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A model for resistance welding including phase transitions and Joule heating

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Abstract

In this paper we introduce a new model for solid-liquid phase transitions triggered by Joule heating as they arise in the case of resistance welding of metal parts. The main novelties of the paper are the coupling of the thermistor problem with a phase field model and the consideration of phase dependent physical parameters through a mixture ansatz.

The PDE system resulting from our modelling approach couples a strongly nonlinear heat equation, a non-smooth equation for the the phase parameter (standing for the local proportion of one of the two phases) with quasistatic electric charge conservation law. We prove existence of weak solutions in the 3D case, while the regularity result and the uniqueness of solution is stated only in the 2D case. Indeed, uniqueness for the three dimensional system is still an open problem.

1 Introduction

This paper is concerned with the analysis of the initial boundary-value problem for the following PDE system:

$$\theta_t + \ell\chi_t - \operatorname{div}(\kappa(\theta,\chi)\nabla\theta) = \sigma(\theta,\chi)|\nabla\varphi|^2 \quad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\operatorname{div}(\sigma(\theta,\chi)\nabla\varphi) = 0 \quad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\chi_t - \nu\Delta\chi + \beta(\chi) + \gamma(\chi) \ni \frac{\ell}{\vartheta_c}\theta \quad \text{in } \Omega\times(0,T), \tag{1.3}$$

coupled with the following initial-boundary conditions:

$$\mathbf{n}\cdot\kappa(\theta,\chi)\nabla\theta + \alpha\theta = \alpha\vartheta_{ext}, \quad \partial_{\mathbf{n}}\chi = 0 \quad \text{on } \Gamma\times(0,T), \tag{1.4}$$

$$\mathbf{n}\cdot\sigma(\theta,\chi)\nabla\varphi = u \quad \text{on } \Gamma_N\times(0,T), \tag{1.5}$$

$$\varphi = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{1.6}$$

$$\mathbf{n}\cdot\sigma(\theta,\chi)\nabla\varphi = 0 \quad \text{on } \Gamma\setminus(\Gamma_N\cup\Gamma_D)\times(0,T), \tag{1.7}$$

$$\theta(0) = \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{1.8}$$

This PDE system describes phase transitions phenomena triggered by Joule heating, occurring in a bounded, connected domain $\Omega\subset\mathbb{R}^N$ ($N\le 2$), with Lipschitz continuous boundary $\Gamma := \partial\Omega$ (Γ_D, $\Gamma_N \subset \Omega$ Γ_D, $\Gamma_N \neq \emptyset$), during a time interval $[0,T]$. The state variables are the relative temperature θ of the system, the electrical potential φ, and the order parameter χ, standing for the local proportion of one of the two phases. In the melting-solidification process we shall have $\chi = 0$ in the solid phase and $\chi = 1$ in the liquid phase.

The particular application we have in mind is the resistance welding of metal. Figure 1 depicts the special case of resistance spot welding. Here, two sheet metals are pressed together by two electrodes, the so-called welding tongs. Then electric current

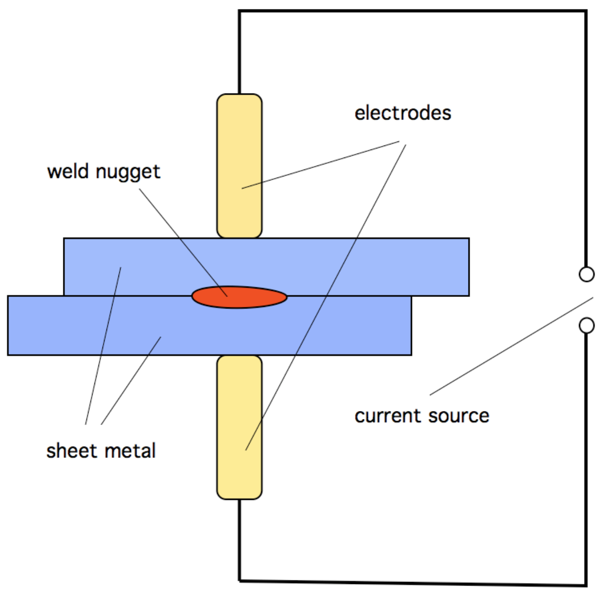

Figure 1: Sketch of resistance spot welding.

is transmitted through electrodes and sheet metals. Owing to a significantly higher resistivity in the contact area of the two sheet metals this region is heated up quickly caused by the Joule effect. In turn a weld nugget develops and starts to grow. After the current is switched off, the weld nugget solidifies leading to a lasting weld joint between both parts.

In our model we describe the parts to be joined as one workpiece Ω. The effect of higher resistivity at the contact surface is taken care of by assuming an explicit dependence of the electrical conductivity σ on the space variable (cf. Hyp. 3.1 (ii)). The temperature evolution is governed by the internal energy balance equation (1.1), where Δ is the Laplace operator (with respect to the space variables), κ and σ (both depending on the space variable x and on θ and χ) represent the (positive) thermal and electrical conductivity, respectively, ℓ stands for the latent heat of the phase change process and ϑ_c for the critical temperature. Equation (1.2), ruling the evolution of the electrical potential φ, is the conservation equation of the electrical charge, while equation (1.3) rules the evolution of the variable χ and it is derived from a particular choice of the free-energy functional (cf. formula (2.1) in Section 2).

In particular, ν is a positive interfacial energy coefficient and the potential $W = \widehat{\beta}+\widehat{\gamma}$ in (1.3) is given by the sum of a smooth non convex function $\widehat{\gamma}$, whose derivative is denoted by γ, and of a convex function $\widehat{\beta}$ possibly with bounded domain, $\beta = \partial\widehat{\beta}$ stands for its subdifferential in the sense of convex analysis (cf., e.g., [6]). The inclusion sign in (1.3) accounts for the fact that β may be multivalued. Typical examples of functionals which we can include in our analysis are the logarithmic potential

$$W(r) := r\ln(r) + (1-r)\ln(1-r) - c_1 r^2 - c_2 r - c_3 \quad \forall\, r \in (0,1), \tag{1.9}$$

where c_1 and c_2 are positive constants, as well as the sum of the indicator function $I_{[0,1]}$ with a non convex $\widehat{\gamma}$. According, e.g, to [10, 11], (1.9) is particularly relevant in the case of solid-liquid phase transitions in metals.

Regarding the boundary conditions, we linearize the radiative heat transfer taking place in reality by choosing a third type boundary condition for θ, where α stands for the (non-negative) surface heat transfer coefficient, and ϑ_{ext} represents the surrounding temperature. We impose Neumann homogenous boundary conditions on χ, as usual, while, having in mind the welding application we choose mixed type boundary conditions on φ, with Γ_D, $\Gamma_N \subset \Gamma$, $\Gamma_D, \Gamma_N \neq \emptyset$.

The state system turns out to be highly nonlinear and non-standardly coupled. In the rest of the paper we first derive the PDE system from the basic principles of thermodynamics. Next, we will prove the existence of at least a solution for a suitable formulation of the 3D problem (1.1–1.8) in case of a general potential W (possibly also multivalued). Then, we will use the regularity results for parabolic and elliptic equations obtained in [17, 18] to prove further regularity properties of our solutions (in particular the continuity of the θ and χ components), as well as continuous dependence of solutions with respect to the data u, θ_0, χ_0, which could be fundamental, e.g., in the study of optimal control problems associated to our system.

Unfortunately, these regularity results are up to now available only in the 2D case ($N = 2$) and in case of a regular potential W (e.g., the *standard double well potential* $W(\chi) = \chi^3 - \chi$).

An early approach to model resistance welding based on an enthalpy formulation of the Stefan problem but disregarding the Joule heating part can be found in [3]. It is impossible to review the vast literature on Joule heating without phase transitions. In [2] the thermistor problem with temperature dependent heat conductivities and for the 3D case is studied, but the authors do not allow for mixed boundary conditions and non-smooth domains. Periodic solutions of the thermistor problem are discussed in [4], [19] is devoted to the investigation of state constrained optimal control of the thermistor problem. Finally, we quote a recent paper accounting for a coupling of the thermistor problem with viscoelastic effects but also disregarding phase transitions [21]. In [26] the enthalpy formulation of the Stefan problem is considered in combination with Joule heating. The literature related to phase-field models without Joule effects (cf. (1.1), (1.3) in case $\nabla\varphi = 0$) is also very wide. Without any attempt to be exhaustive we can quote here the book [7] (and the references therein) and the pioneering modelling and analytical works of [8] and [20, 23].

Finally, our aim here is to establish a new model of solid-liquid phase transitions triggered by Joule heating. Its main novelties are twofold. To our knowledge this is the first study of a coupling of Joule heating with a phase field approach. Moreover, we can allow for phase dependent physical quantities through a mixture ansatz ($\sigma = \sigma(x, \theta, \chi)$ and $\kappa = \kappa(x, \theta, \chi)$, cf. Hyp. 3.1).

We prove existence of weak solutions for the corresponding PDE system and study the problem of regularity and uniqueness of solutions. The main mathematical difficulties here are concerned with the presence of the quadratic contribution in the gradient of the electric field φ in the heat equation. The regularity of φ (whose evolution is ruled by the quasistatic equation (1.2)) does not allow us to prove existence of strong solutions in the three dimensional case. Uniqueness in this framework is also an open problem. Only in the 2D case, indeed, we are able to prove the existence of more regular solutions, applying the regularity results of [17, 18] to (1.2), which lead to the proof of uniqueness of solutions.

Plan of the paper. The paper is organized as follows: after deriving PDEs from the basic principles of thermo-mechanics in Section 2, the system is discussed in Section 3, where existence of weak solutions for a suitable formulation of system (1.1–1.8) is proved in the 3D case ($N = 3$) and, in case of more regular data and for $N = 2$, regularity results for the associated solutions as well as their uniqueness are obtained.

2 The model

In this section we derive from the basic law of thermodynamics and then by linearization the PDE system (1.1–1.8) with which we deal in the present contribution.

The thermistor problem. The heat produced in a conductor by an electrical current leads to the so-called *thermistor problem* (cf. [2]) which couples equations (1.1) and (1.2). These two equations follow from the conservation laws

$$\operatorname{div} \mathcal{I} = 0, \quad E_t + \operatorname{div} \mathcal{Q} = \mathcal{I} \cdot \mathcal{E},$$

where E stands for the internal energy of the system, $\mathcal{I}$ denotes the current density, $\mathcal{Q}$ the heat flux, $\mathcal{E}$ the electric field. Note that, by standard Helmholtz relations, we have that $E = F + \vartheta S$, F and S being, respectively, the local free energy and the entropy of the system, and ϑ the absolute temperature of the system. Note that E and S are linked to each other by the classical relation

$$S = -\frac{\partial F}{\partial \vartheta}.$$

We choose now the form of the local free energy functional in agreement, e.g., with [7] (cf. also [15] for another approach to phase transitions)

$$F[\vartheta, \chi] = c_V \vartheta (1 - \log \vartheta) + \vartheta \left(\widehat{\beta}(\chi) + \widehat{\jmath}(\chi) + \nu \frac{|\nabla \chi|^2}{2} \right) + \ell \chi, \tag{2.1}$$

being $\widehat{\beta}$ and $\widehat{\jmath}$ two nonlinear and possibly non-smooth functions (the sum of the two can have the form of a double well potential), c_V the specific heat which we will take equal to 1 in the following for simplicity. This leads to the following form for E and S

$$S = \log \vartheta - \widehat{\beta}(\chi) - \widehat{\jmath}(\chi) - \nu \frac{|\nabla \chi|^2}{2}, \quad E = \vartheta + \ell \chi. \tag{2.2}$$

Moreover, using the Ohm and Fourier laws, respectively

$$\mathcal{I} = -\tilde{\sigma}(\vartheta, \chi) \nabla \varphi, \tag{2.3}$$

$$\mathcal{Q} = -\tilde{\kappa}(\vartheta, \chi) \nabla \vartheta, \tag{2.4}$$

we get a PDE system similar to (1.1–1.2), where the quadratic contribution on the right hand side in (1.1) is due to Joule effect. Note that in this framework it seems meaningful to consider a χ-dependence in the electrical and thermal conductivities $\tilde{\sigma}$ and $\tilde{\kappa}$ which can be considerably different in the two phases. Typical expressions for $\tilde{\sigma}$ and $\tilde{\kappa}$ are given by a classical mixture ansatz, e.g.,

$$\tilde{\sigma}(\vartheta,\chi) = \chi\tilde{\sigma}_1(\vartheta) + (1-\chi)\tilde{\sigma}_2(\vartheta), \quad \tilde{\kappa}(\vartheta,\chi) = \chi\tilde{\kappa}_1(\vartheta) + (1-\chi)\tilde{\kappa}_2(\vartheta),$$

with possibly different $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\kappa}_1$, $\tilde{\kappa}_2$.

The phase equation. The order parameter dynamics is assumed in the form

$$\mu(\vartheta)\chi_t \in -\delta_\chi \mathcal{F}[\vartheta,\chi]\,, \tag{2.5}$$

with a factor $\mu(\vartheta) > 0$, where we denote

$$\mathcal{F}[\vartheta,\chi] \;=\; \int_\Omega F(\vartheta,\chi)\,dx\,,$$

and where $\delta_\chi \mathcal{F}$ stands for the variational derivative of $\mathcal{F}$ with respect to the variable χ. The inclusion sign in (2.5) accounts for the fact that $\mathcal{F}$ may contain components that are not Fréchet differentiable, but convex, and the derivative can be interpreted as the subdifferential, which may be multivalued. Condition (2.5) is based on the assumption that the system tends to move towards local minima of the free energy with a speed proportional to $1/\mu(\vartheta)$. Using (2.1), and choosing $\mu(\vartheta) = \vartheta$, we can rewrite (2.5) as

$$\chi_t - \nu\Delta\chi + \beta(\chi) + j(\chi) + \frac{\ell}{\vartheta} \ni 0 \quad \text{in } \Omega\times(0,T), \tag{2.6}$$

where j denotes here the derivative of $\widehat{j}$.

Finally, in order to deduce (1.3) from (2.6), we observe that we are considering a material which at the equilibrium temperature ϑ_c is converted from a lower temperature phase into a higher one or viceversa. Hence, we can introduce the quantity $\theta = \vartheta - \vartheta_c$ as new state variable and linearize the kinetic equation (2.6) with respect to θ. In this way we obtain

$$\chi_t - \nu\Delta\chi + \beta(\chi) + j(\chi) + \ell\left(1 - \frac{\theta}{\vartheta_c}\right) \ni 0 \quad \text{in } \Omega\times(0,T),$$

which corresponds exactly to our inclusion (1.3) with the choice $\gamma(\chi) = j(\chi) + \ell$. Finally, we define

$$\begin{aligned}\sigma(\theta,\chi) &= \tilde{\sigma}(\theta+\vartheta_c,\chi),\\ \kappa(\theta,\chi) &= \tilde{\kappa}(\theta+\vartheta_c,\chi)\end{aligned}$$

to arrive at (1.1–1.3). The PDE system coupling (1.1) and (1.3), with constant φ, is well-known in the literature as the Caginalp phase-field system (cf. [8]). The reader can refer to [20, 23] for the analytical results related to its well-posedness. Although the Caginalp model is often used to describe the melting and solidification of metals, we would like to point out that the temperature range covered by resistance welding is more than 1000^o C. Hence it seems questionable if the linearization leading to the Caginalp model is valid in this case. Indeed, let us note that more sophisticated models could be employed here, like the Penrose-Fife model of phase transitions (cf. [25]). This choice would lead to a singular (in ϑ) phase equation (cf. (2.6)), which cannot be directly handled with our analysis. The main difficulties would come from both the singular parts (in ϑ), from the presence of a non constant conductivity κ and from the quadratic nonlinearity on the right hand side of (1.1). This is still an open and interesting problem. We can quote here the recent contribution [12] in which the case of a *nonlocal* phase-field model with non constant heat conductivity and specific heat has been treated. In [12] the existence of solutions has been proved for a system where no electrical current φ is present and the model is *nonlocal* because the Laplacian of χ in the phase equation has been substituted by a nonlocal operator.

3 Well-posedness

In this section we will first give a rigorous formulation of the PDE system (1.1–1.8) and we will list our assumptions on the data. Then, we will state our main results concerning existence and uniqueness of solutions as well as their continuous dependence on the data. Finally, in the last subsection, we will detail the proofs.

3.1 Main results

In this subsection we introduce a suitable variational formulation of our PDE system as well as our precise assumptions on the data, in order to state our main existence-uniqueness result.

Let us first introduce some notation: denote with the symbol $\mathcal{B}(X;Y)$ the space of linear and bounded operators from X into Y, being X and Y two generic Banach spaces, with the convention $\mathcal{B}(X;Y) =: \mathcal{B}(X)$ in case $Y = \mathbb{R}$. We consider a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitzian boundary, $N = 1, 2, 3$, $T > 0$ is a fixed final time, and for $t \in (0, T]$ we denote $Q_t = \Omega \times (0, t)$. Let Γ_N be an open part (of positive measure) of $\Gamma := \partial\Omega$, Γ_D a closed part (of positive measure) of Γ such that Γ_N and Γ_D are disjoint sets. In addition, the set $\overline{\Gamma \setminus \Gamma_D} \cap \Gamma_D$ is finite and no connected component of Γ_D consists of a single point. Moreover, let the symbol $H^1_D(\Omega)$ denote the closure of $\{\psi_{|\Omega} : \psi \in C_0^\infty(\mathbb{R}^N),\ \mathrm{supp}(\psi) \cap \Gamma_D = \emptyset\}$ in $H^1(\Omega)$ and (for $q \in (2, +\infty)$) let $W^{1,q}_D(\Omega)$ denote the closure of $\{\psi_{|\Omega} : \psi \in C_0^\infty(\mathbb{R}^N),\ \mathrm{supp}(\psi) \cap \Gamma_D = \emptyset\}$ in $W^{1,q}(\Omega)$. We use the notation $W^{2,p}_{\mathbf{n}} = \{v \in W^{2,p}(\Omega) \mid \mathbf{n} \cdot \nabla v|_{\partial\Omega} = 0\}$, $\|\cdot\|_p$ for the norms in $L^p(\Omega)$, $p \in [1, +\infty]$. Finally, we denote by $\langle \cdot, \cdot \rangle$ the duality product between $H^1(\Omega)$ and its dual or by $H^1_D(\Omega)$ and $H^{-1}_D(\Omega)$.

The following assumptions on the data are supposed to hold.

Hypothesis 3.1. *Assume that there exist positive constants L_κ, κ_0, κ_1, L_σ, σ_0, σ_1 such that*

(i) $\kappa(x, \theta, \chi) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{B}(\mathbb{R}^N)$ *is bounded and measurable with respect to x for all $\theta, \chi \in \mathbb{R}$ and Lipschitz continuous with respect to θ and χ for a.a. $x \in \Omega$,* *and for all* $\widetilde{\theta}, \theta, \widetilde{\chi}, \chi \in \mathbb{R}$*, it holds true*

$$\|\kappa(x, \widetilde{\theta}, \widetilde{\chi}) - \kappa(x, \theta, \chi)\|_{\mathcal{B}(\mathbb{R}^N)} \leq L_\kappa \left(|\widetilde{\theta} - \theta| + |\widetilde{\chi} - \chi| \right).$$

Moreover, for all θ, $\chi \in \mathbb{R}$ *and a.a.* $x \in \Omega$, κ *is a symmetric matrix satisfying*

$$\inf_{\theta,\chi\in\mathbb{R}} \operatorname{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{N} \kappa_{ij}(x, \theta, \chi)\xi_i\xi_j \geq \kappa_0 \|\xi\|^2_{\mathbb{R}^N} \quad \forall \xi \in \mathbb{R}^N,$$
$$\sup_{\theta,\chi\in\mathbb{R}} \|\kappa(x, \theta, \chi)\|_{L^\infty(\Omega;\mathcal{B}(\mathbb{R}^N))} \leq \kappa_1;$$

(ii) $\sigma(x, \theta, \chi) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{B}(\mathbb{R}^N)$ *is bounded and measurable with respect to* x *for all* $\theta, \chi \in \mathbb{R}$ *and Lipschitz continuous with respect to* θ *and* χ *for a.a.* $x \in \Omega$*, and for all* $\widetilde{\theta}, \theta, \widetilde{\chi}, \chi \in \mathbb{R}$*, it holds true*

$$\|\sigma(x, \widetilde{\theta}, \widetilde{\chi}) - \sigma(x, \theta, \chi)\|_{\mathcal{B}(\mathbb{R}^N)} \leq L_\sigma \left(|\widetilde{\theta} - \theta| + |\widetilde{\chi} - \chi| \right).$$

Moreover, for all θ, $\chi \in \mathbb{R}$ *and a.a.* $x \in \Omega$, σ *is a symmetric matrix satisfying*

$$\inf_{\theta,\chi\in\mathbb{R}} \operatorname{ess\,inf}_{x\in\Omega} \sum_{i,j=1}^{N} \sigma_{ij}(x, \theta, \chi)\xi_i\xi_j \geq \sigma_0 \|\xi\|^2_{\mathbb{R}^N} \quad \forall \xi \in \mathbb{R}^N,$$
$$\sup_{\theta,\chi\in\mathbb{R}} \|\sigma(x, \theta, \chi)\|_{L^\infty(\Omega;\mathcal{B}(\mathbb{R}^N))} \leq \sigma_1;$$

(iii) $\widehat{\beta} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is a proper, convex, and lower semicontinuous function,* $D(\widehat{\beta})$ *denotes its domain;*

(iv) $\widehat{\gamma} \in C^{1,1}(\mathbb{R})$*;*

(v) $\vartheta_{ext} \in L^\infty(0, T; L^\infty(\Gamma))$*;*

(vi) $\alpha \in L^2(\Gamma)$ *with* $\int_\Gamma \alpha^2 \, ds > 0$ *and* $\alpha \geq 0$ *a.e. on* Γ*;*

(vii) $\theta_0 \in L^2(\Omega)$*;*

(viii) $\chi_0 \in H^1(\Omega)$, $\widehat{\beta}(\chi_0) \in L^1(\Omega)$.

We continue stating a precise formulation of the system (1.1–1.8) and the definition of the associated weak solutions. Hence, we define, for every coefficient function $\rho \in L^\infty(\Omega; \mathcal{B}(\mathbb{R}^N))$, the operator $-\operatorname{div}(\rho\nabla) : H^1_D(\Omega) \to H^{-1}_D(\Omega)$ as

$$\langle -\operatorname{div}(\rho\nabla w), z\rangle := \int_\Omega \rho\nabla w\nabla z \, dx\,, \quad w,\, z \in H^1_D(\Omega). \tag{3.1}$$

Moreover, let us denote by $\widetilde{\alpha}$ the $L^2(\Gamma)$ function $\alpha(t)\vartheta_{ext}(t)$, while the function $u \in L^\infty(0, T; L^2(\Gamma))$ will be interpreted as an element $\widetilde{u} \in L^\infty(0, T; H^{-1}_D(\Omega))$ by setting

$$\langle \widetilde{u}(t), v\rangle := \int_{\Gamma_N} u(t)v \, ds, \quad v \in H^1_D(\Omega)\,,$$

for almost all $t \in (0, +\infty)$. Finally, we introduce the realization of the Laplace operator with homogeneous Neumann boundary conditions as

$$A : H^1(\Omega) \to (H^1(\Omega))', \qquad \langle Au, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{for } u, v \in H^1(\Omega). \tag{3.2}$$

We are now in the position to give a precise definition of *weak solutions* to (1.1–1.8) in which we take $\ell = \vartheta_c = 1$ for simplicity and without any loss of generality.

Definition 3.2. *Let u be a given function in $L^\infty(0,T;L^2(\Gamma_N))$. We define as a weak solution of (1.1–1.8) the triple (θ, φ, χ) and the selection ξ satisfying*

$$\theta \in H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega)); \tag{3.3}$$

$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)); \tag{3.4}$$

$$\chi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C^0([0,T];H^1(\Omega)); \tag{3.5}$$

$$\xi \in L^2(Q_T), \tag{3.6}$$

and the equations

$$\langle \theta_t + \chi_t, v\rangle + \int_\Omega \kappa(x,\theta,\chi)\nabla\theta\nabla v \, dx + \int_\Gamma \alpha\theta v \, ds = \int_\Omega \sigma(x,\theta,\chi)|\nabla\varphi|^2 v \, dx + \int_{\Gamma_N} \widetilde{\alpha} v \, ds \quad \forall v \in H^1(\Omega) \quad \text{and a.e. in } (0,T), \tag{3.7}$$

$$-\operatorname{div}(\sigma(x,\theta,\chi)\nabla\varphi) = \widetilde{u} \quad \text{in } H^{-1}_D(\Omega) \quad \text{and a.e. in } (0,T), \tag{3.8}$$

$$\chi_t + A\chi + \xi + \gamma(\chi) = \theta \quad \text{a.e. in } Q_T, \tag{3.9}$$

where

$$\xi \in \beta(\chi) \qquad \text{a.e. in } Q_T, \tag{3.10}$$

coupled with the following initial conditions:

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{3.11}$$

The main results we will prove in next sections are the following ones.

Theorem 3.3. *Let Hypothesis 3.1 hold true and suppose that u is given in the space $L^\infty(0,T;L^2(\Gamma_N))$. Then, there exists at least a solution of (1.1–1.8) in the sense of Definition 3.2.*

Theorem 3.4. *Let Hypothesis 3.1 hold true and u be given in $L^\infty(0,T;L^2(\Gamma_N))$, $\theta_0, \chi_0 \in C(\overline{\Omega})$. Suppose that the spatial dimension is $N = 2$. Then, there exists a solution of (1.1–1.8) in the sense of Definition 3.2 and an index $q \in (2,4)$ such that the following regularity properties hold true:*

$$\theta \in H^1(0,T;H^{-1,q}_\Omega(\Omega)) \cap L^2(0,T;W^{1,q}(\Omega)) \hookrightarrow C^0(\overline{\Omega} \times [0,T]); \tag{3.12}$$

$$\varphi \in L^\infty(0,T;W^{1,q}_D(\Omega)); \tag{3.13}$$

$$\chi \in C^0(\overline{\Omega} \times [0,T]). \tag{3.14}$$

Let moreover $W(:= \widehat{\beta} + \widehat{\gamma})$ *satisfy the following assumption*

$$W \in C^2_{\mathrm{loc}}(\mathbb{R}), \quad |W''(r)| \le c_{Lip} \quad \forall r \in \mathcal{B}(\mathbb{R})\,. \tag{3.15}$$

Then, such a solution is also unique and depends continuously on the data u, θ_0, *and* χ_0. *Finally, let Hypothesis 3.1 hold true,* $N = 2$, u *be given in the space* $L^\infty(0,T;L^2(\Gamma_N))$, *and assume that there exists an index* $\eta_0 \in (5,+\infty)$ *such that:*

$$\theta_0 \text{ is Hölder continuous in } \overline{\Omega}; \tag{3.16}$$

$$\chi_0 \in [L^{\eta_0}(\Omega), W^{2,\eta_0}_{\mathbf{n}}(\Omega)]_{1-1/\eta_0,\eta_0}\,, \tag{3.17}$$

then there exists a sufficiently small $\eta > 0$ *such that the solution has the further regularity properties:*

$$\theta \in C^{0,\eta}([0,T];C^{0,\eta}(\overline{\Omega})); \tag{3.18}$$

$$\chi \in C^{0,\eta}([0,T];C^{0,\eta}(\overline{\Omega}))\,. \tag{3.19}$$

3.2 Proofs

In this subsection we give the proofs of our main Theorems 3.3, 3.4. We will denote the positive constants hereafter by the same symbol C_i, $i = 1,2,\ldots$. We will specify their dependence on the problem data any time it will be necessary.

3.2.1 Proof of Theorem 3.3

In order to prove the existence of weak solutions (in the sense of Definition 3.2), we first recall the following preliminary results. The first theorem we state turns out to be a particular case of [5, Thm. 2.1] (cf. also [20] and [23]).

Theorem 3.5. *Let* $f \in L^2(0,T;(H^1(\Omega)')$, *Hyp. 3.1 (i), (iii)–(iv), (vii)–(viii) hold true. Let* $\bar{k} \in L^\infty(Q_T)$ *be such that* $0 < \kappa_0 \le \bar{k} \le \kappa_1$ *a.e. Then, there exists a unique couple* (θ,χ) *and a selection* ξ *satisfying the regularity properties* (3.3), (3.5), (3.6), *the relations (3.9–3.10) and the equation*

(3.20)
$$\langle \theta_t + \chi_t, v\rangle + \int_\Omega \bar{k}(x,t)\nabla\theta\nabla v\,dx + \int_\Gamma \alpha\theta v\,ds = f \quad \forall v \in H^1(\Omega) \quad \textit{and a.e. in } (0,T).$$

Moreover, there exists a positive constant C_1 *depending on the data of the problem, but not on* f, *such that the following estimate holds true for all* $t \in [0,T]$:

$$\begin{aligned} \|\theta(t)\|_2^2 &+ \int_0^t \|\theta(s)\|^2_{H^1(\Omega)}\,ds + \int_0^t \|\chi_t(s)\|_2^2\,ds + \|\chi(t)\|^2_{H^1(\Omega)} \\ &\le C_1\left(1 + \int_0^t \|\chi_t\|^2_{L^2(0,s;L^2(\Omega))}\,ds + \int_0^t \langle f(s),\theta(s)\rangle\,ds\right)\,. \end{aligned} \tag{3.21}$$

In [16] it is possible to find a proof for the following result.

Theorem 3.6. *Let $(v,w) \in L^2(Q_T) \times L^2(Q_T)$, $u \in L^\infty(0,T;L^2(\Gamma_N))$, and assume Hyp. 3.1 (ii). Then, there exists a unique φ complying with the regularity property (3.4) and the equation:*

$$-\operatorname{div}(\sigma(x,v,w)\nabla\varphi) = \widetilde{u} \quad \text{in } (H^1(\Omega))', \quad \text{a.e. in } (0,T). \tag{3.22}$$

Moreover, there exists a positive constant C_2, depending on the data of the problem, but not on (v,w), such that

$$\int_\Omega |\nabla\varphi(x,t)|^2\,dx \le C_2 \quad \text{for all } t \in [0,T]. \tag{3.23}$$

It is clear that, given $(u,v) \in (L^2(Q_T))^2$, for all $\zeta \in H^1(\Omega)$,

$$-\langle \operatorname{div}(\sigma(x,v,w)\nabla\varphi), \zeta\rangle = \langle \widetilde{u}, \zeta\rangle + \int_\Omega \sigma(v,w)\varphi\nabla\varphi\cdot\nabla\zeta\,dx \tag{3.24}$$

defines an element $f \in L^2(0,T;(H^1(\Omega))')$. Let us proceed (in the spirit of [2]) considering the following map F carrying $(L^2(Q_T))^2$ into itself, which associates to the couple (v,w) the solution (θ,χ) to (3.9), (3.20) (given by Thm. 3.5) with datum f defined as above and $\bar{k} = \kappa(x,v,w)$. In the following, we would like to apply a Schauder fixed point argument to F.

First, by (3.21), we have

$$\|\theta(t)\|_2^2 + \int_0^t \|\theta(s)\|_{H^1(\Omega)}^2\,ds + \int_0^t \|\chi_t(s)\|_2^2\,ds + \|\chi(t)\|_{H^1(\Omega)}^2$$
$$\le C_1\left(1 + \int_0^t \|\chi_t\|_{L^2(0,s;L^2(\Omega))}^2\,ds + \int_0^t \langle\widetilde{u},\theta(s)\rangle\,ds + \int_0^t\int_\Omega \sigma(x,v,w)\varphi\nabla\varphi\cdot\nabla\theta\,dx\,ds\right).$$

By Schwarz and Young inequalities, using Hyp. 3.1 (ii), and the boundedness of φ (inferred by Them. 3.6), we deduce

$$\|\theta(t)\|_2^2 + \int_0^t \|\theta(s)\|_{H^1(\Omega)}^2\,ds + \int_0^t \|\chi_t(s)\|_2^2\,ds + \|\chi(t)\|_{H^1(\Omega)}^2$$
$$\le C_1\left(1 + \int_0^t \|\chi_t\|_{L^2(0,s;L^2(\Omega))}^2\,ds + \int_0^t \|\widetilde{u}(s)\|_{H_D^{-1}(\Omega)}^2\,ds + \int_0^t \|\nabla\varphi(s)\|_2^2\,ds\right).$$

Using estimate (3.23) together with a standard Gronwall lemma (cf. [6, Lemma A4, p. 156]), we obtain

$$\|\theta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} + \|\chi\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \le C_3, \tag{3.25}$$

for some positive constant C_3, depending on the data of the problem, but not on (v,w). Hence, from (3.20), we easily deduce (note that $\bar{k}\nabla\theta \in L^2(Q_T)$)

$$\|\theta_t\|_{L^2(0,T;(H^1(\Omega))')} \le C_4\,. \tag{3.26}$$

Moreover, applying standard regularity results for elliptic equations, we also get

$$\|\chi\|_{L^2(0,T;H^2(\Omega))} \le C_5\,. \tag{3.27}$$

Hence, taking R sufficiently large, F maps the ball B_R in $(L^2(Q_T))^2$ of center 0 and radius R in itself. Moreover, the space

$$\{(\theta,\chi) \in (L^2(0,T;H^1(\Omega)))^2 \,:\, (\theta_t,\chi_t) \in (L^2(0,T;(H^1(\Omega))'))^2\}$$

is compactly embedded in $(L^2(Q_T))^2$ (cf. [27, Cor. 4, Sec. 8]). Hence, in order to employ a Schauder fixed point argument, we only need to prove the continuity of F in $(L^2(Q_T))^2$. Consider a sequence (v_n,w_n) in B_R converging to (v,w) in $(L^2(Q_T))^2$ as n tends to $+\infty$. Define φ_n as in (3.22) and f_n as in (3.24). Let $(\theta_n,\chi_n) = F(v_n,w_n)$. We have to show that

$$(\theta_n,\chi_n) \to (\theta,\chi) = F(v,w) \quad \text{in } B_R \quad \text{as } n \nearrow \infty\,. \tag{3.28}$$

In order to prove that, let us consider the difference between the equations (3.20) and (3.20) with θ_n in place of θ. Test it by $(\theta-\theta_n)$. Take the difference between the equations (3.9) and (3.9) with χ_n in place of χ. Test it by $(\chi-\chi_n)_t$. Sum up the two resulting equations and integrate the result over $(0,t)$, $t \in (0,T]$. In this way one gets

(3.29)
$$\begin{aligned}
&\|(\theta-\theta_n)(t)\|_2^2 + \int_0^t \|(\theta-\theta_n)(s)\|_{H^1(\Omega)}^2\,ds + \|\nabla(\chi-\chi_n)(t)\|_2^2 + \int_0^t \|(\chi-\chi_n)_t(s)\|_2^2\,ds \\
&\le C_6\Big(\int_0^t \langle (f-f_n)(s),(\theta-\theta_n)(s)\rangle\,ds - \int_0^t\int_\Omega (\gamma(\chi)(s)-\gamma(\chi_n)(s))(\chi-\chi_n)_t(s)\,dx\,ds \\
&\quad + \int_0^t\int_\Omega (\kappa(x,v,w)-\kappa(x,v_n,w_n))\nabla\theta\nabla(\theta-\theta_n)\,dx\,ds\Big).
\end{aligned}$$

Let us start from the second integral on the right hand side of (3.35). We use here Hyp. 3.1 (iv), getting

$$\begin{aligned}
&-C_2\int_0^t\int_\Omega (\gamma(\chi)(s)-\gamma(\chi_n)(s))(\chi-\chi_n)_t(s)\,dx\,ds \\
&\le \frac{1}{2}\int_0^t \|(\chi-\chi_n)_t(s)\|_2^2\,ds + C_7\int_0^t \|(\chi-\chi_n)(s)\|_2^2\,ds\,.
\end{aligned} \tag{3.30}$$

Regarding the first integral on the right-hand side of (3.29), we use (3.24) to obtain

$$\begin{aligned}
&C_6\int_0^t \langle (f-f_n)(s),(\theta-\theta_n)(s)\rangle\,ds \\
&= C_6\int_0^t\int_\Omega \left(\sigma(x,v,w)\varphi\nabla\varphi - \sigma(x,v_n,w_n)\varphi_n\nabla\varphi_n\right)\nabla(\theta_n-\theta)\,dx\,ds \\
&\le \frac{1}{2}\int_0^t \|\nabla(\theta-\theta_n)(s)\|_2^2\,ds \\
&\quad + C_8\int_0^t \|(\sigma(x,v,w)\varphi\nabla\varphi - \sigma(x,v_n,w_n)\varphi_n\nabla\varphi_n)(s)\|_2^2\,ds\,.
\end{aligned} \tag{3.31}$$

Following [2, pp. 1132–1133], we can prove that the last integral in (3.31) tends to 0 when $n \nearrow \infty$. Indeed, we can rewrite it as

$$\begin{aligned}
&\int_0^t \|(\sigma(x,v_n,w_n)\varphi_n\nabla\varphi_n - \sigma(x,v,w)\varphi\nabla\varphi)(s)\|_2^2\,ds \\
&\le \int_0^t \|(\sigma(x,v_n,w_n)\varphi_n\nabla\varphi_n - \sigma(x,v_n,w_n)\varphi_n\nabla\varphi)(s)\|_2^2\,ds \\
&\quad + \int_0^t \|(\sigma(x,v_n,w_n)\varphi_n\nabla\varphi - \sigma(x,v_n,w_n)\varphi\nabla\varphi)(s)\|_2^2\,ds \\
&\quad + \int_0^t \|(\sigma(x,v_n,w_n)\varphi\nabla\varphi - \sigma(x,v,w)\varphi\nabla\varphi)(s)\|_2^2\,ds \\
&= I_1 + I_2 + I_3\,.
\end{aligned}$$

Then, using Hyp. 3.1, (ii) and Thm. 3.6, we get

$$\begin{aligned}
&I_1 \le C_9 \int_0^t \|\nabla(\varphi_n - \varphi)(s)\|_2^2\,ds; \\
&I_2 \le C_{10} \int_0^t \int_\Omega |\varphi_n - \varphi(s)|^2 |\nabla\varphi|^2\,dx\,ds; \\
&I_3 \le C_{11} \int_0^t \int_\Omega |(\sigma(x,v_n,w_n) - \sigma(x,w,v))(s)|^2 |\nabla\varphi(s)|^2\,dx\,ds\,.
\end{aligned}$$

Using (3.23) and the fact that (w_n, v_n) is contained in a relative compact set of B_R, which implies that, at least for a subsequence of n, which we do not relable for the reader's convenience, $(v_n, w_n) \to (v, w)$ a.e. in Q_T, we get, by means of the Lebesgue theorem, that $I_3 \to 0$. Next, using (3.22), we deduce

$$\int_\Omega \sigma(x,v_n,w_n)\nabla\varphi_n\nabla(\varphi_n - \varphi)\,dx = \int_\Omega \sigma(x,v,w)\nabla\varphi\nabla(\varphi_n - \varphi)\,dx$$

and

$$\int_\Omega \sigma(x,v_n,w_n)|\nabla(\varphi_n - \varphi)|^2\,dx = \int_\Omega (\sigma(x,v,w) - \sigma(x,v_n,w_n))\nabla\varphi\nabla(\varphi_n - \varphi)\,dx,$$

which entails, together with Hyp. 3.1 (ii),

$$I_1 \le C_{12} \int_0^t \int_\Omega |(\sigma(x,v,w) - \sigma(x,v_n,w_n))(s)|^2 |\nabla\varphi(s)|^2\,dx\,ds \to 0\,. \tag{3.32}$$

By means of Poincaré inequality, this implies (cf. (3.1))

$$\int_0^t \|(\varphi_n - \varphi)(s)\|_2^2\,ds \to 0$$

and so, up to a subsequence of $n \nearrow \infty$, $\varphi_n \to \varphi$ a.e. in Q_T. Then, the Lebesgue theorem gives the desired convergence $I_2 \to 0$. Finally, we can treat the last integral on the right hand side of (3.29) as follows

$$\int_0^t\int_\Omega(\kappa(x,v,w)-\kappa(x,v_n,w_n))\nabla\theta\nabla(\theta-\theta_n)\,dx\,ds \le \frac{1}{2}\int_0^t \|\theta_n-\theta\|^2_{H^1(\Omega)}\,ds$$
$$+C\int_0^t \|(\kappa(x,v_n,w_n)-\kappa(x,v,w))|\nabla\theta|\|_2^2\,ds$$
$$\le \frac{1}{2}\int_0^t \|\theta_n-\theta\|^2_{H^1(\Omega)}\,ds$$
$$+C\int_0^t\int_\Omega |\kappa(x,v_n,w_n)-\kappa(x,v,w)|^2|\nabla\theta|^2\,dx\,ds$$

and the last integral tends to zero as $n\nearrow\infty$ because $|\nabla\theta|\in L^1(Q_T)$ and $|\kappa(x,v_n,w_n)-\kappa(x,v,w)|^2\to 0$ a.e. because $(v_n,w_n)\to(v,w)$ a.e. in Q_T and we obtain the result applying the Lebesgue theorem. Collecting estimates (3.29–3.32), using a standard Gronwall lemma (cf. [6, Lemma A4, p. 156]), we get the desired convergence $(\theta_n,\chi_n)\to(\theta,\chi)$ in $(L^2(Q_T))^2$. According to Theorems 3.5 and 3.6 the limit is independent of the extracted subsequences, hence the convergence holds for the whole sequence (θ_n,χ_n). This completes the proof of Theorem 3.3.

3.2.2 Proof of Theorem 3.4

In this subsection, we proceed proving Theorem 3.4.

Proof of the regularity results (3.12–3.14). Using [19, Lemma 3.9], we immediately get $\varphi\in L^\infty(0,T;W^{1,q}_D(\Omega))$ for some $q\in(2,4)$, which is exactly the desired regularity property (3.13). Then, applying the maximal regularity results for parabolic equations to (3.7), we obtain (3.12), while (3.14) just follows from (3.5) by applying e.g. [1, Ch. III, Thm. 4.10.2] (cf. also [19, Rem. 3.15, Lemma 3.17, pp. 8,9]).

Proof of uniqueness. We continue now proving uniqueness of solutions and the Lipschitz continuous dependence of the solutions from the data. In order to perform this estimate, we need the following inequalities. The first one is a particular case of the well-known Gagliardo-Nirenberg inequality (cf. [24, p. 125]), which, in dimension $N=2$, reads as:

$$\|w\|_{2q/(q-2)}\le C_{GL}\|w\|_2^{1-(2/q)}\left(\|w\|_2^2+\|\nabla w\|_2^2\right)^{1/q}\quad \forall w\in H^1(\Omega),\quad \forall q>2\,, \tag{3.33}$$

and for some positive constant C_{GL}, while the second one is the following Young inequality

$$ab\le\epsilon a^{q/2}+C_\epsilon b^{q/(q-2)}\quad\forall a,\,b,\,\epsilon>0\,. \tag{3.34}$$

Take now two solutions $(\theta_i,\varphi_i,\chi_i)$, $i=1,2$ of (3.7–3.11) in the sense of Def. 3.2, enjoying the regularity properties (3.12–3.14) (cf. Thm. 3.4), and corresponding to the data $u_i\in L^\infty(0,T;L^2(\Gamma_N))$, $\theta_0^i,\chi_0^i\in C^0(\bar\Omega)$ and to the same datum $\tilde\alpha$ in (3.7). Use the following notation

$$\bar\theta=\theta_1-\theta_2,\quad \bar\varphi=\varphi_1-\varphi_2,\quad \bar\chi=\chi_1-\chi_2\,.$$

Take the differences of equations (3.7–3.9) written for the two solutions and test them, respectively, by $\bar\theta$, $\bar\varphi$, $\bar\chi_t$, sum up the resulting equations and integrate the result between 0 and $t \in (0,T]$. Add to both sides the term

$$\frac{1}{2}\|\bar\chi(t)\|_2^2 \le \frac{1}{2}\|\chi_0^1-\chi_0^2\|_2^2+\int_0^t\int_\Omega |\bar\chi_t\bar\chi|\,dx\,ds \le \frac{1}{2}\|\chi_0^1-\chi_0^2\|_2^2+\frac{1}{4}\int_0^t \|\bar\chi_t\|_2^2\,ds+\int_0^t \|\bar\chi\|_2^2\,ds.$$

In this way, we get

$$\begin{aligned}(3.35)\quad &\frac{1}{2}\|\bar\theta(t)\|_2^2+\int_0^t\int_\Gamma \alpha\bar\theta^2\,dx\,ds+\int_0^t\int_\Omega \sigma(x,\theta_2,\chi_2)|\nabla\bar\varphi|^2\,dx\,ds+\frac{3}{4}\int_0^t \|\bar\chi_t\|_2^2\,ds\\ &+\frac{1}{2}\|\bar\chi(t)\|_{H^1(\Omega)}^2=\sum_{i=4}^{9} I_i+\frac{1}{2}\|\theta_0^1-\theta_0^2\|_2^2+\frac{1}{2}\|\chi_0^1-\chi_0^2\|_{H^1(\Omega)}^2+\int_0^t \|\bar\chi\|_2^2\,ds\end{aligned}$$

where the I_i's are estimated as follows. Using Hyp. 3.1 (ii) and (3.13), (3.33), (3.34), we have the following inequalities

$$\begin{aligned}(3.36)\quad I_4 :&= \int_0^t\int_\Omega (\sigma(x,\theta_1,\chi_1)-\sigma(x,\theta_2,\chi_2))\,|\nabla\varphi_1|^2\bar\theta\,dx\,ds\\ &\le C_{13}\int_0^t \left(\|\bar\theta\|_{2q/(q-2)}^2+\|\bar\chi\|_{2q/(q-2)}^2\right)\|\nabla\varphi_1\|_q^2\,ds\\ &\le C_{13}\int_0^t \|\bar\theta\|_2^{2(q-2)/q}\left(\|\bar\theta\|_2^2+\|\nabla\bar\theta\|_2^2\right)^{2/q}\,ds\\ &\quad+C_{13}\int_0^t \|\bar\chi\|_2^{2(q-2)/q}\left(\|\bar\chi\|_2^2+\|\nabla\bar\chi\|_2^2\right)^{2/q}\,ds\\ &\le \frac{\kappa_0}{6}\int_0^t \|\nabla\bar\theta\|_2^2\,ds+C_{14}\int_0^t\left(\|\bar\theta\|_2^2+\|\bar\chi\|_{H^1(\Omega)}^2\right)\,ds.\end{aligned}$$

Using Hyp. 3.1 (ii) and (3.13), (3.33), (3.34), we obtain, similarly to (3.36),

$$\begin{aligned}(3.37)\quad I_5 &:= \int_0^t\int_\Omega \sigma(x,\theta_2,\chi_2)\left(|\nabla\varphi_1|^2-|\nabla\varphi_2|^2\right)\bar\theta\,dx\,ds\\ &\le \frac{\sigma_0}{6}\int_0^t \|\nabla\bar\varphi\|_2^2\,ds+C_{15}\int_0^t \|\bar\theta\|_{2q/(q-2)}^2\,ds\\ &\le \frac{\sigma_0}{6}\int_0^t \|\nabla\bar\varphi\|_2^2\,ds+\frac{\kappa_0}{6}\int_0^t \|\nabla\bar\theta\|_2^2\,ds+C_{16}\int_0^t \|\bar\theta\|_2^2\,ds,\\ (3.38)\quad I_6 &:= \int_0^t\int_\Omega (\sigma(x,\theta_1,\chi_1)-\sigma(x,\theta_2,\chi_2))\,\nabla\varphi_1\nabla\bar\varphi\,dx\,ds\\ &\le C_{17}\int_0^t \left(\|\bar\theta\|_{2q/q-2}+\|\bar\chi\|_{2q/q-2}\right)\|\nabla\varphi_1\|_q\|\nabla\bar\varphi\|_2\,ds\\ &\le \frac{\sigma_0}{6}\int_0^t \|\nabla\bar\varphi\|_2^2\,ds+\frac{\kappa_0}{6}\int_0^t \|\nabla\bar\theta\|_2^2\,ds+C_{18}\int_0^t\left(\|\bar\theta\|_2^2+\|\bar\chi\|_{H^1(\Omega)}^2\right)\,ds.\end{aligned}$$

Moreover, using Schwarz and Poincaré inequality together with boundary conditions (1.6) (cf. also (3.1)), we get

$$(3.39)\quad I_7 := \int_0^t \langle\tilde u_1-\tilde u_2,\bar\varphi\rangle\,ds \le \frac{\sigma_0}{6}\int_0^t \|\nabla\bar\varphi\|_2^2\,ds+C_{19}\|\tilde u_1-\tilde u_2\|_{L^2(0,t;H_D^{-1}(\Omega))}^2.$$

Invoking (3.14) together with assumption (3.15), we obtain

$$
(3.40)\quad I_8 := -\int_0^t \int_\Omega (W'(\chi_1) - W'(\chi_2))\bar{\chi}_t \, dx\, ds \le C_{20}\int_0^t \|\bar{\chi}\|_2 \|\bar{\chi}_t\|_2 \, ds \le \frac{1}{4}\int_0^t \|\bar{\chi}_t\|_2^2 \, ds + C_{21}\int_0^t \|\bar{\chi}\|_2^2 \, ds.
$$

Finally, the last integral in (3.35) can be estimated using Hyp. 3.1 (i), (3.12), (3.33), and (3.34) as follows

(3.41)

$$
\begin{aligned}
I_9 &:= -\int_0^t \int_\Omega (\kappa(x,\theta_1,\chi_1)\nabla\theta_1 - \kappa(x,\theta_2,\chi_2)\nabla\theta_2)\,\nabla(\theta_1-\theta_2)\, dx\, ds \\
&= -\int_0^t \int_\Omega \big(\kappa(x,\theta_1,\chi_1)\nabla\bar{\theta} + (\kappa(x,\theta_1,\chi_1) - \kappa(x,\theta_2,\chi_2))\nabla\theta_2\big)\,\nabla\bar{\theta}\, dx\, ds \\
&\le -\int_0^t \int_\Omega \kappa(x,\theta_1,\chi_1)|\nabla\bar{\theta}|^2 \, dx\, ds + \int_0^t \big(\|\bar{\theta}\|_{2q/q-2} + \|\bar{\chi}\|_{2q/q-2}\big)\, \|\nabla\theta_2\|_q \|\nabla\bar{\theta}\|_2 \, ds \\
&\le -\int_0^t \int_\Omega \kappa_0|\nabla\bar{\theta}|^2 \, dx\, ds + \frac{\kappa_0}{6}\int_0^t \|\nabla\bar{\theta}\|_2^2 \, ds \\
&\quad + C_{22}\int_0^t \big(\|\bar{\theta}\|_{2q/q-2}^2 + \|\bar{\chi}\|_{2q/q-2}^2\big)\, \|\nabla\theta_2\|_q^2 \, ds \\
&\le -\int_0^t \int_\Omega \kappa_0|\nabla\bar{\theta}|^2 \, dx\, ds + \frac{\kappa_0}{3}\int_0^t \|\nabla\bar{\theta}\|_2^2 \, ds + C_{23}\int_0^t (\|\bar{\theta}\|_2^2 + \|\bar{\chi}\|_{H^1(\Omega)}^2)\, ds\,.
\end{aligned}
$$

Collecting (3.35–3.41), using Hyp. 3.1 to get a lower bound for the second and the third integral in (3.35), and a standard Gronwall lemma (cf. [6, Lemma A4, p. 156]), we obtain the desired continuous dependence (of solutions with respect to data) estimate:

$$
\begin{aligned}
&\|\bar{\theta}(t)\|_2^2 + \int_0^t \|\bar{\theta}\|_{H^1(\Omega)}^2 \, ds + \int_0^t \|\nabla\bar{\varphi}\|_2^2 \, ds + \int_0^t \|\bar{\chi}_t\|_2^2 \, ds + \|\nabla\bar{\chi}(t)\|_2^2 \\
&\le C_{24}\left(\|\theta_0^1 - \theta_0^2\|_2^2 + \|\nabla(\chi_0^1 - \chi_0^2)\|_2^2 + \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,t;H_D^{-1}(\Omega))}^2\right)\,,
\end{aligned}
$$

entailing also uniqueness of solutions.

Proof of the regularity (3.18–3.19). We use the fact that the right hand side in (3.7) is bounded at least in $L^2(Q_T)$, assumptions (3.16), and apply, e.g., the regularity result [14, Lemma 3.3], getting the desired estimate

$$
|\theta|_{C^{0,\eta}([0,T];C^{0,\eta}(\overline{\Omega}))} \le C_{25}\,,
$$

for some $\eta \in (0,1)$. The same argument applies to the χ-component (satisfying (3.9)). Using assumptions (3.15–3.17), we get the same estimate on χ, i.e.

$$
|\chi|_{C^{0,\eta}([0,T];C^{0,\eta}(\overline{\Omega}))} \le C_{26}\,,
$$

for some $\eta \in (0,1)$. These are classical results: the case of the Dirichlet boundary conditions can be found in the monograph by Ladyzhenskaya et al. [22, Chapter V, Theorem 1.1], the proof adapted to the Neumann boundary conditions is given by DiBenedetto [13, Chapter III, Theorem 1.3 and Remark 1.1]. This concludes the proof of Theorem 3.4.

References

[1] H. Amann: Linear and quasilinear parabolic problems, Birkhäuser, Basel, 1995.

[2] S.N. Antonsev, M. Chipot: The thermistor problem: existence, smoothness, uniqueness, blowup, Siam. J. Math. Anal., **25** (1994), 1128–1156.

[3] D.R. Atthey: A Finite Difference Scheme for Melting Problems, IMA J. Appl. Math. 1974 **13** (1974), 353–366.

[4] M. Badii: Existence of periodic solutions for the quasi-static thermoelastic thermistor problem, NoDEA Nonlinear Differential Equations Appl. **16** (2009), 1–15.

[5] G. Bonfanti, F. Luterotti: Global solution to a phase-field model with memory and quadratic nonlinearity, Adv. Math. Sci. Appl., **9** (1999), 523–538.

[6] H. Brezis: Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematics Studies, no. 5., North-Holland Publishing Co., 1973.

[7] M. Brokate, J. Sprekels: Hysteresis and Phase Transitions, Appl. Math. Sci. 121 Springer, New York (1996).

[8] G. Caginalp: An analysis of a phase-field model of a free boundary, Arch. Rational Mech. Anal., **92** (1986), 205–245.

[9] J.W. Cahn, *On spinodal decomposition*, Acta Metall., **9** (1961), 795–801.

[10] J.W. Cahn, S.M. Allen, *A microscopic theory for domain wall motion and its sperimental verification in Fe-Al alloy domain growth kinetics*, Journal de Physique C7, **38** (1977), 51–54.

[11] J. Cahn, J. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., **28** (1958), 258–267.

[12] P. Colli, P. Krejčí, E. Rocca, J. Sprekels, A nonlocal multiphase transition system with non-constant specific heat and heat conductivity, work in progress (2010).

[13] E. DiBenedetto: Degenerate Parabolic Equations, Springer-Verlag Berlin Heidelberg New York, 1993.

[14] E. Feireisl, H. Petzeltovà, E. Rocca: Existence of solutions to some models of phase changes with microscopic movements, Math. Meth. Appl. Sci., **32** (2009), 1345?-1369.

[15] M. Frémond: Non-smooth thermomechanics, Springer-Verlag, Berlin, 2002.

[16] D. Gilbarg, N.S. Trudinger: Elliptic partial differential equations of second order, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[17] K. Gröger: A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations, Math. Ann., **283** (1989), 679–687.

[18] K. Gröger: $W^{1,p}$-estimates of solutions to evolution equations corresponding to nonsmooth second order elliptic differential operators, Nonlinear Analysis, **18** (1992), 569–577.

[19] D. Hömberg, C. Meyer, J. Rehberg, W. Ring: Optimal control for the thermistor problem, SIAM Journal on Control and Optimization, **48** (2010), 3449–3481.

[20] N. Kenmochi, M. Niezgódka: Evolution systems of nonlinear variational inequalities arising phase change problems, Nonlin. Anal., **22** (1994), 1163–1180.

[21] K.L. Kuttler, M. Shillor, J.R. Fernández, Existence for the thermoviscoelastic thermistor problem, Differ. Equ. Dyn. Syst., **16** (2008), 309–332.

[22] A. Ladyženskaja, O.A.A. Solonnikov, V.A.A. Uralceva: Lineinye i kvazilineinye uravneniya parabolicheskogo tipa I Izdat. "Nauka", Moscow, 1968.

[23] Ph. Laurençot: Long-time behaviour for a model of phase-field type, Proc. Roy. Soc. Edinburgh, **126A** (1996), 167–185.

[24] L. Nirenberg: On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa (3), **13** (1959), 115–162.

[25] O. Penrose, P.C. Fife, *Thermodynamically consistent models of phase field type for the kinetics of phase transitions*, Phys. D, **43** (1990), 44–62.

[26] P. Shi, M. Shillor, X. Xu: Existence of a solution to the Stefan problem with Joule's heating, J. Diff. Eqns, **105** (1993), 239–263.

[27] J. Simon: Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl., **146**, no. 4 (1987), 65–96.