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Noncompactness of Integral Operators Modeling Diffuse-Gray Radiation in Polyhedral and Transient Settings

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Abstract

While it is well-known that the standard integral operator K of (stationary) diffuse-gray radiation, as it occurs in the radiosity equation, is compact if the domain of radiative interaction is sufficiently regular, we show noncompactness of the operator if the domain is polyhedral. We also show that a stationary operator is never compact when reinterpreted in a transient setting. Moreover, we provide new proofs, which do not use the compactness of K, for 1 being a simple eigenvalue of K for connected enclosures, and for $I - (1 - \epsilon)K$ being invertible, provided the emissivity ϵ does not vanish identically.

1 Introduction

Accounting for diffuse-gray radiative heat transfer is important for the accurate modeling of processes that involve heat transfer at high temperatures through transparent or semi-transparent media with nonspecular surfaces, crystal growth from melt or vapor being two examples [DNR+90, KPS04]. The physical modeling of radiative heat transfer is well-understood [SC78, Mod93], where diffuse-gray radiative heat transfer between points on the surface Σ of a transparent cavity are described by the radiosity equation involving the linear integral operator K (see (1.1) and (1.2) below for details). In [Met99, LT01, Dru08, Dru10, Amo10a, Amo10b], the L^p theory of the radiosity equation is employed to couple the radiosity equation to the heat equation in both stationary and transient settings, developing the corresponding existence theory. More regular solutions of the radiosity equation are desirable in the context of optimal control [MPT06, MY09]. Under suitable hypotheses, one can obtain solutions in Sobolev spaces and in spaces of continuous functions (see [Han02] and references therein).

If Σ is sufficiently regular (e.g. at least $C^{1,\alpha}$), then K is known to be compact on $L^p(\Sigma)$ [Tii97b, Dru08]. In [LT01], the compactness of K is used to show that, if Σ is the surface of a connected enclosure, then the eigenvalue 1 of K is simple [LT01, Lem. 1(iv)] and $I - (1 - \epsilon)K$ is invertible for a nonvanishing emissivity ϵ [LT01, Lem. 2]. These results are then exploited to couple the radiosity equation to the heat equation and to develop the corresponding existence theory.

However, in applications, the involved domains often lack the regularity needed to obtain compactness of K. In Th. 2 below, we will show that K is *noncompact* for polyhedral domains. However, we will also show in Theorems 3 and 5 how to obtain the abovementioned properties of K without employing compactness.

In Section 3, we prove that K as well as other stationary bounded linear operators can never be compact if reinterpreted as time-dependent operators in a transient setting, which has sometimes been incorrectly assumed in the literature (cf. remark after the proof of Th. 8).

For the space domain $\Omega \subseteq \mathbb{R}^3$, we assume it consists of two parts Ω_s and Ω_g, where Ω_s represents an opaque solid and Ω_g represents a transparent gas. More precisely, we assume

(A-1) $\overline{\Omega} = \overline{\Omega}_s \cup \overline{\Omega}_g$, $\Omega_s \cap \Omega_g = \emptyset$, and each of the sets Ω, Ω_s, Ω_g, is a nonvoid, bounded, open subset of $\mathbb{R}^3$ such that the interface surface $\Sigma := \overline{\Omega}_s \cap \overline{\Omega}_g$ is Lipschitz and piecewise C^1, i.e. Σ can be partitioned into finitely many C^1-surfaces.

(A-2) Ω_g is enclosed by Ω_s, i.e. $\Sigma = \partial\Omega_g$ (see Fig. 1).

We will state additional hypotheses on the domains where needed.

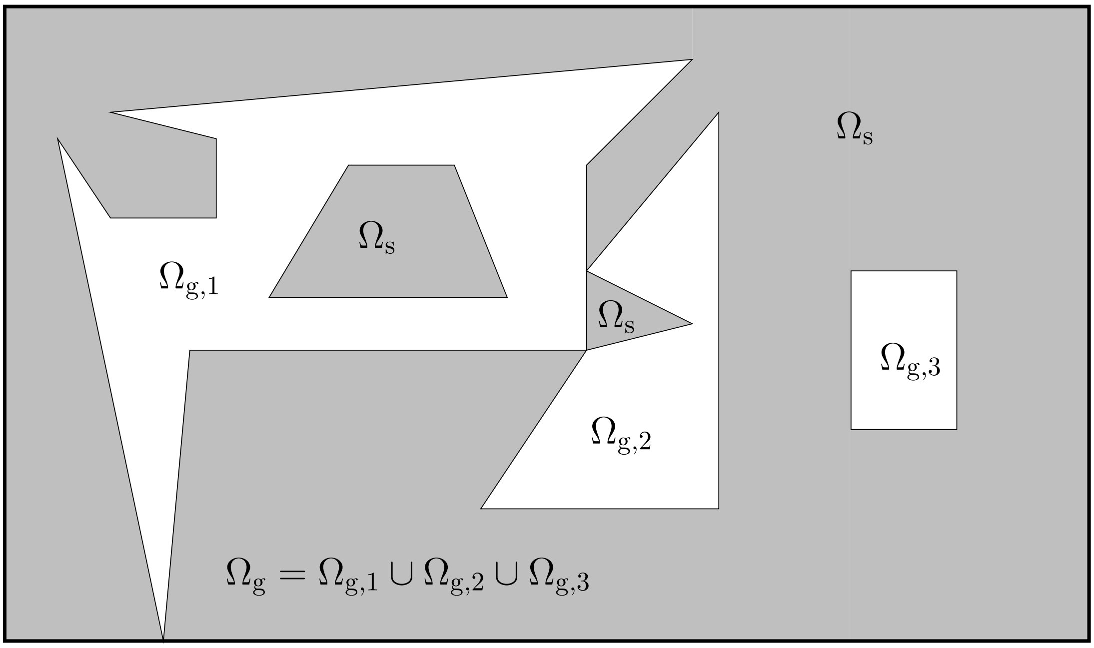

Figure 1: Possible shape of a 2-dimensional section through the 3-dimensional domain $\overline{\Omega} = \overline{\Omega}_s \cup \overline{\Omega}_g$. Here, Ω_g has the 3 connected components $\Omega_{g,1}$, $\Omega_{g,2}$, $\Omega_{g,3}$. The picture illustrates that $\Sigma = \partial\Omega_g$ can have fewer connected components than Ω_g. Note also that, according to (A-2), Ω_g is engulfed by Ω_s, which can not be seen in the 2-dimensional section.

The abovementioned radiosity equation modeling diffuse-gray radiative heat transfer between points on the surface Σ reads

$$\big(I - (1-\epsilon)K\big)(R) = \epsilon\sigma\theta^4, \tag{1.1}$$

where I denotes the identity operator, θ represents absolute temperature, ϵ represents the emissivity of the solid, $\sigma \in \mathbb{R}^+$ represents the Boltzmann radiation constant, $R =$

$R(\theta)$ represents radiosity, and K is the linear integral operator defined by

$$K(\rho)(x) := \int_\Sigma V(x,y)\,\omega(x,y)\,\rho(y)\,\mathrm{d}y \quad \text{for a.e. } x \in \Sigma, \tag{1.2}$$

$$\omega(x,y) := \frac{\big(\mathbf{n}(y)\cdot(x-y)\big)\,\big(\mathbf{n}(x)\cdot(y-x)\big)}{\pi\big((y-x)\cdot(y-x)\big)^2} \quad \text{for a.e. } (x,y) \in \Sigma\times\Sigma, \tag{1.3}$$

$$V(x,y) := \begin{cases} 0 & \Sigma\cap]x,y[\neq\emptyset, \\ 1 & \Sigma\cap]x,y[=\emptyset \end{cases} \quad \text{for } (x,y)\in\Sigma\times\Sigma, \tag{1.4}$$

ω denoting the view factor, V denoting the visibility factor (being 1 if, and only if, x and y are mutually visible), and $\mathbf{n}$ denoting the outer unit normal to the solid domain Ω_s, existing almost everywhere on the assumed Lipschitz boundary.

Theorem 1. *Assume* (A-1) *and* (A-2)*.*

(a) *The kernel $V\omega$ of K is almost everywhere nonnegative (actually positive for $V(x,y) = 1$), symmetric, and*

$$\int_\Sigma V(z,y)\,\omega(z,y)\,\mathrm{d}y = 1 \quad \textit{for a.e. } z\in\Sigma. \tag{1.5}$$

Moreover, if Ω_s and Ω_g are polyhedral, then (1.5) *actually holds* for every $z\in\Sigma$, *where one can choose each of the finitely many possible values of $\mathbf{n}(z)$ if z belongs to more than one face of Σ.*

(b) *For each $1\le p\le\infty$, the operator $K : L^p(\Sigma) \longrightarrow L^p(\Sigma)$ given by* (1.2) *is well-defined, linear, bounded, and positive.*

Proof. See [Tii97a, Lem. 1] and [Tii97b, Lem. 2]. ■

2 Noncompactness of K for Polyhedral Domains

Theorem 2. *Let $p\in[1,\infty]$ and assume* (A-1) *and* (A-2)*. If Ω_s and Ω_g are polyhedral, then $K : L^p(\Sigma)\longrightarrow L^p(\Sigma)$ is not compact.*

Proof. For the sake of readability and briefness, we present the proof for $\Omega :=]-3,3[^3$, $\Omega_g :=]0,2[\times]-1,1[\times]0,2[$, $\Omega_s := \Omega\setminus\overline{\Omega}_g$, and leave the adaptation to general polyhedral domains to the reader. For each $k\in\mathbb{N}$, we define the following subsets A_k^+ and A_k^- of $\Sigma = \partial\Omega_g$ (see Fig. 2):

$$\begin{aligned} A_k^+ &:= \{0\}\times[-1/(2k),1/(2k)]\times[1/(2k),1/k], \\ A_k^- &:= [1/(2k),1/k]\times[-1/(2k),1/(2k)]\times\{0\}. \end{aligned} \tag{2.1}$$

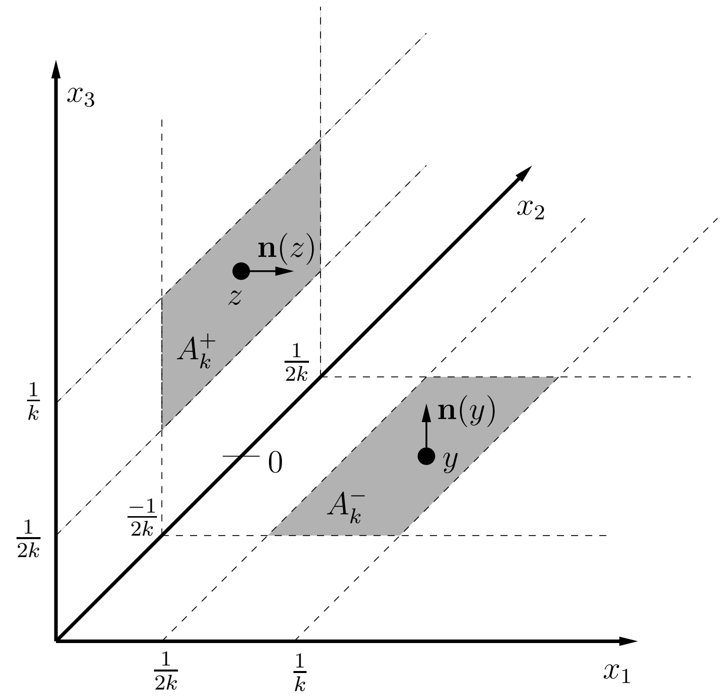

Figure 2: Illustrating the construction for the proof of Th. 2.

For each $(z, y) \in A_k^+ \times A_k^-$, we obtain

$$\mathbf{n}(z) \cdot (y - z) = y_1 \geq \frac{1}{2k}, \quad \mathbf{n}(y) \cdot (z - y) = z_3 \geq \frac{1}{2k}. \tag{2.2}$$

Since, on the other hand, $\|z - y\|_2 \leq \frac{\sqrt{3}}{k}$, we estimate

$$\omega(z, y) = \frac{\mathbf{n}(z) \cdot (y - z)\, \mathbf{n}(y) \cdot (z - y)}{\pi\, \|z - y\|_2^4} \geq \frac{k^2}{36\, \pi} \quad \text{for each } (z, y) \in A_k^+ \times A_k^-. \tag{2.3}$$

Fix $1 \leq p < \infty$, and define, for each $k \in \mathbb{N}$,

$$f_k : \Sigma \longrightarrow \mathbb{R}, \quad f_k(y) := k^{2/p}\, \chi_{A_k^-}(y). \tag{2.4}$$

Then $\|f_k\|_{L^p(\Sigma)}^p = \frac{1}{2}$ and $\{f_k : k \in \mathbb{N}\}$ is bounded. Clearly, we also have pointwise convergence $f_k(y) \to 0$ for each $y \in \Sigma$. In particular, if a subsequence of f_k were to converge to f in $L^p(\Sigma)$, then $f = 0$. However, on the other hand, for each $z \in A_k^+$:

$$(K(f_k))(z) = \int_\Sigma \omega(z, y)\, f_k(y)\, \mathrm{d}y \geq \frac{k^{2/p}\, k^2 \text{ meas}(A_k^-)}{36\, \pi} = \frac{k^{2/p}}{72\, \pi}. \tag{2.5}$$

In consequence,

$$\|K(f_k)\|_{L^p(\Sigma)}^p \geq \frac{k^2}{72^p\, \pi^p} \text{ meas}(A_k^+) = \frac{1}{2 \cdot 72^p\, \pi^p} > 0, \tag{2.6}$$

showing that $K(f_k)$ does not converge to 0 in $L^p(\Sigma)$. For $1 < p < \infty$, the pointwise convergence $f_k \to 0$ implies weak convergence $f_k \rightharpoonup 0$ in $L^p(\Sigma)$ and $K(f_k) \not\to 0$ shows that K is not compact. For $p = 1$, consider the sequence $g_k := f_{2^k}$. Since, for $k \neq l$, $\mathrm{int}(A^-_{2^k})$ and $\mathrm{int}(A^-_{2^l})$ are disjoint, for each $z \in A^+_{2^k} \cup A^+_{2^l}$:

$$\big|(K(g_k))(z) - (K(g_l))(z)\big| = \int_{A^+_{2^k}} \omega(z,y)\, g_k(y)\,\mathrm{d}y + \int_{A^+_{2^l}} \omega(z,y)\, g_l(y)\,\mathrm{d}y \geq \frac{(2^k+2^l)^2}{72\,\pi}. \tag{2.7}$$

Since $\mathrm{meas}(A^+_{2^k} \cup A^+_{2^l}) = \frac{1}{2}\,(1/2^{2k} + 1/2^{2l})$, this shows, analogous to (2.6), that $\|K(g_k) - K(g_l)\|_{L^1(\Sigma)} \geq \frac{1}{144\,\pi} > 0$ for $l \neq k$ and that $K\{g_k : k \in \mathbb{N}\}$ is closed, but not compact, hence K is not compact. Finally, for $p = \infty$, consider the sequence $h_k := \chi_{A^-_{2^k}}$. Then $B := \{h_k : k \in \mathbb{N}\}$ is bounded in $L^\infty(\Sigma)$ and the disjointness of the $\mathrm{int}(A^-_{2^k})$ implies that $K(B)$ is closed, but not compact. ∎

The following result was essentially proved as Lemma 1(iv) of [LT01] for the case where K is compact. We provide a direct proof that does not hinge on the compactness of K.

Theorem 3. *Assume* (A-1) *and* (A-2). *If $\rho \in L^1(\Sigma)$ and $K(\rho) = \rho$, then ρ is constant on each Σ_k, where $\Sigma_k := \partial\Omega_{\mathrm{g},k}$ is the boundary of the connected component $\Omega_{\mathrm{g},k}$ of Ω_{g} (cf. Fig. 1). In particular, if Ω_{g} is connected, then the eigenvalue 1 of $K : L^p(\Sigma) \longrightarrow L^p(\Sigma)$ is simple ($p \in [1,\infty]$, note $L^p(\Sigma) \subseteq L^1(\Sigma)$).*

Proof. Since $V(y,z) = 0$ if y, z lie in different Σ_k, we may assume Ω_{g} is connected without loss of generality. We first show f takes only one sign on Σ. Introducing the sets $\Sigma^+ := \{z \in \Sigma : f(z) \geq 0\}$ and $\Sigma^- := \{z \in \Sigma : f(z) < 0\}$, for $z \in \Sigma$:

$$f(z) = K(f)(z) = \int_{\Sigma^+} V(z,y)\,\omega(z,y)\,f(y)\,\mathrm{d}y + \int_{\Sigma^-} V(z,y)\,\omega(z,y)\,f(y)\,\mathrm{d}y\,. \tag{2.8}$$

After integrating over Σ^+:

$$\begin{aligned}\int_{\Sigma^+} f(z)\,\mathrm{d}z &= \int_{\Sigma^+} f(z) \left(\int_{\Sigma^+} V(z,y)\,\omega(z,y)\,\mathrm{d}y\right) \mathrm{d}z \\ &\quad + \int_{\Sigma^-} f(z) \left(\int_{\Sigma^+} V(z,y)\,\omega(z,y)\,\mathrm{d}y\right) \mathrm{d}z \\ &\leq \int_{\Sigma^+} f(z)\,\mathrm{d}z + \int_{\Sigma^-} f(z) \left(\int_{\Sigma^+} V(z,y)\,\omega(z,y)\,\mathrm{d}y\right) \mathrm{d}z\,. \end{aligned} \tag{2.9}$$

From the definition of Σ^-, we obtain

$$\int_{\Sigma^+} V(z,y)\,\omega(z,y)\,\mathrm{d}y = 0 \quad \text{for a.e. } z \in \Sigma^-.$$

From Th. 1(a), we conclude $V(z,y) = 0$ for almost every $z \in \Sigma^-$ and almost every $y \in \Sigma^+$ (Σ^- and Σ^+ are mutually invisible), implying $\mathrm{meas}(\Sigma^-) = 0$ or $\mathrm{meas}(\Sigma^+) = 0$ due to the assumed connectedness of Ω_{g}. If we now let $M := \mathrm{meas}(\Sigma)^{-1} \int_\Sigma f$ denote the mean of f and $\tilde{f} := f - M$, then

$$K(\tilde{f}) = K(f) - K(M) \overset{(1.5)}{=} f - M = \tilde{f}, \tag{2.10}$$

and we know $\tilde{f}$ takes only one sign on Σ, i.e. f is constant and equal to M. ∎

One is usually interested in solving the radiosity equation to obtain R as a function of θ, for example to formulate the coupled conductive-radiative heat flux through Σ, with θ remaining as the only unknown quantity. It is thus desirable to establish the invertibility of $I-(1-\epsilon)K$. It was proved for compact K in [LT01, Lem. 2]. In the following Th. 5, we present a proof that works for polyhedral domains, where we know that compactness is not available. We start by establishing injectivity merely using (A-1) and (A-2).

Lemma 4. *Let $p \in [1,\infty]$, and assume* (A-1), (A-2). *If $\epsilon \in L^\infty(\Sigma)$ with values in $[0,1]$ is such that, for each connected component $\Omega_{g,k}$ of Ω_g (cf. Fig. 1), there exists $M_k \subseteq \Sigma_k := \partial\Omega_{g,k}$ such that M_k has positive surface measure and $\epsilon > 0$ on M_k, then the operator*

$$\big(I-(1-\epsilon)K\big) : L^p(\Sigma) \longrightarrow L^p(\Sigma) \tag{2.11}$$

is injective on $L^p(\Sigma)$.

Proof. As $L^p(\Sigma) \subseteq L^1(\Sigma)$, it suffices to consider $p=1$. Let $f \in L^1(\Sigma)$. From

$$f = (1-\epsilon)\,K(f), \tag{2.12}$$

we trivially conclude that f vanishes in the set $\Sigma' := \{z \in \Sigma : \epsilon(z) = 1\}$. We extend this conclusion to $\Sigma^* := \{z \in \Sigma : \epsilon(z) > 0\}$ by noting

$$\int_{\Sigma\setminus\Sigma'} \left|\frac{f}{1-\epsilon}\right| = \|K(f)\|_{L^1(\Sigma\setminus\Sigma')} \leq \|f\|_{L^1(\Sigma\setminus\Sigma')}. \tag{2.13}$$

Thus, letting $\Sigma_k^* := \Sigma^* \cap \Sigma_k$, we obtain $(1-\epsilon)(K_k^*)(f) = f$, where

$$K_k^* \in \mathcal{L}\big(L^1(\Sigma_k\setminus\Sigma_k^*), L^1(\Sigma_k\setminus\Sigma_k^*)\big), \quad K_k^*(\rho)(x) := \int_{\Sigma_k\setminus\Sigma_k^*} V(x,y)\,\omega(x,y)\,\rho(y)\,\mathrm{d}y\,. \tag{2.14}$$

Moreover, it is $M_k \subseteq \Sigma_k^*$, i.e. the hypothesis $\operatorname{meas}(M_k) > 0$ implies $\|K_k^*\| < 1$ via Hölder's inequality (cf. [Tii97b, Lem. 2], $\Sigma_k \setminus \Sigma_k^*$ is not an enclosure). In consequence, $(1-\epsilon)(K_k^*)(f) = f$ implies $f = 0$ a.e. on Σ_k, concluding the proof. ∎

For the proof that $I-(1-\epsilon)K$ is also surjective, we make use of the following technical condition (A-3). In Lem. 6 we will show that Ω_s and Ω_g being polyhedral is sufficient for (A-3) to hold.

(A-3) There exists $r_0 > 0$ such that

$$\operatorname*{ess\,sup}_{z\in\Sigma} \int_{B_{r_0}(z)} V(z,y)\,\omega(z,y)\,\mathrm{d}y \,<1, \tag{2.15}$$

where $B_{r_0}(z) := \{y \in \Sigma : \|z-y\|_2 < r_0\}$.

Theorem 5. *Assume the hypotheses of Lem. 4 plus* (A-3)*. Then $I-(1-\epsilon)K$ has an inverse in $\mathcal{L}(L^p(\Sigma),L^p(\Sigma))$.*

Proof. We have to show that, for each $g \in L^p(\Sigma)$, the equation

$$\big(I-(1-\epsilon)K\big)(f) = g \tag{2.16}$$

has a unique solution $f \in L^p(\Sigma)$. Let $r_0 > 0$ be such that (2.15) holds and define the auxiliary operators

$$(K_1(f))(z) := \int_{B_{r_0}(z)} V(z,y)\,\omega(z,y)\,f(y)\,\mathrm{d}y\,, \tag{2.17a}$$

$$(K_2(f))(z) := \int_{\Sigma\setminus B_{r_0}(z)} V(z,y)\,\omega(z,y)\,f(y)\,\mathrm{d}y\,. \tag{2.17b}$$

From K, both K_1 and K_2 inherit the property of being bounded linear operators from $L^p(\Sigma)$ into itself. From Hölder's inequality and (2.15), we obtain $\|K_1\|_{\mathcal{L}(L^p(\Sigma),L^p(\Sigma))} < 1$, i.e. $I-(1-\epsilon)K_1$ is invertible via the Neumann series in $\mathcal{L}(L^p(\Sigma),L^p(\Sigma))$. As $K = K_1+K_2$, applying the inverse $\big(I-(1-\epsilon)K_1\big)^{-1}$ to (2.16) yields the equivalent equation

$$\Big(I-\big(I-(1-\epsilon)K_1\big)^{-1}(1-\epsilon)K_2\Big)(f) = \big(I-(1-\epsilon)K_1\big)^{-1}(g). \tag{2.18}$$

With the abbreviation $H := \big(I-(1-\epsilon)K_1\big)^{-1}(1-\epsilon)K_2$, we write the left-hand side of (2.18) as $I-H$. If we can show that H is compact and $I-H$ is injective, then $I-H$ is invertible by the Riesz-Schauder theorem and we are done. The integral operator K_2 is compact, as its kernel

$$k_2(z,y) = \chi_{\Sigma\setminus B_{r_0}(z)}(y)V(z,y)\,\omega(z,y) \tag{2.19}$$

is uniformly bounded by $1/(\pi r_0^2)$. In consequence, H is also compact. It remains to prove that $I-H$ is one-to-one, i.e. that 0 is the only solution to the homogeneous version of (2.18). As (2.18) and (2.16) are equivalent, Lem. 4 completes the proof. ■

Lemma 6. *Assuming* (A-1), (A-2), *and that $\Omega_\mathrm{s}, \Omega_\mathrm{g}$ are polyhedral is sufficient for* (A-3) *to hold.*

Proof. Seeking a contradiction, assume there does not exist $r_0 > 0$ such that (2.15) holds. Then there is a sequence (z_n) in Σ such that $\int_{B_{1/n}(z_n)} V(z_n,y)\,\omega(z_n,y)\,\mathrm{d}y \ge 1-1/n$ for each $n \in \mathbb{N}$. Thus, according to (1.5):

$$\int_{\Sigma\setminus B_{1/n}(z_n)} V(z_n,y)\,\omega(z_n,y)\,\mathrm{d}y \le 1/n. \tag{2.20}$$

Fatou's lemma now implies

$$\liminf_{n\to\infty} V(z_n,y)\,\omega(z_n,y) = \liminf_{n\to\infty} \chi_{\Sigma\setminus B_{1/n}(z_n)}(y)V(z_n,y)\,\omega(z_n,y) = 0 \quad \text{for a.e. } y \in \Sigma. \tag{2.21}$$

As Ω_s, Ω_g are polyhedral, the outer unit normal $\mathbf{n}(z)$ takes only finitely many values on Σ. As Σ is also compact, there must exist $z^* \in \Sigma$ and ξ in the range of $\mathbf{n}$ such that, for a subsequence (not relabelled), $z_n \to z^*$ and $\mathbf{n}(z_n) \to \xi$. Thus, for almost every $y \in \Sigma$ such that $V(z^*, y) = 1$, we obtain

$$\xi \cdot (y - z^*)\, \mathbf{n}(y) \cdot (z^* - y) = \liminf_{n\to\infty} \mathbf{n}(z_n) \cdot (y - z_n)\, \mathbf{n}(y) \cdot (z_n - y) = 0. \tag{2.22}$$

However, (2.22) is in contradiction to $\int_\Sigma V(z^*, y)\, \omega(z^*, y)\, \mathrm{d}y = 1$, which must hold according to the polyhedral case of Th. 1(a) (z^* might lie in the intersection of several faces of Σ, but $\mathbf{n}(z_n) \to \xi$ guarantees that ξ is the outer unit normal of one of these faces). ■

3 Noncompactness of K for Transient Settings

Theorem 7. *For each $1 \le p < \infty$, the operator*

$$\tilde{K} : L^p(0, T, L^p(\Sigma)) \longrightarrow L^p(0, T, L^p(\Sigma)), \quad (\tilde{K}(\rho))(t) := K(\rho(t)), \tag{3.1}$$

is noncompact.

Actually, Th. 7 is a corollary of the following general result that shows reinterpreting a nontrivial bounded linear operator $K : X \longrightarrow Y$ between normed vector spaces X and Y in a transient setting can *never* result in a compact operator $\tilde{K} : L^p(0, T, X) \longrightarrow L^p(0, T, Y)$:

Theorem 8. *Let X and Y be normed vector spaces, and let $K : X \longrightarrow Y$ be a bounded linear operator. Then, for each $1 \le p < \infty$,*

$$\tilde{K} : L^p(0, T, X) \longrightarrow L^p(0, T, Y), \quad (\tilde{K}(\rho))(t) := K(\rho(t)), \tag{3.2}$$

defines a bounded linear operator. If there is $x_0 \in X$ such that $K(x_0) \neq 0$, then $\tilde{K}$ is noncompact.

Proof. The continuity of K implies $\tilde{K}$ preserves Bochner measurability; the inequality

$$\|\tilde{K}(f)\|^p_{L^p(0,T,Y)} = \int_0^T \|\tilde{K}(f)(t)\|^p_Y \, \mathrm{d}t \le \|K\|^p \, \|f\|^p_{L^p(0,T,X)} \tag{3.3}$$

shows that $\tilde{K}$ maps $L^p(0, T, X)$ into $L^p(0, T, Y)$ and is bounded.

Now assume there is $x_0 \in X$ such that $K(x_0) \neq 0$, let $y_0 := K(x_0)$, $\delta := \|x_0\|_X \in \mathbb{R}^+$, and $\epsilon := \|y_0\|_Y \in \mathbb{R}^+$. Fix $p \in [1, \infty[$. For each $n \in \mathbb{N}$, let

$$I_n :=]T\, 2^{-n}, T\, 2^{-n+1}[\tag{3.4}$$

and define $f_n \in \mathcal{S}(0, T, X)$ as follows:

$$f_n : [0, T] \longrightarrow X, \quad f_n(t) := 2^{\frac{n}{p}}\, T^{\frac{-1}{p}}\, x_0\, \chi_{I_n}(t). \tag{3.5}$$

Note that $(I_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint measurable subsets of $[0,T]$ such that $\mathrm{meas}(I_n) = 2^{-n}\,T$. Thus, for each $n \in \mathbb{N}$,

$$\|f_n\|^p_{L^p(0,T,X)} = \int_{I_n} \|f_n(t)\|^p_X \,\mathrm{d}t \;= 2^n\,T^{-1}\,\|x_0\|^p_X\;\mathrm{meas}(I_n) = \delta^p, \tag{3.6}$$

showing that the set $B := \{f_n : n \in \mathbb{N}\}$ is bounded. Next, it will be shown that $\tilde{K}[B]$ is closed and noncompact. To that end, for $m \neq n$, one computes

$$\begin{aligned}\big\|\tilde{K}(f_n) - \tilde{K}(f_m)\big\|^p_{L^p(0,T,Y)} &= \int_0^T \big\|K\big(f_n(t)\big) - K\big(f_m(t)\big)\big\|^p_Y \,\mathrm{d}t \\ = \int_{I_m} \Big\|2^{\frac{m}{p}}\,T^{\frac{-1}{p}}\,y_0\Big\|^p_Y \,\mathrm{d}t \;+ \int_{I_n} \Big\|2^{\frac{n}{p}}\,T^{\frac{-1}{p}}\,y_0\Big\|^p_Y \,\mathrm{d}t \;&= 2\,\epsilon^p,\end{aligned} \tag{3.7}$$

i.e. the distance between any two distinct elements of $\tilde{K}[B]$ is identical and positive, implying that $\tilde{K}[B]$ is noncompact and closed, showing $\tilde{K}$ is noncompact. ■

In [LT01, Lem. 11], it is incorrectly assumed that stationary compact linear operators are compact when reinterpreted as time-dependent operators in a transient setting. [LT01, Lem. 11] claims the pseudomonotonicity of a certain transient operator, which is then used to prove existence to a transient heat equation with radiative coupling. For a different existence proof not founding on [LT01, Lem. 11], see [Dru10].

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