Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

Homogenization of elastic waves in fluid-saturated porous media using the Biot model

Alexander Mielke[1,2], Eduard Rohan[3]

submitted: February 22, 2012

[1] Weierstraß-Institut
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: alexander.mielke@wias-berlin.de

[2] Humboldt-Universität zu Berlin
Institut für Mathematik
Rudower Chaussee 25
12489 Berlin
Germany

[3] University of West Bohemia, Department of Mechanics
Faculty of Applied Sciences
Univerzitní 8
30614 Plzeň
Czech Republic
E-Mail: rohan@kme.zcu.cz

No. 1688

Berlin 2012

2010 *Mathematics Subject Classification.* 35B27, 74F10, 76M50, 76S05.

Key words and phrases. Two-scale homogenization, porous media, acoustic waves, elastodynamics, Darcy's law, seepage.

The research of E.R. was supported by the European Regional Development Fund (ERDF), project "NTIS – New Technologies for Information Society", European Centre of Excellence, CZ.1.05/1.1.00/02.0090, and in part by the Czech Scientific Foundation project GACR P101/12/2315. The research of A.M. was partially supported by DFG through the SFB 910, subproject A5 "Pattern formation in systems with multiple scales".

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider periodically heterogeneous fluid-saturated poroelastic media described by the Biot model with inertia effects. The weak and semistrong formulations for displacement, seepage and pressure fields involve three equations expressing the momentum and mass balance and the Darcy law. Using the two-scale homogenization method we obtain the limit two-scale problem and prove the existence and uniqueness of its weak solutions. The Laplace transformation in time is used to decouple the macroscopic and microscopic scales. It is shown that the seepage velocity is eliminated form the macroscopic equations involving strain and pressure fields only. The plane harmonic wave propagation is studied using an example of layered medium. Illustrations show some influence of the orthotropy on the dispersion phenomena.

1 Introduction

Wave propagation in fluid-saturated porous (elastic) materials (FSPM) has been addressed in many publications since the pioneering works of Frenkel (1944) and Biot (1956). Especially the occurrence of the fast and slow compressional waves became the issue frequently discussed in the contexts of various particular models and their engineering applications, namely in geomechanics, seismology, or in biomechanics.

The FSPM models which we have in mind are relevant to the scale where individual fluid-filled pores are not distinguishable so that at any point of the bulk material both the solid and fluid phases are present according to the volume fractions. In particular we consider the following system of PDEs proposed by Biot in [BiW57], cf. [AlW05]also other references which use this model,

$$
\begin{aligned}
-\nabla \cdot (\mathbf{I\!D} \boldsymbol{\epsilon}(\boldsymbol{u})) + \nabla(\boldsymbol{\alpha} p) + \bar{\rho} \ddot{\boldsymbol{u}} + \rho^f \dot{\boldsymbol{w}} = \boldsymbol{f} \ , \\
\rho^f \ddot{\boldsymbol{u}} + \rho^w \dot{\boldsymbol{w}} + \boldsymbol{K}^{-1} \boldsymbol{w} + \nabla p = 0 \\
\boldsymbol{\alpha} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) + \operatorname{div} \boldsymbol{w} + \frac{1}{\mu} \dot{p} = 0 \ ,
\end{aligned}
\tag{1.1}
$$

consisting of the momentum equation $(1.1)_1$, the Darcy law $(1.1)_2$, relating the total fluid pressure to the seepage velocity $\boldsymbol{w}$, and the fluid volume conservation $(1.1)_3$. Although we explain the structure of this model in detail in Section 2, for comprehension we introduce the notation: by dot we indicate the time derivative, $\boldsymbol{u}$ is the displacement field describing the solid skeleton kinematics, $\boldsymbol{\epsilon}(\boldsymbol{u})$ is the small strain tensor, $\mathbf{I\!D}$ is the skeleton elasticity tensor, p is the static fluid pressure, $\boldsymbol{w}$ describes the relative fluid velocity w.r.t. the solid skeleton, see (2.2), ρ^f is the fluid density, density $\rho^w = \phi_0^{-1} \rho^f$ involving the fluid volume fraction ϕ_0 is relevant to the seepage acceleration, $\boldsymbol{K}$ is the permeability, $\boldsymbol{\alpha}$ is the Biot effective stress coefficient (tensor), μ is the Biot modulus reflecting compressibility of the fluid and of the skeleton. All the material parameters listed above are defined for a given porous solid structure – the skeleton – which is defined at the "microscopic scale".

Our aim in this paper is to derive the homogenized model of such FSPM which are heterogeneous at the "mesoscopic scale". We shall consider all the material parameters involved in (1.1) vary periodically with the spatial position and pursue the asymptotic behavior of the model while the frequency of these periodic oscillations grows up to infinity with $\varepsilon \to 0$, where ε is the scale parameter expressing the ratio between the characteristic lengths of the "mesoscopic" and the "macroscopic" scales. Since in the "mathematical" context of the homogenization methods the heterogeneities are defined w.r.t. the *microscopic scale*, in what follows by the microscopic scale we mean the "physical mesoscopic scale" which was referred to above.

Obviously, the wave propagation in FSPM has been addressed by many authors; apart of Biot's pioneering works [BiW57], the acceleration fronts were discussed in [Cou04], cf. [HeU05]. Slow and fast compression waves were studied e.g. in [Bar95], see also [Sch01, ABG09], the waves on rough interfaces in the context homogenization of the Biot medium was presented in [GiO03]. Acoustics in deformable porous media were investigated from the first principles of the fluid-structure interaction in [AuB94] in the context of double porosity, see also [ABG09].

The *homogenized* model (HM) obtained using the two-scale convergence method is similar in the structure with the *original* "ε-model" (OM) in the sense that all the material parameters have their effective counterparts relevant for homogenized medium. The major difference is in the *fading memory dynamics* of HM arising from the homogenized permeability by virtue of the inertia effects in the Darcy law, cf. [All92b, Hor97]; here the effect of the fluid pressure gradient, ∇p is extended by the fluid acceleration, $\rho^f \dot{\boldsymbol{v}}^f = \rho^w \dot{\boldsymbol{w}} + \rho^f \ddot{\boldsymbol{u}}$, thus, giving rise to the first-order dynamics and to the time-convolutions. In this paper we do not report this phenomenon in detail. We present the HM in its Laplace-transformed, or Fourier-transformed form which is pertinent for an analysis of incident harmonic waves; it can be shown easily that with respects to harmonic waves propagations the structure of OM and HM is the same; the only difference is in the use of the effective parameters in the latter case.

The key role in deriving the a priori estimates of solutions $(\boldsymbol{u}, \boldsymbol{w}, p)$ to (1.1), equipped with suitable boundary and initial conditions, is performed by the mechanic energy

$$E(\boldsymbol{u}, \boldsymbol{w}, p) = \frac{1}{2} \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{I\!D} \boldsymbol{\epsilon}(\boldsymbol{u}) + \int_\Omega \frac{1}{2\mu} |p|^2 + \frac{1}{2} \int_\Omega (\dot{\boldsymbol{u}}, \boldsymbol{w}) \cdot \begin{pmatrix} \overline{\rho} & \rho^f \\ \rho^f & \rho^w \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{u}} \\ \boldsymbol{w} \end{pmatrix} ,$$

which obeys the 1st law of thermodynamics: $\dot{E}(\boldsymbol{u}, \boldsymbol{w}, p) = W(\dot{\boldsymbol{u}}) - D(\boldsymbol{w})$, where W is the external power and $D(\boldsymbol{w})$ is the dissipation associated with the viscose flow in the porous structure, see (2.13). We shall assume such material parameters that E is positive definite.

The paper is organized, as follows. In Section 2 we describe briefly the physical background of model (1.1); we specify the boundary conditions, discuss possible initial conditions and, on introducing convenient functional spaces, we arrive at the weak formulation of the initial-boundary value problem for the FSPM with varying coefficients. For its solutions, in Section 3, the a priori estimates uniform w.r.t. scale parameter ε are derived, so that the two-scale convergence results can be used to obtain the limit model, as reported in Section 4. In Section 5 the homogenized model is established for the limit problem transformed by the Laplace transformation in time. This operation allows us to split multiplicatively the two-scale limit functions in the macroscopic part and the part representing the characteristic response of the microstructure, so that the homogenized material coefficients can be introduced. In this paper we do not describe application of the inverse Laplace transformation to obtain the homogenized problem defined in time – this will be issued in a forthcoming publication. Instead we illustrate the model properties in the context of the plane harmonic waves propagation. For illustration, in Section 6.2 we present analytical formulae for computing the homogenized material coefficients in the layered medium constituted by commuting two different materials. In our setting the materials have similar properties, although we intend to extend to modeling for a strongly heterogeneous media with fractures, or with high contrasts in permeability, cf. [BGS05]. For this the dual porosity ansatz [GrR07, RoC10] can be used.

2 Model construction

2.1 Mechanical modeling

General material considered, two phases, volume fractions, phenomenological approach

Volume fractions and mass The model is based on the concept of the volume fractions of the two phases constituting the saturated medium: denoting by ϕ_0 the *reference* porosity, i.e. the reference fluid volume content, the average mass of the mixture is

$$\overline{\rho} = \rho^s(1 - \phi_0) + \rho^f \phi_0 \ , \tag{2.1}$$

where ρ^s and ρ^f are the solid and fluid densities, respectively.

State variables and kinematics The state of the porous medium is described in terms of the solid skeleton displacements $\boldsymbol{u}(t,x)$, the fluid velocity $\boldsymbol{v}^f(t,x)$ and the fluid (static) pressure $p(t,x)$. However, it is often customary to replace $\boldsymbol{v}^f$ by the seepage velocity

$$\boldsymbol{w} = \phi_0(\boldsymbol{v}^f - \dot{\boldsymbol{u}}) \ , \tag{2.2}$$

which describes the effective fluid relative velocity (w.r.t. the solid). The skeleton deformation is described by the small strain tensor, i.e. the symmetric gradient of the displacements:

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) \equiv \nabla^S \boldsymbol{u} \ , \quad \boldsymbol{\epsilon}(\boldsymbol{u}) = (\epsilon_{ij}) \ , \quad i,j = 1,2,3 \ . \tag{2.3}$$

2.1.1 Constitutive laws

The Biot model involves three essential constitutive laws: 1) the relationship between the drained solid skeleton "macroscopic" deformation $\boldsymbol{\epsilon}(t,x)$, the fluid pressure in pores $p(t,x)$ and the total stress $\boldsymbol{\sigma}(t,x)$, 2) the relationship between the variation of the fluid content, $\zeta(t,x)$, (dimensionless scalar variable), skeleton (macroscopic) deformation, and the fluid pressure, 3) the Darcy law relating the seepage velocity, $\boldsymbol{w}(t,x)$, with "dynamic fluid pressure", i.e. the static part $p(t,x)$ and the fluid inertia part (see below).

Anisotropic poroelastic constitutive equations In an elastic fluid-saturated porous media, the total stress tensor $\boldsymbol{\sigma}(t,x)$ is linearly related to the skeleton strains $\boldsymbol{\epsilon}(t,x)$ of the porous solid and to the pore pressure $p(t,x)$.

$$\boldsymbol{\sigma} = \mathbb{D}\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\alpha} p \tag{2.4}$$

where $\mathbb{D} = (D_{ijkl})$, $i,j,k,l = 1,...,3$ is the fourth-order tensor which is the drained anisotropic elasticity tensor of the porous matrix, and $\boldsymbol{\alpha} = (\alpha_{ij})$ is the symmetric second-order tensor called the *Biot effective stress coefficient tensor*; it can be reduced to a scalar variable in the isotropic medium case.

Fluid pressure in the compressible medium. The second constitutive equation is the relation between the fluid content, strain and pore pressure:

$$p = \mu(\zeta - \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\boldsymbol{u})), \quad \text{hence } \dot{\zeta} = \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) + \frac{1}{\mu}\dot{p} \;, \tag{2.5}$$

where the scalar constant μ is called the Biot's modulus. If both the fluid and solid parts are incompressible (in the latter case the notion of incompressibility is related to structural properties of the skeleton, not only to the "pure solid" material), then $\mu \to +\infty$, so that p becomes the Lagrange multiplier of the overall incompressibility constraint.

Darcy's law. The generalized Darcy's law is reads as (see (2.2)):

$$\begin{aligned} \boldsymbol{w} &= -\boldsymbol{K}(\nabla p + \rho^f \dot{\boldsymbol{v}}^f) \\ &= -\boldsymbol{K}\left(\nabla p + \rho^f(\ddot{\boldsymbol{u}} + \phi_0^{-1}\dot{\boldsymbol{w}})\right) \;, \end{aligned} \tag{2.6}$$

where the symmetric second-order tensor $\boldsymbol{K}=(K_{ij})$, $i,j = 1,...,3$ stands for the anisotropic permeability tensor which is disproportional to the fluid dynamic viscosity. We remark that the Darcy law can be derived by homogenizing the Stokes flow in a given microstructure, see e.g. [Hor97, All89], so that it has not the phenomenological character. We neglect the pore tortuosity effects related to the fluid inertia, cf. [Cou04]; to consider this effect, the poroelastic coefficients can be adjusted and all the results can easily be adapted, see e.g. [NNS10].

2.1.2 Balance equations

The two-phase saturated medium must obey *the mass conservation* and *the balance of momentum.* Since we do not describe any thermal processes and its influence on the material parameters, the energy balance need not be considered explicitly in the model. Nevertheless, the total mechanic energy and the dissipation are natural quantities allowing us to obtain the a priori estimates.

The volume conservation expresses the change in the fluid content (in the solid skeleton) due to the seepage,

$$\dot{\zeta} = -\mathrm{div}\boldsymbol{w} \;. \tag{2.7}$$

The momentum balance is related to the skeleton, where fluid presents the added mass. Neglecting the tortuosity (microscopic) effects, the total rate of momentum is derived from the reference motion of the skeleton, giving $\overline{\rho}\ddot{\boldsymbol{u}}$ and the relative motion of the fluid part, giving $\rho^f\dot{\boldsymbol{w}}$. Thus, using the total stress, we have

$$-\nabla\cdot\boldsymbol{\sigma} = \boldsymbol{f} - (\overline{\rho}\ddot{\boldsymbol{u}} + \rho^f\dot{\boldsymbol{w}}) \;, \tag{2.8}$$

where $\boldsymbol{f}(t,x)$ are the volume forces, as related to the mean density, $\overline{\rho}$.

The final set of equations describing the dynamics of the Biot medium is obtained substituting (2.4) and $(2.5)_2$ into (2.8) and (2.7), respectively, and collecting them with the Darcy law, (2.6)

$$\begin{aligned} -\nabla\cdot(\boldsymbol{I\!D}\boldsymbol{\epsilon}(\boldsymbol{u})) + \nabla\cdot(\boldsymbol{\alpha}p) + \overline{\rho}\ddot{\boldsymbol{u}} + \rho^f\dot{\boldsymbol{w}} &= \boldsymbol{f} \;, \\ \rho^f\ddot{\boldsymbol{u}} + \phi_0^{-1}\rho^f\dot{\boldsymbol{w}} + \boldsymbol{K}^{-1}\boldsymbol{w} + \nabla p &= 0 \\ \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) + \mathrm{div}\boldsymbol{w} + \frac{1}{\mu}\dot{p} &= 0 \;. \end{aligned} \tag{2.9}$$

2.1.3 Mechanical energy and dissipation

The mechanical energy consists of the *potential energy* U related to the material elastic deformations and of the kinetic energy K. We introduce the *generalized state* as the quadruplet $\boldsymbol{q}(t,x) = (\boldsymbol{u}, \boldsymbol{w}, p, \dot{\boldsymbol{u}})(t,x)$, then the total mechanical energy distributed in domain Ω is

$$\begin{aligned} E(\boldsymbol{q}) &= U(\boldsymbol{u}, p) + K(\dot{\boldsymbol{u}}, \boldsymbol{w}) \;, \\ \text{where} \quad U(\boldsymbol{u}, p) &= \frac{1}{2}\int_\Omega \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{I\!D}\boldsymbol{\epsilon}(\boldsymbol{u}) + \int_\Omega \frac{1}{2\mu}|p|^2 \;, \\ K(\dot{\boldsymbol{u}}, \boldsymbol{w}) &= \frac{1}{2}\int_\Omega (\dot{\boldsymbol{u}}, \boldsymbol{w}) \cdot \mathbb{M} \begin{pmatrix} \dot{\boldsymbol{u}} \\ \boldsymbol{w} \end{pmatrix} \;, \end{aligned} \tag{2.10}$$

where

$$\mathbb{M} = \begin{pmatrix} \overline{\rho} & \rho^f \\ \rho^f & \phi_0^{-1}\rho^f \end{pmatrix} \;. \tag{2.11}$$

For readers convenience we demonstrate that K defined above is the kinetic energy of solid and fluid parts. Indeed, the density of the kinetic energy κ is

$$\begin{aligned} \kappa &= \frac{1}{2}\left((1-\phi_0)\rho^s|\dot{\boldsymbol{u}}|^2 + \phi_0\rho^f|\boldsymbol{v}^f|^2\right) = \frac{1}{2}\left((1-\phi_0)\rho^s|\dot{\boldsymbol{u}}|^2 + \phi_0\rho^f|\phi_0^{-1}\boldsymbol{w} + \dot{\boldsymbol{u}}|^2\right) \\ &= \frac{1}{2}\left((1-\phi_0)\rho^s + \phi_0\rho^f\right)|\dot{\boldsymbol{u}}|^2 + \frac{1}{2}\left(\phi_0^{-1}\rho^f|\boldsymbol{w}|^2 + \rho^f 2\boldsymbol{w}\dot{\boldsymbol{u}}\right) \\ &= \frac{1}{2}(\dot{\boldsymbol{u}}, \boldsymbol{w}) \cdot \mathbb{M} \cdot \begin{pmatrix} \dot{\boldsymbol{u}} \\ \boldsymbol{w} \end{pmatrix} \;. \end{aligned} \tag{2.12}$$

The dissipation is related to the fluid viscosity η which is embedded in the Darcy law by virtue of the permeability: $|\boldsymbol{K}^{-1}| \approx \eta$. Therefore the total dissipation power associated with the seepage in Ω is

$$D(\boldsymbol{w}) = \int_\Omega \boldsymbol{w} \cdot (\boldsymbol{K})^{-1} \cdot \boldsymbol{w} \;. \tag{2.13}$$

Now we can express the 1st law of thermodynamics: the rate of the mechanical energy is given by the power of external forces lowered by the dissipation power:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}E(\boldsymbol{q}(t)) = \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega - D(\boldsymbol{w}(t)) \;. \tag{2.14}$$

This equality has the central role in Section 3.2, where we derive the a priori estimates of $\boldsymbol{q}$ in a norm equivalent to $E(\boldsymbol{q})$.

2.2 Mathematical formulation

The porous fluid-saturated medium occupies an open bounded domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary $\partial\Omega$. We discus possible boundary conditions, then we shall comment on the choice of the initial conditions and finally introduce functional spaces for the weak formulation of the problem

2.2.1 Boundary conditions

In general, we may consider the following split of boundary $\partial\Omega$ ("up to the zero measure manifolds"):

$$\begin{aligned} \partial\Omega &= \partial_\sigma\Omega \cup \partial_u\Omega\ , \quad \partial_\sigma\Omega \cap \partial_u\Omega = \emptyset\ , \\ \partial\Omega &= \partial_w\Omega \cup \partial_p\Omega\ , \quad \partial_w\Omega \cap \partial_p\Omega = \emptyset\ . \end{aligned} \tag{2.15}$$

The following mixed boundary conditions can be prescribed:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^{BC} && \text{on } \partial_u\Omega\ , \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{g}^{BC} && \text{on } \partial_\sigma\Omega\ , \\ p &= p^{BC} && \text{on } \partial_p\Omega\ , \\ \boldsymbol{n}\cdot\boldsymbol{w} &= w_n^{BC} && \text{on } \partial_w\Omega\ , \end{aligned} \tag{2.16}$$

where $\boldsymbol{n} = (n_i)$ is the unit normal outward to Ω and by superscript $\square^{BC}$ we denote prescribed quantities on subsets of $\partial\Omega$.

Among all possible combinations of defining functions $\square^{BC}$ in (2.16), there are two cases of interest (we do not consider a Robin-Newton condition):

- **Undrained body** – discussed in literature, whenever $w_n^{BC} = 0$ on $\partial\Omega$, whereas $\partial_p\Omega = \emptyset$. This presents the homogeneous Neumann condition for an elliptic problem for p (related to the reduced system defined in terms of $\boldsymbol{u}$ and p).
- **Drained body** – whenever $\partial_p\Omega \neq \emptyset$, so that w_n^{BC} and p^{BC} can be prescribed on disjoint parts of $\partial\Omega$. As a special case homogeneous Dirichlet condition is considered, $p^{BC} = 0$ on entire $\partial\Omega$.

2.2.2 Initial conditions and steady states

In order to set a convenient (natural) *initial* conditions of the dynamic problem we can use either of the following alternatives:

1. **Unloaded (stress free) state.** This assumption featured by vanishing pressure gradient ($\nabla p = 0$) is guaranteed clearly when

$$\boldsymbol{u}(0,\cdot) = 0\ , p(0,\cdot) = 0 \quad \text{in } \Omega\ . \tag{2.17}$$

   According to this "null" state at $t = 0$ consistent boundary conditions must be considered including vanishing external volume forces $\boldsymbol{f}(0,\cdot) = 0$ or boundary tractions $\boldsymbol{g}(0,\cdot) = 0$ on $\partial_\sigma\Omega$, if considered.

2. **Steady state.** All $\boldsymbol{u}, \boldsymbol{w}, p$ are constant in time for $t \leq 0$. In this case the steady state will depend on the particular boundary conditions prescribed.

We shall now explain how the steady state solution to (2.9) is computed for the drained and undrained body.

Steady states. Clearly (2.9) reduces for $t \leq 0$ to

$$\begin{aligned} -\nabla \cdot \mathbb{D}\boldsymbol{\epsilon}(\boldsymbol{u}^{\text{sts}}) + \nabla \cdot (\boldsymbol{\alpha} p^{\text{sts}}) = \boldsymbol{f}^{\text{sts}} \quad & \text{in } \Omega \, , \\ \nabla \cdot \boldsymbol{K} \nabla p^{\text{sts}} = 0 \quad & \text{in } \Omega \, , \end{aligned} \tag{2.18}$$

where the boundary conditions (2.16) are to be satisfied by $(\boldsymbol{u}^{\text{sts}}, p^{\text{sts}})$. Let us first assume the *drained body*; $(2.18)_2$ possesses a unique p^{sts}, so that a unique $\boldsymbol{u}^{\text{sts}}$ is computed by solving another elliptic problem arising from $(2.18)_1$.

Second, let us consider an *undrained body*. In this case the homogeneous Neumann problem, i.e. $(2.18)_2$ with $\partial_p \Omega = \emptyset$ and $w_n^{BC} = 0$ on $\partial\Omega$, is satisfied by any constant pressure p_0. Then $(2.18)_1$ can be resolved for any p_0 and any "suitable" combination of boundary conditions $(2.16)_{1,2}$ imposed on $\partial_\sigma \Omega \cup \partial_u \Omega$ (obviously, the equilibrium of all external loads must be ensured as the solvability condition when $\partial\Omega_u = \emptyset$).

From the physical point of view, however, we should take into account fluid content which remains constant in an *undrained experiment*. Let us integrate $(2.9)_3$ in Ω and in time interval $]t_0, t[$, this yields

$$\int_\Omega \boldsymbol{\alpha} : (\boldsymbol{\epsilon}(\boldsymbol{u}(t,\cdot)) - \boldsymbol{\epsilon}(\boldsymbol{u}(t_0,\cdot))\,) + \int_{t_0}^t \int_{\partial\Omega} \boldsymbol{w} \cdot \boldsymbol{n} \, d\Gamma + \int_\Omega \frac{1}{\mu}(p(t,\cdot) - p(t_0,\cdot)) = 0 \, . \tag{2.19}$$

Further assuming the "zero" state at t_0 and steady state at t characterized by a constant pressure p_0^{sts}, due to impermeability of entire $\partial\Omega$ we arrive at the following condition:

$$\int_\Omega \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\boldsymbol{u}^{\text{sts}}) + p_0^{\text{sts}} \int_\Omega \frac{1}{\mu} = 0 \, . \tag{2.20}$$

This constraint supplements $(2.18)_1$ which, thus, possesses a unique pressure p_0^{sts} according to the load applied during the undrained experiment. Integral condition (2.20) presents the relationship between the "macroscopic" skeleton strains and the capability of the microstructure to accumulate fluid; the system $(2.18)_1$ and (2.20) leads to a symmetric weak formulation.

2.3 Weak formulation and preliminary assumptions

Problem (2.9) defined point-wise in time-space domain $[0,T] \times \Omega$ can be reformulated in a weak sense using Sobolev spaces on Ω. For this we shall need the following bilinear forms

$$\begin{aligned} a_\Omega(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega [\mathbb{D}\boldsymbol{\epsilon}(\boldsymbol{u})] : \boldsymbol{\epsilon}(\boldsymbol{v}) & b_\Omega(p, \boldsymbol{v}) &= \int_\Omega p \, \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\boldsymbol{v}) \\ c_\Omega(\boldsymbol{w}, \boldsymbol{v}) &= \int_\Omega [\boldsymbol{K}^{-1}\boldsymbol{w}] \cdot \boldsymbol{v} & d_\Omega(p, q) &= \int_\Omega \frac{1}{\mu} pq \, , \\ \bar{\varrho}_\Omega(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega [\rho^s(1-\phi_0) + \phi_0 \rho^f] \boldsymbol{u} \cdot \boldsymbol{v} & \varrho_\Omega^f(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \rho^f \phi_0 \boldsymbol{u} \cdot \boldsymbol{v} \\ \varrho_\Omega^w(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \rho^f \phi_0^{-1} \boldsymbol{u} \cdot \boldsymbol{v} & \langle p, q \rangle_\Omega &= \int_\Omega pq \, . \end{aligned} \tag{2.21}$$

We assume that all material coefficients involved in these expressions are defined in $L^\infty(\Omega)$ and they are bounded form below and above; in particular we require existence of real positive constants c_ρ, C_ρ, $0 < c_\phi, C_\phi < 1$, c_D, C_D, c_K, C_K, c_μ, C_μ and c_α, C_α such that ρ^f, ρ^s, ϕ_0, μ, D_{ijkl}, $\alpha_{ij}, K_{ij} \in L^\infty(\Omega)$ and

$$
\begin{aligned}
c_\rho < \rho^f(x) < C_\rho \;, \quad c_\rho < \rho^s(x) < C_\rho \;, \\
0 < c_\phi < \phi_0(x) < C_\phi < 1 \;, \quad c_\mu < \mu(x) < C_\mu \;, \\
c_D|\boldsymbol{\epsilon}|^2 \leq \boldsymbol{\epsilon} : \boldsymbol{I\!D}(x)\boldsymbol{\epsilon} \;, \quad \boldsymbol{\epsilon} : \boldsymbol{I\!D}(x)\boldsymbol{\epsilon}' \geq C_D|\boldsymbol{\epsilon}||\boldsymbol{\epsilon}'| \;, \\
c_\alpha|\boldsymbol{\xi}|^2 \leq \boldsymbol{\alpha}(x) : \boldsymbol{\xi} \otimes \boldsymbol{\xi} \;, \quad \boldsymbol{\alpha}(x) : \boldsymbol{\xi} \otimes \boldsymbol{\eta} \leq C_\alpha|\boldsymbol{\xi}||\boldsymbol{\eta}| \;, \\
c_K|\boldsymbol{\xi}|^2 \leq \boldsymbol{K}(x) : \boldsymbol{\xi} \otimes \boldsymbol{\xi} \;, \quad \boldsymbol{K}(x) : \boldsymbol{\xi} \otimes \boldsymbol{\eta} \leq C_K|\boldsymbol{\xi}||\boldsymbol{\eta}| \;,
\end{aligned}
\tag{2.22}
$$

holds a.e. in Ω, where $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^3$ and $\boldsymbol{\epsilon}, \boldsymbol{\epsilon}' \in \mathbb{R}^{3\times 3}_{\text{sym}}$.

Lemma 1 *Matrix $\mathbb{M}$ defined in (2.11) is positive definite a.e. in Ω, and there exist C, $c > 0$ such that for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{L}^2(\Omega)$ we have*

$$
\begin{aligned}
C \left(\|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} \right) \;&\geq\; K(\boldsymbol{v}, \boldsymbol{w}) \\
= \frac{1}{2}\left(\bar{\varrho}_\Omega(\boldsymbol{v}, \boldsymbol{v}) + \varrho^w_\Omega(\boldsymbol{w}, \boldsymbol{w}) \right) &+ \varrho^f_\Omega(\boldsymbol{w}, \boldsymbol{v}) \geq c \left(\|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} \right) \;.
\end{aligned}
\tag{2.23}
$$

Proof: For $0 < \phi_0 < 1$ the determinant $\det(\mathbb{M}) = \rho^f \rho^s \phi_0^{-1}(1 - \phi_0) > 0$. Then (2.23) is the consequence of (2.10). □

Boundedness and ellipticity of the quadratic forms Due to properties of the material parameters $\boldsymbol{I\!D}$, see (2.22), and the boundary conditions (2.16), we can apply Korn's and Poincaré's inequalities to obtain

$$
c_P c_a \|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} \leq c_a \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \leq a_\Omega(\boldsymbol{u}, \boldsymbol{u}) \leq C_a \|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C_a \|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} \tag{2.24}
$$

Similarly, due to boundedness and positivity of $\mu, \boldsymbol{K}$, see (2.22),

$$
\begin{aligned}
c_\mu \|p\|^2_{L^2(\Omega)} \leq d_\Omega(p, p) \leq C_\mu \|p\|^2_{L^2(\Omega)} \;, \\
c_c \|\boldsymbol{w}\|^2_{\mathbf{L}^2(\Omega)} \leq D(\boldsymbol{w}) \equiv c_\Omega(\boldsymbol{w}, \boldsymbol{w}) \leq C_c \|\boldsymbol{w}\|^2_{\mathbf{L}^2(\Omega)} \;,
\end{aligned}
\tag{2.25}
$$

where $c_c = 1/C_K, C_c = 1/c_K$, see (2.13). As the consequence, there exist $C, c > 0$ such that for any $\boldsymbol{u}, \boldsymbol{w} \in \mathbf{L}^2(\Omega)$ and $p \in L^2(\Omega)$,

$$
\begin{aligned}
c \left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|p\|^2_{L^2(\Omega)} \right) &\leq U(\boldsymbol{u}, p) \\
&= \frac{1}{2}\left(a_\Omega(\boldsymbol{u}, \boldsymbol{u}) + d_\Omega(p, p) \right) \leq C \left(\|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|p\|^2_{L^2(\Omega)} \right) \;.
\end{aligned}
\tag{2.26}
$$

Hence, recalling Lemma 1, the energy E defined in (2.10) is positive definite quadratic form in $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{w}, p, \dot{\boldsymbol{u}})$.

Various weak formulations Although the precise definition of weak solutions to problem (2.9) will be given in Section 3.1, we introduce here variants of "weak formulations" with respect to space, but still point-wise with respect to time. For the sake of simplicity (but without loss of generality), from now on we shall consider the "undrained" boundary conditions, i.e. $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ on entire $\partial\Omega$. Upon multiplying all equations in (2.9) with test functions $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}$ and $\tilde{p}$ and integrating in Ω, we obtain the following several possible weak formulations. To formulate them we introduce the space

$$\mathbf{H}_0(\mathrm{div},\Omega) := \{\, \boldsymbol{v} \in \mathbf{L}^2(\Omega) \,|\, \mathrm{div}\boldsymbol{w} \in L^2(\Omega),\ \ \boldsymbol{w}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega \,\},$$

which will be equipped with the norm $\|\boldsymbol{w}\|_{\mathbf{H}_0(\mathrm{div},\Omega)} = \|\boldsymbol{w}\|_{\mathbf{L}^2(\Omega)} + \|\mathrm{div}\boldsymbol{w}\|_{L^2(\Omega)}$.

1. Find a triplet $(\boldsymbol{u}, \boldsymbol{w}, p) : [0,T] \to \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Omega)$, which satisfy, for a.a. $t \in [0,T]$,

$$\begin{aligned}
\bar{\varrho}_\Omega(\ddot{\boldsymbol{u}}(t), \tilde{\boldsymbol{v}}) + \varrho^f_\Omega(\dot{\boldsymbol{w}}(t), \tilde{\boldsymbol{v}}) + a_\Omega(\boldsymbol{u}(t), \tilde{\boldsymbol{v}}) - b_\Omega(p(t), \tilde{\boldsymbol{v}}) &= \langle \boldsymbol{f}(t), \tilde{\boldsymbol{v}} \rangle_\Omega \ , \\
\varrho^f_\Omega(\ddot{\boldsymbol{u}}(t), \tilde{\boldsymbol{w}}) + \varrho^w_\Omega(\dot{\boldsymbol{w}}(t), \tilde{\boldsymbol{w}}) + c_\Omega(\boldsymbol{w}(t), \tilde{\boldsymbol{w}}) - \langle p(t), \mathrm{div}\tilde{\boldsymbol{w}} \rangle_\Omega &= 0 \ , \\
b_\Omega(\tilde{p}, \dot{\boldsymbol{u}}(t)) - \langle \boldsymbol{w}(t), \nabla\tilde{p} \rangle_\Omega + d_\Omega(\dot{p}(t), \tilde{p}) &= 0 \ ,
\end{aligned} \tag{2.27}$$

for all $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{p}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0(\mathrm{div},\Omega) \times H^1(\Omega)$. Let us remark, that the functional spaces for solutions and those for test functions are different, so the formulation is "non-symmetric" in a sense.

2. If more regularity of solutions is required, namely $p \in H^1(\Omega)$, we may integrate by parts $\langle \mathrm{div}\tilde{\boldsymbol{w}}, p(t,\cdot) \rangle_\Omega = -\langle \tilde{\boldsymbol{w}}, \nabla p(t,\cdot) \rangle_\Omega$ and define the following problem:
Find $(\boldsymbol{u}, \boldsymbol{w}, p) : [0,T] \to \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0(\mathrm{div},\Omega) \times H^1(\Omega)$, such that for a.a. $t \in [0,T]$

$$\begin{aligned}
\bar{\varrho}_\Omega(\ddot{\boldsymbol{u}}(t), \tilde{\boldsymbol{v}}) + \varrho^f_\Omega(\dot{\boldsymbol{w}}(t), \tilde{\boldsymbol{v}}) + a_\Omega(\boldsymbol{u}(t), \tilde{\boldsymbol{v}}) - b_\Omega(p(t), \tilde{\boldsymbol{v}}) &= \langle \boldsymbol{f}(t), \tilde{\boldsymbol{v}} \rangle_\Omega \ , \\
\varrho^f_\Omega(\ddot{\boldsymbol{u}}(t), \tilde{\boldsymbol{w}}) + \varrho^w_\Omega(\dot{\boldsymbol{w}}(t), \tilde{\boldsymbol{w}}) + c_\Omega(\boldsymbol{w}(t), \tilde{\boldsymbol{w}}) + \langle \nabla p(t), \tilde{\boldsymbol{w}} \rangle_\Omega &= 0 \ , \\
b_\Omega(\tilde{p}, \dot{\boldsymbol{u}}(t)) + \langle \mathrm{div}\boldsymbol{w}(t), \tilde{p} \rangle_\Omega + d_\Omega(\dot{p}(t), \tilde{p}) &= 0 \ ,
\end{aligned} \tag{2.28}$$

for all $(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{p}) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) \times L^2(\Omega)$.

3. There is also an intermediate and more symmetric form, where we ask for $\boldsymbol{w}(t), \tilde{\boldsymbol{w}} \in \mathbf{H}_0(\mathrm{div},\Omega)$ and $p(t), \tilde{p} \in L^2(\Omega)$. For this, the second equation of (2.27) and the third equation from (2.28) are used. Such "symmetric" formulations are especially useful in numerical calculations, namely for application of mixed finite elements [BrF91].

3 Existence and a priori estimates

3.1 Well-posedness of the problem

We first give two different notions of solutions which are necessary to deal with the very particular coupling between the hyperbolic elastic wave equation for $\boldsymbol{u}$ and the dissipative Darcy's law. The *weak solutions* are just given in the energy space. For the *semistrong solutions* we impose more regularity for the seepage velocity $\boldsymbol{w}$ and the pressure p.

To explain why the notions announced just above will be introduced, let us first consider a nonstandard (simplified) subsystem of (2.9) for $(\boldsymbol{w}, p)$ which is given in the form

$$\begin{aligned}
&\dot{\boldsymbol{w}} + \boldsymbol{w} + \nabla p = 0 \ , \quad \dot{p} + \mathrm{div}\boldsymbol{w} = 0 \quad \text{ in } \Omega \ , \\
&\boldsymbol{w}\cdot\boldsymbol{n} = 0 \quad \text{ on } \partial\Omega \ .
\end{aligned} \tag{3.1}$$

For the sake of simplicity we made all material constants equal to 1. Obviously, for smooth solutions we have the energy-dissipation relation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_\Omega\left(\frac{1}{2}|\boldsymbol{w}|^2+\frac{1}{2}|p|^2\right)=-\int_\Omega|\boldsymbol{w}|^2\,.$$

Moreover, (3.1) contains the damped wave equation $\ddot{p}+\dot{p}-\Delta p=0$ for the pressure and the ODE $\dot{\boldsymbol{w}}_{\mathrm{sol}}+\boldsymbol{w}_{\mathrm{sol}}=0$, where $\boldsymbol{w}_{\mathrm{sol}}$ is the solenoidal part of $\boldsymbol{w}$.

Our notions of weak and semistrong solutions are guided by the following abstract approach to (3.1), which we write as $\dot{z}=\mathtt{L}z$ where $z=(\boldsymbol{w},p)\in\mathrm{H}_0:=\mathbf{L}^2(\Omega)\times L^2(\Omega)$ and $\mathtt{L}:\mathrm{dom}(\mathtt{L})=\mathrm{H}_1\to\mathrm{H}_0$, $z\mapsto(\boldsymbol{w}-\nabla p,-\mathrm{div}\boldsymbol{w})$ with $\mathrm{H}_1=\mathbf{H}_0(\mathrm{div},\Omega)\times H^1(\Omega)$. Using the dense and continuous embedding $\mathrm{H}_1\hookrightarrow\mathrm{H}_0$ we have $\mathrm{H}_0\cong\mathrm{H}_0^*\hookrightarrow\mathrm{H}_{-1}:=\mathrm{H}_1^*$.

A function $z\in C^0(]t_1,t_2[;\mathrm{H}_0)$ is called *weak solution* of (3.1) if

$$\int_{t_1}^{t_2}\big(\langle z(t),\mathtt{L}^*\tilde{z}(t)\rangle_{\mathrm{H}_0}+\langle z(t),\dot{\tilde{z}}(t)\rangle_{\mathrm{H}_0}\big)\,\mathrm{d}t=0 \tag{3.2}$$

for all $\tilde{z}\in C_c^1(]t_1,t_2[;\mathrm{H}_0)\cap C_c^0(]t_1,t_2[;\mathrm{H}_1)$. A weak solution z of (3.1) is called *strong solution*, if additionally $z\in C^1(]t_1,t_2[;\mathrm{H}_0)\cap C^0(]t_1,t_2[;\mathrm{H}_1)$. Then, (3.2) implies

$$\int_{t_1}^{t_2}\langle\mathtt{L}z(t)-\dot{z}(t),\tilde{z}(t)\rangle_{\mathrm{H}_0}\mathrm{d}t=0$$

for all $\tilde{z}\in C_c^0(]t_1,t_2[;\mathrm{H}_0)$, i.e. $\dot{z}=\mathtt{L}z$ is satisfied for all $t\in\,]t_1,t_2[$.

We now return to the full coupled system for $(\boldsymbol{u},\boldsymbol{w},p)$ with general material data. We introduce the spaces

$$\begin{aligned}&\mathbb{H}_0=\mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega)\times L^2(\Omega)\,, &&\mathbb{H}_1=\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0(\mathrm{div},\Omega)\times H^1(\Omega)\\ &\mathbb{Q}_0=\mathbb{H}_0\times\mathbf{L}^2(\Omega)\,, &&\mathbb{Q}_1=\mathbb{H}_1\times\mathbf{L}^2(\Omega)\end{aligned}$$

and the bilinear forms

$$\begin{aligned}\mathbb{M}_\Omega\left(\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{w}\end{pmatrix},\begin{pmatrix}\tilde{\boldsymbol{v}}\\ \tilde{\boldsymbol{w}}\end{pmatrix}\right)&=\bar{\varrho}_\Omega(\boldsymbol{v},\tilde{\boldsymbol{v}})+\varrho_\Omega^f(\boldsymbol{v},\tilde{\boldsymbol{w}})+\varrho_\Omega^f(\boldsymbol{w},\tilde{\boldsymbol{v}})+\varrho_\Omega^w(\boldsymbol{w},\tilde{\boldsymbol{w}})\,,\\ \mathbb{A}_\Omega\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{w}\\ p\end{pmatrix},\begin{pmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{w}}\\ \tilde{p}\end{pmatrix}\right)&=a_\Omega(\boldsymbol{u},\tilde{\boldsymbol{u}})-b_\Omega(p,\tilde{\boldsymbol{u}})+b_\Omega(\tilde{p},\boldsymbol{u})+c_\Omega(\boldsymbol{w},\tilde{\boldsymbol{w}})+d_\Omega(p,\tilde{p})\,.\end{aligned}$$

Both forms are well-defined on $\mathbb{H}_0$. While $\mathbb{M}_\Omega(\cdot,\ \cdot)$ is symmetric, $\mathbb{A}_\Omega(\cdot,\ \cdot)$ has a non-trivial antisymmetric part involving $b_\Omega(\cdot,\cdot)$.

Considering time-dependent test functions in (2.27) , integrating in time over $[0,T]$ and then integrating by parts with respect to t leads to our notion of weak solutions:

Definition 1 Weak solution. *A function* $(\boldsymbol{u},\boldsymbol{w},p)\in C^0([0,T],\ \mathbb{H}_0)$ *is called* weak solution *of (2.27) with initial data* $(\boldsymbol{u}(0),\boldsymbol{w}(0),p(0),\dot{\boldsymbol{u}}(0))=\boldsymbol{q}^0\equiv(\boldsymbol{u}^0,\boldsymbol{w}^0,p^0,\boldsymbol{u}^1)\in\mathbb{Q}_0$, *if* $\boldsymbol{u}\in C^1([0,T],\mathbf{L}^2(\Omega))$ *and if we have*

$$\begin{aligned}0=\int_0^T\Big[\mathbb{A}_\Omega&\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{w}\\ p\end{pmatrix},\begin{pmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{w}}\\ -\dot{\tilde{p}}\end{pmatrix}\right)-\mathbb{M}_\Omega\left(\begin{pmatrix}\dot{\boldsymbol{u}}\\ \boldsymbol{w}\end{pmatrix},\begin{pmatrix}\dot{\tilde{\boldsymbol{u}}}\\ \dot{\tilde{\boldsymbol{w}}}\end{pmatrix}\right)\\ &-\langle p,\mathrm{div}\tilde{\boldsymbol{w}}\rangle_\Omega-\langle\boldsymbol{w},\nabla\tilde{p}\rangle_\Omega-\langle\boldsymbol{f},\tilde{\boldsymbol{u}}\rangle_\Omega\Big]\mathrm{d}t\\ &+\mathbb{M}_\Omega\left(\begin{pmatrix}\boldsymbol{u}^1\\ \boldsymbol{w}^0\end{pmatrix},\begin{pmatrix}\tilde{\boldsymbol{u}}(0)\\ \tilde{\boldsymbol{w}}(0)\end{pmatrix}\right)+b_\Omega\big(\tilde{p}(0),\boldsymbol{u}^0\big)+d_\Omega\big(p^0,\tilde{p}(0)\big)\,.\end{aligned} \tag{3.3}$$

for all $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}, \tilde{p}) \in C^1([0,T],\ \mathbb{H}_1)$ *with* $(\tilde{\boldsymbol{u}}(T), \tilde{\boldsymbol{w}}(T), \tilde{p}(T)) = 0$

A solution is called semistrong solution *of (2.28), if it is a weak solution and if additionally* $(\boldsymbol{u}, \boldsymbol{w}, p) \in L^\infty([0,T],\ \mathbb{H}_1)$ *satisfies*

$$\begin{aligned} 0 = \int_0^T \Big[\mathbb{A}_\Omega \left(\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{w} \\ p \end{pmatrix}, \begin{pmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{w}} \\ -\dot{\tilde{p}} \end{pmatrix} \right) - \mathbb{M}_\Omega \left(\begin{pmatrix} \dot{\boldsymbol{u}} \\ \boldsymbol{w} \end{pmatrix}, \begin{pmatrix} \dot{\tilde{\boldsymbol{u}}} \\ \dot{\tilde{\boldsymbol{w}}} \end{pmatrix} \right) \\ + \langle \nabla p, \tilde{\boldsymbol{w}} \rangle_\Omega + \langle \mathrm{div} \boldsymbol{w}, \tilde{p} \rangle_\Omega - \langle \boldsymbol{f}, \tilde{\boldsymbol{u}} \rangle_\Omega \Big] \mathrm{d}t \\ + \mathbb{M}_\Omega \left(\begin{pmatrix} \boldsymbol{u}^1 \\ \boldsymbol{w}^0 \end{pmatrix}, \begin{pmatrix} \tilde{\boldsymbol{u}}(0) \\ \tilde{\boldsymbol{w}}(0) \end{pmatrix} \right) + b_\Omega(\tilde{p}(0), \boldsymbol{u}^0) + d_\Omega(p^0, \tilde{p}(0)) \ . \end{aligned} \tag{3.4}$$

for all $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}, \tilde{p}) \in C^1([0,T],\ \mathbb{H}_0)$ *with* $(\tilde{\boldsymbol{u}}(T), \tilde{\boldsymbol{w}}(T), \tilde{p}(T)) = 0$.

We use the name "semistrong", since only the part $(\boldsymbol{w}, p)$ has increased spatial regularity, whereas $\boldsymbol{u}$ remains in $\mathbf{H}^1_0(\Omega)$. The semistrong form is the most convenient one for passing to the two-scale limit.

Next we will show existence and uniqueness of weak solutions via the Galerkin approximation and *a priori* estimates. We will obtain semistrong solutions by showing that additional temporal regularity implies higher regularity of $(\boldsymbol{w}, p)$. This is contained in the following result.

Proposition 1 *Assume that* $(\boldsymbol{u}, \boldsymbol{w}, p) \in C^0([0,T], \mathbb{H}_0)$ *is a weak solution with the additional time regularity* $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{w}}, \dot{p}) \in L^\infty([0,T], \mathbb{H}_0)$ *and* $\ddot{\boldsymbol{u}} \in L^\infty([0,T], \mathbf{L}^2(\Omega))$. *Then,* $(\boldsymbol{u}, \boldsymbol{w}, p, \dot{\boldsymbol{u}}) \in L^\infty([0,T]; \mathbb{Q}_1)$ *and* $(\boldsymbol{u}, \boldsymbol{w}, p)$ *is a semistrong solution.*

Proof: Because of the higher temporal regularity, we can integrate by parts with respect to time in (3.3) removing $\dot{\tilde{\boldsymbol{u}}}$, $\dot{\tilde{\boldsymbol{w}}}$ and $\dot{\tilde{p}}$. Hence, we conclude that (2.27) is satisfied for a.a. $t \in [0,T]$. Since the test functions $\tilde{\boldsymbol{w}}, \tilde{p}$ in $(2.27)_2$ and $(2.27)_3$ are arbitrary, $(2.9)_2$ and $(2.9)_3$ hold in the distributional sense, namely

$$\begin{aligned} \nabla p &= -\rho^f \ddot{\boldsymbol{u}} - \phi_0^{-1} \rho^f \dot{\boldsymbol{w}} - \boldsymbol{K}^{-1} \boldsymbol{w} \in \mathbf{L}^2(\Omega) \ , \\ \mathrm{div} \boldsymbol{w} &= \frac{1}{\mu} \dot{p} - \boldsymbol{\alpha} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) \in \mathbf{L}^2(\Omega) \ . \end{aligned} \tag{3.5}$$

Hence, we obtain $p \in L^\infty([0,T]; H^1(\Omega))$ and $\mathrm{div} \boldsymbol{w} \in L^\infty([0,T]; L^2(\Omega))$ giving $(\boldsymbol{u}, \boldsymbol{w}, p, \dot{\boldsymbol{u}}) \in L^\infty([0,T]; \mathbb{Q}_1)$, as desired for semistrong solutions. Since we can now integrate by pars the "middle terms" in (3.3), we conclude that (2.28) holds. □

For the following existence theory it is important to introduce the relationship between the energy E, as introduced in (2.10), and the weak solutions.

Proposition 2 *All weak solution satisfy the energy balance:*

$$E(\boldsymbol{q}(t)) + \int_0^t c_\Omega(\boldsymbol{w}(s), \boldsymbol{w}(s))\, \mathrm{d}s = E(\boldsymbol{q}(0)) + \int_0^t \langle \boldsymbol{f}(s), \dot{\boldsymbol{u}}(s) \rangle_\Omega\, \mathrm{d}s \qquad \forall t \in [0,T] \ . \tag{3.6}$$

Proof: Assume first that the solution to (3.3) is regular enough in time, e.g. $(\boldsymbol{u}, \boldsymbol{w}, p) \in C^2([0,T], \mathbb{H}_0)$. Then, (2.27) holds for all t. Moreover, as in the proof of Proposition 1 we have $(\boldsymbol{u}, \boldsymbol{w}, p) \in C^0([0,T]; \mathbb{H}_1)$ and may choose the test functions $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}, \tilde{p}) = (\dot{\boldsymbol{u}}, \boldsymbol{w}, p)$.

Integrating by parts in $\langle \nabla p, \boldsymbol{w}\rangle_\Omega$ leads to cancelation of the off-diagonal terms upon summation the three equations in (2.27). Further integrating in time over $[t_1, t_2] \subset [0, T]$ we obtain

$$\int_{t_1}^{t_2} (a_\Omega(\boldsymbol{u}, \dot{\boldsymbol{u}}) + d_\Omega(\dot{p}, p) + c_\Omega(\boldsymbol{w}, \boldsymbol{w}))\mathrm{d}t + \int_{t_1}^{t_2} \mathbb{M}_\Omega\left(\begin{pmatrix}\ddot{\boldsymbol{u}}\\ \dot{\boldsymbol{w}}\end{pmatrix}, \begin{pmatrix}\dot{\boldsymbol{u}}\\ \boldsymbol{w}\end{pmatrix}\right)\mathrm{d}t = \int_{t_1}^{t_2} \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega \,\mathrm{d}t\,.$$

It can be rewritten using the definitions of potential and kinetic energy (cf. (2.10)) yielding

$$\int_{t_1}^{t_2} (\frac{\mathrm{d}}{\mathrm{d}\,t} U(\boldsymbol{u}, p) + c_\Omega(\boldsymbol{w}, \boldsymbol{w}))\mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\,t} K(\dot{\boldsymbol{u}}, \boldsymbol{w})\mathrm{d}t = \int_{t_1}^{t_2} \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega \,\mathrm{d}t\,.$$

Hence, using $E(\boldsymbol{q}) = U(\boldsymbol{u}, p) + K(\dot{\boldsymbol{u}}, \boldsymbol{w})$ we arrive at

$$E(\boldsymbol{q}(t_2)) - E(\boldsymbol{q}(t_1)) + \int_{t_1}^{t_2} c_\Omega(\boldsymbol{w}, \boldsymbol{w})\,\mathrm{d}t = \int_{t_1}^{t_2} \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega \,\mathrm{d}t\,,$$

so that the assertion follows for $t_1 = 0$.

Next we consider the case of general weak solutions without additional temporal smoothness. For this we first note that the regularity of weak solutions $\boldsymbol{q}(t)$ is sufficient to evaluate all terms in the energy balance (3.6). We approximate the general solution $\boldsymbol{q}$ by temporal convolution setting $\boldsymbol{q}^\delta(t) = \int_\mathbb{R} \boldsymbol{q}(s)\psi_\delta(t{-}s)\mathrm{d}s$, where $\psi_\delta(r) = \psi(r/\delta)/\delta$ with $\psi \geq 0$, $\psi \in C^2(\mathbb{R})$, $\mathrm{sppt}\psi = [-1, 1]$, and $\int_\mathbb{R} \psi(r)\mathrm{d}r = 1$. We similarly define $\boldsymbol{f}^\delta$ and find easily that $\boldsymbol{q}^\delta$ is a weak solution with for the right-hand side $\boldsymbol{f}^\delta$ having the additional smoothness $\boldsymbol{q}^\delta \in C^2([\delta, T{-}\delta], \mathbb{Q}_0)$. Thus the energy balance holds for $\boldsymbol{q}^\delta$ on $[t_1, t_2] \subset [\delta, T{-}\delta]$.

The continuity of $\boldsymbol{q}$ implies $\boldsymbol{q}^\delta(t) \to \boldsymbol{q}(t)$ in $\mathbb{Q}_0$ for $\delta \to 0$ for all $t \in\,]0, T[$. Since the energy and all other terms in (3.6) are continuous on $\mathbb{Q}_0$, the energy balance for $\boldsymbol{q}$ holds for all $[t_1, t_2] \subset\,]0, T[$. We can now pass to the limit $t_1 \to 0$ and continuity gives the desired result. □

Proposition 3 *For each $(\boldsymbol{u}^0, \boldsymbol{w}^0, p^0) \in \mathbb{H}_0$ and $\boldsymbol{u}^1 \in \mathbf{L}^2(\Omega)$ there exists a unique weak solution $(\boldsymbol{u}, \boldsymbol{w}, p) \in C^0([0, T]; \mathbb{H}_0)$ with $\boldsymbol{u} \in C^1([0, T], \mathbf{L}^2(\Omega))$ satisfying the initial conditions, i.e. $(\boldsymbol{u}(0), \boldsymbol{w}(0), p(0)) = (\boldsymbol{u}^0, \boldsymbol{w}^0, p^0)$ and $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}^1$.*

Proof: We chose an increasing sequence $(\mathbb{H}^N)_{N\in\mathbb{N}}$ of finite dimensional subspaces $\mathbb{H}^N \subset \mathbb{H}_1$ such that $\bigcup_{N\in\mathbb{N}} \mathbb{H}^N$ is dense in $\mathbb{H}_0$. Let $\mathtt{P}^N : \mathbb{H}_0 \to \mathbb{H}^N$ be the orthogonal projectors, $\mathtt{P}^N = \mathrm{diag}(\mathtt{P}^N_u, \mathtt{P}^N_w, \mathtt{P}^N_p)$. Then there is a unique weak solution $(\boldsymbol{u}^N, \boldsymbol{w}^N, p^N) : [0, T] \to \mathbb{H}^N$ of (3.3), where $\mathbb{H}_0$ and $\mathbb{H}_1$ are replaced by $\mathbb{H}^N$, for the initial conditions $(\boldsymbol{u}^N(0), \boldsymbol{w}^N(0), p^N(0)) = \mathtt{P}^N(\boldsymbol{u}^0, \boldsymbol{w}^0, p^0)$ and $\dot{\boldsymbol{u}}^N(0) = \mathtt{P}^N \boldsymbol{u}^1$. All arguments of the above Proposition 2 work and we obtain the energy balance (3.6). Proceeding as in "Estimation – 1st step" below (cf. (3.15) to (3.17)) and using the boundedness and ellipticity as in (2.24) to (2.26) we obtain uniform bounds for $\boldsymbol{q}^N = (\boldsymbol{u}^N, \boldsymbol{w}^N, p^N, \dot{\boldsymbol{u}}^N)$ in $L^\infty([0, T]; \mathbb{Q}_0)$. After choosing a suitable subsequence, we obtain a weak* limit $\boldsymbol{q} \in L^\infty([0, T]; \mathbb{Q}_0)$, which is a weak solution. To see this, note that $\boldsymbol{q}^N$ satisfies (3.3) if the test functions are restricted to $\mathbb{H}_1^N$ and the initial conditions are adjusted. Keeping the test functions fixed in $\mathbb{H}_1^M$ and letting $N \to \infty$ the result follows, since the projected initial conditions converge strongly to the correct initial data. Since our system generates a strongly continuous contraction semigroup (cf. [Paz83, ReR93]) with respect to the energy norm defined by

E, the classical semigroup theory yields $\boldsymbol{q} \in C^0([0,T];\mathbb{Q}_0)$ and the existence of a weak solution is established.

According to Proposition 2 all weak solutions satisfy the energy estimate $E(\boldsymbol{q}(t)) \leq E(\boldsymbol{q}(0)) + \int_0^t \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega \,\mathrm{d}t$. Hence, if for a given initial solutions we have two solutions, then the difference $\overline{\boldsymbol{q}} = \boldsymbol{q}_1 - \boldsymbol{q}_2$ is a weak solution with right-hand side $\boldsymbol{f} \equiv 0$. Using $\overline{\boldsymbol{q}}(0) = 0$ we obtain $E(\overline{\boldsymbol{q}}(t)) = 0$ for all t and conclude $\boldsymbol{q}_1(t) = \boldsymbol{q}_2(t)$ for all $t \in [0,T]$, which is the desired uniqueness. $\square$

Our next result shows that under additional assumptions on the right hand side $\boldsymbol{f}$ and the initial data we also have the semistrong solutions.

Proposition 4 *Given the assumptions of Prop. 3, assume further* $\boldsymbol{f} \in W^{1,1}([0,T];\mathbf{L}^2(\Omega))$ *and that the initial data* $\boldsymbol{q}^0 = (\boldsymbol{u}^0, \boldsymbol{w}^0, p^0, \boldsymbol{u}^1) \in \mathbb{Q}_0$ *satisfy*

$$(\boldsymbol{u}^0, \boldsymbol{w}^0, p^0) \in \mathbb{H}_1 , \quad \boldsymbol{u}^1 \in \mathbf{H}^1_0(\Omega) , \quad \mathrm{div}(\boldsymbol{D}\boldsymbol{\epsilon}(\boldsymbol{u}^0) + p^0\boldsymbol{\alpha}) \in \mathbf{L}^2(\Omega) . \tag{3.7}$$

Then the unique weak solution satisfies $\boldsymbol{q} \in C^1([0,T];\mathbb{Q}_0) \cap C^0([0,T];\mathbb{Q}_1)$*, and, hence,* $\boldsymbol{q}$ *is a semistrong solution.*

Proof: The idea is to differentiate formally the solution $\boldsymbol{q}$ w.r.t. time which will become a weak solution of (2.9) with right hand side $\dot{\boldsymbol{f}} \in L^1([0,T];\mathbf{L}^2(\Omega))$ and suitable initial conditions. We insert the initial conditions $\boldsymbol{q}^0 = (\boldsymbol{u}^0, \boldsymbol{w}^0, p^0, \boldsymbol{u}^1)$ into (2.9), which makes possible to compute $\boldsymbol{q}^1 = (\boldsymbol{u}^1, \boldsymbol{w}^1, p^1, \boldsymbol{u}^2) \in \mathbb{Q}_0$ where $(\boldsymbol{w}^1, p^1, \boldsymbol{u}^2) = (\dot{\boldsymbol{w}}(0), \dot{p}(0), \ddot{\boldsymbol{u}}(0))$. From $(2.9)_1$ and $(2.9)_2$ we obtain $\boldsymbol{u}^2$ and $\boldsymbol{w}^1$ upon solving

$$\begin{aligned} \bar\rho \boldsymbol{u}^2 + \rho^f \boldsymbol{w}^1 &= \boldsymbol{f} - \mathrm{div}(\boldsymbol{I\!D}\boldsymbol{\epsilon}(\boldsymbol{u}^0) + p^0\boldsymbol{\alpha}) \in \mathbf{L}^2(\Omega) , \\ \rho^f \boldsymbol{u}^2 + \rho^w \boldsymbol{w}^1 &= -\boldsymbol{K}^{-1}\boldsymbol{w}^0 - \nabla p^0 \in \mathbf{L}^2(\Omega) . \end{aligned} \tag{3.8}$$

Thus, we find $\boldsymbol{u}^2, \boldsymbol{w}^1 \in \mathbf{L}^2(\Omega)$. Similarly, the third equation in (2.9) gives

$$p^1 = -\mu \left(\boldsymbol{\alpha} : \boldsymbol{\epsilon}(\boldsymbol{u}^1) + \mathrm{div}\boldsymbol{w}^0\right) \in \mathbf{L}^2(\Omega) . \tag{3.9}$$

Now we employ Proposition 3 to construct the weak solution $\hat{\boldsymbol{q}} \in C^0([0,T];\mathbb{Q}_0)$ for the right hand side $\dot{\boldsymbol{f}} \in L^1([0,T];\mathbf{L}^2(\Omega))$ with initial conditions $\hat{\boldsymbol{q}}(0) = \boldsymbol{q}^1$. It is then easy to show that $\boldsymbol{q}(t) = \boldsymbol{q}^0 + \int_0^t \hat{\boldsymbol{q}}(\tau)\mathrm{d}\tau$ is a weak solution of the original problem with right hand side $\boldsymbol{f}$ and initial condition $\boldsymbol{q}(0) = \boldsymbol{q}^0$. Hence, we conclude $\boldsymbol{q} \in C^1([0,T];\mathbb{Q}_0)$, as desired. Applying Proposition 1 shows that $\boldsymbol{q}$ is also a semistrong solution. $\square$

3.2 A priori estimates uniform in ε

We employ the energy-equivalent norm applicable to $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{w}, p, \boldsymbol{v}) \in \mathbb{Q}_0$ given by

$$|\!|\!|\boldsymbol{q}|\!|\!|^2_{\mathbb{Q}_0} := \|\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{w}\|^2_{\mathbf{L}^2(\Omega)} + \|p\|^2_{L^2(\Omega)} . \tag{3.10}$$

The total energy introduced in (2.10) can be rewritten using the quadratic forms from (2.27) via

$$\begin{aligned} E^\varepsilon(\boldsymbol{q}) &= U^\varepsilon(\boldsymbol{u}, p) + K^\varepsilon(\boldsymbol{v}, \boldsymbol{w}) \\ &= \frac{1}{2}\Big(a^\varepsilon_\Omega(\boldsymbol{u}, \boldsymbol{u}) + d^\varepsilon_\Omega(p,p) + \bar\varrho^\varepsilon_\Omega(\boldsymbol{v}, \boldsymbol{v}) + 2\varrho^{f,\varepsilon}_\Omega(\boldsymbol{w}, \boldsymbol{v}) + \varrho^{w,\varepsilon}_\Omega(\boldsymbol{w}, \boldsymbol{w})\Big) , \end{aligned} \tag{3.11}$$

where from now on we add the small parameter $\varepsilon > 0$ to indicate that the material parameters $\rho^f, \rho^s, \phi_0, \mu, \boldsymbol{I\!D}, \boldsymbol{\alpha}$ and $\boldsymbol{K}$ may depend on ε as well as on the material point $x \in \Omega$. Our main assumption is that

$$\text{the estimates (2.22) hold uniformly with respect to } \varepsilon \in]0,1] \ . \tag{3.12}$$

In the application we have in mind and treat in Section 4, the material parameters oscillate with the spatial coordinates on a length scale ε, namely $\boldsymbol{I\!D}^\varepsilon(x) = \boldsymbol{I\!D}_{\text{per}}(\frac{x}{\varepsilon})$ where $\boldsymbol{I\!D}_{\text{per}}(\cdot)$ is 1-periodic in each coordinate direction (or Y-periodic, see Section 4). However, this periodicity is not needed for the present purpose, we only rely on (3.12).

Using (3.11) and properties of the bilinear forms, see (2.24), (2.25) and (2.23), we find a constant $C_E > 1$ such that

$$\frac{1}{C_E} \|\!|\boldsymbol{q}|\!\|^2_{\mathbb{Q}_0} \le E^\varepsilon(\boldsymbol{q}) \le C_E \|\!|\boldsymbol{q}|\!\|^2_{\mathbb{Q}_0} \tag{3.13}$$

for all $\boldsymbol{q} \in \mathbb{Q}_0$ and $\varepsilon \in]0,1]$.

Energy identity As the result of Proposition 2 we have the relation for all $t \in [0,T]$:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} E^\varepsilon(\boldsymbol{q}(t)) + c^\varepsilon_\Omega(\boldsymbol{w}(t), \boldsymbol{w}(t)) = \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}(t)\rangle_\Omega \ . \tag{3.14}$$

Estimation – 1st step Integration in (3.14) over $[0,t]$ and dropping the non-negative dissipation term $c^\varepsilon_\Omega(\boldsymbol{w},\boldsymbol{w}) \ge 0$, we get the estimate:

$$\begin{aligned} E^\varepsilon(\boldsymbol{q}(t)) &\le E^\varepsilon(\boldsymbol{q}(0)) + \int_0^t \langle \boldsymbol{f}, \dot{\boldsymbol{u}}\rangle_\Omega \le E^\varepsilon(\boldsymbol{q}(0)) + \|\dot{\boldsymbol{u}}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \|\boldsymbol{f}\|_{L^1(0,T;\mathbf{L}^2(\Omega))} \\ &\le E^\varepsilon(\boldsymbol{q}(0)) + \frac{\lambda}{2} \|\dot{\boldsymbol{u}}\|^2_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2\lambda} \|\boldsymbol{f}\|^2_{L^1(0,T;\mathbf{L}^2(\Omega))} \ , \end{aligned} \tag{3.15}$$

where $\lambda > 0$ is arbitrary. Using (3.13) and choosing $\lambda = 1/C_E$ we find

$$E^\varepsilon(\boldsymbol{q}(t)) \le E^\varepsilon(\boldsymbol{q}(0)) + \frac{1}{2} \|E^\varepsilon(\boldsymbol{q})\|_{C^0(0,T)} + \frac{C_E}{2} \|\boldsymbol{f}\|^2_{L^1(0,T;\mathbf{L}^2(\Omega))} \ , \tag{3.16}$$

for all $t \in [0,T]$. Hence, taking the supremum over $[0,T]$ at the left hand side and using (3.13) once again, we obtain the following uniform *a priori* estimates for all weak solutions of the ε-dependent version of (3.3), namely

$$\|\!|\boldsymbol{q}|\!\|_{C^0(0,T;\mathbb{Q}_0)} \le 2C_E \|\!|\boldsymbol{q}(0)|\!\|_{\mathbb{Q}_0} + C_E \|\boldsymbol{f}\|_{L^1(0,T;\mathbf{L}^2(\Omega))} \ . \tag{3.17}$$

We emphasize that the constant C_E is independent of ε, thus the estimate is uniform with respect to ε.

Estimate of the first derivative – 2nd step Following the arguments of the proof of Proposition 4 and using the assumption on the initial conditions and $\boldsymbol{f} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega))$, we can use that $\tilde{\boldsymbol{q}} := \dot{\boldsymbol{q}}$ is also a weak solution of (3.3). Hence it satisfies (3.17) as well, namely

$$\|\!|\dot{\boldsymbol{q}}|\!\|_{C^0(0,T;\mathbb{Q}_0)} \le 2C_E \|\!|\dot{\boldsymbol{q}}(0)|\!\|_{\mathbb{Q}_0} + C_E \|\dot{\boldsymbol{f}}\|_{L^1(0,T;\mathbf{L}^2(\Omega))} \ . \tag{3.18}$$

Let us note that $\dot{\boldsymbol{q}}(0,\cdot)$ represents the rate of change of the initial state, including the skeleton acceleration $\ddot{\boldsymbol{u}}(0)$, see below. The purpose of estimate (3.18) is to obtain uniform estimates for semistrong solutions, i.e. for $\|\!|\boldsymbol{q}(t)|\!\|_{\mathbb{Q}_1}$. Thus, we need to obtain also ε-independent bounds for $p(t) \in H^1(\Omega)$ and $\mathrm{div}\boldsymbol{w}(t) \in L^2(\Omega)$. For this we use the relations (3.5) together with (3.12) and (3.18) and find an ε-independent constant C_2 such that for all $\varepsilon \in]0,1]$ and all semistrong solutions $\boldsymbol{q}$ of (3.3) we have

$$\|\nabla p\|_{C^0(0,T;\mathbf{L}^2(\Omega))} + \|\mathrm{div}\boldsymbol{w}\|_{C^0(0,T;L^2(\Omega))} \le C_2 \left(\|\!|\boldsymbol{q}(0)|\!\|_{\mathbb{Q}_0} + \|\!|\dot{\boldsymbol{q}}(0)|\!\|_{\mathbb{Q}_0} + \|\boldsymbol{f}\|_{W^{1,1}(0,T;\mathbf{L}^2(\Omega))} \right) . \tag{3.19}$$

Estimation of the initial velocity $\dot{\boldsymbol{q}}(0)$ – 3rd step In (3.19) we need ε-independent bounds on all the right hand side terms. In particular, we need to show that there exist a family of initial velocities $\dot{\boldsymbol{q}}^\varepsilon(0) = (\dot{\boldsymbol{u}}^\varepsilon(0), \dot{\boldsymbol{w}}^\varepsilon(0), \dot{p}^\varepsilon(0), \ddot{\boldsymbol{u}}^\varepsilon(0)) \in \mathbb{Q}_0$. For this we need to impose suitable assumptions on the initial state $\boldsymbol{q}^\varepsilon(0)$ and specify how it is related to the imposed loads $\boldsymbol{f}(0)$ and $\dot{\boldsymbol{f}}(0)$. This is associated with the compatibility conditions in hyperbolic systems or to "gentle-start conditions" in some nonlinear evolutionary systems. Such compatibility can be investigated using equations (3.8) and (3.9) which determine $\dot{\boldsymbol{q}}(0)$ uniquely, if $\boldsymbol{q}(0)$ and $\boldsymbol{f}(0)$ are given. Thus, uniform *a priori* bounds on $\dot{\boldsymbol{q}}(0)$ can be obtained is several circumstances:

1. Let $\boldsymbol{f}(0) = 0$ and $\boldsymbol{q}(0) = 0$, then immediately one obtains $\dot{\boldsymbol{q}}(0) = 0$ as well.

2. Let $\boldsymbol{f}(0) \neq 0$, while $\boldsymbol{q}(0) = 0$; it corresponds to "no loading in past", i.e. $\boldsymbol{q}(t) = 0$, $\boldsymbol{f}(t) = 0$ for $t < 0$, and a step increase of the load at $t = 0$. From (3.8) and (3.9) we get immediately

$$\mathbb{M} \begin{pmatrix} \boldsymbol{u}^2 \\ \boldsymbol{w}^1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}(0) \\ 0 \end{pmatrix} \quad \text{and } p^1 = 0 \,, \tag{3.20}$$

   where $\mathbb{M}$ is given in (2.11). Thus, due to Lemma 1, $\dot{\boldsymbol{q}}(0) \in \mathbb{Q}_0$ is given uniquely and we obtain an ε-uniform estimate on $\|\!|\dot{\boldsymbol{q}}(0)|\!\|_{\mathbb{Q}_0}$ due to (3.12).

3. Let the continuum be in a steady state for $t < 0$, $\boldsymbol{q}^{\mathrm{sts}}(0) = (\boldsymbol{u}^{\mathrm{sts}}, \boldsymbol{w}^{\mathrm{sts}}, p^{\mathrm{sts}}, 0)$ where $(\boldsymbol{u}^{\mathrm{sts}}, p^{\mathrm{sts}})$ is computed by (2.18) and $\boldsymbol{w}^{\mathrm{sts}} = -\boldsymbol{K}^{-1}\nabla p^{\mathrm{sts}}$. By standard elliptic theory we obtain ε-independent *a priori* estimates for $\boldsymbol{q}^{\mathrm{sts}} = \boldsymbol{q}(0)$, on calculating E and using (3.13). Then for a given $\dot{\boldsymbol{u}}(0) \in \mathbf{H}^1_0(\Omega)$ we can solve for $\dot{\boldsymbol{q}}(0)$ via (3.8) and (3.9), where the right hand side are uniformly bounded in ε, therefore we obtain the desired estimate for $\dot{\boldsymbol{q}}(0)$. Let us note that also in this case any step change of loads $\boldsymbol{f}$ at $t = 0$ is admitted, as far as $\boldsymbol{f}^{\mathrm{sts}} - \boldsymbol{f}(0)$ remains bounded in $\mathbf{L}^2(\Omega)$.

Summary of the a priori estimation We conclude this section by the crucial a priori estimate that is independent of ε under the assumption that (3.12) holds. As the consequence of (3.17), (3.18) and (3.19), we find C such that

$$\|\boldsymbol{q}\|_{C^1([0,T];\mathbb{Q}_0)} + \|\boldsymbol{q}\|_{C^0([0,T];\mathbb{Q}_1)} \le C\big(\|\!|\boldsymbol{q}(0)|\!\|_{\mathbb{Q}_0} + \|\!|\dot{\boldsymbol{q}}(0)|\!\|_{\mathbb{Q}_0} + \|\boldsymbol{f}\|_{W^{1,1}([0,T];\mathbf{L}^2(\Omega))} \big) \tag{3.21}$$

for all $\varepsilon \in]0,1]$ and all semistrong solutions.

4 Two-scale convergence

In this section we obtain the main result of the paper, namely the two-scale limit problem of the homogenized medium, and prove the existence and uniqueness of weak solutions in Theorem 6.

4.1 Microstructure and periodicity assumptions

From now on we assume that all material parameters $\rho^f, \rho^s, \phi_0, \mu, \boldsymbol{\alpha}, \boldsymbol{K}$ and $\mathbb{D}$ are periodic functions. The following definition of $\mathbb{D}^\varepsilon$ holds in analogy for the other material parameters:

$$\mathbb{D}^\varepsilon(x) = \mathbb{D}_{\text{per}}(\frac{x}{\varepsilon}) \ , \tag{4.1}$$

where $\mathbb{D}_{\text{per}} : \mathbb{R}^3 \to \mathbb{R}^{3\times3\times3\times3}$ is 1-periodic function in each component, i.e. $\mathbb{D}_{\text{per}}(y+k) = \mathbb{D}_{\text{per}}(y)$ for all $y \in \mathbb{R}^3$ and $k \in \mathbb{Z}^3$.

We define the unit *reference* cell $Y = [0,1[^3$ where the material parameters are defined; in fact, we assume $\mathbb{D}_{\text{per}} \in L^\infty(Y)$. Thus all the material parameters (i.e. not only $\mathbb{D}_{\text{per}}(y)$) are just L^∞ functions defined for $y \in Y$ with no other restriction with respect to the periodicity which is ensured just by construction (4.1).

For a fixed $\varepsilon \in]0,1]$, each point $x \in \Omega$ will be associated with a copy of Y: we define a microscopic cell $Y^\varepsilon(x) = \varepsilon(Y+k)$ where $k = \left[\frac{1}{\varepsilon}x\right]$ is corner point of this cell. Further we introduce $y = \frac{1}{\varepsilon}(x - \varepsilon k)$, the "microscopic coordinate", i.e. the relative position with respect to the corner point. Here $[z] = ([z_1], [z_2], [z_3])$ denotes the vector-valued Gauß bracket, i.e. $[z_1]$ is the largest integer smaller than or equal to z_1. Often $\varepsilon\left[\frac{1}{\varepsilon}x\right]$ is called the lattice coordinate and the periodic material is defined as a "periodic lattice generated by εY".

Using the coordinate split $x = \varepsilon\left[\frac{1}{\varepsilon}x\right] + \varepsilon y$ we can write any function of x as a two-scale function of (x,y), which will be used in the asymptotic analysis when $\varepsilon \to 0$. Following the approach [CDG08], for a function $\varphi : \Omega \to \mathbb{R}$ we define its periodic unfolding $\mathcal{T}_\varepsilon(\varphi) : \Omega \times Y \to \mathbb{R}$ via

$$\mathcal{T}_\varepsilon(\varphi)(x,y) = \begin{cases} \varphi(\varepsilon\left[\frac{1}{\varepsilon}x\right] + \varepsilon y) & \text{if } \varepsilon\left[\frac{1}{\varepsilon}x\right] + \varepsilon Y \subset \Omega \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4.2}$$

For $x \in \Lambda^\varepsilon = \{x \in \Omega | \varepsilon\left[\frac{1}{\varepsilon}x\right] + \varepsilon Y \not\subset \Omega\}$ we have $\mathcal{T}_\varepsilon(\varphi)(x,y) = 0$, but note that $|\Lambda^\varepsilon| = O(\epsilon)$ for Lipschitz domains as considered here. The following useful properties hold, [CDG08]

$$\mathcal{T}_\varepsilon(\phi\psi) = \mathcal{T}_\varepsilon(\phi)\mathcal{T}_\varepsilon(\psi) \quad \text{for all measurable } \phi, \psi \ , \tag{4.3a}$$

$$\int_\Omega \psi \mathrm{d}x = \int_{\Omega\times Y} \mathcal{T}_\varepsilon(\phi)\mathrm{d}x\mathrm{d}y + O(\varepsilon) \quad \text{for all } \psi \in L^1(\Omega) \ , \tag{4.3b}$$

where $O(\varepsilon) \to 0$ with $\varepsilon \to 0$, due to $|\Lambda^\varepsilon| \to 0$; this is the "unfolding criterion for integrals", see [CDG08].

Through the next paper we use $Y_\#$ to indicate the torus $\mathbb{R}^3/\mathbb{Z}^3$. In this way, all functional spaces of Y-periodic functions can be introduced quite naturally; for instance in (4.5), $\psi(x,\cdot) \in C^\infty(Y_\#)$ is a Y-periodic function which lies in $C^\infty(\mathbb{R}^3)$. In particular, by $H^1(Y_\#)$ we refer to the space of Y-periodic functions, i.e. comprising all functions $g \in L^2(Y)$ whose Y-periodic extension lies in $H^1_{\text{loc}}(\mathbb{R}^3)$. The norm is given by $\|g\|_{H^1(Y_\#)} = (\int_Y(|g|^2 + |\nabla g|^2))^{1/2}$. Further we shall employ $H^1_{\text{av}}(Y_\#) = \{\phi \in H^1(Y_\#) | \int_Y \phi = 0\}$.

4.2 Basic facts on two-scale convergence

For a sequence $\{\varphi^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ we say that φ^ε two-scale convergence to $\Phi \in L^2(\Omega\times Y)$, and write $\varphi^\varepsilon \overset{2}{\rightharpoonup} \Phi$, if

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon) \rightharpoonup \Phi \quad \text{ in } L^2(\Omega\times Y)\ . \tag{4.4}$$

For bounded sequences, which we will only have in mind, this definition is equivalent to the classical one proposed in [Ngu89, All92c]

$$\begin{aligned} \varphi^\varepsilon \overset{2}{\rightharpoonup} \Phi \quad \iff \quad & \forall \psi \in C^\infty(\Omega; C^\infty(Y_\#)) : \\ & \int_\Omega \phi^\varepsilon(x)\psi(x,\frac{1}{\varepsilon}x)\mathrm{d}x \to \int_{\Omega\times Y} \Phi(x,y)\psi(x,y)\mathrm{d}x\mathrm{d}y\ . \end{aligned} \tag{4.5}$$

Since $\|\mathcal{T}_\varepsilon(\varphi)\|_{L^2(\Omega\times Y)} \leq \|\varphi\|_{L^2(\Omega)}$ by construction, every bounded sequence $\{\varphi^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ has a subsequence converging in the two-scale sense. In [Vis04, MiT07] this notion is called weak two-scale convergence, while strong two-scale convergence is defined via strong convergence in (4.4). The latter notion is important for fully nonlinear systems, but here we can adhere to the classical notions as our system is linear.

The next result concerns multiplication with periodic coefficients. If $m^\varepsilon(x) = m_{\mathrm{per}}(\frac{1}{\varepsilon}x)$, then

$$\phi^\varepsilon \overset{2}{\rightharpoonup} \Phi \quad \Longrightarrow \quad m^\varepsilon\phi^\varepsilon \overset{2}{\rightharpoonup} m_{\mathrm{per}}\Phi\ , \tag{4.6}$$

where $(m_{\mathrm{per}}\Phi)(x,y) = m_{\mathrm{per}}(y)\Phi(x,y)$. This follows easily using (4.3a).

For a function $\phi \in L^2(\Omega)$ we denote by $\mathtt{X}_Y\phi \in L^2(\Omega\times Y)$ the constant extension over Y: $(\mathtt{X}_Y\phi)(x,y) = \phi(x)$. We then have

$$\phi^\varepsilon \to \phi \text{ in } L^2(\Omega) \quad \Longrightarrow \quad \phi^\varepsilon \overset{2}{\rightharpoonup} \mathtt{X}_Y\phi\ . \tag{4.7a}$$

$$\phi^\varepsilon \overset{2}{\rightharpoonup} \Phi \text{ and } \phi^\varepsilon \rightharpoonup \phi^0 \quad \Longrightarrow \quad \phi^0 = \fint_Y \Phi(\cdot,y)\mathrm{d}y. \tag{4.7b}$$

The symbol $\fint_Y$ denotes the average operator $|Y|^{-1}\int_Y$ (although in our case $|Y| = 1$). Statement (4.7a) means that strong convergence in $L^2(\Omega)$ implies that the two-scale limit has no microscopic fluctuations, while (4.7b) shows that the weak limit ϕ^0 is just the average over the fluctuations. To obtain (4.7b) simply use functions $\psi(x)$ in (4.5).

The fundamental results on the two-scale convergence is the following. If $\{\phi^\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(Y)$, then there exists a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ and functions $\varphi \in H^1(\Omega)$ and $\Phi^1 \in L^2(\Omega, H^1_{\mathrm{av}}(Y_\#))$ such that

$$\phi^{\varepsilon_k} \to \varphi \quad \text{in } L^2(\Omega)\ , \quad \phi^{\varepsilon_k} \rightharpoonup \varphi \quad \text{in } H^1(\Omega)\ , \quad \nabla\phi^{\varepsilon_k} \overset{2}{\rightharpoonup} \mathtt{X}_Y(\nabla_x\varphi) + \nabla_y\Phi^1\ . \tag{4.8}$$

The following result is less known, see [Vis04, Thm. 7.2] for a similar result with $\Omega = \mathbb{R}^d$. We first define the space (recalling the default Y-periodicity property induced by $Y_\#$):

$$\mathbf{H}_0(\mathrm{div},\Omega,Y_\#) = \big\{ W \in \mathbf{L}^2(\Omega\times Y_\#) \,\big|\, \mathrm{div}_x \fint_Y \boldsymbol{W} \in \mathbf{L}^2(\Omega),\ \mathrm{div}_y\,\boldsymbol{W} = 0 \text{ in } \Omega\times Y_\# \big\}\ . \tag{4.9}$$

Proposition 5 *Assume that $\{\boldsymbol{w}^\varepsilon\}_{\varepsilon>0}$ is bounded in $\mathbf{H}_0(\mathrm{div},\Omega)$ such that*

$$\begin{aligned} &\boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}^0 \text{ in } \mathbf{L}^2(\Omega)\ , && \mathrm{div}\boldsymbol{w}^\varepsilon \rightharpoonup d^0 \text{ in } L^2(\Omega)\ , \\ &\boldsymbol{w}^\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{W}\ , && \mathrm{div}\boldsymbol{w}^\varepsilon \overset{2}{\rightharpoonup} D\ . \end{aligned}$$

Then, we have $\boldsymbol{W} \in \mathbf{H}_0(\mathrm{div}, \Omega, Y_\#)$ *and*

$$\boldsymbol{w}^0 = \fint_Y \boldsymbol{W}(x,y)\mathrm{d}y \qquad \text{and} \qquad d^0 = \mathrm{div}_x \boldsymbol{w}^0 = \fint_Y D(x,y)\mathrm{d}y \,. \tag{4.10}$$

In [Vis04, Thm. 7.2] there is a slightly stronger characterization, namely $D = \mathrm{div}_x \boldsymbol{w}_0 + \mathrm{div}_y \boldsymbol{\psi}$ with $\boldsymbol{\psi} \in L^2(\Omega; \mathbf{H}^1_\#(Y))$.

Proof: The relations in (4.10) follow from (4.7b). The y-divergence of $\boldsymbol{W}$ is obtained by testing with suitable functions. Using $\mathcal{T}_\varepsilon\big(\frac{\partial}{\partial x} v\big) = \frac{1}{\varepsilon}\frac{\partial}{\partial y}\mathcal{T}_\varepsilon(v)$ and (4.3b) we obtain, for all $\Psi^1 \in L^2(\Omega; H^1_\#(Y))$, the identities

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \int_\Omega \mathrm{div}\boldsymbol{w}^\varepsilon(x)\varepsilon\Psi^1(x, \tfrac{1}{\varepsilon}x)\mathrm{d}x = -\lim_{\varepsilon\to 0}\int_\Omega \boldsymbol{w}^\varepsilon\big(\varepsilon\nabla_x\Psi^1 + \nabla_y\Psi^1\big)\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\boldsymbol{w}^\varepsilon)\cdot\nabla_y\Psi^1 = -\int_{\Omega\times Y}\boldsymbol{W}\cdot\nabla_y\Psi^1 = \int_{\Omega\times Y}\Psi^1 \mathrm{div}_y\boldsymbol{W} \,. \end{aligned}$$

This proves $0 = \mathrm{div}_y \boldsymbol{W}$. Since $\mathbf{H}_0(\mathrm{div}, \Omega)$ is a closed subspace of $\mathbf{L}^2(\Omega)$ (cf. [Tem84]) we have $\boldsymbol{w}^0 = \fint_Y \boldsymbol{W} \in \mathbf{L}^2(\Omega)$, and the assertion is established. □

4.3 Passing to the limit with $\varepsilon \to 0$

We now consider our system (3.3) with ε-periodic coefficients according to (4.1). We denote the corresponding solutions by $\boldsymbol{q}^\varepsilon : [0,T] \to \mathbb{Q}_0$. Moreover, as in the end of Section 3.2, see "the 3rd step of the estimation", we use initial conditions $\boldsymbol{q}^\varepsilon(0)$ such that

$$\|\boldsymbol{q}^\varepsilon(0)\|_{\mathbb{Q}_0} + \|\dot{\boldsymbol{q}}^\varepsilon(0)\|_{\mathbb{Q}_0} \le C \,.$$

We emphasize that here and in the sequel all constants C do not depend on ε, unless they are labeled with ε. Hence, our a priori estimates (3.21) show that for a given $\boldsymbol{f} \in W^{1,1}(0,T;\mathbf{L}^2(\Omega))$ the semistrong solutions $(\boldsymbol{u}^\varepsilon, \boldsymbol{w}^\varepsilon, p^\varepsilon)$ satisfy

$$\|\boldsymbol{q}^\varepsilon\|_{C^0(0,T;\mathbb{Q}_1)} + \|\boldsymbol{q}^\varepsilon\|_{C^1(0,T;\mathbb{Q}_0)} \le C \,. \tag{4.11}$$

Hence, we obtain boundedness in $L^\infty(0,T;L^2(\Omega))$ or $L^\infty(0,T;\mathbf{L}^2(\Omega))$ of the quantities $\boldsymbol{u}^\varepsilon$, $\dot{\boldsymbol{u}}^\varepsilon$, $\ddot{\boldsymbol{u}}^\varepsilon$, $\nabla\boldsymbol{u}^\varepsilon$, $\nabla\dot{\boldsymbol{u}}^\varepsilon$, $\boldsymbol{w}^\varepsilon$, $\dot{\boldsymbol{w}}^\varepsilon$, $\mathrm{div}\boldsymbol{w}^\varepsilon$, p^ε, $\dot{p}^\varepsilon$, and ∇p^ε. Applying the unfolding operator $\mathcal{T}_\varepsilon$ to the relevant sequences we obtain boundedness in $L^\infty(0,T;L^2(\Omega\times Y))$ which allows us to extract weakly* convergent subsequences. For this we introduce the following two-scale counterparts to $\mathbb{H}_0$, $\mathbb{H}_1$, $\mathbb{Q}_0$, and $\mathbb{Q}_1$:

$$\begin{aligned} &\boldsymbol{\mathcal{H}}_0(\Omega, Y_\#) := \mathbf{H}^1_0(\Omega)\times L^2(\Omega\times\mathbf{H}^1_{\mathrm{av}}(Y_\#))\times\mathbf{L}^2(\Omega\times Y_\#)\times L^2(\Omega)\,, \\ &\boldsymbol{\mathcal{H}}_1(\Omega, Y_\#) := \mathbf{H}^1_0(\Omega)\times L^2(\Omega\times\mathbf{H}^1_{\mathrm{av}}(Y_\#))\times\mathbf{H}_0(\mathrm{div},\Omega,Y_\#)\times L^2(\Omega)\times L^2(\Omega;\mathbf{H}^1_{\mathrm{av}}(Y_\#))\,, \\ &\boldsymbol{\mathcal{Q}}_0(\Omega, Y_\#) := \boldsymbol{\mathcal{H}}_0(\Omega, Y_\#)\times\mathbf{L}^2(\Omega)\,, \quad \boldsymbol{\mathcal{Q}}_1(\Omega, Y_\#) := \boldsymbol{\mathcal{H}}_1(\Omega, Y_\#)\times\mathbf{L}^2(\Omega)\,. \end{aligned}$$

We also introduce the convergence notions

$$\begin{aligned} &\text{(i)}\ \ \boldsymbol{q}^\varepsilon = (\boldsymbol{u}^\varepsilon, \boldsymbol{w}^\varepsilon, p^\varepsilon, \boldsymbol{v}^\varepsilon) \overset{2,\boldsymbol{\mathcal{Q}}_0}{\rightharpoonup} \boldsymbol{Q} := (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \boldsymbol{v}) \quad \text{and} \\ &\text{(ii)}\ \boldsymbol{q}^\varepsilon = (\boldsymbol{u}^\varepsilon, \boldsymbol{w}^\varepsilon, p^\varepsilon, \boldsymbol{v}^\varepsilon) \overset{2,\boldsymbol{\mathcal{Q}}_1}{\rightharpoonup} (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1, \boldsymbol{v}) \end{aligned} \tag{4.12}$$

defined via the following conditions

$$(4.12)\ (i) \Leftrightarrow \quad \begin{cases} \boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u} \text{ in } \mathbf{H}^1_0(\Omega)\ , \quad p^\varepsilon \rightharpoonup p \text{ in } L^2(\Omega)\ , \quad \boldsymbol{v}^\varepsilon \rightharpoonup \boldsymbol{v} \text{ in } \mathbf{L}^2(\Omega)\ , \\ \nabla \boldsymbol{u}^\varepsilon \overset{2}{\rightharpoonup} \nabla_x \boldsymbol{u} + \nabla_y \boldsymbol{U}^1\ , \quad \boldsymbol{w}^\varepsilon \overset{2}{\rightharpoonup} \boldsymbol{W}\ ; \end{cases}$$

$$(4.12)\ (ii) \Leftrightarrow \quad \begin{cases} (\boldsymbol{u}^\varepsilon, \boldsymbol{w}^\varepsilon, p^\varepsilon, \boldsymbol{v}^\varepsilon) \overset{2,\boldsymbol{\mathcal{Q}}_0}{\rightharpoonup} (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \boldsymbol{v})\ , \\ \operatorname{div}\boldsymbol{w}^\varepsilon \rightharpoonup \operatorname{div}_x \fint_Y \boldsymbol{W} \text{ in } L^2(\Omega)\ , \quad \nabla p^\varepsilon \overset{2}{\rightharpoonup} \nabla_x p + \nabla_y P^1\ . \end{cases} \tag{4.13}$$

Using the bound (4.11) there exists a subsequence (not denoted explicitly) such that

$$\begin{aligned} &(\boldsymbol{u}^\varepsilon, \mathcal{T}_\varepsilon(\nabla \boldsymbol{u}^\varepsilon), \mathcal{T}_\varepsilon(\boldsymbol{w}^\varepsilon), p^\varepsilon, \dot{\boldsymbol{u}}^\varepsilon) \overset{*}{\rightharpoonup} (\boldsymbol{u}, \nabla_x \boldsymbol{u} + \nabla_y \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}}) \text{ in } W^{1,\infty}([0,T]; \boldsymbol{\mathcal{Q}}_0(\Omega\times Y_\#)), \\ &(\mathcal{T}_\varepsilon(\boldsymbol{w}^\varepsilon), \mathcal{T}_\varepsilon(\nabla p^\varepsilon)) \overset{*}{\rightharpoonup} (\boldsymbol{W}, \nabla_x p + \nabla_y P^1) \text{ in } L^\infty([0,T]; \mathbf{H}_0(\operatorname{div}, \Omega\times Y_\#))\times \mathbf{L}^2(\Omega\times Y_\#))\ . \end{aligned} \tag{4.14}$$

Here weak* convergence $\boldsymbol{y}^\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{y}$ in $W^{1,\infty}([0,T]; \boldsymbol{Y})$ means that $\boldsymbol{y}^\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{y}$ in $L^\infty([0,T]; \boldsymbol{Y})$ and $\dot{\boldsymbol{y}}^\varepsilon \overset{*}{\rightharpoonup} \dot{\boldsymbol{y}}$ in $L^\infty([0,T]; \boldsymbol{Y})$. Using $\boldsymbol{y}^\varepsilon(t) = \frac{1}{T}\int_0^t s\dot{\boldsymbol{y}}^\varepsilon(s)\mathrm{d}s + \frac{1}{T}\int_t^T (s-T)\dot{\boldsymbol{y}}^\varepsilon(s)\mathrm{d}s + \frac{1}{T}\int_0^T \dot{\boldsymbol{y}}^\varepsilon(s)\mathrm{d}s$, we obtain the pointwise convergence for all $t \in [0,T]$, namely

$$\boldsymbol{q}^\varepsilon(t) = (\boldsymbol{u}^\varepsilon(t), \boldsymbol{w}^\varepsilon(t), p^\varepsilon(t), \boldsymbol{v}^\varepsilon(t)) \overset{2,\boldsymbol{\mathcal{Q}}_0}{\rightharpoonup} \boldsymbol{Q}(t) = (\boldsymbol{u}(t), \boldsymbol{U}^1(t), \boldsymbol{W}(t), p(t), \boldsymbol{v}(t))\ . \tag{4.15}$$

The next step of the asymptotic analysis is to identify the limit $\boldsymbol{Q}$ as the solution of a corresponding two-scale problem. For passing to the limit with $\varepsilon \to 0$ in formulation (3.4), we introduce two-scale bilinear forms corresponding to those in (2.21):

$$\begin{aligned} &a^{\nabla_{xy}}_{\Omega\times Y}((\boldsymbol{u}, \boldsymbol{U}), (\boldsymbol{v}, \boldsymbol{V})) = \int_\Omega \fint_Y [\boldsymbol{\epsilon}^x(\boldsymbol{v}) + \boldsymbol{\epsilon}^y(\boldsymbol{V})] : \boldsymbol{I\!D}[\boldsymbol{\epsilon}^x(\boldsymbol{u}) + \boldsymbol{\epsilon}^y(\boldsymbol{U})]\ , \\ &b^{\nabla_{xy}}_{\Omega\times Y}(p,(\boldsymbol{v},\boldsymbol{V})) = \int_\Omega \fint_Y p\,\boldsymbol{\alpha} : [\boldsymbol{\epsilon}^x(\boldsymbol{v}) + \boldsymbol{\epsilon}^y(\boldsymbol{V})]\ , \quad c_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) = \int_\Omega \fint_Y \boldsymbol{W}\cdot \boldsymbol{K}^{-1}\tilde{\boldsymbol{W}}\ , \\ &d_{\Omega\times Y}(p,\tilde{p}) = \int_\Omega \fint_Y \frac{1}{\mu} p\tilde{p}\ , \qquad \varrho^f_{\Omega\times Y}(\boldsymbol{W}, \boldsymbol{v}) = \int_\Omega \fint_Y \rho^f \boldsymbol{W}\cdot \boldsymbol{v}\ , \\ &\bar{\varrho}_{\Omega\times Y}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \fint_Y \bar{\rho}\boldsymbol{u}\cdot\boldsymbol{v}\ , \qquad \varrho^w_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) = \int_\Omega \fint_Y \rho^f \phi_0^{-1} \boldsymbol{W}\cdot\tilde{\boldsymbol{W}}\ , \end{aligned}$$

where according to (4.1) all the coefficients $\boldsymbol{I\!D}$, $\boldsymbol{K}$, $\boldsymbol{\alpha}$, μ, ρ^f, and $\bar{\rho}$ depend on $y \in Y_\#$. Using these unfolded bilinear forms we define the vector-valued bilinear forms $\mathbb{A}_{\Omega\times Y}$ and $\mathbb{M}_{\Omega\times Y}$, as follows

$$\begin{aligned} \mathbb{A}_{\Omega\times Y}\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{U}\\ \boldsymbol{W}\\ p\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{U}}\\ \tilde{\boldsymbol{W}}\\ \tilde{p}\end{pmatrix}\right) &= a^{\nabla_{xy}}_{\Omega\times Y}\Big((\boldsymbol{u}, \boldsymbol{U}), \Big(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}\Big)\Big) - b^{\nabla_{xy}}_{\Omega\times Y}\Big(p, \Big(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}\Big)\Big) \\ &\quad + b^{\nabla_{xy}}_{\Omega\times Y}(\tilde{p}, (\boldsymbol{u}, \boldsymbol{U})) + c_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) + d_{\Omega\times Y}(p,\tilde{p})\ , \end{aligned}$$

$$\mathbb{M}_{\Omega\times Y}\left(\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{W}\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{v}}\\ \tilde{\boldsymbol{W}}\end{pmatrix}\right) = \bar{\varrho}_{\Omega\times Y}(\boldsymbol{v}, \tilde{\boldsymbol{v}}) + \varrho^f_{\Omega\times Y}(\boldsymbol{W}, \tilde{\boldsymbol{v}}) + \varrho^f_{\Omega\times Y}\Big(\tilde{\boldsymbol{W}}, \boldsymbol{v}\Big) + \varrho^w_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big)\ .$$

In analogy to (3.4) we now define a notion of solutions for the two-scale system.

Definition 2 Weak and semistrong two-scale solutions. *A function* $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p) : [0,T] \to \boldsymbol{\mathcal{H}}_0(\Omega, Y_\#)$ *is called a* weak two-scale solution*, if*

$$(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}}) \in C^0([0,T]; \boldsymbol{\mathcal{Q}}_0(\Omega, Y_\#)) \;, \text{ and} \tag{4.16a}$$

$$\begin{aligned}\int_0^T \Big[\mathbb{A}_{\Omega\times Y}\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{U}^1\\ \boldsymbol{W}\\ p\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{U}}\\ \tilde{\boldsymbol{W}}\\ -\dot{\tilde{p}}\end{pmatrix}\right) - \mathbb{M}_{\Omega\times Y}\left(\begin{pmatrix}\dot{\boldsymbol{u}}\\ \boldsymbol{W}\end{pmatrix}, \begin{pmatrix}\dot{\tilde{\boldsymbol{u}}}\\ \dot{\tilde{\boldsymbol{W}}}\end{pmatrix}\right)\\ - \left\langle p, \mathrm{div}_x \fint_Y \tilde{\boldsymbol{W}}\right\rangle_\Omega - \left\langle \boldsymbol{W}, \nabla_x\tilde{p} + \nabla_y \tilde{P}\right\rangle_{\Omega\times Y} - \langle \boldsymbol{f}, \tilde{\boldsymbol{u}}\rangle_\Omega\Big]\mathrm{d}t \\ = -\mathbb{M}_{\Omega\times Y}\left(\begin{pmatrix}\dot{\boldsymbol{u}}(0)\\ \boldsymbol{W}(0)\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{u}}(0)\\ \tilde{\boldsymbol{W}}(0)\end{pmatrix}\right) - b^{\nabla_{xy}}_{\Omega\times Y}\big(\tilde{p}(0), \big(\boldsymbol{u}(0), \boldsymbol{U}^1(0)\big)\big) - d_{\Omega\times Y}(p(0), \tilde{p}(0))\end{aligned} \tag{4.16b}$$

for all $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p}, \tilde{P}) \in C^1([0,T], \boldsymbol{\mathcal{H}}_1(\Omega, Y_\#))$ *with* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p}, \tilde{P})(T) = 0$*. The initial conditions* $(\boldsymbol{u}(0), \boldsymbol{U}^1(0), \boldsymbol{W}(0), p(0))$ *will be treated as given data.*

A function $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1) : [0,T] \to \boldsymbol{\mathcal{H}}_1(\Omega, Y_\#)$ *is called a* semistrong two-scale solution*, if* $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p)$ *is a weak two-scale solution,*

$$\boldsymbol{Q} = (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1, \dot{\boldsymbol{u}}) \in L^\infty([0,T]; \boldsymbol{\mathcal{Q}}_1(\Omega, Y_\#)) \;, \tag{4.17a}$$

$$\begin{aligned}\int_0^T \Big[\mathbb{A}_{\Omega\times Y}\left(\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{U}^1\\ \boldsymbol{W}\\ p\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{u}}\\ \tilde{\boldsymbol{U}}\\ \tilde{\boldsymbol{W}}\\ -\dot{\tilde{p}}\end{pmatrix}\right) - \mathbb{M}_{\Omega\times Y}\left(\begin{pmatrix}\dot{\boldsymbol{u}}\\ \boldsymbol{W}\end{pmatrix}, \begin{pmatrix}\dot{\tilde{\boldsymbol{u}}}\\ \dot{\tilde{\boldsymbol{W}}}\end{pmatrix}\right)\\ + \left\langle \nabla_x p + \nabla_y P^1, \; \tilde{\boldsymbol{W}}\right\rangle_{\Omega\times Y} + \left\langle \mathrm{div}_x \fint_Y \boldsymbol{W}, \tilde{p}\right\rangle_\Omega - \langle \boldsymbol{f}, \tilde{\boldsymbol{u}}\rangle_\Omega\Big]\mathrm{d}t \\ = -\mathbb{M}_{\Omega\times Y}\left(\begin{pmatrix}\dot{\boldsymbol{u}}(0)\\ \boldsymbol{W}(0)\end{pmatrix}, \begin{pmatrix}\tilde{\boldsymbol{u}}(0)\\ \tilde{\boldsymbol{W}}(0)\end{pmatrix}\right) - b^{\nabla_{xy}}_{\Omega\times Y}\big(\tilde{p}(0), \big(\boldsymbol{u}(0), \boldsymbol{U}^1(0)\big)\big) - d_{\Omega\times Y}(p(0), \tilde{p}(0))\end{aligned} \tag{4.17b}$$

for all $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p}) \in C^1([0,T], \boldsymbol{\mathcal{H}}_0(\Omega, Y_\#))$ *with* $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p})(T) = 0$*. (Note that* $\mathrm{div}_y \boldsymbol{W} = 0$ *is imposed through* $\boldsymbol{W} \in \mathbf{H}_0(\mathrm{div}, \Omega, Y_\#)$ *because of* $\boldsymbol{Q}(t) \in \boldsymbol{\mathcal{Q}}_1(\Omega, Y_\#)$ *for a.a.* $t \in [0,T]$*.)*

In Proposition 6 below we will show that (4.17b) has for each $(\boldsymbol{u}(0), \boldsymbol{U}(0), \boldsymbol{W}(0), p(0)) \in \mathbb{Q}(\Omega, Y)$ at most one solution, which satisfies an energy balance in analogy with (3.14). Before proving this we provide the main result on the two-scale convergence that resembles similar results for simpler wave equations in [Mie08].

Theorem 1 Two-scale convergence. *Assume that the material parameters of (3.4) are periodic as in (4.1) and that* $(\boldsymbol{u}^\varepsilon, \boldsymbol{w}^\varepsilon, p^\varepsilon) \in C^0([0,T]; \mathbb{H}_1)$ *are semistrong solutions (3.4) for a fixed* $\boldsymbol{f} \in W^{1,1}([0,T], \mathbf{L}^2(\Omega))$ *and initial conditions satisfying*

$$\sup_{\varepsilon\in]0,1]} \big(\|\boldsymbol{q}^\varepsilon(0)\|_{\mathbb{Q}_0} + \|\dot{\boldsymbol{q}}^\varepsilon(0)\|_{\mathbb{Q}_0}\big) < \infty \;. \tag{4.18}$$

Moreover, assume

$$\boldsymbol{q}^\varepsilon(0) = (\boldsymbol{u}^\varepsilon(0), \boldsymbol{w}^\varepsilon(0), p^\varepsilon(0), \dot{\boldsymbol{u}}^\varepsilon(0)) \overset{2,\boldsymbol{\mathcal{Q}}_0}{\rightharpoonup} (\boldsymbol{u}(0), \boldsymbol{U}^1(0), \boldsymbol{W}(0), p(0), \dot{\boldsymbol{u}}(0)) \;, \tag{4.19}$$

then there exists $P^1 \in L^\infty([0,T]; L^2(\Omega; H^1_{\mathrm{av}}(Y_\#)))$ *such that we have the convergence*

$$\begin{aligned}
&(\boldsymbol{u}^\varepsilon(t), \boldsymbol{w}^\varepsilon(t), p^\varepsilon(t), \dot{\boldsymbol{u}}^\varepsilon(t)) \overset{2,\mathcal{Q}_0}{\rightharpoonup} (\boldsymbol{u}(t), \boldsymbol{U}^1(t), \boldsymbol{W}(t), p(t), \dot{\boldsymbol{u}}(t)) \text{ for all } t \in [0,T] \, , \\
&\mathrm{div}\boldsymbol{w}^\varepsilon \overset{*}{\rightharpoonup} \mathrm{div} \fint_Y \boldsymbol{W} \text{ in } L^\infty([0,T]; L^2(\Omega)) \, , \\
&\mathcal{T}_\varepsilon(\nabla p^\varepsilon) \overset{*}{\rightharpoonup} \nabla_x p + \nabla_y P^1 \text{ in } L^\infty([0,T], \mathbf{L}^2(\Omega \times Y_\#)) \, ,
\end{aligned} \tag{4.20}$$

where $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1)$ *is the unique semistrong two-scale solution of (4.17b) for the initial conditions* $(\boldsymbol{u}(0), \boldsymbol{W}(0), p(0))$.

Proof: The a priori estimates (4.14) guarantee the existence of a converging subsequence in the sense of (4.14). Thus, the obtained limits satisfy (4.17a) and (4.16a), and the convergences (4.20) hold.

To obtain (4.17b) we pass to the limit for $\varepsilon \to 0$ in (3.4) by choosing suitable test functions $(\tilde{\boldsymbol{u}}^\varepsilon, \tilde{\boldsymbol{w}}^\varepsilon, \tilde{p}^\varepsilon)$ constructed according to the desired (limit) test functions $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p}, \tilde{P})$ in (4.17b). Since it is sufficient to test (4.17b) on a dense set, we may chose smooth enough $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p})$ and set

$$\tilde{\boldsymbol{u}}^\varepsilon(t,x) = \tilde{\boldsymbol{u}}(t,x) + \varepsilon \tilde{\boldsymbol{U}}^1(t,x,\tfrac{1}{\varepsilon}x) \, , \quad \tilde{\boldsymbol{w}}^\varepsilon(t,x) = \tilde{\boldsymbol{W}}(t,x,\tfrac{1}{\varepsilon}x) \, , \quad \tilde{p}^\varepsilon(t,x) = \tilde{p}(t,x) \, . \tag{4.21}$$

Inserting these test functions into (3.4) and using the convergences established in (4.14) and (4.15) allows us to pass to the limit. In fact, employing the definition (4.5) for two-scale convergence, the multiplication property (4.6) and (4.15) we obtain for all $t \in [0,T]$ the limits

$$\begin{aligned}
&a^\varepsilon_\Omega(\boldsymbol{u}^\varepsilon, \tilde{\boldsymbol{v}}^\varepsilon) \to a^{\nabla_{xy}}_{\Omega\times Y}\Big((\boldsymbol{u}, \boldsymbol{U}^1), (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{V}}^1)\Big) \, , && b^\varepsilon_\Omega(p^\varepsilon, \tilde{\boldsymbol{v}}^\varepsilon) \to b^{\nabla_{xy}}_{\Omega\times Y}\Big(p, (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{V}}^1)\Big) \, , \\
&b^\varepsilon_\Omega(\tilde{q}^\varepsilon, \boldsymbol{u}^\varepsilon) \to b^{\nabla_{xy}}_{\Omega\times Y}(q, (\boldsymbol{u}, \boldsymbol{U}^1)) \, , && c^\varepsilon_\Omega(\boldsymbol{w}^\varepsilon, \tilde{\boldsymbol{w}}^\varepsilon) \to c_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) \, , \\
&d^\varepsilon_\Omega(p^\varepsilon, \tilde{q}^\varepsilon) \to d_{\Omega\times Y}(p, \tilde{q}) \, , && \varrho^{f,\varepsilon}_\Omega(\dot{\boldsymbol{u}}^\varepsilon, \tilde{\boldsymbol{v}}^\varepsilon) \to \varrho^f_{\Omega\times Y}(\dot{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) \, , \\
&\bar{\varrho}^\varepsilon_\Omega(\dot{\boldsymbol{u}}^\varepsilon, \tilde{\boldsymbol{v}}^\varepsilon) \to \bar{\varrho}_{\Omega\times Y}(\dot{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) \, , && \varrho^{w,\varepsilon}_\Omega(\boldsymbol{w}^\varepsilon, \tilde{\boldsymbol{w}}^\varepsilon) \to \varrho^w_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) \, .
\end{aligned}$$

Moreover (4.20) provides the limits

$$\begin{aligned}
\int_0^T \langle \nabla p^\varepsilon, \tilde{\boldsymbol{w}}^\varepsilon \rangle_\Omega \, \mathrm{d}t &\to \int_0^T \Big\langle \nabla_x p + \nabla_y P^1, \, \tilde{\boldsymbol{W}} \Big\rangle_{\Omega\times Y} \mathrm{d}t \, , \\
\int_0^T \langle \mathrm{div}_x \boldsymbol{w}^\varepsilon, \tilde{p}^\varepsilon \rangle_\Omega \, \mathrm{d}t &\to \int_0^T \langle \mathrm{div}_x \boldsymbol{W}, \, \tilde{p} \rangle_{\Omega\times Y} \, \mathrm{d}t \, .
\end{aligned}$$

Thus, we conclude that the limit $\boldsymbol{Q} = (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1, \dot{\boldsymbol{u}})$ satisfies (4.17).

So far, we only showed convergence along a subsequence. However, in Prop. 6 we show that there is at most one semistrong two-scale solution for given $(\boldsymbol{u}(0), \boldsymbol{W}(0), p(0), \dot{\boldsymbol{u}}(0))$. Hence, we know that any convergent subsequence has to converge to the same limit. This implies that the whole family converges, and the theorem is established.

$\square$

4.4 Existence and Uniqueness for the two-scale limit model

In (4.17b) we have introduced the two-scale model, which now can be rewritten in a more detailed form comprising 5 equations for unknowns $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, P^1)$. To discuss the structure of the equation we assume temporarily that the solutions are sufficiently smooth in time, so that we can write down the equations for each $t \in [0,T]$.

$$
\begin{aligned}
\bar{\varrho}_{\Omega\times Y}(\ddot{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) + \varrho^f_{\Omega\times Y}\Big(\dot{\boldsymbol{W}}, \tilde{\boldsymbol{u}}\Big) + a^{\nabla_{xy}}_{\Omega\times Y}\big((\boldsymbol{u}, \boldsymbol{U}^1), (\tilde{\boldsymbol{u}}, 0)\big) - b^{\nabla_{xy}}_{\Omega\times Y}(p, (\tilde{\boldsymbol{u}}, 0)) &= \langle \boldsymbol{f}, \tilde{\boldsymbol{u}}\rangle_\Omega \ , \\
a^{\nabla_{xy}}_{\Omega\times Y}\Big((\boldsymbol{u}, \boldsymbol{U}^1), \Big(0, \tilde{\boldsymbol{U}}\Big)\Big) - b^{\nabla_{xy}}_{\Omega\times Y}\Big(p, \Big(0, \tilde{\boldsymbol{U}}\Big)\Big) &= 0 \ , \\
\varrho^f_{\Omega\times Y}\Big(\ddot{\boldsymbol{u}}, \tilde{\boldsymbol{W}}\Big) + \varrho^w_{\Omega\times Y}\Big(\dot{\boldsymbol{W}}, \tilde{\boldsymbol{W}}\Big) + c_{\Omega\times Y}\Big(\boldsymbol{W}, \tilde{\boldsymbol{W}}\Big) + \Big\langle \nabla_x p + \nabla_y P^1, \ \tilde{\boldsymbol{W}}\Big\rangle_{\Omega\times Y} &= 0 \ , \\
b^{\nabla_{xy}}_{\Omega\times Y}\Big(\tilde{p}, \Big(\dot{\boldsymbol{u}}, \dot{\boldsymbol{U}}^1\Big)\Big) - \langle \boldsymbol{W}, \nabla_x \tilde{p}\rangle_{\Omega\times Y} + d_{\Omega\times Y}(\dot{p}, \tilde{p}) &= 0 \ .
\end{aligned}
\tag{4.22}
$$

Note that the four equations correspond to the four independent test-functions $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}}, \tilde{p})$, i.e. just one non-vanishing per each equation. Further we recall the constraint $\mathrm{div}_y \boldsymbol{W} = 0$ which is imposed by $\boldsymbol{W}(t,\cdot) \in \mathbf{H}_0(\mathrm{div}, \Omega, Y_\#)$. These equations are used in Section 5.

Recall that the simplified subproblem (3.1) involved only $(\boldsymbol{w}, p)$, which presents a simplified form of the full system (3.4). To obtain the analogous structural insight as for the latter, we consider a subproblem of (4.22) that is the two-scale version of (3.1), namely

$$
\begin{aligned}
\dot{\boldsymbol{W}} + \boldsymbol{W} + \nabla_x p + \nabla_y P^1 &= 0 \ , \\
\dot{p} + \mathrm{div}_x\Big(\fint_Y \boldsymbol{W}\Big) &= 0 \ , \\
\mathrm{div}_y \boldsymbol{W} &= 0 \ .
\end{aligned}
\tag{4.23}
$$

Hence, P^1 is the microscopic pressure associated with the microscopic constraint $\mathrm{div}_y \boldsymbol{W} = 0$, while p is the macroscopic pressure associated with $\mathrm{div}_x(\fint_Y \boldsymbol{W})$ by $(4.23)_2$.

The second equation, $(4.22)_2$ shows that $\boldsymbol{U}^1(t)$ can be calculated using instantaneous "macroscopic" data, namely $\boldsymbol{U}^1(t) = \mathfrak{L}(\boldsymbol{u}(t), p(t))$, so there is no dynamics of microscopic fluctuation represented by $\boldsymbol{U}^1$, since there is no $\dot{\boldsymbol{U}}$. Consequently, the time rate $\dot{\boldsymbol{U}}^1$ is simply given via the same mapping, $\dot{\boldsymbol{U}}^1 = \mathfrak{L}(\dot{\boldsymbol{u}}, \dot{p})$.

In order to derive an existence theory for (4.17b), or its equivalent (4.22), we use the fact that there is a natural energy balance. For $\boldsymbol{Q} = (\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}})$ we define the energy

$$
\begin{aligned}
\mathcal{E}(\boldsymbol{Q}) =& \frac{1}{2} a^{\nabla_{xy}}_{\Omega\times Y}\big((\boldsymbol{u}, \boldsymbol{U}^1), (\boldsymbol{u}, \boldsymbol{U}^1)\big) + \frac{1}{2} d_{\Omega\times Y}(p,p) \\
&+ \frac{1}{2}\Big(\bar{\varrho}_{\Omega\times Y}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}}) + 2\varrho^f_{\Omega\times Y}(\boldsymbol{W}, \dot{\boldsymbol{u}}) + \varrho^w_{\Omega\times Y}(\boldsymbol{W}, \boldsymbol{W})\Big) \ .
\end{aligned}
$$

Then, every weak two-scale solution $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p)$ of (4.22) satisfies the energy balance

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}}) + c_{\Omega\times Y}(\boldsymbol{W}, \boldsymbol{W}) = \langle \boldsymbol{f}, \dot{\boldsymbol{u}}\rangle_\Omega \ . \tag{4.24}
$$

The proof is analogous to that of Prop. 2: we first show it for solutions $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}}) \in C^2([0,T]; \boldsymbol{\mathcal{Q}}_0(\Omega, Y_\#))$, which are even semistrong two-scale solutions. Then, the general result follows by approximation via temporal smoothing. Since $\mathcal{E}$ is a positive definite quadratic form in $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p, \dot{\boldsymbol{u}})$, we obtain immediately uniqueness of solutions, except for the two-scale pressure P^1. However, $P^1(t) \in L^2(\Omega; H^1_{\rm av}(Y_\#))$ can be determined uniquely from $(4.22)_3$, once the solution $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p)$ is determined.

We follow the Galerkin approach; the construction of weak, or semistrong solution now follows as in Section 3.1 by choosing suitable finite-dimensional subspace $\mathbb{H}^N(\Omega, Y)$ of $\mathbb{H}(\Omega, Y)$. All finite dimensional approximations will still satisfy the energy balance (4.24) and, therefore, we can obtain uniform *a priori* bounds. Taking the limit $N \to \infty$, where $\cup_{N\in\mathbb{N}}\mathbb{H}^N(\Omega, Y)$ is dense in $\mathbb{H}(\Omega, Y)$, we obtain the desired solutions. All this is summarized in the following result.

Proposition 6 *For each initial state $(\boldsymbol{u}(0), \boldsymbol{W}(0), p(0), \dot{\boldsymbol{u}}(0)) \in \boldsymbol{\mathcal{Q}}_0(\Omega, Y_\#)$ and each $f \in L^1([0,T]; \mathbf{L}^2(\Omega))$ problem (4.17b) has a unique weak two-scale solution $(\boldsymbol{u}, \boldsymbol{U}^1, \boldsymbol{W}, p)$. This solution satisfies the energy balance (4.24) and, for all $t \in [0,T]$, we have $\boldsymbol{U}^1(t) = \mathfrak{L}(\boldsymbol{u}(t), p(t))$, where the bounded linear operator $\mathfrak{L} : \mathbf{H}^1_0(\Omega) \times \mathbf{L}^2(\Omega) \to L^2(\Omega; \mathbf{H}^1_\#(Y))$ is defined via $(4.22)_2$.*

If additionally, $f \in W^{1,1}([0,T]; \mathbf{L}^2(\Omega))$, $(\boldsymbol{u}(0), \boldsymbol{U}^1(0), \boldsymbol{W}(0), p(0)) \in \boldsymbol{\mathcal{H}}_1(\Omega, Y_\#)$,

$$\dot{\boldsymbol{u}}(0) \in \mathbf{H}^1_0(\Omega)\ , \quad \textit{and} \quad \operatorname{div}\Big(\boldsymbol{D}\big(\boldsymbol{\epsilon}^x(\boldsymbol{u}(0)) + \boldsymbol{\epsilon}^y(\boldsymbol{U}^1(0))\big) + p(0)\boldsymbol{\alpha}\Big) \in \mathbf{L}^2(\Omega)\ ,$$

then the weak two-scale solution extended by the uniquely determined pressure corrector P^1 is a semistrong two-scale solution.

5 Scale decoupling for the Laplace-transformed model

In (4.22) the equations involve both the macroscopic and the two-scale functions. Due to linearity of the problem, it is possible to express the two-scale functions as the time convolutions of the macroscopic fields $\boldsymbol{\epsilon}(\boldsymbol{u})$ and p with the corrector basis functions which satisfy auxiliary microscopic problems and constitute the homogenized coefficients involved in the macroscopic problem. To derive the homogenized equations efficiently, we apply the Laplace transformation, as in [GrR07, RoC10].

5.1 Microscopic problems

They are obtained from (4.22) upon substituting there special combinations of the two-scale test functions $(\tilde{\boldsymbol{U}}, \tilde{\boldsymbol{W}})$ whereby $(\tilde{\boldsymbol{u}}, \tilde{p}) \equiv 0$. Thus, we introduce autonomous microscopic problems for the corrector basis functions. We shall use the following bilinear forms

$$\begin{aligned}
a_Y\left(\boldsymbol{U},\, \tilde{\boldsymbol{U}}\right) &= \fint_Y \boldsymbol{\epsilon}^y(\tilde{\boldsymbol{U}}) : \mathbb{D}(y)\boldsymbol{\epsilon}^y(\boldsymbol{U})\ , \quad & b_Y\left(P,\, \boldsymbol{U}\right) &= \fint_Y P\boldsymbol{\alpha}(y) : \boldsymbol{\epsilon}^y(\boldsymbol{U})\ ,\\
c_Y\left(\boldsymbol{W},\, \tilde{\boldsymbol{W}}\right) &= \fint_Y \tilde{\boldsymbol{W}} \cdot \boldsymbol{K}^{-1}(y)\, \boldsymbol{W}\ , & \varrho^w_Y\left(\boldsymbol{W},\, \tilde{\boldsymbol{W}}\right) &= \fint_Y \rho^f(y)\phi_0^{-1}(y)\, \boldsymbol{W} \cdot \tilde{\boldsymbol{W}}\ .
\end{aligned} \tag{5.1}$$

5.1.1 Time-independent corrector basis functions

In (4.22) we choose $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{W}}, \tilde{p}) \equiv 0$, which leads to

$$
\begin{aligned}
a_{\Omega\times Y}^{\nabla_{xy}}\Big(\big(\boldsymbol{u}, \boldsymbol{U}^1\big), \big(0, \tilde{\boldsymbol{U}}\big)\Big) - b_{\Omega\times Y}^{\nabla_{xy}}\Big(P, \big(0, \tilde{\boldsymbol{U}}\big)\Big) = 0 \;, \forall \tilde{\boldsymbol{U}}^1 \in L^2(\Omega; \mathbf{H}^1(Y_\#)) \;, \\
\text{hence, a.e. in } \Omega \qquad\qquad (5.2) \\
a_Y\left(\boldsymbol{U}^1 + \boldsymbol{\Pi}^{rs}\epsilon_{rs}(\boldsymbol{u}),\ \tilde{\boldsymbol{V}}\right) - b_Y\left(p,\ \tilde{\boldsymbol{V}}\right) = 0 \quad \forall \tilde{\boldsymbol{V}} \in \mathbf{H}^1(Y_\#) \;.
\end{aligned}
$$

By virtue of the linearity we can introduce the corrector basis functions $\boldsymbol{\chi}^{rs}, \boldsymbol{\chi}^* \in \mathbf{H}^1(Y_\#)$, $r, s = 1, 2, 3$ to express the displacement fluctuations

$$
\boldsymbol{U}^1(x, y) = \boldsymbol{\chi}^{rs}(y)\epsilon_{rs}^x(\boldsymbol{u}) + \boldsymbol{\chi}^*(y)p(x) \;, \tag{5.3}
$$

On substituting (5.3) into (5.2), the following local problems:

1. Find $\boldsymbol{\chi}^{rs} \in \mathbf{H}^1_{\rm av}(Y_\#)$ such that

$$
a_Y\left(\boldsymbol{\chi}^{rs} + \boldsymbol{\Pi}^{rs},\ \tilde{\boldsymbol{V}}\right) = 0 \quad \forall \tilde{\boldsymbol{V}} \in \mathbf{H}^1(Y_\#) \;, \tag{5.4}
$$

   where $\boldsymbol{\Pi}^{rs} = (\Pi_i^{rs}) = (y_s\delta_{ir})$.

2. Find $\boldsymbol{\chi}^* \in \mathbf{H}^1_{\rm av}(Y_\#)$ such that

$$
a_Y\left(\boldsymbol{\chi}^*,\ \tilde{\boldsymbol{V}}\right) = b_Y\left(1,\ \tilde{\boldsymbol{V}}\right) \quad \forall \tilde{\boldsymbol{V}} \in \mathbf{H}^1(Y_\#) \;. \tag{5.5}
$$

5.1.2 Time-variant corrector basis functions

Taking the test functions $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{U}}, \tilde{p}) \equiv 0$, equation $(4.22)_2$ presents the local problem, where $\boldsymbol{W}$ is "driven" by the macroscopic acceleration, $\ddot{\boldsymbol{u}}$ and the macroscopic pressure gradient, $\nabla_x p^0$ (due to $(4.23)_1$). Since $(4.22)_2$ is the evolutionary equation for $\boldsymbol{W}$, the two scales are coupled in time and their decoupling is more complicated than in the "static case". In order to separate the scales, we apply the Laplace transform $\mathcal{L} : f(t) \mapsto \underset{*}{f}(\lambda)$, where λ is the Laplace-transformed time variable. Alternatively we shall use the notation $\mathcal{L}\{f\} \equiv \underset{*}{f}$. Assuming the "zero" initial conditions, for a.a. $x \in \Omega$ we have

$$
\varrho_Y^f\left(1,\ \tilde{\boldsymbol{W}}\right) \cdot \lambda^2 \underset{*}{\boldsymbol{u}} + \varrho_Y^w\left(\lambda \underset{*}{\boldsymbol{W}},\ \tilde{\boldsymbol{W}}\right) + c_Y\left(\underset{*}{\boldsymbol{W}},\ \tilde{\boldsymbol{W}}\right) + \nabla_x \underset{*}{p} \cdot \fint_Y \tilde{\boldsymbol{W}} = 0 \;, \tag{5.6}
$$

for all $\tilde{\boldsymbol{W}} \in \mathbf{H}_0(\mathrm{div}, Y_\#) = \{\boldsymbol{v} \in \mathbf{L}^2(Y_\#)|\ \mathrm{div}_y \boldsymbol{v} = 0 \text{ in } Y\}$.

Now the split of $\underset{*}{\boldsymbol{W}}$ can be defined in terms of two corrector functions $\boldsymbol{\varsigma}$ and $\boldsymbol{\pi}$:

$$
\begin{aligned}
\underset{*}{\boldsymbol{W}} &= \lambda^3 \underset{*}{\boldsymbol{\varsigma}}^k \underset{*}{u}_k + \lambda \underset{*}{\boldsymbol{\pi}}^l \partial_l^x \underset{*}{p} \;, \\
\text{hence} \quad \boldsymbol{W}(t, x, y) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\, t}\boldsymbol{\varsigma}^k(t - s, y)\frac{\mathrm{d}^2}{\mathrm{d}\, s^2}u_k(s, x)\, \mathrm{d}s + \int_0^t \frac{\mathrm{d}}{\mathrm{d}\, t}\boldsymbol{\pi}^k(t - s, y)\partial_k^x p(s, x)\, \mathrm{d}s \;.
\end{aligned} \tag{5.7}
$$

Eq. (5.6) is satisfied whenever the following auxiliary problems hold:

1. find $\underset{*}{\boldsymbol{\varsigma}}(\lambda,\cdot) \in \mathbf{H}_0(\mathrm{div}, Y_\#)/\mathbb{R}$ such that (for $k = 1,2,3$)

$$\lambda \varrho_Y^w \left(\underset{*}{\boldsymbol{\varsigma}}^k,\ \tilde{\boldsymbol{W}}\right) + c_Y\left(\underset{*}{\boldsymbol{\varsigma}}^k,\ \tilde{\boldsymbol{W}}\right) = -\frac{1}{\lambda}\varrho_Y^f\left(1,\ \tilde{W}_k\right) \quad \forall \tilde{\boldsymbol{W}} \in \mathbf{H}_0(\mathrm{div}, Y_\#)\ , \tag{5.8}$$

2. find $\underset{*}{\boldsymbol{\pi}}(\lambda,\cdot) \in \mathbf{H}_0(\mathrm{div}, Y_\#)/\mathbb{R}$ such that (for $k = 1,2,3$)

$$\lambda \varrho_Y^w \left(\underset{*}{\boldsymbol{\pi}}^k,\ \tilde{\boldsymbol{W}}\right) + c_Y\left(\underset{*}{\boldsymbol{\pi}}^k,\ \tilde{\boldsymbol{W}}\right) = -\frac{1}{\lambda}\ \fint_Y \tilde{W}_k \quad \forall \tilde{\boldsymbol{W}} \in \mathbf{H}_0(\mathrm{div}, Y_\#)\ , \tag{5.9}$$

Remark 1. If the fluid is microscopically homogeneous, i.e. $\rho^f = \rho^f(x)$ is constant w.r.t. $y \in Y$, then just one auxiliary problem has to be solved (see the definition of $\varrho_Y^f(\cdot,\ \cdot)$); in such situation, instead of (5.7), we define

$$\begin{aligned} \underset{*}{\boldsymbol{W}} &= \lambda \underset{*}{\boldsymbol{\pi}}^k \left(\lambda^2 \rho^f \underset{*}{u}_k + \partial_k^x \underset{*}{p}\right)\ , \\ \text{hence} \quad \boldsymbol{W}(t,x,y) &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}\,t}\boldsymbol{\pi}^k(t-s,y)\left(\rho^f \frac{\mathrm{d}^2}{\mathrm{d}\,s^2}u_k(s,x) + \partial_k^x p(s,x)\right)\,\mathrm{d}s\ . \end{aligned} \tag{5.10}$$

$\triangle$

From now on we shall assume that the fluid density, ρ^f, is independent of the microscopic coordinate, i.e. $\rho^f = \rho^f(x)$. Therefore only (5.9) is the auxiliary microscopic problem.

In order to get rid of the functional space constraints $\boldsymbol{\pi}(t,\cdot)$, $\tilde{\boldsymbol{W}} \in \mathbf{H}_0(\mathrm{div}, Y_\#)$, we can reformulate (5.9) as the saddle point problem, on introducing the Lagrange multiplier $\underset{*}{\eta}(\lambda,\cdot) \in L^2(Y_\#)$

$$\begin{aligned} \lambda \varrho_Y^w \left(\underset{*}{\boldsymbol{\pi}}^k,\ \tilde{\boldsymbol{W}}\right) + c_Y\left(\underset{*}{\boldsymbol{\pi}}^k,\ \tilde{\boldsymbol{W}}\right) + \left\langle \underset{*}{\eta}^k, \mathrm{div}_y \tilde{\boldsymbol{W}}\right\rangle_Y &= -\frac{1}{\lambda}\ \fint_Y \tilde{W}_k \quad \forall \tilde{\boldsymbol{W}} \in \mathbf{H}_\#(\mathrm{div}, Y)\ , \\ \left\langle \tilde{q},\ \mathrm{div}_y \underset{*}{\boldsymbol{\pi}}^k \right\rangle_Y = 0 &\qquad \forall \tilde{q} \in L^2(Y_\#)\ , \end{aligned} \tag{5.11}$$

Assuming more regularity, namely $\mathcal{L}\{\eta\}(\lambda,\cdot) \in H^1_\#(Y)$, we can integrate by parts in $(5.11)_1$ to obtain

$$\left(\lambda \phi_0^{-1}\rho^f \boldsymbol{I} + \boldsymbol{K}^{-1}\right)\underset{*}{\boldsymbol{\pi}}^k - \nabla_y \underset{*}{\eta}^k = -\lambda^{-1}\boldsymbol{1}^k \qquad \text{a.e. in } Y\ , \tag{5.12}$$

hence we can express

$$\underset{*}{\boldsymbol{\pi}}^k = \boldsymbol{F}(\lambda)\nabla_y(\underset{*}{\eta}^k - \frac{1}{\lambda}y_k)\ , \tag{5.13}$$

where $\boldsymbol{F}(\lambda)$ is the ("frequency-dependent") permeability:

$$\boldsymbol{F}(\lambda) = [\lambda \phi_0^{-1}\rho^f \boldsymbol{I} + \boldsymbol{K}^{-1}]^{-1}\ . \tag{5.14}$$

On substituting (5.13) in $(5.11)_2$ we get

$$\mathrm{div}_y\left[\boldsymbol{F}(\lambda)\nabla_y(\underset{*}{\eta}^k - \frac{1}{\lambda}y_k)\right] = 0 \qquad \text{a.e. in } Y\ , \tag{5.15}$$

which yields the following weak formulation of the microscopic Laplace-transformed problem: find $\underset{*}{\eta}^k \in H^1_{\mathrm{av}}(Y_\#)$ such that

$$\int\!\!\!\!\!\!-_Y \left[\boldsymbol{F}(\lambda)\nabla_y(\underset{*}{\eta}^k - \frac{1}{\lambda}y_k)\right] \cdot \nabla_y \tilde{q} = 0 \quad \forall \tilde{q} \in H^1(Y_\#) \;. \tag{5.16}$$

Then we can evaluate the characteristic seepage (5.13); by virtue of the term $\lambda \underset{*}{\boldsymbol{\pi}}^k$ involved in (5.10) we define

$$\boldsymbol{\mathcal{K}}(\lambda) = (\mathcal{K}_{ij}(\lambda)) = -\left(\lambda \int\!\!\!\!\!\!-_Y \underset{*}{\pi}_i^j\right) \tag{5.17}$$

which is symmetric, i.e. $\mathcal{K}_{ij} = \mathcal{K}_{ji}$. This can be proved using symmetry of $\boldsymbol{F}$ and using (5.16) with $\tilde{q} := \underset{*}{\eta}^l$, as follows:

$$\begin{aligned}\mathcal{K}_{lk} &\equiv -\lambda^2 \int_Y \left[\boldsymbol{F}\nabla_y(\underset{*}{\eta}^k - \frac{1}{\lambda}y_k)\right] \cdot \frac{1}{\lambda}\nabla_y y_l = \lambda^2 \int_Y \left[\boldsymbol{F}\nabla_y(\underset{*}{\eta}^k - \frac{1}{\lambda}y_k)\right] \cdot \nabla_y(\underset{*}{\eta}^l - \frac{1}{\lambda}y_l) \\ &= -\lambda^2 \int_Y \left[\boldsymbol{F}\nabla_y(\underset{*}{\eta}^l - \frac{1}{\lambda}y_l)\right] \cdot \frac{1}{\lambda}\nabla_y y_k \equiv \mathcal{K}_{kl} \;.\end{aligned} \tag{5.18}$$

Moreover, $\boldsymbol{\mathcal{K}}$ is positive definite for any real $\lambda > 0$, as follows by the second equality.

5.2 Macroscopic problem and effective parameters

In this section we derive the Laplace-transformed form of the limit "homogenized" problem which is decoupled of the local (microscopic) problems. For this we begin with (4.22) where we substitute $\tilde{\boldsymbol{U}} \equiv 0$, $\tilde{\boldsymbol{W}} \equiv 0$ and apply the Laplace transformation; thus we get:

$$\begin{aligned}&\lambda^2 \bar{\varrho}_{\Omega\times Y}\big(\underset{*}{\boldsymbol{u}}, \boldsymbol{v}\big) + \lambda \varrho^f_{\Omega\times Y}\big(\underset{*}{\boldsymbol{W}}, \boldsymbol{v}\big) \\ &+ a^{\nabla_{xy}}_{\Omega\times Y}\big((\underset{*}{\boldsymbol{u}}, \underset{*}{\boldsymbol{U}}^1), (\boldsymbol{v}, 0)\big) - b^{\nabla_{xy}}_{\Omega\times Y}\Big(\underset{*}{p}, (\boldsymbol{v}, 0)\Big) = \Big\langle \underset{*}{\boldsymbol{f}}, \boldsymbol{v}\Big\rangle_\Omega \;, \\ &\lambda b^{\nabla_{xy}}_{\Omega\times Y}\big(q, (\underset{*}{\boldsymbol{u}}, \underset{*}{\boldsymbol{U}}^1)\big) + \big\langle q, \operatorname{div} \underset{*}{\boldsymbol{W}}\big\rangle_{\Omega\times Y} + \lambda d_{\Omega\times Y}\Big(\underset{*}{p}, q\Big) = 0 \;,\end{aligned} \tag{5.19}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_0(\Omega)$ and $q \in L^2(\Omega)$. On substituting there $\underset{*}{\boldsymbol{U}}^1$ and $\underset{*}{\boldsymbol{W}}$ by the corresponding expressions(5.3) and (5.10),(5.17) involving the corrector basis functions, we can rewrite each individual l.h.s. term in $(5.19)_1$, as follows

$$\begin{aligned}\lambda^2 \bar{\varrho}_{\Omega\times Y}\big(\underset{*}{\boldsymbol{u}}, \boldsymbol{v}\big) &= \lambda^2 \int_\Omega \int\!\!\!\!\!\!-_Y \bar{\rho} \cdot \underset{*}{\boldsymbol{u}}\boldsymbol{v} \\ \lambda \varrho^f_{\Omega\times Y}\big(\underset{*}{\boldsymbol{W}}, \boldsymbol{v}\big) &= \int_\Omega \rho^f \boldsymbol{v} \cdot \int\!\!\!\!\!\!-_Y \underset{*}{\boldsymbol{W}} = \int_\Omega \lambda \rho^f \boldsymbol{v} \cdot \Big(\lambda^2 \rho^f \underset{*}{\boldsymbol{u}} + \nabla \underset{*}{p}\Big) \cdot \boldsymbol{\mathcal{W}}(\lambda) \;, \\ a^{\nabla_{xy}}_{\Omega\times Y}\big((\underset{*}{\boldsymbol{u}}, \underset{*}{\boldsymbol{U}}^1), (\boldsymbol{v}, 0)\big) &= \int_\Omega a_Y\left(\underset{*}{\boldsymbol{U}}^1 + \boldsymbol{\Pi}^{kl} \epsilon^x_{kl}(\underset{*}{\boldsymbol{u}}), \boldsymbol{\Pi}^{ij}\right) \epsilon^x_{ij}(\boldsymbol{v}) \\ &= \int_\Omega \Big(a_Y\left(\boldsymbol{\chi}^{kl} + \boldsymbol{\Pi}^{kl}, \boldsymbol{\Pi}^{ij}\right) \epsilon^x_{kl}(\underset{*}{\boldsymbol{u}}) + a_Y\left(\boldsymbol{\chi}^*, \boldsymbol{\Pi}^{ij}\right) \underset{*}{p}\Big) \epsilon^x_{ij}(\boldsymbol{v}) \;, \\ b^{\nabla_{xy}}_{\Omega\times Y}\Big(\underset{*}{p}, (\boldsymbol{v}, 0)\Big) &= \int_\Omega \underset{*}{p} \epsilon^x_{ij}(\boldsymbol{v}) \int\!\!\!\!\!\!-_Y \alpha_{ij} \;.\end{aligned} \tag{5.20}$$

In analogy, we rewrite all terms involved in $(5.19)_2$

$$\begin{aligned}
\lambda b_{\Omega\times Y}^{\nabla_{xy}}\big(q, (\underset{*}{\boldsymbol{u}}, \underset{*}{\boldsymbol{U}}^1)\big) &= \lambda \int_\Omega q \left(b_Y \left(1, \boldsymbol{\chi}^{ij} + \boldsymbol{\Pi}^{ij}\right) \epsilon_{ij}^x(\underset{*}{\boldsymbol{u}}) + b_Y \left(1, \boldsymbol{\chi}^*\right) \underset{*}{p} \right) \;, \\
\big\langle q, \operatorname{div} \underset{*}{\boldsymbol{W}} \big\rangle_{\Omega\times Y} &= \int_\Omega q \operatorname{div}_x \left(\boldsymbol{\mathcal{W}}(\lambda) \left(\lambda^2 \rho^f \underset{*}{\boldsymbol{u}} + \nabla_x \underset{*}{p} \right) \right) \;, \\
\lambda d_{\Omega\times Y}\left(\underset{*}{p}, q\right) &= \lambda \int_\Omega q \underset{*}{p} \fint_Y \frac{1}{\mu} \;.
\end{aligned} \tag{5.21}$$

Now, collecting the terms associating the same combinations between $(\boldsymbol{\epsilon}(\underset{*}{\boldsymbol{u}}), \underset{*}{p})$ on one hand and $(\boldsymbol{\epsilon}(\boldsymbol{v}), q)$ on the other hand, we may introduce *homogenized coefficients* $\boldsymbol{\mathcal{D}} = (\mathcal{D}_{ijkl})$, $\boldsymbol{\mathcal{A}} = (\mathcal{A}_{ij})$, $\boldsymbol{\mathcal{B}} = (\mathcal{B}_{ij})$ and $\mathcal{Q}$, which are independent of λ:

$$\begin{aligned}
\mathcal{D}_{ijkl} &= a_Y \left(\boldsymbol{\chi}^{kl} + \boldsymbol{\Pi}^{kl}, \boldsymbol{\Pi}^{ij} \right) = a_Y \left(\boldsymbol{\chi}^{kl} + \boldsymbol{\Pi}^{kl}, \boldsymbol{\chi}^{ij} + \boldsymbol{\Pi}^{ij} \right) \;, \\
\mathcal{A}_{ij} &= \fint_Y \alpha_{ij} - a_Y \left(\boldsymbol{\chi}^*, \boldsymbol{\Pi}^{ij} \right) \;, \\
\mathcal{B}_{ij} &= b_Y \left(1, \boldsymbol{\chi}^{ij} + \boldsymbol{\Pi}^{ij} \right) \;, \\
\mathcal{Q} &= \fint_Y \frac{1}{\mu} + b_Y \left(1, \boldsymbol{\chi}^* \right) \;.
\end{aligned} \tag{5.22}$$

In $(5.22)_1$ the symmetric expression follows easily from (5.4), thus we obtain positive definiteness of $\boldsymbol{\mathcal{D}}$. Using (5.5) one verifies $\mathcal{Q} > 0$. Further we define the homogenized mass tensor

$$\boldsymbol{\mathcal{M}}(\lambda) = \fint_Y \bar{\rho} \boldsymbol{I} - \lambda (\rho^f)^2 \boldsymbol{\mathcal{K}}(\lambda) \;. \tag{5.23}$$

By virtue of the symmetry of the original system (3.4) (up to the terms coupling the pressure and the seepage) one expects a similar feature in the homogenized model. Indeed, it results from the following relationship:

$$\mathcal{A}_{ij} = \mathcal{B}_{ij} \;, \qquad \text{whereby } \mathcal{A}_{ij} = \mathcal{A}_{ji} \;. \tag{5.24}$$

This can be obtained using (5.4)-(5.5). We substitute $\boldsymbol{w} := \boldsymbol{\chi}^*$ in (5.4), hence

$$a_Y \left(\boldsymbol{\chi}^{rs}, \boldsymbol{\chi}^* \right) = -a_Y \left(\boldsymbol{\Pi}^{rs}, \boldsymbol{\chi}^* \right) \;.$$

Above the l.h.s. can be rewritten using (5.5) with substituted $\boldsymbol{w} := \boldsymbol{\chi}^{rs}$, which leads to (5.24):

$$\begin{aligned}
\mathcal{A}_{rs} - \fint_Y \alpha_{rs} &= -a_Y \left(\boldsymbol{\Pi}^{rs}, \boldsymbol{\chi}^* \right) = a_Y \left(\boldsymbol{\chi}^{rs}, \boldsymbol{\chi}^* \right) = b_Y \left(1, \boldsymbol{\chi}^{rs} \right) \\
&= b_Y \left(1, \boldsymbol{\chi}^{rs} \right) + b_Y \left(1, \boldsymbol{\Pi}^{rs} \right) - \fint_Y \alpha_{rs} \\
&= \mathcal{B}_{rs} - \fint_Y \alpha_{rs} \;.
\end{aligned} \tag{5.25}$$

We can now state the *macroscopic Laplace-transformed problem for the homogenized medium* which arise from (5.19) where we use expressions (5.20),(5.21) and substitute the coefficients (5.22) and (5.23): Given external loads $\boldsymbol{f} = W^{1,1}(0,T;\mathbf{L}^2(\Omega))$, for any $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$ find pair $(\underset{*}{\boldsymbol{u}}(\lambda,\cdot), \underset{*}{p}(\lambda,\cdot)) \in \mathbf{H}_0^1(\Omega) \times H^1(\Omega)$ such that

$$
\begin{aligned}
\lambda^2 \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\mathcal{M}}(\lambda) \underset{*}{\boldsymbol{u}} + \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{v}) : \boldsymbol{\mathcal{D}} \boldsymbol{\epsilon}(\underset{*}{\boldsymbol{u}}) - \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{v}) : \boldsymbol{\mathcal{A}} \underset{*}{p} - \int_\Omega \lambda \boldsymbol{v} \cdot \left(\rho^f \boldsymbol{\mathcal{K}}(\lambda)\right) \nabla \underset{*}{p} = \int_\Omega \underset{*}{\boldsymbol{f}} \cdot \boldsymbol{v} \, , \\
\lambda \int_\Omega q \boldsymbol{\mathcal{A}} : \boldsymbol{\epsilon}(\underset{*}{\boldsymbol{u}}) - \lambda^2 \int_\Omega q \nabla \cdot \left(\rho^f \boldsymbol{\mathcal{K}}(\lambda) \underset{*}{\boldsymbol{u}}\right) - \int_\Omega q \nabla \cdot \Big(\boldsymbol{\mathcal{K}}(\lambda) \nabla \underset{*}{p}\Big) + \lambda \int_\Omega q \mathcal{Q} \underset{*}{p} = 0 \, ,
\end{aligned}
\tag{5.26}
$$

for all $\boldsymbol{v} \in \mathbf{H}_0^1(\Omega)$ and $q \in L^2(\Omega)$.

5.2.1 Operator forms

System (5.26) can be rewritten in the operator form which emphasizes the block structure of the homogenized equations and underlines their mechanical interpretation. First we present the differential form:

$$
\begin{aligned}
\lambda^2 \boldsymbol{\mathcal{M}}(\lambda) \underset{*}{\boldsymbol{u}} - \nabla \cdot \boldsymbol{\mathcal{D}} \boldsymbol{\epsilon}(\underset{*}{\boldsymbol{u}}) - \nabla \cdot \Big(\boldsymbol{\mathcal{A}} \underset{*}{p}\Big) - \lambda \left(\rho^f \boldsymbol{\mathcal{K}}(\lambda)\right) \nabla \underset{*}{p} = \underset{*}{\boldsymbol{f}} \text{ , a.e. in } \Omega \, , \\
\lambda \boldsymbol{\mathcal{A}} : \boldsymbol{\epsilon}(\underset{*}{\boldsymbol{u}}) - \lambda^2 \nabla \cdot \left(\rho^f \boldsymbol{\mathcal{K}}(\lambda) \underset{*}{\boldsymbol{u}}\right) - \nabla \cdot \Big(\boldsymbol{\mathcal{K}}(\lambda) \nabla \underset{*}{p}\Big) + \lambda \mathcal{Q} \underset{*}{p} = 0 \text{ , a.e. in } \Omega \, .
\end{aligned}
\tag{5.27}
$$

We can present the macroscopic wave equation in an *operator form.* For this, let us define $\boldsymbol{q}_{\mathcal{L}}(\lambda) = (\underset{*}{\boldsymbol{u}}, \underset{*}{p})$ and consider $(5.27)_2$ divided by λ; hence, in the corresponding *inverse Laplace-transformed* problem, the volume conservation is integrated in time, which yields

$$
\left[\lambda^2 \mathbb{M}_{\mathcal{L}}(\lambda) + \lambda \mathbb{D}_{\mathcal{L}}(\lambda) + \mathbb{G} + \mathbb{K}\right] \boldsymbol{q}_{\mathcal{L}} = \boldsymbol{f}_{\mathcal{L}} \, , \tag{5.28}
$$

where operator matrices $\mathbb{M}_{\mathcal{L}}(\lambda)$, $\mathbb{D}_{\mathcal{L}}(\lambda)$, $\mathbb{G}$ and $\mathbb{K}$ are introduced, as follows

$$
\begin{aligned}
\mathbb{M}_{\mathcal{L}}(\lambda) &= \begin{pmatrix} \boldsymbol{\mathcal{M}}(\lambda) & 0 \\ 0 & 0 \end{pmatrix} , &
\mathbb{K} &= \begin{pmatrix} -\nabla^S \cdot \left(\boldsymbol{\mathcal{D}} \nabla^S \circ\right) & 0 \\ 0 & \mathcal{Q} \end{pmatrix} , \\
\mathbb{D}_{\mathcal{L}}(\lambda) &= \begin{pmatrix} 0 & -\rho^f \boldsymbol{\mathcal{K}}(\lambda) \nabla \circ \\ -\nabla \cdot \left(\rho^f \boldsymbol{\mathcal{K}}(\lambda) \circ\right) & -\lambda^{-2} \nabla \cdot (\boldsymbol{\mathcal{K}}(\lambda) \nabla \circ) \end{pmatrix} , &
\mathbb{G} &= \begin{pmatrix} 0 & \nabla \cdot (\mathcal{A} \circ) \\ \mathcal{A} : \nabla^S \circ & 0 \end{pmatrix} , \\
\boldsymbol{q}_{\mathcal{L}}(\lambda) &= \begin{pmatrix} \underset{*}{\boldsymbol{u}} \\ \underset{*}{p} \end{pmatrix} , &
\underset{*}{\boldsymbol{f}} &= \begin{pmatrix} \underset{*}{\boldsymbol{f}} \\ 0 \end{pmatrix} .
\end{aligned}
\tag{5.29}
$$

5.2.2 Existence and uniqueness results

Due to Proposition 6 we know that the limit two-scale problem imposed in $\Omega \times Y \times]0,T]$ possesses a unique solution. Indeed, recalling the arguments of Proposition 3, also for the two-scale limit problem the Galerkin approximations lead to constructing a contraction semigroup. Let us recall that in this section we described the scale-decoupling procedure which using the Laplace transformation and the micro-macro multiplicative split of all two-scale functions introducing the corrector basis functions as solutions to the microscopic problems. This step allows for solving the two-scale problem in part. Therefore, the existence and uniqueness for problem (5.28) follows from Proposition 6 since we can use the general result from the semigroup theory: due to the Hill–Yosida Theorem, cf. [EnN00], we know that the existence of the contraction semigroup guarantees the existence of the resolvent operator of the Laplace-transformed two-scale problem.

6 Propagation of plane harmonic waves

We shall consider harmonic plane incident wave propagating in direction $\boldsymbol{n}$

$$
\begin{aligned}
\boldsymbol{u}(t,x) &= \bar{\boldsymbol{u}} \exp\{\mathrm{i}\omega t + \boldsymbol{\kappa} \cdot x\} \;, \\
p(t,x) &= \bar{p} \exp\{\mathrm{i}\omega t + \boldsymbol{\kappa} \cdot x\} \;,
\end{aligned}
\tag{6.1}
$$

where ω is the imposed (real) frequency, $\boldsymbol{\kappa} = \varkappa \boldsymbol{n}$ and $\varkappa$ is the complex wave number, $\varkappa = \varkappa^R + \mathrm{i}\varkappa^I$. Clearly, $2\pi/\varkappa^I$ gives the length of propagating waves, whereas $\varkappa^R$ gives the damping (attenuation). Note that, according to the ansatz (6.1), the wave with $\varkappa^I > 0$ are running in the opposite direction w.r.t. $\boldsymbol{n}$. Therefore, the wave is attenuated in the direction of its propagation appears, if $\varkappa^R > 0$; in general, $\varkappa^R \varkappa^I > 0$ means the "positive damping".

For the dispersion analysis pursued below we consider model (5.28),(5.29), where all the λ-dependent expressions are interpreted as the Fourier transforms, substituting there $\lambda := \mathrm{i}\omega$. In what follows we denote the Fourier transformation of $v(t)$ by $\widehat{v}(\mathrm{i}\omega)$. For a given harmonic loading $\widehat{\boldsymbol{f}}$ the transformed solution $(\widehat{\boldsymbol{u}}, \widehat{p})$ must satisfy the following equations imposed in Ω:

$$
\begin{aligned}
-\omega^2 \boldsymbol{\mathcal{M}}(\mathrm{i}\omega)\widehat{\boldsymbol{u}} - \nabla \cdot \boldsymbol{I\!D} \nabla^S \widehat{\boldsymbol{u}} + \nabla^S \cdot \boldsymbol{\mathcal{A}} \widehat{p} - \mathrm{i}\omega\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega) \nabla \widehat{p} &= \widehat{\boldsymbol{f}} \\
\boldsymbol{\mathcal{A}} : e(\widehat{\boldsymbol{u}}) - \mathrm{i}\omega\rho^f \nabla \cdot \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\widehat{\boldsymbol{u}} + \mathcal{Q}\widehat{p} + \frac{\mathrm{i}}{\omega} \nabla \cdot \boldsymbol{\mathcal{K}}(\mathrm{i}\omega) \nabla \widehat{p} &= 0 \;,
\end{aligned}
\tag{6.2}
$$

where $\boldsymbol{\mathcal{M}}(\mathrm{i}\omega) = \left(\bar{\rho}_Y I - \mathrm{i}\omega(\rho^f)^2 \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\right)$ is the mass tensor and $\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)$ is called the *dynamic permeability*, since it is associated with the seepage velocity depending also on the inertia term, see $(5.10)_2$. Note that $\mathcal{L}^{-1}\{\mathcal{K}_{kl}\}(t) = -\fint_Y \frac{\mathrm{d}}{\mathrm{d}\,t} \pi_l^k(t)$, thus $\mathcal{K}_{kl} = -\mathrm{i}\omega \fint_Y \widehat{\pi_l^k}$, where $\widehat{\pi_l^k}$ can be computed using (5.13) with $\lambda = \mathrm{i}\omega$. It can be shown easily that for a *homogeneous material* $\underset{*}{\eta}^k \equiv 0$ as the consequence of (5.14) and (5.16), thereby we have

$$
\boldsymbol{\mathcal{K}}(\mathrm{i}\omega) = \boldsymbol{F}(\mathrm{i}\omega) = \left(\boldsymbol{K}^{-1} + \mathrm{i}\omega\rho^f \phi_0^{-1} \boldsymbol{I}\right)^{-1} \;. \tag{6.3}
$$

The amplitudes of the propagating plane wave are obtained from (6.2), where we substitute $(\widehat{\boldsymbol{u}}, \widehat{p}) = (\bar{\boldsymbol{u}}, \bar{p}) \exp\{\varkappa \boldsymbol{n} \cdot x\}$, $\widehat{\boldsymbol{f}} = 0$, assuming that all homogenized coefficients are independent of x. This yields

$$
\begin{pmatrix}
-\omega^2 \left(\bar{\rho}_Y I - \mathrm{i}\omega(\rho^f)^2 \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\right) - \varkappa^2 \boldsymbol{I\!D}(\boldsymbol{n} \otimes \boldsymbol{n}) & (\varkappa \boldsymbol{\mathcal{A}} - \mathrm{i}\omega\varkappa\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega))\boldsymbol{n} \\
(\varkappa \boldsymbol{\mathcal{A}} \cdot \boldsymbol{n} - \mathrm{i}\omega\varkappa\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\boldsymbol{n})^T & \mathcal{Q} + \frac{\mathrm{i}}{\omega}\varkappa^2 \boldsymbol{\mathcal{K}}(\mathrm{i}\omega) : (\boldsymbol{n} \otimes \boldsymbol{n})
\end{pmatrix}
\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{p} \end{pmatrix}
=
\begin{pmatrix} \boldsymbol{0} \\ 0 \end{pmatrix}
\tag{6.4}
$$

In the Appendix we analyze the wave modes associated with (6.4) and verify the well known existence [ABG09] of 2 compressional modes and $n-1$ shear modes, where n is spatial dimension, see Proposition 7. To obtain a general dispersion relationship, from (6.4), assuming the lower diagonal entry does not vanish (in the Appendix, this hypothesis is proved to be sufficient for computing all modes), we can eliminate the pressure

$$
\bar{p} = -\varkappa \frac{\boldsymbol{n} \cdot (\boldsymbol{\mathcal{A}} - \mathrm{i}\omega\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega))\bar{\boldsymbol{u}}}{\mathcal{Q} + \frac{\mathrm{i}}{\omega}\varkappa^2 \boldsymbol{\mathcal{K}}(\mathrm{i}\omega) : (\boldsymbol{n} \otimes \boldsymbol{n})} \;. \tag{6.5}
$$

The reduced system now may be written in the following form (we employ $\boldsymbol{\mathcal{M}}(\mathrm{i}\omega) = \big(\bar{\rho}_Y I - \mathrm{i}\omega(\rho^f)^2 \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\big)\big)$

$$\left[\boldsymbol{\mathcal{M}}(\mathrm{i}\omega) + \gamma^2 \boldsymbol{\mathcal{D}}(\boldsymbol{n}\otimes\boldsymbol{n}) + \gamma^2 \frac{\big(\boldsymbol{\mathcal{A}} - \mathrm{i}\omega\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\big)\boldsymbol{n}\big) \otimes \big(\boldsymbol{\mathcal{A}} - \mathrm{i}\omega\rho^f \boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\big)\boldsymbol{n}\big)}{\mathcal{Q} + \gamma^2 \mathrm{i}\omega \boldsymbol{\mathcal{K}}(\mathrm{i}\omega) : (\boldsymbol{n}\otimes\boldsymbol{n})}\right] \bar{\boldsymbol{u}} = \boldsymbol{0} \;, \tag{6.6}$$

where $\gamma = \varkappa/\omega$ was introduced (note $1/\Im(\gamma) = \omega/\varkappa^I$ is the phase velocity). Below we shall use identity (6.6) to analyze the wave dispersion.

Remark 2. For *homogeneous medium* (i.e. there is no heterogeneous microstructure) one obtains (6.6) where the homogenized coefficients $\boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{A}}, \mathcal{Q}$ are replaced directly by $\boldsymbol{I\!D}, \boldsymbol{\alpha}, 1/\mu$, respectively, and $\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)$ by $\boldsymbol{F}(\mathrm{i}\omega)$, as explained in (6.3).

$\triangle$

6.1 Dispersion analysis

In order to analyze dispersion in the homogenized medium, i.e. the relationship between ω and $\varkappa$, we rewrite (6.6) in the matrix form, on introducing the following 2nd order tensors:

$$\begin{aligned}
T(\mathrm{i}\omega) &= \mathcal{Q}\left(\bar{\rho}_Y I - \mathrm{i}\omega\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)(\rho^f)^2\right) \;,\\
S(\mathrm{i}\omega) &= \mathcal{Q}\boldsymbol{\mathcal{D}}(\boldsymbol{n}\otimes\boldsymbol{n}) + \mathrm{i}\omega\boldsymbol{\mathcal{K}}(\mathrm{i}\omega) : (\boldsymbol{n}\otimes\boldsymbol{n})\,\left[\bar{\rho}_Y I - \mathrm{i}\omega\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)(\rho^f)^2\right]\\
&\quad + \big(\boldsymbol{\mathcal{A}} - \mathrm{i}\omega\rho^f\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\big)\,\boldsymbol{n} \otimes \big(\boldsymbol{\mathcal{A}} - \mathrm{i}\omega\rho^f\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\big)\,\boldsymbol{n} \;,\\
R(\mathrm{i}\omega) &= \boldsymbol{\mathcal{D}}(\boldsymbol{n}\otimes\boldsymbol{n})\,\mathrm{i}\omega\boldsymbol{\mathcal{K}}(\mathrm{i}\omega) : (\boldsymbol{n}\otimes\boldsymbol{n}) \;.
\end{aligned} \tag{6.7}$$

Using the notation just introduced, (6.6) attains the form

$$\left[R(\mathrm{i}\omega)\gamma^4 + S(\mathrm{i}\omega)\gamma^2 + T(\mathrm{i}\omega)\right] \cdot \bar{\boldsymbol{u}} = \boldsymbol{0} \;, \tag{6.8}$$

from where we can obtain the characteristic equation for γ^2.

A possible way of computing γ for a given ω is to introduce the substitution $\bar{\boldsymbol{v}} := -\gamma^2 R\bar{\boldsymbol{u}}$ and to solve the following eigenvalue problem, where $\Lambda = 1/\gamma^2$:

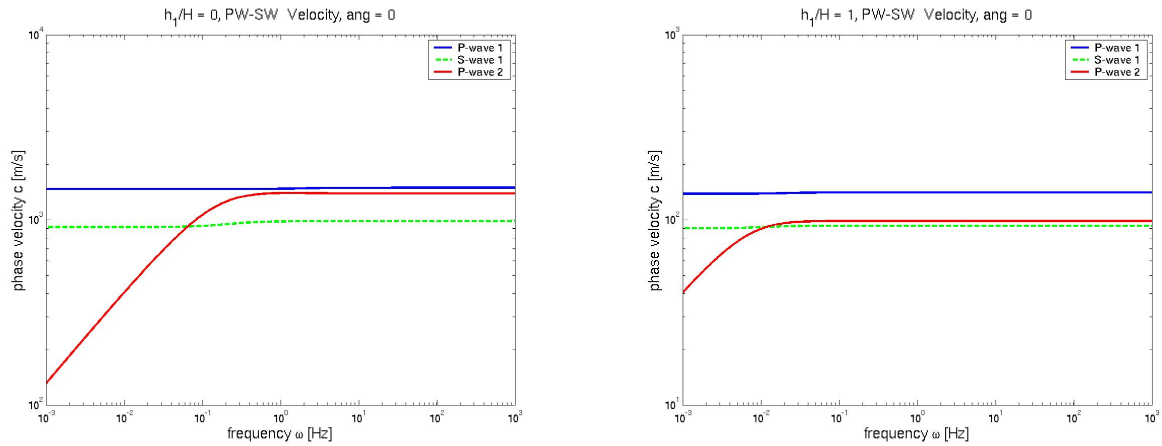
$$-\begin{pmatrix} S & -I \\ R & 0 \end{pmatrix}\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \end{pmatrix} = \Lambda \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{\boldsymbol{v}} \end{pmatrix} \;, \tag{6.9}$$

which yields $\varkappa = \pm\omega(\sqrt{\Lambda})^{-1}$, where the sign determines the wave orientation. Thus, we obtain four complex eigenvalues $\varkappa$ and the associated eigenvectors, namely the $\bar{\boldsymbol{u}}$, which determine the wave polarization. However, by virtue of the mode analysis, as discussed in the Appendix, one of the eigenvalues corresponds to vanishing lower diagonal entry in (6.4) – in this case (6.5) is not defined, thus, the eigenvalue is false.

For better understanding the asymptotics of the dispersion $\gamma = \gamma(\omega)$ for $\omega \to 0$ and $\omega \to \infty$, it is important to analyze the terms $\mathcal{M}_{ij} = \big(\bar{\rho}_Y - (\rho^f)^2 g_{ij}\big)$ which is the mass, see (5.29); for homogeneous isotropic material one may analyze the equivalent mass term $M_{11} = \big(\bar{\rho} - \mathrm{i}\omega(\rho^f)^2 F_{11}(\mathrm{i}\omega)\big)$. Easy calculations reveal the following:

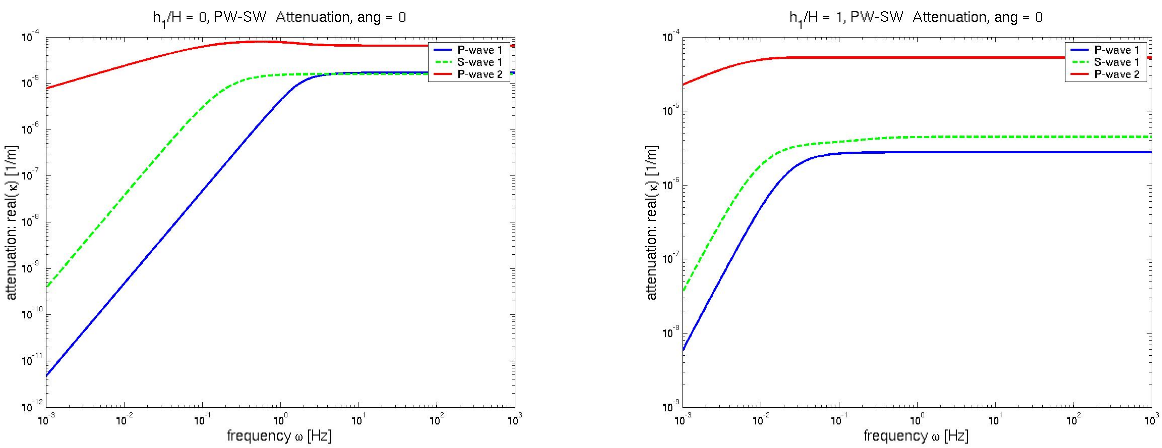
	mass $\Re(M_{11}) \to$	damping contribution
$\omega \to 0$	$\bar{\rho}$	$-\mathrm{i}\omega(\rho^f)^2 K_{11}$
$\omega \to \infty$	$(1-\phi_0)\rho^s$	0

Thus, for the high frequency regime only the solid phase mass induces the inertia effects, while in the low frequency regime both the fluid and the solid participate.



Pure material 1 Pure material 2

Figure 1: Phase velocities $\omega/\varkappa^I$ for the pure materials. Computed using (6.21) and (6.22).



Pure material 1 Pure material 2

Figure 2: Attenuation $\varkappa^R$ for the pure materials. Computed using (6.21) and (6.22).

6.2 Homogenized laminated structure

To describe how the specific type of heterogeneities contributes to the model anisotropy, we shall consider the plane stress problem in infinite medium where plane wave of a given frequency ω propagate in direction $\boldsymbol{n}$. We shall derive the particular effective parameters and illustrate numerically the phenomena of the wave *dispersion and attenuation.*

Let us consider material heterogeneities as generated by parameter $a(y)$, $y \in Y =]0,1[^2$,

$$a(y) = \bar{a}(1 + \psi_a(y_1)) \quad y_1 \in [0,1] \; , \quad -1 < \psi_a \text{ a.e. in } [0,1] \; , \tag{6.10}$$

where $\bar{a}$ is a given real constant and $\psi_a \in L^\infty(0,1)$.

In order to compute components $\mathcal{D}_{ijkl}$, $\mathcal{A}_{ij}$ $(i,j,k,l = 1,2)$ and $\mathcal{Q}$, one needs to solve (5.4) and (5.5) for correctors $\chi_i^{kl}(y_1)$ and $\chi_i^*(y_1)$. These equations can be reduced a priori, since the corrector functions cannot depend on y_2. Thus, due to the material coefficients being independent of y_2 and due to the ellipticity of the corrector problems, one has to solve

$$\begin{aligned} -\frac{\partial}{\partial y_1}\left(D_{i111}(y_1)\frac{\partial \chi_1^{rs}}{\partial y_1} + D_{i121}(y_1)\frac{\partial \chi_2^{rs}}{\partial y_1} + D_{i1rs}(y_1)\right) &= 0 \; , \quad i,r,s = 1,2 \; , \\ -\frac{\partial}{\partial y_1}\left(D_{i111}(y_1)\frac{\partial \chi_1^{*}}{\partial y_1} + D_{i121}(y_1)\frac{\partial \chi_2^{*}}{\partial y_1} - \alpha_{i1}\right) &= 0 \; , \quad i = 1,2 \; , \end{aligned} \tag{6.11}$$

where D_{ijkl} and α_{ij} are given in the form of (6.10). Straightforward calculations lead to the desired gradients of the corrector functions (the Einstein summation convention for repeated indices is applied)

$$\frac{\partial \chi_k^{rs}}{\partial y_1} = -(D_{i1k1})^{-1}\left(D_{i1rs} - c_i^{rs}\right) \; , \qquad \frac{\partial \chi_k^{*}}{\partial y_1} = (D_{i1k1})^{-1}\left(\alpha_{i1} + c_i^{*}\right) \; , \tag{6.12}$$

for $i,k = 1,2$, where, due to the periodic boundary conditions,

$$\begin{aligned} c_i^{rs} &= \left(\int_0^1 (D_{i1j1})^{-1} dy_1\right)^{-1} \int_0^1 \left((D_{j1k1})^{-1} D_{k1rs}\right) dy_1 \; , \\ c_i^{*} &= -\left(\int_0^1 (D_{i1j1})^{-1} dy_1\right)^{-1} \int_0^1 \left((D_{j1k1})^{-1} \alpha_{k1}\right) dy_1 \; . \end{aligned} \tag{6.13}$$

Now the effective parameters can be evaluated:

$$\begin{aligned} \mathcal{D}_{ijkl} &= \int_0^1 D_{ijrl}(y_1)\left(\delta_{rk} + \delta_{l1}\frac{\partial \chi_r^{kl}}{\partial y_1}\right) dy_1 \\ \mathcal{A}_{ij} &= \int_0^1 \left(\alpha_{ij} - D_{ijr1}(y_1)\frac{\partial \chi_r^{*}}{\partial y_1}\right) dy_1 \; , \\ \mathcal{Q} &= \int_0^1 \left(\alpha_{k1}\frac{\partial \chi_k^{*}}{\partial y_1} + \frac{1}{\mu(y_1)}\right) dy_1 \; . \end{aligned} \tag{6.14}$$

Further we compute $\mathcal{K}(\omega)$, as introduced in (5.17) through (5.16) and (5.13). We consider a diagonal permeability tensor $K_{ij}(y_1)$, i.e. $K_{ij} = 0$ for $i \neq j$, so that (5.15) reduces to

$$-\frac{\partial}{\partial y_1}\left(F_{11}(y_1,\lambda)\left(\frac{\partial \underset{*}{\eta}^1}{\partial y_1} - \frac{1}{\lambda}\right)\right) = 0 \; , \qquad \underset{*}{\eta}^2 \equiv 0 \; , \tag{6.15}$$

where the second identity results from the independence of F_{22} on y_2; tensor F_{ij} has only the diagonal entries

$$F_{ii}(y_1,\lambda) = ((K_{ii}(y_1)^{-1} + \lambda\rho^f\phi_0^{-1})^{-1}\;, \quad i = 1,2 \quad \text{no summation over } i\;. \tag{6.16}$$

Due to the periodic boundary conditions, one obtains

$$\begin{aligned} \frac{\partial \overset{*}{\eta}{}^1}{\partial y_1} &= \frac{1}{\lambda} + \frac{c_K(\lambda)}{F_{11}(y_1,\lambda)}\;, \\ \text{where}\quad c_K(\lambda) &= -\left(\lambda\int_0^1 (F_{11}(y_1,\lambda))^{-1}\,dy_1\right)^{-1}\;. \end{aligned} \tag{6.17}$$

Hence, using (5.13), the non-vanishing seepage corrector gradients are

$$\overset{*}{\pi}{}_1^1 = F_{11}(y_1,\lambda)\left(\frac{\partial \overset{*}{\eta}{}^1}{\partial y_1} - \frac{1}{\lambda}\right) = c_K(\lambda)\;, \qquad \overset{*}{\pi}{}_2^2 = -\frac{F_{22}(y_1,\lambda)}{\lambda}\;, \tag{6.18}$$

while $\overset{*}{\pi}{}_1^2 = \overset{*}{\pi}{}_2^1 \equiv 0$. It is worth noting that $\overset{*}{\pi}{}_2^2$ depends on y_1, whereas $\overset{*}{\pi}{}_1^1$ is a constant, i.e. independent of y_1 (in entire cell Y). This reveals existence of *nonuniform* seepage flows in the y_2 direction.

Now using (5.17) we can express the homogenized permeability for $\lambda := \mathrm{i}\omega$

$$\begin{aligned} \mathcal{K}_{11}(\mathrm{i}\omega) &= -\mathrm{i}\omega\, c_K(\mathrm{i}\omega) = \left(\int_0^1 \frac{1}{F_{11}(y_1,\mathrm{i}\omega)}\,dy_1\right)^{-1}\;, \\ \mathcal{K}_{22}(\mathrm{i}\omega) &= \int_0^1 F_{22}(y_1,\mathrm{i}\omega)\,dy_1\;, \end{aligned} \tag{6.19}$$

obviously, $\mathcal{K}_{ij} = 0$ for $i \neq j$.

Having computed the effective parameters $\mathcal{K}_{ij}(\mathrm{i}\omega)$, $\mathcal{D}_{ijkl}$, $\mathcal{A}_{ij}$ and $\mathcal{Q}$, we can establish the dispersion equation (6.8) and solve the associated eigenvalue problem (6.9). Then, for *complex* eigenpairs $(\varkappa, \bar{\boldsymbol{u}})$ we can compute the eigenpressures and eigenseepage. First, using (6.5) with (5.17), we compute $\bar{p}$, then we employ $(5.10)_1$, replacing there λ by $\mathrm{i}\omega$, and (5.17), to obtain the averaged seepage velocity

$$\bar{\boldsymbol{w}} := \fint_Y \overline{\boldsymbol{W}} = -\boldsymbol{\mathcal{K}}(\mathrm{i}\omega)\left(\varkappa\boldsymbol{n}\bar{p} - \omega^2\rho^f\bar{\boldsymbol{u}}\right)\;. \tag{6.20}$$

6.3 Pressure and shear waves for transversal plane waves

For the laminated structure we can simplify the general dispersion relationship (6.8), assuming transversal wave propagation, i.e. $\boldsymbol{n} = (1,0)$, see (6.10). Dispersion curves $\omega \mapsto \varkappa$ can then be computed by solving scalar quadratic equations in complex variables, as will now be explained. Due to diagonality of $\mathcal{A}$ and $\mathcal{K}$, which is obvious from (6.30) and (6.19) because of the isotropy of both the materials, (6.8) becomes an equation with the diagonal matrix. Therefore, $\bar{\boldsymbol{u}}^{\parallel} = (1,0)$ and $\bar{\boldsymbol{u}}^{\perp} = (0,1)$ are two eigenvectors, whereby each of them yields two eigenvalues.

Let us first consider polarization $\bar{\boldsymbol{u}}^{\perp}$ (we recall $\bar{\boldsymbol{u}}^{\perp}\cdot\boldsymbol{n} = 0$), so that we analyze the *shear waves* (S-waves). From (6.5) we get $\bar{p} = 0$. i.e. the S-waves do not induce any

	K_B [GPa]	G [GPa]	α [1]	ϕ_0 [1]	μ [GPa]	K [m^2/(Pa.s)]
mat. 1	0.6	0.8	0.4231	0.1	0.9660	0.01
mat. 2	6	8	0.8462	0.2	9.6595	0.001

Table 1: Material parameters, K_B is the bulk modulus of the drained skeleton, G is the solid shear modulus, K and α are the diagonal entries of $\boldsymbol{K}$ and $\boldsymbol{\alpha}$, respectively.

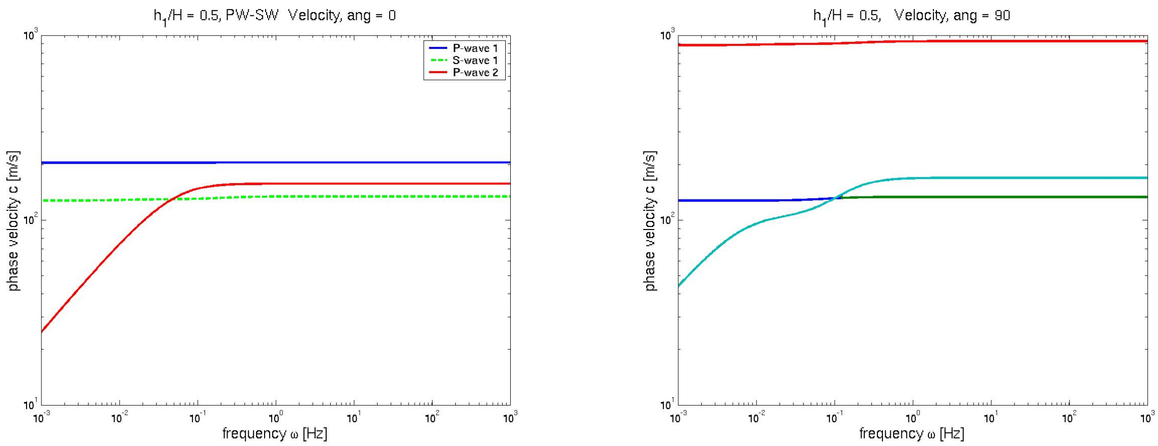


Figure 3: Composite $h_1/h_2 = 1$, phase velocity $c = \omega/\varkappa^I$. Left: $\beta = 0°$ (computed using (6.21) and (6.22)), right: $\beta = 90°$ (computed by solving (6.8)).

pressure oscillations. De to the above mentioned diagonality, from (6.8) we obtain to the following single equation which yields two γ_S^2 for any $\omega > 0$ (we denote $g_{ij} = \mathrm{i}\omega\mathcal{K}_{ij}(\mathrm{i}\omega)$, thus $\mathcal{M}_{22} = \bar{\rho}_Y - (\rho^f)^2 g_{22}$),

$$\left(\gamma_{S_1}^2 \mathcal{D}_{2211} + \mathcal{M}_{22}\right)\left(\gamma_{S_2}^2 g_{11} + \mathcal{Q}\right) = 0\ . \tag{6.21}$$

Note that only $\gamma_{S_1}^2$ is associated with a shear wave, as for $\gamma_{S_2}^2$ the assumption used to eliminate $\bar{p}$ in (6.5) is not satisfied. Therefore, there is only one S-wave.

In analogy, we can compute dispersion curves for the *pressure waves* (P-waves) by solving the equation obtained from (6.8) for polarization $\bar{\boldsymbol{u}}^{\parallel} \parallel \boldsymbol{n}$, which now becomes

$$\gamma_P^4 g_{11} D_{1111} + \gamma_P^2 \left(\mathcal{Q} D_{1111} + g_{11}\left(\bar{\rho}_Y - (\rho^f)^2 g_{11}\right) + (\alpha_{11} - \rho^f g_{11})^2\right) + \mathcal{Q}(\bar{\rho}_Y - (\rho^f)^2 g_{11}) = 0\ . \tag{6.22}$$

Thus we obtain two waves $\gamma_{P_{1,2}}^2$, the "slow" one the "fast" one, which induce pressure oscillations.

The dispersion properties of the homogenized medium will be illustrated on a laminated structure consisting of two inter-commuting layers h_1, h_2, $h_1 + h_2 = H$, filled by two different isotropic materials. A harmonic plane wave is given by frequency ω and by the angle of incidence $\beta \in [0, \pi/2]$ which determines the propagation direction $\boldsymbol{n} = (\cos\beta, \sin\beta)$; let us note $\beta = 0$ is the orientation of the lamination.

The two isotropic materials are specified in Table 1. The two-dimensional problem was considered for the plane stress conditions. In Figs. 1 and 2, using the "log-log" axes labeling, we display the phase velocities and attenuation for both the pure materials. Apparently, for frequencies higher than 10Hz there is no dispersion. Solving (6.9) for $\omega \in [10^{-3}, 10^3]$Hz, 4 curves $\omega \mapsto \varkappa$ are obtained – two P-waves, one S-wave and one

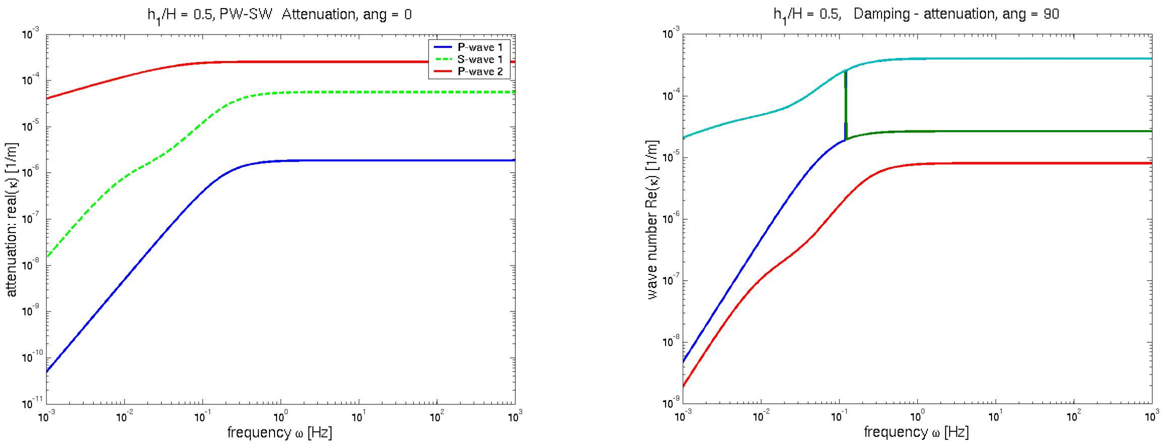


Figure 4: Composite $h_1/h_2 = 1$, attenuation $\varkappa^R$. Left: $\beta = 0°$ (computed using (6.21) and (6.22)), right: $\beta = 90°$ (computed by solving (6.8)).

"false" curve which, however, merge with one of the remaining three. It can be seen that for very small ω the dispersion for one P-wave attains the form $v \approx \sqrt{\omega}$.

In Figs. 3 and 4 we depict the dispersion behavior for the layered composite material with thicknesses $h_1 = h_2 = 0.5H$. To see the homogenized material orthotropy, we considered the directions $\beta = 0°$ and $\beta = 90°$. (Obviously, for $\beta = 0°$ calculations by either solving (6.8), or obtained from (6.21) and (6.22) give the same result.)

Appendix

Wave dispersion analysis in the upscaled Biot medium

We analyze existence of plane waves with polarizations given as non-trivial solutions to (6.4), where $\boldsymbol{n}$ is the direction of propagation. For the sake of brevity, we introduce the following notation:

$$
\begin{aligned}
\mathbf{T}(\gamma^2) &:= \boldsymbol{\mathcal{M}}(\mathrm{i}\omega) + \gamma^2 \boldsymbol{\mathcal{D}} : \boldsymbol{n} \otimes \boldsymbol{n} \;, \\
\mathbf{b} &:= (\boldsymbol{\mathcal{A}} - \rho^f \mathrm{i}\omega \boldsymbol{\mathcal{K}}) \cdot \boldsymbol{n} \;, \\
a(\gamma^2) &:= \mathcal{Q} + \gamma^2 \mathrm{i}\omega \boldsymbol{\mathcal{K}} : \boldsymbol{n} \otimes \boldsymbol{n} \;.
\end{aligned}
\tag{6.23}
$$

Now (6.4) reads, as follows

$$
\begin{pmatrix} \mathbf{T}(\gamma^2) & \frac{\gamma}{\omega}\mathbf{b} \\ \gamma\omega\mathbf{b}^T & a(\gamma^2) \end{pmatrix} \cdot \begin{pmatrix} \bar{\boldsymbol{u}} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ 0 \end{pmatrix} . \tag{6.24}
$$

Proposition 7 *For any $\omega > 0$ there are just two pressure waves (associated with model (6.4)) and $n-1$ shear waves with "$\mathbf{b}$-orthogonal" (complex) polarizations in $\mathbb{C}^n$, $n = 2, 3$.*

Proof: (1) Let us assume $a(\gamma^2) \neq 0$, then the pressure amplitude can be expressed $\bar{p} = \gamma\omega\mathbf{b}^T\bar{\boldsymbol{u}}/a(\gamma^2)$ and substituted into the first equation in (6.24) which, thus, yields

$$
\left(a(\gamma^2)\mathbf{T}(\gamma^2) + \gamma^2\mathbf{b}\mathbf{b}^T\right)\bar{\boldsymbol{u}} = 0 \,. \tag{6.25}
$$

Nontrivial solutions $\bar{\boldsymbol{u}}$ exist in the following cases.
(1.i) If $\bar{\boldsymbol{u}} \cdot \mathbf{b} = 0$, then condition $\det(\mathbf{T}(\gamma^2)) = 0$ holds for a complex γ^2, so that there are 2 waves (in 3D, but only one wave in 2D) with polarization $\bar{\boldsymbol{u}} \in \mathrm{Ker}(\mathbf{T}(\gamma^2))$ constraint by $\bar{\boldsymbol{u}} \perp \mathbf{b}$. They are called *shear* waves, being characterized by $\bar{p} = 0$, which means that no volume changes are induced.
(1.ii) If $\bar{\boldsymbol{u}} \cdot \mathbf{b} \neq 0$, then (6.25) presents a biquadratic equation for γ^2. Two waves with $\bar{p} \neq 0$ are obtained, therefore, being called the *pressure* waves.

(2) Now we consider $a(\gamma^2) = 0$ and explore existence of nontrivial solutions to (6.24).
(2.i) Let $\det(\mathbf{T}(\gamma^2)) \neq 0$, then using the Schur complement, (6.24) reduces to $\gamma^2 \mathbf{b}^T [\mathbf{T}(\gamma^2)]^{-1} \mathbf{b} \bar{p} = 0$. Since $\mathbf{T}(\gamma^2)$ is regular, $\bar{p} = 0$ (for $|\gamma| > 0$), hence also $\bar{\boldsymbol{u}} = 0$, so that there is no propagating wave.
(2.ii) Let $\det(\mathbf{T}(\gamma^2)) = 0$. First, assume $\mathbf{b} \perp \mathrm{Ker}\mathbf{T}(\gamma^2)$, then there exists a non-vanishing $\bar{\boldsymbol{u}} \in \mathrm{Ker}\mathbf{T}(\gamma^2)$, such that $\bar{\boldsymbol{u}} \cdot \mathbf{b} = 0$ and (6.24) holds with $\bar{p} = 0$. In this case we recover the case of shear waves, as discussed above. Second, assume $\mathbf{b} \not\perp \mathrm{Ker}\mathbf{T}(\gamma^2)$. To satisfy (6.24), we need $\bar{\boldsymbol{u}} \notin \mathrm{Ker}\mathbf{T}(\gamma^2)$ such that $\mathbf{b}$ is in the range of $\mathbf{T}(\gamma^2)$. Supposing existence of $p \in \mathbb{C}$ such that we have $\bar{p}\mathbf{b} = \mathbf{T}(\gamma^2)\bar{\boldsymbol{u}}$, simultaneously we would require $0 = \bar{\boldsymbol{u}}^T \mathbf{b} = (1/\bar{p}) \bar{\boldsymbol{u}}^T \mathbf{T}(\gamma^2) \bar{\boldsymbol{u}}$, which holds only provided $\bar{\boldsymbol{u}} \in \mathrm{Ker}\mathbf{T}(\gamma^2)$. Thus, we reached a contradiction and conclude that all solutions are trivial. $\square$

Homogenized coefficients for laminated structure

Here we offer the homogenization formulas for computing $\mathcal{D}_{ijkl}, \mathcal{A}_{ij}$ and $\mathcal{Q}$ written a matrix notation which can be easily interpreted for the Matlab implementation.

In our example $Y \equiv]0, 1[$ is just the interval. Then the following matrix representation for $\boldsymbol{I\!D} = (D_{ijkl})$, $\boldsymbol{A} = (\alpha_{ij})$ and $\boldsymbol{K} = (K_{ij})$ is employed:

$$
\begin{aligned}
\mathbf{D}(y) = E(y)\bar{\mathbf{D}} \;, \quad \mathbf{a}(y) &= \alpha(y)\bar{\mathbf{a}} \;, \quad \bar{\mathbf{a}} = [\bar{\alpha}_{11}, \bar{\alpha}_{22}, \bar{\alpha}_{12}] \;, \\
\mathbf{k}(y) &= K(y)\bar{\mathbf{k}} \;, \quad \bar{\mathbf{k}} = [\bar{K}_{11}, \bar{K}_{22}, \bar{K}_{12}] \;,
\end{aligned}
\tag{6.26}
$$

where $\bar{\mathbf{D}} \in \mathbb{R}^{3\times 3}$, $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{k} \in \mathbb{R}^3$ and $E(y), \alpha(y), K(y)$ are defined according to (6.10). We assume isotropic elasticity with a constant Poisson ratio, i.e. independent of $y \in Y$, so that matrix $\bar{\mathbf{D}}$ is constituted by components D_{ijkl}/E, E being the Young modulus, so that on introducing the strain vector $\mathbf{e} : \boldsymbol{v} \to \mathbb{R}^3$, the stress is $\mathbf{s} = \mathbf{D}\mathbf{e}$, where

$$
\mathbf{e}(\boldsymbol{v}) = [\epsilon_{11}(\boldsymbol{v}), \epsilon_{22}(\boldsymbol{v}), 2\epsilon_{12}(\boldsymbol{v})]^T \;, \quad \mathbf{s} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^T \;. \tag{6.27}
$$

To shorten formulae we use the standard abbreviation of double indices ij (the Voigt notation) denoted by $\hat{k}$, $k = 1, 2, 3$, where the relationship is: $\hat{1} = 11$, $\hat{2} = 22$ and $\hat{3} = 12$. Further we employ the colon ":" in a Matlab sense to indicate a (sorted) sublist of $\{1, 2, 3\}$, e.g. $(:) = (1:3) = \{1, 2, 3\}$. Therefore, $\mathbf{D}_{(3,1:3)} = [D_{\hat{3}\hat{1}}, D_{\hat{3}\hat{2}}, D_{\hat{3}\hat{3}}] = [D_{1211}, D_{1222}, D_{1212}]$. Further we introduce $\mathbf{i} = \{1, 3\}$, so that $\mathbf{D}_{(\mathbf{i},:)} = (D_{\hat{i}\hat{k}})$, $i = 1, 2$, $k = 1, 2, 3$ is 2-by-3 matrix.

Using this matrix notation we may express the constants defined in (6.13)

$$
\begin{aligned}
\boldsymbol{c}_{(\mathbf{i},:)} &= (\fint_Y E^{-1}(y))^{-1} \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})} \bar{\mathbf{D}}^{-1}_{(\mathbf{i},\mathbf{i})} \bar{\mathbf{D}}_{(\mathbf{i},:)} = (\fint_Y E^{-1}(y))^{-1} \bar{\mathbf{D}}_{(\mathbf{i},:)} \;, \\
\boldsymbol{c}^*_{(\mathbf{i})} &= -(\fint_Y E^{-1}(y))^{-1} \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})} \bar{\mathbf{D}}^{-1}_{(\mathbf{i},\mathbf{i})} \bar{\mathbf{a}}_{(\mathbf{i})} \fint_Y \alpha(y) E^{-1}(y) = -\frac{\fint_Y \alpha(y) E^{-1}(y)}{\fint_Y E^{-1}(y)} \bar{\mathbf{a}}_{(\mathbf{i})} \;,
\end{aligned}
\tag{6.28}
$$

consequently (6.12) can be written as

$$\begin{aligned} \partial_1^y \boldsymbol{\chi}_{(\mathbf{i},:)} &= -\bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})}^{-1} \left(\bar{\mathbf{D}}_{(\mathbf{i},:)} - E^{-1}(y) \boldsymbol{c}_{(\mathbf{i},:)} \right) \ , \\ \partial_1^y \boldsymbol{\chi}_{(\mathbf{i})}^* &= \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})}^{-1} \left(\bar{\mathbf{a}}_{(\mathbf{i})} \alpha(y) + \boldsymbol{c}_{(\mathbf{i})}^* \right) E^{-1}(y) \ . \end{aligned} \tag{6.29}$$

Now the matrix forms of the homogenized tensors can be computed; let us denote $\mathbf{D}_{(k,l)}^{\text{eff}} = \mathcal{D}_{\hat{k}\hat{l}}$ and $\mathbf{A}_{(k)}^{\text{eff}} = \mathcal{A}_{\hat{k}}$, then (6.14) reads as

$$\begin{aligned} \mathbf{D}_{(:,:)}^{\text{eff}} &= \bar{\mathbf{D}}_{(:,:)} \fint_Y E(y) - \bar{\mathbf{D}}_{(:,\mathbf{i})} \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})}^{-1} \bar{\mathbf{D}}_{(\mathbf{i},:)} \left(\fint_Y E(y) - (\fint_Y E^{-1}(y))^{-1} \right) \ , \\ \mathbf{A}_{(:)}^{\text{eff}} &= \bar{\mathbf{a}}_{(:)} \fint_Y \alpha(y) + \bar{\mathbf{D}}_{(:,\mathbf{i})} \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})}^{-1} \bar{\mathbf{a}}_{(\mathbf{i})} \left(\fint_Y \alpha(y) - \frac{\fint_Y \alpha(y) E^{-1}(y)}{\fint_Y E^{-1}(y)} \right) \ , \\ \mathcal{Q} &= \fint_Y \frac{1}{\mu(y)} + \bar{\mathbf{a}}_{(\mathbf{i})}^T \bar{\mathbf{D}}_{(\mathbf{i},\mathbf{i})}^{-1} \bar{\mathbf{a}}_{(\mathbf{i})} \left(\fint_Y \frac{\alpha(y)^2}{E(y)} - \frac{\left(\fint_Y \alpha(y) E^{-1}(y) \right)^2}{\fint_Y E^{-1}(y)} \right) \ . \end{aligned} \tag{6.30}$$

References

[Abd06] M. Abellan and R. de Borst. Wave propagation and localisation in softening two-phase medium. *Comp. Meth. Appl. Mech. Engrg.*, 195, 5011–5019, 2006.

[ABG09] J. L. Auriault, C. Boutin, and C. Geindreau. *Homogenization of Coupled Phenomena in Heterogenous Media.* John Wiley & Sons, 2009.

[All89] G. Allaire. Homogenization of the stokes flow in a connected porous medium. *Asymptotic Analysis*, 2, 203–222, 1989.

[All92a] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23, 1482–1518, 1992.

[All92b] G. Allaire. Homogenization of the unsteady stokes equation in porous media. *Progress in PDE: Calculus of variation, applications, Pitman Research notes in mathematics Series*, 267, 109–123, 1992.

[All92c] G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23, 1482–1518, 1992.

[AlW05] B. Albers and K. Wilmanski. On modeling acoustic waves in saturated poroelastic media. *J. Statist. Phys.*, 131, 873–996, 2005.

[AuB94] J. L. Auriault and C. Boutin. Deformable porous media with double porosity. *Transport in Porous Media*, 14, 143–162, 1994.

[Aur91] J. L. Auriault. Dynamic behaviour of porous media. In J. B. a. M. Y. C. (eds.), editor, *Transport Processes in Porous Media*, pages 471–519. Kluwer Academic Publishers, 1991.

[Bar95] J. P. Bardet. The damping of saturated poroelastic soils during steady-state vibrations. *Appl. Math. Comput.*, 67, 3–31, 1995.

[BeW00] J. Berryman and H. Wang. Elastic wave propagation and attenuation in a double-porosity dual-permeability medium. *Rock Mechanics and Mining Sciences*, 37, 63–78, 2000.

[BGS05] M. Brajanovski, B. Gurevich, and M. Schoenberg. A model of p-wave attenuation and dispersion in a porous medium permeated by aligned fractures. *Geophysics J. Int.*, 163, 372–384, 2005.

[BiW57] M. A. Biot and D. G. Willis. The elastic coefficients of the theory of consolidation. *J. Appl. Mech.*, 79, 594–601, 1957.

[BrF91] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer, 1991.

[CDG08] D. Cioranescu, A. Damlamian, and G. Griso. The periodic unfolding method in homogenization. *SIAM Journal on Mathematical Analysis*, 40(4), 1585–1620, 2008.

[Cha11] A. Chakraborty. An analytical homogenization method for heterogeneous porous materials. *International Journal of Solids and Structures*, 2011. In Press.

[Cou04] O. Coussy. *Poromechanics*. John Wiley & Sons, 2004.

[EnN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Springer, 2000.

[GiO03] R. Gilbert and M. Ou. Acoustic wave propagation in a composite of two different poroelastic materials with a very rough periodic interface: a homogenization approach. *Int. Jour. Multiscale Comput. Engrg.*, 1, 2003.

[GrR07] G. Griso and E. Rohan. On the homogenization of a diffusion-deformation problem in strongly heterogeneous media. *Ricerche mat.*, 56, 161–188, 2007.

[HeU05] C. Helmich and J. Ulm. Microporodynamics of bones: prediction of the "frenkel-biot" slow compressional wave. *Jour. Eng. Mech.*, 131, 918–927, 2005.

[Hor97] U. Hornung. *Homogenization and Porous Media*. Springer-Verlag, Berlin, 1997.

[Mie08] A. Mielke. Weak-convergence methods for Hamiltonian multiscale problems. *Discr. Cont. Dynam. Systems Ser. A*, 20(1), 53–79, 2008.

[MiT07] A. Mielke and A. M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM J. Math. Analysis*, 39(2), 642–668, 2007.

[Ngu89] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3), 608–623, 1989.

[NNS10] V.-H. Nguyen, S. Naili, and V. Sansalone. Simulation of ultrasonic wave propagation in anisotropic cancellous bone immersed in fluid. *Wave Motion*, 47, 117–129, 2010.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[QMD02] S. Quiligotti, G. A. Maugin, and F. Dell'isola. Wave motions in unbounded poroelastic solids infused with compressible fluids. *Zeitschrift fr angewandte Mathematik und Physik*, 53(6), 1110–1138, 2002.

[ReR93] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1993.

[RoC10] E. Rohan and R. Cimrman. Two-scale modelling of tissue perfusion problem using homogenization of dual prous media. *Int. Jour. for Multiscale Comput. Engrg.*, 8, 81–102, 2010.

[Sch01] M. Schanz. *Wave propagation in viscoelastic and poroelastic continua: a boundary element approach.* Springer, 2001.

[Tem84] R. Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, third edition, 1984.

[Vis04] A. Visintin. Some properties of two-scale convergence. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 15, 93–107, 2004.