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Modelling, Analysis and Simulation of Stochastic Innovation Diffusion

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Abstract

The well-known BASS model for description of diffusion of innovations has been extensively investigated within deterministic framework. One of the basic processes in modelling of these diffusions concerns with the propagation through word of mouth which is inherently nonlinear. As a more realistic modelling, the diffusion of an innovation in the presence of uncertainty is generally formulated in terms of nonlinear stochastic differential equations (SDEs). At first we discuss well-posedness, regularity (boundedness) and uniqueness of solutions of these SDEs. However, an explicit expression for analytical solution itself is not available. Accordingly one has to resort to numerical solution of SDEs for studying various aspects like the time-development of growth patterns, exit frequencies, mean passage times and impact of advertising policies. In this respect we present some basic aspects of numerical analysis of these random extensions of the BASS model, e.g. numerical regularity and mean square convergence. Therein the problem of numerical movement within reasonable boundaries (numerical solution on bounded manifolds) plays a significant role, in particular on intervals with reflecting or absorbing barriers, whereas the discretization of the state space (continuous time version of the set of possible adopters of the innovation) is circumvented. Such a study brings out salient features of the stochastic models (as e.g. boundedness of equilibria, faster initial adoption or earlier peak sales in comparison with deterministic model). To this end we shall provide discrete time estimations of the moment evolution and pathwise solutions based on Balanced implicit methods (see Mil'shtein et al. (1992)).

Keywords. Innovation diffusion; Bass model; Stochastic differential equations; Algebraic constraints; Regularity; Lyapunov–type methods; Numerical methods; Balanced implicit methods; Numerical regularization; Boundedness; Mean square convergence.

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1. Introduction to General Model

Nonlinear innovation diffusions play a significant role in Marketing and Social Sciences, e.g. see [3], [10] or [15]. One of the commonly accepted models is the BASS MODEL. Bass [4] suggested to model how a product, technology, news, ideas, rumours, etc. diffuse in a given deterministic media. This model admits to describe the number of adoptions X_t in terms of nonlinear differential equations. Bass model is purely deterministic and only lives within algebraic constraints by its own nature. In reality of Marketing Sciences, this diffusion of innovation undergoes environmental and parametric noise which can be also interpreted as an expression of a certain amount of uncertainty in modelling real phenomena. Stochastic generalizations of Bass model in terms of nonlinear Stochastic Differential Equations (SDEs), e.g. see Karmeshu et al. [7], have recently led to

$$dX_t = [(p + \frac{q}{M}X_t)(M - X_t)]\,dt + \sigma X_t^{\alpha}(M - X_t)^{\beta}\,dW_t \tag{1}$$

driven by a given standard Wiener process W_t, starting in $X_0 \in \mathbb{D} = [0, M] \subset \mathbb{R}^1$, where p, q, M, σ are positive and α, β nonnegative real parameters. There p can be understood as coefficient of innovation, q as coefficient of immitation and M as total adoption size. However, in view of marketing issues, model (1) makes only sense within deterministic algebraic constraints. This fact generally leads to Stochastic Differential Algebraic Equations (SDAEs) with nonanticipating boundary conditions. In the following exposition we study **regularity** (boundedness on $\mathbb{D}$) of both exact and numerical solutions of (1). Eventually we construct a numerical solution which exclusively possesses values in $\mathbb{D}$ and is mean square converging with order $\gamma = 0.5$ towards the exact solution. Note that usual numerical methods as most–used Euler method fail to live a.s. on bounded domain $\mathbb{D}$ for any choice of constant step sizes (for examples see [18]). Besides, socalled 'higher order methods', as systematically developed in [21], can not be applied to general model (1), since they require too much boundedness and smoothness on drift and diffusion coefficients of SDEs, what is not given within the general framework of model (1).

2. Analytical Nonregularity and Regularity

At first we recall the notion of regularity of continuous time stochastic processes. Let $\mathbb{D} \subset \mathbb{R}^d$ be a fixed closed domain. Note, for simplicity, we exclusively consider deterministic domains $\mathbb{D}$ in this exposition.

Definition 1. A continuous time stochastic process $\{X(t), t \geq 0\}$ is called **regular on $\mathbb{D}$** (or regular with respect to $\mathbb{D}$) iff $\mathbb{P}\,(X(t) \in \mathbb{D}) = 1\ \forall t \geq 0$, otherwise **nonregular with respect to $\mathbb{D}$.**

2.1. Nonregularity of Additive Noise. Intuitively clear, at first we rigorously show that exact solution of SDEs (1) with additive noise (i.e. $\alpha = \beta = 0$) leaves the bounded domain $\mathbb{D} = [0, M]$. That is, one has to impose algebraic constraints on SDEs (1) which leads to formulation of SDAEs. For this purpose we make use of STOCHASTIC LYAPUNOV–TYPE METHODS. Note that there is an alternative to them given by Feller (see [6]), however only for classification of boundary conditions in $\mathbb{R}^1$. Let $\tau^{s,x} = \tau^{s,x}(\mathbb{D})$ be the random time of first exit of stochastic process $X(t)$ from domain $\mathbb{D}$, starting in $X(s) = x \in \mathbb{D}$ at initial time $s \in [0, +\infty)$. In passing, this random variable is a stopping time with respect to natural filtration $\mathcal{F}_t = \sigma(\{W_u : 0 \leq u \leq t\})$.

Theorem 1. *Assume that $\{X(t), t \geq 0\}$ satisfies SDE (1) with exponent $\alpha = \beta = 0$ and initial value $X(0) = x \in \mathbb{D} = [0, M]$.*

Then $\{X(t), t \geq 0\}$ is nonregular with respect to $\mathbb{D}$. More precisely speaking,

$$\forall x \in \mathbb{D} \quad \exists K = K(x) \, : \, 0 \leq K(x) < +\infty \quad \forall s \geq 0 \quad \forall t > s + K(x)$$

$$\mathbb{P}\left(\tau^{s,x}(\mathbb{D}) < t\right) \; > 0\,. \tag{2}$$

Proof. Define drift $a(x) = (p + \frac{q}{M}x)(M - x)$ and diffusion $b(x) = \sigma$. Introduce the Lyapunov function $V(x) = 1 + x^2, x \in \mathbb{D}$. Note, equation (1) is well–defined, has unique and bounded solution up to random time $\tau^{s,x}(\mathbb{D})$, due to Lipschitz continuity and (linear) boundedness of drift $a(x)$ and diffusion $b(x)$ on $\mathbb{D}$. Let $\mathcal{L}$ denote the generator of diffusion process X_t, i.e.

$$\mathcal{L} \; = \; \frac{\partial}{\partial t} + a(x)\frac{\partial}{\partial x} + \frac{1}{2}b^2(x)\frac{\partial^2}{\partial x^2}\,.$$

Then it holds $V(x) \in C^2(\mathbb{D}), 1 \leq V(x) \leq 1 + M^2$ and $\mathcal{L}V(x) \geq c \cdot V(x)$ where $c = \sigma^2/(1 + M^2)$ and $x \in \mathbb{D}$. Fix initial time $s \geq 0$ and introduce new Lyapunov function $W(t, x) := \exp(-c(t - s))V(x)$ for all $t \geq s \geq 0, x \in \mathbb{D}$. It follows $\mathcal{L}W(t, x) \geq 0$. After applying Dynkin's formula (Itô formula) one obtains $\mathbb{E}\, W(t, X_t) \geq V(x)$ where $X_s = x \in \mathbb{D}$ is deterministic! This fact leads to

$$t - s \; \leq \; \frac{1}{c}\ln\left(\frac{\sup_{y\in\mathbb{D}} V(y)}{V(x)}\right) \; = \; \frac{1 + M^2}{\sigma^2}\ln\left(\frac{1 + M^2}{1 + x^2}\right) \; =: \; K(x) \quad \forall t \geq s \geq 0\,.$$

Therefore $\mathbb{E}\left(\tau^{s,x}(\mathbb{D}) - s\right) \leq K(x)$. Returning to the key assertion of Theorem 1, we finally notice

$$\begin{aligned} 1 - \mathbb{P}\left(\tau^{s,x}(\mathbb{D}) < t\right) \; &= \; \mathbb{P}\left(\tau^{s,x}(\mathbb{D}) - s \geq t - s\right) \\ &\leq \; \frac{1}{t-s}\mathbb{E}\left(\tau^{s,x}(\mathbb{D}) - s\right) \; \leq \; \frac{K(x)}{t-s} \; < \; 1 \quad \text{if } t > s + K(x)\,. \; \diamond \end{aligned}$$

2.2. Regularity of Some Parametric SDEs. Otherwise there is a quite general class of SDEs (1) which provides regular stochastic processes with respect to domain $\mathbb{D} = [0, M]$.

Theorem 2. *Under the presumption $\alpha \geq 0, \beta \geq 1$, the stochastic process $\{X(t), t \geq 0\}$ governed by (1) is regular on $\mathbb{D} = [0, M]$, i.e. it holds $\mathbb{P}\left(X(t) \in [0, M]\right) \; = \; 1 \quad \forall t \geq 0$. Moreover, regularity on $\mathbb{D}$ implies boundedness, uniqueness and Markov property of the exact solution of (1).*

Remark . The proof of Theorem 2 is done using a method described in Khas'minskij [8]. However, the 'art of this technique' consists of finding an appropriate Lyapunov function!

Proof. Define drift $a(x) = (p + \frac{q}{M}x)(M - x)$ and diffusion $b(x) = \sigma x^\alpha (M - x)^\beta$. Take sequence of open domains $\mathbb{D}_n := (\exp(-n), M - \exp(-n)), n \in \mathbb{N}$. Then, equation (1) is well–defined, has unique, bounded and Markovian solution up to random time $\tau^{s,x}(\mathbb{D}_n)$, due to Lipschitz continuity and (linear) boundedness of drift $a(x)$ and diffusion $b(x)$ on $\mathbb{D}_n$. Now, use Lyapunov function $V(x)$ defined on $\mathbb{D}$ via

$$V(x) \; = \; 1 + x^2 + \ln(1 + \frac{1}{x}) + \ln(1 + \frac{1}{M - x})\,.$$

Fix initial time $s \geq 0$ and introduce new Lyapunov function $W(t, x)$ via $W(t, x) \; = \; \exp(-c(t - s))V(x)$ for all $(t, x) \in [s, +\infty) \times \mathbb{D}$ where $c = \sigma^2/(1 + M^2)$. Then $V(x), W(t, x) \in C^{1,2}([s, +\infty) \times \overline{\mathbb{D}_n})$. Define Itô–differential operator (generator of diffusion process X_t)

$$\mathcal{L} \; := \; \frac{\partial}{\partial t} + a(x)\frac{\partial}{\partial x} + \frac{1}{2}b^2(x)\frac{\partial^2}{\partial x^2}\,.$$

$$\text{It holds} \qquad V(x) \geq 1, \quad \inf_{y\in\mathbb{D}\setminus\mathbb{D}_n} V(y) > n, \quad \mathcal{L}V(x) \leq c \cdot V(x) \qquad \forall x \in \mathbb{D}\,.$$

Therefore one may conclude that $\mathcal{L}W(t,x) \leq 0$, since $\mathcal{L}V(x) \leq c \cdot V(x)$. Introduce $\tau_n = \min(\tau^{s,x}(\mathbb{D}_n), t)$. After applying Dynkin's formula (Itô formula), one finds that $\mathbb{E}\, W(\tau_n, X_{\tau_n}) \leq V(x)$ ($X_s = x$ is deterministic!), hence

$$\mathbb{E}\, \exp(c(t-\tau_n))V(X_{\tau_n}) \;\leq\; \exp(c(t-s))V(x)\,.$$

Using this fact, one estimates

$$\begin{aligned}
\mathbb{P}\,(\tau^{s,x}(\mathbb{D}_n) < t) \;&=\; \mathbb{P}\,(\tau_n < t) = \mathbb{E}\, I\!I_{\tau_n < t} \leq \mathbb{E}\, \exp(c(t-\tau_n)) \cdot \frac{V(X_{\tau^{s,x}(\mathbb{D}_n)})}{\inf_{y \in \mathbb{D}\setminus\mathbb{D}_n} V(y)} \cdot I\!I_{\tau_n < t} \\
&\leq\; \mathbb{E}\, \exp(c(t-\tau_n)) \cdot \frac{V(X_{\tau_n})}{\inf_{y\in\mathbb{D}\setminus\mathbb{D}_n} V(y)} \;\leq\; \exp(c(t-s)) \cdot \frac{V(x)}{\inf_{y\in\mathbb{D}\setminus\mathbb{D}_n} V(y)} \\
&\leq\; \exp(c(t-s)) \cdot \frac{V(x)}{n} \quad \underset{n\to+\infty}{\longrightarrow} \quad 0\,,
\end{aligned}$$

for all fixed $t \in [s, +\infty)$, where $I\!I_{(.)}$ represents the indicator function of subscribed random set. Consequently

$$\mathbb{P}\,(\tau^{s,x}(\mathbb{D}) < t) \;=\; \lim_{n\to+\infty} \mathbb{P}\,(\tau^{s,x}(\mathbb{D}_n) < t) \;=\; 0\,.$$

Eventually, uniqueness and Markov property of the solution X_t is obtained by a result from Khas'minskij [8]. ◇

3. Numerical Regularization via Balanced Implicit Methods

Numerical regularization is generally aiming at construction of convergent and appropriately bounded numerical solutions for SDEs. In anology to regularity of continuous time processes we introduce the notion of regularity of discrete time stochastic processes.

Definition 2. A random sequence $(Y_i)_{i\in\mathbb{N}}$ is called **regular on** (or regular with respect to given domain) $\mathbb{D} \subset \mathbb{R}^d$ iff $\mathbb{P}\,(Y_i \in \mathbb{D}) = 1\ \forall i \in \mathbb{N}$, otherwise **nonregular**.

Throughout this paper we only consider such random sequences which have a direct link to numerical solution of SDEs. That is that one interpretes random values Y_i as values of an approximation Y for exact solution X at times $t_i \in [0,T]$. For example, **Balanced Implicit Methods (BIMs)** (see [12]) provide schemes to construct such sequences. For other methods and details, e.g. see [2], [9], [11], [13], [14], [17], [19], [20] and [21]. BIMs turn out to be somehow efficient to guarantee both convergence towards exact solution and some algebraic constraints on numerical solutions, i.e. to guarantee numerical regularity. For general exposition in this respect, see [18]. The following BIM solves the problem of numerical regularization on the bounded domain $\mathbb{D} = [0, M]$, provided that $\alpha \geq 1, \beta \geq 1$. Take

$$\begin{aligned}
Y_{n+1} \;&=\; Y_n + (p + \frac{q}{M}Y_n)(M - Y_n)\Delta_n + \sigma Y_n^{\alpha}(M-Y_n)^{\beta}\Delta W_n \\
&\quad + \sigma K Y_n^{\alpha-1}(M-Y_n)^{\beta-1}|\Delta W_n|(Y_n - Y_{n+1}),
\end{aligned} \tag{3}$$

where $K = K(M)$ is an appropriate positive constant and $Y_0 \in \mathbb{D} = [0, M]$ (a.s.). Then it holds

Theorem 3. *The random sequence $(Y_n)_{n\in\mathbb{N}}$ governed by (3) is regular on $\mathbb{D} = [0,M]$ if*

$$Y_0 \in [0,M](a.s.),\ K(M) \geq M > 0,\ \alpha \geq 1,\ \beta \geq 1,\ 0 < \Delta_n \leq \frac{1}{p+q} \quad (\forall n \in \mathbb{N})\,.$$

Proof. Use induction on $n \in \mathbb{N}$. Then, after explicit rewriting of (3), one finds the following estimation of an upper bound

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{(p + \frac{q}{M} Y_n)\Delta_n + \sigma Y_n^{\alpha}(M - Y_n)^{\beta - 1}\Delta W_n}{1 + \sigma K(M) Y_n^{\alpha - 1}(M - Y_n)^{\beta - 1}|\Delta W_n|}(M - Y_n) \\ &= Y_n + \rho \cdot (M - Y_n) \ \leq \ M \, , \end{aligned}$$

since $\rho \leq 1$ if $K(M) \geq M$ and $\Delta_n \leq 1/(p+q)$. Otherwise, nonnegativity of Y_{n+1} follows from

$$Y_{n+1} \ = \ \frac{Y_n + (p + \frac{q}{M} Y_n)(M - Y_n)\Delta_n + \sigma Y_n^{\alpha}(M - Y_n)^{\beta - 1}((M - Y_n)\Delta W_n + K(M)|\Delta W_n|)}{1 + \sigma K(M) Y_n^{\alpha - 1}(M - Y_n)^{\beta - 1}|\Delta W_n|}$$

if $K = K(M) \geq M$. Consequently, we have $\mathbb{P}\ (0 \leq Y_n \leq M) \ = \ 1 \quad \forall n \in \mathbb{N}$. $\diamond$

Remark. The boundedness of this sequence of numerical values turns out to be essential for both the interpretability within the framework of Marketing issues and the proof of rates of convergence. Note, STOCHASTIC ADAPTATION of step sizes would form an alternative to fixed constant step size selection (as above) within simulation studies. However, then one has to find a truncation procedure to guarantee finiteness of corresponding algorithms, in particular for long term runs on computers!
Sequence $(Y_n)_{n \in \mathbb{N}}$ following (3) with conditions of Theorem 3 and $\alpha \in [0,1)$ is also regular on $\mathbb{D}$. However, the weights $c(x) = \sigma K(M) x^{\alpha - 1}(M - x)^{\beta - 1}$ are unbounded functions on $\mathbb{D}$ in this case. Then one obtains boundedness of numerical increments, but we suspect to loose convergence speed with such methods (Open question – Who knows the right answer?)!

In addition to boundedness, we can show convergence of numerical sequences towards the exact solution as mesh of discretization tends to zero. For this purpose, mean square convergence is examined along any sequence $(\eta = \eta^{\Delta}([0,T]))_{\Delta > 0}$ of discretizations of fixed, finite time–intervals $[0,T]$ when maximum step size

$$\Delta \ := \ \max \Delta_i \ = \ \max\{|t_{i+1} - t_i| : t_i, t_{i+1} \in \eta\}$$

tends to zero. The CRITERION OF NUMERICAL MEAN SQUARE CONVERGENCE is given by

$$\forall T > 0 \quad \exists K = K(T) \quad \forall \Delta < \delta \quad \forall \eta = \eta^{\Delta}([0,T]) \qquad \sup_{t_i \in \eta} \mathbb{E}\, ||X(t_i) - Y_i||^2 \ \leq \ K\, \Delta^{2\gamma} \tag{4}$$

where γ is said to be the **order** (rate) **of mean square convergence** of numerical sequence $(Y_i)_{i \in \mathbb{N}}$ (numerical method, scheme, solution).

Theorem 4. *The numerical sequence $(Y_n)_{n \in \mathbb{N}}$ governed by (3) is mean square converging with order $\gamma = 0.5$ towards the exact solution $X(t)$ of (1), at least when $\alpha, \beta \geq 1$ and $Y_0 = X(0) \in \mathbb{D} = [0, M]$ (a.s.).*

Proof. Take any $\eta^{\Delta}([0,T])$ with $\Delta \leq \delta = \min\{1, 1/(p+q)\}$. Note that drift and diffusion coefficient of SDE (1) are bounded and Lipschitz continuous on $\mathbb{D} = [0, M]$. Thus, in view of Theorem 2, classical requirements for existence and uniqueness of strong solution of SDEs are satisfied, cf. [1], [5], [6], [16]. Let $X_{t,x}(t+h)$ denote the solution of SDE (1) at time $t+h$ where $0 \leq h \leq \Delta$, starting in $X(t) = X_{t,x}(t) = x$ at any time $t \in [0, T-h]$, for any $x \in \mathbb{D}$. In a similar notation, let $Y_{t,x}(t+h)$ be the integral representation of one–step approximation belonging to scheme (3), starting in $Y_{t,x}(t) = x$ at any time $t \in [0, T-h]$. Now verify

$$|\mathbb{E}\,(X_{t,x}(t+h) - Y_{t,x}(t+h))| \ \leq \ K_1\, \Delta^{p_1} \qquad \text{and} \tag{5}$$

$$\mathbb{E}\, |X_{t,x}(t+h) - Y_{t,x}(t+h)|^2 \;\leq\; K_2\, \Delta^{2p_2} \tag{6}$$

where $p_1 \geq 1.5 = p_2 + 0.5 > p_2 = 1.0$ as local rates. Finally apply a generalization of Mil'shtein's main convergence theorem (see appendix) to the case of SDEs on bounded manifolds in order to obtain

$$\forall t \in [0,T] \quad \mathbb{E}\, |X_{0,x}(t) - Y_{0,x}(t)|^2 \;\leq\; K_3\, \Delta^{2p} \tag{7}$$

where $p = 0.5$ as global rate (order) of mean square convergence. Note that K_1, K_2 and K_3 are only positive real constants which may depend on the finite terminal time T, but not on intermediate time t, not on Δ or h. $\diamond$

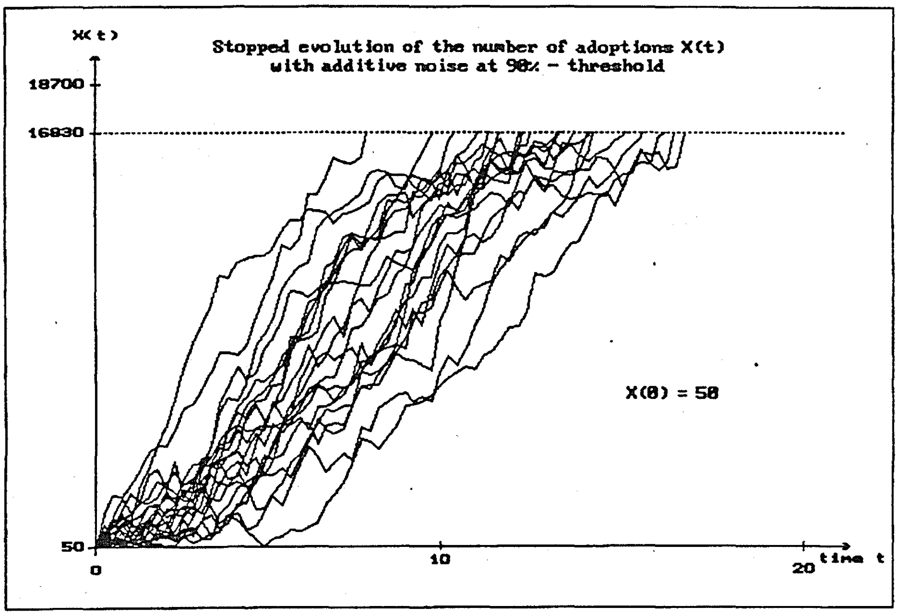

Figure 1. Stopped trajectories for additive noise with large intensity $\sigma = 1000$.

4. Simulation Results

In this section we carry out some simulation studies using parameters which are close to those in Mahajan and Wind [10]. There, for example for a data set belonging to a sale of room air conditioners, one has estimated parameters as $p \;=\; 0.0094$ and $q \;=\; 0.3748$, based on the maximum adoption $M = 1.87 \cdot 10^7$. Throughout the simulations we will use the same coefficients p of innovation and q of immitation, but we slightly reduce the total size of possible adoptions to $M = 18700$, just for computational simplification. The initial adoption is set to be $x_0 = 50$ and $x_0 = 187$, resp. For this parameter constellation, in figures 1 and 2 temporal evolutions of dynamics (1) are plotted. Figure 1 displays a collection of trajectories for the model with additive noise (i.e. $\alpha = \beta = 0$) stopped at 90 % levels. Figure 2 shows the mean evolution of adoption and its confidence intervals using Stratonovich and Itô interpretation compared with that of deterministic adoption. We observe a significant difference between different calculis and to deterministic adoption process. Thus the choice of stochastic calculi is very sensitive for SDEs on bounded manifolds. In principle, one notices a faster initial adoption under stochasticity compared to that of deterministic model. Besides, one can prove that 'stochastic equilibration' (i.e. process in which steady states are asymptotically reached) within Itô interpretation takes place below deterministic equilibrium $x^* = M$. Note, this is converse to that of Stratonovich interpretation, due to the positive difference of drift functions. In passing, stochasticity also leads to earlier time T^* of inflection (= time–

point where derivative of adoption takes its maximum), i.e. earlier peak sales (in mean sense), which represent an important assertion for the marketing process and strategies.

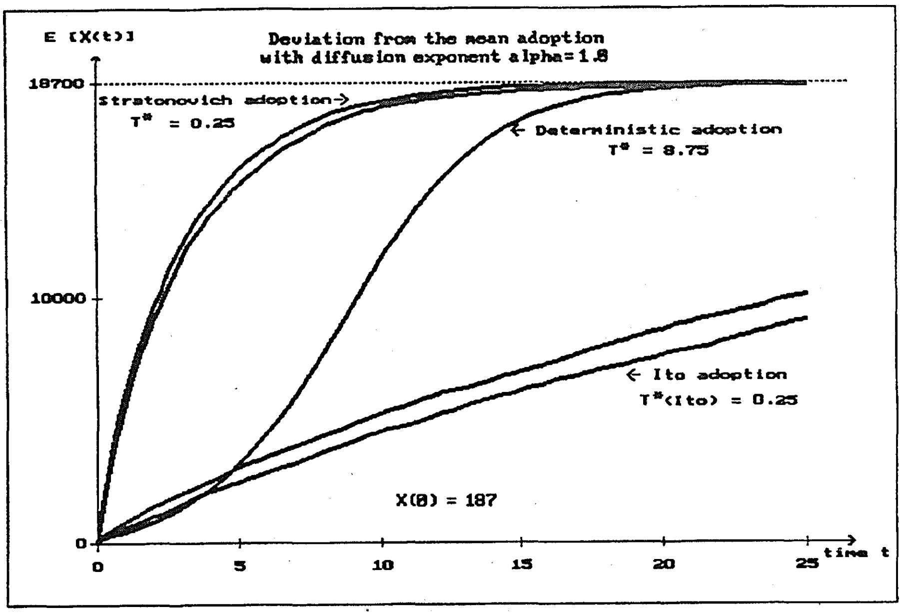

Figure 2. Mean adoption, confidence intervals and times T^* of inflection with exponent $\alpha = \beta = 1.0$ and $\sigma = 0.02$.

5. Appendix: A Generalization of Main Mean Square Convergence Theorem

Under classical requirements on boundedness and smoothness of drift and diffusion coefficients, there is a general theorem of Mil'shtein [11] which admits to verify global rates (orders) of mean square convergence. This statement is given when both discrete and continuous time stochastic processes are living on the whole $\mathbb{R}^d$, i.e. without any algebraic constraints. In a straight forward way one can generalize this theorem to the following one in case of algebraic restrictions. Let $\mathbb{D}$ be any closed subdomain of $\mathbb{R}^d$. Fix a finite terminal time $T > 0$. Let $X_{t,x}(t+h)$ and $Y_{t,x}(t+h)$ be the integral representations of exact and numerical solution as above, resp. Without loss of generality, $||\cdot||$ denotes Euclidean vector norm.

Theorem 5. *Assume that* $\mathbb{E}\ ||X(0)||^2 < +\infty$ *and there are real, positive constants* $K_1 = K_1(T), K_2 = K_2(T)$, $p_1 \geq p_2 + 0.5, p_2 \geq 0.5$ *such that for all* $x \in \mathbb{D}$, *for all* h *with* $0 \leq h \leq \delta \leq 1$, *for all* $t \in [0, T-h]$ *it holds*

$$\mathbb{P}\ (X_{t,x}(t+h) \in \mathbb{D}) \ = \ \mathbb{P}\ (Y_{t,x}(t+h) \in \mathbb{D}) \ = \ 1, \tag{8}$$

$$||\mathbb{E}\ (X_{t,x}(t+h) - Y_{t,x}(t+h))|| \ \leq \ K_1\, h^{p_1} \qquad \textit{and} \tag{9}$$

$$\mathbb{E}\ ||X_{t,x}(t+h) - Y_{t,x}(t+h)||^2 \ \leq \ K_2\, h^{2p_2} \tag{10}$$

Then the numerical solution Y^{Δ} *belonging to one-step approximation* $Y_{t,x}(t+h)$ *is mean square converging with global order* $p = p_2 - 0.5$ *towards the exact solution of SDE* $dX_t = a(t, X_t)\,dt + \sum_{j=1}^{m} b^j(t, X_t) dW_t^j$ *under linear-polynomial growth and Lipschitz continuity of its drift* $a(t,x)$ *and diffusion functions* $b^j(t,x)$ *on* $\mathbb{D}$, *where* $(W_t^j)_{j=1,2,...,m}$ *are* m *mutually independent Wiener processes (also independent of random variable* $X(0) = X_0 \in \mathbb{D}$*).*

Proof. Analogously to that in [11]. ◇

6. Summary and Remarks

For adequate modelling of diffusion of innovation **in Marketing Sciences**, for stochastic interest rates in **Mathematical Finance**, for population dynamics in **Biology** (cf. concept of permanence), etc., one has to consider the **problem of regular stochastic processes**, both in continuous and discrete time. In general it leads to the mathematical treatment of Stochastic Differential Algebraic Systems (**SDASs**). Classification of boundary conditions and **stochastic Lyapunov–Type Methods** are the right tools to study and explain the behaviour of stochastic dynamics with algebraic constraints. For adequate numerical treatment, the class of Balanced Implicit Methods (**BIMs**) seems to be quite promising in order to guarantee both boundedness, stability and convergence with acceptable rates of consistency and convergence.

Further studies concerning (stochastic) diffusion of innovation can easily bring out interesting marketing issues, e.g. effects of pulsing policies (i.e. pulsing advertisement by pulsing parameters p and q). Some mathematical clarification of well–posedness, regularity and adequate numerical solution remains open for future research (e.g. when $\alpha, \beta \in (0,1)$). Besides, a comparison with real data is necessary for practical evaluation of the herein suggested models. Another interesting task would be to clarify the problem '**Stochastic versus Deterministic Modelling**'. Also, a generalization to multi–dimensional models, to more complex domains $\mathbb{D} \subset \mathbb{R}^d$ and to incorporation of stochastic boundary conditions is left to future. Some of the open questions will be touched in a forthcoming paper by Karmeshu and Schurz (1995).

Summarizing results, it is definitively worth to **consider uncertainty in models of Marketing Sciences**, not only for replication of the very erratic behaviour of nature.

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