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Quasiconvexity equals rank-one convexity for isotropic sets of 2x2 matrices

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Abstract

Let K be a given compact set of real $2{\times}2$ matrices that is isotropic, meaning invariant under the left and right action of the special orthogonal group. Then we show that the quasiconvex hull of K coincides with the rank-one convex hull (and even with the lamination convex hull of order 2). In particular, there is no difference between quasiconvexity and rank-one convexity for K. This is a generalization of a known result for connected sets.

1 Introduction

We study quasiconvexity in the calculus of variations. Morrey [Mor52] introduced it as the essential property for functions in the context of sequentially weakly lower semicontinuity for multiple integrals. He also conjectured that quasiconvexity is a "non-local" property, which was later shown to be true by Kristensen [Kri99]. At the heart of Kristensen's proof lies Šverák's counterexample of a rank-one convex function that fails to be quasiconvex [Šve92]. However, this counterexample works only in the case of an underlying space $\mathbb{M}^{m\times n}$ with $m \geq 3$, $n \geq 2$. Müller [Mül99a] showed that rank-one convexity implies quasiconvexity on diagonal $2{\times}2$ matrices. The general situation in $\mathbb{M}^{2\times 2}$ remains unknown.

Closely related to the quasiconvexity for functions is the corresponding concept for sets. Basically, quasiconvex sets are lower-level sets of quasiconvex continuous functions. We focus on quasiconvexity for isotropic sets in $\mathbb{M}^{2\times 2}$ and prove the following result (see Theorem 7.2).

Theorem (Equivalence)**.** *Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given compact and isotropic set. Then K is lamination convex if and only if K is quasiconvex.*

As long as the set K is connected, there is even equivalence between lamination convexity and polyconvexity. This was shown by Conti et al. [CDLMR03] and, before, by Cardaliaguet and Tahraoui [CT00, CT02a, CT02b] in the case when K contains only matrices with non-negative determinant. Conti et al. [CDLMR03] give also an example of a disconnected K that is lamination convex but not polyconvex. In addition, we will characterize the structure of the quasiconvex hull of K. Our main result reads (see Theorem 7.3)

Theorem (Characterization of K^{qc})**.** *Let $K \subseteq \mathbb{M}^{2\times 2}$ be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2.*

The paper is organized as follows:

In Section 2 we will fix the notations and recall definitions of the convexity notions that are used later on. Preliminaries can be found in Section 3 and 4. Then we refine a result by Conti et al. [CDLMR03] for connected K in Section 5. Section 6 is dedicated to the closed lamination convex hull K^{clc} and its structure. The key observation is that the principle structure of K^{clc} is already determined by the lamination convex hull of order one. In Section 7 we deal with the equivalence of lamination convexity and quasiconvexity. The main step is to show that what is disconnected in K^{clc} remains so in K^{pc}. Then we apply a deep result by Faraco and Székelyhidi [FS08] saying that the quasiconvex hull for the support of a homogeneous gradient Young measure is connected.

2 Functions, measures, and hulls

We are going to recall some convexity notions that play an important role in this paper. Our focus lies on dimension 2. A detailed discussion, also for higher dimensions, can be found in Dacorogna [Dac89, 4.1], Ball [Bal77] and Müller [Mül99b].

We denote by $\mathbb{M}^{2\times 2}$ the vector space of all real 2×2 matrices equipped with the Euclidean structure of $\mathbb{R}^4$. The corresponding matrix norm is denoted by $|.|$, the identity matrix by I. Let $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a given continuous function. Then f is *convex* if for every $A, B \in \mathbb{M}^{2\times 2}$ we have

$$\forall \lambda \in [0,1] \quad f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B). \tag{1}$$

The function f is *polyconvex* if there exists a convex function $g\colon \mathbb{R}^5 \to \mathbb{R}$ such that for every $A \in \mathbb{M}^{2\times 2}$ we have $f(A) = g(A, \det(A))$, where $\det(A)$ denotes the determinant of A. We will often use that for every real number $\alpha \in \mathbb{R}$ the function $\alpha \det$ is polyconvex. The function f is *quasiconvex* (in the sense of Morrey [Mor52]), if for every $A \in \mathbb{M}^{2\times 2}$ and every smooth function $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ with compact support we have

$$0 \leq \int_{\mathbb{R}^2} \big(f(A + \mathrm{D}\phi(x)) - f(A)\big)\mathrm{d}x.$$

The function f is *rank-one convex* if (1) holds for every $A, B \in \mathbb{M}^{2\times 2}$ that are *rank-one connected*, meaning $A - B$ equals the tensor product $a \otimes b$ for some vectors $a, b \in \mathbb{R}^2$. Polyconvexity and rank-one convexity were introduced by Ball [Bal77].

With the help of the convexity notions for functions, we now define the convexity notions for sets. Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given set and $A \in \mathbb{M}^{2\times 2}$ a matrix. Then A lies in the *polyconvex hull* of K and we write $A \in K^{\mathrm{pc}}$ whenever $f(A) \leq \sup\{f(B) \,|\, B \in K\}$ holds for every polyconvex function $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$. The set K is called *polyconvex* whenever $K = K^{\mathrm{pc}}$ holds. The *quasiconvex hull* and the *rank-one convex hull* as well as *quasiconvexity* and *rank-one convexity* for sets are defined correspondingly.

We will give an alternative characterization in the case of compact sets. Therefore, denote by $\mathcal{P}_0(\mathbb{M}^{2\times 2})$ the set of all compactly supported probability measures that are defined over the Borel sets of $\mathbb{M}^{2\times 2}$. Let $\nu \in \mathcal{P}_0(\mathbb{M}^{2\times 2})$ be a given element. We write $\bar{\nu}$ for its mean value and $\mathrm{supp}(\nu)$ for its support, meaning the compliment of the set $\cup\{U \subseteq \mathbb{M}^{2\times 2} \,|\, \nu(U)=0 \wedge U \text{ open}\}$. In addition, let $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a continuous function. Then the following pairing is finite and well-defined

$$\langle \nu, f\rangle = \int_{\mathbb{M}^{2\times 2}} f(A)\mathrm{d}\nu(A).$$

We define the sets $\mathcal{P}^{\mathrm{pc}}$, $\mathcal{P}^{\mathrm{qc}}$ and $\mathcal{P}^{\mathrm{rc}}$. A probability measure $\nu \in \mathcal{P}_0(\mathbb{M}^{2\times 2})$ lies in $\mathcal{P}^{\mathrm{pc}}$ ($\mathcal{P}^{\mathrm{qc}}$ or $\mathcal{P}^{\mathrm{rc}}$) if and only if Jensen's inequality $f(\bar{\nu}) \leq \langle \nu, f\rangle$ is fulfilled for every polyconvex (quasiconvex or rank-one convex) continuous function $f\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$. Kinderlehrer and Pedregal [KP91] show that every $\nu \in \mathcal{P}^{\mathrm{qc}}$ is a homogenous gradient Young measure. Whereas every $\nu \in \mathcal{P}^{\mathrm{rc}}$ is a laminate, see Pedregal [Ped93].

Remark 2.1. *Let $K \in \mathbb{M}^{2\times 2}$ be a given compact set. Then the set K^{pc} coincides with $\{\bar{\nu} \mid \nu{\in}\mathcal{P}^{\mathrm{pc}} \wedge \operatorname{supp}(\nu){\subseteq}K\}$ and K^{qc} as well as K^{rc} can be characterized in a corresponding way.*

As in Müller and Šverák [MŠ96], K is called *lamination convex* if for every rank-one connected $A, B \in K$ and every real number $\lambda \in [0,1]$ we have that $\lambda A + (1{-}\lambda)B$ lies in K. The *closed lamination convex hull* K^{clc} is the intersection of all closed lamination convex subsets in $\mathbb{M}^{2\times 2}$ that contain K. Note that $\{A,B\}^{\mathrm{clc}}$ equals $\{\lambda A{+}(1{-}\lambda)B \mid \lambda{\in}[0,1]\}$ and, hence, is a connected set if $A, B \in \mathbb{M}^{2\times 2}$ are rank-one connected. Otherwise $\{A,B\}^{\mathrm{clc}} = \{A,B\}$ is disconnected. Here we call a given set $S \subseteq \mathbb{M}^{2\times 2}$ *connected* if there is no way to write S as the union of two disjoint nonempty relatively-open subsets of S. Moreover, we set $K^{\mathrm{lc},1} = \cup\{\{A,B\}^{\mathrm{clc}} \mid A, B{\in}K\}$ as well as $K^{\mathrm{lc},2} = (K^{\mathrm{lc},1})^{\mathrm{lc},1}$, which are called the *lamination convex hulls of order one and two*, respectively. We would like to remark that, in general, the set K^{clc} and the lamination convex hull of K (which is not defined here) are different as has been shown by Kolář [Kol03]. The previous definitions together with the hierarchy of convexity notions on the level of functions imply that

$$K \subseteq K^{\mathrm{lc},1} \subseteq K^{\mathrm{lc},2} \subseteq K^{\mathrm{clc}} \subseteq K^{\mathrm{rc}} \subseteq K^{\mathrm{qc}} \subseteq K^{\mathrm{pc}}.$$

Finally, we denote by $\operatorname{cc}(K)$ the set of all connected components (meaning maximal connected subsets) of K.

3 Compatible isotropic sets

We give a characterization of compatible isotropic sets. The general result for $\mathbb{M}^{n\times n}$, $n \geq 1$, is due to Šilhavý [Šil01, Pro. 3.1]. In our case $\mathbb{M}^{2\times 2}$, this was already done by Aubert and Tahraoui [AT87, Thé. 2.8], if only for matrices with non-negative determinant. The proofs of Lemma 3.1 and Lemma 3.2 are given for the convenience of the reader.

We call a set $M \subseteq \mathbb{M}^{2\times 2}$ *isotropic* whenever it is invariant under the left and right action of the special orthogonal group $\mathrm{SO}(2)$, meaning $M = M^{\mathrm{iso}}$ where

$$M^{\mathrm{iso}} = \{QAR \mid Q, R \in \mathrm{SO}(2) \wedge A \in M\}.$$

Here we consider $\mathrm{SO}(2)$ as a subset of $\mathbb{M}^{2\times 2}$ so that the group action becomes just matrix multiplication. The following notation works well in the context of isotropic sets and has been used before by many authors. Let $A \in \mathbb{M}^{2\times 2}$ be a given matrix, then we define $\lambda(A) = (\lambda_1(A), \lambda_2(A)) \in \mathbb{R}^2$ as the only pair of real numbers such that $\{|\lambda_1(A)|, \lambda_2(A)\}$ is the set of singular values of A and, in addition, $|\lambda_1(A)| \leq \lambda_2(A)$ as well as $\det(A) = \lambda_1(A)\lambda_2(A)$ holds. In fact, we have that

$$\{A\}^{\mathrm{iso}} = \{B\}^{\mathrm{iso}} \quad \Leftrightarrow \quad \lambda(A) = \lambda(B) \quad \Leftrightarrow \quad (|A| = |B| \ \wedge\ \det(A) = \det(B)).$$

We say that two subsets $M_1, M_2 \subseteq \mathbb{M}^{2\times 2}$ are *compatible* whenever there exist rank-one connected matrices $A_1 \in M_1$ and $A_2 \in M_2$. Otherwise M_1 and M_2 are called *incompatible*.

Lemma 3.1. *Let $A \in \mathbb{M}^{2\times 2}$ be a given matrix. Then $\{A\}^{\mathrm{iso}}$ and $\mathrm{SO}(2)$ are compatible if and only if $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$ holds.*

Proof. Assume that $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$. Then the following matrices are rank-one connected: $I \in \mathrm{SO}(2)$ and

$$I + \begin{pmatrix} \lambda_1(A)\lambda_2(A) - 1 & \sqrt{(1-\lambda_1(A)^2)(\lambda_2(A)^2-1)} \\ 0 & 0 \end{pmatrix} \in \{A\}^{\mathrm{iso}}.$$

Now assume that $\{A\}^{\mathrm{iso}}$ and $\mathrm{SO}(2)$ are compatible. Then there exist vectors $a, b \in \mathbb{R}^2$ and a matrix $C \in \{A\}^{\mathrm{iso}}$ such that $C = I + a\otimes b$. We know that $\det(C) = 1 + \langle a, b\rangle$ and $|C|^2 = 2 + 2\langle a, b\rangle + |a|^2|b|^2$. Together with the Cauchy-Schwarz inequality, we obtain the estimate $|C|^2 - \det(C)^2 - 1 \geq 0$. This implies that

$$\lambda_1(A)^2 + \lambda_2(A)^2 - \lambda_1(A)^2\lambda_2(A)^2 - 1 = (1-\lambda_1(A)^2)(\lambda_2(A)^2 - 1) \geq 0.$$

Hence, we must have $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$. □

Lemma 3.2. *Let $A, B \in \mathbb{R}^{2\times 2}$ be given matrices. Then $\{A\}^{\mathrm{iso}}$ and $\{B\}^{\mathrm{iso}}$ are compatible if and only if $|\lambda_1(A)| \leq \lambda_2(B)$ and, at the same time, $|\lambda_1(B)| \leq \lambda_2(A)$.*

Proof. Clearly, the lemma is true for $\det(A) = \det(B) = 0$. By symmetry, we can and we will assume that $\det(B) > 0$ for the rest of the proof. If necessary, we replace A and B by $-A$ and $-B$, respectively. In particular, we then have $0 < \lambda_1(B)$.

First, we start with $|\lambda_1(A)| \leq \lambda_2(B)$ and $|\lambda_1(B)| \leq \lambda_2(A)$. Then we obtain the inequality $|\lambda_1(A)/\lambda_2(B)| \leq 1 \leq \lambda_2(A)/\lambda_1(B)$. By Lemma 3.1, we conclude that the sets $\{C\}^{\mathrm{iso}}$ and $\mathrm{SO}(2)$ are compatible where $C = \mathrm{diag}(\lambda_1(A)/\lambda_2(B), \lambda_2(A)/\lambda_1(B))$. Hence, there exist a rotation $R \in \mathrm{SO}(2)$ and vectors $a, b \in \mathbb{R}^2$ such that $R + a\otimes b = C$. If we multiply both sides from the right by $\mathrm{diag}(\lambda_2(B), \lambda_1(B))$, we get

$$R\,\mathrm{diag}(\lambda_2(B), \lambda_1(B)) + a\otimes b = \mathrm{diag}(\lambda_1(A), \lambda_2(A)).$$

This shows that $\{A\}^{\mathrm{iso}}$ and $\{B\}^{\mathrm{iso}}$ are compatible.

Second, we start with $\{A\}^{\mathrm{iso}}$ and $\{B\}^{\mathrm{iso}}$ being compatible. Then we can write

$$R\,\mathrm{diag}(\lambda_2(B), \lambda_1(B)) + a\otimes b = \mathrm{diag}(\lambda_1(A), \lambda_2(A))Q$$

for some rotations $R, Q \in \mathrm{SO}(2)$ and vectors $a, b \in \mathbb{R}^2$. Multiplying both sides from the right by $\mathrm{diag}(1/\lambda_2(B), 1/\lambda_1(B))$, we see that $\mathrm{SO}(2)$ and the set $\{D\}^{\mathrm{iso}}$ are compatible where

$$D = \mathrm{diag}(\lambda_1(A), \lambda_2(A))Q\,\mathrm{diag}(1/\lambda_2(B), 1/\lambda_1(B)). \tag{2}$$

Hence, by Lemma 3.1, we must have $|\lambda_1(D)| \leq 1 \leq \lambda_2(D)$. This implies, in particular, that we can fix a vector $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$ such that $|Dx_0| = 1$.

The rest of the proof is by contradiction. Suppose that $|\lambda_1(A)| > \lambda_2(B)$. In view of (2), we obtain the inequality $|Dx_0| \geq |\lambda_1(A)|/\lambda_2(B) > 1$. Now suppose that $|\lambda_1(B)| > \lambda_2(A)$. Then we have $|Dx_0| \leq \lambda_2(A)/|\lambda_1(B)| < 1$. In both cases, we get a contradiction to the choice of x_0. □

In Figure 1(a), you see a given set $\{A\}^{\mathrm{iso}}$ and the region of all $\{B\}^{\mathrm{iso}}$ such that $\{A\}^{\mathrm{iso}}$ and $\{B\}^{\mathrm{iso}}$ are compatible.

The lemma and remark are taken from Conti et al. [CDLMR03, Lem. 2.2, Rem. 2].

Lemma 3.3. *Let $c \in \mathbb{R}\backslash\{0\}$ be a real number. Then the functions $\varphi_c^\pm \colon \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ given by*

$$\varphi_c^\pm(A) = \lambda_2(A) \pm \lambda_1(A) - \det(A)/c$$

are polyconvex. The same holds for the functions $\varphi_0^\pm = -\det$.

Proof. The lemma follows from the convexity of the functions $\lambda_2 \pm \lambda_1$, which in turn is proved by the explicit computation

$$\lambda_2(A) \pm \lambda_1(A) = \sqrt{|A|^2 \pm 2\det(A)} = \sqrt{(A_{11} \pm A_{22})^2 + (A_{21} \mp A_{12})^2}.$$

The functions $-\det/c$ as well as $-\det$ are polyconvex by definition. □

Remark 3.4. *Let $A \in \mathbb{M}^{2\times 2}$ be given. Consider the matrices $A_+, A_- \in \mathbb{M}^{2\times 2}$ defined via*

$$A_\pm = \begin{pmatrix} |\det(A)|^{1/2} & \pm\sqrt{|A|^2 - 2|\det(A)|} \\ 0 & |\det(A)|^{-1/2}\det(A) \end{pmatrix}.$$

The matrices A_+ and A_- are rank-one connected and $A_+, A_- \in \{A\}^{\mathrm{iso}}$ holds. Thus, the matrix $(A_+ + A_-)/2 = \operatorname{diag}(|\det(A)|^{1/2}, |\det(A)|^{-1/2}\det(A))$ as well as every other matrix $B \in \mathbb{M}^{2\times 2}$ with $\det(A) = \det(B)$ and $\lambda_2(B) \leq \lambda_2(A)$ lies in $(\{A\}^{\mathrm{iso}})^{\mathrm{lc},1}$.

4 Lamination convex sets

We will introduce the sets $L_\alpha^\pm$, L_β^0, $\triangle_\pm(\alpha,\beta)$ and $\triangle_0(\beta)$. With the help of these sets, the proof of our results is becoming much simpler.

The following lemma can be used to construct compact lamination convex sets.

Lemma 4.1. *Let $\alpha, \beta \geq 0$ be given real numbers. Then the following three sets are closed, isotropic and lamination convex*

$$L_\alpha^\pm = \{A \in \mathbb{M}^{2\times 2} \mid \alpha \leq \pm\lambda_1(A)\}, \quad L_\beta^0 = \{A \in \mathbb{M}^{2\times 2} \mid \lambda_2(A) \leq \beta\}.$$

Proof. By definition, the sets L^+_α, L^-_α and L^0_β are closed as well as isotropic. The set L^0_β is even convex, in fact, we have that $L^0_\beta = \{A\in\mathbb{M}^{2\times 2} \mid \|A\|_{\rm s}\leq\beta\}$ where $\|.\|_{\rm s}$ denotes the spectral norm. Since for L^-_α we can exploit the fact $L^-_\alpha = -L^+_\alpha$, it remains to show that L^+_α is lamination convex. Suppose that this is not the case. Then there exist rank-one connected matrices $A_1, A_2 \in \mathbb{M}^{2\times 2}$ and a real number $\mu \in [0,1]$ such that $\lambda_1(A_1), \lambda_1(A_2) \geq \alpha$ and $\alpha_0 = \lambda_1(\mu A_1 + (1{-}\mu)A_2) < \alpha$. On the one hand, since α_0 is a singular value of the matrix $\mu A_1 + (1{-}\mu)A_2$, there exist a normalized vector $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$ and a rotation $R \in \mathrm{SO}(2)$ such that

$$x_0^t R(\mu A_1 + (1{-}\mu)A_2)x_0 = \alpha_0.$$

On the other hand, we know that $|x_0^t R A_i x_0| \geq \lambda_1(A_i) \geq \alpha$ for $i = 1,2$. We conclude that $x_0^t R A_1 x_0$ and $x_0^t R A_2 x_0$ have different signs. Hence, we can fix a real number $\mu_0 \in [0,1]$ such that $x_0^t R(\mu_0 A_1 + (1{-}\mu_0)A_2)x_0 = 0$ and $\det(\mu_0 A_1 + (1{-}\mu_0)A_2) = 0$. This forms a contradiction, since the function $-\det$ is rank-one convex (even polyconvex) and $-\det(A_i) \leq -\alpha^2 < 0$ holds for $i = 1,2$. □

For given non-negative real numbers $\alpha, \beta \geq 0$ we consider the following isotropic and compact (possibly empty) sets

$$\begin{aligned} \triangle_\pm(\alpha,\beta) &= \{A \in \mathbb{M}^{2\times 2} \mid \alpha \leq \pm\lambda_1(A) \wedge \lambda_2(A) \leq \beta\}, \\ \triangle_0(\beta) &= \{A \in \mathbb{M}^{2\times 2} \mid \lambda_2(A) \leq \beta\}. \end{aligned}$$

We collect some properties of these sets.

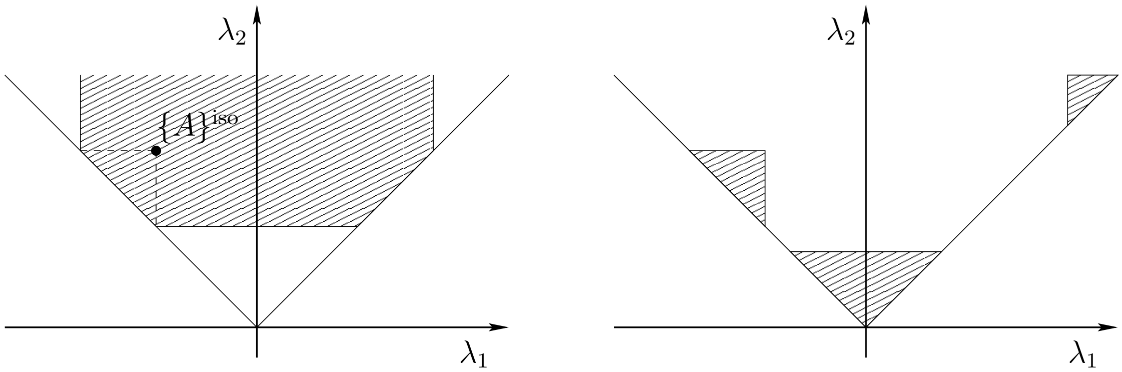
Lemma 4.2. *The sets $\triangle_+(\alpha,\beta)$, $\triangle_-(\alpha,\beta)$ as well as $\triangle_0(\beta)$ are compact, isotropic and lamination convex. Consider the matrices $A_1^\pm = \mathrm{diag}(\pm\alpha,\alpha)$, $A_2^\pm = \mathrm{diag}(\pm\alpha,\beta)$ and $A_3^\pm = \mathrm{diag}(\pm\beta,\beta)$. Then we have $\triangle_\pm(\alpha,\beta) = (\{A_1^\pm\}^{\rm iso}\cup\{A_2^\pm\}^{\rm iso}\cup\{A_3^\pm\}^{\rm iso})^{\rm clc}$ as well as $\triangle_0(\beta) = (\{A_3^-\}^{\rm iso}\cup\{A_3^+\}^{\rm iso})^{\rm clc}$.*

Proof. The sets $\triangle_+(\alpha,\beta)$, $\triangle_-(\alpha,\beta)$ as well as $\triangle_0(\beta)$ can be written as the intersection of L^+_α, L^-_α and L^0_β from Lemma 4.1, which implies the first part. The second part exploits that $\{A_1^\pm\}^{\rm iso}$ and $\{A_2^\pm\}^{\rm iso}$, $\{A_2^\pm\}^{\rm iso}$ and $\{A_3^\pm\}^{\rm iso}$ as well as $\{A_3^-\}^{\rm iso}$ and $\{A_3^+\}^{\rm iso}$ are compatible, see Lemma 3.2. □

Let $Z \subseteq \mathbb{M}^{2\times 2}$ be a given compact and isotropic set. Using the pair $\sigma(Z) = (\sigma_1(Z), \sigma_2(Z))$ given by $\sigma_1(Z) = \min\{|\lambda_1(A)| \mid A\in Z\}$ and $\sigma_2(Z) = \max\{\lambda_2(A) \mid A\in Z\}$, we define the set $Z^\triangle \subseteq \mathbb{M}^{2\times 2}$ (see Figure 1(b)) via

$$Z^\triangle = \begin{cases} \triangle_\pm(\sigma(Z)) & \text{if } \ \forall A \in Z \ \pm\lambda_1(A) > 0 \\ \triangle_0(\sigma_2(Z)) & \text{otherwise.} \end{cases} \tag{3}$$

In view of Lemma 4.2, we obtain $Z \subseteq Z^{\rm clc} \subseteq Z^\triangle$.



(a) Region of compatibility for $\{A\}^{\text{iso}}$ (b) Different types of $Z^{\triangle}$

Figure 1: Subsets in the cone $\{|\lambda_1| \leq \lambda_2\}$

5 A refinement for the connected case

Conti et al. [CDLMR03] show that polyconvexity and lamination convexity are the same for isotropic compact subsets of $\mathbb{M}^{2\times 2}$ that are connected. Their idea can be used to prove a bit more. In order to see that, we will sketch their proof and give the details where minor changes are necessary.

Theorem 5.1. *Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given isotropic and compact set and $Z \in \operatorname{cc}(K^{\mathrm{lc},1})$ a connected component. Then $Z^{\mathrm{lc},1}$ is polyconvex.*

Proof. Let $Z \in \operatorname{cc}(K^{\mathrm{lc},1})$ be an arbitrary but fixed connected component. Then we have

$$Z^{\mathrm{lc},1} \supseteq \{B \in \mathbb{M}^{2\times 2} \mid \exists C \in Z^{\mathrm{lc},1} \ \det(B) = \det(C) \wedge |\lambda_1(B)| = \lambda_2(B)\}. \tag{4}$$

In fact, set $d_1 = \min\{\det(B) \mid B \in Z\}$ and $d_2 = \max\{\det(B) \mid B \in Z\}$. By definition, the set $\{B \in \mathbb{M}^{2\times 2} \mid d_1 \leq \det(B) \leq d_2\}$ is polyconvex and, hence, $Z^{\mathrm{lc},1}$ is a subset of it. The connectedness of Z together with Remark 3.4 implies that every matrix $B \in \mathbb{M}^{2\times 2}$ with $d_1 \leq \det(B) \leq d_2$ and $|\lambda_1(B)| = \lambda_2(B)$ lies in $Z^{\mathrm{lc},1}$.

We show that for every matrix $A \in \mathbb{M}^{2\times 2} \setminus Z^{\mathrm{lc},1}$ there is a polyconvex function $\varphi \colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ that separates A from $Z^{\mathrm{lc},1}$, meaning $\varphi(A) > \max\{\varphi(B) \mid B \in Z^{\mathrm{lc},1}\}$. In order to do that, we follow Conti et al. [CDLMR03]. They show that it is sufficient to check every $A \in \mathbb{M}^{2\times 2} \setminus Z^{\mathrm{lc},1}$ such that $A = \operatorname{diag}(\sigma_1, \sigma_2)$ holds for some real numbers $0 \leq \sigma_1 \leq \sigma_2$. Fix such a matrix A. If $\sigma_1 = \sigma_2$ holds, the set $\{B \in \mathbb{M}^{2\times 2} \mid \det(B) = \det(A)\}$ does not intersect $Z^{\mathrm{lc},1}$. Otherwise (4) yields that A must lie in $Z^{\mathrm{lc},1}$, a contradiction. Thus, the connectedness of $Z^{\mathrm{lc},1}$ implies that we can either put $\varphi = \det$ or $\varphi = -\det$ and are done.

Assume that $\sigma_2 > \sigma_1$. Given a real number $c \in [-\sigma_2, \sigma_2]$, they consider the level set

$$L_c = \begin{cases} \{B \in \mathbb{M}^{2\times 2} \mid \varphi_c^-(B) = \varphi_c^-(A)\} & \text{for} \quad c \in [-\sigma_2, \sigma_1[\\ \{B \in \mathbb{M}^{2\times 2} \mid \varphi_c^+(B) = \varphi_c^+(A)\} & \text{for} \quad c \in [\sigma_1, \sigma_2], \end{cases} \tag{5}$$

see Lemma 3.3 for the definition of $\varphi_c^{\pm}$. They show that there exists a polyconvex φ that separates A from $Z^{\mathrm{lc},1}$ whenever at least one of the L_c does not intersect $Z^{\mathrm{lc},1}$. In fact, by a nice argument, they can reduce this further. Let $\widetilde{Z} \subseteq \mathbb{M}^{2\times 2}$ be any compact, connected and isotropic set. They prove that there exists one $L_{\widetilde{c}}$ that does not intersect $\widetilde{Z}$ if for every $c \in [-\sigma_2, \sigma_2]$ at least one of the sets $L_c^{>} \cap \widetilde{Z}$ and $L_c^{<} \cap \widetilde{Z}$ is empty, where $L_c^{>}$ and $L_c^{<}$ are the connected components of $L_c \setminus \{A\}^{\mathrm{iso}}$. This can be used for $\widetilde{Z} = Z^{\mathrm{lc},1}$. Fix $c \in [-\sigma_2, \sigma_2]$ and suppose that both sets $L_c^{>} \cap Z^{\mathrm{lc},1}$ and $L_c^{<} \cap Z^{\mathrm{lc},1}$ are non-empty. Then there show that A must lie in $\{B, C\}^{\mathrm{lc},1}$ for some rank-one connected matrices $B \in L_c^{>} \cap Z^{\mathrm{lc},1}$ and $C \in L_c^{<} \cap Z^{\mathrm{lc},1}$. This forms a contradiction as long as $Z^{\mathrm{lc},1}$ is lamination convex and, hence, completes their proof.

In our case, we use the following argument. We still have $A \in \{B, C\}^{\mathrm{lc},1} \subseteq Z^{\mathrm{lc},2}$ and conclude that $d_1 \leq \det(A) \leq d_2$. Connectedness of Z implies that there exists a matrix $A' \in Z$ with $\det(A') = \det(A)$. We know that $A \notin Z^{\mathrm{lc},1}$ holds and, hence, we conclude that $\lambda_2(A') < \lambda_2(A)$ by Remark 3.4. A simple computation shows that

$$\forall c \in [-\sigma_2, \sigma_2] \quad \varphi_c^{\pm}(A) \geq \varphi_c^{\pm}(A'). \tag{6}$$

If for every $c \in [-\sigma_2, \sigma_2]$ at least one of the sets $L_c^{>} \cap Z$ and $L_c^{<} \cap Z$ is empty, then we use the above argument for $\widetilde{Z} = Z$. Hence, we can fix a real number $\widetilde{c} \in [-\sigma_2, \sigma_2]$ such that $L_{\widetilde{c}}$ does not intersect Z. Let $\varphi \in \{\varphi_{\widetilde{c}}^{+}, \varphi_{\widetilde{c}}^{-}\}$ be the function that defines $L_{\widetilde{c}}$ in (5). Then connectedness of Z implies that either $\varphi(A) < \min\{\varphi(B) \mid B \in Z\}$ or $\varphi(A) > \max\{\varphi(B) \mid B \in Z\}$. In view of (6), the second alternative must hold, meaning φ separates A from Z. Polyconvexity of φ implies that φ also separates A from $Z^{\mathrm{lc},1}$ and we are done. Now if there exists a real number $c \in [-\sigma_2, \sigma_2]$ such that both sets $L_c^{>} \cap Z$ and $L_c^{<} \cap Z$ are non-empty, then, as before, there exist $B \in L_c^{>} \cap Z$ and $C \in L_c^{<} \cap Z$ such that A lies in $\{B, C\}^{\mathrm{lc},1}$. But $\{B, C\}^{\mathrm{lc},1}$ is contained in $Z^{\mathrm{lc},1}$ and, hence, we must have $A \in Z^{\mathrm{lc},1}$, a contradiction. ☐

6 Closed lamination convex hull

We are going to characterize the closed lamination convex hull of an isotropic and compact set of 2×2 matrices. The key ingredients are the following two lemmas. The first shows that the laminates of order one fully describe the topology of the closed lamination convex hull.

Lemma 6.1. *Let $K \subseteq \mathbb{M}^{2\times 2}$ be compact and isotropic. Let $Z_1, Z_2 \in \mathrm{cc}(K^{\mathrm{lc},1})$ be arbitrary but fixed connected components with $Z_1 \neq Z_2$. Then $(Z_1)^{\triangle}$ and $(Z_2)^{\triangle}$ are incompatible and so are $(Z_1)^{\mathrm{clc}}$ and $(Z_2)^{\mathrm{clc}}$ as well as Z_1 and Z_2.*

Proof. Since K is compact, so are the sets $K^{\mathrm{lc},1}$, Z_1 and Z_2. The compact and isotropic sets given via $K_i = Z_i \cap K$ fulfill $(K_i)^{\mathrm{clc}} = (Z_i)^{\mathrm{clc}}$ for $i = 1, 2$. In addition, the sets K_1 and K_2 are incompatible. Otherwise there exist rank-one connected matrices $B_1 \in K_1$ and $B_2 \in K_2$ such that $\{B_1, B_2\}^{\mathrm{clc}}$ connects Z_1 and Z_2, which forms a contradiction.

We know that $Z_i \subseteq (Z_i)^{\mathrm{clc}} \subseteq (Z_i)^{\triangle}$ as well as $(K_i)^{\mathrm{clc}} \subseteq (K_i)^{\triangle}$ and, hence, $(K_i)^{\triangle} = (Z_i)^{\triangle}$ holds for $i = 1, 2$. It suffices to prove that $(K_1)^{\triangle}$ and $(K_2)^{\triangle}$ are incompatible. Without loss of generality, we set $\sigma_2(K_1) \leq \sigma_2(K_2)$. We distinguish two cases. First, suppose that $\sigma_1(K_2) > \sigma_2(K_1)$. Then, by Lemma 3.2, the sets $(K_1)^{\triangle}$ and $(K_2)^{\triangle}$ are incompatible. Second, suppose that $\sigma_1(K_2) \leq \sigma_2(K_1)$. Fix matrices $B_1 \in K_1$ and $B_2, B_2' \in K_2$ such that $\lambda_2(B_1) =$

$\sigma_2(K_1)$, $|\lambda_1(B_2)| = \sigma_1(K_2)$ and $\lambda_2(B_2') = \sigma_2(K_2)$. Since K_1 and K_2 are incompatible, so are $\{B_1\}$ and $\{B_2\}$ as well as $\{B_1\}$ and $\{B_2'\}$. We conclude that

$$|\lambda_1(B_2)| \leq \lambda_2(B_2) < |\lambda_1(B_1)| \leq \lambda_2(B_1) < |\lambda_1(B_2')| \leq \lambda_2(B_2').$$

But then the set K_2 decomposes into at least two incompatible subsets and, hence, Z_2 is not connected. This is a contradiction. □

The next lemma gives a candidate for the closed lamination convex hull.

Lemma 6.2. *Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given compact and isotropic set. Then the set $T = \cup\{Z^{\mathrm{clc}} \mid Z \in \mathrm{cc}(K^{\mathrm{lc},1})\}$ is compact, lamination convex and contains K.*

Proof. By definition, T contains K. We show that T is compact. Let $A_1, A_2, \ldots$ be a given sequence in T. Since T is a bounded set, we can and we will assume that $A_k \to A$ in $\mathbb{M}^{2\times 2}$ holds for some matrix $A \in \mathbb{M}^{2\times 2}$. If necessary, we replace $A_1, A_2, \ldots$ by a subsequence. Let $Z_1, Z_2, \cdots \in \mathrm{cc}(K^{\mathrm{lc},1})$ be the sequence of connected components such that $A_k \in (Z_k)^{\mathrm{clc}} \subseteq (Z_k)^{\triangle}$ holds for every $k = 1, 2, \ldots$. First, suppose that there exists a real number $\epsilon > 0$ such that for every $k = 1, 2, \ldots$ we have $|(Z_k)^{\triangle}| \geq \epsilon$ where $|.|$ denotes the Lebesgue measure of a set. Boundedness of T implies that there exists a connected component $Z_0 \in \mathrm{cc}(K^{\mathrm{lc},1})$ and a subsequence (not relabeled) such that $Z_k = Z_0$ for every k. Since the set $(Z_0)^{\mathrm{clc}}$ is compact, A lies in $(Z_0)^{\mathrm{clc}} \subseteq T$. Second, suppose that there is no such $\epsilon > 0$ as before. Then there exists a subsequence (not relabeled) such that $|(Z_k)^{\triangle}| \to 0$ holds. In view of (3), this means that

$$\sup\{|B_1 - B_2| \mid B_1 \in (Z_k)^{\triangle} \ \wedge \ B_2 \in Z_k\} \to 0.$$

We take any sequence $A_1', A_2', \ldots$ in $K^{\mathrm{lc},1}$ such that $A_k' \in Z_k$ for every $k = 1, 2, \ldots$. Then we must have $A_k' \to A$ and, hence, compactness of $K^{\mathrm{lc},1}$ implies that $A \in K^{\mathrm{lc},1} \subseteq T$.

Now we show that T is lamination convex. Let $A_1, A_2 \in T$ be given matrices. First, suppose that $A_1, A_2 \in Z^{\mathrm{clc}}$ for some $Z \in \mathrm{cc}(K^{\mathrm{lc},1})$. Then we have $\{A_1, A_2\}^{\mathrm{clc}} \subseteq Z^{\mathrm{clc}} \subseteq T$. Second, suppose that $A_i \in Z_i^{\mathrm{clc}}$ for $i = 1, 2$ such that $Z_1, Z_2 \in \mathrm{cc}(K^{\mathrm{lc},1})$ and $Z_1 \neq Z_2$. We know from Lemma 6.1 that $(Z_1)^{\mathrm{clc}}$ and $(Z_2)^{\mathrm{clc}}$ are incompatible and so are $\{A_1\}$ and $\{A_2\}$. We conclude that $\{A_1, A_2\}^{\mathrm{clc}} = \{A_1, A_2\} \subseteq T$. □

Finally, we are in the position to characterize the closed lamination convex hull.

Theorem 6.3 (Characterization of K^{clc})**.** *Let $K \subseteq \mathbb{M}^{2\times 2}$ be compact and isotropic. Then its closed lamination convex hull is given by $K^{\mathrm{clc}} = K^{\mathrm{lc},2}$.*

Proof. Let $T \subseteq \mathbb{M}^{2\times 2}$ be as in Lemma 6.2. On the one hand, we know that $K^{\mathrm{clc}} = (K^{\mathrm{lc},1})^{\mathrm{clc}} \supseteq T$. On the other hand, we have shown in Lemma 6.2 that the set T is lamination convex, compact and contains K. We conclude that $K^{\mathrm{clc}} = T$.

Let $Z \in \mathrm{cc}(K^{\mathrm{lc},1})$ be a connected component. Then $Z^{\mathrm{lc},1}$ is polyconvex as an application of Theorem 5.1. In particular, we have $Z^{\mathrm{clc}} = Z^{\mathrm{lc},1}$ and, hence, $K^{\mathrm{clc}} \subseteq K^{\mathrm{lc},2}$. Since the other inclusion holds by definition, we conclude that $K^{\mathrm{clc}} = K^{\mathrm{lc},2}$. □

7 Quasiconvex hull

We show the equivalence of quasiconvexity and lamination convexity for isotropic compact subsets of $\mathbb{M}^{2\times 2}$. We rely on a result by Faraco and Székelyhidi [FS08].

The next lemma deals with the case of two connected components.

Lemma 7.1. *Let $S \subseteq \mathbb{M}^{2\times 2}$ be a given compact set (not necessarily isotropic) and $0 \leq \beta_1 < \alpha_2 \leq \beta_2$ real numbers. If $S \subseteq \triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ holds and S^{qc} is connected, then one of the sets $S \cap \triangle_0(\beta_1)$ and $S \cap \triangle_+(\alpha_2, \beta_2)$ must be empty.*

Proof. By rescaling the matrix space $\mathbb{M}^{2\times 2}$, if necessary, we can and we will assume that there exists a positive real number $\epsilon > 0$ such that

$$\beta_1 \leq 1-\epsilon < 1+\epsilon \leq \alpha_2 \leq \beta_2. \tag{7}$$

Lemma 3.3 implies that $f = \varphi_1^+ - 1$, with $f(A) = \lambda_1(A) + \lambda_2(A) - \det(A) - 1$, is a polyconvex function. In particular, the set $P = \{A \in \mathbb{M}^{2\times 2} \mid f(A) \leq -\epsilon^2\}$ is polyconvex by definition. We are going to show that $\triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ is a subset of P. Consider the matrices $A_1 = \mathrm{diag}(-\beta_1, \beta_1)$, $A_2 = \mathrm{diag}(\beta_1, \beta_1)$, $A_3 = \mathrm{diag}(\alpha_2, \alpha_2)$, $A_4 = \mathrm{diag}(\alpha_2, \beta_2)$ and $A_5 = \mathrm{diag}(\beta_2, \beta_2)$. Since, in addition, P is isotropic, Lemma 4.2 implies that it is sufficient to show that $A_i \in P$ for $i = 1, \ldots, 5$. However this can be tested easily if we make use of (7) and the fact that for every $A \in \mathbb{M}^{2\times 2}$ we have $f(A) = (1 - \lambda_1(A))(\lambda_2(A) - 1)$.

We have shown that $S \subseteq \triangle_0(\beta_1) \cup \triangle_+(\alpha_2, \beta_2)$ is a subset of P. Since the set P is polyconvex, the quasiconvex hull S^{qc} is also contained in P. Yet the identity matrix lies not in P. As a consequence of Remark 3.4, for every matrix $A \in S^{\mathrm{qc}}$ we must have $\det(A) \neq 1$. We know that S^{qc} is connected and, in addition, $\det < 1$ holds in $\triangle_0(\beta_1)$ and $\det > 1$ in $\triangle_+(\alpha_2, \beta_2)$. Hence, one of the sets $S \cap \triangle_0(\beta_1)$ and $S \cap \triangle_+(\alpha_2, \beta_2)$ must be empty. □

Now we are going to prove our result about the equivalence of lamination convexity and quasiconvexity.

Theorem 7.2 (Equivalence)**.** *Let $K \subseteq \mathbb{M}^{2\times 2}$ be a given compact and isotropic set. Then K is lamination convex if and only if K is quasiconvex.*

Proof. We only have to show one implication. Assume that K is lamination convex and let $\nu \in \mathcal{P}^{\mathrm{qc}}$ be a fixed homogenous gradient Young measure with support $S = \mathrm{supp}(\nu) \subseteq K$. By Remark 2.1, we need to show that $\bar{\nu} \in K$. Let $\mathcal{Z}$ be the set of all connected components $Z \in \mathrm{cc}(K)$ such that $Z \cap S$ is non-empty. First, suppose that there exists only one such connected component, meaning $\mathcal{Z} = \{Z\}$. Since Z is isotropic, lamination convex, compact and connected, Theorem 5.1 implies that Z is quasiconvex (even polyconvex). Hence, $\bar{\nu}$ must lie in $Z \subseteq K$.

Second, suppose that S is distributed over more than one connected component. By compactness arguments, we can fix $Z_1, Z_2 \in \mathcal{Z}$ that are extremal in the following sense. For every

$Z \in \mathcal{Z}$ we have $\sigma_2(Z_1) \leq \sigma_2(Z) \leq \sigma_2(Z_2)$. Up to symmetry, there are only three different cases: $(Z_2)^{\triangle} = \triangle_+(\alpha_2, \beta_2)$ and either $(Z_1)^{\triangle} = \triangle_0(\beta_1)$, $(Z_1)^{\triangle} = \triangle_+(\alpha_1, \beta_1)$ or $(Z_1)^{\triangle} = \triangle_-(\alpha_1, \beta_1)$ for some reals $0 \leq \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2$.

We fix real numbers $\tilde{\beta}, \tilde{\alpha} \in \mathbb{R}$ such that $\beta_1 < \tilde{\beta} < \tilde{\alpha} < \alpha_2$ holds as well as

$$S \subseteq \triangle_0(\tilde{\beta}) \cap \triangle_+(\tilde{\alpha}, \beta_2). \tag{8}$$

In order to see that this can be done, let $\epsilon > 0$ be a given real number. Recall that $K^{\mathrm{lc},1} = K$ holds and, hence, Lemma 6.1 implies that elements in $\mathcal{Z}$ are pairwise incompatible. We fix $\tilde{\beta}(\epsilon), \tilde{\alpha}(\epsilon) \in \mathbb{R}$ such that $\alpha_2 - \epsilon < \tilde{\beta}(\epsilon) < \tilde{\alpha}(\epsilon) < \alpha_2$ holds and, in addition, for every $Z \in \mathcal{Z}$ we have either $\sigma_2(Z) < \tilde{\beta}(\epsilon)$ or $\sigma_1(Z) > \tilde{\alpha}(\epsilon)$. Suppose that $\tilde{\beta}(\epsilon)$ and $\tilde{\alpha}(\epsilon)$ fail to fulfill (8) for every $\epsilon > 0$. Then there must be a sequence $A_1, A_2, \ldots$ in S such that $\lambda_1(A_k) < 0$ holds for every $k = 1, 2, \ldots$ and $\lambda_2(A_k) \to \alpha_2$. By compactness of the set S, we can fix a cluster point $A_0 \in S$ of this sequence. On the one hand, we know that $\lambda_1(A_0) \leq 0$ and, hence, $A_0 \notin (Z_2)^{\triangle} = \triangle_+(\alpha_2, \beta_2)$. On the other hand, $\lambda_2(A_0) = \alpha_2$ implies that $\{A_0\}$ and $(Z_2)^{\triangle}$ are compatible. By Lemma 6.1, we must have $(Z_0)^{\triangle} = (Z_2)^{\triangle}$ where Z_0 is given by $A_0 \in Z_0 \in \mathcal{Z}$. This is a contradiction.

A result by Faraco and Székelyhidi [FS08, Cor. 3] implies that S^{qc} is connected. As a consequence of Lemma 7.1, one of the sets $S \cap \triangle_0(\tilde{\beta})$ and $S \cap \triangle_+(\tilde{\alpha}, \beta_2)$ must be empty. This forms a contradiction. Hence, it is impossible that S is distributed over more than one connected component. ☐

Our next result can be used to compute the quasiconvex hull.

Theorem 7.3 (Characterization of K^{qc})**.** *Let $K \subseteq \mathbb{M}^{2\times 2}$ be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2.*

Proof. Clearly, the set K^{clc} is compact, isotropic and lamination convex. Theorem 6.3 and Theorem 7.2 imply that $K^{\mathrm{lc},2} = K^{\mathrm{clc}} = K^{\mathrm{qc}}$ holds. ☐

References

[AT87] Gilles Aubert and Rabah Tahraoui. Sur la faible fermeture de certains ensembles de contraintes en elasticite non-lineaire plane. *Arch. Rational Mech. Anal.*, 97:33–58, 1987. 10.1007/BF00279845.

[Bal77] John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1977.

[CDLMR03] Sergio Conti, Camillo De Lellis, Stefan Müller, and Mario Romeo. Polyconvexity equals rank-one convexity for connected isotropic sets in $\mathbb{M}^{2\times 2}$. *C. R. Acad. Sci. Paris, Sér. I*, 337(4):233–238, 2003.

[CT00] Pierre Cardaliaguet and Rabah Tahraoui. Sur l'équivalence de la 1-rang convexité et de la polyconvexité des ensembles isotropiques de $\mathbb{R}^{2\times 2}$. *C. R. Acad. Sci. Paris, Sér. I*, 331:851–856, 2000.

[CT02a] Pierre Cardaliaguet and Rabah Tahraoui. Equivalence between rank-one convexity and polyconvexity for isotropic sets of $\mathbb{R}^{2\times 2}$ (part I). *Nonlin. Anal.*, 50(8):1179–1199, 2002.

[CT02b] Pierre Cardaliaguet and Rabah Tahraoui. Equivalence between rank-one convexity and polyconvexity for isotropic sets of $\mathbb{R}^{2\times 2}$ (part II). *Nonlin. Anal.*, 50(8):1201–1239, 2002.

[Dac89] Bernard Dacorogna. *Direct Methods in the Calculus of Variations*. Springer-Verlag, Berlin, 1989.

[FS08] Daniel Faraco and László Székelyhidi. Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. *Acta Math.*, 200:279–305, 2008. 10.1007/s11511-008-0028-1.

[Kol03] Jan Kolář. Non-compact lamination convex hulls. *Ann. I. H. Poincaré – AN*, 20(3):391–403, 2003.

[KP91] David Kinderlehrer and Pablo Pedregal. Characterizations of Young measures generated by gradients. *Arch. Rational Mech. Anal.*, 115:329–365, 1991.

[Kri99] Jan Kristensen. On the non-locality of quasiconvexity. *Ann. I. H. Poincaré – AN*, 16(1):1–13, 1999.

[Mor52] Charles B. Morrey, Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2:25–53, 1952.

[MŠ96] Stefan Müller and Vladimír Šverák. Attainment results for the two-well problem by convex integration. In Jürgen Jost, editor, *Geometric Analysis and the Calculus of Variations*, pages 239–251. International Press, Cambridge, MA, 1996.

[Mül99a] Stefan Müller. Rank-one convexity implies quasiconvexity on diagonal matrices. *Internat. Math. Res. Notices*, 20:1087–1095, 1999.

[Mül99b] Stefan Müller. Variational models for microstructure and phase transitions. In *Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996)*, pages 85–210. Springer, Berlin, 1999.

[Ped93] Pablo Pedregal. Laminates and microstructure. *Eur. J. Appl. Math.*, 4:121–149, 1993.

[Šil01] Miroslav Šilhavý. Rotationally invariant rank 1 convex functions. *Appl. Math. Optim.*, 44(1):1–15, 2001.

[Šve92] Vladimír Šverák. Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(1-2):185–189, 1992.