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Strong synchronization of weakly interacting oscillons

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Abstract

We study interaction of well-separated oscillating localized structures (oscillons). We show that oscillons emit weakly decaying dispersive waves, which leads to formation of bound states due to subharmonic synchronization. We also show that in optical applications the Andronov-Hopf bifurcation of stationary localized structures leads to a drastic increase in their interaction strength.

Investigation of localized structures arising in physical systems of various nature is an important subject of nonlinear science. Lately much attention has been attracted to the so-called dissipative solitons [1, 2]. Their formation requires a balance of energy gain and dissipation, which makes the dissipative solitons more stable to perturbations and, therefore, more attractive for practical applications (e.g. for optical information processing) than the classical solitons of integrable/Hamiltonian equations. Exact analytical expressions for dissipative solitons are rarely available, so qualitative methods become especially important in their study. An interesting problem which can be treated by qualitative methods is the interaction of dissipative solitons [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. While most of the studies here were focused on the case of stationary solitons, in this letter we analyze interaction of dissipative solitons which *oscillate in time*.

It is well known that a stationary soliton can exhibit instabilities which lead to various dynamical regimes. One of the simplest and most frequently encountered between these instabilities is the Andronov-Hopf (AH) bifurcation resulting in undamped pulsations of the soliton's parameters, such as amplitude, width, etc. [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 16, 1]. Here we show that the transition from stationary to an oscillating soliton (oscillon) leads to formation of various new types of multisoliton bound states. In particular, the AH bifurcation of stationary optical pulses results in a considerable increase of their interaction strength.

Although the approach we use is general, to illustrate the enhancement of the solitons interaction, we consider a specific model equation (Lugiato-Lefever model [28]):

$$\partial_t a = (i+\epsilon)\,\partial_{xx} a - (\gamma + i\theta)\,a + ia|a|^2 + p. \tag{1}$$

The equation describes formation of transverse patterns in Kerr cavity [26] or "temporal cavity solitons" in fibers [27]. Here, a is the electric field envelope, γ is the cavity decay rate, θ is the cavity detuning, and p describes the external coherent pumping. Spatial filtering (gain dispersion) ϵ is typically quite small in optical applications. So, this coefficient was omitted in Ref. [28] where only stationary regimes were studied. However, as we will see, it plays an important role in the interaction of oscillons.

The soliton in Eq. (1) is asymptotic to a nonzero stationary value $a_s = u_s + iv_s$ which depends on p. As the pumping parameter p increases above the critical value $p_{_{AH}}$, the soliton undergoes

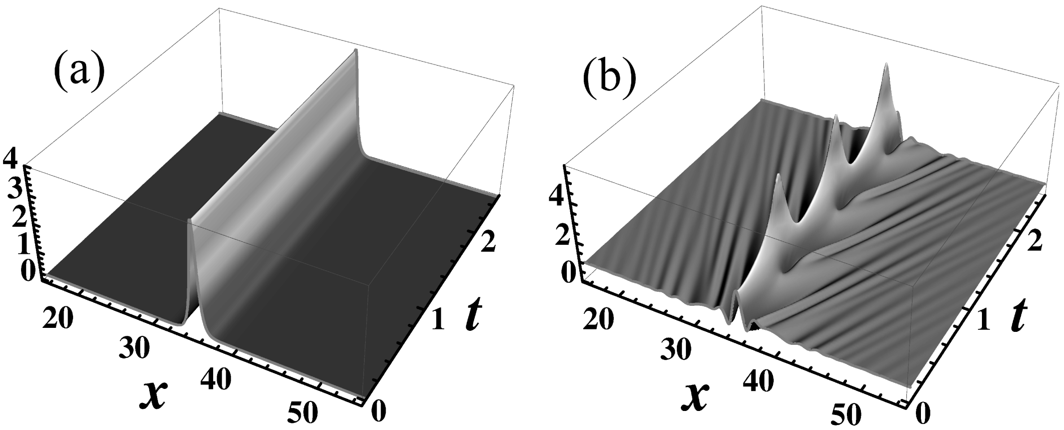

Figure 1: Stationary (a) and oscillating (b) solitons of Eq. (1) obtained for $p = 1.5$ and $p = 2$, respectively. In all the figures, the other parameters are $\epsilon = 0.02$, $\gamma = 1$, $\theta = 5.75$.

an AH bifurcation [26]. It is seen from Fig. 1 that after the AH bifurcation the soliton starts to radiate weakly decaying dispersive waves.

Results of numerical investigation of the two-soliton interaction are given in Fig. 2. Below the bifurcation threshold $p = p_{_{AH}}$, the distance between two well-separated solitons stays constant, i.e. the strength of interaction between stationary solitons is negligible on the chosen spatial scale (Fig. 2a). This is in agreement with the experimental findings of Ref. [27] where for solitons in a coherently driven fiber cavity the effective stop of the interaction is reported as the intersoliton distance exceeds a certain threshold (the saddle steady state of the soliton interaction equation, in our interpretation). Above the onset of self-oscillations the picture changes drastically (Fig. 2b): the oscillons visibly move and form numerous bound states distinguished by the intersoliton distance and the difference in the oscillation phases.

The phenomenon can be understood if we note that the strength of interaction between two well-separated solitons is determined mostly by the rate of soliton's tail exponential decay. As the numerics show, above the AH bifurcation threshold the tail decay rate becomes much slower (Fig. 3). Indeed, for stationary solitons this decay rate is determined by a single exponent that dominates the tail. In contrast to that, for the solitons oscillating in time at the frequency Ω, each frequency $n\Omega$ determines its own spatial decay rate. When it is slow for non-zero frequencies, the oscillon can be effectively seen as emitting decaying linear waves. For small spatial filtering ϵ, a higher modulation frequency corresponds to a lower wave dissipation rate, hence oscillons interact by exchanging waves on subharmonic frequencies (see Fig. 6).

In order to find the dispersion relation which determines the behavior of the oscillon tails, we add a small perturbation to the stationary homogeneous solution, i.e. we let $a = a_s + \delta u + i\delta v$ in Eq. (1). Then, we separate real and imaginary parts of this equation, linearize the resulting system for the variables δu and δv, and apply the Laplace transform in x along with the Fourier transform in t. This gives a biquadratic equation:

$$\begin{aligned} &\{i\omega - \lambda^2(i+\epsilon) + [\gamma + i(\theta - 2|a_s|^2)]\} \times \\ &\quad \times \{i\omega + \lambda^2(i-\epsilon) + [\gamma - i(\theta - 2|a_s|^2)]\} = |a_s|^4. \end{aligned} \tag{2}$$

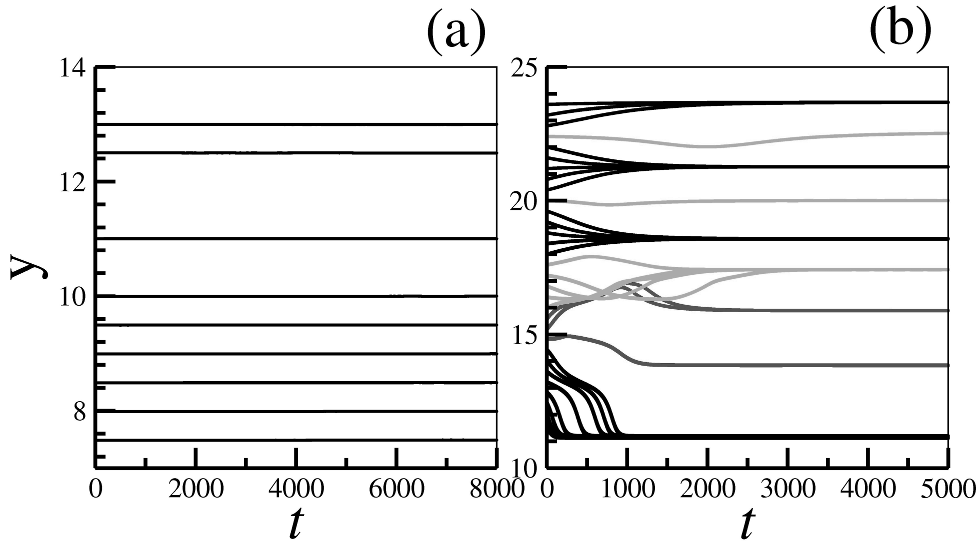

Figure 2: The intersoliton distance y vs. time t. a) $p = 1.5$: two stationary solitons below the AH bifurcation threshold. b) $p = 2$: oscillating solitons after the AH bifurcation; black shows solutions converging to inphase oscillating soliton bound states, grey – to antiphase bound states, light gray – to the bound states with the oscillation phase difference $\approx \pi/2$.

The two solution branches with $\mathrm{Re}\lambda < 0$ determine the spatial decay rates corresponding to the oscillations in time with the frequency $\omega = n\Omega$. These two branches are related by $\lambda_1(\omega) = \lambda_2^*(\omega)$. As a result, we obtain the following asymptotic behavior for the tail:

$$a - a_s \sim \sum_n b_n e^{\lambda(n\Omega)x + in\Omega t} + c_n e^{\lambda^*(n\Omega)x - in\Omega t}. \tag{3}$$

This expression describes the dispersive waves which are radiated by the oscillon, see Fig. 1b.

One should expect that at sufficiently large x (the distance from the soliton center) the exponent with $\max_n \mathrm{Re}\lambda(n\Omega)$ dominates in expansion (3). At $\epsilon = 0$ we have $\mathrm{Re}\lambda(\omega) \to 0$ as $\omega \to \infty$ for one of the two solution branches of Eq. (2). Therefore, at small ϵ the maximal value of $\mathrm{Re}\lambda$ is achieved at large n (see Fig. 3), i.e. the oscillating part of the tail indeed prevails over the stationary one. This is illustrated in Fig. 4. The tails of the stationary soliton decay fast with $Re\lambda \approx -2.2$ which is in a good agreement with the solution of Eq. (2) at $\omega = 0$. For the oscillatory soliton we see a much slower decay with at least two exponents contributing: $\mathrm{Re}\lambda_1 \approx -0.47$ and $\mathrm{Re}\lambda_2 \approx -0.20$, which correspond to the subharmonic frequencies with $n = 1$ and $n = 2$, respectively. Since the maximum of the dispersion relation in Fig. 3 is very flat, the value of $\mathrm{Re}\lambda_2$ is already close to the maximum and the contribution of the higher subharmonics into the tail is suppressed (on the given spatial scale) due to the exponentially fast decay of the Fourier coefficients b_n, c_n.

In order to understand details of the oscillon interaction, let us derive the oscillon interaction equations. By plugging $a = a_s + A$ into Eq. (1), we can write it as

$$\partial_t \mathbf{A} = \mathcal{L}\mathbf{A} + \mathbf{f}(\mathbf{A}). \tag{4}$$

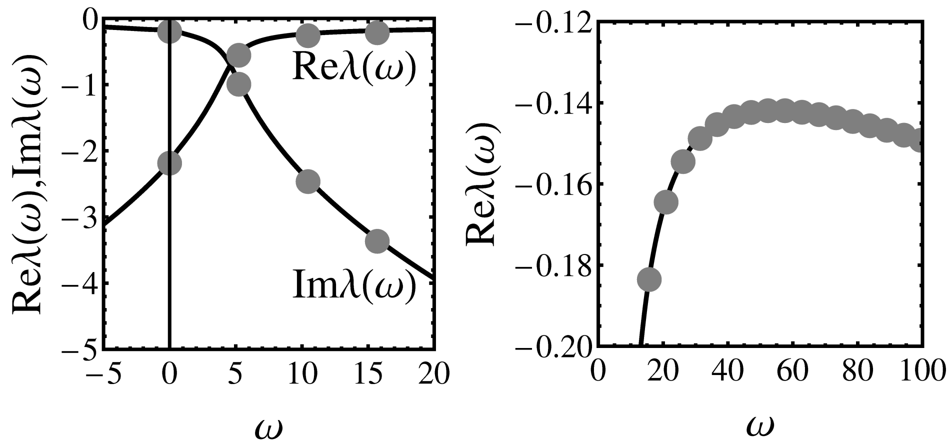

Figure 3: Dispersion relation for the oscillon tails at $p = 2$. a) real and imaginary parts of $\lambda(\omega)$ obtained by solving Eq. (2). b) the maximum of the real part of λ is very flat. Gray dots correspond to $\omega = n\Omega$, where $\Omega \approx 5.2$.

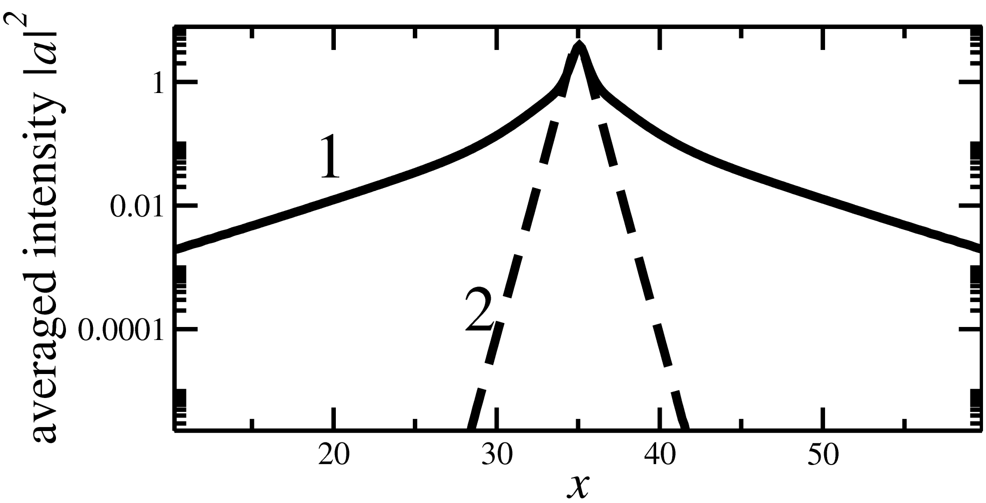

Figure 4: Solitons in the logarithmic scale. Tail of the (averaged over the period) oscillating soliton ($p = 2.0$, curve 1) decays much slower than the tail of the stationary soliton ($p = 1.5$, curve 2). Two exponents corresponding to $\mathrm{Re}\lambda_1 \approx -0.47$ and $\mathrm{Re}\lambda_2 \approx -0.20$ dominate the oscillon tail.

where $\mathcal{L}$ is a linear differential operator with constant coefficients, $\mathbf{f}(0) = 0$, $\mathbf{f}'(0) = 0$, and $\mathbf{A} = (\mathrm{Re}A, \mathrm{Im}A)^T$. Let $\mathbf{A}_0\,(x,t) = \mathbf{A}_0\,(-x,t)$ be a symmetric oscillon solution of Eq. (1), so $\mathbf{A}_0\,(x,t) \to 0$ as $x \to \pm\infty$ and $\mathbf{A}_0\,(x,t) = \mathbf{A}_0\,(x,t+T)$ where $T = 2\pi/\Omega$. Since Eq. (4) is invariant with respect to space and time shifts, the neutral mode equation $-\partial_t\boldsymbol{\psi} + \mathcal{L}\boldsymbol{\psi} + f'(A_0)\boldsymbol{\psi} = 0$ has two natural solutions $\boldsymbol{\xi} = \partial_x\boldsymbol{A}_0$ and $\boldsymbol{\eta} = \partial_t\boldsymbol{A}_0$. We assume there are no other critical modes, i.e. we are sufficiently above the AH instability threshold. The adjoint neutral modes $\boldsymbol{\psi} = \boldsymbol{\xi}^\dagger$(odd in x) and $\boldsymbol{\psi} = \boldsymbol{\eta}^\dagger$ (even in x) satisfy $\partial_t\boldsymbol{\psi} + \mathcal{L}^\dagger\boldsymbol{\psi} + [f'(A_0)]^\dagger\boldsymbol{\psi} = 0$ with the normalization condition $\int_0^T dt \int_{-\infty}^{+\infty} dx\ \boldsymbol{\xi}^\dagger \cdot \boldsymbol{\xi} = \int_0^T dt \int_{-\infty}^{+\infty} dx\ \boldsymbol{\eta}^\dagger \cdot \boldsymbol{\eta} = T$.

We look for the solution of Eq. (4) in the form of two interacting oscillons plus a small correction:

$$\mathbf{A} = \mathbf{A}_1\,(x,t) + \mathbf{A}_2\,(x,t) + \boldsymbol{\chi}(x,t),$$

where $\boldsymbol{\chi}$ is small, $\mathbf{A}_{1,2} = \mathbf{A}_0\left(x - y_{1,2}, t - \tau_{1,2}/\Omega\right)$, and $y_{1,2}$ and $\tau_{1,2}$, the coordinates and the oscillation phases of the solitons, are slowly varying functions of time. By performing asymptotic expansions, similar to what is done for stationary solitons [3, 12, 13, 14, 15], we obtain the leading order approximation for the oscillon interaction equation:

$$T\frac{dy_{1,2}}{dt} = \int_0^T\int_{-\infty}^{+\infty}\boldsymbol{\xi}^\dagger_{1,2}\cdot\mathbf{S}\ dxdt,\quad 2\pi\frac{d\tau_{1,2}}{dt} = \int_0^T\int_{-\infty}^{+\infty}\boldsymbol{\eta}^\dagger_{1,2}\cdot\mathbf{S}\ dxdt,$$

where $\boldsymbol{\psi}_{1,2} = \boldsymbol{\psi}\left(x - y_{1,2}, t - \tau_{1,2}/\Omega\right)$ (here $\boldsymbol{\psi} = \boldsymbol{\xi}^\dagger$ or $\boldsymbol{\psi} = \boldsymbol{\eta}^\dagger$), and $\mathbf{S} = \mathbf{f}\left(\mathbf{A}_1 + \mathbf{A}_2\right) - \mathbf{f}\left(\mathbf{A}_1\right) - \mathbf{f}\left(\mathbf{A}_2\right)$. We assume the oscillons are well-separated, i.e. $y_2 - y_1$ is large, so the overlap function $\mathbf{S}$ is small. The oscillon tails decay fast, so $\mathbf{S} \approx \mathcal{M}_1\mathbf{A}_2$ at $x < y_*$ and $\mathbf{S} \approx \mathcal{M}_2\mathbf{A}_1$ at $x > y_*$, where $\mathcal{M}_{1,2} = \mathcal{L} - \partial_t + \mathbf{f}'(\mathbf{A}_{1,2})$, and $y_* = (y_1 + y_2)/2$ is the middle point of the two-oscillon configuration. Since $\boldsymbol{\xi}_j^\dagger$ and $\boldsymbol{\eta}_j^\dagger$ are localized near $x = y_j$, we have

$$T\frac{dy_j}{dt} \approx \int_0^T\int_{I_j}\boldsymbol{\xi}^\dagger_j\cdot\mathcal{M}_j\mathbf{A}_k\ dxdt,\quad 2\pi\frac{d\tau_j}{dt} \approx \int_0^T\int_{I_j}\boldsymbol{\eta}^\dagger_j\cdot\mathcal{M}_j\mathbf{A}_k\ dxdt,$$

where $k = 3 - j$, and $I_1 = [-\infty, y_*]$, $I_2 = [y_*, +\infty]$. Now, using the relations $\mathcal{M}_j^\dagger\boldsymbol{\xi}_j^\dagger = 0$, $\mathcal{M}_j^\dagger\boldsymbol{\eta}_j^\dagger = 0$, one takes the integrals with respect to x. In our case, where the operator $\mathcal{L}$ is defined by Eq. (1), we finally obtain

$$T\frac{dy_j}{dt} = (-1)^j\int_0^T\Big(\partial_x\boldsymbol{\xi}^\dagger_j E\mathbf{A}_k - \boldsymbol{\xi}^\dagger_j E\partial_x\mathbf{A}_k\Big)_{x=y_*}dt,$$

$$2\pi\frac{d\tau_j}{dt} = (-1)^j\int_0^T\Big(\partial_x\boldsymbol{\eta}^\dagger_j E\mathbf{A}_k - \boldsymbol{\eta}^\dagger_j E\partial_x\mathbf{A}_k\Big)_{x=y_*}dt,$$

where $E = \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix}$. As we see the evolution of the interacting oscillons is, to the first order, determined by the asymptotics of their tails and of the adjoint neutral modes $\boldsymbol{\xi}^\dagger$ and $\boldsymbol{\eta}^\dagger$ and does not depend on the specific form of the nonlinearity f. Plugging the asymptotic formula (3) in the obtained interaction equations, we find

$$\left\{\begin{aligned} \frac{dy}{dt} &= \sum_{n=-\infty}^{\infty} B_n e^{-\alpha_n y}\sin\left(\beta_n y + \Theta_{1n}\right)\cos\left(n\tau\right), \\ \frac{d\tau}{dt} &= \sum_{n=-\infty}^{\infty} C_n e^{-\alpha_n y}\cos\left(\beta_n y + \Theta_{2n}\right)\sin\left(n\tau\right), \end{aligned}\right. \tag{5}$$

where $\alpha_n = -\mathrm{Re}\ \lambda(n\Omega)$, $\beta_n = \mathrm{Im}\ \lambda(n\Omega)$, $y = y_2 - y_1$, $\tau = \tau_2 - \tau_1$, and the coefficients B, C, θ are expressed via the Fourier coefficients b_n and c_n and analogous coefficients of the asymptotic expansions for $\boldsymbol{\xi}^\dagger$ and $\boldsymbol{\eta}^\dagger$.

Main contribution to the sums in Eqs. (5) is typically made by a small number of exponents which correspond to the minimal values of α_n and, at moderate y, to the maximal values of B_n, C_n. Consider the case where only one term dominates in the sum. If it corresponds to the zero harmonics $n = 0$ (i.e. $\alpha_0 < \min_{n\neq 0} \alpha_n$), then the y-equation does not, to the leading order, depend on τ. Then, the distance between the oscillons behaves like in the stationary case [3, 13, 14, 15]: at $\beta_0 \neq 0$ stable bound states are formed near $\beta_0 y + \theta_{10} = \pi(2k+1)$, independently of the value of the phase difference τ. Possible phase synchronization effects appear on a much longer time scale and are governed by non-zero harmonics.

If the dominating exponent corresponds to a non-zero harmonic, $\alpha_N = \min_n \alpha_n$, $N \neq 0$, then the oscillon interaction equations reduce to

$$\left\{ \begin{array}{l} \dfrac{dy}{dt} = Be^{-\alpha_N y} \sin\left(\beta_N y + \theta_{1N}\right) \cos\left(N\tau\right), \\ \dfrac{d\tau}{dt} = Ce^{-\alpha_N y} \cos\left(\beta_N y + \theta_{2N}\right) \sin\left(N\tau\right). \end{array} \right. \tag{6}$$

When $N = 1$, Eqs. (6) coincide with those derived in [12, 15] for the interaction of stationary solitons in complex Ginzburg-Landau (CGL) type models (unlike Eq. (1), CGL-equations have a phase-shift symmetry $a \to ae^{i\phi}$, so $\tau = \phi_2 - \phi_1$ describes there the difference between the corresponding phases of the stationary solitons). By borrowing the results of the analysis of the stationary solitons interaction in CGL [9, 10, 11, 12, 16, 15], we find that Eqs. (6) at $N = 1$ have three different sets of steady states synchronized with the phase differences $\tau = 0, \pi, \pm\pi/2$. Depending on the parameter values, Eqs. (6) demonstrate two different types of dynamical behavior [12]. If $BC \cos\left(\theta_{21} - \theta_{11}\right) > 0$, the only attractors are inphase and antiphase bound states. On the contrary, for $BC \cos\left(\theta_{21} - \theta_{11}\right) < 0$, the inphase and antiphase bound state are unstable, and solutions of Eqs. (6) oscillate around the $\pm\pi/2$ out-of-phase bound states. In the full system, the inphase/antiphase oscillon bound states are preserved, while the phase shift for the out-of-phase bound states can slightly differ from $\pi/2$ (since higher order corrections destroy the reversibility of Eqs. (6)). The phase portrait for the case $N > 1$ is formally recovered from that for $N = 1$ by rescaling τ. However, a novel phenomenon of the subharmonic oscillon synchronization emerges: stable bound states with the phase differences $\tau \approx \pi k/\left(2N\right)$ become possible.

The results of numerical simulations of two-oscillon interaction in Eqs. (1) are presented in Fig. 5. It is seen from this figure that when the oscillon separation is sufficiently small we have only inphase and antiphase stable bound states, which is typical for Eqs. (6) with $N = 1$ and $BC \cos\left(\theta_{2N} - \theta_{1N}\right) > 0$. However, at larger oscillon separations, stable bound states with the phase difference around $\pi/2$ appear (see Fig. 6) and the phase portrait becomes consistent with Eqs. (6) in the case $N = 2$. In particular, the sequence of inphase bound states becomes equidistant with the increment ≈ 1.3, close to π/β_2.

To explain this, recall that a single exponent is not sufficient for the description of the oscillon tail asymptotics of Eq. (1) for the chosen set of parameters. The oscillon tail shown in Fig. 4 contains at least two decaying exponents which correspond to $n = 1$ and $n = 2$. By retaining

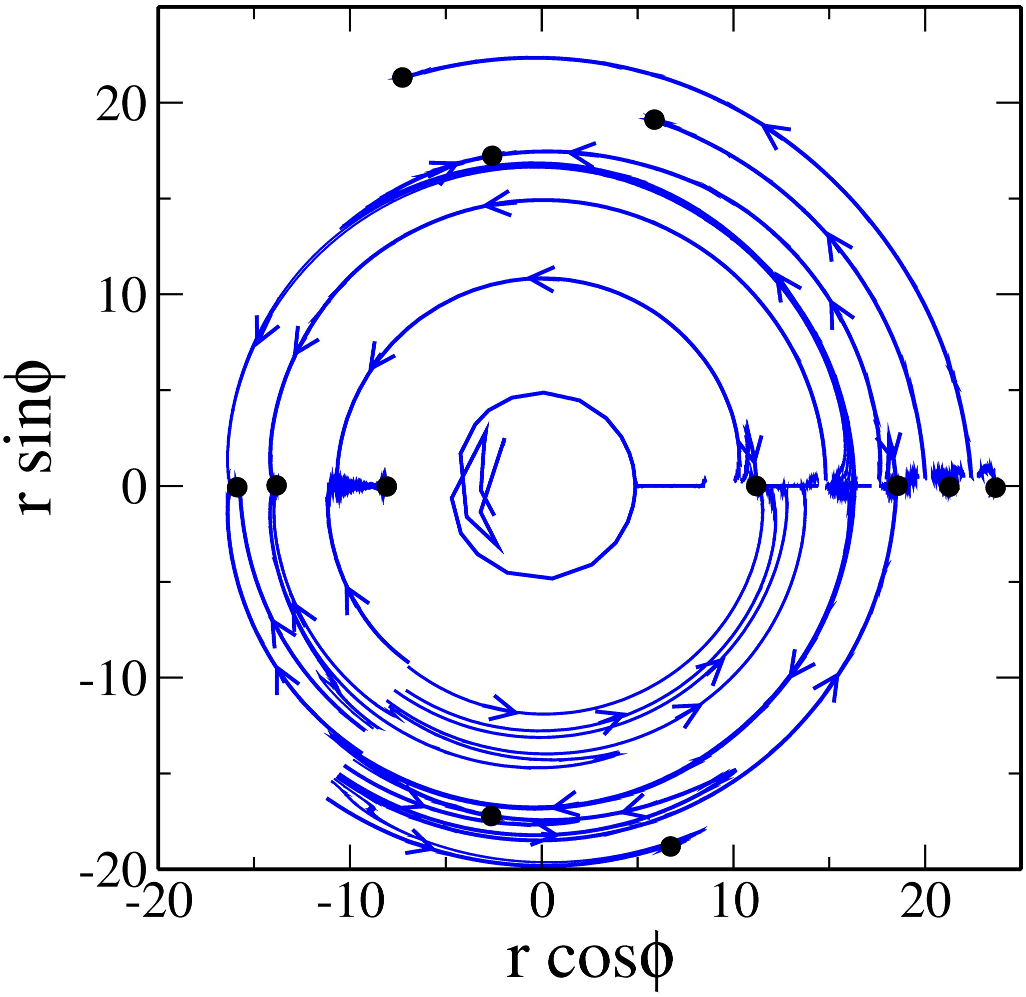

Figure 5: Poincare map for the evolution of two interacting oscillons ($p = 2.0$). For various initial conditions, consecutive values of the intersoliton distance y and the phase difference τ are shown at the time moments the amplitude of the left oscillon in the pair takes its maximal value. Large dots indicate stable oscillon bound states. At distances $y > 4.8$ the discrete trajectories closely follow continuous lines, as predicted by Eqs. (5), while at smaller distances y the theory of weak ocillon interaction is not applicable.

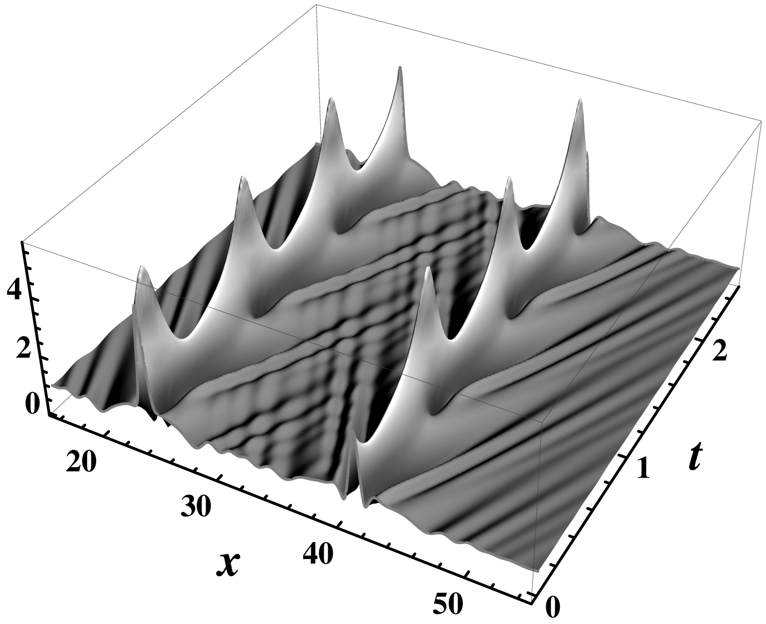

Figure 6: Bound state of two oscillons with the oscillation phase difference $\tau \approx \pi/2$.

the two corresponding terms in Eqs. (5) we obtain:

$$\begin{cases} \dfrac{dy}{dt} = B_1 e^{-\alpha_1 y} \sin\left(\beta_1 y + \Theta_{11}\right) \cos\tau + \\ \qquad\quad + B_2 e^{-\alpha_2 y} \sin\left(\beta_2 y + \Theta_{12}\right) \cos 2\tau, \\ \dfrac{d\tau}{dt} = C_1 e^{-\alpha_1 y} \cos\left(\beta_1 y + \Theta_{21}\right) \sin\tau + \\ \qquad\quad + C_2 e^{-\alpha_2 y} \cos\left(\beta_2 y + \Theta_{22}\right) \sin 2\tau. \end{cases} \tag{7}$$

Since $\alpha_1 > \alpha_2$, the terms with 2τ begin to dominate in these equations with the increase of the oscillon separation y. The phase portrait shown in Fig. 5 is consistent with Eqs. (7) for $B_1/B_2 \approx C_1/C_2 \approx 0.02$. At even larger distances numerical simulations reveal stable bound states with $\tau \approx \pm\pi/3, 2\pi/3$ which should correspond to higher subharmonics coming into play.

To conclude, we have shown that the transition from stationary to the oscillating solitons can lead to a drastic enhancement of the soliton interaction strength. Especially, this is true in many optical applications where the spectral filtering coefficient ϵ is typically small: in this case the high frequency linear waves emitted by the oscillons have a low dissipation rate and, therefore, are the main agent of the weak interaction. Different bound states of oscillons are distinguished by the distance between them and oscillations phase difference, i.e. they correspond to different oscillon synchronization regimes. We have found that synchronization of subharmonics is a typical phenomenon here.

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