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Representation of hysteresis operators for vector-valued continuous monotaffine input functions by functions on strings

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Abstract

In Brokate-Sprekels-1996, it was shown that scalar-valued hysteresis operators for scalar-valued continuous piecewise monotone input functions can be uniquely represented by functionals defined on the set of all finite alternating strings of real numbers. Using this representation, various properties of these hysteresis operators were investigated.

In this work, it is shown that a similar representation result can be derived for hysteresis operators dealing with inputs in a general topological linear vector space. Introducing a new class of functions, the so-called *monotaffine* functions, which can be considered as a vector generalization of monotone scalar functions, and the convexity triple free strings on a vector space as a generalization of the alternating strings allows to formulate the corresponding representation result.

As an example for the application of the representation result, a vectorial formulation of the second and third Madelung rule are discussed.

1 Introduction

In [2, 4], Brokate and Sprekels derived a representation formula for hysteresis operators acting on scalar-valued continuous piecewise monotone input functions by considering functionals acting on alternating strings. This representation result was used in a number of papers, see, e.g., [2, 3, 4, 10, 12, 13, 14, 15, 20, 21, 24, 26, 28], to define and to investigate hysteresis operators. For example, in [4, Sec. 2.7], a set of conditions for the string representation of a hysteresis operator was formulated that are satisfied, if, and only if, the operator is a Preisach operator.

In the current work, hysteresis operators dealing with inputs in a general topological vector space are investigated. To be able to formulate a corresponding representation result, a generalization of the notion of monotone functions for vector-valued functions is introduced by considering the composition of a monotone and an affine function, which will be denoted as *monotaffine* functions. In the light of the observation that a string of real numbers is an alternating string as defined in [4] if and only if no element in the string can be written as the convex combination of its its predecessor and its successor, this condition is used to define convexity triple free strings of elements of the vector space as a generalization of alternating strings.

These preparations allow to formulate and to prove an extension of to the representation result in [2, 4] to hysteresis operators acting on vector-valued continuous piecewise monotaffine input functions. It will be shown that these operators can be generated by considering functions acting on convexity triple free strings of elements of the vector space. The author of this paper has already presented these definitions in [18], and the representation result has been announced there without proof.

This result will be used to investigate the second and third Madelung rules by generalizing the notion of Madelung deletion introduced in [2, 4].

The paper is organized as follows: In Section 2, some fundamental definitions will be presented; in Section 3 the representation result by Brokate and Sprekels will be briefly recalled, and the extension of this result to hysteresis operators with vector-valued inputs, being the main result of this paper, will be presented. The monotaffine functions and the convexity triple free strings will be introduced in Section 4. Section 5 contains the proof of the main result of the paper. In Section 6, a generalization of the Madelung deletion is formulated. Some examples for hysteresis operators and their representation by functions on strings will be shown in Section 7.

2 Fundamental Definitions

Let $T > 0$ denote some final time. Let X be some topological linear vector space, let Y be some nonempty set, and let $\mathrm{Map}\,([0,T],Y) := \{v : [0,T] \to Y\}$.

The following notations correspond to the ones in [4, Def. 2.2.2]:

2.1 Definition. Let a function $u : [0,T] \to \mathbb{R}$ be given. Let $t_a, t_b \in [0,T]$ with $t_a < t_b$ be given.

a) The function u is denoted as *(strictly) increasing on* $[t_a,t_b]$ if for all $s,t \in [t_a,t_b]$ with $s < t$ holds $u(s) \leq u(t)$ (resp. $u(s) < u(t)$).

b) The function u is denoted as *(strictly) decreasing on* $[t_a,t_b]$ if for all $s,t \in [t_a,t_b]$ with $s < t$ holds $u(s) \geq u(t)$ (resp. $u(s) > u(t)$).

c) The function u is denoted as monotone on $[t_a,t_b]$ if u is increasing on $[t_a,t_b]$ or/and decreasing on $[t_a,t_b]$.

Following the monographs [4, 19, 27], it is defined:

2.2 Definition. Let $\mathcal{H} : D(\mathcal{H})(\subseteq \mathrm{Map}\,([0,T],X)) \to \mathrm{Map}\,([0,T],Y)$ with $D(\mathcal{H}) \neq \emptyset$ be some operator.

a) The operator $\mathcal{H}$ is denoted as *hysteresis operator*, if it is causal and rate-independent according to the following definitions.

b) The operator $\mathcal{H}$ is said to be *causal* or to have the *Volterra property*, if for every $v,w \in D(\mathcal{H})$ and every $t \in [0,T]$ it holds: If $v(\tau) = w(\tau)$ is satisfied for all $\tau \in [0,t]$ then it follows that $\mathcal{H}[v](t) = \mathcal{H}[w](t)$.

c) The operator $\mathcal{H}$ is called *rate-independent*, if for every $v \in D(\mathcal{H})$ and every admissible time-transformation $\alpha : [0,T] \to [0,T]$ (see Def. 2.3 below) with $v \circ \alpha \in D(\mathcal{H})$ it holds that $\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t))$ for all $t \in [0,T]$.

2.3 Definition. A function $\alpha : [0,T] \to [0,T]$ is an admissible time transformation if and only if $\alpha(0) = 0$, $\alpha(T) = T$, α is continuous, and α is increasing (not necessary strictly increasing).

3 Representation results for hysteresis operators

3.1 Representation results for hysteresis operators with scalar-valued inputs

In [2, 4], Brokate and Sprekels investigated hysteresis operators for scalar-valued continuous piecewise monotone input functions.

Following [4, Def. 2.2.3], it is defined:

3.1 Definition. Let S_A denote the set of all finite alternating strings of real numbers, i.e. ,

$$S_A := \{(v_0, v_1, \dots, v_n) \in \mathbb{R}^{n+1} \,|\, n \geq 1,\ (v_{i+1} - v_i)(v_i - v_{i-1}) < 0, \quad \forall\, 1 \leq i < n\}. \tag{3.1}$$

Following [4, p. 34], it is considered

3.2 Definition. Let a function $u : [0,T] \to \mathbb{R}$ be given.

a) A partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ is a *monotonicity partition of* $[0,T]$ *for* u if u is monotone on $[t_{i-1}, t_i]$ for all $i = 1, \dots, n$.

b) u is denoted as *piecewise monotone* if there exists a monotonicity partition of $[0,T]$ for u.

c) If u is piecewise monotone the *standard monotonicity partition of* $[0,T]$ *for* u is the uniquely defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ so that t_i is the maximal number in $]t_{i-1}, T]$ with u being monotone on $[t_{i-1}, t_i]$ for all $i = 1, \dots, n$.

3.3 Definition. Let $\mathrm{M_{pm}}[0,T]$ be the set of all piecewise monotone functions from $[0,T]$ to $\mathbb{R}$, and let $\mathrm{C_{pm}}[0,T]$ be the set of all continuous piecewise monotone functions from $[0,T]$ to $\mathbb{R}$, i.e.:

$$\mathrm{M_{pm}}[0,T] := \big\{u : [0,T] \to \mathbb{R} \,|\, u \text{ is piecewise monotone}\big\}, \tag{3.2}$$

$$\mathrm{C_{pm}}[0,T] := \big\{u \in \mathrm{M_{pm}}[0,T] \,|\, u \text{ is continuous }\big\}. \tag{3.3}$$

Combining the representation result in [4, Pro. 2.2.5] with [4, Rem. 2.2.6, (2.18), (2.19), Def. 2.2.8, Prop. 2.2.9], one gets the following definition and the following theorem:

3.4 Definition. Let a function $G : S_A \to \mathbb{R}$ be given.

a) For $u \in \mathrm{M_{pm}}[0,T]$ let $\mathcal{H}_G^{\mathcal{GEN}}[u] : [0,T] \to \mathbb{R}$ be defined by considering the standard monotonicity partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ for u and defining

$$\mathcal{H}_G^{\mathcal{GEN}}[u](t) := G\left(u(t_0), u(t)\right), \quad \forall\, t \in [t_0, t_1], \tag{3.4a}$$

$$\mathcal{H}_G^{\mathcal{GEN}}[u](t) := G\left(u(t_0), \dots, u(t_{i-1}), u(t)\right), \qquad \forall\, t \in]t_{i-1}, t_i],\ i = 2, \dots, n. \tag{3.4b}$$

b) The mapping $\mathrm{M_{pm}}[0,T] \ni u \mapsto \mathcal{H}_G[u]$ is *the operator on* $\mathrm{M_{pm}}[0,T]$ *generated by* G. Its restriction to $\mathrm{C_{pm}}[0,T]$ is denoted as *the operator on* $\mathrm{C_{pm}}[0,T]$ *generated by* G.

3.5 Theorem. *a) Let a function $G : S_A \to \mathbb{R}$ be given. The operator on $\mathrm{M}_{\mathrm{pm}}[0,T]$ generated by G and the operator on $\mathrm{C}_{\mathrm{pm}}[0,T]$ generated by G are hysteresis operators.*

b) For every hysteresis operator $\mathcal{B} : \mathrm{C}_{\mathrm{pm}}[0,T] \to \mathrm{Map}\,([0,T],\mathbb{R})$ there exists a unique functional $G : S_A \to \mathbb{R}$ such that $\mathcal{B}$ is the operator on $\mathrm{C}_{\mathrm{pm}}[0,T]$ generated by G.

3.6 Remark.

a) For a given hysteresis operator $\mathcal{H} : \mathrm{C}_{\mathrm{pm}}[0,T] \to \mathrm{Map}\,([0,T],\mathbb{R})$ the function $G : S_A \to \mathbb{R}$ so that $\mathcal{B}$ is the operator on $\mathrm{C}_{\mathrm{pm}}[0,T]$ generated by G can be determined by evaluating $\mathcal{B}$ for the linear interpolates considered in [4, p. 34], i.e. , for the piecewise affine functions as in Def. 4.17 with $X = \mathbb{R}$.

b) The representation result yields that for evaluating a hysteresis operator for continuous piecewise monotone input it is sufficient to memorize the local maxima and minima of the input, and one does not need to keep track of the details of the input between these extrema.

c) The representation result allows to formulate conditions for a hysteresis operator on $\mathrm{C}_{\mathrm{pm}}[0,T]$ as conditions for the functional on S_A generating the operator, and to invesigate these conditions. There is a number of conditions which can be easiear be formulated and be checked for the string representation, e.g., the forgetting according to some deletion rule. Examples can be found in [2, 4].

3.2 Representation result for hysteresis operators with vector-valued inputs

Now, the main result of this paper is presented. The exact formulation of the addaptions to vector-valued inputs replacing the definitions used above will be presented in Section 4.

The set of all alternating strings considered above is replaced by the set $S_F(X)$ of all convexity triple free string of elements of X that will be defined in Def. 4.1.

The piecewise monotone functions are replaced by piecewise monotaffine functions, see Def. 4.9. The set $\mathrm{M}_{\mathrm{pm}}[0,T]$ is replaced by the set $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ of all piecewise monotaffine functions from $[0,T]$ to X, see also Def. 4.9. The set $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ of all of continuous functions in $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$, being also introduced in Def. 4.9, replaces $\mathrm{C}_{\mathrm{pm}}[0,T]$.

The standard monotonicity partition is replaced by the standard monotaffinicity decomposition defined in Def. 4.12.

3.7 Definition. Let some function $G : S_F(X) \to Y$ be given.

a) Let $u \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ and let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard monotaffinicity decomposition of $[0,T]$ for u be given. Now, $\mathcal{H}_G^{\mathcal{GEN}}[u] : [0,T] \to Y$ is defined by requesting that the equations in (3.4) are satisfied, using that Lemma 4.14 yields that the right-hand sides in the equations in (3.4) are well defined.

b) The mapping $\mathcal{H}_G^{\mathcal{GEN}} : \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\left([0,T],Y\right)$ that is defined by a) is denoted as the *operator on* $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ *generated by* G or as the *hysteresis operator on* $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ *generated by* G.

c) The mapping $\mathcal{H}_G^{\mathcal{GEN},\mathrm{C}} : \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\left([0,T],Y\right)$ is defined as the restriction of $\mathcal{H}_G^{\mathcal{GEN}}$ to $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ and is denoted as the *operator on* $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ *generated by* G or as the *hysteresis operator on* $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ *generated by* G.

3.8 Theorem. *a) Let some function $G : S_F(X) \to Y$ be given. The operator $\mathcal{H}_G^{\mathcal{GEN}}$ on $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by G and the operator $\mathcal{H}_G^{\mathcal{GEN},\mathrm{C}}$ on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by G are hysteresis operators.*

b) For every hysteresis operator $\mathcal{G} : \mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\left([0,T],Y\right)$ it holds: There is a unique function $G : S_F(X) \to Y$ such that $\mathcal{G}$ is the hysteresis operator on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by G. This function is the string function $F_{\mathcal{G}}^{\mathrm{gen}} : S_F(X) \to Y$ generated by $\mathcal{G}$ according to Def. 4.18.

3.9 Remark.

a) If one needs to evaluate a hysteresis operator $\mathcal{H}$ acting on all continuous piecewise monotaffine functions, then it is sufficient to keep track of the positions of the changes of direction of the input function, and to use these values as input for the string function $F_{\mathcal{G}}^{\mathrm{gen}} : S_F(X) \to Y$ generated by $\mathcal{H}$ (see Def. 4.18).

b) If a function $G : S_F(X) \to Y$ satisfies appropriate locally uniform continuity conditions, one can extend the hysteresis operator $\mathcal{H}_G^{\mathcal{GEN},\mathrm{C}}$ on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by G to a hysteresis operator on $\mathrm{C}\left([0,T];X\right)$. Details can be found in the forthcoming thesis [16].

c) As it is done for hysteresis operatos with scalar inputs in [2, 4, 21], one can investigated properties of a hysteresis operator with vector-valued input by considering the string function generated by the operator. See, for example, Section 6, Section 7 and the forthcoming thesis [16].

d) The results in Theorem 3.8 are also satisfied, if X is replaced by some convex supset of X. Details, and considerations for dealing with more general subsets of X can be found in the forthcoming thesis [16].

e) If one is dealing with a hysteresis operator $\mathcal{H}$ acting on all monotaffine functions, then one can consider the string function $F_{\mathcal{H}}^{\mathrm{gen}} : S_F(X) \to Y$ generated by $\mathcal{H}$ and the restriction $\mathcal{H}^{\mathrm{C}}$ of $\mathcal{H}$ to $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ that is also the hysteresis operator on $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by $F_{\mathcal{H}}^{\mathrm{gen}}$.

   It can be shown (see the forthcoming paper [17]) that the hysteresis operator on $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by $F_{\mathcal{H}}^{\mathrm{gen}}$ is just the restriction to $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ of the arclen-extension of $\mathcal{H}^{\mathrm{C}}$ to $BV([0,T],X)$ considered in [25]. These operator may be different from $\mathcal{H}$.

For example, the hysteresis operator

$$\mathcal{H} : \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\,([0,T],X)\,, \tag{3.5}$$

$$\mathcal{H}[u](t) = \begin{cases} u(t), & \text{if the restriction of } u \text{ to [0,t] is continuous,} \\ 0_X, & \text{otherwise,} \end{cases} \tag{3.6}$$

is well defined and its restriction to $\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)$ is just the identity. Hence, we see that the string function $F^{\mathrm{gen}}_{\mathcal{H}} : S_F(X) \to X$ generated by $\mathcal{H}$ satisfies $F^{\mathrm{gen}}_{\mathcal{H}}(v_0,\ldots,v_n) = v_n$. The hysteresis operator on $\mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ generated by $F^{\mathrm{gen}}_{\mathcal{H}}$ maps every piecewise monotaffine functions to itself, and is therefore different from $\mathcal{H}$.

Further investigation for hysteresis operators dealing with discontinuous inputs and their representation by function on strings can be found in the forthcoming paper [17] and in the forthcoming thesis [16].

4 Monotaffine functions and convexity triple free strings

4.1 Convexity triple free strings

The observation that a string of real numbers is an alternating string as defined in Def. 3.1, i.e. , as in [2, 4], if and only if no element in the string can be written as the convex combination of its its predecessor and its successor has been the reason for the following definition of convexity triple free strings.

4.1 Definition. a) A *string of elements of* X is any $(v_0,\ldots,v_n) \in X^{n+1}$ with $n \in \mathbb{N}$.

b) A string $(v_0,\ldots,v_n)$ of elements of X is a *convexity triple free string of elements of* X if $v_i \notin \mathrm{conv}(v_{i-1},v_{i+1})$ for all $i = 1,\ldots,n-1$, with

$$\mathrm{conv}(x,w) := \{(1-\lambda)x + \lambda w \mid \lambda \in [0,1]\}\,, \quad \forall\, x,w \in X. \tag{4.1}$$

c) Let $S_F(X)$ denote the set of all convexity triple free string of elements of X.

4.2 Remark. A string $(v_0,\ldots,v_n) \in \mathbb{R}^{n+1}$ is an alternating string according to Def. 3.1, i.e. , to [4, Def. 2.2.3], if it is a convexity triple free strings of elements of $\mathbb{R}$ according to the definition above.

Hence it holds $S_F(\mathbb{R}) = S_A$.

4.2 Monotaffine functions

To define an appropriate generalization of monotonicity for scalar function for functions with values in the vector space X, the composition of a **monot**one with an **affine** function is considered and leads to a **monotaffine** function according to the following definition.

4.3 Definition. Let some $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and some function $u : [0, T] \to X$ be given.

a) u is denoted as *affine on* $[t_1, t_2]$ if

$$u(t) = \frac{t_2 - t}{t_2 - t_1} u(t_1) + \frac{t - t_1}{t_2 - t_1} u(t_2), \quad \forall\, t \in [t_1, t_2]. \tag{4.2}$$

b) u is denoted as *monotaffine on* $[t_1, t_2]$ if there exists a monotone increasing (not necessary strictly increasing) function $\beta : [t_1, t_2] \to [0, 1]$ such that

$$u(t) = (1 - \beta(t))u(t_1) + \beta(t)u(t_2), \quad \forall\, t \in [t_1, t_2]. \tag{4.3}$$

4.4 Remark. It is easy to see that a scalar valued function is monotone on some closed interval if and only if it is monotaffine according to the above definition.

4.5 Remark. Let some $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and some function $u : [0, T] \to X$ be given. If u is monotaffine on $[t_1, t_2]$, then it holds for all $s_1, s_2 \in [t_1, t_2]$ with $s_1 < s_2$ that

a) u is monotaffine on $[s_1, s_2]$,

b) It holds

$$u([s_1, s_2]) \subseteq \mathrm{conv}(u(s_1), u(s_2)) \subseteq \mathrm{conv}(u(t_1), u(t_2)) \tag{4.4}$$

with $\mathrm{conv}(\cdot, \cdot)$ defined as in (4.1).

4.6 Lemma. *Let some $t_1, t_2, t_3 \in [0, T]$ with $t_1 < t_2 < t_3$ and some function $u : [0, T] \to X$ be given so that u is monotaffine on $[t_1, t_2]$ and monotaffine on $[t_2, t_3]$. Then it holds: u is monotaffine on $[t_1, t_3]$ if and only if $u(t_2) \in \mathrm{conv}(u(t_1), u(t_3))$.*

Proof. $\Longrightarrow$**:** If u is monotaffine on $[t_1, t_3]$, then there is some $\beta : [t_1, t_3] \to [0, 1]$ such that (4.3) holds with t_2 replaced by t_3. Considering the resulting equation for $t = t_2$, we have

$$u(t_2) = (1 - \beta(t_2))u(t_1) + \beta(t_2)u(t_3) \in \mathrm{conv}(u(t_1), u(t_3)). \tag{4.5}$$

$\Longleftarrow$**:** If $u(t_2) \in \mathrm{conv}(u(t_1), u(t_3))$, then there is some $\lambda \in [0, 1]$ such that

$$u(t_2) = (1 - \lambda)u(t_1) + \lambda u(t_3). \tag{4.6}$$

Since u is monotaffine on $[t_1, t_2]$ and monotaffine on $[t_2, t_3]$, there exist increasing functions $\beta_1 : [t_1, t_2] \to [0, 1]$ and $\beta_2 : [t_2, t_3] \to [0, 1]$ such that

$$u(t) = (1 - \beta_1(t))u(t_1) + \beta_1(t)u(t_2), \quad \forall\, t \in [t_1, t_2], \tag{4.7}$$
$$u(t) = (1 - \beta_2(t))u(t_2) + \beta_2(t)u(t_3), \quad \forall\, t \in [t_2, t_3]. \tag{4.8}$$

Defining now $\gamma : [t_1, t_3] \to [0, 1]$ by

$$\gamma(t) = \begin{cases} \lambda\beta_1(t), & \text{if} \quad t < t_2, \\ \lambda, & \text{if} \quad t = t_2, \\ \lambda + \beta_2(t)(1 - \lambda), & \text{if} \quad t > t_2, \end{cases} \tag{4.9}$$

we have an increasing function. Recalling (4.6)–(4.8), we deduce that

$$u(t) = (1 - \gamma(t))u(t_1) + \gamma(t)u(t_3), \quad \forall\, t \in [t_1, t_3]. \tag{4.10}$$

Hence, it is shown that u is monotaffine on $[t_1, t_3]$.

□

4.7 Corollary. *Let some $t_1, t_2, t_3, t_4 \in [0, T]$ with $t_1 < t_2 < t_3 < t_4$ and some function $u : [0, T] \to X$ be given, so that u is monotaffine on $[t_1, t_3]$ and monotaffine on $[t_2, t_4]$. Then it follows that u is monotaffine on $[t_1, t_4]$.*

Proof. Since u is monotaffine on $[t_2, t_4]$ and $t_3 \in [t_2, t_4]$, it holds that $u(t_3) \in \operatorname{conv}(u(t_2), u(t_4))$ and that u is monotaffine on $[t_3, t_4]$. Recalling that u is monotaffine on $[t_1, t_3]$ and using Lemma 4.6 yields that u is is monotaffine on $[t_1, t_4]$. □

4.8 Corollary. *Let some $t_1, t_2, t_3, t_4 \in [0, T]$ with $t_1 < t_2 < t_3 \le t_4$ and some function $u : [0, T] \to X$ be given, so that u is monotaffine on $[t_1, t_2]$ and monotaffine on $[t_2, t_4]$. Moreover, assume that $u(t_2) \notin \operatorname{conv}(u(t_1), u(t_4))$. Then it holds that u is not monotaffine on $[t_1, t_3]$.*

Proof. For a proof by contradiction, assume that u is monotaffine on $[t_1, t_3]$. Hence, Corollary 4.7 yields that u is monotaffine on $[t_1, t_4]$. This implies that $u(t_2) \in \operatorname{conv}(u(t_1), u(t_4))$, which is a contradiction. □

4.3 Piecewise monotaffine function

The following definition generalizes Def. 3.2:

4.9 Definition. a) A function $u : [0, T] \to X$ is denoted as *piecewise monotaffine* if there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that u is monotaffine on $[t_{i-1}, t_i]$ for all $i = 1, \ldots, n$.

b) Let $\mathrm{M}_{\text{pw.m.a.}}([0, T]; X)$ be the set of all piecewise monotaffine functions from $[0, T]$ to X, i.e., $\mathrm{M}_{\text{pw.m.a.}}([0, T]; X) := \{u : [0, T] \to X \mid u \text{ is piecewise monotaffine}\}$.

c) Let $\mathrm{C}_{\text{pw.m.a.}}([0, T]; X)$ be the set of all continuous, piecewise monotaffine functions form $[0, T]$ to X, i.e., $\mathrm{C}_{\text{pw.m.a.}}([0, T]; X) := \mathrm{M}_{\text{pw.m.a.}}([0, T]; X) \cap \mathrm{C}([0, T]; X)$.

4.10 Remark. It holds $\mathrm{C}_{\text{pw.m.a.}}([0, T]; \mathbb{R}) = \mathrm{C}_{\text{pm}}[0, T]$.

In Fig. 1, the graph of a function $u \in \mathrm{C}_{\text{pw.m.a.}}([0, T]; \mathbb{R}^2)$ is shown and the values $u(t_i)$ of the function computed for the corresponding standard monotaffinicity partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ for u are marked.

4.11 Lemma. *Let $u \in \mathrm{M}_{\text{pw.m.a.}}([0, T]; X)$ and let $s_a \in [0, T[$ be given. Then there exists a maximal element in the following set:*

$$M := \big\{ s \in]s_a, T] \mid u \textit{ is monotaffine on } [s_a, s] \big\}. \tag{4.11}$$

Proof. Thanks to the assumption, there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that u is monotaffine on $[t_{i-1}, t_i]$ for all $i = 1, \ldots, n$. Hence, there is some $j \in \{1, \ldots, n\}$ such that $s_a \in [t_{j-1}, t_j[$. It follows that u is monotaffine on $[s_a, t_j]$.

- It $j = n$, then u is monotaffine on $[s_a, t_j] = [s_a, T]$. Therefore, we see that T is the maximal number in the set M.
- It $j < n$ and $u(t_j) \notin \operatorname{conv}(u(s_a), u(t_{j+1}))$, then if follows by Corollary 4.8 that, for all $t \in]t_j, t_{j+1}]$, it holds that u is not monotaffine on $[s_a, t]$. Therefore, t_j is the maximal element in the set M.
- It $j < n$ and $u(t_j) \in \operatorname{conv}(u(s_a), u(t_{j+1}))$, then there exist some maximal $k \in \{j + 1, \ldots, n\}$ such that $u(t_i) \in \operatorname{conv}(u(s_a), u(t_{i+1}))$ holds for all $i = j, \ldots, k - 1$. Applying Lemma 4.6 for $i = j, \ldots, k - 1$, we deduce that u is monotaffine on $[s_a, t_{i+1}]$ for all $i = j, \ldots, k - 1$.

  Hence, is follows that u is monotaffine on $[s_a, t_k]$.

  - If $t_k = T$, then it follows that T is the maximal element in the set M defined in (4.11).
  - If $t_k < T$, then it follows that $k < n$ and that $u(t_k) \notin \operatorname{conv}(u(s_a), u(t_{k+1}))$. For all $t \in]t_k, t_{k+1}]$, we use Corollary 4.8 and deduce that u is not monotaffine on $[s_a, t]$

    Therefore, t_k is the maximum of the set M defined in (4.11).

□

The above lemma allows to formulate the following definition:

4.12 Definition. Let $u \in \mathrm{M}_{\mathrm{pw.m.a.}}([0, T]; X)$ be given.

The *standard monotaffinicity partition of* $[0, T]$ *for* u is the uniquely defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ so that, for all $i = 1, \ldots, n$, it holds that

$$t_i := \max\big\{s \in]t_{i-1}, T] \mid u \text{ is monotaffine on } [t_{i-1}, s]\big\}. \tag{4.12}$$

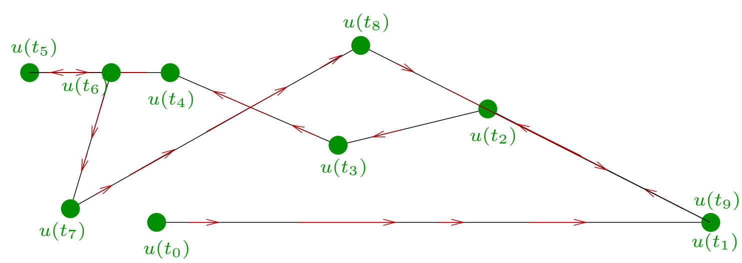

Figure 1: Graph of a monotaffine function mapping $[0, T]$ to $\mathbb{R}^2$ and values of $u(t_i)$ for the standard monotaffinicity partition $0 = t_0 < t_1 < \cdots < t_9 = T$ of $[0, T]$ for u.

To prepare further investigations, the following lemma is formulated:

4.13 Lemma. *Let $u \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ be given and let $\alpha : [0,T] \to [0,T]$ be an admissible time transformation.*

a) It holds $u \circ \alpha \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$.

b) Let the standard monotaffinicity partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ for u be given. Let $s_0, \ldots, s_n \in [0,T]$ be defined by $s_0 = 0$ and

$$s_i := \max\{s \in [0,t] \,|\, \alpha(s) = t_i\}, \quad \forall\, i = 1, \ldots, n. \tag{4.13}$$

Then it follows that $0 = s_0 < s_1 < \cdots < s_n = T$ is the standard monotaffinicity partition of $[0,T]$ for $u \circ \alpha$.

Proof. Using induction, it remains to show that for all $i = 1, \ldots, n$ it holds that $s_{i-1} < s_i$, that $u \circ \alpha$ is monotaffine on $[s_{i-1}, s_i]$, and that for all $\overline{s} \in]s_{i-1}, T]$ such that $u \circ \alpha$ is monotaffine on $[s_{i-1}, \overline{s}]$ it holds that $\overline{s} \leq s_i$.

1 Since $\alpha(s_{i-1}) = t_{i-1} < t_i = \alpha(s_i) = t_i$ and α is an admissible time transformation of $[0,T]$, we have $s_{i-1} < s_i$ and

$$\alpha([s_{i-1}, s_i]) = [t_{i-1}, t_i]. \tag{4.14}$$

2 Since u is monotaffine on $[t_{i-1}, t_i]$ there exists an increasing (not necessary strictly increasing) function $\beta : [t_{i-1}, t_i] \to [0,1]$ such that

$$u(t) = (1 - \beta(t))u(t_{i-1}) + \beta(t)u(t_i), \quad \forall\, t \in [t_{i-1}, t_i]. \tag{4.15}$$

Thanks to (4.14), we can define $\gamma : [s_{i-1}, s_i] \to [0,1]$ by $\gamma := \beta \circ \alpha\big|_{[s_{i-1}, s_i]}$. We see that for all $s \in [s_{i-1}, s_i]$ holds:

$$\begin{aligned} u \circ \alpha(s) = u\,(\alpha(s)) &= (1 - \beta(\alpha(s)))u(t_{i-1}) + \beta(\alpha(s))u(t_i) \\ &= (1 - \gamma(s))u \circ \alpha(s_{i-1}) + \gamma(s)u \circ \alpha(s_i). \end{aligned} \tag{4.16}$$

Hence, we see that $u \circ \alpha$ is monotaffine on $[s_{i-1}, s_i]$.

3 Consider any $\overline{s} \in]s_{i-1}, T]$ such that $u \circ \alpha$ is monotaffine on $[s_{i-1}, \overline{s}]$. Hence, there is an increasing function $\overline{\gamma} : [s_{i-1}, \overline{s}] \to [0,1]$ such that

$$u \circ \alpha(s) = (1 - \overline{\gamma}(s))\, u \circ \alpha(s_{i-1}) + \overline{\gamma}(s) u \circ \alpha(\overline{s}), \quad \forall\, s \in [s_{i-1}, \overline{s}]. \tag{4.17}$$

Since α and $\overline{\gamma}$ are monotone increasing functions, and α is also continuous, it is easy to see that $\overline{\beta} : [t_{i-1}, \alpha(\overline{s})] \to [0,1]$ with

$$\overline{\beta}(t) := \begin{cases} 0, & \text{if} \quad t = t_{i-1}, \\ \overline{\gamma}\,(\max\{s \in [s_{i-1}, \overline{s}] \,|\, \alpha(s) = t\}), & \text{otherwise} \end{cases} \tag{4.18}$$

defines an increasing function. A straightforward calculation, using that α is continuous and that (4.16) holds, yields that (4.15) is satisfied with t_i replaced by $\alpha(\overline{s})$ and β replaced by $\overline{\beta}$. Hence, it is proved that u is monotaffine on $[t_i, \alpha(\overline{s})]$. The definition of t_i yields that $\alpha(\overline{s}) \leq t_i = \alpha(s_i)$. Recalling the monotonicity of α, we deduce that $\overline{s} \leq s_i$.

□

4.4 Connections between piecewise monotaffine functions and strings

4.14 Lemma. *Let $u \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ be given. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard monotaffinicity partition of $[0,T]$ for u. Then it holds.*

$$(u(t_0), u(t)) \in S_F(X), \quad \forall\, t \in [t_0, t_1], \tag{4.19a}$$
$$(u(t_0), u(t_1), \ldots, u(t_{i-1}), u(t)) \in S_F(X), \quad \forall\, t \in]t_{i-1}, t_i], \quad i = 2, \ldots, n. \tag{4.19b}$$

Proof. Def. 4.1 yields that $X \times X \subset S_F(X)$. Hence, we see that the first assertion follows. To prove the second assertion by a contradiction argument, assume that there exists some $t \in]t_{i-1}, t_i]$ and some $i \in \{2, \ldots, n\}$ such that $(u(t_0), u(t_1), \ldots, u(t_{i-1}), u(t)) \notin S_F(X)$. Then there is some $k \in \{1, \ldots, i-1\}$ such that

$$u(t_k) \in \operatorname{conv}\left(u(t_{k-1}), u(\tilde{t_k})\right) \tag{4.20}$$

with

$$\tilde{t}_k := \begin{cases} t_{k+1}, & \text{if } k < i-1, \\ t, & \text{if } k = i-1. \end{cases} \tag{4.21}$$

Since $\tilde{t}_k \in]t_k, t_{k+1}]$ and u is monotaffine on $[t_k, t_{k+1}]$, it holds therefore by Remark 4.5 that u is monotaffine on $[t_k, \tilde{t}_k]$. Using that u is moreover monotaffine on $[t_{k-1}, t_k]$ and that (4.20) holds, we deduce by recalling Lemma 4.6 that u is monotaffine on $[t_{k-1}, \tilde{t}_k]$.

The definition of the standard monotaffinicity partition of $[0,T]$ for u yields that therefore $\tilde{t}_k$ must be smaller or equal to t_k. This is a contradiction since $\tilde{t}_k > t_k$. $\square$

4.15 Definition. Let $V = (v_0, \ldots, v_n) \in S_F(X)$ be given. The function $\pi_{\mathrm{pw.af.}}[V] : [0,T] \to \operatorname{conv}\{v_0, \ldots, v_n\}$ is defined as the piecewise affine function that is equal to v_i at time $\frac{i}{n}T$ for all $i = 0, \ldots, n$ and that is affine on $\left[\frac{i-1}{n}T, \frac{i}{n}T\right]$ for all $i = 1, \ldots, n$. Hence, for all $i = 1, \ldots, n$, it holds that

$$\pi_{\mathrm{pw.af.}}[V](t) = \left(i - \frac{n}{T}t\right) v_{i-1} + \left(\frac{n}{T}t - (i-1)\right) v_i\,, \quad \forall\, t \in \left[\frac{i-1}{n}T, \frac{i}{n}T\right]. \tag{4.22}$$

4.16 Lemma. *Let $V = (v_0, \ldots, v_n) \in S_F(X)$ be given. Then it holds that $0 = \frac{0}{n}T < \frac{1}{n}T < \cdots < \frac{i}{n}T < \cdots < \frac{n}{n}T = T$ is the standard monotaffinicity partition of $[0,T]$ for $\pi_{\mathrm{pw.af.}}[V]$.*

Proof. It holds that u is monotaffine on $\left[\frac{i-1}{n}T, \frac{i}{n}T\right]$ for all $i = 1, \ldots, n$. For all $i = 1, \ldots, n-1$ it holds

$$u\left(\frac{i}{n}T\right) = v_i \notin \operatorname{conv}(v_{i-1}, v_{i+1}) = \operatorname{conv}\left(u\left(\frac{i-1}{n}T\right), u\left(\frac{i+1}{n}T\right)\right), \tag{4.23}$$

such that recalling Corollary 4.8 yields that $\frac{i}{n}T$ is the maximal number t in $[\frac{i-1}{n}, T]$ with u being monotaffine on $\left[\frac{i-1}{n}T, t\right]$. Moreover, since u is monotaffine on $\left[\frac{n-1}{n}T, \frac{n}{n}T\right]$, $T = \frac{n}{n}T$ is the maximal number $t \in [\frac{i-1}{n}, T]$ with u being monotaffine on $\left[\frac{n-1}{n}T, t\right]$. $\square$

4.5 Functions acting on strings generated from operators

4.17 Definition. Let $\mathrm{C}_{\mathrm{pw.af.}}([0,T];X)$ be the set of all piecewise affine functions.

4.18 Definition. Let $\mathcal{H} : D(\mathcal{H})(\subseteq \mathrm{Map}\,([0,T],X)) \to \mathrm{Map}\,([0,T],Y)$ be an operator so that all piecewise affine functions belong to its domain of definition, i.e. , so that $\mathrm{C}_{\mathrm{pw.af.}}([0,T];X) \subseteq D(\mathcal{H})$. Now, the *string function* $F_{\mathcal{H}}^{\mathrm{gen}} : S_F(X) \to Y$ *generated by* $\mathcal{H}$ is defined by

$$F_{\mathcal{H}}^{\mathrm{gen}}(V) := \mathcal{H}[\pi_{\mathrm{pw.af.}}[V]](T), \quad \forall\, V \in S(X). \tag{4.24}$$

5 Proof of Theorem 3.8

The following proofs and lemmas have been inspirated by considerations for operatos with scalar inputs and strings with elements in $\mathbb{R}$ in [4, Sec. 2].

5.1 Proof of assertion a) in Theorem 3.8

Proof. Let some function $G : S_F(X) \to Y$ be given.

To prove that the mapping $\mathcal{H}_G^{\mathcal{GEN}} : \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X) \to \mathrm{Map}\,([0,T],Y)$ defined in Def. 3.7 is a hysteresis operator, we have to check that the operator is causal and rate-independent.

1 *Proof of causality:* Let $v, w \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ and $t \in [0,T]$ be given so that $v(\tau) = w(\tau)$ is satisfied for all $\tau \in [0,t]$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the standard monotaffinicity decomposition of $[0,T]$ for v and let $0 = s_0 < s_1 < \cdots < s_m = T$ be the standard monotaffinicity decomposition of $[0,T]$ for w.

- If $t \in [t_0, t_1]$, then we know that w is monotaffine on $[t_0, t]$, and therefore $s_1 \geq t$. Using now (3.4a), we observe that

$$\mathcal{H}_G^{\mathcal{GEN}}[v](t) = G\,(v(t_0), v(t)) = G\,(w(t_0), w(t)) = \mathcal{H}_G^{\mathcal{GEN}}[w](t). \tag{5.1}$$

- If $t \in]t_{i-1}, t_i]$ for some $i \in \{2, \ldots, n\}$, then we have $i \leq m$ and $t_k = s_k$ for all $k = 0, \ldots, i-1$. Recalling now (3.4b), we see that

$$\begin{aligned}\mathcal{H}_G^{\mathcal{GEN}}[v](t) &= G\,(v(t_0), \ldots, v(t_{i-1}), v(t)) = G\,(v(s_0), \ldots, v(s_{i-1}), v(t)) \\ &= G\,(w(s_0), \ldots, w(s_{i-1}), w(t)) = \mathcal{H}_G^{\mathcal{GEN}}[w](t).\end{aligned} \tag{5.2}$$

Hence, it is shown that $\mathcal{H}_G^{\mathcal{GEN}}$ is causal.

2 *Proof of rate-independency:* Let $u \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ and let $\alpha : [0,T] \to [0,T]$ be an admissible time transformation. Thanks to Lemma 4.13 it holds $u \circ \alpha \in \mathrm{M}_{\mathrm{pw.m.a.}}([0,T];X)$ and for the standard monotaffinicity partitions $0 = s_0 < s_1 < \cdots < s_n = T$ of $[0,T]$ for $u \circ \alpha$ holds: $0 = \alpha(s_0) < \alpha(s_1) < \cdots < \alpha(s_n) = T$ is the standard monotaffinicity partitions of $[0,T]$ for u, and it holds $\alpha(s) > t_i$ for all $s \in]t_i, T]$ and all $i = 0, \ldots, n-1$.

- For all $s \in [s_0, s_1]$, it holds $\alpha(s) \in [t_0, t_1]$. In the light of (3.4a), we see that
$$\mathcal{H}_G^{\mathcal{GEN}}[u \circ \alpha](s) = G\left(u \circ \alpha(s_0), u \circ \alpha(s)\right) = G\left(u(t_0), u(\alpha(s))\right) = \mathcal{H}_G^{\mathcal{GEN}}[u](\alpha(s)).$$
- For all $i = 2, \dots, n-1$ and all $s \in [s_{i-1}, s_i]$, it holds $\alpha(s) \in]t_{i-1}, t_i]$. Recalling now (3.4b), we deduce that
$$\begin{aligned}\mathcal{H}_G^{\mathcal{GEN}}[u \circ \alpha](s) &= G\big(u \circ \alpha(s_0), \dots, u \circ \alpha(s_{i-1}), u \circ \alpha(t)\big) \\ &= G\big(u(t_0), \dots, u(i_{i-1}), u(\alpha(s))\big) = \mathcal{H}_G^{\mathcal{GEN}}[u](\alpha(s)).\end{aligned}$$

Thus, it is shown that $\mathcal{H}_G^{\mathcal{GEN}}$ is a hysteresis operator. Hence, this holds also for its restriction $\mathcal{H}_G^{\mathcal{GEN},\mathrm{C}}$ to $\mathrm{C}_{\text{pw.m.a.}}([0,T];X)$. $\square$

5.2 Preparations of proof of assertion b) in Theorem 3.8

5.1 Lemma. *Let a hysteresis operator $\mathcal{H} : D(\mathcal{H}) \left(\subseteq \mathrm{Map}\left([0,T],X\right)\right) \to \mathrm{Map}\left([0,T],Y\right)$ be given. Let $u \in D(\mathcal{H})$ and $t_1, t_2 \in [0,T]$ be given, so that u is constant on $[t_1, t_2]$.*

Then it follows that $\mathcal{H}[u]$ is also constant on $[t_1, t_2]$.

Proof. Let $\alpha_1, \alpha_2 : [0,T] \to [0,T]$ be defined by
$$\alpha_1(t) = \begin{cases} t, & \text{if} \quad t \le t_1, \\ t_1 & \text{if} \quad t_1 < t \le \frac{t_1+t_2}{2}, \\ 2t - t_2, & \text{if} \quad \frac{t_1+t_2}{2} < t \le t_2, \\ t, & \text{if} \quad t_2 < t, \end{cases} \tag{5.3}$$
$$\alpha_2(t) = \begin{cases} t, & \text{if} \quad t \le t_1, \\ 2t - t_1, & \text{if} \quad t_1 < t \le \frac{t_1+t_2}{2}, \\ t_2 & \text{if} \quad \frac{t_1+t_2}{2} < t \le t_2 \\ t, & \text{if} \quad t_2 < t, \end{cases} \tag{5.4}$$
for all $t \in [0,T]$. Hence, we see that α_1 and α_2 are admissible time transformation on $[0,T]$, differing only on $[t_1, t_2]$. Both function map this interval to itself. Since u is constant on this interval, we have $D(\mathcal{H}) \ni u = u \circ \alpha_1 = u \circ \alpha_2$.

Because of the rate-independency of $\mathcal{H}$, it holds, for all $t \in \left[t_1, \frac{t_1+t_2}{2}\right]$:
$$\mathcal{H}[u](t_1) = \mathcal{H}[u]\left(\alpha_1(t)\right) = \mathcal{H}[u \circ \alpha_1](t) = \mathcal{H}[u \circ \alpha_2](t) = \mathcal{H}[u]\left(\alpha_2(t)\right). \tag{5.5}$$
Since $\alpha_2\left(\left[t_1, \frac{t_1+t_2}{2}\right]\right) = [t_1, t_2]$, we have shown that $\mathcal{H}$ is constant on $[t_1, t_2]$. $\square$

5.2 Lemma. *Let some function $H : S_F(X) \to Y$ be given. Following Def. 3.7 and Def. 4.18 to create $\mathcal{H} := \mathcal{H}_H^{\mathcal{GEN},\mathrm{C}} : \mathrm{C}_{\text{pw.m.a.}}([0,T];X) \to \mathrm{Map}\left([0,T],Y\right)$ and $F_{\mathcal{H}}^{\text{gen}} = F_{\mathcal{H}_H^{\mathcal{GEN},\mathrm{C}}}^{\text{gen}} : S_F(X) \to Y$, respectively, it holds*
$$F_{\mathcal{H}_H^{\mathcal{GEN},\mathrm{C}}}^{\text{gen}} = H. \tag{5.6}$$

Proof. Let $V = (v_0, \dots, v_n) \in S_F(X)$ be given. Combining Def. 4.18, Def. 3.7, and Lemma 4.16, we observe that

$$\begin{aligned} F^{\mathrm{gen}}_{\mathcal{H}}(V) &= \mathcal{H}[\pi_{\mathrm{pw.af.}}[V]](T) = \mathcal{H}^{\mathcal{GEN},\mathrm{C}}_{H}[\pi_{\mathrm{pw.af.}}[V]](T) \\ &= H\left(\pi_{\mathrm{pw.af.}}[V]\left(\frac{0}{n}T\right), \pi_{\mathrm{pw.af.}}[V]\left(\frac{1}{n}T\right), \dots, \pi_{\mathrm{pw.af.}}[V]\left(\frac{n}{n}T\right)\right) \\ &= H(v_0, \dots, v_n) = H(V). \end{aligned}$$

Hence, it follows that (5.6) is proved. □

5.3 Lemma. *Let two hysteresis operators $\mathcal{G}, \mathcal{H} : \mathrm{C}_{\mathrm{pw.m.a.}}([0,T]; X) \to \mathrm{Map}\left([0,T], Y\right)$ be given.*

Then the following assertions are equivalent:

a)

$$\mathcal{H} = \mathcal{G}. \tag{5.7}$$

b)

$$\mathcal{H}[u](T) = \mathcal{G}[u](T), \quad \forall\, u \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T]; X). \tag{5.8}$$

c) For the functions $F^{\mathrm{gen}}_{\mathcal{G}}, F^{\mathrm{gen}}_{\mathcal{H}} : S_F(X) \to Y$ defined as in Def. 4.18 holds

$$F^{\mathrm{gen}}_{\mathcal{G}} = F^{\mathrm{gen}}_{\mathcal{H}}. \tag{5.9}$$

Proof. **a) $\Longrightarrow$ c)** clear.

b) $\Longrightarrow$ a) Assume that (5.8) is true.

Let $u \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T]; X)$ and $s \in [0,T]$ be arbitrary. Defining $\tilde{u}_s : [0,T] \to X$ by

$$\tilde{u}_s(t) := u\left(\min(t,s)\right), \tag{5.10}$$

we have a function $\tilde{u}_s \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T]; X)$ that is constant on $[s,T]$ and that coincides with u on $[0,s]$. Thanks to the causality of the operators, Lemma 5.1, and (5.8), we deduce that

$$\mathcal{H}[u](s) = \mathcal{H}[\tilde{u}_s](s) = \mathcal{H}[\tilde{u}_s](T) = \mathcal{G}[\tilde{u}_s](T) = \mathcal{G}[\tilde{u}_s](s) = \mathcal{G}[u](s). \tag{5.11}$$

Hence, a) is proved.

c) $\Longrightarrow$ b) Assume that (5.9) is satisfied.

Let $u \in \mathrm{C}_{\mathrm{pw.m.a.}}([0,T]; X)$ be given. Let $0 = t_0 < t_1 < \dots < t_n = T$ be the standard monotaffinicity partition of $[0,T]$ for u. It holds that u is monotaffine on $[t_{i-1}, t_i]$ for all $i = 1, \dots, n$.

For every $i = 1, \dots, n$, there exists some monotone increasing function $\beta_i : [t_{i-1}, t_i] \to [0,1]$ such that

$$\begin{aligned} u(t) &= (1 - \beta_i(t))u(t_{i-1}) + \beta_i(t)u(t_i) \\ &= u(t_{i-1}) + \beta_i(t)\left(u(t_i) - u(t_{i-1})\right), \quad \forall\, t \in [t_{i-1}, t_i]. \end{aligned} \tag{5.12}$$

- If $u(t_i) - u(t_{i-1}) \neq 0$, then it follows that $\beta_i(t_{i-1}) = 0$ and that $\beta_i(t_i) = 1$. Since $\beta_i : [t_{i-1}, t_i] \to [0,1]$ is an increasing function, we see that the one-sided limits for β_i exists for all $t \in [t_{i-1}, t_i]$. Using that the multiplication of scalars and vectors is continuous, that u is continuous, and that (5.12) holds, we deduce that one-sided limits are equal to $\beta_i(t)$ for all $t \in [t_{i-1}, t_i]$. Hence, it is proved that β_i is continuous.
- If $u(t_i) - u(t_{i-1}) = 0$, then we can replace β_i by the affine function satisfying $\beta_i(t_{i-1}) = 0$ and $\beta_i(t_i) = 1$. Also after this replacement, (5.12) is satisfied.

Defining now $\alpha : [0,T] \to [0,T]$ by

$$\alpha(t) = \begin{cases} 0, & \text{if} \quad t = 0, \\ \frac{i-1}{n}T + \beta_i(t)\frac{T}{n}, & \text{if} \quad t \in]t_{i-1}, t_i], \quad i \in \{1, \ldots, n\}, \end{cases} \tag{5.13}$$

we deduce that we have constructed an admissible time transformation with $\alpha(t_i) = \frac{i}{n}T$ for all $i = 0, \ldots, n$.

Let $V := (u(t_0), u(t_1), \ldots, u(t_n))$. Considering $\pi_{\text{pw.af.}}[V] : [0,T] \to X$ as in Def. 4.18, we have

$$\pi_{\text{pw.af.}}[V] \circ \alpha(0) = \pi_{\text{pw.af.}}[V]\,(0) = u(t_0), \tag{5.14}$$

and, for all $i = 1, \ldots, n$ and all $t \in]t_{i-1}, t_i]$, it holds that

$$\begin{aligned} \pi_{\text{pw.af.}}[V] \circ \alpha(t) &= \pi_{\text{pw.af.}}[V] \left(\frac{i-1}{n}T + \beta_i(t)\frac{T}{n} \right) \\ &= (1 - \beta_i(t))\, u(t_{i-1}) + \beta_i(t) u(t_i) = u(t). \end{aligned} \tag{5.15}$$

Hence, we have $\pi_{\text{pw.af.}}[V] \circ \alpha = u$ and therefore, thanks to the rate-independence of $\mathcal{H}$,

$$\begin{aligned} \mathcal{H}[u](T) &= \mathcal{H}[\pi_{\text{pw.af.}}[V] \circ \alpha](T) = \mathcal{H}[\pi_{\text{pw.af.}}[V]](\alpha(T)) \\ &= \mathcal{H}[\pi_{\text{pw.af.}}[V]](T) = F^{\text{gen}}_{\mathcal{H}}(V). \end{aligned} \tag{5.16}$$

Since the rate-independence of $\mathcal{G}$ leads to a analogous equation for $\mathcal{G}$, we see that (5.9) yields that (5.8) is satisfied.

$\square$

5.3 Proof of assertion b) in Theorem 3.8

Proof. Let some hysteresis operator $\mathcal{H} : \mathrm{C}_{\text{pw.m.a.}}([0,T]; X) \to \mathrm{Map}\,([0,T], Y)$ be given. Let $G := F^{\text{gen}}_{\mathcal{H}}$ with $F^{\text{gen}}_{\mathcal{H}} : S_F(X) \to Y$ being defined as in Def. 4.18. Now, it is to prove that $\mathcal{H}^{\mathcal{GEN},\mathrm{C}}_{G} = \mathcal{H}$ and that this is the only function with this property.

1 Thanks to Lemma 5.2, it holds that $F^{\text{gen}}_{\mathcal{H}^{\mathcal{GEN},\mathrm{C}}_{G}} = G = F^{\text{gen}}_{\mathcal{H}}$. Recalling Lemma 5.3, we deduce that $\mathcal{H}^{\mathcal{GEN},\mathrm{C}}_{G} = \mathcal{H}$.

2 Let any function $E : S_F(X) \to Y$ with $\mathcal{H} = \mathcal{H}_E^{\mathcal{GEN},\mathrm{C}}$ be given. Thanks to the Lemma 5.2, we have

$$G = F_{\mathcal{H}}^{\mathrm{gen}} = F_{\mathcal{H}_E^{\mathcal{GEN},\mathrm{C}}}^{\mathrm{gen}} = E.$$

□

6 Madelung rules, Madelung deletion and forgetting according do Madelung deletion

For describing the relations between applied fields and generated magnetization observed during magnetization experiments, Madelung formulated in [22] some rules. The following translations of these rules can be found in [4] on p. 27:

i. Any curve Γ_1 emanating form a turning point A of the input-output graph is uniquely determined by the coordinates of A.

ii. If any point B on the curve Γ_1 becomes a new turning point, then the curve Γ_2 originating at B leads back to the point A.

iii. if the curve Γ_2 is continued beyond th point A, then it coincides with the continuation of the curve Γ which led to the point A before the Γ_1–Γ_2–cycle was traversed.

The second Madelung rule is also denoted as *return point memory* property, the combination of second and third as *wiping out* property.

To investigate, if the second and the third Madelung rules are satisfied for an hysteresis operator with scalar inputs, it has been checked, if the string representation the operator is invariant to the so-called *Madelung deletion rule*, see [2, Sec. 4] or [4, Def. 2.6.1]. The following definitions generalizes this consideration to the case of strings with elements of the vector space X.

6.1 Definition. Let $V = (v_0, \dots, v_n) \in X^{n+1}$ and $W = (w_0, \dots, w_{n-2}) \in X^{n-1}$ with $n \in \mathbb{N}$ and $n > 2$ be given.

W is denoted as *the result of a Madelung deletion in* V, if there is some $j \in \{1, \dots, n-2\}$ such that

$$W = (v_0, \dots, v_{j-1}, v_{j+2}, \dots, v_n), \quad \text{i.e.,}$$
$$w_i = v_i, \quad \forall\, i = 0, \dots, j-1, \qquad w_{i'} = v_{i'+2}, \quad \forall\, i' = j, \dots, n-2, \tag{6.1a}$$
$$\operatorname{conv}(v_j, v_{j+1}) \subseteq \operatorname{conv}(v_{j-1}, v_{j+2}), \tag{6.1b}$$
$$v_j \notin \operatorname{conv}(v_{j-1}, v_{j+1}), \quad v_{j+1} \notin \operatorname{conv}(v_j, v_{j+2}). \tag{6.1c}$$

6.2 Remark. Let $n \in \mathbb{N}$ with $n > 2$ be given.

a) Let $(v_0,\dots,v_n)\in X^{n+1}$ and $j\in\{1,\dots,n-2\}$ be given. If (6.1b) and (6.1c) are satisfied, then it holds that

$$v_j \in \operatorname{conv}(v_{j+1},v_{j+2}), \quad v_{j+1}\in \operatorname{conv}(v_{j-1},v_j), \tag{6.2a}$$
$$\operatorname{conv}(v_{j-1},v_{j+2}) = \operatorname{conv}(v_{j-1},v_j)\cup \operatorname{conv}(v_{j+1},v_{j+2}), \tag{6.2b}$$
$$\operatorname{conv}(v_j,v_{j+1}) = \operatorname{conv}(v_{j-1},v_j)\cap \operatorname{conv}(v_{j+1},v_{j+2}). \tag{6.2c}$$

b) Let $V\in X^{n+1}\cap S_F(X)$ be given. If $W\in X^{n-1}$ is the result of a Madelung deletion in V, then it holds $W\in S_F(X)$.

The following definition is inspired by a more abstract definition for hysteresis operators with scalar inputs in [2, Def. 4.4] and in [4, Def. 2.6.2 and Def. 2.7.1].

6.3 Definition. a) Let a function $G: S_F(X)\to Y$ be given. The *function G forgets according to the Madelung deletion*, if for all $V\in X^{n+1}$ and $W\in X^{n-1}$ with $n\in\mathbb{N}$ and $n>2$ holds: If W is the result of a Madelung deletion in V, then $G(V)=G(W)$ is satisfied.

b) Let some hysteresis operator $\mathcal{H}:\mathrm{C}_{\mathrm{pw.m.a.}}([0,T];X)\to \operatorname{Map}([0,T],Y)$ be given. The *operator $\mathcal{H}$ forgets according to the Madelung deletion*, if the string function $F^{\mathrm{gen}}_{\mathcal{H}}: S_F(X)\to Y$ generated by $\mathcal{H}$ forgets according to the Madelung deletion.

7 Examples

7.1 Vectorial relay

Let N be some natural number. Let $\|\cdot\|$ denote the Euclidian norm on $\mathbb{R}^N$.

The *vectorial relay operator* introduced by Della Torre, Pinzaglia, and Cardelli, see [5, 6, 7, 8, 9], is also denoted as Della Torre-Pinzaglia-Cardelli model or as DPC model. Considering this operator for the special case that the critical surface is the boundary $\partial B_r(\xi)$ of the open ball $B_r(\xi)\subseteq\mathbb{R}^N$ with radius $r>0$ around some $\xi\in\mathbb{R}^N$ leads to the vectorial relay operator investigated in [11, 20, 21]:

$$\mathcal{R}_{\xi,r}:\partial B_1(0_X)\times \mathrm{C}\left([0,T];\mathbb{R}^N\right)\to \operatorname{Map}\left([0,T],\mathbb{R}^N\right), \tag{7.1a}$$

$$\mathcal{R}_{\xi,r}[\eta^0,u](t) := \begin{cases}\zeta_\xi(u(t)), & \text{if } \|u(t)-\xi\|\ge r,\\ \eta^0, \quad \text{if} \quad \|u(s)-\xi\|<r, \quad \forall s\in[0,t], & \\ \zeta_\xi\Big(u\left(\max\{s\in[0,t]\mid \|u(s)-\xi\|\ge r\}\right)\Big), & \text{otherwise,}\end{cases} \tag{7.1b}$$

with

$$\zeta_\xi:\mathbb{R}^N\setminus\{\xi\}\to\partial B_1(0_{\mathbb{R}^N}), \quad \zeta_\xi(v) := \frac{v-\xi}{\|v-\xi\|}. \tag{7.2}$$

The vectorial relay operator is a hysteresis operator.

Considering Figure 2, we observe that for all $\eta_0\in\partial B_1(0_X)$ and all $u\in \mathrm{C}\left([0,T];\mathbb{R}^N\right)$ it holds:

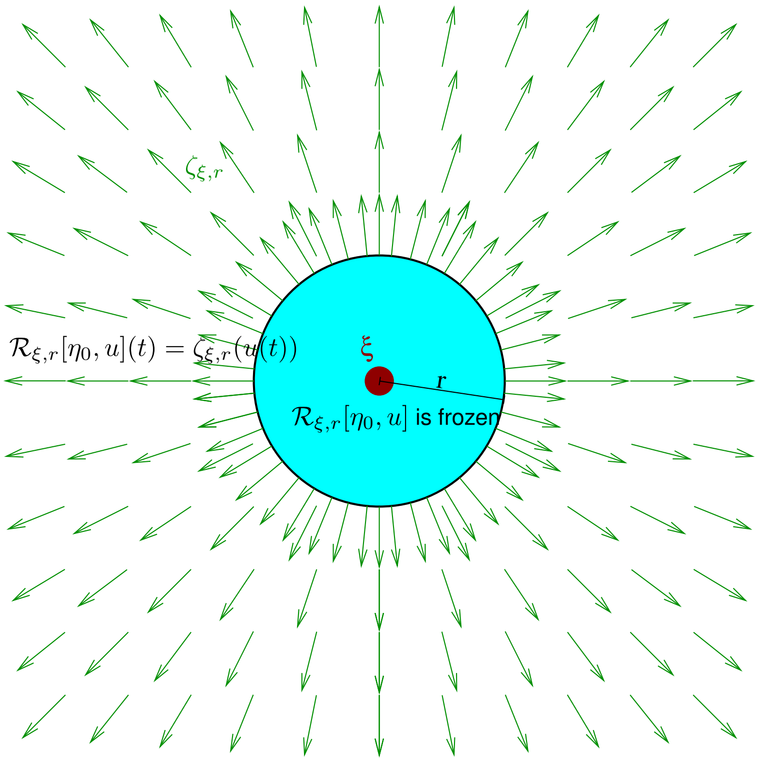

Figure 2: Values of the vectorial relay. (Figure follows inspiration from Löschner-Greenberg 2008.)

- $\mathcal{R}_{\xi,r}[\eta_0, u] = \zeta_\xi(u(t))$ if $u(t)$ is not in the blue circle in the Figure 2.
- If $u(t)$ enters the blue circle, then $\mathcal{R}_{\xi,r}[\eta_0, u]$ becomes constant, i.e. , $\mathcal{R}_{\xi,r}[\eta_0, u]$ "freezes".

7.1 Lemma. *Let $\xi \in \mathbb{R}^N$, $r > 0$, and $\eta_0 \in \partial B_1(0_X)$ be given.*

a) The string function generated by the relay operator $\mathcal{R}_{(\xi,r)}[\eta_0, \cdot]$ is equal to $G_{\mathcal{R},(\xi,r),\eta_0} : S_F(\mathbb{R}^N) \to \mathbb{R}^N$ with

$$G_{\mathcal{R},(\xi,r),\eta_0}(v_0, v_1, \ldots, v_n)$$
$$:= \begin{cases} \zeta_\xi(v_n), & \text{if} \quad \|v_n - \xi\| \geq r, \\ \eta^0, & \text{if} \quad \|v_i - \xi\| < r \quad \forall\, i = 0, 1, \ldots, n, \\ \zeta_\xi\left((1-\chi_r)v_k + \chi_r v_{k+1}\right) & \text{with} \\ \qquad k := \max\{i \in \{0, \ldots, n-1\} \,\big|\, \|v_i - \xi\| \geq r\}, & \\ \qquad \chi_r \in [0,1] \text{ being the (unique) solution to} & \\ \qquad\quad \|(1-\chi_r)v_k + \chi_r v_{k+1} - \xi\| = r, & \\ \qquad \text{otherwise} & \end{cases} \tag{7.3}$$

with ζ_ξ as in (7.2).

b) *The relay operator $\mathcal{R}_{(\xi,r)}[\eta_0,\cdot]$ forgets according to the Madelung deletion.*

Proof. a) Let $V=(v_0,v_1,\ldots,v_n)\in S_F(\mathbb{R}^N)$ be given.

- If $\|v_n-\xi\|\geq r$, then it holds $\|\pi_{\text{pw.af.}}[V](T)-\xi\|\geq r$. Recalling now (7.1) yields that
$$\mathcal{R}_{(\xi,r)}[\eta_0,\pi_{\text{pw.af.}}[V]](T)=\zeta_\xi\left(\pi_{\text{pw.af.}}[V](T)\right)=\zeta_\xi\left(v_n\right).$$
- If $\|v_i-\xi\|<r$ for all $i=0,\ldots,n$, then we can use the triangle inequality to justify that the same inequality holds for all convex combinations of v_{i-1} and v_i for all $i=1,\ldots,n$. This allows to conclude that $\|\pi_{\text{pw.af.}}[V](t)-\xi\|<r$ for all $t\in[0,T]$. Owing to (7.1), we deduce that
$$\mathcal{R}_{(\xi,r)}[\eta_0,\pi_{\text{pw.af.}}[V]](T)=\eta_0.$$
- Otherwise, we see that the formula for k in (7.3) generates a unique defined k satisfying
$$\|v_k-\xi\|\geq r,\qquad \|v_i-\xi\|<r,\quad \forall\, i=k+1,\ldots,n. \tag{7.4}$$
It follows that
$$\left\|\pi_{\text{pw.af.}}[V]\left(\frac{k}{n}T\right)-\xi\right\|=\|v_k-\xi\|\geq r, \tag{7.5}$$
and, with an argumentation similar to the one used above, we deduce that
$$\|\pi_{\text{pw.af.}}[V](t)-\xi\|<r,\quad \forall\, t\in[\frac{k+1}{n}T,T].$$
Hence, it holds
$$\frac{k}{n}T\in\left\{s\in[0,t]\,\middle|\,\|\pi_{\text{pw.af.}}[V](s)-\xi\|\geq r\right\}\subseteq\left[0,\frac{k+1}{n}T\right[. \tag{7.6}$$
Since $\|v_k-\xi\|\geq r>\|v_{k+1}-\xi\|$ and $\left(\mathbb{R}^N,\|\cdot\|\right)$ is strict convex, it follows that there exists a unique solution $\chi_r\in[0,1[$ to
$$\|(1-\chi_r)v_k+\chi_r v_{k+1}-\xi\|=r. \tag{7.7}$$
It holds
$$\max\left\{s\in[0,t]\,\middle|\,\|\pi_{\text{pw.af.}}[V](s)-\xi\|\geq r\right\}=(1-\chi_r)\frac{k}{n}T+\chi_r\frac{k+1}{n}T. \tag{7.8}$$
Recalling now (7.1) and the definition of $\pi_{\text{pw.af.}}$ in Def. 4.15 yields that
$$\mathcal{R}_{(\xi,r)}[\eta_0,\pi_{\text{pw.af.}}[V]](T)=\zeta_\xi\big((1-\chi_r)v_k+\chi_r v_{k+1}\big). \tag{7.9}$$

b) Let $V=(v_0,\ldots,v_n)\in\left(\mathbb{R}^N\right)^{n+1}$ and $W=(w_0,\ldots,w_{n-2})\in\left(\mathbb{R}^N\right)^{n-1}$ with $n\in\mathbb{N}$ and $n>2$ be given, such that W is the result of a Madelung deletion in V. Then there exists some $j\in\{1,\ldots,n-2\}$ such that (6.1) holds.

If $\|v_n - \xi\| \geq r$ holds or if $\|v_i - \xi\| < r$ holds for all $i = 0, 1, \ldots, n$, then we obtain by recalling (7.3) and (6.1a) that $G_{\mathcal{R},(\xi,r),\eta_0}[V] = G_{\mathcal{R},(\xi,r),\eta_0}[W]$ is proved.

Otherwise, let χ_r and k be defined as in (7.3). Then we see that (7.4) and

$$G_{\mathcal{R},(\xi,r),\eta_0}[V] = \zeta_\xi\big((1-\chi_r)v_k + \chi_r v_{k+1}\big) \tag{7.10}$$

are satisfied. If $k < j-1$ holds or if $k \geq j+2$ holds, then we conclude by using (7.3) and (6.1a) that $G_{\mathcal{R},(\xi,r),\eta_0}[V] = G_{\mathcal{R},(\xi,r),\eta_0}[W]$.

Hence, it remains to consider the situations $k = j-1$, $k = j$, and $k = j+1$.

- Assume that $k = j-1$.

  In the light of (6.1a) and (7.4), we see that

$$w_k = v_k = v_{j-1}, \quad w_{k+1} = w_j = v_{j+2}, \tag{7.11a}$$
$$\|w_k - \xi\| = \|v_k - \xi\| \geq r, \tag{7.11b}$$
$$\|w_i - \xi\| = \|v_{i+2} - \xi\| < r, \quad \forall\, i = k, \ldots, n-2. \tag{7.11c}$$

  Thanks to (6.1b), there is some $\lambda \in [0,1]$ such that

$$v_{k+1} = v_j = (1-\lambda)v_{j-1} + \lambda v_{j+2}. \tag{7.12}$$

  Recalling (7.11), we deduce that

$$\begin{aligned} &(1-\lambda\chi_r)\, w_k + \lambda\chi_r w_{k+1} = (1-\chi_r)\, w_k + \chi_r\left((1-\lambda)w_k + \lambda w_{k+1}\right) \\ =\ &(1-\chi_r)\, v_k + \chi_r\left((1-\lambda)v_{j-1} + \lambda v_{j+2}\right) = (1-\chi_r)\, v_k + \chi_r v_{k+1}. \end{aligned} \tag{7.13}$$

  Utilizing the definition of χ_r in (7.3), it follows that

$$\|(1-\lambda\chi_r)\, w_k + \lambda\chi_r w_{k+1} - \xi\| = \|(1-\chi_r)\, v_k + \chi_r v_{k+1} - \xi\| = r. \tag{7.14}$$

  Thanks to $\lambda\chi_r \in [0,1[$, (7.3), (7.11), (7.13), and (7.10), we have

$$\begin{aligned} G_{\mathcal{R},(\xi,r),\eta_0}[W] &= \zeta_\xi\big((1-\lambda\chi_r)\, w_k + \lambda\chi_r w_{k+1}\big) \\ &= \zeta_\xi\big((1-\chi_r)\, v_k + \chi_r v_{k+1}\big) = G_{\mathcal{R},(\xi,r),\eta_0}[V]. \end{aligned} \tag{7.15}$$

- We will now show that $k \neq j$. Recalling (6.2a), we deduce that

$$v_j \in \operatorname{conv}\left(v_{j+1}, v_{j+2}\right). \tag{7.16}$$

  Hence, we can conclude from the triangle inequality that

$$\|v_j - \xi\| \leq \max\left(\|v_{j+1} - \xi\|, \|v_{j+2} - \xi\|\right). \tag{7.17}$$

  Utilizing (7.4), we deduce that if $k = j$ held, then the above inequality would imply that $r < r$, i.e., it would lead to a contradiction.

  Therefore, if follows that $k \neq j$.

- Assume that $k = j + 1$. Because of (6.1b) there exists some $\lambda \in [0, 1]$ such that

$$v_k = v_{j+1} = (1 - \lambda)v_{j-1} + \lambda v_{j+2}. \tag{7.18}$$

Using (7.4) and the triangle inequality, we deduce that

$$r \leq \|v_k - \xi\| \leq (1 - \lambda)\|v_{j-1} - \xi\| + \lambda \|v_{j+2} - \xi\|. \tag{7.19}$$

Since $\|v_{j+2} - \xi\| = \|v_{k+1} - \xi\| < r$ on account of (7.4), we see that $\|v_{j-1} - \xi\| \geq r$ and $\lambda \in [0, 1[$ must hold. Defining now

$$\beta := \lambda + (1 - \lambda)\chi_r, \tag{7.20}$$

we have a number in $[0, 1[$. Recalling (6.1a), we deduce that

$$\begin{aligned}(1 - \beta)w_{j-1} + \beta w_j &= (1 - \chi_r)(1 - \lambda)v_{j-1} + (\lambda + \chi_r - \lambda\chi_r)v_{j+2} \\ = (1 - \chi_r)\left((1 - \lambda)v_{j-1} + \lambda v_{j+1}\right) &+ \chi_r v_{j+2} = (1 - \chi_r)v_k + \chi_r v_{k+1}.\end{aligned} \tag{7.21}$$

Hence, we have

$$\|(1 - \beta)w_{j-1} + \beta w_j - \xi\| = r. \tag{7.22}$$

Since $\|w_i - \xi\| = \|v_{i+1} - \xi\| < r$ for $i = j, \ldots, n - 2$, and $\|w_{j-1} - \xi\| \geq r$, we deduce by recalling (7.3), (7.21), and (7.10) that $G_{\mathcal{R},(\xi,r),\eta^0}[W] = G_{\mathcal{R},(\xi,r),\eta^0}[V]$.

□

7.2 Vectorial Preisach operator

Let a *Preisach density function* $\omega : \mathbb{R}^N \times [0, \infty[\to \mathbb{R}$ with $\omega \in L^1(\mathbb{R}^N \times [0, \infty[)$ be given. Let some *measurable initial state* $\eta^0 : \mathbb{R}^N \times [0, \infty[\to \partial B_1(0_{\mathbb{R}^N})$ be given.

Considering, as for the scalar Preisach operator, a weighted superposition of relays, one obtains the *vectorial Preisach operator* considered in [11, 20, 21]. Omitting the dependence on the initial state in the notation, we consider the vectorial Preisach operator

$$\begin{aligned}\mathcal{PR}^{vec} &: \mathrm{C}\left([0, T]; \mathbb{R}^N\right) \to \mathrm{Map}\left([0, T], \mathbb{R}^N\right), \\ \mathcal{PR}^{vec}[u](t) &:= \int_0^\infty \int_{\mathbb{R}^N} \omega(\xi, r)\mathcal{R}_{(\xi,r)}[\eta^0(\xi, r), u](t)\, \mathrm{d}\xi\, \mathrm{d}r.\end{aligned}$$

Using the Lemma derived for the relay operator, we see that

7.2 Lemma. *a) The string function generated by the vectorial Preisach operator* $\mathcal{PR}^{vec}$ *is*

$$\begin{aligned}G_{\mathcal{PR},vec} &: S_F(X) \to \mathbb{R}, \\ G_{\mathcal{PR},vec}(V) &= \int_0^\infty \int_{\mathbb{R}^N} \omega(\xi, r) G_{\mathcal{R},(\xi,r),\eta^0(\xi,r)}(V)\, \mathrm{d}\xi\, \mathrm{d}r.\end{aligned}$$

b) The vectorial Preisach operator $\mathcal{PR}^{vec}$ *forgets according to the Madelung deletion.*

7.3 Generalized vectorial relay

Many vectorial relay considered in the literature can be rewritten as an operator of the following form:

For a nonempty, open subset O of X, a nonempty set Y, and a function $\zeta : X \setminus O \to Y$, we consider the following *generalized relay operator*:

$$\widetilde{\mathcal{R}}_{X,O,Y,\zeta} : Y \times \mathrm{C}\left([0,T];X\right) \to \mathrm{Map}\left([0,T],Y\right), \tag{7.23a}$$

$$\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta,u](t) := \begin{cases} \zeta(u(t)), & \text{if} \quad u(t) \notin O, \\ \eta, \quad \text{if} \quad u([0,t]) \subseteq O, \\ \zeta\left(u\Big(\max\big\{s \in [0,t] \,\big|\, u(s) \notin O\big\}\Big)\right), & \text{otherwise.} \end{cases} \tag{7.23b}$$

The generalized relay operator is a hysteresis operator.

Examples:

- A vectorial relay was introduced by Mayergoyz (see [23]) for inputs with values in $\mathbb{R}^2$. In the formulation as in [27, IV (5.1)], one has to consider the scalar relay $\mathcal{R}_{a,b} : \mathrm{C}\left([0,T];\mathbb{R}\right) \to \{-1,1\}$ with thresholds $a < b$ and some angle $\theta \in [0,2\pi[$. Let $e_\theta = (\cos(\theta),\sin(\theta))$ be the unit vector in $\mathbb{R}^2$ pointing in the direction corresponding to this angle. The corresponding vector relay operator $\mathcal{R}_{a,b,\theta} : \mathrm{C}\left([0,T];\mathbb{R}^2\right) \to \mathrm{Map}\left([0,T],\partial B_1\left(0_{\mathbb{R}^2}\right)\right)$ is for $u \in \mathrm{C}\left([0,T];\mathbb{R}^2\right)$ defined by

  $$\mathcal{R}_{a,b,\theta}[u] := \mathcal{R}_{a,b}[u \cdot e_\theta]e_\theta. \tag{7.24}$$

  A straightforward computations yields that

  $$\mathcal{R}_{a,b,\theta} = \widetilde{\mathcal{R}}_{\mathbb{R}^2,O,\partial B_1\left(0_{\mathbb{R}^2}\right),\zeta} \tag{7.25}$$

  with

  $$O := \{v \in \mathbb{R}^2 \,|\, v \cdot e_\theta \in]a,b[\}, \tag{7.26a}$$

  $$\zeta : \mathbb{R}^2 \setminus O \to \partial B_1(0_{\mathbb{R}^2}), \tag{7.26b}$$

  $$\zeta(v) := \begin{cases} e_\theta, & \text{if} \quad v \cdot e_\theta \geq b, \\ -e_\theta, & \text{if} \quad v \cdot e_\theta \leq a. \end{cases} \tag{7.26c}$$

- Let K be a convex, compact subset of $\mathbb{R}^2$. Let $\eta_0 \in \mathbb{R}^2$ be given. Let $\mathrm{Proj}_K : \mathbb{R}^2 \to K$ be the projection to K. The corresponding rotating model $\mathcal{H}_K : \mathbb{R}^2 \times \mathrm{C}\left([0,T];\mathbb{R}^2\right) \to \mathrm{Map}\left([0,T],\mathbb{R}^2\right)$ considered in [27, IV (5.10), (5.11)] is the hysteresis operator mapping $u \in \mathrm{C}\left([0,T];\mathbb{R}^2\right)$ to the function $\mathcal{H}_K[\eta,u] : [0,T] \to K$ with

  $$\mathcal{H}_K[\eta,u](0) := \begin{cases} \mathrm{Proj}_K(\eta_0), & \text{if} \quad u(0) \in K, \\ \mathrm{Proj}_K(u(0)), & \text{if} \quad u(0) \notin K, \end{cases} \tag{7.27}$$

and, for all $t \in]0, T]$,

$$\mathcal{H}_K[\eta, u](t) = \begin{cases} \mathrm{Proj}_K(u(t)), & \text{if} \quad u(t) \notin K, \\ \mathcal{H}_K[\eta, u](0), & \text{if} \quad u(t) \in K, \quad u([0,t]) \cap \partial K = \emptyset, \\ u\left(\max\{\tau \in]0,t] \,\middle|\, u(\tau) \in \partial K\}\right) & \text{otherwise.} \end{cases} \tag{7.28}$$

Hence, we see that

$$\mathcal{H}_K[\eta, \cdot] = \widetilde{\mathcal{R}}_{\mathbb{R}^2, \mathrm{int}K, K, \rho_K}[\mathrm{Proj}_K(\eta), \cdot] \tag{7.29}$$

with $\mathrm{int}(K)$ being the interior of K and ρ_K being the restriction of Proj_K to the set $\mathbb{R}^2 \setminus \mathrm{int}(K)$.

- Let K be a convex, compact subset of a Hilbert space H. Let $n_{\partial K} : \partial K \to \partial B_1(0_X)$ be some choice for an outer normal to K on ∂K, i.e. , it holds

$$\langle n_{\partial K}(y), y - z\rangle \geq 0 \quad \forall y \in \partial K,\ z \in K. \tag{7.30}$$

The suggestion for an operator of relay type in [21, p. 34] is modified by requiring that if the value of the input belongs to the boundary ∂K of K, then the output of the the relay should to the corresponding value of $n_{\partial K}$ at this point. This leads to $\widetilde{\mathcal{R}}_{H, \mathrm{int}(K), \partial B_1(0_H), \phi_K}$ with $\phi_K : H \setminus \mathrm{int}(K) \to \partial B_1(0_H)$ defined by

$$\phi_K(x) := \begin{cases} n_{\partial K}(x), & \text{if} \quad x \in \partial K, \\ \frac{x - \mathrm{Proj}_K(x)}{\|x - \mathrm{Proj}_K(x)\|}, & \text{otherwise.} \end{cases} \tag{7.31}$$

- Assume that $(X, \|\cdot\|)$ is some normed space. To formulate the Della Torre-Pinzaglia-Cardelli vectorial relay, see [5, 6, 8, 9], one has to consider a critical surface S in X, being the boundary of some open, bounded domain O, and a point $\xi \in O$, e.g., the center of O. This relay can now be rewriten as $\widetilde{\mathcal{R}}_{X, O, \partial B_1(0_X), \zeta_{O,\xi}}$, with

$$\zeta_{O,\xi} : \mathbb{R}^N \setminus O \to \partial B_1(0_X), \quad \zeta_{O,\xi}(v) := \frac{v - \xi}{\|v - \xi\|}. \tag{7.32}$$

- For $r > 0$, $\xi \in \mathbb{R}^N$, $\zeta_{B_r(\xi),\xi}$ as in (7.32), and the vectorial relay operator $\mathcal{R}_{\xi,r}$ defined in (7.1), it holds that

$$\mathcal{R}_{\xi,r} = \widetilde{\mathcal{R}}_{\mathbb{R}^N, B_r(\xi), \partial B_1(0_{\mathbb{R}^N}), \zeta_{B_r(\xi),\xi}}. \tag{7.33}$$

- Considering two open subsets O_0, O_1 of $\mathbb{R}^2$ so that $O_0 \cup O_1 = \mathbb{R}^2$. The relay defined in [1, Def. 2.1] is equal to $\widetilde{\mathcal{R}}_{\mathbb{R}^2, O_0 \cap O_1, \{0,1\}, \phi_{BK}}$ with

$$\phi_{BK} : \mathbb{R}^2 \setminus (O_0 \cap O_1) \to \{0, 1\}, \tag{7.34a}$$

$$\phi_{BK}(x) := \begin{cases} 0, & \text{if} \quad x \in O_0, \\ 1, & \text{if} \quad x \in O_1. \end{cases} \tag{7.34b}$$

7.3 Lemma. *Let a nonempty, open subset O of X, a nonempty set Y, a function $\zeta : X \setminus O \to Y$ and $\eta \in Y$ be given. Let $\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\cdot,\cdot]$ be the generalized vectorial relay defined in* (7.23).

a) *The string function generated by generalized vectorial relay operator $\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta,\cdot]$ is $G_{\mathcal{R},X,O,\zeta,\eta} : S_F(X) \to Y$ defined by*

$$G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(v_0, v_1, \dots, v_n) = \begin{cases} \zeta(v_n), & \text{if} \quad v_n \notin O, \\ \eta, & \text{if} \quad \operatorname{conv}(v_i, v_{i+1}) \subseteq O \quad \forall\, i = 0, \dots, n-1, \\ \zeta\left((1-\chi_{\max})v_k + \chi_{\max} v_{k+1}\right) & \text{with} \\ \qquad k := \max\bigl\{i \in \{0, \dots, n-1\} \bigm| \operatorname{conv}(v_i, v_{i+1}) \not\subseteq O\bigr\}, \\ \qquad \chi_{\max} := \max\bigl\{\chi \in [0,1[\bigm| (1-\chi)v_k + \chi v_{k+1} \notin O\bigr\}, \\ \qquad \text{otherwise.} \end{cases} \tag{7.35}$$

b) *The generalized vectorial relay operator $\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta,\cdot]$ forgets according to the Madelung deletion.*

Proof. a) Let $V = (v_0, v_1, \dots, v_n) \in S_F(X)$ be given.

- If $v_n \notin O$, then it holds $\pi_{\text{pw.af.}}[V](T) = v_n \notin O$. Invoking (7.23) yields that

$$\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta, \pi_{\text{pw.af.}}[V]](T) = \zeta\left(\pi_{\text{pw.af.}}[V](T)\right) = \zeta\left(v_n\right).$$

- If $\operatorname{conv}(v_i, v_{i+1}) \subseteq O$ for all $i = 0, \dots, n-1$, then it holds $\pi_{\text{pw.af.}}[V](t) \in O$ for all $t \in [0, T]$. Recalling now (7.23) yields that

$$\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta, \pi_{\text{pw.af.}}[V]](T) = \eta.$$

- Otherwise, we see that the formula for k in (7.35) generates the unique defined k satisfying

$$\operatorname{conv}(v_k, v_{k+1}) \not\subseteq O, \quad \operatorname{conv}(v_i, v_{i+1}) \subseteq O, \quad \forall\, i = k+1, \dots, n-1. \tag{7.36}$$

  It follows that

$$\pi_{\text{pw.af.}}[V]\left(\left[\frac{k}{n}T, \frac{k+1}{n}T\right]\right) \not\subseteq O, \qquad \pi_{\text{pw.af.}}[V]\left(\left[\frac{k+1}{n}T, T\right]\right) \subseteq O. \tag{7.37}$$

  We apply (7.36) to show that the set

$$M := \bigl\{\chi \in [0,1[\bigm| (1-\chi)v_k + \chi v_{k+1} \notin O\bigr\} \tag{7.38}$$

  is not empty. Let $\chi_{\max} := \sup M$. Using that the multiplication with a scalar and the addition is continuous, we see that there is a sequence in the complement of O converging to $(1-\chi_{\max})v_k + \chi_{\max} v_{k+1}$. Since O is an open set, we conclude that $(1-\chi_{\max})\,v_k + \chi_{\max}v_{k+1} \notin O$. Using that (7.37) yields that $v_{k+1} \in O$, we see that $\chi_{\max}$ belongs to M and can be defined as in (7.35). Moreover, for $t^* := (1-\chi_{\max})\,\frac{k}{n}T + \chi_{\max}\frac{k+1}{n}T$ it holds that

$$\pi_{\text{pw.af.}}[V]\,(t^*) \notin O, \quad \pi_{\text{pw.af.}}[V]\left(\left]t^*, \frac{k+1}{n}T\right]\right) \subseteq O\,. \tag{7.39}$$

Combining this with (7.23) , we infer that

$$\widetilde{\mathcal{R}}_{X,O,Y,\zeta}[\eta, \pi_{\text{pw.af.}}[V]](T) = \zeta\left(\pi_{\text{pw.af.}}[V]\,(t^*)\right) = \zeta\left((1-\chi_{\max})v_k + \chi_{\max}v_{k+1}\right). \tag{7.40}$$

b) Let $V = (v_0, \ldots, v_n) \in X^{n+1}$ and $W = (w_0, \ldots, w_{n-2}) \in X^{n-1}$ with $n \in \mathbb{N}$ and $n > 2$ be given. Assume that W is the result of a Madelung deletion in V. Then there is some $j \in \{1, \ldots, n-2\}$ such that (6.1) holds.

- If $v_n \notin O$ holds, then we obtain by recalling (7.35) and (6.1a) that

$$G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(V) = \zeta(v_n) = \zeta(w_{n-2}) = G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(W).$$

- If $\operatorname{conv}(v_i, v_{i+1}) \subseteq O$ holds for all $i = 0, 1, \ldots, n-1$, then we obtain by (6.1a) and (6.2b) that $\operatorname{conv}(w_i, w_{i+1}) \subseteq O$ holds for all $i = 0, \ldots, n-2$. Recalling now (7.35) yields that $G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(V) = \eta = G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(W)$.
- Otherwise, let $\chi_{\max}$ and k be defined as in (7.35), and let

$$v^* := (1-\chi_{\max})v_k + \chi_{\max}v_{k+1}. \tag{7.41}$$

Then, we see that (7.36) and

$$G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(V) = \zeta\left(v^*\right) \tag{7.42}$$

are satisfied.
If $k < j-1$ holds or if $k \geq j+2$ holds, we conclude by using (7.35) and (6.1a) that $G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(V) = G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(W)$.
Recalling (6.2c), we deduce that

$$\operatorname{conv}(v_j, v_{j+1}) \subseteq \operatorname{conv}(v_{j+1}, v_{j+2}). \tag{7.43}$$

Utilizing (7.36), we see that

$$\operatorname{conv}(v_k, v_{k+1}) \nsubseteq O, \quad \operatorname{conv}(v_{k+1}, v_{k+2}) \subseteq O. \tag{7.44}$$

Hence, we see that if $k = j$ held, then on one hand $\operatorname{conv}(v_j, v_{j+1})$ would be, as a subset of $\operatorname{conv}(v_{j+1}, v_{j+2}) = \operatorname{conv}(v_{k+1}, v_{k+2})$, a subset of O, on the other hand $\operatorname{conv}(v_j, v_{j+1}) = \operatorname{conv}(v_k, v_{k+1})$ would be no subset of O. Hence, it follows that $k \neq j$.
Hence, it remains to consider the situations $k = j-1$ and $k = j+1$.

Utilizing now (6.1a) and (6.2b), we deduce that

$$\begin{aligned} &\operatorname{conv}(w_{j-1}, w_j) = \operatorname{conv}(v_{j-1}, v_{j+2}) \\ = &\operatorname{conv}(v_{j-1}, v_j) \cup \operatorname{conv}(v_{j+1}, v_{j+2}) \supseteq \operatorname{conv}(v_k, v_{k+1}). \end{aligned} \tag{7.45}$$

We apply (7.36) and (6.1a) to decude that

$$\operatorname{conv}(w_{j-1}, w_j) \nsubseteq O, \quad \operatorname{conv}(w_i, w_{i+1}) \subseteq O, \quad \forall\, i = j, \ldots, n-3. \tag{7.46}$$

Applying (7.35), see that for $\beta \in [0,1[$ and $w^* \in X$ defined by

$$\beta := \max\big\{\chi \in [0,1[\,\big|\,(1-\chi)w_{j-1} + \chi w_j \notin O\big\}, \tag{7.47}$$
$$w^* := (1-\beta)w_{j-1} + \beta w_j\,, \tag{7.48}$$

it holds that

$$G_{\widetilde{\mathcal{R}},X,O,\zeta,\eta}(W) = \zeta\left(w^*\right). \tag{7.49}$$

Recalling (6.1a) yields that

$$\beta = \max\big\{\chi \in [0,1[\,\big|\,(1-\chi)v_{j-1} + \chi v_{j+2} \notin O\big\}, \tag{7.50}$$
$$w^* = (1-\beta)v_{j-1} + \beta v_{j+2} \notin O. \tag{7.51}$$

Hence, it remains to show that $v^* = w^*$ is satisfied.

- Assume that $k = j-1$.
  Thanks to (7.45), there is some $\lambda \in [0,1]$ such that

$$v_{k+1} = (1-\lambda)v_{j-1} + \lambda v_{j+2}. \tag{7.52}$$

We recall now (7.36) to prove that $\operatorname{conv}(v_{j+1}, v_{j+2}) = \operatorname{conv}(v_{k+2}, v_{k+3}) \subseteq O$ and that $v_{k+1} \in O$. Since $w^* \notin O$, we can conclude form (6.2b) and (7.52) that

$$\begin{aligned} w^* &\in \operatorname{conv}(v_{j-1}, v_{j+2}) \setminus O = \big(\operatorname{conv}(v_{j-1}, v_j) \cup \operatorname{conv}(v_{j+1}, v_{j+2})\big) \setminus O \\ &= \operatorname{conv}(v_{j-1}, v_j) \setminus O \subseteq \operatorname{conv}(v_{j-1}, v_{k+1}) \setminus \{v_{k+1}\} \\ &= \operatorname{conv}(v_{j-1}, (1-\lambda)v_{j-1} + \lambda v_{j+2}) \setminus \{(1-\lambda)v_{j-1} + \lambda v_{j+2})\}\,. \end{aligned} \tag{7.53}$$

Utilizing now (7.51) and (7.52) yields that $\beta < \lambda$.
Recalling (7.52), we deduce that for all $\chi \in [0,1]$ holds:

$$\begin{aligned} &\left(1-\lambda\chi\right) v_{j-1} + \lambda\chi v_{j+2} \\ = &\left(1-\chi\right) v_{j-1} + \chi\left(\left(1-\lambda\right) v_{j-1} + \lambda v_{j+2}\right) = \left(1-\chi\right) v_k + \chi v_{k+1}. \end{aligned} \tag{7.54}$$

Combining the definition of $\chi_{\max}$ in (7.35) and (7.41) implies that

$$\left(1-\lambda\chi_{\max}\right) v_{j-1} + \lambda\chi_{\max} v_{j+2} = \left(1-\chi_{\max}\right) v_k + \chi_{\max} v_{k+1} = v^* \notin O, \tag{7.55}$$

$$\left(1-\lambda\chi\right) v_{j-1} + \lambda\chi v_{j+2} = \left(1-\chi\right) v_k + \chi v_{k+1} \in O, \quad \forall \chi \in]\chi_{\max}, 1[\,. \tag{7.56}$$

From (7.50), it follows that

$$\lambda\chi_{\max} \leq \beta \notin]\chi_{\max}\lambda, \lambda[. \tag{7.57}$$

Since it has already been shown that $\beta < \lambda$ holds, we deduce that $\lambda\chi_{\max} = \beta$. Applying (7.55) and (7.51), it follows that $v^* = w^*$ is proved.

- Assume that $k = j + 1$.
  Thanks to (7.45), there is some $\phi \in [0, 1]$ such that

$$v_k = (1 - \phi)v_{j-1} + \phi v_{j+2}. \tag{7.58}$$

Since (7.44) yields that $v_k \neq v_{k+1} = v_{j+2}$, it follows that $\phi \in [0, 1[$.
For $\chi \in [0, 1[$ and $\gamma_\chi := \phi + (1 - \phi)\chi \in [0, 1[$, we apply (7.58) to derive

$$\begin{aligned}(1 - \gamma_\chi)v_{j-1} + \gamma_\chi v_{j+2} &= (1 - \chi)(1 - \phi)v_{j-1} + (\phi + \chi - \phi\chi)v_{j+2} \\ = (1 - \chi)\left((1 - \phi)v_{j-1} + \phi v_{j+2}\right) + \chi v_{j+2} &= (1 - \chi)v_k + \chi v_{k+1}.\end{aligned} \tag{7.59}$$

Recalling the definition of $\chi_{\max}$ in (7.35) and (7.41), we deduce that

$$(1 - \gamma_{\chi_{\max}})\, v_{j-1} + \gamma_{\chi_{\max}} v_{j+2} = (1 - \chi_{\max})v_k + \chi_{\max} v_{k+1} = v^* \notin O, \tag{7.60}$$

$$(1 - \gamma_\chi v_{j-1}) + \gamma_\chi v_{j+2} = (1 - \chi)v_k + \chi v_{k+1} \in O, \quad \forall \chi \in]\chi_{\max}, 1[\,. \tag{7.61}$$

Since $\{\gamma_\chi \,|\, \chi \in]\chi_{\max}, 1[\,\} =]\gamma_{\chi_{\max}}, 1[$, we can conclude from (7.50) that $\beta = \gamma_{\chi_{\max}}$. Using now (7.51) and (7.60) yields $v^* = w^*$.

$\square$

7.4 Remark. Many other hysteresis operators do not forget according to the Madelung deletion, for examples, most vectorial play and stop operators. Details can be found in the forthcoming thesis [16].

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