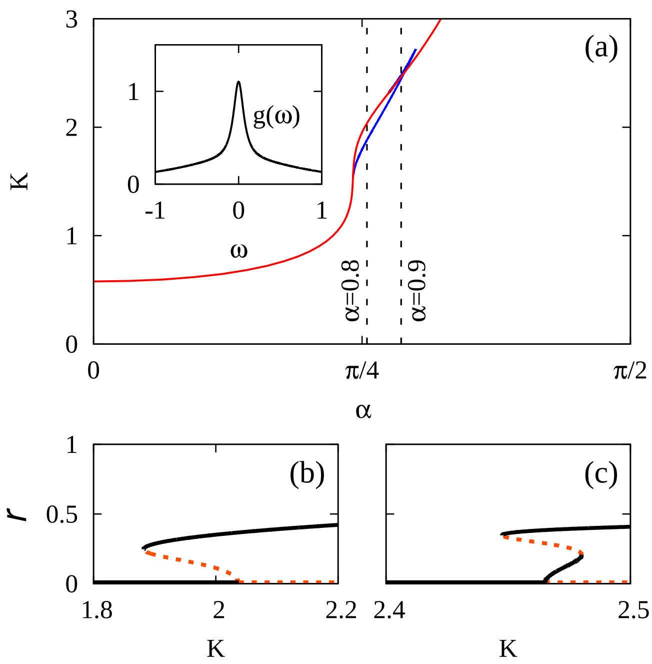

Figure 3: (color online) (a): Bifurcation diagram for a frequency distribution given as a superposition of two Lorentzians (see insert panel or formulae (21), (22) with $\delta = 0.075$ and $\tau = 0.8$). Red/gray: stability boundary of the completely incoherent state. Blue/dark: fold of partially synchronized states. (b) and (c): Examples of synchronization transitions with bistability ($\alpha = 0.8$ and $\alpha = 0.9$ as indicated by dashed lines in panel (a)). Stable and unstable branches are marked as in Fig. 1.

In this form, it can be interpreted as a complex function that uniquely determines the system parameters $\alpha,\ K$ from the solution parameters $\Omega,\ p$. Remarkably, the degeneracies

$$\chi(p,\Omega) := \det\begin{pmatrix} \mathrm{Re}\, H_p & \mathrm{Im}\, H_p \\ \mathrm{Re}\, H_\Omega & \mathrm{Im}\, H_\Omega \end{pmatrix} = 0. \tag{20}$$

of this functional dependence provide immediately the set of bifurcations in the system parameters $\alpha,\ K$.

Bifurcations in the synchronization transitions. We will use now this general result to study the bifurcations in the synchronization transitions for various choices of the frequency distribution. It turns out, that for a simple Gaussian, Lorentzian or finitely supported constant distributions there are no bifurcations and the critical coupling rate, given by (13), is a monotonously increasing function of $\alpha \in (0, \pi/2)$. However, for slightly more complex but still unimodal frequency distributions, e.g. superposition of two Lorentzians, piecewise linear, or step functions, one can find bifurcations leading to non-trivial synchronization transitions. As an example, we will use two parameter families of distributions, constructed by an interpolation

$$g(\omega; \delta, \tau) = \tau g_1(\omega) + (1-\tau) g_\delta(\omega), \tag{21}$$

between two distributions of the same type g_σ, but different widths indicated by the index. Pos-

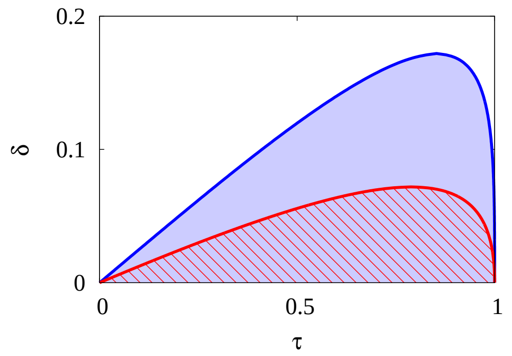

Figure 4: (color online) Bifurcation diagram for the two parameter family of frequency distributions (21), (22). Hatched region: parameters of frequency distributions that allow for the completely incoherent state to regain stability with increasing K. Shaded region: parameters of frequency distributions that allow for a bistability.

sible choices for g_σ are Lorentzians (cf. Fig 3)

$$g_\sigma(\omega) = \frac{1}{\pi}\frac{\sigma}{\omega^2+\sigma^2} \tag{22}$$

or normalized triangular distributions (cf. Fig 2)

$$g_\sigma(\omega) := \begin{cases} (\sigma-|\omega|)/\sigma^2 & \text{for} \quad |\omega| \leq \sigma, \\ 0 & \text{for} \quad |\omega| > \sigma \end{cases} . \tag{23}$$

At first, we calculated from (13) the critical coupling, where the completely incoherent state changes its stability, given by gray (online red) lines in Figs. 2 and 3(a). In the classical scenario, there is a single change of stability for each fixed α and the critical coupling strength increases with increasing α. However, for suitably chosen $g(\omega)$ this curve folds and multiple transitions occur, i.e. incoherence regains stability for increasing coupling. Then, we use (20) to identify the locations, where the synchronization transition curves fold (cf. Fig. 1) and coexistence of stable solutions emerges (dark (online blue) lines in Figs. 2 and 3(a)). Finally, in Fig. 4 we present for the two parameter family (21) of Lorenzians the parameter regions where folded synchronization transitions and multiple changes of stability of the incoherent states occur for suitably chosen parameter α.

Conclusions. Employing a frequency dependent version of the Ott-Antonsen approach, we performed to a large extent analytical bifurcation analysis, which unveiled several nonuniversal synchronization transitions that can occur for certain unimodal frequency distributions and suitably chosen values of the phase-lag parameter α. The results were confirmed by numerical simulations of the original Sakaguchi-Kuramoto system. In contrast to earlier studies, where an immediate reduction to finite dimensions has been performed, our approach allowed for a direct exploration of the infinite dimensional space of frequency distributions. The obtained results strongly underline that, as already pointed out in [22], the behavior in the general class of frequency distributions is not always covered by the easily tractable case of a simple Lorentzian distribution, which has been believed in many studies to provide the generic scenario for unimodal distributions. Indeed, already for the simple classical Sakaguchi-Kuramoto system with unimodal and symmetric frequency distribution a nonuniversal behavior can be found.

References

[1] H. Sakaguchi and Y. Kuramoto, Prog. Theor. Phys. **76**, 576 (1986).

[2] A. Winfree, *The Geometry of Biological Time* (Springer, Berlin Heidelberg New York, 2001).

[3] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin Heidelberg New York, 1984).

[4] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization, a universal concept in nonlinear sciences* (Cambridge University Press, Cambridge, 2001).

[5] P. A. Tass, *Phase Resetting in Medicine and Biology* (Springer, Berlin Heidelberg New York, 1999).

[6] S. Watanabe and S. H. Strogatz, Phys. Rev. Lett. **70**, 2391 (1993).

[7] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).

[8] M. C. Cross, A. Zumdieck, R. Lifshitz, and J. L. Rogers, Phys. Rev. Lett. **93**, 224101 (2004).

[9] G. Filatrella, A. H. Nielsen, and N. F. Pedersen, Eur. Phys. J. B **61**, 485 (2008).

[10] A. Pikovsky and M. Rosenblum, Phys. Rev. Lett. **101**, 264103 (2008).

[11] H. Hong and S. H. Strogatz, Phys. Rev. Lett. **106**, 054102 (2011).

[12] E. Montbrio and D. Pazo, Phys. Rev. Lett. **106**, 254101 (2011).

[13] S. H. Strogatz, Physica D **143**, 1 (2000).

[14] J. A. Acerbrón, L. L. Bonilla, C. J. Pérez-Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005).

[15] E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So, and T. M. Antonsen, Phys. Rev. E **79**, 026204 (2009).

[16] Y. Kuramoto and D. Battogtokh, Nonlinear Phenom. Complex Syst. **5**, 380 (2002).

[17] D. M. Abrams and S. H. Strogatz, Phys. Rev. Lett. **93**, 174102 (2004).

[18] C. R. Laing, Physica D **238**, 1569 (2009).

[19] E. A. Martens, C. R. Laing, and S. H. Strogatz, Phys. Rev. Lett. **104**, 044101 (2010).

[20] M. Wolfrum, O. E. Omel'chenko, S. Yanchuk, and Y. L. Maistrenko, Chaos **21**, 013112 (2011).

[21] M. Wolfrum and O. E. Omel'chenko, Phys. Rev. E **84**, 015201 (2011).

[22] L. F. Lafuerza, P. Colet, and R. Toral, Phys. Rev. Lett. **105**, 084101 (2010).

[23] E. Ott and T. M. Antonsen, Chaos **18**, 037113 (2008).

[24] E. Ott and T. M. Antonsen, Chaos **19**, 023117 (2009).

[25] E. Ott, B. R. Hunt, and T. M. Antonsen, Chaos **21**, 025112 (2011).