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Elastic scattering by unbounded rough surfaces

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Abstract

We consider the two-dimensional time-harmonic elastic wave scattering problem for an unbounded rough surface, due to an inhomogeneous source term whose support lies within a finite distance above the surface. The rough surface is supposed to be the graph of a bounded and uniformly Lipschitz continuous function, on which the elastic displacement vanishes. We propose an upward propagating radiation condition (angular spectrum representation) for solutions of the Navier equation in the upper half-space above the rough surface, and establish an equivalent variational formulation. Existence and uniqueness of solutions at arbitrary frequency is proved by applying a priori estimates for the Navier equation and perturbation arguments for semi-Fredholm operators.

1 Introduction

This paper is concerned with the mathematical analysis of time-harmonic elastic wave scattering problems for unbounded rough surfaces. By the phrase *rough surface*, we will denote in this paper a surface which is a (usually non-local) perturbation of an infinite plane surface such that the surface lies within a finite distance of the original plane. Rough surface scattering problems for acoustic, electromagnetic and elastic waves are of interest to physicists, engineers and applied mathematicians since many years due to their wide range of applications in optics, acoustics, radio-wave propagation, seismology and radar techniques (see, e.g., [1, 15, 22, 23, 39, 42, 43]). In particular, diffraction phenomena for elastic waves propagating in unbounded periodic and non-periodic structures have many applications in geophysics and seismology. For instance, the problem of elastic pulse transmission and reflection through the earth is fundamental to the investigation of earthquakes and the utility of controlled explosions in search for oil and ore bodies (see, e.g., [1, 29, 30, 40] and the references therein).

The mathematical analysis of acoustic and electromagnetic rough surface scattering problems that can be modeled by the Helmholtz equation has mainly been developed by Chandler-Wilde and his collaborators over the last fifteen years. Via the integral equation method, the well-posedness of the Dirichlet boundary value problem for an impenetrable rough surface in $\mathbb{R}^2$ is proved by Chandler-Wilde & Ross [17] and by Chandler-Wilde & Zhang [18], and the well-posedness of the corresponding problem in $\mathbb{R}^3$ has been established only recently by Chandler-Wilde, Heinemeyer & Potthast [12] and by Thomas [41, Chapter 5]; see also [19, 20, 37] for the integral equation method applied to the scattering by rough interfaces and inhomogeneous layers. Using variational methods, Chandler-Wilde and Monk [14] are able to prove the well-posedness of the Dirichlet problem in two and three dimensions for much more general boundaries. The approach proposed in [14] leads to explicit bounds on the solution and has been extended to rather general acoustic scattering problems, including problems of scattering by impedance surfaces and by inhomogeneous layers in a half-space; see, e.g. [16, 35, 41]. A recently developed variational approach in weighted Sobolev spaces covers the problem of plane wave incidence for two-dimensional sound-soft rough surfaces, whereas in the 3D case incident spherical and cylindrical waves can be treated; see Chandler-Wilde & Elschner [11]. Based on the variational formulation proposed in [14], rigorous numerical methods using finite elements combined with the perfectly matched layer (PML) technique or with the finite section method have been developed and analyzed for acoustic scattering by sound-soft rough surfaces; see [15, 11].

Despite significant progress made for the Helmholtz equation, relatively little analysis for the Navier and Maxwell equations in unbounded non-periodic structures has been carried out. A rigorous mathematical analysis on existence and uniqueness of solutions is given by Arens in [6, 7] for $C^{1,\alpha}$-smooth rough surfaces via the boundary integral equation method, which generalizes the solvability results in [18, 19, 20] for acoustic waves to the elastic case. Moreover, an upward propagating radiation condition (UPRC) is proposed in [6] based on the elastic Green's tensor of the Dirichlet boundary value problem for the Navier equation in a half-space. Note that the classical Kupradze radiation condition (e.g. [34]) is not appropriate in the case of unbounded rough surfaces. Concerning the variational approach applied to electromagnetic rough surface scattering problems modeled by the full Maxwell system, we refer to the recent publications [36] by Li, Wu & Zheng where existence and uniqueness is established for an incident magnetic or electric dipole in a lossy medium, and to Haddar & Lechleiter [31] in the more challenging case of a penetrable dielectric layer.

In contrast to the general case of unbounded rough surfaces, there is already a vast literature on the variational approach applied to acoustic and electromagnetic scattering by periodic diffractive structures (diffraction gratings) and locally perturbed plane scatterers (cavities); see e.g. Ammari, Bao & Wood [3], Bao & Dobson [9], Bonnet-Bendhia & Starling [10], Elschner & Schmidt [26], Elschner, Hinder, Penzel & Schmidt [27], Elschner & Yamamoto [28] and Kirsch [33]. In the case of elastic scattering by periodic surfaces, the variational approach is established by Elschner & Hu in [24, 25] for the boundary value problems of the first, second, third and fourth kind as well as for transmission problems with non-smooth interfaces in $\mathbb{R}^n$ ($n = 2, 3$). We note that the assumptions made in all of these papers lead to a variational formulation over a bounded domain, so that compact imbedding arguments can be applied and the sesquilinear form that arises satisfies a Gårding inequality which considerably simplifies the mathematical arguments. We also refer to Arens [4] and [5] for the well-posedness of the two-dimensional elastic scattering problem for smooth (C^2) diffraction gratings, where the existence proof is based on the boundary integral equation method.

In this paper we assume that the rough surface is invariant along the x_3-direction, so that the three-dimensional elastic scattering problem can be reduced to a two-dimensional problem in the (x_1, x_2)-plane. A rough surface in this sense always means its cross section by the (x_1, x_2)-plane. Our aim is to study the two-dimensional elastic wave scattering problem for an unbounded rough surface, due to an inhomogeneous source term whose support lies within some finite distance above the surface. This paper is closest in its methods and results to those of Chandler-Wilde & Monk [14], Elschner & Yamamoto [28] and Elschner & Hu [24], where the well-posedness of acoustic and elastic scattering problems for rough surfaces and diffraction gratings is established using variational methods. Compared to the acoustic case studied in [14], the elasticity problem appears to be more complicated because of the coexistence of compressional and shear waves that propagate at different speeds. What differs dramatically from the Helmholtz equation is that, in contrast to our previous work [24, 25] on diffraction gratings, the Dirichlet-to-Neumann map for the Navier equation does not have a definite real part. This gives rise to essential difficulties in extending the method in [14] to the elastic case (see Remark 3). In this paper, we suppose that the rough surface is the graph of a bounded and uniformly Lipschitz continuous function. Such a geometric assumption imposed on the rough surface is weaker than the condition used in [6, 7] (i.e., uniform Hölder continuity) but stronger than that in [14]. Under this assumption, we are able to establish a priori estimates for the scalar functions $\operatorname{div} u$ and $\operatorname{curl} u$ on the rough surface and on an infinite layer of finite thickness above the surface where u denotes a solution of the Navier equation on that layer. Based on the bounds for $\operatorname{div} u$ and $\operatorname{curl} u$, we finally derive an a priori estimate for u that leads to uniqueness and existence of solutions to our elastic scattering problem.

The paper is organized as follows. In Section 2 we present the formulation of our scattering problem. Moreover, based on the Helmholtz decomposition of the elastic displacement, we derive the radiation condition for elastic waves in a half-space above the rough surface. This condition is proved to be equivalent to the UPRC proposed by Arens in [6] and extends the UPRC for acoustic waves (see [8, 14, 17]) to elastic scattering. These radiation conditions are often used in a formal manner in the literature (e.g. [21, 22]), that, above the rough surface and the support of the source term, the solution can be represented in integral form (the angular spectrum representation) as a superposition of upward travelling and evanescent plane waves. In Section 3 we establish an equivalent variational formulation involving the Dirichlet-to-Neumann map, and present our main solvability result (Theorem 1). In Section 4, we prove the coercivity of the sesquilinear form corresponding to the variational formulation for small frequencies. Together with a perturbation result on semi-Fredholm operators (Lemma 3) and our main a priori estimate for the Navier equation (Lemmas 5 and 8), this leads to the unique solvability of our scattering problem at arbitrary frequency. This a priori estimate will be first justified for a smooth rough surface in Section 5.1 and then extended to a Lipschitz surface in Section 5.2 by approximation arguments.

We finish this section by introducing some notation that will be used throughout the paper. Denote by $(\cdot)^\top$ the transpose of a 1×2 vector in $\mathbb{C}^2$. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for $\mathbf{a} \in \mathbb{C}^2$, let $|\mathbf{a}|$ denote its Euclidean norm. For a matrix $M = (m_{ij}) \in \mathbb{C}^{2\times 2}$, $||M||$ denotes the norm defined by $||M|| := \max_{i,j} |m_{ij}|$. The symbol $\mathbf{a} \cdot \mathbf{b}$ stands for the inner product $a_1 b_1 + a_2 b_2$ of $\mathbf{a} = (a_1, a_2)^\top$, $\mathbf{b} = (b_1, b_2)^\top \in \mathbb{C}^2$. Standard L^2-based Sobolev spaces defined in a domain Ω or on a surface Γ are denoted by $H^s(\Omega)$ or $H^s(\Gamma)$ for $s \in \mathbb{R}$.

2 The boundary value problem and radiation condition

In this section, we present the mathematical formulation of the two-dimensional elastic wave scattering problem for rough surfaces. Let $D \subset \mathbb{R}^2$ be an unbounded connected open set such that, for some constants $f_- < f_+$,

$$U_{f_+} \subset D \subset U_{f_-}, \quad U_h := \{x = (x_1, x_2) : x_2 > h\}.$$

For $h > f_+$, let $\Gamma_h := \{x \in \mathbb{R}^2 : x_2 = h\}$ and $S_h := D \backslash \overline{U}_h$. The variational problem will be posed on the open set S_h which lies between the rough surface $\Gamma := \partial D$ and the line Γ_h. Throughout the paper we fix the constants $f_-,\ f_+$, and assume that Γ is the graph of a uniformly Lipschitz continuous function f ($f \in C^{0,1}$), i.e.,

$$\Gamma = \{x \in \mathbb{R}^2 : x_2 = f(x_1),\ x_1 \in \mathbb{R}\},$$

and that there exists a constant $L > 0$ such that

$$|f(x_1) - f(x_2)| \le L\,|x_1 - x_2|, \quad \text{for all} \quad x_1\,,\, x_2 \in \mathbb{R}\,. \tag{2.1}$$

Given an inhomogeneous source term $g \in L^2(D)^2$ whose support lies within a finite distance above Γ, we wish to seek the elastic displacement $u = (u_1, u_2)^\top$ such that

$$(\Delta^* + \omega^2)u = g \quad \text{in} \quad D\,, \quad \Delta^* := \mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,, \tag{2.2}$$

$$u = 0 \quad \text{on} \quad \Gamma\,, \tag{2.3}$$

with (2.2) understood in a distributional sense, and such that u satisfies an appropriate radiation condition. Here ω denotes the angular frequency, and the Lamé constants λ and μ are fixed throughout the paper and satisfy $\lambda > 0,\ \lambda + \mu > 0$. Note that in (2.2) we have assumed for simplicity that the mass density of the elastic medium in D is equal to one. In the following we will derive a new upward propagating radiation condition (UPRC) for elastic waves based on the UPRC for acoustic waves in [14].

Let $\mathcal{F}v$ denote the Fourier transform of v defined by

$$\hat{v}(\xi) = \mathcal{F}\, v(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-it\xi)\, v(t)\, dt\,, \quad \xi \in \mathbb{R}\,,$$

with the inverse transform given by

$$v(t) = \mathcal{F}^{-1}\, \hat{v}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(it\xi)\, \hat{v}(\xi)\, d\xi\,, \quad t \in \mathbb{R}\,.$$

Note that $\mathcal{F}$ is an isometry of $L^2(\mathbb{R})$ onto itself. Since the support of g is bounded in x_2-direction, we can choose a number $h > f_+$ such that $\mathrm{supp}\,(g)$ is contained in $\overline{S}_h$. We want to derive a representation for u in U_h in terms of $u|_{\Gamma_h}$. Define the compressional and shear wave numbers by

$$k_p := \omega/\sqrt{2\mu+\lambda}\,, \quad k_s := \omega/\sqrt{\mu}$$

respectively. Since u satisfies the homogeneous Navier equation in U_h, it can be decomposed into a sum of its compressional and shear parts:

$$u = \frac{1}{i}(\mathrm{grad}\,\varphi + \overrightarrow{\mathrm{curl}}\,\psi) \quad \text{with} \quad \varphi := -\frac{i}{k_p^2}\,\mathrm{div}\,u\,, \quad \psi := \frac{i}{k_s^2}\,\mathrm{curl}\,u\,, \tag{2.4}$$

where the two curl operators in $\mathbb{R}^2$ are defined by

$$\mathrm{curl}\,u := \partial_1 u_2 - \partial_2 u_1\,, \quad u = (u_1, u_2)^\top \quad \text{and} \quad \overrightarrow{\mathrm{curl}}\,v := (\partial_2 v, -\partial_1 v)^\top\,,$$

with $\partial_j := \partial/\partial x_j, j = 1, 2$. The scalar functions φ and ψ satisfy the homogeneous Helmholtz equations

$$(\Delta + k_p^2)\,\varphi = 0 \quad \text{and} \quad (\Delta + k_s^2)\,\psi = 0 \quad \text{in} \quad U_h\,. \tag{2.5}$$

Applying the Fourier transform to (2.5) with respect to x_1, we obtain, for $(\xi, x_2) \in U_h$,

$$\begin{aligned} \partial_2^2\hat{\varphi}(\xi, x_2) + \gamma_p^2\,\hat{\varphi}(\xi, x_2) = 0\,, &\quad \text{with} \quad \gamma_p = \gamma_p(\xi) := \sqrt{k_p^2 - \xi^2}\,, \\ \partial_2^2\hat{\psi}(\xi, x_2) + \gamma_s^2\,\hat{\psi}(\xi, x_2) = 0\,, &\quad \text{with} \quad \gamma_s = \gamma_s(\xi) := \sqrt{k_s^2 - \xi^2}\,. \end{aligned}$$

Throughout the paper the branch cut of a complex square root is always chosen such that its imaginary part is non-negative, i.e., $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ if $|\xi| > k$. As the field u above the support of g should be the superposition of outgoing plane waves, we seek solutions to the above equations in the form

$$\begin{aligned} \hat{\varphi}(\xi, x_2) &= P(\xi)\, \exp\left(i(x_2 - h)\,\gamma_p(\xi)\right), \\ \hat{\psi}(\xi, x_2) &= S(\xi)\, \exp\left(i(x_2 - h)\,\gamma_s(\xi)\right), \quad (\xi, x_2) \in U_h\,, \end{aligned} \tag{2.6}$$

for some $P(\xi)\,,\ S(\xi) \in L^2(\mathbb{R})$. Note that, for fixed $x_2 > h$, the exponential functions in (2.6) are rapidly decaying as $|\xi| \to \infty$. Taking the Fourier transform of (2.4) with respect to x_1 gives

$$\begin{aligned} \hat{u}_1(\xi, x_2) &= \xi\, \hat{\varphi}(\xi, x_2) + \frac{1}{i}\, \partial_2\hat{\psi}(\xi, x_2)\,, \\ \hat{u}_2(\xi, x_2) &= \frac{1}{i}\, \partial_2\hat{\varphi}(\xi, x_2) - \xi\, \hat{\psi}(\xi, x_2)\,, \quad (\xi, x_2) \in U_h. \end{aligned} \tag{2.7}$$

Inserting (2.6) into (2.7) and setting $x_2 = h$ yields

$$\begin{pmatrix} \hat{u}_{h,1}(\xi) \\ \hat{u}_{h,2}(\xi) \end{pmatrix} = \begin{pmatrix} \xi & \gamma_s \\ \gamma_p & -\xi \end{pmatrix} \begin{pmatrix} P(\xi) \\ S(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R}, \quad u_h = \begin{pmatrix} u_{h,1} \\ u_{h,2} \end{pmatrix} := u|_{\Gamma_h}, \tag{2.8}$$

which implies that

$$\begin{pmatrix} P(\xi) \\ S(\xi) \end{pmatrix} = \frac{1}{\xi^2 + \gamma_p \gamma_s} \begin{pmatrix} \xi & \gamma_s \\ \gamma_p & -\xi \end{pmatrix} \begin{pmatrix} \hat{u}_{h,1}(\xi) \\ \hat{u}_{h,2}(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R}. \tag{2.9}$$

Note that the function in the denominator in (2.9) satisfies the bounds

$$C_1 \leq |\xi^2 + \gamma_p(\xi)\, \gamma_s(\xi)| \leq C_2, \quad \xi \in \mathbb{R}, \tag{2.10}$$

where the constants $C_1, C_2 > 0$ only depend on ω (for fixed Lamé parameters λ and μ). More precisely, we have $\xi^2 + \gamma_p(\xi)\, \gamma_s(\xi) \sim (k_p^2 + k_s^2)/2$ as $|\xi| \to \infty$, and it follows from the proof of Lemma 2 below that (2.10) holds with $C_1 = k_p^2$, $C_2 = k_s^2$. Inserting (2.9) into (2.6) and then inserting the representations of $\hat{\varphi}(\xi, x_2), \hat{\psi}(\xi, x_2)$ into (2.7), we finally obtain the Fourier transform of u with respect to x_1 in U_h given by

$$\hat{u}(\xi, x_2) = (e^{i(x_2-h)\,\gamma_p(\xi)}\, M_p(\xi) + e^{i(x_2-h)\,\gamma_s(\xi)}\, M_s(\xi))\, \hat{u}_h(\xi), \quad (\xi, x_2) \in U_h,$$

with the matrices $M_p(\xi)$ and $M_s(\xi)$ defined by

$$M_p(\xi) := \frac{1}{\xi^2 + \gamma_p \gamma_s} \begin{pmatrix} \xi^2 & \xi\gamma_s \\ \xi\gamma_p & \gamma_p\gamma_s \end{pmatrix}, \quad M_s(\xi) := \frac{1}{\xi^2 + \gamma_p \gamma_s} \begin{pmatrix} \gamma_p\gamma_s & -\xi\gamma_s \\ -\xi\gamma_p & \xi^2 \end{pmatrix}.$$

Taking the inverse Fourier transform of $\hat{u}(\xi, x_2)$, we arrive at the following representation for u,

$$u(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(e^{i(x_2-h)\,\gamma_p(\xi)}\, M_p(\xi) + e^{i(x_2-h)\,\gamma_s(\xi)}\, M_s(\xi)\right) \hat{u}_h(\xi)\, e^{ix_1\xi}\, d\xi, \quad x_2 > h, \tag{2.11}$$

in terms of the Fourier transform of $u(x_1, h)$. The formula (2.11) is just the upward propagating radiation condition (UPRC) that we are going to use in the following sections. The right hand side of (2.11) can be interpreted as a superposition (in integral form) of upward propagating plane compressional and shear waves corresponding to $|\xi| \leq k_p$ and $|\xi| \leq k_s$ respectively, and evanescent plane waves corresponding to $|\xi| > k_p$ and $|\xi| > k_s$ respectively. Since each element of $M_p(\xi)\, \exp(i(x_2 - h)\gamma_p(\xi))$ and $M_s(\xi)\, \exp(i(x_2 - h)\gamma_s(\xi))$ is uniformly bounded in $\xi \in \mathbb{R}$, the integral (2.11) exists in the Lebesgue sense for all $x \in U_h$ when $u_h \in L^2(\Gamma_h)^2$ so that $\hat{u}_h \in L^2(\mathbb{R})^2$.

Taking into account the relations (2.8) and (2.9) between $\hat{u}_h$ and $(P(\xi), S(\xi))^\top$, we may rewrite the UPRC (2.11) as

$$\begin{aligned} u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Bigg\{ &\begin{pmatrix} \xi \\ \gamma_p(\xi) \end{pmatrix} P(\xi) e^{i(x_2-h)\,\gamma_p(\xi)} \\ &+ \begin{pmatrix} \gamma_s(\xi) \\ -\xi \end{pmatrix} S(\xi) e^{i(x_2-h)\,\gamma_s(\xi)} \Bigg\}\, e^{ix_1\xi}\, d\xi, \quad x_2 > h, \end{aligned} \tag{2.12}$$

in terms of the Fourier transforms $P(\xi)$ and $S(\xi)$ of the functions

$$\varphi(x_1, h) = (-i/k_p^2)\, \mathrm{div}\, u(x_1, h), \quad \psi(x_1, h) = (i/k_s^2)\, \mathrm{curl}\, u(x_1, h),$$

respectively; see (2.6).

Remark 1 *(i) The equivalence of the UPRC (2.11) with that proposed in [6] can be seen as follows. Let $\mathcal{F}_{y_1\to\xi}$ denote the Fourier transform with respect to the variable y_1, and let $\Psi_k(x,y)$ denote the free space fundamental solution of the Helmholtz equation $(\Delta + k^2)u = 0$ given by $\Psi_k(x,y) := (i/4)H_0^{(1)}(k|x-y|)$, with $H_0^{(1)}$ being the Hankel function of the first kind of zero order. Since*

$$\mathcal{F}_{y_1\to\xi}\Big(\frac{\partial\Psi(x,y)}{\partial y_2}\Big|_{y_2=h}\Big) = (2\sqrt{2\pi})^{-1}\exp(i(x_2-h)\sqrt{k^2-\xi^2}+ix_1\xi),$$

the UPRC (2.12) can be rewritten as

$$\begin{aligned} u(x) &= \frac{1}{i\sqrt{2\pi}}\Big\{\mathrm{grad}_x\int_{\mathbb{R}} e^{i(x_2-h)\,\gamma_p(\xi)+ix_1\xi}\,P(\xi)\,d\xi + \overrightarrow{\mathrm{curl}}_x\int_{\mathbb{R}} e^{i(x_2-h)\,\gamma_s(\xi)+ix_1\xi}\,S(\xi)\,d\xi\Big\} \\ &= \frac{2}{i}\Big\{\mathrm{grad}_x\int_{\Gamma_h}\frac{\partial\Psi_{k_p}(x,y)}{\partial y_2}\,\varphi(y)\,ds(y) + \overrightarrow{\mathrm{curl}}_x\int_{\Gamma_h}\frac{\partial\Psi_{k_s}(x,y)}{\partial y_2}\,\psi(y)\,ds(y)\Big\}, \end{aligned} \qquad (2.13)$$

for $x_2 > h$, which is equivalent to the UPRC proposed by Arens; see [6, Theorem 3.7 (ii)].

(ii) We say that u is α-quasiperiodic with the phase-shift α if

$$u(x_1+2\pi,x_2) = \exp(i2\pi\alpha)\,u(x_1,x_2)\,,\quad (x_1,x_2)\in D\,.$$

If u is quasi-periodic and the profile function f is 2π-periodic, the UPRC (2.13) for a bounded solution u is equivalent to the commonly used Rayleigh expansion radiation condition in U_h; see [6, Remark 3.8]. Therefore our UPRC (2.11) also generalizes the Rayleigh expansion in the case of a periodic surface Γ. For the uniqueness and existence of quasi-periodic solutions in grating diffraction problems, we refer to [24, 25] concerning the variational approach in $\mathbb{R}^n$ $(n=2,3)$, and to [6, 7] where the integral equation method and the Rayleigh expansion radiation condition are used for the Navier equation in $\mathbb{R}^2$.

To state the boundary value problem, for $h > f_+$, we introduce the energy space V_h as the closure of $C_0^\infty(S_h\cup\Gamma_h)^2$ in the norm

$$||u||_{V_h} = \Big(||\nabla u||^2_{L^2(S_h)^2} + ||u||^2_{L^2(S_h)^2}\Big)^{1/2}.$$

Boundary value problem (BVP): Given $g\in L^2(D)^2$, with $\mathrm{supp}\,(g)\subset\overline{S}_h$ for some $h > f_+$, find $u\in H^1_{loc}(D)^2$ such that $u|_{S_a}\in V_a$ for every $a > f_+$, the Navier equation

$$(\Delta^* + \omega^2)u = g \quad \text{in} \quad D \qquad (2.14)$$

holds in a distributional sense, and the radiation condition (2.11) is satisfied with $u_h := u|_{\Gamma_h}\in H^{1/2}(\Gamma_h)^2$ (from the trace theorem).

Remark 2 *We note that the solutions of (BVP) do not depend on the choice of h since the arguments of [14, Remark 2.1] for the Helmholtz equation can be easily adapted to our elastic case. More precisely, if u is a solution to (BVP) for one value $h > f_+$ for which $\mathrm{supp}(g)\subset\overline{S}_h$, then u is a solution for all $H > f_+$ having this property. Note that if the UPRC (2.11) holds for some $h > f_+$, then it holds for all larger values of h; see Lemma 1 below. To show that (2.11) also holds for every $H < h$ such that $H > f_+$ and $\mathrm{supp}(g)\subset\overline{S}_H$, the uniqueness result of Theorem 1 below can be applied.*

3 The Dirichlet-to-Neumann map and variational formulation

We now derive an equivalent variational formulation of the boundary value problem (BVP) in the space V_h, which involves the Dirichlet-to-Neumann operator on the artificial boundary Γ_h. We introduce the generalized stress (or traction) operator on $\partial S_h = \Gamma \cup \Gamma_h$ defined by

$$T_{a,b}u = (\mu + a)\,\partial_{\mathbf{n}} u + b\,\mathbf{n}\,\mathrm{div}\,u - a\,\tau\,\mathrm{curl}\,u\,, \tag{3.1}$$

where $\mathbf{n} = (n_1, n_2)^\top$ denotes the exterior unit normal, $\tau := (-n_2, n_1)^\top$ is the tangential vector, and a and b are real numbers satisfying $a + b = \lambda + \mu$. Throughout this paper we choose $a = 0\,,\ b = \lambda + \mu$, so that

$$Tu := T_{0,\lambda+\mu}u = \mu\,\partial_{\mathbf{n}} u + (\lambda + \mu)\,\mathbf{n}\,\mathrm{div}\,u \quad \text{on} \quad \partial S_h\,. \tag{3.2}$$

With this choice, the first Betti formula reads as follows

$$\begin{aligned} -\int_{S_h} (\Delta^* + \omega^2) w \cdot \overline{v}\,dx = &\int_{S_h} \big(\mathcal{E}(w, \overline{v}) - \omega^2 w \cdot \overline{v}\big)\,dx \\ &- \int_{\partial S_h} \overline{v} \cdot Tw\,ds, \quad w, v \in H^2(S_h)^2\,, \end{aligned} \tag{3.3}$$

where the bar indicates the complex conjugate, and $\mathcal{E}(\cdot,\cdot)$ is the symmetric bilinear form defined by

$$\begin{aligned} \mathcal{E}(w, v) \;\; := \;\; & (\lambda + 2\mu)\,(\partial_1 w_1\;\partial_1 v_1 + \partial_2 w_2\;\partial_2 v_2) + \mu\,(\partial_2 w_1\;\partial_2 v_1 + \partial_1 w_2\;\partial_1 v_2) \\ & + (\lambda + \mu)\,(\partial_1 w_1\;\partial_2 v_2 + \partial_2 w_2\;\partial_1 v_1) \end{aligned} \tag{3.4}$$

in accordance with the stress operator (3.2). Moreover, we obviously have the coercivity estimate

$$\begin{aligned} \int_{S_h} \mathcal{E}(v, \overline{v})\,dx &= \mu\,||\nabla v||^2_{L^2(S_h)^2} + (\lambda + \mu)\,||\mathrm{div}\;v||^2_{L^2(S_h)^2} \\ &\geq \mu\,||\nabla v||^2_{L^2(S_h)^2}\,, \quad v \in H^1(S_h)^2\,. \end{aligned} \tag{3.5}$$

Note that the normal on Γ_h takes the form $\mathbf{n} = e_2 := (0,1)^\top$, so that

$$Tu = \begin{pmatrix} \mu\partial_2 & 0 \\ (\lambda + \mu)\,\partial_1 & (\lambda + 2\mu)\,\partial_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{on} \quad \Gamma_h\,. \tag{3.6}$$

To introduce the Dirichlet-to-Neumann map on Γ_h for our scattering problem, we further define the matrices

$$T_p(\xi) := i\begin{pmatrix} \mu\,\gamma_p(\xi) & 0 \\ (\lambda + \mu)\,\xi & (\lambda + 2\mu)\,\gamma_p(\xi) \end{pmatrix}, \quad T_s(\xi) := i\begin{pmatrix} \mu\,\gamma_s(\xi) & 0 \\ (\lambda + \mu)\,\xi & (\lambda + 2\mu)\,\gamma_s(\xi) \end{pmatrix}. \tag{3.7}$$

Consider $v \in C_0^\infty(\Gamma_h)^2$ and extend it to a function $u \in C^\infty(\overline{U}_h)^2$ using the UPRC (2.11) with $u_h = v$. Then, applying the stress operator (3.6) on $\Gamma_{h+\epsilon}\,,\ \epsilon > 0$, to the representation (2.11), letting $\epsilon \to 0$ and using (3.7), we obtain the relation

$$\mathcal{T}v(x_1, h) := \lim_{\epsilon \to 0^+} Tu(x_1, h + \epsilon) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} M(\xi)\,\hat{u}_h(\xi)\,\exp(ix_1\xi)\,d\xi\,, \quad x_1 \in \mathbb{R}\,, \tag{3.8}$$

where the matrix $M(\xi) = M(\xi, \omega) := T_p(\xi)\,M_p(\xi) + T_s(\xi)\,M_s(\xi)$ takes the form

$$M(\xi, \omega) = \frac{i}{\xi^2 + \gamma_p\gamma_s}\begin{pmatrix} \omega^2\gamma_p & -\xi\omega^2 + \xi\mu\,(\xi^2 + \gamma_p\gamma_s) \\ \xi\omega^2 - \xi\mu\,(\xi^2 + \gamma_p\gamma_s) & \omega^2\gamma_s \end{pmatrix}. \tag{3.9}$$

The operator $\mathcal{T} = \mathcal{T}(\omega)$ defined in (3.8) can be represented as

$$\mathcal{T}(\omega)v = \mathcal{F}^{-1}(M(\xi,\omega)\,\mathcal{F}v)\,, \quad v \in C_0^\infty(\Gamma_h)^2\,. \tag{3.10}$$

This operator, which will prove to be a Dirichlet-to-Neumann map on Γ_h, extends to a bounded linear map from $H^{1/2}(\Gamma_h)^2$ to $H^{-1/2}(\Gamma_h)^2$. This follows from the definition of the Sobolev spaces $H^s(\Gamma_h)$, $s \in \mathbb{R}$, as the completion of $C_0^\infty(\Gamma_h)$ in the norm

$$||v||_{H^s(\Gamma_h)} := \left(\int_{\mathbb{R}} (1+\xi^2)^s\,|\mathcal{F}v(\xi)|^2\,d\xi\right)^{1/2}\,,$$

and the relations $\gamma_p(\xi)\,,\,\gamma_s(\xi) \sim i|\xi|$ as $|\xi| \to \infty$ and (2.10). In fact, these relations imply the bound

$$|M(\xi,\omega)z|^2 \le ||M(\xi,\omega)||^2\,|z|^2 \le C(\omega)\,(1+\xi^2)\,|z|^2\,, \tag{3.11}$$

with some $C(\omega) > 0$ uniformly in $\xi \in \mathbb{R}$ and $z \in \mathbb{C}^2$. Moreover, the matrix $M(\xi,\omega)$ (and thus the operator $\mathcal{T}(\omega)$) depend continuously on $\omega \in \mathbb{R}^+$. Furthermore,

$$||M(\xi,\omega) - M(\xi,\omega_1)||^2/(1+\xi^2) \to 0 \quad \text{as} \quad \omega \to \omega_1$$

holds uniformly in $\xi \in \mathbb{R}$. Thus the operator $\mathcal{T}(\omega)$ is continuous with respect to ω in operator norm, i.e.,

$$||\mathcal{T}(\omega) - \mathcal{T}(\omega_1)||_{H^{1/2}(\Gamma_h)^2 \to H^{-1/2}(\Gamma_h)^2} \to 0 \quad \text{as} \quad \omega \to \omega_1. \tag{3.12}$$

We next follow [14] to establish an equivalent variational formulation for the boundary value problem (BVP). It is well known that, for $H > h \ge f_+$, the trace operators $\gamma_+ : H^1(U_h \backslash U_H)^2 \to H^{1/2}(\Gamma_h)^2$ and $\gamma_- : V_h \to H^{1/2}(\Gamma_h)^2$ are continuous such that $\gamma_\pm u$ coincides with the restriction of u to Γ_h if u is C^∞. The following lemma extends some results of [14, Lemma 2.2] to the elastic case.

Lemma 1 *If the UPRC (2.11) holds with $u_h \in H^{1/2}(\Gamma_h)^2$, then $u \in H^1(U_h \backslash U_H)^2 \cap C^2(U_h)^2$ for every $H > h$, $(\Delta^* + \omega^2)\,u = 0$ in U_h, $\gamma_+ u = u_h$, and*

$$-\int_{\Gamma_h} \overline{v}\cdot\mathcal{T}\gamma_+ u\,ds + \omega^2\int_{U_h} u\cdot\overline{v}\,dx - \int_{U_h}\mathcal{E}(u,\overline{v})\,dx = 0, \quad v \in C_0^\infty(D)^2. \tag{3.13}$$

Furthermore, for all $H > h$, the restrictions of u and ∇u to Γ_H are in $L^2(\Gamma_H)^2$, and the UPRC (2.11) holds with h replaced by H.

Since Lemma 1 can be proved analogously to [14, Lemma 2.2], we omit its proof. Now suppose that u is a solution to the boundary value problem (BVP). Then $u|_{S_H} \in V_H$ for every $H \ge h$ and, since the equation (2.2) holds in a distributional sense, we have

$$\int_D \left(g\cdot\overline{v} + \mathcal{E}(u,\overline{v}) - \omega^2 u\cdot\overline{v}\right)\,dx = 0\,, \quad v \in C_0^\infty(D)^2\,.$$

Making use of the identity (3.13) and the fact that $\gamma_+ u = \gamma_- u$ on Γ_h, we arrive at

$$\int_{S_h}\left(g\cdot\overline{v} + \mathcal{E}(u,\overline{v}) - \omega^2 u\cdot\overline{v}\right)\,dx - \int_{\Gamma_h}\overline{v}\cdot\mathcal{T}\gamma_- u\,ds = 0\,, \quad v \in C_0^\infty(D)^2\,. \tag{3.14}$$

From the denseness of $C_0^\infty(S_h \cup \Gamma_h)^2$ in V_h and the continuity of γ_- and $\mathcal{T}$, it follows that the above equation holds for all $v \in V_h$. Thus the function $w := u|_{S_h}$ satisfies the variational equation

$$B(w,v) = -\int_{S_h} g\cdot\overline{v}\,dx\,, \quad v \in V_h\,, \tag{3.15}$$

with the sesquilinear form $B : S_h \times S_h \to \mathbb{C}$ defined by

$$B(w,v) = \int_{S_h} \left(\mathcal{E}(w,\overline{v}) - \omega^2 w \cdot \overline{v} \right) \, dx - \int_{\Gamma_h} \gamma_- \overline{v} \cdot \mathcal{T} \gamma_- w \, ds \, . \tag{3.16}$$

Conversely, if $u \in V_h$ is a solution to the variational problem (3.15) for some $h > f_+$, we define u in U_h by the right hand side of (2.11) with $u_h := \gamma_- u$. Then, by Lemma 1, $u \in H^1(U_h \backslash U_H)^2 \cap C^2(U_h)^2$ for every $H > h$ and $\gamma_+ u = \gamma_- u$ on Γ_h, implying that $u|_{S_H} \in V_H$ for every $H > f_+$. Moreover, it follows from (3.13) and (3.15) that the equation (2.2) holds in a distributional sense, with g extended by zero from U_h to D. Thus the variational problem (3.15) is equivalent to the boundary value problem (BVP).

Furthermore, we note that if u is a solution to the boundary value problem (BVP), then (3.14) implies that

$$\mathcal{T} \gamma_- u = Tu \quad \text{on} \quad \Gamma_h \, , \tag{3.17}$$

where T is the stress operator on Γ_h defined in (3.7). Thus $\mathcal{T}$ is actually the Dirichlet-to-Neumann map on Γ_h of our scattering problem.

Since the operator $\mathcal{T} : H^{1/2}(\Gamma_h)^2 \to H^{-1/2}(\Gamma_h)^2$ is bounded, the sesquilinear form B defined in (3.16) is bounded on the energy space V_h. Thus the form $B(\cdot,\cdot)$ obviously generates a continuous linear operator $\mathcal{B}(\omega) : V_h \to V_h^*$ such that

$$B(w,v) = (\mathcal{B}(\omega) w, v)_{S_h} \, , \quad w, v \in V_h \, , \tag{3.18}$$

where V_h^* denotes the dual of the space V_h with respect to the duality $(\cdot,\cdot)_{S_h}$ extending the scalar product in $L^2(S_h)^2$. In this paper we also consider the following more general problem: given $h > f_+$, $\mathcal{G} \in V_h^*$ and a fixed frequency $\omega > 0$, find $u \in V_h$ such that

$$\mathcal{B}(\omega) u = \mathcal{G} \, . \tag{3.19}$$

Note that equation (3.19) covers our variational problem (3.15) when the right hand side $\mathcal{G} \in V_h^*$ is defined specifically as the functional

$$\mathcal{G}(v) := - \int_{S_h} g \cdot \overline{v} \, dx \, , \quad v \in V_h \, ,$$

which satisfies the bound

$$||\mathcal{G}||_{V_h^*} = \sup_{||v||_{V_h} = 1} |\mathcal{G}(v)| \le ||g||_{L^2(S_h)^2} \, ,$$

where g is a source term with support in $\overline{S}_h$.

The main theorem of this paper can now be stated as follows.

Theorem 1 *For any $\omega > 0$, the variational problem (3.19) is uniquely solvable, and the solution satisfies the bound*

$$||u||_{V_h} \le C \, ||\mathcal{G}||_{V_h^*} \, , \quad C = C(\omega) > 0 \, ,$$

where the constant C does not depend on u and $\mathcal{G}$. In particular, the boundary value problem (BVP) is uniquely solvable, and the solution satisfies the estimate

$$||u||_{V_h} \le C \, ||g||_{L^2(S_h)^2} \, .$$

4 Analysis of the variational problem for small frequencies

For a matrix $M \in \mathbb{C}^{2\times 2}$, let Re $M := (M + M^*)/2$, and we shall write Re $M > 0$ if Re M is positive-definite. Here M^* is the adjoint of M with respect to the scalar product $(\cdot,\cdot)_{\mathbb{C}^2}$ in $\mathbb{C}^2$. To study the form B defined in (3.16) for small frequencies, we need the following properties of the matrix $M(\xi,\omega)$ defined in (3.9).

Lemma 2 *(i) For $|\xi| > k_s$, we have $\mathrm{Re}\,(-M(\xi,\omega)) > 0$ for every fixed frequency $\omega > 0$.*
(ii) There exists a sufficiently small frequency $\omega_0 > 0$ such that the estimate

$$|(\mathrm{Re}\, M(\xi,\omega)\,z,z)_{\mathbb{C}^2}| \le C\,\omega\,|z|^2\,,\quad z\in\mathbb{C}^2\,,\quad |\xi|\le k_s\,,\quad \omega\in(0,\omega_0] \tag{4.1}$$

holds for some constant $C > 0$ independent of ω, ξ and z.

Proof. (i) For $|\xi| > k_s > k_p$, we have $\gamma_p = i|\gamma_p|, \gamma_s = i|\gamma_s|$ and $\rho(|\xi|) := |\xi|^2 + \gamma_p\gamma_s = |\xi|^2 - |\gamma_p||\gamma_s| > 0$. Hence,

$$\mathrm{Re}\,(-M(\xi,\omega)) = \frac{1}{\rho(|\xi|)}\begin{pmatrix} \omega^2\,|\gamma_p| & -i\,[-\xi\,\omega^2+\xi\,\mu\,\rho(|\xi|)] \\ i\,[-\xi\,\omega^2+\xi\,\mu\,\rho(|\xi|)] & \omega^2|\gamma_s| \end{pmatrix}.$$

To prove the first assertion, we only need to verify that $\det\,(-\mathrm{Re}\,M(\xi,\omega)) > 0$ for all $|\xi| > k_s$, where det$(\cdot)$ denotes the determinant of a matrix. By the definition of $\rho(\cdot)$, it is easy to see that

$$\begin{aligned}\det\,(-\mathrm{Re}\,M(\xi,\omega))\,\rho(|\xi|)^2 &= \omega^4\,|\gamma_p|\,|\gamma_s| - \xi^2\,(\omega^2-\mu\,\rho(|\xi|))^2 \\ &= \rho(|\xi|)\,(-\omega^4 + 2\xi^2\,\mu^2\,k_s^2 - \xi^2\,\mu^2\,\rho(|\xi|)),\end{aligned}$$

which leads to

$$\begin{aligned}\det\,(-\mathrm{Re}\;M(\xi,\omega)) &= \left[-\omega^4 + 2\xi^2\,\mu^2\,k_s^2 - \xi^2\,\mu^2\,\rho(|\xi|)\right]/\rho(|\xi|) \\ &= \left[\mu^2\,k_s^2\,(\xi^2-k_s^2) + \xi^2\,\mu^2(k_s^2-\rho(|\xi|))\right]/\rho(|\xi|) \\ &\ge \left[\xi^2\,\mu^2(k_s^2-\rho(|\xi|))\right]/\rho(|\xi|).\end{aligned}$$

For $|\xi| > k_s$, it holds that

$$\rho'(|\xi|) = 2|\xi| - |\xi|\left(\frac{|\gamma_p|}{|\gamma_s|} + \frac{|\gamma_s|}{|\gamma_p}|\right) < 0,$$

implying that $k_s^2 - \rho(|\xi|) > k_s^2 - \rho(k_s) = 0$. Thus we have $\det\,(-\mathrm{Re}\;\;M(\xi,\omega)) > 0$ and $\mathrm{Re}\,(-M(\xi,\omega)) > 0$ for all $|\xi| > k_s$.

(ii) We first consider the case $|\xi| \le k_p < k_s$, where we have $\gamma_p = \sqrt{k_p^2 - |\xi|^2} \ge 0$, $\gamma_s = \sqrt{k_s^2-|\xi|^2} > 0$ and $\rho(|\xi|) := |\xi|^2 + \gamma_p\gamma_s > 0$. Then

$$\mathrm{Re}\,(-M(\xi,\omega)) = \frac{1}{\rho(|\xi|)}\begin{pmatrix} 0 & -i\,[-\xi\,\omega^2+\xi\,\mu\,\rho(|\xi|)] \\ i\,[-\xi\,\omega^2+\xi\,\mu\,\rho(|\xi|)] & 0 \end{pmatrix}. \tag{4.2}$$

Again one can check that $\rho'(|\xi|) \le 0$ for $0 < |\xi| \le k_p$, implying that $\rho(|\xi|) \ge \rho(k_p) = k_p^2$ and $\rho(|\xi|) \le \rho(0) = k_p\,k_s$. It follows that the inequality

$$\left|[-\xi\,\omega^2+\xi\,\mu\,\rho(|\xi|)]/\rho(|\xi|)\right| \le |\xi|\,\omega^2/\rho(|\xi|) + |\xi|\,\mu \le C(\lambda,\mu)\,\omega$$

holds for some positive constant $C(\lambda,\mu)$ as $\omega\to 0^+$. Thus we obtain

$$|(\mathrm{Re}\, M(\xi,\omega)\, z, z)_{\mathbb{C}^2}| \le C(\lambda,\mu)\,\omega\,|z|^2\,,\quad z\in\mathbb{C}^2\,,\quad |\xi|<k_p\,,\quad \omega\in(0,\omega_0] \tag{4.3}$$

for some sufficiently small frequency $\omega_0>0$.

We now consider the case $k_p<|\xi|\le k_s$. In this case there holds $\gamma_s=\sqrt{k_s^2-\xi^2}>0$, $\gamma_p = i\sqrt{\xi^2-k_p^2}$ and $\rho(|\xi|)=\xi^2+i\,|\gamma_p|\,\gamma_s$, with the bounds $|\gamma_p|^2$, $|\gamma_s|^2\le k_s^2-k_p^2$ and $k_p^2<|\rho(\xi)|\le k_s^2$. It can be derived from these bounds that each element of the matrix $\mathrm{Re}\,(-M(\xi,\omega))$ can be bounded by $C(\lambda,\mu)\omega$ for some constant $C(\lambda,\mu)>0$ as $\omega\to 0^+$. Thus the inequality (4.3) remains true for $k_p<|\xi|\le k_s$. □

By the Plancherel identity and the definition of the operator $\mathcal{T}$, for all $u\in V_h$ we have

$$\begin{aligned}\int_{\Gamma_h}\mathcal{T}\gamma_-u\cdot\gamma_-\overline{u}\,ds &= \int_{\mathbb{R}}\mathcal{F}(\mathcal{T}u_h)\cdot\mathcal{F}(\overline{u}_h)\,d\xi\\ &= \int_{|\xi|>k_s} M(\xi)\,\hat{u}_h\cdot\overline{\hat{u}}_h\,d\xi + \int_{|\xi|\le k_s} M(\xi)\,\hat{u}_h\cdot\overline{\hat{u}}_h\,d\xi\,,\end{aligned} \tag{4.4}$$

with the matrix $M(\xi)=M(\xi,\omega)$ defined in (3.9) and $u_h=\gamma_-u=u|_{\Gamma_h}$. Together with Lemma 2 and the trace theorem, the identity (4.4) implies that

$$-\int_{\Gamma_h}\mathcal{T}\gamma_-u\cdot\gamma_-\overline{u}\,ds \ge -C\,\omega\int_{|\xi|\le k_s}|\hat{u}_h(\xi)|^2\,d\xi \ge -C\,\omega\,||\gamma_-u||^2_{L^2(\Gamma_h)^2}\ge -\tilde{C}\,\omega\,||u||^2_{V_h} \tag{4.5}$$

with $C,\tilde{C}$ being some positive constants independent of u and ω.

Remark 3 *In contrast to the case of the scalar Helmholtz equation, the Dirichlet-to-Neumann map $\mathcal{T}$ for the Navier equation does not have a definite real part, which can be seen from the matrix (4.2) for $|\xi|<k_p$. We note that this leads to essential difficulties in establishing a priori estimates of solutions (see Lemma 4 below), and that the approach of using the generalized Lax-Milgram lemma in Chandler-Wilde & Monk [14] cannot be straightforwardly extended to the elastic case. However, in the periodic case one can decompose $\mathrm{Re}\,(-\mathcal{T})$ into the sum of a positive-definite operator and a finite dimensional operator. This decomposition combined with compact imbedding arguments applied to one periodic cell leads to the strong ellipticity of the corresponding sesquilinear form, and thus existence simply follows from uniqueness via the Fredholm alternative. However, the compact imbedding of H^1 into L^2 does not hold for the unbounded domain S_h.*

Remark 4 *With our selection of the stress operator $T:=T_{0,\lambda+\mu}$, we observe that k_s is the (explicit) lower bound of the numbers κ such that $\mathrm{Re}\,(-M(\xi,\omega))$ is positive definite for all $|\xi|>\kappa$. The results of Lemma 2 can be extended to the case where the matrix $M(\xi,\omega)$ in (3.9) is defined via an arbitrary generalized stress operator $T_{a,b}$ with $a+b=\lambda+\mu$ and $a,\ b\in\mathbb{R}$. In particular, Lemma 2 (i) then holds for $|\xi|>\kappa$ with some sufficiently large $\kappa>k_s$; see also [24, 25] where the usual stress operator $T_{\lambda,\mu}$ has been used in the cases of two-dimensional and three-dimensional periodic structures.*

Moreover, defining the Dirichlet-to-Neumann map $\mathcal{T}$ via the generalized stress operator $T_{a,b}$ and replacing (3.4) with the corresponding expression

$$\begin{aligned}\mathcal{E}(w,v) \;:=\; & (\lambda+2\mu)\,(\partial_1w_1\,\partial_1v_1+\partial_2w_2\,\partial_2v_2)+\mu\,(\partial_2w_1\,\partial_2v_1+\partial_1w_2\,\partial_1v_2)\\ & +a\,(\partial_1w_1\,\partial_2v_2+\partial_2w_2\,\partial_1v_1)+b\,(\partial_2w_1\,\partial_1v_2+\partial_1w_2\,\partial_2v_1)\end{aligned}$$

in the sesquilinear form (3.16), we arrive at a variational equation that is equivalent to equation (3.19).

Using Lemma 2 we can now establish the V_h-ellipticity of the sesquilinear form (3.16) for small frequencies, which implies the existence of a unique solution to equation (3.19) in this case.

Theorem 2 *Let $\mathcal{B}(\omega)$ be the operator defined in (3.18). Then there exists a sufficiently small frequency $\omega_0 > 0$ such that the bounded inverse operator $\mathcal{B}(\omega)^{-1} : V_h^* \to V_h$ of $\mathcal{B}$ exists for all $\omega \in (0, \omega_0]$.*

Proof. From the inequalities (3.5), (4.5) and the definition (3.16) of the sesquilinear form $B(\cdot, \cdot)$, it follows that

$$\mathrm{Re}\, B(u,u) \geq \mu\, ||\nabla u||^2_{L^2(S_h)^2} - \omega^2\, ||u||^2_{L^2(S_h)^2} - \tilde{C}\,\omega\, ||u||^2_{V_h}\,, \quad u \in V_h\,, \tag{4.6}$$

where the constant $\tilde{C} > 0$ is independent of u and ω. Recalling [14, Lemma 3.4] that

$$||u||^2_{L^2(S_h)^2} \leq C_1 ||\partial_2 u||^2_{L^2(S_h)^2} \leq C_1 ||\nabla u||^2_{L^2(S_h)^2}\,, \quad u \in V_h\,, \tag{4.7}$$

we arrive at the bound

$$||\nabla u||^2_{L^2(S_h)^2} \geq C_2 ||u||^2_{V_h}\,, \quad u \in V_h\,,$$

where the constants $C_1,\, C_2 > 0$ are independent of u and ω. Therefore, combining (4.6) and (4.7) we obtain the uniform estimate

$$\mathrm{Re}\, B(u,u) \geq C_3 ||u||^2_{V_h}\,, \quad u \in V_h\,, \quad \omega \in (0, \omega_0]$$

for a sufficiently small frequency $\omega_0 > 0$. By the Lax-Milgram lemma, $\mathcal{B}(\omega)^{-1} : V_h^* \to V_h$ exists with the bound $||\mathcal{B}(\omega)^{-1}||_{V_h^* \to V_h} \leq C_3^{-1/2}$. $\square$

5 Analysis of the variational formulation at arbitrary frequency

We now turn to analyzing the operator equation (3.19) for an arbitrary frequency $\omega > 0$, which covers the variational problem (3.15) as a special case. Our solvability result for (3.19) is a direct consequence of Lemma 3 below on the perturbation of semi-Fredholm operators which is known but will be presented for the reader's convenience.

Let X, Y be Banach spaces equipped with the norms $||\cdot||_X$ and $||\cdot||_Y$ repectively, and let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from X to Y. Denote by $N(\mathcal{B})$ and $R(\mathcal{B})$ the kernel and range repectively of an operator $\mathcal{B} \in \mathcal{L}(X,Y)$. Recall that $\mathcal{B} : X \to Y$ is semi-Fredholm if it has a closed range and at least one of the defect numbers $\dim N(\mathcal{B})$, $\mathrm{codim}\, R(\mathcal{B})$ is finite, where dim and codim stand for the dimension and codimension of a linear set respectively. If both of them are finite, then $\mathcal{B}$ is called Fredholm. The index of a semi-Fredholm operator is defined by $\mathrm{ind}\,(\mathcal{B}) = \dim N(\mathcal{B}) - \mathrm{codim}\, R(\mathcal{B})$.

Lemma 3 *Assume that $\{\mathcal{B}(\omega) : \omega \in \mathbb{R}^+\} \subset \mathcal{L}(X,Y)$, and that the operator $\mathcal{B}(\omega)$ is continuous with respect to ω in the sense that $||\mathcal{B}(\omega) - \mathcal{B}(\omega_1)||_{X \to Y} \to 0$ as $w \to w_1$, for all $\omega_1 \in \mathbb{R}^+$. Suppose further that*

(i) *$||\mathcal{B}(\omega)(u)||_Y \geq C(\omega)\, ||u||_X$ with some constant $C(\omega) > 0$ for each $\omega \in \mathbb{R}^+$,*

(ii) *there is a number $\omega_0 > 0$ such that the bounded inverse of $\mathcal{B}(\omega)$ exists for all $\omega \in (0, \omega_0]$.*

Then the operator $\mathcal{B}(\omega)$ is invertible for all $\omega \in \mathbb{R}^+$, and the norm of its inverse operator satisfies the bound $||\mathcal{B}(\omega)^{-1}||_{Y\to X} \le C(\omega)^{-1}\,,\ \omega \in \mathbb{R}^+$.

Proof. It follows from condition (i) that the operators $\mathcal{B}(\omega),\ \omega \in \mathbb{R}^+$, are all semi-Fredholm. Using a classical stability results for semi-Fredholm operators (see e.g. [32, Theorem 19.1.5]) and the continuity of $\mathcal{B}(\omega)$ with respect to ω, we have

$$\mathrm{ind}\,(\mathcal{B}(\omega_1)) = \mathrm{ind}\,(\mathcal{B}(\omega_2)), \quad \text{provided } |\omega_1 - \omega_2| \text{ is sufficiently small.}$$

This implies that the number ind $(\mathcal{B}(\omega))$ is constant (either a finite number or $-\infty$) for all $\omega \in \mathbb{R}^+$. However, from condition (ii), it follows that

$$\text{dim}\, N(\mathcal{B}(\omega)) = \text{codim}\, R(\mathcal{B}(\omega)) = 0\,,$$

and thus that $\mathrm{ind}\,(\mathcal{B}(\omega)) = 0$ for all $\omega \in (0, \omega_0]$. Hence $\mathrm{ind}\,(\mathcal{B}(\omega)) = 0$ for all $\omega \in \mathbb{R}^+$. Again using condition (i), we obtain $\mathrm{codim}\, R(\mathcal{B}(\omega)) = 0$ for all $\omega \in \mathbb{R}^+$, which is equivalent to the surjectivity of $\mathcal{B}(\omega)$. Therefore $\mathcal{B}(\omega)^{-1}$ always exists with the bound $||\mathcal{B}(\omega)^{-1}||_{Y\to X} \le C(\omega)^{-1}$. □

To apply Lemma 3, we take $X = V_h\,,\ Y = V_h^*$, and define $\mathcal{B}(\omega)$ as the operator in (3.18), which is continuous with respect to $\omega \in \mathbb{R}^+$ in operator norm; see (3.12) and (3.18). It obviously remains to verify the estimate

$$||u||_{V_h} \le C(\omega)\, ||\mathcal{G}||_{V_h^*}\,, \quad \text{for all} \quad u \in V_h\,, \quad \mathcal{G} := \mathcal{B}(\omega)u \in V_h^*\,, \tag{5.1}$$

for each $\omega \in \mathbb{R}^+$. Analogously to [14, Lemma 4.4], we establish an auxiliary lemma which reduces the problem of justifying (5.1) to that of proving an a priori bound for solutions of the variational equation (3.15), which is a special case of the equation (3.19). Note that the extension of [14, Lemma 4.4] to the elastic case is not trivial, due to the lack of a definite real part of the Dirichlet-to-Neumann map $\mathcal{T}$.

Lemma 4 *The bound (5.1) holds if there exists $C_0 = C_0(\omega) > 0$ such that*

$$||u||_{V_h} \le C_0\, ||\tilde{g}||_{V_h} \tag{5.2}$$

for all $u \in V_h$ and $\tilde{g} \in V_h$ satisfying the equation $\mathcal{B}(\omega)u = \tilde{g}$.

Proof. Consider the operator $\mathcal{B}_\alpha := \mathcal{B} + \alpha I\, :\, V_h \to V_h^*$, where $\alpha > 0$ and I is the identity operator. We claim that $\mathcal{B}_\alpha$ is invertible provided $\alpha > 0$ is sufficiently large. To see this, we will verify that $\mathrm{Re}\,(\mathcal{B}_\alpha u, u)_{S_h} \ge C_1\, ||u||^2_{V_h}$ for some constant $C_1 > 0$ independent of u, where the sesquilinear form corresponding to $\mathcal{B}_\alpha$ is given by (cf. (3.16))

$$(\mathcal{B}_\alpha u, v)_{S_h} = \int_{S_h} \left(\mathcal{E}(u,\overline{v}) + (\alpha - \omega^2) u\cdot\overline{v}\right)\, dx - \int_{\Gamma_h} \gamma_-\overline{v}\cdot \mathcal{T}\gamma_- u\, ds, \quad u, v \in V_h\,.$$

Using (3.5), (4.4) and Lemma 2 (i), we find

$$\mathrm{Re}\,(\mathcal{B}_\alpha u, u)_{S_h} \ge \mu\, ||\nabla u||^2_{L^2(S_h)^2} + (\alpha - \omega^2)\, ||u||^2_{L^2(S_h)^2} - \int_{|\xi|\le k_s} M(\xi,\omega)\, \hat{u}_h \cdot \overline{\hat{u}}_h\, d\xi\,, \tag{5.3}$$

where $u_h := \gamma_- u$. We estimate the last integral in (5.3) as follows. By (3.11),

$$\Big|\int_{|\xi|<k_s} M(\xi,\omega)\, \hat{u}_h\cdot\overline{\hat{u}}_h\, d\xi\Big| \le \tilde{C}(\omega) \int_{|\xi|<k_s} |\xi|\, |\hat{u}_h|^2\, d\xi \le C_1(\omega) \int_{|\xi|<k_s} (1+|\xi|)\, |\hat{u}_h|^2\, d\xi\,.$$

Since the bound

$$\frac{(1+|\xi|)\,l}{\sqrt{\xi^2+l^2}} = \frac{(1+|\xi|)}{\sqrt{(\xi/l)^2+1}} \leq 1+k_s$$

holds for all $|\xi| \leq k_s$, $l \in \mathbb{R}^+$, we have

$$\begin{aligned} \Big| \int_{|\xi|<k_s} M(\xi,\omega)\, \hat{u}_h \cdot \overline{\hat{u}}_h \, d\xi \Big| &\leq C_1(\omega)\,(1+k_s) l^{-1} \int_{\mathbb{R}} \sqrt{l^2+\xi^2}\, |\hat{u}_h(\xi)|^2 d\,\xi \\ &\leq C_1(\omega)\,(1+k_s)\, l^{-1} \Big(l^2\, ||u||^2_{L^2(S_h)^2} + ||\nabla u||^2_{L^2(S_h)^2} \Big), \end{aligned} \tag{5.4}$$

where the last inequality follows from the trace estimate in [14, Lemma 3.4]. Combining (5.4) and (5.3) yields

$$\mathrm{Re}\,(\mathcal{B}_\alpha u, u)_{S_h} \geq (\mu - C_2\, l^{-1})\, ||\nabla u||^2_{L^2(S_h)^2} + (\alpha - \omega^2 - C_2\, l)\, ||u||^2_{L^2(S_h)^2}, \quad l > 0,$$

with $C_2 = C_2(\omega) := C_1(\omega)(1+k_s)$. Choosing some $l = l_0 > C_2/\mu$ and then $\alpha > \omega^2 + C_2\, l_0$, we arrive at the V_h-ellipticity of the operator $\mathcal{B}_\alpha$.

Now we choose a sufficiently large number $\alpha > 0$ such that the problem

$$\mathcal{B}_\alpha u = \mathcal{G}, \quad \mathcal{G} \in V_h^*$$

has always a unique solution $u = u_\alpha \in V_h$, which satisfies the estimate

$$||u_\alpha||_{V_h} \leq C_3 ||\mathcal{G}||_{V_h^*}\,, \tag{5.5}$$

with some constant $C_3 > 0$ independent of $\mathcal{G}$. Suppose that $u \in V_h$ is a solution of

$$\mathcal{B} u = \mathcal{G}\,, \quad \mathcal{G} \in V_h^*\,.$$

Defining $w = u - u_\alpha$, we then see that

$$(\mathcal{B}w, v)_{S_h} = (\mathcal{B}u, v)_{S_h} - (\mathcal{B}u_\alpha, v)_{S_h} = -\alpha (u_\alpha, v)_{S_h}\,, \quad v \in V_h\,.$$

By the assumption (5.2) and the bound (5.5), it holds that

$$||w||_{V_h} \leq C_0\, \alpha\, ||u_\alpha||_{V_h} \leq C_0\, C_3\, \alpha\, ||\mathcal{G}||_{V_h^*}\,,$$

leading to the estimate

$$||u||_{V_h} \leq ||w||_{V_h} + ||u_\alpha||_{V_h} \leq C(\omega)\; ||\mathcal{G}||_{V_h^*}\,,$$

with some $C(\omega) > 0$ independent of u and $\mathcal{G}$. □

We turn now to establishing the crucial a priori estimate (5.2). This will be done in subsection 5.1 when the rough surface Γ is given by the graph of a bounded C^∞ function f with a uniform Lipschitz constant, and in subsection 5.2 for a bounded and uniformly Lipschitz continuous function f.

5.1 A priori estimate for smooth rough surfaces

Suppose that Γ is the graph of a C^∞ function f satisfying (2.1). Let $u \in V_h$ be a solution of the variational problem

$$B(u,v) = (\mathcal{B}u, v)_{S_h} = -\int_{S_h} \tilde{g} \cdot \overline{v}\, dx\,, \quad v \in V_h\,, \tag{5.6}$$

where $h > f_+$, $\tilde{g} \in V_h$, and $\mathcal{B} = \mathcal{B}(\omega)$ is defined in (3.18). Then u satisfies the inhomogeneous Navier equation

$$(\Delta^* + \omega^2)\, u = \tilde{g} \quad \text{in} \quad S_h \tag{5.7}$$

in a distributional sense, with the boundary conditions (cf. (3.17))

$$u = 0 \quad \text{on} \quad \Gamma\,, \qquad Tu = \mathcal{T}\gamma_- u \quad \text{on} \quad \Gamma_h\,.$$

The following lemma is crucial for proving Theorem 1 in the case of smooth rough surfaces.

Lemma 5 *Assume that Γ is given by the graph of a C^∞ function f satisfying (2.1), and that $u \in V_h$ is a solution of the problem (5.6). Then there exists a constant $C_0 > 0$ only depending on ω, h and the Lipschitz constant L of f such that $||u||_{V_h} \le C_0\, ||\tilde{g}||_{V_h}$.*

In the following we extend a solution u of (5.6) to D by the UPRC (2.11) with $u_h := \gamma_- u \in H^{1/2}(\Gamma_h)^2$. It follows from Lemma 1 and standard elliptic regularity that $u \in H^2(U_h \backslash U_H)^2$ for all $H > h$. Thus we have $\nabla u|_{\Gamma_h} \in H^{1/2}(\Gamma_h)^2$. Moreover, the UPRC (2.11) holds with h replaced by H; see Remark 2.

Our proof of Lemma 5 relies heavily on the use of Rellich identities for both the Helmholtz and Navier equations in an infinite layer of finite width. Motivated by the existence and uniqueness proofs for elastic scattering by periodic surfaces (cf. [24, 25]) and acoustic scattering by rough surfaces (cf. [14, 16, 41]), we first derive an a priori estimate for the traces of the functions $\mathrm{div}\, u$ and $\mathrm{curl}\, u$ on the rough surface Γ using a Rellich identity for the Navier equation. Then we extend the estimates of [28, Lemma 5.2] for the Helmholtz equation to the case of non-periodic rough surfaces and obtain bounds for the L^2 norms of $\mathrm{div}\, u$ and $\mathrm{curl}\, u$ on S_H and Γ_H for $H > h$.. These bounds combined with another Rellich identity for the Navier equation lead to the desired estimate in Lemma 5 when f is a smooth function.

Lemma 6 *Suppose that $f \in C^\infty(\mathbb{R})$ satisfies (2.1), $\tilde{g} \in V_h$, and that $u \in V_h$ is a solution of (5.6). Then there exists a constant $C > 0$ only depending on ω, h and L such that*

$$||\mathrm{div}\, u||^2_{L^2(\Gamma)} + ||\mathrm{curl}\, u||^2_{L^2(\Gamma)} \le C\, ||\tilde{g}||_{L^2(S_h)^2}\; ||\partial_2 u||_{L^2(S_h)^2}.$$

Proof. Following the approach of [14, Section 4], we first derive a Rellich identity for the Navier equation in the unbounded domain S_h. Since $\tilde{g} \in H^1(S_h)^2$ and u vanishes on the rough surface Γ which is C^∞-smooth, by standard elliptic regularity we see that $u \in H^3_{loc}(S_h)^2 \cap H^1(S_h)^2$. For $A \ge 1$, we choose a cut-off function $\chi_A(r) \in C_0^\infty(\mathbb{R}^+)$ with $r = |x|$ such that $\chi_A(r) = 1$ if $r \le A$, $\chi_A(r) = 0$ if $r \ge A+1$, $0 \le \chi_A(r) \le 1$ if $A < r \le A+1$ and that $||\chi_A'||_\infty \le C_1$ for some fixed C_1 independent of A. Using Betti's formula (3.3) with the stress operator T defined in (3.2), integration by parts gives (see [14] for the details in the case of the Helmholtz equation)

$$\begin{aligned}2\,\mathrm{Re} \int_{S_h} \chi_A(r) \partial_2 \overline{u} \cdot \Delta^* u\, dx &= \int_{\partial S_h^{A+1}} \chi_A(r)\{2\,\mathrm{Re}\,(Tu \cdot \partial_2 \overline{u}) - \mathcal{E}(u, \overline{u}) n_2\} ds \\ &+ \int_{S_h^{A+1}\backslash S_h^A} \left(-2\,\mathrm{Re}\,\{\sum_{j=1,2} \mathcal{E}(u, \chi_A(r) e_j)\, \partial_2 \overline{u_j}\} + \mathcal{E}(u, \overline{u})\, \partial_2 \chi_A(r) \right) dx\end{aligned}$$

and

$$2\,\mathrm{Re}\int_{S_h}\omega^2\chi_A(r)u\cdot\partial_2\overline{u}\,dx=\int_{\partial S_h^{A+1}}\omega^2\chi_A(r)|u|^2n_2\,dx-\int_{S_h^{A+1}\backslash S_h^A}\omega^2|u|^2\partial_2\chi_A(r)\,dx,$$

where $S_h^{A+1}:=S_h\cap\{|x|\le A+1\}$ and e_j denotes the unit vector in x_j-direction. Adding up the previous two equalities and letting $A\to+\infty$ yields the following Rellich identity for the Navier equation,

$$2\,\mathrm{Re}\int_{S_h}\partial_2\overline{u}\cdot(\Delta^*+\omega^2)u\,dx=\left(\int_\Gamma+\int_{\Gamma_h}\right)\left\{2\mathrm{Re}(Tu\cdot\partial_2\overline{u})-n_2\mathcal{E}(u,\overline{u})+n_2\omega^2|u|^2\right\}ds\,,\tag{5.8}$$

since the integrals over $S_h^{A+1}\backslash S_h^A$ converge to zero. Noting that $u=0$ and $\partial_\tau u=-n_2\partial_1u+n_1\partial_2u=0$ on Γ, we have

$$n_1\,\partial_2u=n_2\,\partial_1u,\quad\partial_1u=n_1\,\partial_{\mathbf{n}}u\quad\text{and}\quad\partial_2u=n_2\,\partial_{\mathbf{n}}u\quad\text{on}\quad\Gamma,\tag{5.9}$$

from which we derive that

$$n_2\,\mathcal{E}(u,\overline{u})=Tu\cdot\partial_2\overline{u}=n_2\left(\mu\,|\partial_{\mathbf{n}}u|^2+(\lambda+\mu)\,|\mathrm{div}\;u|^2\right)\quad\text{on}\quad\Gamma.\tag{5.10}$$

Hence, by (5.7), (5.8) and (5.10),

$$\begin{aligned}&-\int_\Gamma\left\{n_2\,\mu\,|\partial_{\mathbf{n}}u|^2+n_2\,(\lambda+\mu)\,|\mathrm{div}\;u|^2\right\}ds\\ =\;&\int_{\Gamma_h}\left\{2\,\mathrm{Re}\,(Tu\cdot\partial_2\overline{u})-\mathcal{E}(u,\overline{u})+\omega^2|u|^2\right\}ds-2\,\mathrm{Re}\int_{S_h}\tilde{g}\cdot\partial_2\overline{u}\,dx.\end{aligned}\tag{5.11}$$

Using the Fourier transform of $u_h(x_1)=u|_{\Gamma_h}$ given in (2.8) in terms of $(P(\xi),S(\xi))^\top$, and the Fourier transforms of Tu on Γ_h, $\partial_ju|_{\Gamma_h}$, $j=1,2$, and $\mathrm{div}\;u|_{\Gamma_h}$ given by (cf. (3.8) and (2.8))

$$\begin{aligned}\mathcal{F}(Tu)(\xi)&=\left(T_p(\xi)\begin{pmatrix}\xi\\ \gamma_p\end{pmatrix}+T_s(\xi)\begin{pmatrix}\gamma_s\\ -\xi\end{pmatrix}\right)\begin{pmatrix}P(\xi)\\ S(\xi)\end{pmatrix}\\ &=i\begin{pmatrix}\mu\xi\gamma_p&\omega^2-\mu\xi^2\\ \omega^2-\mu\xi^2&-\mu\xi\gamma_s\end{pmatrix}\begin{pmatrix}P(\xi)\\ S(\xi)\end{pmatrix},\\ \mathcal{F}(\partial_2u|_{\Gamma_h})(\xi)&=i\begin{pmatrix}\xi\gamma_p&\gamma_s^2\\ \gamma_p^2&-\xi\gamma_s\end{pmatrix}\begin{pmatrix}P(\xi)\\ S(\xi)\end{pmatrix},\\ \mathcal{F}(\partial_1u|_{\Gamma_h})(\xi)&=i\begin{pmatrix}\xi^2&\xi\gamma_s\\ \xi\gamma_p&-\xi^2\end{pmatrix}\begin{pmatrix}P(\xi)\\ S(\xi)\end{pmatrix},\\ \mathcal{F}(\mathrm{div}\;u|_{\Gamma_h})(\xi)&=ik_p^2P(\xi)\,,\end{aligned}$$

after some elementary calculations we obtain

$$\begin{aligned}&\int_{\Gamma_h}\left\{2\,\mathrm{Re}\,(\mathcal{T}\gamma_-u\cdot\partial_2\overline{u})-\mathcal{E}(u,\overline{u})+\omega^2|u|^2\right\}ds\\ =\;&2\omega^2\left(\int_{|\xi|<k_p}\gamma_p^2(\xi)\,|P(\xi)|^2\,d\xi+\int_{|\xi|<k_s}\gamma_s^2(\xi)\,|S(\xi)|^2\,d\xi\right)\end{aligned}\tag{5.12}$$

and

$$\mathrm{Im}\int_{\Gamma_h}\mathcal{T}\gamma_-u\cdot\overline{u}ds=\omega^2\left(\int_{|\xi|<k_p}\gamma_p(\xi)\,|P(\xi)|^2\,d\xi+\int_{|\xi|<k_s}\gamma_s(\xi)\,|S(\xi)|^2\,d\xi\right).\tag{5.13}$$

Here $\mathcal{T}$ denotes the Dirichlet-to-Neumann operator (3.10). Using (5.12) and (5.13), and taking the imaginary part of (5.6), we get

$$\begin{aligned} \int_{\Gamma_h} \left\{ 2\,\mathrm{Re}\,(\mathcal{T}\gamma_- u \cdot \partial_2 \overline{u}) - \mathcal{E}(u,\overline{u}) + \omega^2 |u|^2 \right\} ds &\leq 2k_s\, \mathrm{Im} \int_{\Gamma_h} \mathcal{T}\gamma_- u \cdot \overline{u}\, ds \\ &= 2k_s\, \mathrm{Im} \int_{S_h} \tilde{g} \cdot \overline{u}\, dx \,. \end{aligned} \tag{5.14}$$

Combining (5.11) and (5.14), then gives the estimates

$$\begin{aligned} -\int_{\Gamma} \left\{ n_2\, \mu\, |\partial_{\mathbf{n}} u|^2 + n_2\, (\lambda+\mu)\, |\mathrm{div}\; u|^2 \right\} ds &\leq 2k_s\, \mathrm{Im} \int_{S_h} \tilde{g} \cdot \overline{u}\, dx - 2\,\mathrm{Re} \int_{S_h} \tilde{g} \cdot \partial_2 \overline{u}\, dx \\ &\leq C_2\, ||\tilde{g}||_{L^2(S_h)^2} \left(||u||_{L^2(S_h)^2} + ||\partial_2 u||_{L^2(S_h)^2} \right) \\ &\leq C_3\, ||\tilde{g}||_{L^2(S_h)^2}\, ||\partial_2 u||_{L^2(S_h)^2} \,, \end{aligned} \tag{5.15}$$

where the last inequality follows from (4.7) and the constants $C_2,\, C_3$ only depend on ω and h. Recalling that

$$n_2 = -(1+f'(x_1)^2)^{-1/2} \leq -(1+L^2)^{-1/2} < 0 \quad \text{on} \quad \Gamma\,, \tag{5.16}$$

from (5.15) we obtain

$$||\mathrm{div}\; u||^2_{L^2(\Gamma)} + ||\partial_{\mathbf{n}} u||^2_{L^2(\Gamma)} \;\leq C\, ||\tilde{g}||_{L^2(S_h)^2}\, ||\partial_2 u||_{L^2(S_h)^2} \,, \tag{5.17}$$

with $C > 0$ only depending on ω, h and L. Finally, it is easy to check, using $u = 0$ on Γ and the identities in (5.9), that

$$n_2\, |\mathrm{curl}\; u|^2 = n_2\, (|\nabla u|^2 - |\mathrm{div}\; u|^2) = n_2\, (|\partial_{\mathbf{n}} u|^2 - |\mathrm{div}\; u|^2) \quad \text{on} \quad \Gamma\,.$$

Thus $||\mathrm{curl}\; u||^2_{L^2(\Gamma)}$ can also be bounded by the right hand side of (5.17). □

Remark 5 *If $\tilde{g} = 0$ in S_h, then it follows from Lemma 6 that $u = \partial_{\mathbf{n}} u = 0$ on Γ. Thus the uniqueness to (BVP) is a direct consequence of Holmgren's uniqueness theorem if Γ is the graph of a smooth function satisfying (2.1). Furthermore, the uniqueness can be extended to a Lipschitz graph using the approximation arguments from Lemma 8 in subsection 5.2 below.*

To continue the proof of Lemma 5, we now choose some $H > h$ and derive estimates for the L^2 norms of the scalar functions $\mathrm{div}\; u$ and $\mathrm{curl}\; u$ on the artificial boundary Γ_H and the strip S_H. Define the functions (see (2.4))

$$\varphi := -(i/k_p^2)\, \mathrm{div}\; u, \quad \psi := (i/k_s^2)\, \mathrm{curl}\; u, \tag{5.18}$$

where u denotes a solution of problem (5.6) extended to D by the UPRC (2.11). By (5.7) the functions φ and ψ defined in (5.18) satisfy the inhomogeneous Helmholtz equations

$$(\Delta + k^2)\, w = \tilde{g}_k \quad \text{in} \quad S_H\,, \tag{5.19}$$

with $\tilde{g}_k = 0$ in $U_h \backslash U_H$ and

$$\begin{aligned} w = \varphi\,, \quad \tilde{g}_k = -(i/\omega^2)\, \mathrm{div}\; \tilde{g} \quad \text{in} \quad S_h\,, \quad \text{for} \quad k = k_p\,, \\ w = \psi\,, \quad \tilde{g}_k = (i/\omega^2)\, \mathrm{curl}\; \tilde{g} \quad \text{in} \quad S_h\,, \quad \text{for} \quad k = k_s\,. \end{aligned} \tag{5.20}$$

Moreover, for each $c \in [h, H]$, it follows from (2.11) and (2.12) that w satisfies the corresponding upward propagating radiation condition for the Helmholtz equation (see [14])

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(i\sqrt{k^2-\xi^2}\,(x_2 - c) + ix_1\xi)\,\hat{w}_c(\xi)\,d\xi\,, \quad x_2 > c\,, \tag{5.21}$$

where $\hat{w}_c = \mathcal{F}w_c$ denotes the Fourier transform of $w_c = w|_{\Gamma_c}$. Note that we have $\tilde{g}_k \in L^2(S_H)$ and $w \in L^2(S_H) \cap H^2_{loc}(S_H) \cap H^1(U_h \backslash U_H)$ in each case of (5.20), since $u \in V_H$ and $\tilde{g} \in V_h$. On the artificial boundary Γ_H, the Dirichlet-to-Neumann operator for the problem (5.19)–(5.21) takes the form

$$\tilde{\mathcal{T}} v = \mathcal{F}^{-1}\,(i\sqrt{k^2-\xi^2}\,\mathcal{F}v)\,, \quad v \in H^{1/2}(\Gamma_H)\,, \tag{5.22}$$

and is a bounded linear map of $H^{1/2}(\Gamma_H)$ into $H^{-1/2}(\Gamma_H)$; see [14, Lemma 2.4]. It follows from Lemma 6 that we can estimate the L^2 norm of the trace of w on Γ as

$$||w||^2_{L^2(\Gamma)} \le C\,||\tilde{g}||_{L^2(S_H)^2}\,||\partial_2 u||_{L^2(S_H)^2}\,, \tag{5.23}$$

where C only depends on ω, H and L. The following lemma, which is an extension of [28, Lemma 5.2] to the case of non-periodic rough surfaces, is needed to prove corresponding estimates for w on S_H and the trace of w on Γ_H.

Lemma 7 *Let $H > h$, and suppose that $\tilde{g}_k \in L^2(S_H)$, $\tilde{g}_k = 0$ in $U_h \backslash U_H$, and that $w \in L^2(S_H) \cap H^2_{loc}(S_H) \cap H^1(U_h \backslash U_H)$ is a solution of the problem (5.19)–(5.21). Then there holds*

$$||w||_{L^2(\Gamma_H)} \le C\,||w||_{L^2(S_H)} \le \tilde{C}\,(||w||_{L^2(\Gamma)} + ||\tilde{g}_k||_{L^2(S_h)}) \tag{5.24}$$

for some constants C, $\tilde{C} > 0$ only depending on ω, h, H and the Lipschitz constant L of Γ.

Proof. By (5.21) and (5.22), the trace of w on Γ_H satisfies the relation $\partial_2 w = \tilde{\mathcal{T}}\gamma_- w$. To estimate the L^2 norm of w on the strip S_H, we consider the boundary value problem of finding $v \in V_H$ such that

$$(\Delta + k^2)\,v = \overline{w} \quad \text{in} \quad S_H\,, \quad v = 0 \quad \text{on} \quad \Gamma, \quad \partial_2 v = \tilde{\mathcal{T}}\gamma_- v \quad \text{on} \quad \Gamma_H. \tag{5.25}$$

It follows from [14, Lemma 4.6] that problem (5.25) is well-posed, with the unique solution v satisfying the bound

$$||v||_{V_H} \le C_1\,||w||_{L^2(S_H)}, \quad C_1 = C_1(\omega, H) > 0. \tag{5.26}$$

We first prove that $||\partial_{\mathsf{n}} v||^2_{L^2(\Gamma)} \le C_2\,||w||^2_{L^2(S_H)}$ for some constant $C_2 > 0$ only depending on ω, H and the Lipschitz constant L of Γ. This estimate will be verified using the following Rellich identity for the Helmholtz equation,

$$2\,\mathrm{Re}\int_{S_H} \partial_2\overline{v}\,(\Delta v + k^2 v)\,dx = \left(\int_\Gamma + \int_{\Gamma_H}\right)\left\{2\,\mathrm{Re}\,(\partial_{\mathsf{n}} v\,\partial_2\overline{v}) - n_2|\nabla v|^2 + n_2 k^2|v|^2\right\}ds\,, \tag{5.27}$$

which is just the analogue of the identity (5.8) and can be proved in the same way. Furthermore, it holds that (see the proof of Lemma 4.6 in [14])

$$\begin{aligned}\int_{\Gamma_H}\left\{2\,\mathrm{Re}\,(\partial_{\mathsf{n}} v\,\partial_2\overline{v}) - n_2|\nabla v|^2 + n_2 k^2|v|^2\right\}ds &\le 2k\,\mathrm{Im}\int_{\Gamma_H}\overline{v}\,\tilde{\mathcal{T}}\gamma_- v\,ds\\ &\le 2k\,\mathrm{Im}\int_{S_H}\overline{v}\,\overline{w}\,dx,\end{aligned} \tag{5.28}$$

and that

$$
\begin{aligned}
-\int_\Gamma \left\{2\,\mathrm{Re}\,(\partial_{\mathbf{n}} v\,\partial_2\overline{v}) - n_2|\nabla v|^2 + n_2k^2|v|^2\right\} ds &= -\int_\Gamma n_2|\partial_{\mathbf{n}} v|^2\, ds \\
&\geq \frac{1}{\sqrt{1+L^2}}||\partial_{\mathbf{n}} v||^2_{L^2(\Gamma)},
\end{aligned} \tag{5.29}
$$

using the equalities in (5.9) and the bound for n_2 in (5.16). Inserting (5.28) and (5.29) into (5.27) and then using (5.26), we get the estimates

$$
\begin{aligned}
||\partial_{\mathbf{n}} v||^2_{L^2(\Gamma)} &\leq C_3\left\{-2\,\mathrm{Re}\int_{S_H}\overline{w}\,\partial_2\overline{v}\,dx + 2k\,\mathrm{Im}\int_{S_H}\overline{v}\,\overline{w}\,dx\right\} \\
&\leq C_4\,||w||_{L^2(S_H)}||v||_{V_H} \leq C_2||w||^2_{L^2(S_H)}\,,
\end{aligned} \tag{5.30}
$$

where the constants C_3 and C_4 only depend on ω, H and L. We next prove the second inequality in (5.24). Define the cut-off function χ_A as in the proof of Lemma 6. By Green's formula, we then have

$$
\begin{aligned}
&\int_{S_H}\{w\chi_A\,\Delta v - v\,\Delta(\chi_A w)\}\,dx \\
=\;& \int_{S_H^{A+1}}\{w\chi_A\,\Delta v - v\,\Delta(\chi_A w)\}\,dx \\
=\;& \int_{\partial S_H^{A+1}}\{w\chi_A\,\partial_{\mathbf{n}} v - v\,\partial_{\mathbf{n}}(\chi_A w)\}\,ds \\
=\;& \int_{\partial S_H^{A+1}}\{w\chi_A\,\partial_{\mathbf{n}} v - vw\,\partial_{\mathbf{n}}\chi_A - \chi_A\,v\,\partial_{\mathbf{n}} w\}\,ds \\
=\;& \left(\int_{\Gamma_H^{A+1}} + \int_{\Gamma^{A+1}}\right)\{w\chi_A\,\partial_{\mathbf{n}} v - vw\,\partial_{\mathbf{n}}\chi_A - \chi_A\,v\,\partial_{\mathbf{n}} w\}\,ds,
\end{aligned}
$$

where the sets S_H^{A+1}, Γ_H^{A+1}, Γ^{A+1}, $A \geq 1$, are the intersections of S_H, Γ_H and Γ with the disk $\{|x| \leq A+1\}$, respectively. Letting $A \to +\infty$ and making use of the relations $v \in V_H$, $\Delta v \in L^2(S_H)$, $\partial_{\mathbf{n}} v|_\Gamma \in L^2(\Gamma)$ and $w \in L^2(S_H) \cap H^1(U_h\backslash U_H)$, $\nabla w|_{\Gamma_H} \in L^2(\Gamma_H)^2$, we obtain (cf. the proof of Lemma 4.6 in [14])

$$
\begin{aligned}
\int_{S_H}\{w\,\Delta v - v\,\Delta w\}\,dx &= \int_{\Gamma_H}\{w\,\partial_{\mathbf{n}} v - v\,\partial_{\mathbf{n}} w\}\,ds + \int_\Gamma w\,\partial_{\mathbf{n}} v\,ds \\
&= \int_{\Gamma_H}\left\{w\,\tilde{\mathcal{T}}\gamma_- v - v\,\tilde{\mathcal{T}}\gamma_- w\right\}ds + \int_\Gamma w\,\partial_{\mathbf{n}} v\,ds \\
&= \int_\Gamma w\,\partial_{\mathbf{n}} v\,ds\,.
\end{aligned}
$$

Note that $v = 0$ on Γ, and the Dirichlet-to-Neumann operator $\tilde{\mathcal{T}}$ defined in (5.22) is symmetric (see [14, Lemma 3.2]). Thus

$$
\begin{aligned}
\int_{S_H}|w|^2\,dx &= \int_{S_H} w\,(\Delta v + k^2 v)\,dx \\
&= \int_{S_H} v\,(\Delta w + k^2 w)\,dx + \int_\Gamma w\,\partial_{\mathbf{n}} v\,ds \\
&= \int_{S_h} v\,\tilde{g}_k\,dx + \int_\Gamma w\,\partial_{\mathbf{n}} v\,ds.
\end{aligned}
$$

Together with (5.26) and (5.30), this implies, with C_5 only depending on ω, H and L,

$$||w||^2_{L^2(S_H)} \leq ||v||_{L^2(S_H)}\, ||\tilde{g}_k||_{L^2(S_h)} + ||w||_{L^2(\Gamma)}\, ||\partial_{\mathbf{n}} v||_{L^2(\Gamma)}$$
$$\leq C_5\, ||w||_{L^2(S_H)}\, (||\tilde{g}_k||_{L^2(S_h)} + ||w||_{L^2(\Gamma)}).$$

This proves the inequality

$$||w||_{L^2(S_H)} \leq C_5\, (||w||_{L^2(\Gamma)} + ||\tilde{g}_k||_{L^2(S_h)}). \tag{5.31}$$

To prove the first inequality in (5.24), we use the estimate

$$\int_{\Gamma_H} |w|^2\, ds \leq \int_{\Gamma_c} |w|^2\, ds\,, \quad \text{for all } c \in (h, H]\,,$$

which follows from (5.21), (5.22) and the Plancherel identity; see also the proof of Lemma 2.2 in [14]. Thus we obtain the bound

$$(H-h) \int_{\Gamma_H} |w|^2\, dx \leq \int_{U_h \backslash U_H} |w|^2\, ds \leq \int_{S_H} |w|^2\, ds\,. \tag{5.32}$$

Combining (5.31) and (5.32), we then get the desired estimate (5.24). □

To prove the desired bounds of the L^2 norms of $\operatorname{div} u$ and $\operatorname{curl} u$ on S_H and Γ_H in terms of the right hand side of equation (5.7), we now combine the estimates of Lemmas 6 and 7. In fact, applying Lemma 7 to $w = \varphi$ and $w = \psi$ and the corresponding right hand side $\tilde{g}_k$ of (5.19), and then using the bound (5.23), we obtain the estimate

$$||\operatorname{div} u||^2_{L^2(S_H)} + ||\operatorname{curl} u||^2_{L^2(S_H)} \leq C||\tilde{g}||_{V_H} \left(||\tilde{g}||_{V_H} + ||\partial_2 u||_{L^2(S_H)^2}\right). \tag{5.33}$$

Analogously, the estimates (5.24) and (5.23) imply the bound

$$||\operatorname{div} u||^2_{L^2(\Gamma_H)} + ||\operatorname{curl} u||^2_{L^2(\Gamma_H)} \leq C||\tilde{g}||_{V_H} \left(||\tilde{g}||_{V_H} + ||\partial_2 u||_{L^2(S_H)^2}\right). \tag{5.34}$$

Here the constant C only depends on ω, h, H and L.

End of proof of Lemma 5. To deduce the bound (5.2) from the estimates (5.33) and (5.34), we need another Rellich identity for the Navier equation. Note that the Rellich identity (5.8) is not sufficient for our purposes. Using Betti's formula (3.3) on the strip S_H and employing the cut-off function χ_A as in the proof of Lemma 6, integration by parts yields

$$\begin{aligned}
&2\,\mathrm{Re} \int_{S_H} (\Delta^* + \omega^2) u \cdot (x_2 - f_-)\, \partial_2 \overline{u}\, dx \\
&= \int_{S_H} \left\{ \mathcal{E}(u, \overline{u}) - 2\,\mathrm{Re} \left\{ \sum_{j=1,2} \mathcal{E}(u, (x_2 - f_-) e_j)\, \partial_2 \overline{u}_j \right\} - \omega^2 |u|^2 \right\} dx \\
&+ \left(\int_{\Gamma_H} + \int_{\Gamma} \right) \left[-n_2 \mathcal{E}(u, \overline{u}) + 2\,\mathrm{Re}\,(Tu \cdot \partial_2 \overline{u}) + n_2 \omega^2 |u|^2 \right] (x_2 - f_-)\, ds\,;
\end{aligned} \tag{5.35}$$

recall that e_j denotes the unit vector in x_j-direction, and T is the stress operator defined in (3.2). Note that (5.35) extends the Rellich identity used in [14] to the case of elastic rough surface scattering. Using (5.33), (5.34), (5.35) and appropriate estimates of the sesquilinear form B in (5.6), we can now finish the proof of Lemma 5. Recalling (4.4), we rewrite the variational formulation (5.6) on S_H as

$$\begin{aligned}
&\int_{S_H} \left\{ \mathcal{E}(u, \overline{u}) - \omega^2 |u|^2 \right\} dx - \int_{|\xi| > k_s} M(\xi)\, \hat{u}_H(\xi) \cdot \overline{\hat{u}}_H(\xi)\, d\xi \\
&= - \int_{S_H} \tilde{g} \cdot \overline{u}\, dx + \int_{|\xi| \leq k_s} M(\xi)\, \hat{u}_H(\xi) \cdot \overline{\hat{u}}_H(\xi)\, d\xi\,.
\end{aligned} \tag{5.36}$$

where $u_H = u|_{\Gamma_H}$. From (5.36) and Lemma 2 (i),

$$\int_{S_H} \left\{ \mathcal{E}(u,\overline{u}) - \omega^2 |u|^2 \right\} dx \le - \int_{S_H} \tilde{g}\cdot \overline{u}\, dx + \int_{|\xi|\le k_s} M(\xi)\, \hat{u}_H(\xi)\cdot \overline{\hat{u}}_H(\xi)\, d\xi\,. \tag{5.37}$$

Rearranging the terms in (5.35), and then using (5.37) and (5.10), we arrive at

$$\begin{aligned}
&\int_{S_H} 2\,\mathrm{Re}\Big\{ \sum_{j=1,2} \mathcal{E}(u,(x_2-f_-)e_j)\,\partial_2\overline{u}_j \Big\} dx - \int_\Gamma (x_2-f_-)\left\{ \mu\,|\partial_{\mathsf{n}} u|^2 + (\lambda+\mu)\,|\mathrm{div}\, u|^2 \right\} n_2\, ds \\
&= \int_{S_H} \left\{ \mathcal{E}(u,\overline{u}) - \omega^2|u|^2 \right\} dx - 2\,\mathrm{Re}\int_{S_H} (\Delta^* + \omega^2)u\cdot (x_2-f_-)\,\partial_2\overline{u}\, dx \\
&+ (H-f_-)\int_{\Gamma_H} \left\{ 2\,\mathrm{Re}\,(Tu\cdot\partial_2\overline{u}) - \mathcal{E}(u,\overline{u}) + \omega^2|u|^2 \right\} ds \\
&\le \int_{S_H} \left\{ -\tilde{g}\cdot u + 2\,\mathrm{Re}\,(\tilde{g}\cdot\partial_2\overline{u})\,(x_2-f_-) \right\} dx + \int_{|\xi|\le k_s} M(\xi)\,\hat{u}_H(\xi)\cdot\overline{\hat{u}}_H(\xi)\, d\xi \\
&+ (H-f_-)\int_{\Gamma_H} \left\{ 2\,\mathrm{Re}\,(Tu\cdot\partial_2\overline{u}) - \mathcal{E}(u,\overline{u}) + \omega^2|u|^2 \right\} ds\,.
\end{aligned} \tag{5.38}$$

We now estimate the second term on the right hand side of (5.38). Since $||M(\xi)|| \le C_1$ for all $|\xi| < k_s$, with some constant $C_1 > 0$ only depending on ω (cf. Lemma 2 (ii)), there holds

$$\int_{|\xi|\le k_s} M(\xi)\hat{u}_H(\xi)\cdot\overline{\hat{u}}_H(\xi)\, d\xi \le C_1 \int_{|\xi|\le k_s} |\hat{u}_H(\xi)|^2\, d\xi\,. \tag{5.39}$$

Let $P_H(\xi)$, $S_H(\xi)$ denote the Fourier transforms of $(-i/k_p^2)\,\mathrm{div}\, u(x_1,H)$ and $(i/k_s^2)\,\mathrm{curl}\, u(x_1,H)$ respectively, given by (see (2.6))

$$P_H(\xi) = P(\xi)\, \exp(i(H-h)\,\gamma_p(\xi))\,, \quad S_H(\xi) = S(\xi)\, \exp(i(H-h)\,\gamma_s(\xi))\,.$$

Then $\hat{u}_H(\xi)$ is related with $(P_H(\xi), S_H(\xi))^\top$ via the equality (see (2.8))

$$\begin{pmatrix} \hat{u}_{H,1}(\xi) \\ \hat{u}_{H,2}(\xi) \end{pmatrix} = \begin{pmatrix} \xi & \gamma_s \\ \gamma_p & -\xi \end{pmatrix} \begin{pmatrix} P_H(\xi) \\ S_H(\xi) \end{pmatrix}.$$

Thus, from (5.39), (5.34) and the Plancherel identity we get the estimates

$$\begin{aligned}
\int_{|\xi|\le k_s} M(\xi)\hat{u}_H(\xi)\cdot\overline{\hat{u}}_H(\xi)\, d\xi &\le C_2 \int_{|\xi|\le k_s} \left\{ |P_H(\xi)|^2 + |S_H(\xi)|^2 \right\} d\xi \\
&\le C_2 \left(||P_H||^2_{L^2(\mathbb{R})} + ||S_H||^2_{L^2(\mathbb{R})} \right) \\
&\le C_3\, ||\tilde{g}||_{V_H}\, (||\tilde{g}||_{V_H} + ||\partial_2 u||_{L^2(S_H)^2})\,,
\end{aligned} \tag{5.40}$$

where the constants C_2 and C_3 depend on ω, h, H and L.

Furthermore, from the estimates (5.14) and (4.7) we obtain the following bound for the last term of (5.38),

$$\begin{aligned}
\int_{\Gamma_H} \left\{ 2\mathrm{Re}\,(Tu\cdot\partial_2\overline{u}) - \mathcal{E}(u,\overline{u}) + \omega^2|u|^2 \right\} ds &\le 2k_s\, \mathrm{Im}\int_{S_H} \tilde{g}\cdot\overline{u}\, dx \\
&\le 2k_s\, ||\tilde{g}||_{V_H}\, ||u||_{L^2(S_H)^2} \\
&\le C_4\, ||\tilde{g}||_{V_H}\, ||\partial_2 u||_{L^2(S_H)^2}\,,
\end{aligned} \tag{5.41}$$

with $C_4 > 0$ only depending on ω and H. Combining (5.40), (5.41) and (5.38), we then arrive at

$$\int_{S_H} 2\text{Re}\Big\{\sum_{j=1,2} \mathcal{E}(u,(x_2-f_-)e_j)\,\partial_2\overline{u}_j\Big\}\,dx \le C_5\left(||\tilde{g}||^2_{V_H} + ||\tilde{g}||_{V_H}\,||\partial_2 u||_{L^2(S_H)^2}\right), \tag{5.42}$$

where the constant C_5 depends on ω, h, H and L. Note that the second term in (5.38) is non-negative. We further have the easily verified relation

$$\begin{aligned} &\int_{S_H} 2\text{Re}\Big\{\sum_{j=1,2} \mathcal{E}(u,(x_2-f_-)e_j)\,\partial_2\overline{u}_j\Big\}\,dx \\ =\;& 2(\lambda+2\mu)\,||\partial_2 u_2||^2_{L^2(S_H)} + 2\mu\,||\partial_2 u_1||^2_{L^2(S_H)} + 2(\lambda+\mu)\,\text{Re}\int_{S_H} \partial_1 u_1\,\partial_2\overline{u}_2\,dx, \end{aligned}$$

which implies, on choosing $C > 0$ sufficiently large,

$$\begin{aligned} &\int_{S_H} 2\,\text{Re}\Big\{\sum_{j=1,2} \mathcal{E}(u,(x_2-f_-)e_j)\,\partial_2\overline{u}_j\Big\}\,dx + C\,||\text{div}\,u||^2_{L^2(S_H)^2} + C\,||\text{curl}\,u||^2_{L^2(S_H)^2} \\ =\;& [C+2(\lambda+2\mu)]\;||\partial_2 u_2||^2_{L^2(S_H)^2} + C\,||\partial_1 u_1||^2_{L^2(S_H)^2} \\ & +2(C+\lambda+\mu)\,\text{Re}\int_{S_H} \partial_1 u_1\,\partial_2\overline{u}_2\,dx \\ & +(C+2\mu)\,||\partial_2 u_1||^2_{L^2(S_H)^2} + C\,||\partial_1 u_2||^2_{L^2(S_H)^2} - 2C\,\text{Re}\int_{S_H} \partial_1 u_2\,\partial_2\overline{u}_1\,dx \\ \ge\;& C_6\,||\nabla u||^2_{L^2(S_H)^2}\,, \end{aligned} \tag{5.43}$$

where the constant C_6 only depends on the Lamé constants λ and μ. Combining (5.33), (5.42) and (5.43), and using Young's inequality gives

$$||\nabla u||_{L^2(S_H)^2} \le C_7\;||\tilde{g}||_{V_H}\,, \tag{5.44}$$

with $C_7 > 0$ depending on ω, h, H and L. Together with the estimate (4.7) (for the strip S_H), (5.44) finally yields the inequality $||u||_{V_H} \le C_0\;||\tilde{g}||_{V_H}$, hence $||u||_{V_h} \le C_0\;||\tilde{g}||_{V_h}$ in view of $H > h$ and $\tilde{g} = 0$ in $U_h\backslash U_H$. Choosing H, say $H = h+1$, we see that the constant $C_0 > 0$ only depends on ω, h and the Lipschitz constant L. This completes the proof of Lemma 5. □

Let $\mathcal{B} = \mathcal{B}(\omega)$ be the operator defined in (3.18). Combining Lemmas 4 and 5, we now obtain the a priori estimate

$$||\mathcal{B}u||_{V_h^*} \ge C\,||u||_{V_h}\,, \quad u \in V_h\,, \tag{5.45}$$

where the constant C only depends on ω, h and the Lipschitz constant L of the rough surface Γ. Together with Lemma 3 and Theorem 2, the estimate (5.45) then implies Theorem 1 in the case of smooth rough surfaces.

5.2 A priori estimate for Lipschitz rough surfaces

Having established the a priori estimate (5.2) for smooth rough surfaces in subsection 5.1, we now adapt Nečas' method [38, Chap. 5] of approximating a Lipschitz graph by smooth surfaces to justify the a priori estimate (5.2) in the general case. Similar arguments are employed in [28] for the scalar Helmholtz equation and in [24, 25] for the Navier equation in the periodic case.

Lemma 8 *Suppose that Γ is given by the graph of a Lipschitz function f satisfying (2.1), $h > f_+$, $\tilde{g} \in V_h$, and that $u \in V_h$ is a solution of the problem (5.6), i.e.,*

$$B(u,v) = (\mathcal{B}u, v)_{S_h} = -\int_{S_h} \tilde{g} \cdot \overline{v}\, dx, \quad v \in V_h.$$

Then there exists a constant $C_0 > 0$ independent of u and $\tilde{g}$ such that $||u||_{V_h} \le C_0\, ||\tilde{g}||_{V_h}$.

Proof. We first approximate the Lipschitz function f by smooth functions. Choose C^∞-smooth functions f_m such that (see e.g. [41, Theorem 3.10])

$$\begin{aligned}
&\Gamma_m := \{x : x_2 = f_m(x_1), x_1 \in \mathbb{R}\} \subset S_h\,, \quad m \in \mathbb{N}\,,\\
&\sup\{|f_m(x_1) - f(x_1)| : x_1 \in \mathbb{R}\} \to 0, \quad \text{as} \quad m \to \infty\,,\\
&|f_m(x_1) - f_m(x_2)| \le L|x_1 - x_2|\,, \quad \text{for all } x_1, x_2 \in \mathbb{R} \text{ and } m \in \mathbb{N}\,,
\end{aligned}$$

where $L > 0$ is the Lipschitz constant of f (cf. (2.1)). Define the strip S_h^m, the space V_h^m and the operator $\mathcal{B}_m$ as S_h, V_h and $\mathcal{B}$ respectively, with Γ replaced by Γ_m. Then it follows from the proof of Lemma 4 that, for some sufficiently large number $\alpha > 0$, there exists a unique solution $w_m \in V_h^m$ to the problem

$$(\mathcal{B}_m w_m + \alpha w_m, v)_{S_h^m} = \int_{S_h^m} (-\tilde{g} + \alpha u) \cdot \overline{v}\, dx\,, \quad v \in V_h^m\,, \tag{5.46}$$

for all $m \in \mathbb{N}$. Note that $S_h^m \subset S_h$ since $\Gamma_m \subset S_h$. We extend the functions w_m by zero from S_h^m to S_h, and regard V_h^m as a subspace of V_h. Then $w_m \in V_h$ is also the unique solution of the variational problem

$$(\mathcal{B}_\alpha w_m, v)_{S_h} = \int_{S_h} (-\tilde{g}_m + \alpha u_m) \cdot \overline{v}\, dx\,, \quad v \in V_h\,, \tag{5.47}$$

where $\mathcal{B}_\alpha := \mathcal{B} + \alpha I$ and u_m, $\tilde{g}_m \in L^2(S_h)^2$ denote the extensions of $u|_{S_h^m}$ and $\tilde{g}|_{S_h^m}$, respectively, to S_h by zero. Furthermore, the operator $\mathcal{B}_\alpha : V_h \to V_h^*$ is also invertible for the chosen $\alpha > 0$; see the proof of Lemma 4 again. Thus the problem

$$(\mathcal{B}_\alpha w, v)_{S_h} = \int_{S_h} (-\tilde{g} + \alpha u) \cdot \overline{v}\, dx, \quad v \in V_h\,, \tag{5.48}$$

has the unique solution $w = u \in V_h$. By the uniform convergence of the functions f_m to f, we have $u_m \to u$ and $\tilde{g}_m \to \tilde{g}$ in $L^2(S_h)^2$ as $m \to \infty$. Therefore (cf. (5.47) and (5.48)),

$$||u - w_m||_{V_h} \le ||\mathcal{B}_\alpha^{-1}||_{V_h^* \to V_h}\, ||\alpha(u - u_m) + (\tilde{g}_m - \tilde{g})||_{L^2(S_h)^2} \to 0\,, \quad m \to \infty\,,$$

implying that

$$w_m \to u \quad \text{in} \quad V_h, \quad m \to \infty. \tag{5.49}$$

Note that $||v||_{V_h^*} \le ||v||_{L^2(S_h)} \le ||v||_{S_h}$ for all $v \in V_h$.

We rewrite the variational problem (5.46) for w_m as

$$(\mathcal{B}_m w_m, v)_{S_h^m} = \int_{S_h^m} l_m \cdot \overline{v}\, dx\,, \quad v \in V_h^m\,, \quad l_m := -\tilde{g}_m + \alpha(u - w_m).$$

Applying Lemmas 4 and 5 to the operators $\mathcal{B}_m : V_h^m \to (V_h^m)^*$ that correspond to the smooth rough surfaces Γ_m with uniformly bounded Lipschitz constants $L_m \le L$, we obtain the estimates

$$\begin{aligned} ||w_m||_{V_h^m} \le C_1\, ||l_m||_{(V_h^m)^*} \le C_1\, ||l_m||_{L^2(S_h^m)^2} \le C_1\, ||\tilde{g}_m||_{L^2(S_h^m)^2} \\ + C_1\alpha||u - w_m||_{L^2(S_h^m)^2} \end{aligned} \tag{5.50}$$

with some constant $C_1 > 0$ independent of m. Letting $m \to \infty$ in (5.50) and using (5.49), it follows that

$$||u||_{V_h} \le C_1\, ||\tilde{g}||_{L^2(S_h)^2} \le C_1\, ||\tilde{g}||_{V_h}\,,$$

which finishes the proof of Lemma 8. □

Combining Lemmas 4 and 8, we get the a priori estimate (5.45) for a Lipschitz surface Γ, so that Theorem 1 is proven in the general case. We note that, for sound-soft acoustic scattering, existence and uniqueness of solutions has been proved in [14] for a more general class of non-smooth rough surfaces, including a priori estimates of the form (5.45) with an explicit stability constant C in terms of the wave number and the width of the strip S_h. The extension of these more general and precise results to elastic scattering remains an open problem.

We further note that the uniqueness and existence results obtained in this paper can be extended to three-dimensional elastic rough surface scattering problems, including the case of impedance boundary conditions. Moreover, following the variational approach of [11] in appropriate weighted Sobolev spaces, the problem of plane elastic wave incidence in the two-dimensional case and the problem of incident spherical and cylindrical elastic waves in the three-dimensional case can also be treated. These results will be presented in subsequent publications.

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