Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e. V.

Preprint ISSN 0946 – 8633

On the numerical approximation of a viscoelastodynamic problem with unilateral constraints

Dedicated to the memory of J.A.C. Martins

Adrien Petrov[1]

submitted: July 7, 2010

[1] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: petrov@wias-berlin.de

No. 1523
Berlin 2010

2010 *Mathematics Subject Classification.* 35L85, 49J40, 65N30, 73D99, 73V65.

Key words and phrases. Viscoelasticity, Signorini conditions, numerical scheme, variational inequality.

The author acknowledges L. Paoli for reading this work and for providing helpful comments and he is grateful to M. Schatzman for stimulating discussions. The research was supported by the Deutsche Forschungsgemeinschaft through the projet C18 "Analysis and numerics of multidimensional models for elastic phase transformation in a shape-memory alloys" of the Research Center MATHEON.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The present work is dedicated to the study of numerical schemes for a viscoelastic bar vibrating longitudinally and having its motion limited by rigid obstacles at the both ends. Finite elements and finite difference schemes are presented and their convergence is proved. Finally, some numerical examples are reported and analyzed.

1 Introduction

We consider a viscoelastic bar of length L, which, vibrates longitudinally. More precisely, each end of the bar is free to move as long as it does not hit a material obstacle and each obstacle may constrain the displacement of the extremity to be greater than or equal to some number.

The mathematical situation can be described as follows: assume that the bar is made up of an homogeneous viscoelastic material and satisfies the assumptions of the theory of small deformations. Let $u(x,t)$ be the displacement at time t of the material point of spatial coordinate $x \in (0,L)$. Let $f(x,t)$ denote a density of external forces, depending on space and time. Define $\Omega \stackrel{\text{def}}{=} (0,L)$ and let α be a strictly positive number. The mathematical problem is formulated as follows:

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f,\ x \in \Omega,\ t > 0, \tag{1.1}$$

with Cauchy initial data

$$u(\cdot,0) = u_0 \quad \text{and} \quad u_t(\cdot,0) = u_1, \tag{1.2}$$

and unilateral boundary conditions at $x = c_0$ and $x = c_L$, $t > 0$,

$$0 \le (u(0,\cdot)+c_0) \perp -u_x(0,\cdot) - \alpha u_{xt}(0,\cdot) \ge 0, \tag{1.3a}$$

$$0 \le (u(L,\cdot)+c_L) \perp u_x(L,\cdot) + \alpha u_{xt}(L,\cdot) \ge 0, \tag{1.3b}$$

where $(\cdot)_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t}(\cdot)$, $(\cdot)_x \stackrel{\text{def}}{=} \frac{\partial}{\partial x}(\cdot)$ and $c_0, c_L > 0$. The orthogonality has the natural meaning: if we have enough regularity, it means that the product $(u(X,\cdot)+c_X)(u_x(X,\cdot) + \alpha u_{xt}(X,\cdot))$ vanishes almost everywhere on the boundary, with $X = 0, L$. If we do not have enough regularity, the above inequality is integrated on an appropriate set of test functions, yielding a weak formulation for the unilateral condition. From mechanical point of view, (1.3) means that when the bar touches the obstacle in $X = 0$ or $X = L$, its reaction can be only upwards, so that $u_x(0,\cdot) + \alpha u_{xt}(0,\cdot) \le 0$ on the set $\{t : u(0,\cdot) = -c_0\}$ or $u_x(L,\cdot) + \alpha u_{xt}(L,\cdot) \ge 0$ on the set $\{t : u(L,\cdot) = -c_L\}$. When the bar does not touch the obstacle, the end is free to move, namely, we have $u_x(X,\cdot) + \alpha u_{xt}(X,\cdot) = 0$ on the set $\{t : u(X,\cdot) > -c_X\}$, $X = 0, L$. Note that conditions (1.2) are also termed Signorini conditions. We suppose that the initial position u_0 belongs to $\mathrm{H}^2(\Omega)$ and satisfies the compatibility conditions, i.e., $u_0(0,0) = u_0(0) \ge -c_0$ and $u_0(L,0) = u_0(L) \ge -c_L$, the initial velocity u_1 belongs to $\mathrm{H}^1(\Omega)$ and the density of forces f belongs to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$.

Let us describe the weak formulation of the problem. Denote by K the convex set:

$$K \stackrel{\text{def}}{=} \{v \in \mathrm{H}^1(\Omega \times (0,T)) : v_{xt} \in \mathrm{L}^2(\Omega \times (0,T)), v(0,\cdot) \geq -c_0, v(L,\cdot) \geq -c_L\}.$$

This unusual convex set has been devised in order to write a weak formulation of our problem. Since we expect to find a scalar product (u_{xt}, v_x), we require u_{xt} to be square integrable. Thus, the weak formulation associated to (1.1)–(1.3) is obtained by multiplying (1.1) by $v-u$, $v \in K$, and by integrating formally over $\Omega \times (0,\tau)$, $\tau \in [0,T]$. Then, we get the following variational formulation

$$\begin{cases} \text{Find } u \in K \text{ such that for all } v \in K \text{ and for all } \tau \in [0,T], \\ \displaystyle\int_\Omega (u_t(v-u))\Big|_0^\tau \mathrm{d}x - \int_0^\tau \int_\Omega u_t(v_t - u_t)\,\mathrm{d}x\,\mathrm{d}t \\ \displaystyle+ \int_0^\tau \int_\Omega (u_x + \alpha u_{xt})(v_x - u_x)\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^\tau \int_\Omega f(v-u)\,\mathrm{d}x\,\mathrm{d}t. \end{cases} \tag{1.4}$$

The existence result for (1.1)–(1.3) is easily established by penalty method and was already proved by Jarušek et al. [JM*93] in the case of distributed constraints. To do so, the obstacle constraints are penalized, this means that the rigid constraints are replaced by very stiff responses. When the constraint is active, the response is linear and it vanishes when the constraint is not active (see [MaO88]). Thus, the existence of a weak solution is obtained by passing to the limit with respect to the penalty parameter in the variational formulation associated to the penalty problem. The detailed proof can be find in [Pet02, pp. 13–26]. The reader is also referred to [PeS09] for some existence results in higher dimension. Observe that nothing is known about uniqueness.

In the present paper, a family of numerical schemes is defined in Section 2 with help of a variational formulation. Let V_h be a sequence of approximation spaces of $\mathrm{H}^1(\Omega)$, which, can be a space of finite elements. Let K_h be a convex set of elements of V_h. The duality product in $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$ are denoted by $(\cdot,\cdot)$ and by

$$a(u,v) \stackrel{\text{def}}{=} \int_0^L u_x v_x\,\mathrm{d}x, \tag{1.5}$$

respectively. Thus, the family of schemes is defined by

$$\begin{cases} \text{Find } u^{n+1} \in K_h \text{ such that for all } v \in K_h, \\ \displaystyle\left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}, v - u^{n+1}\right) + a\left(\frac{u^{n+1} + u^{n-1}}{2}, v - u^{n+1}\right) \\ \displaystyle+ \alpha a\left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}, v - u^{n+1}\right) \geq \left(f^n, v - u^{n+1}\right), \end{cases} \tag{1.6}$$

with f^n a suitable discretization of f and initial conditions u^0 and u^1 adequately chosen. Thus, we establish that the scheme (1.6) converges, under a stability condition, to a solution of

(1.1)–(1.3). In Section 3, we deal with finite difference schemes given by

$$\begin{cases} \dfrac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2} - \left(\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{2\Delta x^2} + \dfrac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{2\Delta x^2}\right) \\ -\alpha\left(\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{2\Delta t\Delta x^2} - \dfrac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{2\Delta t\Delta x^2}\right) = f_j^n \text{ for } j\in[2,J-1], \\ u_1^{n+1}=\max\left(c_0, \frac{1}{c_\Delta}\left(\frac{2u_1^n-u_1^{n-1}}{\Delta t^2}+\left(\frac{u_2^{n+1}}{2\Delta x^2}+\frac{u_2^{n-1}-u_2^{n-1}}{2\Delta x^2}\right)+\alpha\left(\frac{u_2^{n+1}}{2\Delta t\Delta x^2}-t\frac{u_2^{n-1}-u_1^{n-1}}{2\Delta t\Delta x^2}\right)\right)\right), \\ u_J^{n+1}=\max\left(c_L, \frac{1}{c_\Delta}\left(\frac{2u_J^n-u_J^{n-1}}{\Delta t^2}+\left(\frac{u_{J-1}^{n+1}}{2\Delta x^2}+\frac{u_{J-1}^{n-1}-u_J^{n-1}}{2\Delta x^2}\right)+\alpha\left(\frac{u_{J-1}^{n+1}}{2\Delta t\Delta x^2}-\frac{u_{J-1}^{n-1}-u_J^{n-1}}{2\Delta t\Delta x^2}\right)\right)\right), \end{cases} \tag{1.7}$$

where $c_\Delta \neq 0$ is some constant. The convergence of the scheme (1.7) is proved. Finally, numerical examples in Section 4 illustrate the usage of the provided tools for benchmark examples.

2 Fully discretized finite elements schemes

We consider V_h and H_h two sequences of finite-dimensional subspaces of $\mathrm{H}^1(\Omega)$ and $\mathrm{L}^2(\Omega)$ such that $\mathrm{L}^2(\Omega) = \overline{\bigcup_h H_h}^{\mathrm{L}^2(\Omega)}$ and $\mathrm{H}^1(\Omega) = \overline{\bigcup_h V_h}^{\mathrm{H}^1(\Omega)}$. We assume that Δt is the uniform time step, $t_n = t_0 + n\Delta t$ and $n \leq N(\Delta t)$ with $N(\Delta t) \stackrel{\text{def}}{=} \lfloor \frac{T}{\Delta t} \rfloor$ denotes the greatest integer at the most equal to $\frac{T}{\Delta t}$. Let $K_h \stackrel{\text{def}}{=} \{v \in V_h : v_{xt} \in H_h, v(0) \geq -c_0, v(L) \geq -c_L$ be the sequence of convex sets. We define a fully discrete scheme

$$\begin{cases} \text{Find } u_h^{n+1} \in K_h \text{ such that for all } v \in K_h, \\ \left(\dfrac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2}, v-u_h^{n+1}\right) + a\left(\dfrac{u_h^{n+1}+u_h^{n-1}}{2}, v-u_h^{n+1}\right) \\ +\alpha a\left(\dfrac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, v-u_h^{n+1}\right) \geq \left(f_h^n, v-u_h^{n+1}\right), \end{cases} \tag{2.8}$$

with f_h^n a suitable discretization of f and initial conditions u_h^0 and u_h^1 satisfying

$$\lim_{h\downarrow 0}\left(\|u_h^0-u_0\| + \left|\frac{u_h^1-u_h^0}{\Delta t}-u_1\right|\right) = 0, \tag{2.9}$$

where $|\cdot|$ and $\|\cdot\|$ denote the norms in $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$, respectively. Note that (2.8) can be rewritten in a slightly different but equivalent form. To do so, we define an operator $A_h : V_h \to V_h$ such that

$$\forall v \in V_h : \ (A_h u_h^{n+1}, v) = a(u_h^{n+1}, v), \tag{2.10}$$

and a maximal monotone operator such that

$$\partial\psi_{K_h}(u_h^{n+1}) \stackrel{\text{def}}{=} \begin{cases} \{0\} & \text{if } \ u_h^{n+1} \in \mathrm{int}(K_h), \\ \{w \in K_h : (w, v-u_h^{n+1}) \leq 0, \forall v \in K_h\} & \text{if } \ u_h^{n+1} \in \partial K_h, \\ \emptyset & \text{otherwise}. \end{cases} \tag{2.11}$$

For further details on maximal monotone operators, the reader is referred to [Lio69, Bré73]. Clearly, using (2.10) and (2.11), it follows that (2.8) can be rewritten as

$$\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2} + A_h\frac{u_h^{n+1}+u_h^{n-1}}{2} + A_h\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t} + \partial\psi_{K_h}(u_h^{n+1}) \ni f_h^n. \tag{2.12}$$

The scheme (2.8) is implicit in the constraints. It is equivalent to minimize a coercive and twice differentiable function in a convex set. Thus, for each step u_h^n is unique.

We establish now the convergence of the numerical scheme (2.8). More precisely, following the analogous ideas developed for a wave equation with unilateral constraints in [ScB89], it is possible to prove the Theorem 2.1. To do so, we assume that there exist two strictly positive constants λ, γ such that

$$\forall v \in \mathrm{H}^1(\Omega): \; a(v,v) \geq \gamma \|v\|^2 - \lambda |v|^2. \tag{2.13}$$

In the following, the notations for the constants introduced in the proofs are valid only in the proof.

Theorem 2.1 *Assume that* (2.9) *and* (2.13) *hold. Then the numerical scheme* (2.8) *converges to a solution of* (1.1)–(1.3) *when* Δx *and* Δt *tend to* 0*.*

Proof. First we prove the stability. To do so, we introduce $v \stackrel{\text{def}}{=} u_h^{n-1}$ in (2.8) ,which, leads to the following inequality:

$$\begin{aligned}
&\left(\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2}, u_h^{n-1}-u_h^{n+1}\right) + a\left(\frac{u_h^{n+1}+u_h^{n-1}}{2}, u_h^{n-1}-u_h^{n+1}\right)\\
&+\alpha a\left(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, u_h^{n-1}-u_h^{n+1}\right) \geq \left(f_h^n, u_h^{n-1}-u_h^{n+1}\right).
\end{aligned} \tag{2.14}$$

Note that the identity

$$\frac{1}{\Delta t^2}\left(u_h^{n+1}-2u_h^n+u_h^{n-1}, u_h^{n-1}-u_h^{n+1}\right) = \left|\frac{u_h^{n-1}-u_h^n}{\Delta t}\right|^2 - \left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2,$$

implies that

$$\begin{aligned}
&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2 + \frac{a(u_h^{n+1},u_h^{n+1})}{2} + 2\alpha a\left(\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, \frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}\right)\Delta t\\
&\leq \left|\frac{u_h^{n-1}-u_h^n}{\Delta t}\right|^2 + \frac{a(u_h^{n-1},u_h^{n-1})}{2} + 2\left(f_h^n, \frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}\right)\Delta t.
\end{aligned}$$

Hence we perform a discrete time integration of the above expressions and we obtain

$$\begin{aligned}
&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2 + \frac{a(u_h^{n+1},u_h^{n+1})}{2} + 2\alpha \sum_{m=1}^{n} a\left(\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}, \frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right)\\
&\leq \left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2 + \frac{a(u_h^0,u_h^0)}{2} + 2\sum_{m=1}^{n}\left(f_h^m, \frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right)\Delta t.
\end{aligned} \tag{2.15}$$

By using Cauchy-Schwarz inequality, we find

$$\sum_{m=1}^{n}\left(f_h^m, \frac{u_h^{m+1}+u_h^{m-1}}{2\Delta t}\right)\Delta t \leq \frac{1}{2}\sum_{m=1}^{n}|f_h^m|^2\Delta t + \frac{1}{2}\sum_{m=1}^{n}\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\Delta t. \tag{2.16}$$

Clearly, (2.16) and (2.13) lead to the following inequality

$$\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{a(u_h^{n+1},u_h^{n+1})}{2}+2\alpha\gamma\sum_{m=1}^{n}\left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2\Delta t$$
$$\leq\left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2+(1+2\alpha\lambda)\sum_{m=1}^{n}\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\Delta t+\frac{a(u_h^0,u_h^0)}{2}+\sum_{m=1}^{n}|f_h^m|^2\Delta t.$$

Since $\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\leq\left|\frac{u_h^{m+1}-u_h^m}{\Delta t}\right|^2+\left|\frac{u_h^{m-1}-u_h^m}{\Delta t}\right|^2$, we may infer that

$$\begin{aligned}&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{a(u_h^{n+1},u_h^{n+1})}{2}+2\alpha\gamma\sum_{m=1}^{n}\left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2\Delta t\\&\leq\left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2+\frac{a(u_h^0,u_h^0)}{2}+\sum_{m=1}^{n}|f_h^m|^2\Delta t+2(1+2\alpha\lambda)\sum_{m=0}^{n}\left|\frac{u_h^{m+1}-u_h^m}{\Delta t}\right|^2\Delta t.\end{aligned}\tag{2.17}$$

Thus a discrete Grönwall's lemma implies that there exists $C>0$, independent of h, such that

$$\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{a(u_h^{n+1},u_h^{n+1})}{2}+2\alpha\gamma\sum_{m=1}^{n}\left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2\Delta t\leq C.\tag{2.18}$$

On the other hand, we define an interpolation u_h by

$$u_h(x,t)=u_h^n\frac{(n+1)\Delta t-t}{\Delta t}+u_h^{n+1}\frac{t-n\Delta t}{\Delta t}\quad\text{for}\quad t\in[n\Delta t,(n+1)\Delta t].$$

Therefore by using (2.18), we can extract from the sequence u_h, a subsequence, still denoted by u_h, such that

$$u_h\rightharpoonup u\quad\text{in}\quad \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))\quad\text{weak}\quad *,\tag{2.19a}$$
$$\frac{du_h}{dt}\rightharpoonup\frac{du}{dt}\quad\text{in}\quad \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))\quad\text{weak}\quad *,\tag{2.19b}$$
$$u_h\to u\quad\text{in}\quad \mathrm{C}^{0,\beta}(\Omega\times(0,T))\quad\text{for all}\quad\beta<1/2.\tag{2.19c}$$

In order to prove that the limit u satisfies (1.4), it is necessary to take convenient test functions. It is obvious that u_h belongs to K. Thanks to (2.19), we may deduce that u belongs to K. The elements of K are not smooth enough in time, and they have to be approximated before being projected onto V_h. This projection does not conserve the constraints at $x=0$ and $x=L$, and therefore, the elements of K need another approximation in order to satisfy the constraints strictly. More precisely, let v be an element of K, which, is equal to u for $t\geq T-\varepsilon$. For $\eta\leq\frac{\varepsilon}{4}$, we define

$$v^\eta(x,t)\overset{\text{def}}{=}\begin{cases}u(x,t) & \text{if}\quad t\geq T-\eta,\\ u(x,t)+\dfrac{1}{\eta}\displaystyle\int_t^{t+\eta}(v-u)(x,s)\,\mathrm{d}s+c(\eta)\phi(t) & \text{if}\quad t\leq T-\eta.\end{cases}\tag{2.20}$$

The function ϕ is nonnegative and smooth; it is equal to 1 on $[0,T-\frac{\varepsilon}{2}]$, and it vanishes on $[T-\frac{\varepsilon}{4},T]$. The parameter $c(\eta)$ is chosen as follows:

$$\left|u(L,t)-\frac{1}{\eta}\int_t^{t+\eta}u(L,s)\,\mathrm{d}s\right|\leq\frac{1}{\eta}\int_t^{t+\eta}|u(L,t)-u(L,s)|\,\mathrm{d}s\leq\frac{C}{\eta}\int_0^\eta s^\beta\,\mathrm{d}s=\frac{C\eta^\beta}{\beta+1},$$

where $C \stackrel{\text{def}}{=} \sup_{s\in]t,t+\eta[} \frac{|u(L,t)-u(L,s)|}{(s-t)^\beta}$. We have the inequality

$$\forall t \le T - \frac{\varepsilon}{2}:\ v^\eta(L,t) \ge \frac{1}{\eta}\int_t^{t+\eta} v(L,s)\,\mathrm{d}s - \frac{C\eta^\beta}{\beta+1} + c(\eta)\phi(t).$$

If we choose $c(\eta) = \frac{2C\eta^\beta}{\beta+1}$, we will be sure that

$$\forall t \le T - \frac{\varepsilon}{2}:\ v^\eta(L,t) \ge -c_L + \frac{C\eta^\beta}{\beta+1}.$$

With the same arguments, we obtain

$$\forall t \le T - \frac{\varepsilon}{2}:\ v^\eta(0,t) \ge -c_0 + \frac{C\eta^\beta}{\beta+1}.$$

It is not difficult to check that

$$\forall t \in [T-\tfrac{\varepsilon}{2}, T-\eta]:\ v^\eta(x,t) = u(x,t) + c(\eta)\phi(t),$$

so that v^η belongs to K. Furthermore, the time integration having a smoothing effect, we may easily prove that v^η belongs to $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega))$. We denote by $\mathbf{P}_h$ the projection onto V_h with respect to scalar product of $\mathrm{L}^2(\Omega)$. The sequence $\mathbf{P}_h$ converges in strong operator topology of $\mathrm{L}^2(\Omega)$ to the identity $\mathbf{I}$. Then Sobolev injections imply that there exists a sequence γ_h converging to 0 when h tends to 0 such that

$$\forall z \in \mathrm{H}^1(\Omega):\ \|(\mathbf{P}_h - \mathbf{I})z\|_{\mathrm{C}^0(\Omega)} \le \gamma_h \|z\|.$$

Moreover there exists $C_{\mathbf{P}_h} > 0$ such that

$$\forall v \in \mathrm{H}^1(\Omega):\ \|\mathbf{P}_h v\| \le C_{\mathbf{P}_h}\|v\|.$$

This property is proved by a classical computation. Now, we choose v^η as in (2.20) and we let

$$v_h^n = u_h^{n+1} + \mathbf{P}_h(v^\eta(n\Delta t) - u(n\Delta t)).$$

If we substitute this value for v_h^n in (2.8) and perform a discrete integration, we obtain

$$\begin{aligned} \sum_{n=1}^{N-1}(f_h^n, v_h^n - u_h^{n+1})\Delta t \le &-\left(\frac{u_h^1-u_h^0}{\Delta t}, v_h^0 - u_h^1\right) - \sum_{n=1}^{N-1}\left(\frac{u_h^n-u_h^{n-1}}{\Delta t}, \frac{v_h^n-u_h^{n+1}-v_h^{n-1}+u_h^n}{\Delta t}\right)\Delta t \\ &+ \sum_{n=1}^{N-1} a\left(\frac{u_h^{n+1}+u_h^n}{2}, v_h^n-u_h^{n+1}\right)\Delta t + \alpha\sum_{n=1}^{N-1} a\left(\frac{u_h^{n+1}-u_h^n}{2\Delta t}, v_h^n - u_h^{n+1}\right)\Delta t. \end{aligned} \tag{2.21}$$

The passage to the limit in this expression is obvious. It is enough to show that the total energy of u_h converges to the total energy of u. This is done by a discrete integration of (2.14). □

A similar result to Theorem 2.1 can be obtained in the case where one of the Signorini conditions is replaced by a Neumann or a Dirichlet boundary conditions. Since the proof uses the same ideas as already developed in Theorem 2.1, the verification is let to the reader.

3 Finite difference schemes

Let $\Delta x \stackrel{\text{def}}{=} \frac{L}{J}$ be the space step where J is an integer. We denote by u_j^n the solution of the following finite difference scheme:

$$\begin{cases} \dfrac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}-\left(\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{2\Delta x^2}+\dfrac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{2\Delta x^2}\right) \\ -\alpha\left(\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{2\Delta t\Delta x^2}-\dfrac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{2\Delta t\Delta x^2}\right)=f_j^n \quad \text{for} \quad j\in[2,J-1], \\ u_1^{n+1}=\max\left(c_0,\frac{1}{c_\Delta}\left(\frac{2u_1^n-u_1^{n-1}}{\Delta t^2}+\left(\frac{u_2^{n+1}}{2\Delta x^2}+\frac{u_2^{n-1}-u_1^{n-1}}{2\Delta x^2}\right)+\alpha\left(\frac{u_2^{n+1}}{2\Delta t\Delta x^2}-\frac{u_2^{n-1}-u_1^{n-1}}{2\Delta t\Delta x^2}\right)\right)\right), \\ u_J^{n+1}=\max\left(c_L,\frac{1}{c_\Delta}\left(\frac{2u_J^n-u_J^{n-1}}{\Delta t^2}+\left(\frac{u_{J-1}^{n+1}}{2\Delta x^2}+\frac{u_{J-1}^{n-1}-u_J^{n-1}}{2\Delta x^2}\right)+\alpha\left(\frac{u_{J-1}^{n+1}}{2\Delta t\Delta x^2}-\frac{u_{J-1}^{n-1}-u_J^{n-1}}{2\Delta t\Delta x^2}\right)\right)\right), \end{cases} \tag{3.22}$$

where f_j^n denotes a suitable discretization of f and $c_\Delta \stackrel{\text{def}}{=} \frac{1}{\Delta t^2}-\frac{1}{2\Delta x^2}-\frac{\alpha}{2\Delta t\Delta x^2}$. We assume that $u_0^n=0$ and $c_\Delta \neq 0$. We define now an interpolation in the space of u_j^n and an approximation of the L^2 scalar product. We will see below that this family of schemes have a variational formulation. More precisely, we introduce

$$\varphi_j(x) \stackrel{\text{def}}{=} \begin{cases} 1-\dfrac{|x-j\Delta x|}{\Delta x} & \text{if} \quad x\in\big[(j-1)\Delta x,(j+1)\Delta x\big], \\ 0 & \text{otherwise}, \end{cases}$$

and

$$u_h^n(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} u_j^n\varphi_j(x) \quad \text{and} \quad f_h^n(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} f_j^n\varphi_j(x).$$

Observe that u_h^n and f_h^n belong to V_h, the space of uniform P_1 finite elements with the nodes at the points jh, $j\in[0,J]$. Let $u(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} u_j\varphi_j(x)$ and $v(x) \stackrel{\text{def}}{=} \sum_{j=1}^{J} v_j\varphi_j(x)$, and let us define the scalar product over V_h by

$$\langle u,v\rangle \stackrel{\text{def}}{=} \sum_{j=1}^{J} u_jv_j\Delta x. \tag{3.23}$$

Thus for all u and v belonging to V_h, we have

$$\frac{\Delta x^2}{4}a(u,u) \leq \langle u,u\rangle, \tag{3.24a}$$

$$\langle u,v\rangle-(u,v)=\frac{\Delta x^2}{6}a(u,v)+\frac{\Delta x}{2}u_Jv_J. \tag{3.24b}$$

The detailed proof of (3.24) can be found in the Appendix.

Lemma 3.1 *The finite difference scheme* (3.22) *is equivalent to the following variational inequality*

$$\begin{cases} \text{Find } u_h^{n+1}\in K_h \text{ such that for all } v\in K_h, \\ \left\langle \dfrac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2},v-u_h^{n+1}\right\rangle+a\left(\dfrac{u_h^{n+1}+u_h^{n-1}}{2},v-u_h^{n+1}\right) \\ +\alpha a\left(\dfrac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},v-u_h^{n+1}\right)\geq\left\langle f_h^n,v-u_h^{n+1}\right\rangle. \end{cases} \tag{3.25}$$

Proof. Let $g(x)$ be a function depending on x such that $g(0) = g(L) = 0$. Thus, taking $v = u_h^{n+1} + g(x)$ in (2.8), we find

$$\left\langle \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, g \right\rangle + a\left(\frac{u_h^{n+1} + u_h^{n-1}}{2}, g \right) + \alpha a\left(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, g \right) \geq \langle f_h^n, g \rangle,$$

which, implies the first relation in (3.22). Choosing now v such that $v_J \geq c_L$ and $v_j = u_j^{n+1}$ for $j \in [1, J-1]$, then it follows that

$$\begin{aligned}
\forall v_J \geq c_L : \frac{u_J^{n+1} - 2u_J^n + u_J^{n-1}}{\Delta t^2}(v_J - u_J^{n+1}) - \left(\frac{u_{J-1}^{n+1} - u_J^{n+1}}{2\Delta x^2} + \frac{u_{J-1}^{n-1} - u_J^{n-1}}{2\Delta x^2} \right)(v_J - u_J^{n+1}) \\
- \alpha \left(\frac{u_{J-1}^{n+1} - u_J^{n+1}}{2\Delta x^2} - \frac{u_{J-1}^{n-1} - u_J^{n-1}}{2\Delta x^2 \Delta t} \right)(v_J - u_J^{n+1}) = f_J^n (v_J - u_J^{n+1}).
\end{aligned} \tag{3.26}$$

Therefore it is clear that (3.26) is equivalent to the third relation of (3.22). Finally, choosing v such that $v_1 \geq c_0$ and $v_j = u_j^{n+1}$ for $j = [2, J]$, and proceeding as above, the second relation follows. This proves the lemma. □

Note that the difference between (2.8) and (3.25) is that the scalar product $(\cdot,\cdot)$ is replaced by $\langle \cdot,\cdot \rangle$. Let us introduce the following approximations:

$$u_j^0 \stackrel{\text{def}}{=} u_0(j\Delta x), \tag{3.27a}$$

$$u_1^1 \stackrel{\text{def}}{=} u_1^0 + \frac{2\Delta t}{\Delta x} \int_0^{3/2\Delta x} u_1(x)\,\mathrm{d}x, \tag{3.27b}$$

$$u_j^1 \stackrel{\text{def}}{=} u_j^0 + \frac{\Delta t}{\Delta x} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_1(x)\,\mathrm{d}x \quad \text{for} \quad j \in [2, J-1], \tag{3.27c}$$

$$u_J^1 \stackrel{\text{def}}{=} u_J^0 + \frac{2\Delta t}{\Delta x} \int_{(J-1/2)\Delta x}^{J\Delta x} u_1(x)\,\mathrm{d}x, \tag{3.27d}$$

$$f_j^n \stackrel{\text{def}}{=} \frac{1}{\Delta x \Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \int_{(j-1)\Delta x}^{j\Delta x} f(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.27e}$$

We establish now the convergence of the finite difference scheme (3.22). Note that the proof is quite similar to the proof of convergence of an explicit finite difference scheme for a wave equation with unilateral constraints (see [ScB89]). We assume now that $\zeta \stackrel{\text{def}}{=} \frac{\Delta t}{\Delta x}$ is a fixed number sufficiently small.

Theorem 3.2 *Assume that* (2.20) *and* (3.27) *hold. Then the numerical scheme* (3.22) *converges to a solution of* (1.1)–(1.3) *when* Δx *and* Δt *tend to* 0*.*

Proof. We proceed exactly as in the proof of Theorem 2.1. Let us go into details. Taking $v = u_h^{n-1}$ in (2.8), we get

$$\begin{aligned}
&\left\langle \frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} - u_h^n}{\Delta t} \right\rangle + \frac{1}{2} a\big(u_h^{n+1}, u_h^{n+1}\big) + \frac{\alpha}{2\Delta t} a\big(u_h^{n+1} - u_h^{n-1}, u_h^{n+1} - u_h^{n-1}\big) \\
&\leq \left\langle \frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} - u_h^n}{\Delta t} \right\rangle + \frac{1}{2} a(u_h^{n-1} - u_h^{n-1}) + \langle f_h^n, u_h^{n+1} - u_h^{n-1} \rangle,
\end{aligned}$$

Then we perform a discrete integration and we use a discrete Grönwall's lemma, which, implies that there exists $C>0$, independent of h, such that

$$\left\langle \frac{u_h^{n+1}-u_h^n}{\Delta t}, \frac{u_h^{n+1}-u_h^n}{\Delta t} \right\rangle + \|u_h^n\|^2 \leq C. \tag{3.28}$$

Define an interpolation u_h by

$$u_h(x,t) = u_h^n \, \frac{(n+1)\Delta t - t}{\Delta t} + u_h^{n+1} \, \frac{t-n\Delta t}{\Delta t} \quad \text{for} \quad t \in [n\Delta t, (n+1)\Delta t].$$

Relation (3.28) implies that we can extract from the sequence u_h, a subsequence, still denoted by u_h, such that

$$u_h \rightharpoonup u \quad \text{in} \quad \mathrm{L}^\infty(0,T;\mathrm{H}^1(\Omega)) \quad \text{weak} \quad *, \tag{3.29a}$$

$$\frac{du_h}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega)) \quad \text{weak} \quad *, \tag{3.29b}$$

$$u_h \to u \quad \text{in} \quad \mathrm{C}^{0,\beta}(\Omega\times(0,T)) \quad \text{for all} \quad \beta < 1/2. \tag{3.29c}$$

Obviously u belongs to K. Let $v_h^n = u_h^{n+1} + \mathbf{P}_h(v^\eta(n\Delta t) - u(n\Delta t))$ where $\mathbf{P}_h$ and v^η as defined in the proof of Theorem 2.1. We substitute this value for v_h^n in (2.8) and perform a discrete integration, we obtain

$$\begin{aligned}\sum_{n=1}^{N-1} \langle f_h^n, v_h^n - u_h^{n+1}\rangle \Delta t \leq -\left\langle \frac{u_h^1-u_h^0}{\Delta t}, v_h^0 - u_h^1 \right\rangle - \sum_{n=1}^{N-1} \left\langle \frac{u_h^n - u_h^{n-1}}{\Delta t}, \frac{v_h^n - u_h^{n+1} - v_h^{n-1} + u_h^n}{\Delta t} \right\rangle \Delta t \\ + \sum_{n=1}^{N-1} a\left(\frac{u_h^{n+1}+u_h^n}{2}, v_h^n - u_h^{n+1}\right)\Delta t + \alpha \sum_{n=1}^{N-1} a\left(\frac{u_h^{n+1}-u_h^n}{2\Delta t}, v_h^n - u_h^{n+1}\right)\Delta t.\end{aligned} \tag{3.30}$$

The difference between (2.21) and (3.30) is that the scalar product $(\cdot,\cdot)$ is replaced by the scalar product $\langle\cdot,\cdot\rangle$. Now if we substitute in (3.30) the scalar product $\langle\cdot,\cdot\rangle$ by the scalar product $(\cdot,\cdot)$ and we use (3.24b), the error commited is given by

$$\mathscr{E} \stackrel{\text{def}}{=} \sum_{i=1}^{6} \mathscr{E}_i, \tag{3.31}$$

where

$$\mathscr{E}_1 \stackrel{\text{def}}{=} \frac{\Delta x^2}{6} a\left(\frac{u_h^1-u_h^0}{\Delta t}, v_h^0 - u_h^1\right), \quad \mathscr{E}_2 \stackrel{\text{def}}{=} \frac{\Delta x}{2}\left(\frac{u_J^1-u_J^0}{\Delta t}\right)(v_J^0 - u_J^1), \tag{3.32a}$$

$$\mathscr{E}_3 \stackrel{\text{def}}{=} \frac{\Delta x^2}{6} \sum_{n=1}^{N-1} a\left(\frac{u_h^n-u_h^{n-1}}{\Delta t}, \frac{v_h^n - u_h^{n+1} - v_h^{n-1} + u_h^n}{\Delta t}\right)\Delta t, \tag{3.32b}$$

$$\mathscr{E}_4 \stackrel{\text{def}}{=} \frac{\Delta x}{2} \sum_{n=1}^{N-1} \left(\frac{u_J^n-u_J^{n-1}}{\Delta t}\right)\left(\frac{v_J^n - u_J^{n+1} - v_J^{n-1} + u_J^n}{\Delta t}\right)\Delta t, \tag{3.32c}$$

$$\mathscr{E}_5 \stackrel{\text{def}}{=} \frac{\Delta x^2}{6} \sum_{n=1}^{N-1} a(f_h^n, v_h^n - u_h^{n+1})\Delta t, \quad \mathscr{E}_6 \stackrel{\text{def}}{=} \frac{\Delta x}{2} \sum_{n=1}^{N-1} f_J^n (v_J^n - u_J^{n+1})\Delta t. \tag{3.32d}$$

We evaluate now $\mathscr{E}_i$, $i = 1,\ldots,6$. Concerning the first term, we observe that letting $u_1(0+x) = u_1(0-x)$ and $u_1(L+x) = u_1(L-x)$, (3.24a) can be written as follows

$$\frac{u_j^1 - u_j^0}{\Delta t} = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_1(x)\,\mathrm{d}x \quad \text{for all} \quad j \in [1,J],$$

which, implies that

$$\left|\frac{u_j^1-u_j^0}{\Delta t}\right|^2 \leq \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |u_1(x)|^2\,\mathrm{d}x \quad \text{for all} \quad j\in[1,J]. \tag{3.33}$$

Then, adding (3.33) from $j=1$ to J, we get

$$\left\langle \frac{u_h^1-u_h^0}{\Delta t}, \frac{u_h^1-u_h^0}{\Delta t}\right\rangle = \sum_{j=1}^{J}\left|\frac{u_j^1-u_j^0}{\Delta t}\right|^2 \Delta x \leq \sum_{j=1}^{J}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |u_1(x)|^2\,\mathrm{d}x \leq 2|u_1|^2,$$

which, leads by using (3.24a) that

$$\left\|\frac{u_h^1-u_h^0}{\Delta t}\right\| \leq \frac{2\sqrt{2}|u_1|}{\Delta x}.$$

We may deduce that

$$|\mathscr{E}_1| \leq \frac{\sqrt{2}\Delta x}{3}|u_1|\,\|v_h^0-u_h^1\| = \mathscr{O}(\Delta x). \tag{3.34}$$

On the other hand, there exists $C_2>0$, independent of h, such that

$$|\mathscr{E}_2| \leq C_2\sqrt{\Delta x}|u_1|\,|v_J^0-u_J^1|,$$

since v_J^0 and u_J^1 are bounded independently of h, we infer that

$$|\mathscr{E}_2| = \mathscr{O}\big(\sqrt{\Delta x}\big). \tag{3.35}$$

We evaluate now $\mathscr{E}_3$. We observe by using the projection $\mathbf{P}_h$ introduced in Theorem 2.1 that

$$\begin{aligned} \|v_h^n-u_h^{n+1}-v_h^{n-1}+u_h^n\| &= \|\mathbf{P}_h(v^\eta(n\Delta t)-u(n\Delta t)-v^\eta((n-1)\Delta t)+u((n-1)\Delta t))\| \\ &\leq C\|v^\eta(n\Delta t)-u(n\Delta t)-v^\eta((n-1)\Delta t)+u((n-1)\Delta t)\|. \end{aligned} \tag{3.36}$$

Owing (2.20), we get

$$|\mathscr{E}_3| = \mathscr{O}\big(\sqrt{\Delta x}\big). \tag{3.37}$$

For the bound $|\mathscr{E}_4|$, we use the following inequality

$$|v_J^n-u_J^{n+1}| \leq C_4\frac{\sqrt{\Delta t}}{\eta}\left(\int_{n\Delta t}^{n\Delta t+\eta}\|(v-u)(x,s)\|^2\,\mathrm{d}s\right)^{1/2},$$

where $C_4>0$. Therefore we may deduce that

$$\begin{aligned} |\mathscr{E}_4| \leq{}& \frac{C_4\Delta x\sqrt{\Delta t}}{\eta}\max_n\|u_h^n\|\sqrt{N-1}\Bigg(\left(\sum_{n=1}^{N-1}\int_{n\Delta t}^{n\Delta t+\eta}\|(v-u)(x,s)\|^2\,\mathrm{d}s\right)^{1/2} \\ &+\left(\sum_{n=1}^{N-1}\int_{(n-1)\Delta t}^{(n-1)\Delta t+\eta}\|(v-u)(x,s)\|^2\,\mathrm{d}s\right)^{1/2}\Bigg). \end{aligned}$$

Choosing η such that $\eta\in[l\Delta t,(l+1)\Delta t)$ for all integer l, we may deduce from the above inequality that

$$|\mathscr{E}_4| \leq \frac{2C_4\Delta x}{\eta}\max_n\|u_h^n\|\sqrt{\frac{T}{\Delta t}}\sqrt{\eta+\Delta t}\left(\int_0^T\|(v-u)(x,s)\|^2\,\mathrm{d}s\right)^{1/2} = \mathscr{O}\big(\sqrt{\Delta x}\big). \tag{3.38}$$

The last two terms can be easily estimated, we find

$$|\mathscr{E}_5|+|\mathscr{E}_6| = \mathscr{O}\big(\sqrt{\Delta x}\big). \tag{3.39}$$

Inserting (3.34), (3.35), (3.37), (3.38) and (3.39) into (3.31), we find $|\mathscr{E}| = \mathscr{O}\big(\sqrt{\Delta x}\big)$. Thus, the passage to the limit is done as in the proof of Theorem 2.1. □

4 Numerical examples

We consider the viscoelastodynamic problem with Signorini boundary conditions (1.4) on $\Omega = [0,26]$ in the time interval $(0,T)$, $T = 40$ and its discretization by (2.8). We choose the time step $\Delta t = 0.125$, the space step $\Delta x = \frac{1}{27}$, the initial data $u_0(x) = x(1-x)$ and $u_1(x) = u_0(x)$ and $\alpha = 1$. We performed two numerical experiments for $f(x,t) = 0$, which, are summarized in Figure 1. More precisely, we consider Signorini boundary conditions at the both ends as well as Signorini condition at the one end and Neumann condition at the other end. We did the same experiments for $f(x,t) = \sin(t\sqrt{2})\cos(2x)$, and they are reported in Figure 2. The numerical results show that if the constraint is active, we can observe small oscillations at the boundary. These oscillations do not exist if the constraint is not active.

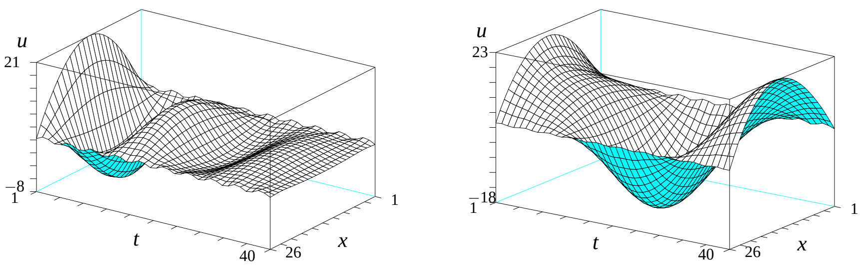

Figure 1: Numerical experiments with Signorini conditions at the both ends and with $f(x,t) = 0$ (left) and $f(x,t) = \sin(t\sqrt{2})\cos(2x)$ (right).

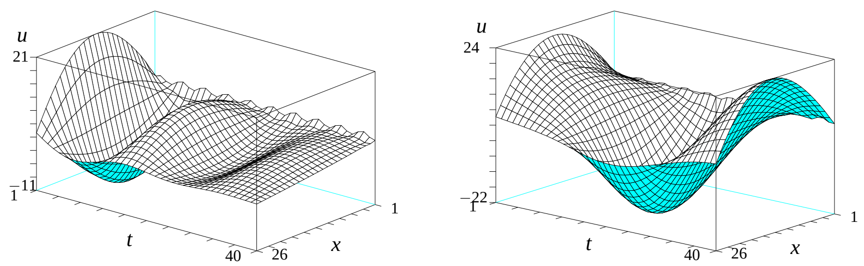

Figure 2: Numerical experiments with Signorini condition at one end and Neumann condition at other end with $f(x,t) = 0$ (left) and $f(x,t) = \sin(t\sqrt{2})\cos(2x)$ (right).

The numerical experiments obtained for $f(x,t) = 0$ and $f(x,t) = \sin(t\sqrt{2})\cos(2x)$ with distributed constraints are summarized in Figure 3. In this case, we consider the following convex set:

$$K \stackrel{\text{def}}{=} \{v \in \mathrm{H}^1(\Omega \times (0,T)) : v_{xt} \in \mathrm{L}^2(\Omega \times (0,T)), v(x,\cdot) \geq c_{0L} \text{ for all } x \in \Omega\}.$$

where $c_{0L} > 0$. We have not treated the mathematical theory of this problem (see [JM*93]), nor its numerical approximation. We give nevertheless the results of these simulations for the reader's information.

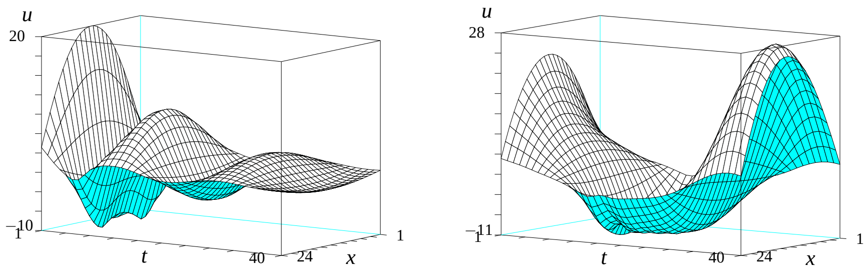

Figure 3: Numerical experiments with distributed Signorini conditions with $f(x,t) = 0$ (left) and $f(x,t) = \sin(t\sqrt{2})\cos(2x)$ (right).

In practical, the resolution is done in a very simple-minded way: at each time step, we check whether the solution of the problem without constraints is admissible; if it is, we advance by one time-step; if it is not, we solve when one or two constraints are active. Thus we have at almost four linear problem to solve per time-step.

Appendix

The aim of this section is to give the proof of (3.24).

Proof. Note that (3.24b) follows from the following inequality

$$|u_{j+1} - u_j|^2 \leq 2\big(u_{j+1}^2 + u_j^2\big). \tag{4.40}$$

Then, adding (4.40) from $j = 1$ to $J-1$ gives

$$\frac{1}{\Delta x}\sum_{j=1}^{J-1}|u_{j+1}-u_j|^2 \leq \frac{2}{\Delta x^2}\sum_{j=1}^{J-1}\big(u_{j+1}^2+u_j^2\big)\Delta x \overset{(3.23)}{\leq} \frac{4}{\Delta x^2}\langle u,u\rangle,$$

which, implies (3.24b). On the other hand, by using (3.23), we get

$$\begin{aligned}\langle u,u\rangle - (u,u) &= \frac{1}{2}\sum_{j=0}^{J-1}\big(u_j^2+u_{j+1}^2\big)\Delta x + \frac{1}{2}u_J^2\Delta x - \frac{1}{3}\sum_{j=0}^{J-1}\big(u_{j+1}^2+u_j^2+u_{j+1}u_j\big)\Delta x\\ &= \frac{1}{6}\sum_{j=0}^{J-1}\big(u_{j+1}-u_j\big)^2\Delta x + \frac{1}{2}u_J^2\Delta x,\end{aligned}$$

which, leads to

$$\langle u,u\rangle - (u,u) = \frac{\Delta x^2}{6}\|u\|^2 + \frac{1}{2}u_J^2\Delta x. \tag{4.41}$$

Therefore by differentiation, we get (3.24a). □

References

[Bré73] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Publishing Co. Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[JM*93] J. Jarušek, J. Málek, J. Nečas and V. Šverák. Variational inequality for a viscous drum vibrating in the presence of an obstacle. *Rend. Mat. Appl. (7)*, 12(4):943–958, 1993.

[Lio69] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod. 1969.

[MaO88] J.A.C. Martins and J.T. Oden. Corrigendum: “Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws” [Nonlinear Anal. **11** (1987), no. 3, 407–428; MR 88e:73025]. *Nonlinear Anal.*, 12(7), 747, 1998.

[Pet02] A. Petrov. *Modélisation mathématique de procédés d’usinage: abrasion et mouillage [Mathematical Modelisation of Maching Process: Abrasion and Wetting]*, Ph.D. thesis, University Claude Bernard, Lyon, pp. 199–2002:13–29, 2002. (Available online at the following address: http://www.wias-berlin.de/people/petrov/pub.html)

[PeS09] A. Petrov and M. Schatzman. Mathematical results on existence for viscoelasto-dynamic problems with unilateral constraints. *SIAM J. Math. Anal*, 40(5):1882–1904, 2008/09.

[ScB89] M. Schatzman and M. Bercovier. Numerical approximation of a wave equation with unilateral constraints. *Math. Comp.*, 53(187):55–79, 1989.